MHT-CET TRIUMPH

MATHEMATICS

Based on Std. XI & XII Syllabus of MHT-CET

HINTS TO MULTIPLE CHOICE QUESTIONS, EVALUATION TESTS & MHT-CET 2019 (6^{th} May, Afternoon) PAPER

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Textbook Chapter No.

02 Trigonometric Functions

Hints

Classical Thinking

1. $\tan\theta = \frac{9}{2}$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{9}{2}$

$\Rightarrow \cos\theta = \frac{2}{9}\sin\theta = \frac{2}{9}\times\frac{3}{4} = \frac{1}{6}$

2. $5\sin\theta = 3 \Rightarrow \sin\theta = \frac{3}{5}$

$\frac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta} = \frac{1+\sin\theta}{1-\sin\theta}$

$= \frac{1+\frac{3}{5}}{1-\frac{3}{5}}$

$= 4$

3. $\frac{\sin\theta}{1-\cot\theta}+\frac{\cos\theta}{1-\tan\theta} = \frac{\sin\theta.\sin\theta}{\sin\theta-\cos\theta}+\frac{\cos\theta.\cos\theta}{\cos\theta-\sin\theta}$

$= \frac{\cos^2\theta-\sin^2\theta}{\cos\theta-\sin\theta}$

$= \cos\theta+\sin\theta$

4. $\sin\theta = -\frac{1}{\sqrt{2}}$ and $\tan\theta = 1$

Since, $\sin\theta$ is –ve and $\tan\theta$ is +ve in third quadrant,

$\therefore$ θ lies in the IIIrd quadrant.

5. Since, $\sin\theta$ is –ve and $\cos\theta$ is +ve

$\therefore$ θ lies in IVth quadrant.

7. $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4} = \left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2-(1)^2$

$= \frac{1}{4}+\frac{1}{4}-1 = -\frac{1}{2}$

8. $x\sin 45^\circ\cos^2 60^\circ = \frac{\tan^2 60^\circ\operatorname{cosec}30^\circ}{\sec 45^\circ\cot^2 30^\circ}$

$\Rightarrow x.\frac{1}{\sqrt{2}}.\frac{1}{4} = \frac{3.2}{\sqrt{2}.3}$

$\Rightarrow \frac{x}{4\sqrt{2}} = \sqrt{2}$

$\Rightarrow x = 8$

9. $\sin\theta = \sqrt{3}\cos\theta$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \sqrt{3}$

$\Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = 60^\circ$

10. $\sin(\alpha-\beta) = \frac{1}{2} = \sin 30^\circ$

$\Rightarrow \alpha-\beta = 30^\circ$ …..(i)

and $\cos(\alpha+\beta) = \frac{1}{2} = \cos 60^\circ$

$\Rightarrow \alpha+\beta = 60^\circ$ …..(ii)

On solving (i) and (ii), we get

$\alpha = 45^\circ$ and $\beta = 15^\circ$

11. $\tan\theta = \frac{20}{21}$

Since $1+\tan^2\theta = \sec^2\theta$

$\therefore$ $\sec^2\theta = 1+\frac{400}{441} = \frac{841}{441}$

$\Rightarrow \sec\theta = \pm\frac{29}{21}$

$\Rightarrow \cos\theta = \pm\frac{21}{29}$

12. $1+\tan^2\theta = \sec^2\theta$

$\Rightarrow 1+\frac{1}{10} = \sec^2\theta$

$\Rightarrow \sec^2\theta = \frac{11}{10} \Rightarrow \sec\theta = \sqrt{\frac{11}{10}}$

….[$\because$ θ lies in the fourth quadrant]

13. $\sec^2\theta = 1+\tan^2\theta = 1+\frac{1}{5} = \frac{6}{5}$

$\Rightarrow \cos^2\theta = \frac{5}{6}$

$\Rightarrow \cos\theta = \frac{\sqrt{5}}{\sqrt{6}}$ ….[$\because$ θ lies in the 1st quadrant]

14. $\sec\theta + \tan\theta = \dfrac{1+\sin\theta}{\cos\theta}$

$= \dfrac{1+\sin\theta}{-\sqrt{1-\sin^2\theta}}$

....[$\because$ θ lies in the second quadrant, $\therefore$ cos θ < 0]

$= \dfrac{-\left(1+\frac{21}{29}\right)}{\sqrt{1-\left(\frac{21}{29}\right)^2}} = -\dfrac{5}{2}$

15. $\sec^4 x - \sec^2 x = \sec^2 x(\sec^2 x - 1)$
$= (1 + \tan^2 x)\tan^2 x$
$= \tan^2 x + \tan^4 x$

16. $\tan^2\theta - \sin^2\theta = \sin^2\theta\left(\dfrac{1}{\cos^2\theta} - 1\right)$
$= \sin^2\theta(\sec^2\theta - 1)$
$= \sin^2\theta\tan^2\theta$
Also $\sec^2\theta\,\text{cosec}^2\theta \neq \sec^2\theta - \text{cosec}^2\theta$, and $\text{cosec}^2\theta + \cot^2\theta \neq \text{cosec}^2\theta\cot^2\theta$

17. $x = \sec\theta + \tan\theta$

$\therefore$ $x + \dfrac{1}{x} = \sec\theta + \tan\theta + \dfrac{1}{\sec\theta + \tan\theta}$

$= \sec\theta + \tan\theta + \dfrac{\sec\theta - \tan\theta}{\sec^2\theta - \tan^2\theta}$

$= \sec\theta + \tan\theta + \sec\theta - \tan\theta$

....[$\because \sec^2\theta - \tan^2\theta = 1$]

$= 2\sec\theta$

18. $\cot x + \tan x = \dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\sin x\cos x}$

$= \dfrac{1}{\sin x\cos x}$

$= \sec x\,\text{cosec}\,x$

19. $\dfrac{\sin^2 20° + \cos^4 20°}{\sin^4 20° + \cos^2 20°}$

$= \dfrac{\sin^2 20° + \cos^2 20°\left(1-\sin^2 20°\right)}{\sin^2 20°\left(1-\cos^2 20°\right) + \cos^2 20°}$

$= \dfrac{1-\sin^2 20°\cos^2 20°}{1-\sin^2 20°\cos^2 20°} = 1$

20. $x = a\cos\theta + b\sin\theta$ ....(i)
and $y = a\sin\theta - b\cos\theta$ ....(ii)

Squaring (i) and (ii) and adding, we get
$x^2 + y^2 = a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta$
$+ a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta$
$\Rightarrow x^2 + y^2 = a^2 + b^2$

21. $x = a\cos^3\theta$

$\Rightarrow \left(\dfrac{x}{a}\right)^{\frac{1}{3}} = \cos\theta$, and ....(i)

$y = b\sin^3\theta$

$\Rightarrow \left(\dfrac{y}{b}\right)^{\frac{1}{3}} = \sin\theta$ ....(ii)

Squaring (i) and (ii) and adding, we get

$\left(\dfrac{x}{a}\right)^{\frac{2}{3}} + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = \cos^2\theta + \sin^2\theta$

$= 1$

22. $\cos x + \cos^2 x = 1$
$\Rightarrow \cos x = \sin^2 x$ ....[$\because 1 - \cos^2 x = \sin^2 x$]
$\therefore$ $\sin^2 x + \sin^4 x = \cos x + \cos^2 x = 1$

23. $\sin x + \sin^2 x = 1$
$\Rightarrow \sin x = \cos^2 x$
$\therefore$ $\cos^8 x + 2\cos^6 x + \cos^4 x$
$= \sin^4 x + 2\sin^3 x + \sin^2 x$
$= (\sin x + \sin^2 x)^2$
$= (1)^2 = 1$

24. $\sec\theta = \dfrac{1}{2}$ is not possible as $|\sec\theta| \geq 1$

25. tan θ can have any value
sin θ and cos θ cannot be numerically greater than 1.
sec θ should be greater than 1.
$\therefore$ Option (D) is the correct answer.

26. Since, $-1 \leq \cos\theta \leq 1$
$\therefore$ $-5 \leq 5\cos\theta \leq 5$
$\Rightarrow -5 + 12 \leq 5\cos\theta + 12 \leq 5 + 12$
$\Rightarrow 7 \leq 5\cos\theta + 12 \leq 17$

Critical Thinking

1. $\dfrac{p\sin\theta - q\cos\theta}{p\sin\theta + q\cos\theta} = \dfrac{p\frac{\sin\theta}{\cos\theta} - q}{p\frac{\sin\theta}{\cos\theta} + q}$

$= \dfrac{p\tan\theta - q}{p\tan\theta + q}$

$= \dfrac{p^2 - q^2}{p^2 + q^2}$ ....$\left[\because \tan\theta = \dfrac{p}{q}\text{(given)}\right]$

2. $\cos^2\theta + \sec^2\theta = (\cos\theta - \sec\theta)^2 + 2 \geq 2$

3. $\sin x + \text{cosec } x = 2$
$\Rightarrow \sin^2 x + 1 = 2\sin x$
$\Rightarrow (\sin x - 1)^2 = 0 \Rightarrow \sin x = 1$
$\therefore$ $\sin^n x + \text{cosec}^n x = \sin^n x + \dfrac{1}{\sin^n x}$
$= (1)^n + \dfrac{1}{(1)^n} = 2$

4. Since, 1 radian = 57° nearly and sin 57° > sin 1°
$\therefore$ sin1 > sin1°

5. Since, 1 radian = 57° nearly
$\therefore$ 2 radians = 114° nearly
Since, 57° lies in I[st] quadrant and 114° lies in II[nd] quadrant.
$\therefore$ tan 1 > 0 and tan 2 < 0
$\therefore$ tan 1 > tan 2

6. $\cos A = \dfrac{\sqrt{3}}{2}$
$\Rightarrow \cos A = \cos 30°$
$\Rightarrow A = 30°$
$\therefore$ $\tan 3A = \tan 90° = \infty$

7. $\tan(A - B) = 1 = \tan\dfrac{\pi}{4}$
$\Rightarrow A - B = \dfrac{\pi}{4}$ ....(i)
and $\sec(A + B) = \dfrac{2}{\sqrt{3}}$
$\Rightarrow A + B = \dfrac{11\pi}{6}$ ....(ii)
From (i) and (ii), we get
$B = \dfrac{19\pi}{24}$

8. $\sin(A + B + C) = 1$
$\Rightarrow A + B + C = 90°$ ....(i)
$\tan(A - B) = \dfrac{1}{\sqrt{3}}$
$\Rightarrow A - B = 30°$ ....(ii)
$\sec(A + C) = 2$
$\Rightarrow A + C = 60°$ ....(iii)
From (i), (ii) and (iii), we get
B = 30°, A = 60°, C = 0°

9. $\cos A = \dfrac{3}{5}$ and $\cos B = \dfrac{4}{5}$
Both A and B lie in the fourth quadrant.
Hence, both sin A and sin B are negative.
$\therefore$ $2\sin A + 4\sin B$
$= -2\sqrt{1-\cos^2 A} - 4\sqrt{1-\cos^2 B}$
$= -2\sqrt{1-\dfrac{9}{25}} - 4\sqrt{1-\dfrac{16}{25}} = -4$

10. $\sec\theta + \tan\theta = \sqrt{3}$ ....(i)
$\Rightarrow \sec\theta - \tan\theta = \dfrac{1}{\sqrt{3}}$ ....(ii)
....[$\because \sec^2\theta - \tan^2\theta = 1$]
Subtracting (ii) from (i), we get
$2\tan\theta = \sqrt{3} - \dfrac{1}{\sqrt{3}}$
$\Rightarrow \tan\theta = \dfrac{1}{\sqrt{3}}$ $\Rightarrow \theta = \dfrac{\pi}{6}$

11. $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\dfrac{(1-\sin\theta)^2}{1-\sin^2\theta}}$
$= \dfrac{1-\sin\theta}{|\cos\theta|}$
$= \dfrac{1-\sin\theta}{-\cos\theta}$ ....$\left[\because \dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2} \Rightarrow \cos\theta < 0\right]$
$= -\sec\theta + \tan\theta$

12. $\sqrt{\left(\dfrac{1-\sin\theta}{1+\sin\theta}\right)} + \sqrt{\left(\dfrac{1+\sin\theta}{1-\sin\theta}\right)}$
$= \dfrac{1-\sin\theta+1+\sin\theta}{\sqrt{1-\sin^2\theta}} = \dfrac{2}{\sqrt{\cos^2\theta}}$
$= \dfrac{2}{-\cos\theta}$ ....[$\because$ θ lies in the 2[nd] quadrant]
$= -2\sec\theta$

13. $\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}} + \sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}$
$= \dfrac{1-\cos\alpha+1+\cos\alpha}{\sqrt{1-\cos^2\alpha}}$
$= \dfrac{2}{-\sin\alpha}$ ....$\left[\because \pi < \alpha < \dfrac{3\pi}{2}\right]$

14. $3\tan A + 4 = 0 \Rightarrow \tan A = -\dfrac{4}{3}$
$1 + \tan^2 A = \sec^2 A$
$\Rightarrow \sec^2 A = 1 + \dfrac{16}{9}$

$\Rightarrow \sec A = \frac{-5}{3}$ ….[∵ θ lies in 2nd quadrant]

$\Rightarrow \cos A = -\frac{3}{5}$

∴ $\sin A = \sqrt{1-\cos^2 A} = \sqrt{1-\frac{9}{25}}$

$= \frac{4}{5}$ ….[∵ A lies in 2nd quadrant]

∴ $2 \cot A - 5 \cos A + \sin A$

$= 2\left(-\frac{3}{4}\right) - 5\left(-\frac{3}{5}\right) + \frac{4}{5}$

$= \frac{23}{10}$

15. $\sec \theta - \tan \theta = \frac{1}{2}$ ….(i)

$\Rightarrow \sec \theta + \tan \theta = 2$ ….(ii)

….[∵ $\sec^2 \theta - \tan^2 \theta = 1$]

Adding (i) and (ii), we get

$2 \sec \theta = \frac{5}{2} \Rightarrow \sec \theta = \frac{5}{4}$

Subtracting (ii) from (i), we get

$2 \tan \theta = \frac{3}{2} \Rightarrow \tan \theta = \frac{3}{4}$

Since, both sec θ and tan θ are positive,

∴ θ lies in the first quadrant.

