

CHAPTER ONE

Beginnings

THE EARLY DAYS

Many findings in archaeology bear witness to some math in the mind of our ancestors. There are many scholarly books on that matter, but we may be content with a few examples. A bone rod, which was discovered in 1937 in Moravia, shows 55 notches in groups of five and is about 30,000 years old. Paintings on cave walls and many engraved objects show various forms of geometric design, more and more sophisticated as one approaches the beginning of the Neolithic period (12,000 years ago, when agriculture began).

Mesopotamian, Egyptian, Indian, Chinese, and Old American civilizations already knew as much math as is taught in the first grades of our high schools. Basic arithmetic, including the four operations, is carefully described in Babylonian tablets and Egyptian papyri. The many exercises accompanying the descriptions show that mathematics was then an empirical science: the examples are very practical, showing, for instance, how one can divide a herd into so many equal parts (quite useful in a case of inheritance or when sharing some plunder). Prime numbers then begin to appear. Mesopotamian, Egyptian, and Chinese scribes knew much geometry. Figures 1.1–1.3 give examples of their knowledge. Figure 1.1 shows how the Egyptians computed the area of a

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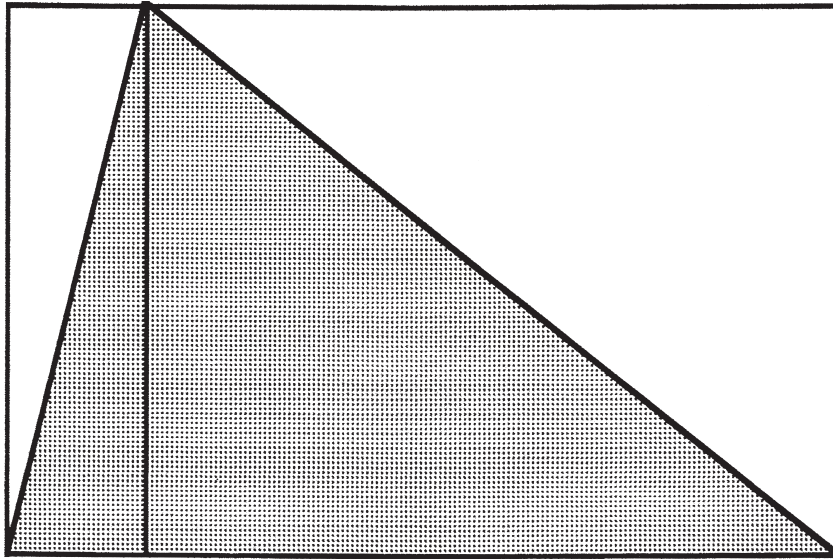
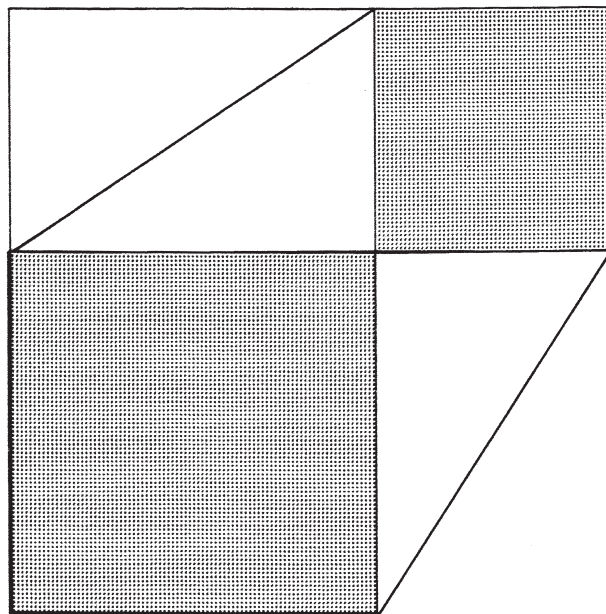
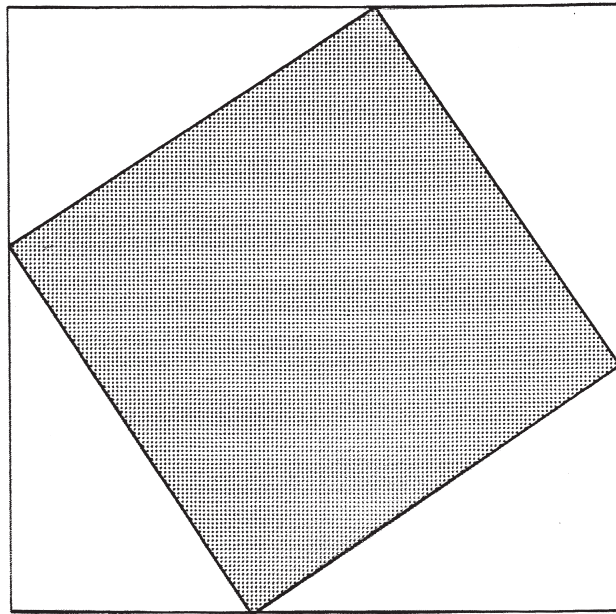