16. $\cos \theta + \sin \theta = \sqrt{2} \cos \theta$

Squaring both sides, we get

∴ $\cos^2 \theta + \sin^2 \theta + 2\sin \theta \cos \theta = 2\cos^2 \theta$

$\Rightarrow 2 \sin \theta \cos \theta = 2\cos^2 \theta - 1$ ….(i)

Now

$(\cos \theta - \sin \theta)^2 = \cos^2 \theta + \sin^2 \theta - 2\sin \theta \cos \theta$

$= 1 - (2 \cos^2 \theta - 1)$

….[From (i)]

$= 2 (1 - \cos^2 \theta)$

$= 2 \sin^2 \theta$

∴ $\cos \theta - \sin \theta = \sqrt{2} \sin \theta$

17. $(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x$

$= 1 - \{(\sin x + \cos x)^2 - (\sin^2 x + \cos^2 x)\}$

$= 1 - (a^2 - 1)$ ….[∵ $\sin x + \cos x = a$]

$= 2 - a^2$

∴ $|\sin x - \cos x| = \sqrt{2-a^2}$

18. $3 \sin \theta + 4 \cos \theta = 5$

Squaring both sides, we get

$9 \sin^2 \theta + 16 \cos^2 \theta + 24 \sin \theta \cos \theta = 25$

$\Rightarrow 9 (1-\cos^2 \theta) + 16 (1-\sin^2 \theta) + 24 \sin \theta \cos \theta = 25$

$\Rightarrow 9 \cos^2 \theta + 16 \sin^2 \theta - 24 \sin \theta \cos \theta = 0$

$\Rightarrow (3 \cos \theta - 4 \sin \theta)^2 = 0$

$\Rightarrow 3 \cos \theta - 4 \sin \theta = 0$

19. $2u_6 - 3u_4$

$= 2 (\cos^6 \theta + \sin^6 \theta) - 3 (\cos^4 \theta + \sin^4 \theta)$

$= 2(1 - 3 \sin^2 \theta \cos^2 \theta) - 3(1 - 2 \sin^2 \theta \cos^2 \theta)$

$= -1$

20. $\sin x + \sin^2 x = 1$

$\Rightarrow \sin x = 1 - \sin^2 x$

$\Rightarrow \sin x = \cos^2 x$

∴ $\cos^{12} x + 3 \cos^{10} x + 3 \cos^8 x + \cos^6 x - 2$

$= \sin^6 x + 3 \sin^5 x + 3 \sin^4 x + \sin^3 x - 2$

$= (\sin^2 x)^3 + 3(\sin^2 x)^2 \sin x + 3(\sin^2 x)(\sin x)^2 + (\sin x)^3 - 2$

$= (\sin^2 x + \sin x)^3 - 2$

$= (1)^3 - 2$ ….[∵ $\sin x + \sin^2 x = 1$]

$= -1$

21. $10 \sin^4 \alpha + 15 \cos^4 \alpha = 6$

$\Rightarrow 10 \sin^4 \alpha + 15 \cos^4 \alpha = 6 (\sin^2 \alpha + \cos^2 \alpha)^2$

$\Rightarrow 10 \tan^4 \alpha + 15 = 6 (\tan^2 \alpha + 1)^2$

$\Rightarrow (2 \tan^2 \alpha - 3)^2 = 0$

$\Rightarrow \tan^2 \alpha = \frac{3}{2}$

∴ $27 \operatorname{cosec}^6 \alpha + 8 \sec^6 \alpha$

$= 27 (1 + \cot^2 \alpha)^3 + 8 (1 + \tan^2 \alpha)^3$

$= 27 \left(1+\frac{2}{3}\right)^3 + 8 \left(1+\frac{3}{2}\right)^3 = 250$

22. $\sec \alpha - \tan \alpha = \frac{1-\sin\alpha}{\cos\alpha}$

$$= \frac{1-\frac{2pq}{p^2+q^2}}{\sqrt{1-\left(\frac{2pq}{p^2+q^2}\right)^2}}$$

$$= \frac{(p-q)^2}{\sqrt{(p^2-q^2)^2}}$$

$$= \frac{(p-q)^2}{p^2-q^2} = \frac{p-q}{p+q}$$

23. $\sec\theta - \tan\theta = \dfrac{a+1}{a-1}$ ….(i)

$\Rightarrow \sec\theta + \tan\theta = \dfrac{a-1}{a+1}$ ….(ii)

….[$\because \sec^2\theta - \tan^2\theta = 1$]

Adding (i) and (ii), we get

$2\sec\theta = \dfrac{a+1}{a-1} + \dfrac{a-1}{a+1}$

$\Rightarrow 2\sec\theta = \dfrac{(a+1)^2 + (a-1)^2}{a^2-1}$

$\Rightarrow 2\sec\theta = \dfrac{2(a^2+1)}{a^2-1}$

$\Rightarrow \sec\theta = \dfrac{a^2+1}{a^2-1}$

$\Rightarrow \cos\theta = \dfrac{a^2-1}{a^2+1}$

24. $\tan^2\theta = \sec^2\theta - 1$

$= \left(x+\dfrac{1}{4x}\right)^2 - 1 = \left(x-\dfrac{1}{4x}\right)^2$

$\Rightarrow \tan\theta = \pm\left(x-\dfrac{1}{4x}\right)$

$\Rightarrow \sec\theta + \tan\theta = x + \dfrac{1}{4x} \pm \left(x-\dfrac{1}{4x}\right)$

$\Rightarrow \sec\theta + \tan\theta = x + \dfrac{1}{4x} + \left(x-\dfrac{1}{4x}\right)$

or

$\sec\theta + \tan\theta = x + \dfrac{1}{4x} - \left(x-\dfrac{1}{4x}\right)$

$\Rightarrow \sec\theta + \tan\theta = 2x$ or $\dfrac{1}{2x}$

25. $\sin^6\left(\dfrac{\pi}{49}\right) + \cos^6\left(\dfrac{\pi}{49}\right) - 1 + 3\sin^2\left(\dfrac{\pi}{49}\right)\cos^2\left(\dfrac{\pi}{49}\right)$

$= \sin^6\left(\dfrac{\pi}{49}\right) + \cos^6\left(\dfrac{\pi}{49}\right)$

$+ 3\sin^2\left(\dfrac{\pi}{49}\right)\cos^2\left(\dfrac{\pi}{49}\right)\left(\sin^2\dfrac{\pi}{49} + \cos^2\dfrac{\pi}{49}\right) - 1$

$= \left(\sin^2\dfrac{\pi}{49} + \cos^2\dfrac{\pi}{49}\right)^3 - 1 = 1 - 1 = 0$

26. $\dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha} = \dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}\cdot\dfrac{1+\cos\alpha+\sin\alpha}{1+\cos\alpha+\sin\alpha}$

$= \dfrac{(1+\sin\alpha)^2 - \cos^2\alpha}{(1+\sin\alpha)(1+\cos\alpha+\sin\alpha)}$

$= \dfrac{1+2\sin\alpha+\sin^2\alpha-(1-\sin^2\alpha)}{(1+\sin\alpha)(1+\cos\alpha+\sin\alpha)}$

$= \dfrac{2\sin\alpha(1+\sin\alpha)}{(1+\sin\alpha)(1+\cos\alpha+\sin\alpha)}$

$= \dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}$

$= x$

27. $1 - \dfrac{\sin^2 y}{1+\cos y} + \dfrac{1+\cos y}{\sin y} - \dfrac{\sin y}{1-\cos y}$

$= \dfrac{1+\cos y-\sin^2 y}{1+\cos y} + \dfrac{(1-\cos^2 y)-\sin^2 y}{\sin y(1-\cos y)}$

$= \dfrac{\cos^2 y+\cos y}{1+\cos y} + \dfrac{\sin^2 y-\sin^2 y}{\sin y(1-\cos y)}$

$= \dfrac{\cos y(1+\cos y)}{1+\cos y} + 0 = \cos y$

28. $\dfrac{2\sin\theta\tan\theta(1-\tan\theta)+2\sin\theta\sec^2\theta}{(1+\tan\theta)^2}$

$= \dfrac{2\sin\theta}{(1+\tan\theta)^2}\{\tan\theta(1-\tan\theta)+\sec^2\theta\}$

$= \dfrac{2\sin\theta}{(1+\tan\theta)^2}(\tan\theta - \tan^2\theta + 1 + \tan^2\theta)$

$= \dfrac{2\sin\theta}{1+\tan\theta}$

29. $\dfrac{\sin^3 A - \cos^3 A}{\sin A - \cos A} + \dfrac{\sin A}{\sqrt{1+\tan^2 A}} - 2\tan A\cot A$

$= (\sin^2 A + \cos^2 A + \sin A\cos A) + \dfrac{\sin A}{|\sec A|} - 2$

$= 1 + \sin A\cos A - \sin A\cos A - 2$

….[$\because$ A is an obtuse angle $\Rightarrow \cos A < 0$]

$= -1$

30. $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = -1$ ….(i)

and $\dfrac{x}{a}\sin\theta - \dfrac{y}{b}\cos\theta = 1$ ….(ii)

Squaring (i) and (ii) and adding, we get

$$\frac{x^2}{a^2}(\sin^2\theta+\cos^2\theta)+\frac{y^2}{b^2}(\sin^2\theta+\cos^2\theta) = 2$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2} = 2$$

31. $x\sin^3\alpha + y\cos^3\alpha = \sin\alpha\cos\alpha$ ....(i)

and $x\sin\alpha - y\cos\alpha = 0$

$\Rightarrow x\sin\alpha = y\cos\alpha$ ....(ii)

$\therefore$ From (i) and (ii), we get

$y\cos\alpha\sin^2\alpha + y\cos^3\alpha = \sin\alpha\cos\alpha$

$\Rightarrow y\cos\alpha(\sin^2\alpha + \cos^2\alpha) = \sin\alpha\cos\alpha$

$\Rightarrow y\cos\alpha = \sin\alpha\cos\alpha$

$\Rightarrow y = \sin\alpha$

$\therefore$ $x = \cos\alpha$

$\therefore$ $x^2 + y^2 = \cos^2\alpha + \sin^2\alpha = 1$

32. $m + n = a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha + 3a\cos^2\alpha\sin\alpha + a\sin^3\alpha$

$= a(\cos\alpha + \sin\alpha)^3$

Similarly, $(m - n) = a(\cos\alpha - \sin\alpha)^3$

$\therefore$ $(m + n)^{2/3} + (m - n)^{2/3}$

$= a^{\frac{2}{3}}\{(\cos\alpha + \sin\alpha)^2 + (\cos\alpha - \sin\alpha)^2\}$

$= a^{\frac{2}{3}}\{2(\cos^2\alpha + \sin^2\alpha)\} = 2a^{\frac{2}{3}}$

33. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$

$$=\frac{\tan^2\alpha}{1+\tan^2\alpha}+\frac{\tan^2\beta}{1+\tan^2\beta}+\frac{\tan^2\gamma}{1+\tan^2\gamma}$$

$$=\frac{x}{1+x}+\frac{y}{1+y}+\frac{z}{1+z}$$

....[Let $x = \tan^2\alpha$, $y = \tan^2\beta$, $z = \tan^2\gamma$]

$$=\frac{(x+y+z)+(xy+yz+zx+2xyz)+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)}$$

$$=\frac{x+y+z+1+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)} = 1$$

34. $$p + q = \frac{2\sin\theta}{1+\sin\theta+\cos\theta}+\frac{\cos\theta}{1+\sin\theta}$$

$$=\frac{2\sin\theta(1+\sin\theta)+\cos\theta(1+\sin\theta+\cos\theta)}{(1+\sin\theta+\cos\theta)(1+\sin\theta)}$$

$$=\frac{2\sin\theta+2\sin^2\theta+\cos\theta+\cos\theta\sin\theta+\cos^2\theta}{1+2\sin\theta+\sin^2\theta+\cos\theta+\cos\theta\sin\theta}$$

$$=\frac{2\sin\theta+\sin^2\theta+(\sin^2\theta+\cos^2\theta)+\cos\theta+\cos\theta\sin\theta}{1+2\sin\theta+\sin^2\theta+\cos\theta+\cos\theta\sin\theta}$$

$\Rightarrow p + q = 1$

35. $xy = (\sec\phi - \tan\phi)(\operatorname{cosec}\phi + \cot\phi)$

$$=\frac{1-\sin\phi}{\cos\phi}\cdot\frac{1+\cos\phi}{\sin\phi}$$

$$xy+1=\frac{1-\sin\phi+\cos\phi-\sin\phi\cos\phi+\sin\phi\cos\phi}{\cos\phi\sin\phi}$$

$$\Rightarrow xy+1=\frac{1-\sin\phi+\cos\phi}{\cos\phi\sin\phi} \quad ....(i)$$

$x - y = (\sec\phi - \tan\phi) - (\operatorname{cosec}\phi + \cot\phi)$

$$=\frac{1-\sin\phi}{\cos\phi}-\frac{1+\cos\phi}{\sin\phi}$$

$$=\frac{\sin\phi-\sin^2\phi-\cos\phi-\cos^2\phi}{\cos\phi\sin\phi}$$

$$=\frac{\sin\phi-\cos\phi-(\sin^2\phi+\cos^2\phi)}{\cos\phi\sin\phi}$$

$$\Rightarrow x-y=\frac{\sin\phi-\cos\phi-1}{\cos\phi\sin\phi} \quad ....(ii)$$

Adding (i) and (ii), we get

$xy+1+(x-y)=0$

$$\Rightarrow x=\frac{y-1}{y+1}$$

36. $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$

$\Rightarrow \sin\theta_1 = \sin\theta_2 = \sin\theta_3 = 1$

....[$\because -1 \le \sin x \le 1$]

$\Rightarrow \theta_1 = \theta_2 = \theta_3 = \frac{\pi}{2}$

$\Rightarrow \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$

37. Given, $(a + b)^2 = 4ab\sin^2\theta$

$\Rightarrow \sin^2\theta = \frac{(a+b)^2}{4ab} \le 1$

$\Rightarrow (a + b)^2 - 4ab \le 0$

$\Rightarrow (a - b)^2 \le 0$

$\Rightarrow a = b$

38. $$12\sin\theta - 9\sin^2\theta = -9\left(\sin^2\theta-\frac{12}{9}\sin\theta\right)$$

$$=-9\left(\sin^2\theta-\frac{4}{3}\sin\theta\right)$$

$$=-9\left[\sin^2\theta-\frac{4}{3}\sin\theta+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2\right]$$

$$=-9\left(\sin\theta-\frac{2}{3}\right)^2+9\times\frac{4}{9}\le 4$$

39. $y = \sin^2\theta + \cos^4\theta$
$\Rightarrow y = \cos^4\theta - \cos^2\theta + 1$
$\Rightarrow y = \left(\cos^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4}$
Now, $0 \le \cos^2\theta \le 1$
$\Rightarrow -\frac{1}{2} \le \cos^2\theta - \frac{1}{2} \le \frac{1}{2}$
$\Rightarrow 0 \le \left(\cos^2\theta - \frac{1}{2}\right)^2 \le \frac{1}{4}$
$\Rightarrow \frac{3}{4} \le \left(\cos^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \le 1$
$\Rightarrow \frac{3}{4} \le y \le 1$

Competitive Thinking

1. $\frac{5\sin\theta - 3\cos\theta}{5\sin\theta + 2\cos\theta} = \frac{5\tan\theta - 3}{5\tan\theta + 2}$
$= \frac{4-3}{4+2}$ ....[$\because$ 5 tan θ = 4 (given)]
$= \frac{1}{6}$

2. $\sin^2\theta + \text{cosec}^2\theta$
$= (\sin\theta + \text{cosec}\,\theta)^2 - 2\sin\theta\,\text{cosec}\,\theta$
$= (2)^2 - 2$ ....[$\because \sin\theta + \text{cosec}\,\theta = 2$ (given)]
$= 4 - 2 = 2$

3. $\sin\theta + \text{cosec}\,\theta = 2$
$\Rightarrow \sin^2\theta + 1 = 2\sin\theta$ ....$\left[\because \text{cosec}\,\theta = \frac{1}{\sin\theta}\right]$
$\Rightarrow \sin^2\theta - 2\sin\theta + 1 = 0$
$\Rightarrow (\sin\theta - 1)^2 = 0$
$\Rightarrow \sin\theta = 1$
$\therefore$ $\sin^{10}\theta + \text{cosec}^{10}\theta = \sin^{10}\theta + \frac{1}{\sin^{10}\theta}$
$= (1)^{10} + \frac{1}{(1)^{10}}$
$= 2$

4. tan A + cot A = 4
Squaring both sides, we get
$\tan^2 A + \cot^2 A + 2\tan A\cot A = 16$
$\Rightarrow \tan^2 A + \cot^2 A = 14$
Again, squaring both sides, we get
$\tan^4 A + \cot^4 A + 2 = 196$
$\Rightarrow \tan^4 A + \cot^4 A = 194$

5. Since, 200° lies in III[rd] quadrant.
$\therefore$ sin 200°, cos 200° are both –ve.
$\therefore$ their sum is –ve.

6. One of the factor of the given expression is cos 90° = 0.
$\therefore$ cos 1°.cos 2°.cos 3°…..cos 179° = 0

7. Given that $x \in \left[0, \frac{\pi}{2}\right]$, $y \in \left[0, \frac{\pi}{2}\right]$
$\sin x + \cos y = 2$
Maximum value of $\sin x = 1 \Rightarrow x = \frac{\pi}{2}$
Maximum value of $\cos y = 1 \Rightarrow y = 0$
$\therefore$ $x + y = \frac{\pi}{2} + 0 = \frac{\pi}{2}$

58. $\sin\theta - \cos\theta = 1$ ...(i)
$\therefore$ $(\sin\theta - \cos\theta)^2 = 1$
$\therefore$ $1 - 2\sin\theta\cos\theta = 1$
$\therefore$ $\sin\theta.\cos\theta = 0$ ...(ii)
Now, $\sin^3\theta - \cos^3\theta$
$= (\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta)$
$= (1)(1 + \sin\theta.\cos\theta)$ ...[From (i)]
$= 1 + 0$ ...[From (ii)]
$= 1$

8. $\text{cosec}^2\theta = 1 + \cot^2\theta$
$= 1 + \frac{9}{16}$ ....$\left[\because \tan\theta = -\frac{4}{3}\right]$
$= \frac{25}{16}$
$\therefore$ $\sin^2\theta = \frac{1}{\text{cosec}^2\theta} = \frac{16}{25} \Rightarrow \sin\theta = \pm\frac{4}{5}$
Both the values are acceptable.
Since, $\tan\theta = -\frac{4}{3}$
i.e., θ lies in 2[nd] or 4[th] quadrant.

9. $\sin\theta = \frac{24}{25}$
$\Rightarrow \cos\theta = \frac{-7}{25}, \tan\theta = \frac{-24}{7}$
....[$\because$ θ lies in the 2[nd] quadrant]
$\therefore$ $\sec\theta + \tan\theta = \frac{-25}{7} - \frac{24}{7} = -7$

10. $\cos\theta = -\sqrt{1 - \left(\frac{2t}{1+t^2}\right)^2}$
....[$\because$ θ lies in the 2[nd] quadrant]

$$= -\sqrt{\frac{(1+t^2)^2 - 4t^2}{(1+t^2)^2}} = -\sqrt{\frac{\left(1-t^2\right)^2}{\left(1+t^2\right)^2}}$$

$$= \frac{-\left|1-t^2\right|}{1+t^2}$$

11. $\sqrt{\operatorname{cosec}^2\alpha + 2\cot\alpha} = \sqrt{1+\cot^2\alpha + 2\cot\alpha}$

$= |\ 1 + \cot\alpha|$

But $\frac{3\pi}{4} < \alpha < \pi$

$\Rightarrow \cot\alpha < -1 \Rightarrow 1 + \cot\alpha < 0$

Hence, $|\ 1 + \cot\alpha\ | = -(1 + \cot\alpha)$

12. $\operatorname{cosec}\theta - \cot\theta = 2017$ …(i)

$\Rightarrow \frac{1}{\operatorname{cosec}\theta - \cot\theta} = \frac{1}{2017}$

$\Rightarrow \frac{\operatorname{cosec}\theta + \cot\theta}{(\operatorname{cosec}\theta - \cot\theta)\ (\operatorname{cosec}\theta + \cot\theta)} = \frac{1}{2017}$

$\Rightarrow \operatorname{cosec}\theta + \cot\theta = \frac{1}{2017}$ …(ii)

Adding (i) and (ii), we get

$2\operatorname{cosec}\theta = 2017 + \frac{1}{2017}$

$\Rightarrow \operatorname{cosec}\theta$ = positive $\Rightarrow \sin\theta$ = positive

Subtracting (i) from (ii), we get

$\Rightarrow 2\cot\theta = \frac{1}{2017} - 2017$

$\Rightarrow \cot\theta$ = negative $\Rightarrow \tan\theta$ = negative

$\therefore$ θ lies in II quadrant.

13. $\operatorname{cosec}\theta - \cot\theta = \frac{1}{2}$ ….(i)

$\therefore$ $\operatorname{cosec}\theta + \cot\theta = 2$ ….(ii)

….[$\because \operatorname{cosec}^2\theta - \cot^2\theta = 1$]

Adding (i) and (ii), we get

$2\operatorname{cosec}\theta = \frac{5}{2} \Rightarrow \sin\theta = \frac{4}{5}$

Subtracting (ii) from (i), we get

$2\cot\theta = \frac{3}{2} \Rightarrow \cot\theta = \frac{3}{4}$

$\Rightarrow \cos\theta = \frac{3}{4}\sin\theta$

$= \frac{3}{4} \times \frac{4}{5} = \frac{3}{5}$

14. $\operatorname{cosec} A + \cot A = \frac{11}{2}$

$\Rightarrow \operatorname{cosec} A - \cot A = \frac{2}{11}$

$\therefore$ $2\cot A = \frac{117}{22} \Rightarrow \tan A = \frac{44}{117}$

15. $\sec\theta + \tan\theta = p$ ….(i)

$\therefore$ $\sec\theta - \tan\theta = \frac{1}{p}$ ….(ii)

$\therefore$ $2\tan\theta = p - \frac{1}{p} \Rightarrow \tan\theta = \frac{p^2 - 1}{2p}$

16. $\tan\theta + \sec\theta = e^x$ …..(i)

$\therefore$ $\sec\theta - \tan\theta = e^{-x}$ …..(ii)

$\therefore$ $2\sec\theta = e^x + e^{-x}$

$\Rightarrow \cos\theta = \frac{2}{e^x + e^{-x}}$

17. $\sin\theta + \cos\theta = 1$

Squaring both sides, we get

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$

$\Rightarrow \sin\theta\cos\theta = 0$

18. $(3\cos A - 5\sin A)^2$

$= 9\cos^2 A + 25\sin^2 A - 30\sin A\cos A$

$= 9(1 - \sin^2 A) + 25(1 - \cos^2 A) - 30\sin A\cos A$

$= 34 - (9\sin^2 A + 25\cos^2 A + 30\sin A\cos A)$

$= 34 - (3\sin A + 5\cos A)^2$

$= 34 - 25$ ....[$\because 3\sin A + 5\cos A = 5$]

$= 9$

19. Given, $\sec\theta = m$ and $\tan\theta = n$

Since, $\sec^2\theta - \tan^2\theta = 1$

$\therefore$ $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$

$\Rightarrow (m - n)(m + n) = 1$

$\Rightarrow m - n = \frac{1}{m+n}$ ....(i)

$\therefore$ $\frac{1}{m}\left\{(m+n) + \frac{1}{m+n}\right\} = \frac{1}{m}(m + n + m - n)$ ….[From (i)]

$= 2$

20. $n(m^2 - 1) = (\sec\theta + \operatorname{cosec}\theta).2\sin\theta\cos\theta$ ….[$\because m^2 = 1 + 2\sin\theta\cos\theta$]

$= \frac{\sin\theta + \cos\theta}{\sin\theta.\cos\theta}.2\sin\theta\cos\theta = 2m$

21. $2y \cos \theta = x \sin \theta$ .....(i)
and $2x \sec \theta - y \operatorname{cosec} \theta = 3$
$\Rightarrow \dfrac{2x}{\cos\theta} - \dfrac{y}{\sin\theta} = 3$
$\Rightarrow 2x \sin\theta - y \cos\theta - 3 \sin\theta \cos\theta = 0$ ....(ii)
Solving (i) and (ii), we get
$y = \sin \theta$ and $x = 2 \cos \theta$
$\therefore$ $x^2 + 4y^2 = 4 \cos^2 \theta + 4 \sin^2 \theta$
$= 4 (\cos^2 \theta + \sin^2 \theta) = 4$

22. $m + n = 2 \tan \theta$, $m - n = 2 \sin \theta$
$\therefore$ $m^2 - n^2 = 4 \tan \theta . \sin \theta$ ....(i)
Also, $4\sqrt{mn} = 4\sqrt{\tan^2\theta - \sin^2\theta}$
$= 4\sin\theta.\tan\theta$ ....(ii)
$\therefore$ From (i) and (ii), we get $m^2 - n^2 = 4\sqrt{mn}$

23. $(\sec \alpha + \tan \alpha)(\sec \beta + \tan \beta)(\sec \gamma + \tan \gamma)$
$= \tan \alpha \tan \beta \tan \gamma$ ....(i)
Let $x = (\sec \alpha - \tan \alpha)(\sec \beta - \tan \beta)$
$(\sec \gamma - \tan \gamma)$ ...(ii)
Multiplying equations (i) and (ii), we get
$(\sec^2\alpha - \tan^2\alpha)(\sec^2\beta - \tan^2\beta)(\sec^2\gamma - \tan^2\gamma)$
$= x(\tan \alpha \tan \beta \tan \gamma)$
$\Rightarrow x = \dfrac{1}{\tan\alpha\tan\beta\tan\gamma}$
$\therefore$ $x = \cot \alpha \cot \beta \cot \gamma$

24. $2 P_6 - 3 P_4 + 1$
$= 2 (\cos^6 \theta + \sin^6 \theta) - 3(\cos^4 \theta + \sin^4 \theta) + 1$
$= 2 (1 - 3 \sin^2 \theta \cos^2 \theta) - 3 [(\sin^2 \theta + \cos^2 \theta)^2$
$- 2 \sin^2 \theta \cos^2 \theta] + 1$
$= 2 - 6 \sin^2 \theta \cos^2 \theta - 3 (1 - 2 \sin^2 \theta \cos^2 \theta) + 1$
$= 0$

25. $(\sec A + \tan A - 1)(\sec A - \tan A + 1) - 2 \tan A$
$= \sec^2 A - \tan^2 A - \sec A \tan A + \sec A$
$+ \sec A \tan A + \tan A - \sec A + \tan A - 1 - 2 \tan A$
$= \sec^2 A - \tan^2 A - 1$
$= 0$ ....$[\because \sec^2 A - \tan^2 A = 1]$

26. $\cos^4 \theta - \sin^4 \theta$
$= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) = \cos^2 \theta - \sin^2 \theta$
$= \cos^2 \theta - (1 - \cos^2 \theta) = 2 \cos^2 \theta - 1$

27. $\sin^6 \theta + \cos^6 \theta + 3 \sin^2 \theta \cos^2 \theta$
$= (\sin^2 \theta + \cos^2 \theta)^3 - 3 \sin^2 \theta \cos^2 \theta$
$+ 3 \sin^2 \theta \cos^2 \theta = 1$

28. $6(\sin^6 \theta + \cos^6 \theta) - 9(\sin^4 \theta + \cos^4 \theta) + 4$
$= 6[(\sin^2 \theta + \cos^2 \theta)^3 - 3 \sin^2 \theta \cos^2\theta]$
$- 9[(\sin^2 \theta + \cos^2 \theta)^2 - 2 \sin^2 \theta \cos^2\theta] + 4$
$= 6(1-3 \sin^2 \theta \cos^2 \theta) - 9(1 - 2 \sin^2 \theta \cos^2 \theta) + 4$
$= 6 - 9 + 4 = 1$

29. $\sin x + \sin^2 x = 1$
$\Rightarrow \sin x = 1 - \sin^2 x$
$\Rightarrow \sin x = \cos^2 x$
$\therefore$ $\cos^{12} x + 3 \cos^{10} x + 3 \cos^8 x + \cos^6 x - 1$
$= \sin^6 x + 3 \sin^5 x + 3 \sin^4 x + \sin^3 x - 1$
$= (\sin^2 x)^3 + 3(\sin^2 x)^2 \sin x + 3 \sin^2 x . \sin^2 x$
$+ (\sin x)^3 - 1$
$= (\sin^2 x + \sin x)^3 - 1 = 1^3 - 1 = 0$

30. $(\cos x + \sin x)^2 + k \sin x \cos x - 1 = 0$
$\Rightarrow \cos^2 x + \sin^2 x + 2 \cos x \sin x$
$+ k \sin x \cos x - 1 = 0$
$\Rightarrow (k + 2) \cos x \sin x = 0$
$\Rightarrow k + 2 = 0 \Rightarrow k = -2$

31. Since, $\sec^2 \theta \geq 1$
$\Rightarrow \dfrac{4xy}{(x+y)^2} \geq 1 \Rightarrow 4xy \geq (x + y)^2$
$\Rightarrow (x - y)^2 \leq 0$
It is possible only when $x = y$ and $x \neq 0$.

32. Since, $\theta \in \left(0, \dfrac{\pi}{4}\right)$
$\therefore$ $\tan \theta < 1$ and $\cot \theta > 1$
Let $\tan \theta = 1 - \lambda_1$ and $\cot \theta = 1 + \lambda_2$, where λ_1 and λ_2 are very small and positive.
Then, $t_1 = (1 - \lambda_1)^{1-\lambda_1}, t_2 = (1 - \lambda_1)^{1+\lambda_2}$
$t_3 = (1 + \lambda_2)^{1-\lambda_1}, t_4 = (1 + \lambda_2)^{1+\lambda_2}$
Clearly, $t_4 > t_3 > t_1 > t_2$

Evaluation Test

1. $\sin A = a \cos B$ and $\cos A = b \sin B$

$\therefore$ $a^2 \cos^2 B + b^2 \sin^2 B = \sin^2 A + \cos^2 A$

$\Rightarrow a^2 \cos^2 B + b^2(1 - \cos^2 B) = 1$

$\Rightarrow \cos^2 B = \dfrac{1-b^2}{a^2-b^2}$ and $\sin^2 B = \dfrac{a^2-1}{a^2-b^2}$

$\Rightarrow \tan^2 B = \dfrac{a^2-1}{1-b^2}$ ....(i)

Again, $\sin A = a \cos B$ and $\cos A = b \sin B$

$\Rightarrow \tan A = \dfrac{a}{b} \cot B$

$\Rightarrow \tan^2 A = \dfrac{a^2}{b^2} \cot^2 B$

$\Rightarrow \tan^2 A = \dfrac{a^2}{b^2}\left(\dfrac{1-b^2}{a^2-1}\right)$ ....[From (i)]

$\therefore$ $(a^2 - 1) \tan^2 A + (1 - b^2) \tan^2 B$

$= \dfrac{a^2}{b^2}(1-b^2)+(a^2-1) = \dfrac{a^2-b^2}{b^2}$

2. Given, $\tan\theta = \dfrac{x\sin\phi}{1-x\cos\phi}$

$\Rightarrow \dfrac{1}{x}\tan\theta - \tan\theta\cos\phi = \sin\phi$

$\Rightarrow \dfrac{1}{x} = \dfrac{\sin\phi + \cos\phi\tan\theta}{\tan\theta}$

Also, $\tan\phi = \dfrac{y\sin\theta}{1-y\cos\theta}$

$\Rightarrow \tan\phi = \dfrac{\sin\theta}{\dfrac{1}{y}-\cos\theta}$

$\Rightarrow \dfrac{1}{y}\tan\phi - \tan\phi\cos\theta = \sin\theta$

$\Rightarrow \dfrac{1}{y}\tan\phi = \sin\theta + \tan\phi\cos\theta$

$\Rightarrow \dfrac{1}{y} = \dfrac{\sin\theta + \tan\phi\cos\theta}{\tan\phi}$

$\therefore$ $\dfrac{x}{y} = \left[\dfrac{\tan\theta}{\sin\phi+\cos\phi\tan\theta}\right]\times\left[\dfrac{\sin\theta+\tan\phi\cos\theta}{\tan\phi}\right]$

$= \dfrac{\tan\theta}{\tan\phi}\left[\dfrac{\sin\theta+\cos\theta\dfrac{\sin\phi}{\cos\phi}}{\sin\phi+\cos\phi\dfrac{\sin\theta}{\cos\theta}}\right]$

$= \dfrac{\tan\theta\cos\theta}{\tan\phi\cos\phi} = \dfrac{\sin\theta}{\sin\phi}$

3. $a \sin^2 x + b \cos^2 x = c$

$\Rightarrow a(1 - \cos^2 x) + b \cos^2 x = c$

$\Rightarrow (b - a)\cos^2 x = c - a$

$\Rightarrow (b - a) = (c - a)(1 + \tan^2 x)$

….[$\because \sec^2\theta = 1 + \tan^2\theta$]

and $b \sin^2 y + a \cos^2 y = d$

$\Rightarrow (a - b)\cos^2 y = d - b$

$\Rightarrow (a - b) = (d - b)(1 + \tan^2 y)$

$\therefore$ $\tan^2 x = \dfrac{b-c}{c-a}$ and $\tan^2 y = \dfrac{a-d}{d-b}$

$\therefore$ $\dfrac{\tan^2 x}{\tan^2 y} = \dfrac{(b-c)(d-b)}{(c-a)(a-d)}$ …..(i)