Figure 1.1. How the early Egyptians computed a triangle area. This drawing shows that the triangle area is half the area of the rectangle built on its base and height.

triangle. Herodotus, the Greek historian, explains that Pharaoh's administration taxed land on the basis of acreage, and one sees how practical this kind of recipe could be. Babylonians knew the so-called Pythagorean theorem (involving the sides of a rectangular triangle), probably on the ground of some drawing like figure 1.2 (figure 1.3, which is similar, is from a later Chinese source). One could also mention good approximations of π , the volume of a pyramid, the solution of second-degree algebraic equations, and a few other items, but they would not add much to the basic statement: much mathematics was known early and always for practical reasons with empirical means.

Figure 1.2. A drawing yielding the Pythagorean theorem. By moving triangles, one concludes that the shaded square area in (a) is the sum of the two square shaded areas in (b).



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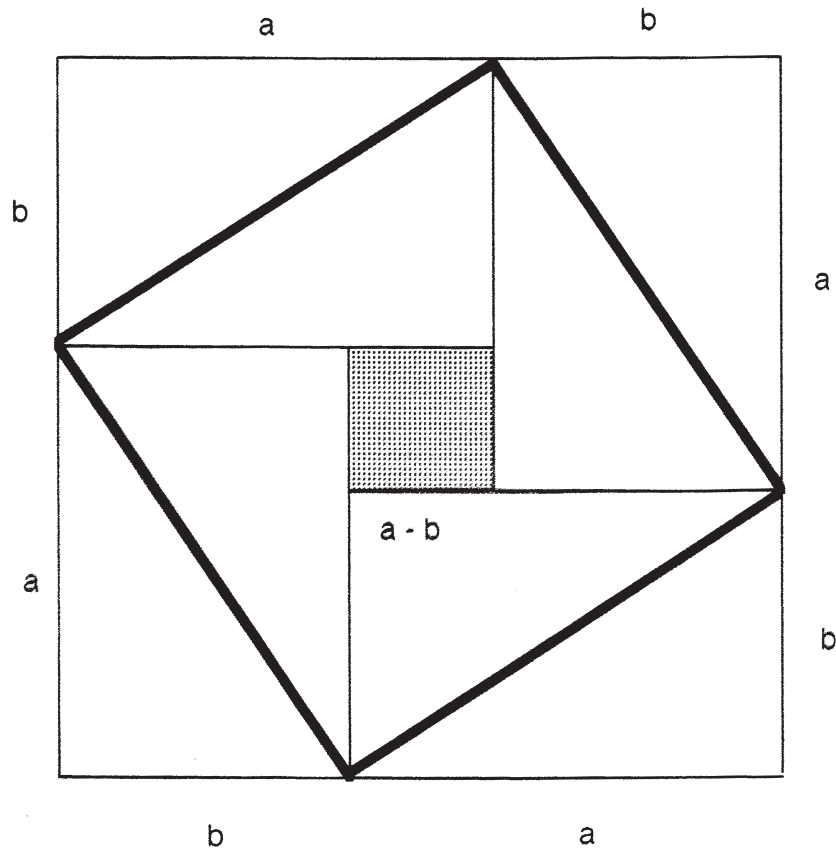


Figure 1.3. A Chinese version of the Pythagorean theorem (around 200 B.C.). If a and b denote the sides of a right triangle, the boundary square has a side $(a+b)$ and the smallest shaded one a side $(a-b)$. The side c of the blank triangle is obviously such that $c^2 = (a-b)^2 + 4(ab/2) = a^2 + b^2$.