But, $a \tan x = b \tan y$

$\Rightarrow \dfrac{\tan x}{\tan y} = \dfrac{b}{a}$ …..(ii)

From (i) and (ii), we get

$\dfrac{b^2}{a^2} = \dfrac{(b-c)(d-b)}{(c-a)(a-d)}$

$\Rightarrow \dfrac{a^2}{b^2} = \dfrac{(c-a)(a-d)}{(b-c)(d-b)}$

4. Given, $\operatorname{cosec}\theta - \sin\theta = a$

$\Rightarrow \dfrac{1-\sin^2\theta}{\sin\theta} = a \Rightarrow \dfrac{\cos^2\theta}{\sin\theta} = a$ ....(i)

and $\sec\theta - \cos\theta = b$

$\Rightarrow \dfrac{1-\cos^2\theta}{\cos\theta} = b \Rightarrow \dfrac{\sin^2\theta}{\cos\theta} = b$ ....(ii)

Squaring (i) and multiplying by (ii), we get

$\cos^3\theta = a^2b$

$\Rightarrow \cos\theta = (a^2b)^{1/3}$ ....(iii)

Squaring (ii) and multiplying by (i), we get

$\sin^3\theta = b^2a$

$\Rightarrow \sin\theta = (b^2a)^{1/3}$ ....(iv)

Squaring (iii) and (iv) and adding, we get

$1 = (a^2b)^{2/3} + (b^2a)^{2/3}$

$\Rightarrow a^{4/3}\, b^{2/3} + b^{4/3}\, a^{2/3} = 1$

$\Rightarrow a^{2/3}\, b^{2/3}\,(a^{2/3} + b^{2/3}) = 1$

5. $\sin x + \sin^2 x + \sin^3 x = 1$

$\Rightarrow \sin x + \sin^3 x = 1 - \sin^2 x$

$\Rightarrow \sin x + \sin^3 x = \cos^2 x$

$\Rightarrow \sin x\,(1 + \sin^2 x) = \cos^2 x$

$\Rightarrow \sin x\,(2 - \cos^2 x) = \cos^2 x$ ....$[\because \sin^2\theta = 1 - \cos^2\theta]$

$\Rightarrow \sin^2 x(2 - \cos^2 x)^2 = \cos^4 x$

$\Rightarrow (1 - \cos^2 x)\,(4 - 4\cos^2 x + \cos^4 x) = \cos^4 x$

$\Rightarrow 4 - 4\cos^2 x + \cos^4 x - 4\cos^2 x + 4\cos^4 x - \cos^6 x = \cos^4 x$

$\Rightarrow \cos^6 x - 4\cos^4 x + 8\cos^2 x = 4$

6. Given, $\cot\theta + \tan\theta = m$

$\Rightarrow \dfrac{1}{\tan\theta} + \tan\theta = m \Rightarrow 1 + \tan^2\theta = m\tan\theta$

$\Rightarrow \sec^2\theta = m\tan\theta$ ....(i)

and $\sec\theta - \cos\theta = n \Rightarrow \sec^2\theta - 1 = n\sec\theta$

$\Rightarrow \tan^2\theta = n\sec\theta$

$\Rightarrow \tan^4\theta = n^2\sec^2\theta$

$\Rightarrow \tan^4\theta = n^2 m\tan\theta$ ....[From (i)]

$\Rightarrow \tan^3\theta = n^2 m$

$\Rightarrow \tan\theta = (n^2 m)^{1/3}$ ....(ii)

Putting (ii) in (i), we get

$\sec^2\theta = m(n^2m)^{1/3}$

Since, $\sec^2\theta - \tan^2\theta = 1$

$\therefore$ $m(mn^2)^{1/3} - (n^2m)^{2/3} = 1$

$\Rightarrow m(mn^2)^{1/3} - n(nm^2)^{1/3} = 1$

7. $\dfrac{\sin^4\theta}{a} + \dfrac{\cos^4\theta}{b} = \dfrac{1}{a+b}$

$\Rightarrow (a+b)\left(\dfrac{\sin^4\theta}{a} + \dfrac{\cos^4\theta}{b}\right) = (\sin^2\theta + \cos^2\theta)^2$

$\Rightarrow \sin^4\theta + \cos^4\theta + \dfrac{b}{a}\sin^4\theta + \dfrac{a}{b}\cos^4\theta$
$= \sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta$

$\Rightarrow \dfrac{b}{a}\sin^4\theta + \dfrac{a}{b}\cos^4\theta - 2\sin^2\theta\cos^2\theta = 0$

$\Rightarrow \left(\sqrt{\dfrac{b}{a}}\sin^2\theta - \sqrt{\dfrac{a}{b}}\cos^2\theta\right)^2 = 0$

$\Rightarrow \sqrt{\dfrac{b}{a}}\sin^2\theta = \sqrt{\dfrac{a}{b}}\cos^2\theta$

$\Rightarrow b\sin^2\theta = a\cos^2\theta$

$\Rightarrow \dfrac{\sin^2\theta}{a} = \dfrac{\cos^2\theta}{b} = \dfrac{\sin^2\theta + \cos^2\theta}{a+b}$

$\Rightarrow \dfrac{\sin^2\theta}{a} = \dfrac{\cos^2\theta}{b} = \dfrac{1}{a+b}$

$\Rightarrow \sin^2\theta = \dfrac{a}{a+b}$ and $\cos^2\theta = \dfrac{b}{a+b}$

Only option (B) does not satisfy these values.
Hence, option (B) is incorrect.

8. $x\sin\theta = y\cos\theta = \dfrac{2z\tan\theta}{1-\tan^2\theta}$

Consider, $x\sin\theta = \dfrac{2z\tan\theta}{1-\tan^2\theta}$

$\Rightarrow x\sin\theta = \dfrac{2z\dfrac{\sin\theta}{\cos\theta}}{\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}}$

$\Rightarrow x\sin\theta = \dfrac{2z\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$

$\Rightarrow x = \dfrac{2z\cos\theta}{\cos^2\theta - \sin^2\theta}$

$\Rightarrow z = \dfrac{x(\cos^2\theta - \sin^2\theta)}{2\cos\theta}$ ....(i)

Similarly, by solving $y\cos\theta = \dfrac{2z\tan\theta}{1-\tan^2\theta}$, we get

$z = \dfrac{y(\cos^2\theta - \sin^2\theta)}{2\sin\theta}$ ....(ii)

From (i) and (ii), we get $\tan\theta = \dfrac{y}{x}$

$\therefore$ $x\sin\theta = \dfrac{2\dfrac{zy}{x}}{1 - \dfrac{y^2}{x^2}} = \dfrac{2xyz}{x^2 - y^2}$

$\Rightarrow \sin\theta = \dfrac{2yz}{x^2 - y^2}$

Similarly, $\cos\theta = \dfrac{2xz}{x^2 - y^2}$

Since, $\sin^2\theta + \cos^2\theta = 1$

$\therefore \quad \left(\dfrac{2yz}{x^2-y^2}\right)^2 + \left(\dfrac{2xz}{x^2-y^2}\right)^2 = 1$

$\Rightarrow \dfrac{4z^2(x^2+y^2)}{(x^2-y^2)^2} = 1$

$\Rightarrow 4z^2(x^2 + y^2) = (x^2 - y^2)^2$

9. $3\cot A = 6\sec B = -2\sqrt{10}$

Consider, $\cot A = -\dfrac{2}{3}\sqrt{10}$

$\Rightarrow \cot^2 A = \dfrac{40}{9}$

$\therefore \quad \operatorname{cosec}^2 A = \dfrac{49}{9}$

$\Rightarrow \operatorname{cosec} A = \dfrac{7}{3} \quad \ldots\left[\because \dfrac{\pi}{2} < A < \pi\right]$

Also, $6\sec B = -2\sqrt{10}$

$\Rightarrow \cos B = -\dfrac{3}{\sqrt{10}}$

$\Rightarrow \sin B = -\sqrt{1-\cos^2 B} \quad \ldots\left[\because \pi < B < \dfrac{3\pi}{2}\right]$

$= -\sqrt{1 - \dfrac{9}{10}} = -\dfrac{1}{\sqrt{10}}$

$\therefore \quad \tan B = \dfrac{\sin B}{\cos B} = \dfrac{1}{3}$

$\therefore \quad \operatorname{cosec} A - \tan B = \dfrac{7}{3} - \dfrac{1}{3}$

$= 2$

10. $x = \displaystyle\sum_{n=0}^{\infty} \cos^{2n}\phi$

$= 1 + \cos^2\phi + \cos^4\phi + \ldots \infty$

$= \dfrac{1}{(1-\cos^2\phi)} \quad \ldots$[Since infinite G.P.]

$= \dfrac{1}{\sin^2\phi}$

$y = \displaystyle\sum_{n=0}^{\infty} \sin^{2n}\phi$

$= 1 + \sin^2\phi + \sin^4\phi + \ldots \infty$

$= \dfrac{1}{(1-\sin^2\phi)} = \dfrac{1}{\cos^2\phi}$

$z = \displaystyle\sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$

$= 1 + \cos^2\phi\sin^2\phi + \cos^4\phi\sin^4\phi + \ldots\infty$

$= \dfrac{1}{(1-\cos^2\phi\sin^2\phi)}$

Now, $xyz = \dfrac{1}{\sin^2\phi\cos^2\phi(1-\cos^2\phi\sin^2\phi)} \quad \ldots$(i)

$\therefore \quad xy + z = \dfrac{1}{\sin^2\phi\cos^2\phi} + \dfrac{1}{1-\cos^2\phi\sin^2\phi}$

$= \dfrac{1}{\sin^2\phi\cos^2\phi(1-\cos^2\phi\sin^2\phi)}$

$= xyz \quad \ldots$[From (i)]

Textbook Chapter No.

03 Trigonometric Functions of Compound Angles

Hints

Classical Thinking

1. $\cos 105° = \cos(60° + 45°)$
$= \cos 60° \cos 45° - \sin 60° \sin 45°$
$= \dfrac{1-\sqrt{3}}{2\sqrt{2}}$

2. $\tan 15° = \tan(45° - 30°)$
$= \dfrac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$
$\ldots\left[\because \tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}\right]$
$= \dfrac{\sqrt{3}-1}{\sqrt{3}+1} \times \dfrac{\sqrt{3}-1}{\sqrt{3}-1}$
$= 2-\sqrt{3}$

3. $\cos 38° \cos 8° + \sin 38° \sin 8°$
$= \cos(38° - 8°) = \cos 30°$

4. $\dfrac{1}{4}(\sqrt{3}\cos 23° - \sin 23°)$
$= \dfrac{1}{2}(\cos 30° \cos 23° - \sin 30° \sin 23°)$
$\ldots\left[\because \cos 30° = \dfrac{\sqrt{3}}{2}, \sin 30° = \dfrac{1}{2}\right]$
$= \dfrac{1}{2}\cos(30° + 23°) = \dfrac{1}{2}\cos 53°$

5. $\tan 5A = \tan(3A + 2A)$
$= \dfrac{\tan 3A + \tan 2A}{1 - \tan 3A \tan 2A}$
$\Rightarrow \tan 5A - \tan 5A \tan 3A \tan 2A$
$= \tan 3A + \tan 2A$
$\Rightarrow \tan 5A - \tan 3A - \tan 2A$
$= \tan 5A \tan 3A \tan 2A$

6. $\tan(57° - 12°) = \tan 45°$
$\Rightarrow \dfrac{\tan 57° - \tan 12°}{1+\tan 57° \tan 12°} = 1$
$\Rightarrow \tan 57° - \tan 12° = 1 + \tan 57° \tan 12°$
$\Rightarrow \tan 57° - \tan 12° - \tan 57° \tan 12° = 1$
$= \tan 45°$

7. $\dfrac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \dfrac{1+\tan 10°}{1-\tan 10°}$
$= \tan(45° + 10°)$
$\ldots[\because \tan 45° = 1]$
$= \tan 55°$

8. $\dfrac{\cos 8° - \sin 8°}{\cos 8° + \sin 8°} = \dfrac{1-\tan 8°}{1+\tan 8°}$
$= \tan(45° - 8°) = \tan 37°$

9. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
$= \dfrac{\frac{a}{a+1} + \frac{1}{2a+1}}{1 - \frac{a}{a+1}\cdot\frac{1}{2a+1}}$
$= \dfrac{2a^2 + a + a + 1}{2a^2 + 2a + a + 1 - a}$
$= \dfrac{2a^2 + 2a + 1}{2a^2 + 2a + 1}$
$= 1 = \tan\dfrac{\pi}{4}$

$\therefore$ $A + B = \dfrac{\pi}{4}$

10. $\cot(A - B) = \dfrac{1}{\tan(A-B)}$
$= \dfrac{1+\tan A \tan B}{\tan A - \tan B}$
$= \dfrac{1}{\tan A - \tan B} + \dfrac{\tan A \tan B}{\tan A - \tan B}$
$= \dfrac{1}{\tan A - \tan B} + \dfrac{1}{\cot B - \cot A}$
$= \dfrac{1}{x} + \dfrac{1}{y}$

11. Since, $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$

$\therefore$ $\cos^2 48° - \sin^2 12° = \cos 60°.\cos 36°$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}\right)$$

$$= \frac{\sqrt{5}+1}{8}$$

12. $$\frac{\cot^2 15° - 1}{\cot^2 15° + 1} = \frac{\frac{\cos^2 15°}{\sin^2 15°} - 1}{\frac{\cos^2 15°}{\sin^2 15°} + 1}$$

$$= \frac{\cos^2 15° - \sin^2 15°}{\cos^2 15° + \sin^2 15°} = \cos(30°)$$

$[\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)]$

$$= \frac{\sqrt{3}}{2}$$

13. $\tan(-945°) = \tan[-(945°)]$

$= -\tan[(2 \times 360° + 225°)]$

$= -\tan(225°)$

$= -\tan 45°$

….$[\because \tan(180° + \theta) = \tan\theta]$

$= -1$

14. $\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \sin 18°.\sin 54°$

$= \sin 18°.\cos 36°$

….$[\because \sin(90° - \theta) = \cos\theta]$

$$= \frac{\sqrt{5}-1}{4}.\frac{\sqrt{5}+1}{4}$$

$$= \frac{1}{4}$$

15. $\sin 15° + \cos 105°$

$= \sin 15° + \cos(90° + 15°)$

$= \sin 15° - \sin 15°$

….$[\because \cos(90° + \theta) = -\sin\theta]$

$= 0$

16. $$\tan(A+B) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}} = 1$$

$\therefore$ $A + B = 45°$

$\Rightarrow 2A = 90° - 2B$

$\Rightarrow \cos 2A = \sin 2B$ ….$[\because \cos(90° - \theta) = \sin\theta]$

17. $\cos 7\theta + \cos\theta = \cos(8\theta - \theta) + \cos\theta$

$= \cos(\pi - \theta) + \cos\theta$

….$[\because 8\theta = \pi\text{(given)}]$

$= -\cos\theta + \cos\theta$

$= 0$

18 Since, $A + C = 180°$ and $B + D = 180°$

$\therefore$ $\cos A + \cos B = \cos(180° - C) + \cos(180° - D)$

$= -(\cos C + \cos D)$

….$[\because \cos(180° - \theta) = -\cos\theta]$

19. Since, ABCD is a cyclic quadrilateral.

$\therefore$ $A + C = 180°$

$\Rightarrow A = 180° - C$

$\Rightarrow \cos A = \cos(180° - C) = -\cos C$

$\Rightarrow \cos A + \cos C = 0$ .....(i)

Also, $B + D = 180°$

$\Rightarrow \cos B + \cos D = 0$ ....(ii)

Subtracting (ii) from (i), we get

$\cos A - \cos B + \cos C - \cos D = 0$

20. $\sin 10° + \sin 20° + \sin 30° + … + \sin 180°$
$+ \sin(180° + 10°) + \sin(180° + 20°) + ….$
$+ \sin(180° + 180°)$

$= 0$ ….$[\because \sin(180° + \theta) = -\sin\theta]$

21. $(\cos 1° + \cos 179°) + (\cos 2° + \cos 178°)$
$+ …. + (\cos 89° + \cos 91°)$
$+ (\cos 90° + \cos 180°)$

$= -1$ ….$[\because \cos(180° - \theta) = -\cos\theta]$

22. $$\sec\left(\frac{7\pi}{2} - A\right) = \sec\left(2\pi + \frac{3\pi}{2} - A\right)$$

$$= \sec\left(\frac{3\pi}{2} - A\right)$$

$= -\text{cosec } A$

23. $$\frac{\cot 54°}{\tan 36°} + \frac{\tan 20°}{\cot 70°}$$

$$= \frac{\cot(90° - 36°)}{\tan 36°} + \frac{\tan 20°}{\cot(90° - 20°)}$$

$= 1 + 1$ ….$[\because \cot(90° - \theta) = \tan\theta]$

$= 2$

24. $$\frac{\cos(90° + \theta)\sec(-\theta)\tan(180° - \theta)}{\sin(360° + \theta)\sec(180° + \theta)\cot(90° - \theta)}$$

$$= \frac{(-\sin\theta)(\sec\theta)(-\tan\theta)}{(\sin\theta)(-\sec\theta)\tan\theta}$$

$= -1$

25. $\sin^2 25° + \sin^2 65° = \sin^2 25° + \sin^2 (90° - 25°)$
$= \sin^2 25° + \cos^2 25°$
....[$\because \sin(90° - \theta) = \cos\theta$]
$= 1$

26. $\sin\frac{7\pi}{8} = \sin\left(\pi - \frac{\pi}{8}\right) = \sin\frac{\pi}{8}$

$\sin\frac{5\pi}{8} = \sin\left(\pi - \frac{3\pi}{8}\right) = \sin\frac{3\pi}{8}$

$\therefore$ $\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8}$

$= 2\left[\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8}\right]$

$= 2\left[\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}\right]$

....$\left[\because \sin\frac{3\pi}{8} = \sin\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos\frac{\pi}{8}\right]$

$= 2$

27. $\cos 2\theta = 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$
$= \frac{1-\tan^2\theta}{1+\tan^2\theta}$

28. $\sin 4\theta = 2\sin 2\theta\cos 2\theta$
$= 2.2\sin\theta\cos\theta(1 - 2\sin^2\theta)$
$= 4\sin\theta(1 - 2\sin^2\theta)\sqrt{1-\sin^2\theta}$

29. $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$\therefore$ $\tan 2\theta + \sec 2\theta = \frac{2t}{1-t^2} + \frac{1+t^2}{1-t^2}$
....[$\because \tan\theta = t$(given)]

$= \frac{(1+t)^2}{(1-t)(1+t)} = \frac{1+t}{1-t}$

30. Given, $\sin A + \cos A = 1$
Squaring on both sides, we get
$(\sin A + \cos A)^2 = 1$
$\Rightarrow 1 + \sin 2A = 1$
$\Rightarrow \sin 2A = 0$

31. $2 + 2\cos 4\theta = 2(1 + \cos 4\theta)$
$= 4\cos^2 2\theta$ ....(i)

$\therefore$ $\sqrt{2+\sqrt{2+2\cos 4\theta}} = \sqrt{2+2\cos 2\theta}$
....[From (i)]
$= \sqrt{2(1+\cos 2\theta)}$
$= \sqrt{4\cos^2\theta}$
$= 2\cos\theta$

32. $1 + \cos^2 2A = (\cos^2 A + \sin^2 A)^2 + (\cos^2 A - \sin^2 A)^2$
$= 2(\cos^4 A + \sin^4 A)$

33. $1 - 2\sin^2\left(\frac{\pi}{4} + \theta\right) = \cos\left(\frac{\pi}{2} + 2\theta\right)$
....$\left[\because \cos 2\theta = 1 - 2\sin^2\theta\right]$
$= -\sin 2\theta$

34. $\sin\theta\cos\theta = \frac{1}{2}(\sin 2\theta)$
Since, $-1 \le \sin 2\theta \le 1$
$\Rightarrow -\frac{1}{2} \le \frac{1}{2}(\sin 2\theta) \le \frac{1}{2}$

$\therefore$ Largest value is $\frac{1}{2}$.

35. $(\sec 2A + 1)\sec^2 A$
$= \left(\frac{1+\tan^2 A}{1-\tan^2 A} + 1\right)(1 + \tan^2 A)$
$= \frac{2(1+\tan^2 A)}{1-\tan^2 A} = 2\sec 2A$

36. $\operatorname{cosec} A - 2\cot 2A\cos A$
$= \frac{1}{\sin A} - \frac{2\cos A\cos 2A}{\sin 2A}$
$= \frac{1}{\sin A} - \frac{2\cos A\cos 2A}{2\sin A\cos A}$
$= \frac{1-\cos 2A}{\sin A} = \frac{2\sin^2 A}{\sin A} = 2\sin A$

37. $\cos 20°\cos 40°\cos 80° = \frac{\sin 2^3\, 20^\circ}{2^3\sin 20^\circ}$
$= \frac{\sin 160^\circ}{8\sin 20^\circ}$
$= \frac{\sin(180° - 20°)}{8\sin 20°}$
$= \frac{1}{8}$

38. $\frac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \frac{\sin\theta + 2\sin\theta\cos\theta}{2\cos^2\theta + \cos\theta}$
$= \frac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)}$
$= \tan\theta$

39. $\sin^3\theta + \cos^3\theta$
$= (\sin\theta + \cos\theta)\left(\cos^2\theta + \sin^2\theta - \frac{\sin 2\theta}{2}\right)$

$= \sqrt{(\sin\theta+\cos\theta)^2}\left(1-\frac{\sin 2\theta}{2}\right)$

$\Rightarrow \sin^3\theta + \cos^3\theta = \sqrt{1+\frac{3}{4}}\left(1-\frac{3}{8}\right)$

$= \frac{\sqrt{7}}{2}\times\frac{5}{8} = \frac{5\sqrt{7}}{16}$

40. Given that, $\cos 3\theta = \alpha\cos\theta + \beta\cos^3\theta$
But, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
$\Rightarrow (\alpha, \beta) = (-3, 4)$

41. Given, $\tan A = \frac{1}{2}$

$\Rightarrow \tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

$= \frac{3.\frac{1}{2}-\frac{1}{8}}{1-3.\frac{1}{4}} = \frac{12-1}{2}$

$= \frac{11}{2}$

42. We have, $x + \frac{1}{x} = 2\cos\theta$

Now, $x^3+\frac{1}{x^3} = \left(x+\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{x}\right)$
$= (2\cos\theta)^3 - 3(2\cos\theta)$
$= 8\cos^3\theta - 6\cos\theta$
$= 2(4\cos^3\theta - 3\cos\theta) = 2\cos 3\theta$

43. $\cos\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1+\cos\alpha}{2}}$

$\ldots\left[\because \pi<\alpha<\frac{3\pi}{2}\Rightarrow\frac{\pi}{2}<\frac{\alpha}{2}<\frac{3\pi}{4}\right]$

Now, $\cos\alpha = -\sqrt{1-\sin^2\alpha}$

$\ldots\left[\because \pi<\alpha<\frac{3\pi}{2}\right]$

$= -\sqrt{1-\frac{9}{25}} = -\frac{4}{5}$

$\therefore \cos\frac{\alpha}{2} = -\sqrt{\frac{1-\frac{4}{5}}{2}} = -\frac{1}{\sqrt{10}}$

44. $\frac{1-t^2}{1+t^2} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \cos\theta$

45. Given that, $\tan\frac{A}{2} = \frac{3}{2}$

$\therefore \frac{1+\cos A}{1-\cos A} = \frac{2\cos^2\frac{A}{2}}{2\sin^2\frac{A}{2}} = \cot^2\frac{A}{2}$

$= \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

46. $\tan\left(\frac{A}{2}\right) = \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)} = \sqrt{\frac{\frac{1-\cos A}{2}}{\frac{1+\cos A}{2}}}$

$= \sqrt{\frac{1-\cos A}{1+\cos A}}$

47. $\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$

$= \frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}$

$\ldots\left[\because \cos\theta=\frac{\tan\beta}{\tan\alpha}\text{(given)}\right]$

$= \frac{\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}}{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}$

$= \frac{\sin\alpha\cos\beta - \sin\beta\cos\alpha}{\sin\alpha\cos\beta + \sin\beta\cos\alpha}$

$= \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$

Critical Thinking

1. $\cos(A + B) = \alpha\cos A\cos B + \beta\sin A\sin B$
But, $\cos(A + B) = \cos A\cos B - \sin A\sin B$
$\Rightarrow \alpha = 1, \beta = -1$

2. $\cos(A + B) = \cos A\cos B - \sin A\sin B$
$= \sqrt{1-\frac{16}{25}}\left(-\frac{12}{13}\right) - \frac{4}{5}\sqrt{1-\frac{144}{169}}$

$= \frac{3}{5}\left(-\frac{12}{13}\right) - \frac{4}{5}\left(-\frac{5}{13}\right)$

....[$\because$ A lies in first quadrant and B lies in third quadrant]

$= -\frac{16}{65}$

3. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$= \frac{1}{\sqrt{10}}\sqrt{1-\frac{1}{5}} + \frac{1}{\sqrt{5}}\sqrt{1-\frac{1}{10}}$

$= \frac{1}{\sqrt{10}}\sqrt{\frac{4}{5}} + \frac{1}{\sqrt{5}}\sqrt{\frac{9}{10}}$

$= \frac{1}{\sqrt{50}}(2+3) = \frac{5}{\sqrt{50}}$

$= \frac{1}{\sqrt{2}}$

$\Rightarrow \sin(A+B) = \sin\frac{\pi}{4}$

$\Rightarrow A + B = \frac{\pi}{4}$

4. $\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}} = 1$

$\tan(2\theta+\phi) = \tan[\theta + (\theta+\phi)]$

$= \frac{\tan\theta + \tan(\theta+\phi)}{1-\tan\theta\tan(\theta+\phi)}$

$= \frac{\frac{1}{2}+1}{1-\frac{1}{2}.1} = 3$

5. $\cos\theta = \frac{8}{17}$ and $0 < \theta < \frac{\pi}{2}$

$\Rightarrow \sin\theta = \sqrt{1-\left(\frac{8}{17}\right)^2}$

$= \frac{15}{17}$

$\cos(30° + \theta) + \cos(45° - \theta) + \cos(120° - \theta)$
$= \cos 30° \cos\theta - \sin 30° \sin\theta + \cos 45° \cos\theta$
$+ \sin 45° \sin\theta + \cos 120° \cos\theta + \sin 120° \sin\theta$

$= \cos\theta\left(\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}-\frac{1}{2}\right) - \sin\theta\left(\frac{1}{2}-\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\right)$

$= \frac{8}{17}\left(\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}-\frac{1}{2}\right) + \frac{15}{17}\left(\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}-\frac{1}{2}\right)$

$= \frac{23}{17}\left(\frac{\sqrt{3}-1}{2}+\frac{1}{\sqrt{2}}\right)$

6. $A + B = \frac{\pi}{4}$

$\Rightarrow \tan(A+B) = \tan\frac{\pi}{4}$

$\Rightarrow \frac{\tan A + \tan B}{1-\tan A\tan B} = 1$

$\Rightarrow \tan A + \tan B + \tan A \tan B = 1$
$\Rightarrow (1 + \tan A)(1 + \tan B) = 2$

7. $A + B = 45°$
$\Rightarrow \tan(A+B) = 1$
$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \frac{1}{\cot A} + \frac{1}{\cot B} = 1 - \frac{1}{\cot A \cot B}$

$\Rightarrow \cot A + \cot B = \cot A \cot B - 1$
$\Rightarrow \cot A \cot B - \cot A - \cot B = 1$
$\Rightarrow \cot A \cot B - \cot A - \cot B + 1 = 2$
$\Rightarrow (\cot A - 1)(\cot B - 1) = 2$

8. $\sin(\alpha-\beta) = \sin[(\theta-\beta) - (\theta-\alpha)]$
$= \sin(\theta-\beta)\cos(\theta-\alpha) - \cos(\theta-\beta)\sin(\theta-\alpha)$
$= ba - \sqrt{1-b^2}\sqrt{1-a^2}$
and $\cos(\alpha-\beta) = \cos[(\theta-\beta) - (\theta-\alpha)]$
$= \cos(\theta-\beta)\cos(\theta-\alpha) + \sin(\theta-\beta)\sin(\theta-\alpha)$
$= a\sqrt{1-b^2} + b\sqrt{1-a^2}$

$\therefore$ $\cos^2(\alpha-\beta) + 2ab\sin(\alpha-\beta)$
$= \left(a\sqrt{1-b^2} + b\sqrt{1-a^2}\right)^2 + 2ab\left(ab - \sqrt{1-a^2}\sqrt{1-b^2}\right)$
$= a^2 + b^2$

9. Let $x - y = \alpha$, $y - z = \beta$ and $z - x = \gamma$,
then $\alpha + \beta + \gamma = 0$
$\Rightarrow \alpha + \beta = -\gamma$
$\Rightarrow \tan(\alpha+\beta) = \tan(-\gamma)$

$\Rightarrow \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = -\tan\gamma$

$\Rightarrow \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

10. Since, $\tan 3A = \frac{\tan A + \tan 2A}{1-\tan A \tan 2A}$

$\Rightarrow \tan 3A - \tan 2A - \tan A$
$= \tan 3A \tan 2A \tan A$

11. $\tan(20° + 40°) = \dfrac{\tan 20° + \tan 40°}{1 - \tan 20° \tan 40°}$

$\Rightarrow \sqrt{3} = \dfrac{\tan 20° + \tan 40°}{1 - \tan 20° \tan 40°}$

$\Rightarrow \sqrt{3} - \sqrt{3}\tan 20° \tan 40° = \tan 20° + \tan 40°$

$\Rightarrow \tan 20° + \tan 40° + \sqrt{3}\tan 20° \tan 40° = \sqrt{3}$

12. $\tan\left(\dfrac{6\pi}{15} - \dfrac{\pi}{15}\right) = \tan\dfrac{\pi}{3}$

$\Rightarrow \dfrac{\tan\dfrac{6\pi}{15} - \tan\dfrac{\pi}{15}}{1 + \tan\dfrac{6\pi}{15}\tan\dfrac{\pi}{15}} = \tan\dfrac{\pi}{3}$

$\Rightarrow \tan\dfrac{6\pi}{15} - \tan\dfrac{\pi}{15} = \sqrt{3} + \sqrt{3}\tan\dfrac{6\pi}{15}\tan\dfrac{\pi}{15}$

$\Rightarrow \tan\dfrac{6\pi}{15} - \tan\dfrac{\pi}{15} - \sqrt{3}\tan\dfrac{6\pi}{15}\tan\dfrac{\pi}{15} = \sqrt{3}$

$\Rightarrow \tan\dfrac{2\pi}{5} - \tan\dfrac{\pi}{15} - \sqrt{3}\tan\dfrac{2\pi}{5}\tan\dfrac{\pi}{15} = \sqrt{3}$

13. $2\tan(A - B) = 2\left(\dfrac{\tan A - \tan B}{1 + \tan A \tan B}\right)$

$= 2\left[\dfrac{2\tan B + \cot B - \tan B}{1 + (2\tan B + \cot B)\tan B}\right]$

…. [$\because \tan A = 2\tan B + \cot B$]

$= 2\left[\dfrac{\tan B + \cot B}{2(1 + \tan^2 B)}\right]$

$= \dfrac{\cot B(\tan^2 B + 1)}{1 + \tan^2 B}$

$= \cot B$

14. $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma)$

$= \sin\alpha + \sin\beta + \sin\gamma - \sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\cos\gamma - \cos\alpha\cos\beta\sin\gamma + \sin\alpha\sin\beta\sin\gamma$

$= \sin\alpha(1 - \cos\beta\cos\gamma) + \sin\beta(1 - \cos\alpha\cos\gamma) + \sin\gamma(1 - \cos\alpha\cos\beta) + \sin\alpha\sin\beta\sin\gamma > 0$