THE BEGINNINGS OF GREEK MATHEMATICS

Babylonian and Egyptian texts show that the early mathematicians were working on bookkeeping, agronomy, architecture, or astronomy, but one does not know what dreams they entertained around their art. Greek mathematics begins, on the contrary, with the flamboyant figure of Pythagoras (c. 580–500 B.C.).

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Had he lived one century earlier he would have come to us as a legendary personage, like Orpheus or Theseus. Myth had already begun haloing him, since he was said to have a thigh made of gold. He founded a quasireligious sect that still existed in Plato's time, 150 years later. Little is known of his doctrine, but he certainly held that "numbers govern everything," whatever that means; he was eagerly interested in mathematics and he is said to have sacrificed an ox when he discovered (or perhaps proved by new means) the Pythagorean theorem on rectangular triangles.

The earliest mathematicians were certainly Pythagoreans and an interesting speculation is mentioned in Bourbaki's *Elements of a History of Mathematics*, which would make math begin like a novel (von Fritz 1945). It may be entertaining to assume this story is true, as follows: One of the main symbols of the Pythagoreans was the pentagram, the stellar regular pentagon that was always treasured by all sorts of mystical groups. They knew how to construct it inside a circle with a compass and a ruler, and the construction shows easily that the ratio of the stellar pentagon side to the radius of the circle is the "golden ratio" $a = (1/2)(\sqrt{5} + 1)$. This ratio played a great role in Greek aesthetics, including painting, sculpture, and architecture, and it was certainly one of the numbers governing the world, according to the Pythagorean doctrine. This doctrine held, however, that the numbers worth considering are integers or made of integers. The golden ratio should therefore have been a quotient $a = p/q$ of two integers p and q . One could divide the two of them by a common divisor until neither was left. And then comes the drama! The obvious relation $a = 1/(a + 1)$ implies that $p(p + q) = q^2$. But that means that p and q must have a common divisor, which is impossible. Something was then wrong in Pythagoras' mystical assertion.

Of course, this story is invented or at best speculative, but one knows for sure that some Pythagoreans investigated the diagonal of a square and proved without any possibility of escape that $\sqrt{2}$ cannot be rational (i.e., a quotient of integers).

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That was a hard blow for Pythagoras' doctrine. It occurred probably early in the fifth century B.C., but one does not know who found it. Did one man discover it, or maybe a woman, since the sect accepted women? Or was it the outcome of long investigations by a group of people? We don't know. We know that a man, Hippasus of Metapontium, was accursed for letting the secret of the result leak out, but that does not mean that he discovered it.

On the other hand, one cannot underestimate the importance of the event. It was certainly the first true theorem, the first one at least that could not be made obvious by means of a clever drawing. It implied to Greek eyes that the mind is able to reach a hidden truth by itself, using only the power of its own thought. It revealed the power of logic, which was still in the turmoil of initial searches. It increased the Greek confidence in the supremacy of ideas and, incidentally, it also led for a long time to disparagement of the empirical approach to science. Last but not least, it showed the value of rigor, which led the way toward Euclid's axiomatic construction of mathematics.

PLATO AND THE PHILOSOPHY OF MATHEMATICS

There is no indication of a specific philosophy of mathematics after the failure of Pythagoras' attempt. The main questions were concerned with general philosophy: what is Being, and should there be non-Being, is there infinity, what are Good and Beauty, what is reason (although people preferred speaking of the nature of ideas), what is life, and many other such issues. Mathematics entered that game with Plato in a rather roundabout way, more as an example than for its own sake, but it was the beginning of a long story.