$\therefore$ $\sin\alpha + \sin\beta + \sin\gamma > \sin(\alpha + \beta + \gamma)$

$\Rightarrow \dfrac{\sin(\alpha + \beta + \gamma)}{\sin\alpha + \sin\beta + \sin\gamma} < 1$

15. $\dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A}$

$= \dfrac{1}{\tan 3A - \tan A} + \dfrac{\tan A \tan 3A}{\tan 3A - \tan A}$

$= \dfrac{1}{\dfrac{\tan 3A - \tan A}{1 + \tan 3A.\tan A}} = \dfrac{1}{\tan 2A} = \cot 2A$

16. $\cos 105° + \sin 105°$

$= \cos(90° + 15°) + \sin(90° + 15°)$

$= \cos 15° - \sin 15°$

$= \cos(45° - 30°) - \sin(45° - 30°)$

$= \dfrac{\sqrt{3}+1}{2\sqrt{2}} - \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

$= \dfrac{2}{2\sqrt{2}}$

$= \dfrac{1}{\sqrt{2}}$

17. $\tan 81° - \tan 63° - \tan 27° + \tan 9°$

$= \{\tan(90° - 9°) + \tan 9°\} - \{\tan(90° - 27°) + \tan 27°\}$

$= (\cot 9° + \tan 9°) - (\cot 27° + \tan 27°)$

$= 2\operatorname{cosec} 18° - 2\operatorname{cosec} 54°$

….[$\because \tan\theta + \cot\theta = 2\operatorname{cosec} 2\theta$]

$= \dfrac{2}{\sin 18°} - \dfrac{2}{\sin 54°}$

$= \dfrac{2}{\sin 18°} - \dfrac{2}{\cos 36°}$

$= \dfrac{2\times 4}{\sqrt{5}-1} - \dfrac{2\times 4}{\sqrt{5}+1}$

$= 8\left[\dfrac{\sqrt{5}+1-\sqrt{5}+1}{\left(\sqrt{5}\right)^2 - 1^2}\right]$

$= 4$

18. $\beta + \gamma = \alpha$

$\Rightarrow \gamma = \alpha - \beta$

$\Rightarrow \tan\gamma = \tan(\alpha - \beta)$

$\Rightarrow \tan\gamma = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$\Rightarrow \tan\gamma = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\cot\alpha}$

….$\left[\because \alpha + \beta = \dfrac{\pi}{2}, \therefore \beta = \dfrac{\pi}{2} - \alpha\right]$

$\Rightarrow \tan\gamma = \dfrac{1}{2}(\tan\alpha - \tan\beta)$

$\Rightarrow \tan\alpha = \tan\beta + 2\tan\gamma$

19. $\tan\left(\dfrac{\pi}{4} + \theta\right)\tan\left(\dfrac{3\pi}{4} + \theta\right)$

$= \tan\left(\dfrac{\pi}{4} + \theta\right)\tan\left(\dfrac{\pi}{2} + \left(\dfrac{\pi}{4} + \theta\right)\right)$

$= \tan\left(\frac{\pi}{4}+\theta\right)\left\{-\cot\left(\frac{\pi}{4}+\theta\right)\right\}$

$\ldots\left[\because \tan\left(\frac{\pi}{2}+\alpha\right) = -\cot\alpha\right]$

$= -1$

20. $\tan(100° + 125°) = \frac{\tan 100° + \tan 125°}{1 - \tan 100° \tan 125°}$

$\Rightarrow \tan 225° = \frac{\tan 100° + \tan 125°}{1 - \tan 100° \tan 125°}$

$\Rightarrow 1 = \frac{\tan 100° + \tan 125°}{1 - \tan 100° \tan 125°}$

$\ldots[\because \tan(180° + 45°) = \tan 45° = 1]$

$\Rightarrow \tan 100° + \tan 125° + \tan 100° \tan 125° = 1$

21. $\frac{\cot A}{1+\cot A} \cdot \frac{\cot B}{1+\cot B}$

$= \frac{1}{(1+\tan A)(1+\tan B)}$

$= \frac{1}{\tan A + \tan B + 1 + \tan A \tan B}$

$= \frac{1}{1 - \tan A \tan B + 1 + \tan A \tan B}$

$\ldots\left[\begin{array}{l}\because \tan(A+B) = \tan 225° \\ \Rightarrow \tan A + \tan B = 1 - \tan A \tan B\end{array}\right]$

$= \frac{1}{2}$

22. Given expression

$= \frac{-\sin(660^\circ)\tan(1050^\circ)\sec(420^\circ)}{\cos(180^\circ + 45^\circ)\operatorname{cosec}(360^\circ - 45^\circ)\cos(360^\circ + 150^\circ)}$

$= \frac{-\sin(7\times 90 + 30^\circ)\tan(3\times 360^\circ - 30^\circ)\sec(360^\circ + 60^\circ)}{(-\cos 45^\circ)(-\operatorname{cosec} 45^\circ)\cos 150^\circ}$

$= \frac{\cos(30^\circ)(-\tan 30^\circ)\sec 60^\circ}{(-\cos 45^\circ)(-\operatorname{cosec} 45^\circ)(-\cos 30^\circ)}$

$= \frac{\cos 30° \tan 30° \sec 60°}{\cos 45° \operatorname{cosec} 45° \cos 30°}$

$= \frac{\frac{1}{\sqrt{3}}\cdot 2}{\frac{1}{\sqrt{2}}\cdot\sqrt{2}}$

$= \frac{2}{\sqrt{3}}$

23. $\cos^2\left(\frac{\pi}{4}-\beta\right) - \sin^2\left(\alpha - \frac{\pi}{4}\right)$

$= \cos\left(\frac{\pi}{4} - \beta + \alpha - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4} - \beta - \alpha + \frac{\pi}{4}\right)$

$= \cos(\alpha - \beta)\cos\left(\frac{\pi}{2} - (\alpha + \beta)\right)$

$= \cos(\alpha - \beta)\sin(\alpha + \beta)$

24. $\cos^2\frac{\pi}{12} + \cos^2\frac{\pi}{4} + \cos^2\frac{5\pi}{12}$

$= \cos^2 15° + \cos^2 45° + \cos^2 75°$

$= \cos^2 15° + \cos^2 75° + \cos^2 45°$

$= \cos^2 15° + \sin^2 15° + \left(\frac{1}{\sqrt{2}}\right)^2$

$\ldots[\because \cos^2\theta = \sin^2(90° - \theta)]$

$= 1 + \frac{1}{2}$

$= \frac{3}{2}$

25. $\cos 2\alpha = \frac{1-t^2}{1+t^2} = \frac{24}{25}$ (Let $t = \tan\alpha$)

$\sin 2\beta = \frac{2T}{1+T^2} = \frac{3}{5}$ (Let $T = \tan\beta$)

$\Rightarrow \cos 2\beta = \frac{4}{5}$

Now, $\sin 4\beta = 2\sin 2\beta\cos 2\beta$

$= 2.\frac{3}{5}.\frac{4}{5}$

$= \frac{24}{25}$

$= \cos 2\alpha$

26. Given that, $\cos\theta = \frac{1}{2}\left(x + \frac{1}{x}\right)$

$\Rightarrow x + \frac{1}{x} = 2\cos\theta$

Now, $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$

$= (2\cos\theta)^2 - 2$

$= 4\cos^2\theta - 2 = 2\cos 2\theta$

$\therefore \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) = \frac{1}{2}\times 2\cos 2\theta = \cos 2\theta$

27. $\sin x + \cos x = \frac{1}{5}$

Squaring both sides

$1 + \sin 2x = \frac{1}{25}$

$\Rightarrow \sin 2x = \frac{-24}{25}$

$\Rightarrow \frac{2\tan x}{1+\tan^2 x} = \frac{-24}{25}$

$\Rightarrow 24 \tan^2 x + 50 \tan x + 24 = 0$

$\Rightarrow 12 \tan^2 x + 25 \tan x + 12 = 0$

$\Rightarrow (3 \tan x + 4)(4 \tan x + 3) = 0$

$\Rightarrow \tan x = \frac{-4}{3}$ or $\frac{-3}{4}$

28. $\cos x + \sin x = \frac{1}{2}$

$\Rightarrow (\cos x + \sin x)^2 = \frac{1}{4}$

$\Rightarrow 1 + \sin 2x = \frac{1}{4}$

$\Rightarrow \sin 2x = -\frac{3}{4}$

$\Rightarrow \frac{2\tan x}{1+\tan^2 x} = -\frac{3}{4}$

$\Rightarrow 3 \tan^2 x + 8 \tan x + 3 = 0$

$\Rightarrow \tan x = \frac{-4 \pm \sqrt{7}}{3}$

29. $\tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right) = \frac{1+\tan\theta}{1-\tan\theta} - \frac{1-\tan\theta}{1+\tan\theta}$

$= \frac{4\tan\theta}{1-\tan^2\theta}$

$= 2\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$= 2\tan 2\theta$

30. $\sec 2\theta = p + \tan 2\theta$

$\Rightarrow \sec 2\theta - \tan 2\theta = p$ ....(i)

$\Rightarrow \sec 2\theta + \tan 2\theta = \frac{1}{p}$ ....(ii)

....[$\because \sec^2 2\theta - \tan^2 2\theta = 1$]

Adding (i) and (ii), we get

$2 \sec 2\theta = p + \frac{1}{p}$

$\Rightarrow \cos 2\theta = \frac{2p}{p^2+1} \Rightarrow 1 - 2\sin^2\theta = \frac{2p}{p^2+1}$

$\Rightarrow 2\sin^2\theta = 1 - \frac{2p}{p^2+1} \Rightarrow 2\sin^2\theta = \frac{(p-1)^2}{p^2+1}$

$\Rightarrow \sin^2\theta = \frac{(p-1)^2}{2(p^2+1)}$

31. Given, $\tan\theta = \frac{1}{7}$, $\sin\phi = \frac{1}{\sqrt{10}}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{50}}$, $\cos\theta = \frac{7}{\sqrt{50}}$, $\cos\phi = \frac{3}{\sqrt{10}}$

$\therefore$ $\cos 2\phi = 2\cos^2\phi - 1 = 2\left(\frac{9}{10}\right) - 1 = \frac{4}{5}$

$\sin 2\phi = 2\sin\phi\cos\phi = 2\left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right) = \frac{3}{5}$

$\therefore$ $\cos(\theta + 2\phi) = \cos\theta\cos 2\phi - \sin\theta\sin 2\phi$

$= \frac{7}{\sqrt{50}}\cdot\frac{4}{5} - \frac{1}{\sqrt{50}}\cdot\frac{3}{5} = \frac{28}{5\sqrt{50}} - \frac{3}{5\sqrt{50}}$

$= \frac{25}{5\sqrt{50}} = \frac{1}{\sqrt{2}}$

$\therefore$ $\theta + 2\phi = 45°$

32. $\cos(\alpha - \beta) = \cos[\theta + \alpha - (\theta + \beta)]$

$= \cos(\theta + \alpha)\cos(\theta + \beta) + \sin(\theta + \alpha)\sin(\theta + \beta)$

$= \sqrt{1-a^2}\sqrt{1-b^2} + ab$

Now, $\cos 2(\alpha - \beta) - 4ab\cos(\alpha - \beta)$

$= 2\cos^2(\alpha - \beta) - 1 - 4ab\cos(\alpha - \beta)$

$= 2\left(\sqrt{1-a^2}\sqrt{1-b^2} + ab\right)^2 - 4ab\left(\sqrt{1-a^2}\sqrt{1-b^2} + ab\right) - 1$

$= 2\{(1-a^2)(1-b^2) + a^2b^2 + 2ab\sqrt{1-a^2}\sqrt{1-b^2}\} - 4ab(\sqrt{1-a^2}\sqrt{1-b^2} + ab) - 1$

$= 2(1 - b^2 - a^2 + a^2b^2) + 2a^2b^2 - 4a^2b^2 - 1$

$= 2(1 - a^2 - b^2) - 1$

$= 1 - 2a^2 - 2b^2$

33. $\tan^2\theta = 2\tan^2\phi + 1$

$\Rightarrow 1 + \tan^2\theta = 2(1 + \tan^2\phi)$

$\Rightarrow \sec^2\theta = 2\sec^2\phi$

$\Rightarrow \cos^2 \phi = 2 \cos^2 \theta$

$\Rightarrow \cos^2 \phi = 1 + \cos 2\theta$

$\Rightarrow \sin^2 \phi + \cos 2\theta = 0$ ....[$\because \sin^2\theta + \cos^2\theta = 1$]

34. $\tan \theta - \cot \theta = a$ and $\sin \theta + \cos \theta = b$

$\therefore$ $(b^2 - 1)^2 (a^2 + 4)$

$= \{(\sin \theta + \cos \theta)^2 - 1\}^2 \{(\tan \theta - \cot \theta)^2 + 4\}$

$= (1 + \sin 2\theta - 1)^2(\tan^2 \theta + \cot^2 \theta - 2 + 4)$

$= \sin^2 2\theta (\operatorname{cosec}^2 \theta + \sec^2 \theta)$

$= 4 \sin^2 \theta \cos^2 \theta \left(\dfrac{1}{\sin^2 \theta} + \dfrac{1}{\cos^2 \theta}\right)$

$= 4$

35. Squaring and adding the given expressions, we get

$x^2 + y^2 = 1 + 1 + 2 \cos (2A - A)$

$\therefore$ $\dfrac{x^2 + y^2 - 2}{2} = \cos A$ .....(i)

Also, $\cos A + 2 \cos^2 A - 1 = y$

$\Rightarrow (\cos A + 1)(2 \cos A - 1) = y$

Putting the value of cos A from (i), we get

$(x^2 + y^2) (x^2 + y^2 - 3) = 2y$

36. $y - z = a(\cos^2 x - \sin^2 x) + 4b \sin x \cos x - c(\cos^2 x - \sin^2 x)$

$= (a - c) \cos 2x + 2b \sin 2x$

$= (a - c)\left(\dfrac{1 - \tan^2 x}{1 + \tan^2 x}\right) + 2b\left(\dfrac{2 \tan x}{1 + \tan^2 x}\right)$

$= (a - c)\left\{\dfrac{1 - 4b^2 / (a - c)^2}{1 + 4b^2 / (a - c)^2}\right\} + 2b\left\{\dfrac{2.2b / (a - c)}{1 + 4b^2 / (a - c)^2}\right\}$

....$\left[\because \tan x = \dfrac{2b}{a - c} \text{(given)}\right]$

$= \dfrac{(a - c)\{(a - c)^2 - 4b^2\} + 8b^2 (a - c)}{(a - c)^2 + 4b^2}$

$\therefore$ $y - z = \dfrac{(a - c)\{(a - c)^2 + 4b^2\}}{(a - c)^2 + 4b^2} = a - c$

$\Rightarrow y \neq z$ ....[$\because a \neq c$]

$y + z = a(\cos^2 x + \sin^2 x) + c(\sin^2 x + \cos^2 x)$

$= a + c$

37. $8 \sin \dfrac{x}{8} \cos \dfrac{x}{2} \cos \dfrac{x}{4} \cos \dfrac{x}{8}$

$= 4 \left(2 \sin \dfrac{x}{8} \cos \dfrac{x}{8}\right) \cos \dfrac{x}{2} \cos \dfrac{x}{4}$

$= 4 \left(\sin \dfrac{x}{4} \cos \dfrac{x}{2} \cos \dfrac{x}{4}\right)$

....[$\because 2 \sin A \cos A = \sin 2A$]

$= 2 \left(2 \sin \dfrac{x}{4} \cos \dfrac{x}{4}\right) \cos \dfrac{x}{2} = 2 \sin \dfrac{x}{2} \cos \dfrac{x}{2}$

$= \sin x$

38. $x = \cos 10° \cos 20° \cos 40°$

$= \dfrac{1}{2 \sin 10^\circ} (2 \sin 10° \cos 10° \cos 20° \cos 40°)$

$= \dfrac{1}{2.2 \sin 10^\circ} (2 \sin 20° \cos 20° \cos 40°)$

$= \dfrac{1}{2.4 \sin 10^\circ} (2 \sin 40° \cos 40°)$

$= \dfrac{1}{8 \sin 10^\circ} (\sin 80°)$

$= \dfrac{1}{8 \sin 10^\circ} (\cos 10°) = \dfrac{1}{8} \cot 10°$

39. Since,

$\cos \theta \cos 2\theta \cos 2^2 \theta \ldots\cos 2^{n-1} \theta = \dfrac{\sin 2^n \theta}{2^n \sin \theta}$

$= \dfrac{\sin(\pi - \theta)}{2^n \sin \theta}$

....$\left[\because \theta = \dfrac{\pi}{2^n + 1} \text{(given)} \Rightarrow 2^n \theta + \theta = \pi \Rightarrow 2^n \theta = \pi - \theta\right]$

$= \dfrac{1}{2^n}$

40. $\cos \dfrac{\pi}{7} \cos \dfrac{2\pi}{7} \cos \dfrac{4\pi}{7} = \left[\dfrac{\sin\left(2^3 . \dfrac{\pi}{7}\right)}{2^3 \sin\left(\dfrac{\pi}{7}\right)}\right]$

$= \dfrac{\sin \dfrac{8\pi}{7}}{8 \sin \dfrac{\pi}{7}}$

$= -\dfrac{1}{8}$ ....$\left[\because \sin \dfrac{8\pi}{7} = \sin\left(\pi + \dfrac{\pi}{7}\right) = -\sin \dfrac{\pi}{7}\right]$

41. $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5} = \frac{\sin\frac{2^4\pi}{5}}{2^4\sin\frac{\pi}{5}}$

$= \frac{\sin\frac{16\pi}{5}}{16\sin\frac{\pi}{5}}$

$= \frac{\sin\left(3\pi+\frac{\pi}{5}\right)}{16\sin\frac{\pi}{5}}$

$= \frac{-\sin\frac{\pi}{5}}{16\sin\frac{\pi}{5}} = -\frac{1}{16}$

42. $\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°} = \frac{\cos 10° - \sqrt{3}\sin 10°}{\sin 10° \cos 10°}$

$= \frac{2\left\{\frac{1}{2}\cos 10° - \frac{\sqrt{3}}{2}\sin 10°\right\}}{\sin 10° \cos 10°}$

$= \frac{2\sin(30° - 10°)}{\frac{1}{2}(\sin 20°)} = 4$

43. $|\tan A| < 1$ and $|A|$ is acute.

$\therefore$ $-\frac{\pi}{4} < A < \frac{\pi}{4} \Rightarrow \cos A > \sin A$

$\therefore$ $\frac{\sqrt{1+\sin 2A}+\sqrt{1-\sin 2A}}{\sqrt{1+\sin 2A}-\sqrt{1-\sin 2A}}$

$= \frac{\sqrt{(\cos A+\sin A)^2}+\sqrt{(\cos A-\sin A)^2}}{\sqrt{(\cos A+\sin A)^2}-\sqrt{(\cos A-\sin A)^2}}$

$= \frac{|\cos A+\sin A|+|\cos A-\sin A|}{|\cos A+\sin A|-|\cos A-\sin A|}$

$= \frac{(\cos A+\sin A)+(\cos A-\sin A)}{(\cos A+\sin A)-(\cos A-\sin A)} = \cot A$

44. Given that, $\tan x = \frac{b}{a}$

$\therefore$ $\sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{1+\frac{b}{a}}{1-\frac{b}{a}}}+\sqrt{\frac{1-\frac{b}{a}}{1+\frac{b}{a}}}$

$= \frac{2}{\sqrt{1-\frac{b^2}{a^2}}} = \frac{2}{\sqrt{1-\tan^2 x}}$

$= \frac{2}{\sqrt{1-\frac{\sin^2 x}{\cos^2 x}}} = \frac{2\cos x}{\sqrt{\cos 2x}}$

45. $\tan(60° + A)\tan(60° - A)$

$= \frac{\sin^2 60° - \sin^2 A}{\cos^2 60° - \sin^2 A}$

$= \frac{\frac{3}{4}-\left(\frac{1-\cos 2A}{2}\right)}{\frac{1}{4}-\left(\frac{1-\cos 2A}{2}\right)} = \frac{3-2+2\cos 2A}{1-2+2\cos 2A}$

$= \frac{2\cos 2A+1}{2\cos 2A-1}$

46. $\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta$

$= 1 - \frac{1}{2}(\sin 2\theta)^2$

Since, $0 \le \sin^2 2\theta \le 1$

$\therefore$ $0 \ge -\frac{1}{2}\sin^2 2\theta \ge -\frac{1}{2}$

$\Rightarrow 1 + 0 \ge 1 - \frac{1}{2}\sin^2 2\theta \ge 1 - \frac{1}{2}$

$\Rightarrow 1 \ge \sin^4\theta + \cos^4\theta \ge \frac{1}{2}$

47. $2\sin^2 x - \cos 2x = 4\sin^2 x - 1$

and $0 \le \sin^2 x \le 1 \Leftrightarrow 0 \le 4\sin^2 x \le 4$

$\Rightarrow -1 \le 4\sin^2 x - 1 \le 3$

48. $3\sin 2\theta = 2\sin 3\theta$

$\Rightarrow 6\sin\theta\cos\theta = 2(3\sin\theta - 4\sin^3\theta)$

Dividing by $2\sin\theta \ne 0$, we get

$3\cos\theta = 3 - 4\sin^2\theta$

$\Rightarrow 3\cos\theta = 3 - 4(1 - \cos^2\theta)$

$\Rightarrow 4\cos^2\theta - 3\cos\theta - 1 = 0$

$\Rightarrow \cos\theta = 1, -\frac{1}{4}$

But, $0 < \theta < \pi$

$\therefore$ $\cos\theta = -\frac{1}{4}$

$\Rightarrow \sin\theta = \sqrt{1-\frac{1}{16}} = \frac{\sqrt{15}}{4}$

49. $\sin 2A = \sin 3A$
$\Rightarrow 2 \sin A \cos A = 3 \sin A - 4 \sin^3 A$
$\Rightarrow \sin A = 0$ or $2 \cos A = 3 - 4 \sin^2 A$
$\Rightarrow A = 0$ or $2 \cos A = 3 - 4 (1 - \cos^2 A)$
$\Rightarrow A = 0$ or $4 \cos^2 A - 2 \cos A - 1 = 0$
$\Rightarrow A = 0$ or $\cos A = \dfrac{2 \pm \sqrt{4+16}}{2\times 4} = \dfrac{1\pm\sqrt{5}}{4}$
$\Rightarrow A = 0°$ or $36°$ ....[$\because 0 \le A \le 90°$]

50. $\tan\theta + \tan\left(\theta + \dfrac{\pi}{3}\right) + \tan\left(\theta + \dfrac{2\pi}{3}\right) = 3$
$\Rightarrow \tan\theta + \left(\dfrac{\tan\theta + \sqrt{3}}{1-\sqrt{3}\tan\theta} + \dfrac{\tan\theta - \sqrt{3}}{1+\sqrt{3}\tan\theta}\right) = 3$
$\Rightarrow \tan\theta + \dfrac{8\tan\theta}{1-3\tan^2\theta} = 3$
$\Rightarrow \dfrac{9\tan\theta - 3\tan^3\theta}{1-3\tan^2\theta} = 3$
$\Rightarrow 3 \tan 3\theta = 3$
$\Rightarrow \tan 3\theta = 1$

51. $\sin\theta = -\dfrac{4}{5}$
$\Rightarrow \cos\theta = \sqrt{1-\dfrac{16}{25}} = \pm\dfrac{3}{5}$
$\Rightarrow \cos\theta = \dfrac{-3}{5}$ ....[$\because$ θ lies in the 3rd quadrant]
Since, $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$
$= \pm\sqrt{\dfrac{1-3/5}{2}} = \pm\sqrt{\dfrac{1}{5}}$
$\therefore$ $\cos\dfrac{\theta}{2} = -\dfrac{1}{\sqrt{5}}$ ....$\left[\because \dfrac{\theta}{2} \text{ lies in the 2}^{\text{nd}} \text{ quadrant}\right]$

52. Given that, $\sec\theta = \dfrac{5}{4}$
Since, $\sec\theta = \dfrac{1+\tan^2\left(\dfrac{\theta}{2}\right)}{1-\tan^2\left(\dfrac{\theta}{2}\right)}$
$\Rightarrow \dfrac{5}{4} = \dfrac{1+\tan^2\left(\dfrac{\theta}{2}\right)}{1-\tan^2\left(\dfrac{\theta}{2}\right)}$
$\Rightarrow 5 - 5\tan^2\left(\dfrac{\theta}{2}\right) = 4 + 4\tan^2\left(\dfrac{\theta}{2}\right)$
$\Rightarrow 9\tan^2\left(\dfrac{\theta}{2}\right) = 1 \Rightarrow \tan\left(\dfrac{\theta}{2}\right) = \dfrac{1}{3}$

53. For $A = 133°, \dfrac{A}{2} = 66.5°$
$\Rightarrow \sin\dfrac{A}{2} > \cos\dfrac{A}{2} > 0$
$\sqrt{1+\sin A} = \sin\dfrac{A}{2} + \cos\dfrac{A}{2}$ ....(i)
and $\sqrt{1-\sin A} = \sin\dfrac{A}{2} - \cos\dfrac{A}{2}$ ....(ii)
Subtracting (ii) from (i), we get
$2\cos\dfrac{A}{2} = \sqrt{1+\sin A} - \sqrt{1-\sin A}$

54. Given, $\cos\theta = \dfrac{3}{5} \Rightarrow \sin\theta = \dfrac{4}{5}$
and $\cos\phi = \dfrac{4}{5} \Rightarrow \sin\phi = \dfrac{3}{5}$
$\therefore$ $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$
$= \dfrac{3}{5}\cdot\dfrac{4}{5} + \dfrac{4}{5}\cdot\dfrac{3}{5} = \dfrac{24}{25}$
But, $2\cos^2\left(\dfrac{\theta-\phi}{2}\right) = 1 + \cos(\theta - \phi) = 1 + \dfrac{24}{25}$
$\therefore$ $\cos^2\left(\dfrac{\theta-\phi}{2}\right) = \dfrac{49}{50}$ $\therefore$ $\cos\left(\dfrac{\theta-\phi}{2}\right) = \dfrac{7}{5\sqrt{2}}$

55. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$
$= \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta + \sin^2\alpha$
$+ \sin^2\beta + 2\sin\alpha\sin\beta$
$= 2\{1 + \cos(\alpha - \beta)\} = 4\cos^2\left(\dfrac{\alpha-\beta}{2}\right)$

56. $\left(\dfrac{\sin 2A}{1+\cos 2A}\right)\left(\dfrac{\cos A}{1+\cos A}\right)$
$= \left(\dfrac{2\sin A\cos A}{2\cos^2 A}\right)\left(\dfrac{\cos A}{1+\cos A}\right)$
$= \dfrac{\sin A}{1+\cos A}$
$= \dfrac{2\sin\dfrac{A}{2}\cos\dfrac{A}{2}}{2\cos^2\dfrac{A}{2}}$
$= \tan\dfrac{A}{2}$

57. $\dfrac{1+\sin A-\cos A}{1+\sin A+\cos A}=\dfrac{2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}$

$=\dfrac{2\sin\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)}{2\cos\frac{A}{2}\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)}$

$=\tan\dfrac{A}{2}$

58. $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1}=\dfrac{\sin A+1-\cos A}{\sin A-1+\cos A}$

$=\dfrac{\sin A+(1-\cos A)}{\sin A-(1-\cos A)}$

$=\dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}+2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}-2\sin^2\frac{A}{2}}$

$=\dfrac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}$

$=\dfrac{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2}{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}$

$=\dfrac{1+\sin A}{\cos A}$

59. $\dfrac{y+1}{1-y}=\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}$

$\Rightarrow \dfrac{y+1}{1-y}=\sqrt{\dfrac{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}}$

$\Rightarrow \dfrac{1+y}{1-y}=\dfrac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\left|\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right|}$

$\Rightarrow \dfrac{1+y}{1-y}=\dfrac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}$

$\ldots\left[\because 0<\theta<\dfrac{\pi}{2}\Rightarrow 0<\dfrac{\theta}{2}<\dfrac{\pi}{4} \Rightarrow \cos\dfrac{\theta}{2}>\sin\dfrac{\theta}{2}\right]$

$\Rightarrow \dfrac{1+y}{1-y}=\dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$

$\Rightarrow y=\tan\dfrac{\theta}{2}$

60. tan A and tan B are the roots of the equation $x^2-ax+b=0$.

$\therefore$ tan A + tan B = a and tan A tan B = b

$\therefore$ $\tan(A+B)=\dfrac{a}{1-b}$

Now, $\sin^2(A+B)=\dfrac{1}{2}\{1-\cos 2(A+B)\}$

$\Rightarrow \sin^2(A+B)=\dfrac{1}{2}\left\{1-\dfrac{1-\tan^2(A+B)}{1+\tan^2(A+B)}\right\}$

$\Rightarrow \sin^2(A+B)=\dfrac{1}{2}\left\{1-\dfrac{1-\frac{a^2}{(1-b)^2}}{1+\frac{a^2}{(1-b)^2}}\right\}$

$\Rightarrow \sin^2(A+B)=\dfrac{1}{2}\left\{\dfrac{2a^2}{a^2+(1-b)^2}\right\}$

$=\dfrac{a^2}{a^2+(1-b)^2}$

Competitive Thinking

1. $\cos\left(\dfrac{\pi}{4}+x\right)+\cos\left(\dfrac{\pi}{4}-x\right)$

$=\cos\dfrac{\pi}{4}\cos x$

$-\sin\dfrac{\pi}{4}\sin x+\cos\dfrac{\pi}{4}\cos x+\sin\dfrac{\pi}{4}\sin x$

$=2\cos\dfrac{\pi}{4}\cos x$

$=\dfrac{2}{\sqrt{2}}\cos x=\sqrt{2}\cos x$

2. $2\sin\left(\theta+\dfrac{\pi}{3}\right)=\cos\left(\theta-\dfrac{\pi}{6}\right)$

$\therefore$ $2\left(\sin\theta\cdot\cos\dfrac{\pi}{3}+\cos\theta\cdot\sin\dfrac{\pi}{3}\right)$

$=\cos\theta.\cos\dfrac{\pi}{6}+\sin\theta.\sin\dfrac{\pi}{6}$

$\therefore \quad 2\left(\frac{\sin\theta}{2} + \frac{\sqrt{3}}{2}\cos\theta\right) = \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta$

$\therefore \quad \sin\theta + \sqrt{3}\cos\theta = 0$

$\tan\theta = -\sqrt{3}$

3. $\cos 15° - \sin 15° = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 15° - \frac{1}{\sqrt{2}}\sin 15°\right)$

$= \sqrt{2}\cos(45° + 15°)$

$= \sqrt{2}\cos 60°$

$= \sqrt{2}\cdot\frac{1}{2} = \frac{1}{\sqrt{2}}$

4. We have, $\sin\theta = \frac{12}{13}$

$\therefore \quad \cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-\left(\frac{12}{13}\right)^2} = \frac{5}{13} \quad \ldots\left[\because 0<\theta<\frac{\pi}{2}\right]$

and $\cos\phi = \frac{-3}{5}$

$\therefore \quad \sin\phi = \sqrt{1-\frac{9}{25}} = \frac{-4}{5} \quad \ldots\left[\because \pi<\phi<\frac{3\pi}{2}\right]$

$\therefore \quad \sin(\theta+\phi) = \sin\theta.\cos\phi + \cos\theta.\sin\phi$

$= \left(\frac{12}{13}\right)\left(\frac{-3}{5}\right) + \left(\frac{5}{13}\right)\left(\frac{-4}{5}\right)$

$= \frac{-36}{65} - \frac{20}{65} = \frac{-56}{65}$

5. $\sin\alpha = \frac{15}{17}$

$\Rightarrow \cos\alpha = -\frac{8}{17} \quad \ldots\left[\because \frac{\pi}{2}<\alpha<\pi\right]$