Plato (c. 428–348 B.C.) knew well the mathematics and the mathematicians of his time, including particularly Eudox, who was his student and friend. One of his main interests was, however, the meaning of ideas. He often took mathematics as a

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paradigm and one of his early dialogues, the *Meno*, gives a good indication of his early views on that matter. At some point in the dialogue, Socrates asks a boy various mathematical questions such as: How much larger is the area of a square when the side is multiplied by two? How big is the square built on the diagonal? The boy is supposed to know nothing, since he was born a slave. . . . Socrates shows himself, however, a very kind inquirer, he gives many clues, he suggests which lines can be drawn to get a hint, and every student would certainly get an A grade with such an examiner. The boy answers correctly, of course: he is led so gently, and a modern reader would simply conclude that he does not lack common sense. That is not Plato's conclusion, however. The answers prove that the boy knew them before he was asked the questions, and the query only helped him to remember them!

This example was typical of Socrates' method, which Plato learned in his youth. The Pythagorean School influenced him later, and mathematics then became a more important key in his philosophy. His main assumption was the existence of two different worlds. There is the world we see with our eyes and in which our body is immersed, a world that can be considered as more or less trivial according to Plato. There is also another world in which perception is replaced by understanding, and mathematics originates from it. Our senses cannot reach that world, which is supposed to be more real than the one we live in, and its inhabitants are immaterial. They are the Forms, or Ideas. They belong to the sphere of divinity, higher than the gods themselves, and they share a common harmony including everything that is "Good."

Plato considered, for instance, that the world of Ideas contains a Form "circle" embodying all the possible circles in the world below, and similarly a Form "triangle." He was much impressed by the fact that mathematical properties can be discovered though nothing hints at them in the definitions, the fact, for instance, that the median lines of a triangle meet at a common point at one-third of their length. That property was

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already there before any worldly triangle had been drawn for the first time. This feeling of a preexistence, of something more real than reality, has always impressed many mathematicians, and one may presume that Eudox, who was one of the greatest mathematicians of all times, confirmed Plato on that point. There are still many believers or would-be believers in this form of Platonism among modern mathematicians, who feel something like the existence of another world where mathematical truth rests.

Plato was aware of an obvious objection to his proposal: How can we, we people made of flesh, who live in this world down here, how can we get in touch with the ideal world where mathematical truth dwells? His answer was that our soul inhabited that world before we were born, and we have memories of it. We may of course forget this answer, but the question itself will remain interesting.

ARISTOTLE AND ABSTRACTION

Aristotle (c. 385–322 B.C.) brought mathematics back down to earth. He considered the mathematical objects, numbers, circles, triangles, and so on, as so many abstractions of real objects, either natural or manmade. Although every line we draw with a stiletto on a waxed plate has a finite breadth and irregularities, our mind can make an abstraction of them, forget them, and consider them as irrelevant. Irrelevance is the motto when somebody worries about giving the same name to two obviously different triangles: everything making them different is inessential. Aristotle in that sense considered the mathematical objects as very close to natural objects, or at least as patterns that are found in reality.

Plato's problem, "Why are there mathematical properties that are not contained in the definitions?" received a new answer: Logic can create new truth, and this kind of property gives a perfect example of its ability. One should not forget

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that the discovery of logic was still recent, and Aristotle was one of its major investigators. Some of his concepts are worth mentioning and, rather than choosing them in his *Logic* or his *Metaphysics* to which I intend to return, I will pick them up in his *Physics*. He says in that book that we cannot really understand something without knowing its first principles. He enters then into various predicaments about Being and non-Being, about motion as a transition from being there to not being there. He states as a principle that every motion must proceed from a permanently active cause (a principle that, by the way, impeded physics for a millennium and a half), so that a moving object is moved by another, which is also moved, and so on, until one must arrive finally at a primary mover, who pushes the sphere on which the stars are nailed in a perfect motion that is necessarily rotation. Aristotle's book is a work of beauty and also an ascetic song of love for nature, *physis*, since love and hate are among its other basic principles: a stone falls because of its love for the earth and smoke rises up for love of the sky.

Philosophers enjoy that book for the tension in its argument and they do not worry that most of its conclusions have turned out to be wrong. Physicists would rather say that it is not a physics book in spite of its title. I wished to mention it however, because of its relation to the main topic of the present book and particularly in view of two significant statements by Aristotle, namely, (i) mathematics relies primarily on an abstraction of reality; (ii) physical reality can be understood only by getting at its first principles. These statements could look like our thesis of physicism in a nutshell, except that mathematics, the principles of physics, and even the meaning of reality have much changed in the meantime.