$\tan\beta = \frac{12}{5}$

$\Rightarrow \sin\beta = \frac{-12}{13}$ and $\cos\beta = \frac{-5}{13}$

$\ldots\left[\because \pi<\beta<\frac{3\pi}{2}\right]$

$\therefore \quad \sin(\beta-\alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha = \frac{171}{221}$

6. $2A = (A+B) + (A-B)$

$\Rightarrow \tan 2A = \frac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B)\tan(A-B)}$

$= \frac{p+q}{1-pq}$

7. Given, $\cos(\alpha+\beta) = \frac{4}{5} \Rightarrow \tan(\alpha+\beta) = \frac{3}{4}$

and $\sin(\alpha-\beta) = \frac{5}{13} \Rightarrow \tan(\alpha-\beta) = \frac{5}{12}$

Now, $\tan 2\alpha = \tan[(\alpha+\beta)+(\alpha-\beta)]$

$= \frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}.\frac{5}{12}} = \frac{56}{33}$

8. Given, $\cos(A-B) = \frac{3}{5}$

$\therefore \quad 5\cos A\cos B + 5\sin A\sin B = 3 \quad \ldots\text{(i)}$

Also, $\tan A\tan B = 2$

$\therefore \quad \sin A\sin B = 2\cos A\cos B \quad \ldots\text{(ii)}$

From (i) and (ii), we get

$\cos A\cos B = \frac{1}{5}$ and $\sin A\sin B = \frac{2}{5}$

9. $\sin\theta = 3\sin(\theta+2\alpha)$

$\Rightarrow \sin(\theta+\alpha-\alpha) = 3\sin(\theta+\alpha+\alpha)$

$\Rightarrow \sin(\theta+\alpha)\cos\alpha - \cos(\theta+\alpha)\sin\alpha$

$= 3\sin(\theta+\alpha)\cos\alpha + 3\cos(\theta+\alpha)\sin\alpha$

$\Rightarrow -2\sin(\theta+\alpha)\cos\alpha = 4\cos(\theta+\alpha)\sin\alpha$

$\Rightarrow \frac{-\sin(\theta+\alpha)}{\cos(\theta+\alpha)} = \frac{2\sin\alpha}{\cos\alpha}$

$\Rightarrow \tan(\theta+\alpha) + 2\tan\alpha = 0$

10. $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$

$\Rightarrow \tan\beta = \frac{n\tan\alpha}{\sec^2\alpha - n\tan^2\alpha}$

$= \frac{n\tan\alpha}{1+\tan^2\alpha-n\tan^2\alpha}$

Now, $\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$= \frac{\tan\alpha - \frac{n\tan\alpha}{1+\tan^2\alpha-n\tan^2\alpha}}{1+\tan\alpha\left(\frac{n\tan\alpha}{1+\tan^2\alpha-n\tan^2\alpha}\right)}$

$= \frac{\tan\alpha+\tan^3\alpha-n\tan^3\alpha-n\tan\alpha}{1+\tan^2\alpha-n\tan^2\alpha+n\tan^2\alpha}$

$= \frac{\tan\alpha\left(1+\tan^2\alpha\right)-n\tan\alpha\left(1+\tan^2\alpha\right)}{\left(1+\tan^2\alpha\right)}$

$= (1-n)(\tan\alpha)$

11. We have, $A - B = \dfrac{\pi}{4}$

$\Rightarrow \tan(A - B) = \tan\dfrac{\pi}{4}$

$\Rightarrow \dfrac{\tan A - \tan B}{1 + \tan A \tan B} = 1$

$\Rightarrow \tan A - \tan B - \tan A \tan B = 1$

$\Rightarrow \tan A - \tan B - \tan A \tan B + 1 = 2$

$\Rightarrow (1 + \tan A)(1 - \tan B) = 2$

$\Rightarrow y = 2$

$\therefore$ $(y + 1)^{y+1} = (2 + 1)^{2+1} = (3)^3 = 27$

12. Since, $\cos\left(\dfrac{\pi}{2}\right) = 0$

$\therefore$ $\cos\left[\left(\dfrac{\pi}{4}+\theta\right)+\left(\dfrac{\pi}{4}-\theta\right)\right] = 0$

$\Rightarrow \cos\left(\dfrac{\pi}{4}+\theta\right)\cos\left(\dfrac{\pi}{4}-\theta\right)$

$- \sin\left(\dfrac{\pi}{4}+\theta\right)\sin\left(\dfrac{\pi}{4}-\theta\right) = 0$

$\Rightarrow \cos\left(\dfrac{\pi}{4}+\theta\right)\cos\left(\dfrac{\pi}{4}-\theta\right) = \sin\left(\dfrac{\pi}{4}+\theta\right)\sin\left(\dfrac{\pi}{4}-\theta\right)$

$\Rightarrow \cot\left(\dfrac{\pi}{4}+\theta\right)\cot\left(\dfrac{\pi}{4}-\theta\right) = 1$

13. Let $\theta = \alpha + \beta$, where $\tan\alpha = \dfrac{1}{2}$, $\tan\beta = \dfrac{1}{3}$

$\therefore$ $\tan\theta = \tan(\alpha + \beta) = \dfrac{\dfrac{1}{2}+\dfrac{1}{3}}{1-\dfrac{1}{2}.\dfrac{1}{3}} = 1 \Rightarrow \theta = \dfrac{\pi}{4}$

14. $\cos P = \dfrac{1}{7} \Rightarrow \sin P = \dfrac{\sqrt{48}}{7}$

$\cos Q = \dfrac{13}{14} \Rightarrow \sin Q = \dfrac{\sqrt{27}}{14}$

$\therefore$ $\cos(P - Q) = \cos P \cos Q + \sin P \sin Q$

$= \dfrac{1}{7}.\dfrac{13}{14} + \dfrac{\sqrt{48}}{7}.\dfrac{\sqrt{27}}{14}$

$= \dfrac{13+36}{98} = \dfrac{1}{2} = \cos 60°$

$\therefore$ $P - Q = 60°$

15. We have, $\sin\alpha = \dfrac{1}{\sqrt{5}}$

$\therefore$ $\cos\alpha = \sqrt{1-\left(\dfrac{1}{\sqrt{5}}\right)^2} = \dfrac{2}{\sqrt{5}}$

and $\sin\beta = \dfrac{3}{5}$

$\therefore$ $\cos\beta = \sqrt{1-\left(\dfrac{3}{5}\right)^2} = \dfrac{4}{5}$

$\therefore$ $\sin(\beta - \alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha$

$= \dfrac{3}{5}\times\dfrac{2}{\sqrt{5}} - \dfrac{4}{5}\times\dfrac{1}{\sqrt{5}} = \dfrac{2}{5\sqrt{5}}$

$= 0.1789$

Now, $\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} = 0.7071$

Since, $0 < 0.1789 < 0.7071$

$\therefore$ $\sin 0 < \sin(\beta - \alpha) < \sin\dfrac{\pi}{4}$

$\Rightarrow 0 < (\beta - \alpha) < \dfrac{\pi}{4}$

16. $\tan\theta_1 = k\cot\theta_2$

$\Rightarrow \dfrac{\tan\theta_1}{\cot\theta_2} = k$

$\Rightarrow \dfrac{\tan\theta_1 + \cot\theta_2}{\tan\theta_1 - \cot\theta_2} = \dfrac{k+1}{k-1}$

$\Rightarrow \dfrac{\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2}{\sin\theta_1\sin\theta_2 - \cos\theta_1\cos\theta_2} = \dfrac{k+1}{k-1}$

$\Rightarrow \dfrac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2} = \dfrac{k+1}{1-k}$

$\Rightarrow \dfrac{\cos(\theta_1 - \theta_2)}{\cos(\theta_1 + \theta_2)} = \dfrac{k+1}{1-k}$

$\Rightarrow \dfrac{\cos(\theta_1 + \theta_2)}{\cos(\theta_1 - \theta_2)} = \dfrac{1-k}{1+k}$

17. $\dfrac{\cos 17^\circ + \sin 17^\circ}{\cos 17^\circ - \sin 17^\circ} = \dfrac{1 + \tan 17^\circ}{1 - \tan 17^\circ}$

$= \dfrac{\tan 45^\circ + \tan 17^\circ}{1 - \tan 45^\circ \tan 17^\circ}$

$= \tan(45° + 17°)$

$= \tan 62°$

18. $\dfrac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ} = \dfrac{1+\tan 9^\circ}{1-\tan 9^\circ}$

$= \tan (45^\circ + 9^\circ)$

$= \tan 54^\circ$

19. We have, $\tan\theta = \dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$

$\Rightarrow \tan\theta = \dfrac{\frac{1}{\sqrt{2}}(\sin\alpha-\cos\alpha)}{\frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)}$

$\Rightarrow \tan\theta = \dfrac{\sin\alpha\cos\frac{\pi}{4}-\cos\alpha\sin\frac{\pi}{4}}{\sin\alpha\sin\frac{\pi}{4}+\cos\alpha\cos\frac{\pi}{4}}$

$\Rightarrow \tan\theta = \dfrac{\sin\left(\alpha-\frac{\pi}{4}\right)}{\cos\left(\alpha-\frac{\pi}{4}\right)}$

$\Rightarrow \tan\theta = \tan\left(\alpha-\frac{\pi}{4}\right)$

$\Rightarrow \theta = \alpha - \frac{\pi}{4} \Rightarrow \alpha = \theta + \frac{\pi}{4}$

$\therefore$ $\sin\alpha + \cos\alpha = \sin\left(\theta+\frac{\pi}{4}\right)+\cos\left(\theta+\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta$

$= \frac{2}{\sqrt{2}}\cos\theta = \sqrt{2}\cos\theta$

and $\sin\alpha - \cos\alpha = \sin\left(\theta+\frac{\pi}{4}\right)-\cos\left(\theta+\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta$

$= \frac{2}{\sqrt{2}}\sin\theta = \sqrt{2}\sin\theta$

20. $\cos^2 45^\circ - \sin^2 15^\circ = \cos (45 + 15)^\circ$

$= \cos (45 - 15)^\circ$

….[$\because \cos^2 A - \sin^2 B = \cos (A+B) \cos(A - B)$]

$= \cos 60^\circ \cos 30^\circ$

$= \dfrac{\sqrt{3}}{4}$

21. $\cos^2\left(\frac{\pi}{6}+\theta\right) - \sin^2\left(\frac{\pi}{6}-\theta\right)$

$= \cos\left(\frac{\pi}{6}+\theta+\frac{\pi}{6}-\theta\right)\cos\left(\frac{\pi}{6}+\theta-\frac{\pi}{6}+\theta\right)$

….[$\because \cos^2 A - \sin^2 B = \cos (A+B) \cos(A - B)$]

$= \cos\frac{2\pi}{6}\cos 2\theta = \frac{1}{2}\cos 2\theta$

22. Let $f(x) = \sin\left(x+\frac{\pi}{6}\right) + \cos\left(x+\frac{\pi}{6}\right)$. Then,

$f(x) = \sqrt{2}\left[\cos\left(x+\frac{\pi}{6}\right)\frac{1}{\sqrt{2}} + \sin\left(x+\frac{\pi}{6}\right)\frac{1}{\sqrt{2}}\right]$

$= \sqrt{2}\left[\cos\left(x+\frac{\pi}{6}\right)\cos\frac{\pi}{4} + \sin\left(x+\frac{\pi}{6}\right)\sin\frac{\pi}{4}\right]$

$= \sqrt{2}\cos\left(x+\frac{\pi}{6}-\frac{\pi}{4}\right)$

….[$\because \cos (A - B) = \cos A \cos B + \sin A \sin B$]

$= \sqrt{2}\cos\left(x-\frac{\pi}{12}\right)$

Since, $-1 \le \cos\left(x-\frac{\pi}{12}\right) \le 1$

$\therefore$ $f(x)$ is maximum, if $x - \frac{\pi}{12} = 0$ i.e., if $x = \frac{\pi}{12}$

23. $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$= \sin45^\circ.\cos30^\circ - \cos45^\circ.\sin30^\circ$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

24. $\sin 75^\circ = \sin(90^\circ - 15^\circ)$

$= \cos 15^\circ$

$= \cos (45^\circ - 30^\circ)$

$= \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

25. $\sin 765^\circ = \sin (720 + 45)^\circ$

$= \sin (4\pi + 45)^\circ$

$= \sin (2\pi + 2\pi + 45^\circ)$

$= \sin 45^\circ$ …[$\because \sin (2\pi + \theta) = \sin \theta$]

$= \dfrac{1}{\sqrt{2}}$

26. $\tan\theta \sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)$

$= \tan\theta \cos\theta \sin\theta$

$= \sin^2\theta$

27. $\cot(45° + \theta)\cot(45° - \theta)$
$= \tan(90° - 45° - \theta)\cot(45° - \theta)$
....[$\because \tan(90° - \theta) = \cot\theta$]
$= \tan(45° - \theta)\cot(45° - \theta) = 1$

28. $\tan 75° - \cot 75°$
$= \tan(90° - 15°) - \cot 75°$
$= \cot 15° - \cot 75°$
$= (2 + \sqrt{3}) - (2 - \sqrt{3})$
$= 2\sqrt{3}$

29. $\tan 50° = \tan(70° - 20°) = \dfrac{\tan 70° - \tan 20°}{1 + \tan 70° \tan 20°}$
$\Rightarrow \tan 50° + \tan 70° \tan 20° \tan 50° = \tan 70° - \tan 20°$
$\Rightarrow \tan 50° + \tan 50° = \tan 70° - \tan 20°$
....[$\because \tan 70° = \cot 20°$]
$\Rightarrow 2\tan 50° + \tan 20° = \tan 70°$

30. $\sec 50° + \tan 50°$
$= \dfrac{1}{\cos 50°} + \tan 50°$
$= \dfrac{\cos 20°}{\cos 20° \cos 50°} + \tan 50°$
$= \dfrac{\sin 70°}{\cos 20° \cos 50°} + \tan 50°$
....[$\because \cos\theta = \sin(90° - \theta)$]
$= \dfrac{\sin(50° + 20°)}{\cos 20° \cos 50°} + \tan 50°$
$= \dfrac{\sin 50° \cos 20° + \cos 50° \sin 20°}{\cos 20° \cos 50°} + \tan 50°$
$= \dfrac{\sin 50° \cos 20°}{\cos 20° \cos 50°} + \dfrac{\cos 50° \sin 20°}{\cos 20° \cos 50°} + \tan 50°$
$= \tan 50° + \tan 20° + \tan 50°$
$= 2\tan 50° + \tan 20°$

31. $2\sec 2\alpha = \tan\beta + \cot\beta$
$\Rightarrow \dfrac{2}{\cos 2\alpha} = \dfrac{\sin\beta}{\cos\beta} + \dfrac{\cos\beta}{\sin\beta}$
$= \dfrac{\sin^2\beta + \cos^2\beta}{\cos\beta\sin\beta} = \dfrac{1}{\cos\beta.\sin\beta}$
$\Rightarrow \cos 2\alpha = \sin 2\beta \Rightarrow \cos 2\alpha = \cos\left(\dfrac{\pi}{2} - 2\beta\right)$
$\Rightarrow 2\alpha = \dfrac{\pi}{2} - 2\beta \quad \Rightarrow 2\alpha + 2\beta = \dfrac{\pi}{2}$
$\Rightarrow \alpha + \beta = \dfrac{\pi}{4}$

32. $\sin(\pi + \theta)\sin(\pi - \theta)\operatorname{cosec}^2\theta$
$= -\sin\theta \sin\theta \dfrac{1}{\sin^2\theta}$
$= -1$

33. Given that, ABCD is a cyclic quadrilateral.
So, $A + C = 180° \Rightarrow A = 180° - C$
$\Rightarrow \cos A = \cos(180° - C) = -\cos C$
$\Rightarrow \cos A + \cos C = 0$ .....(i)
Similarly, $\cos B + \cos D = 0$ ....(ii)
Adding, (i) and (ii), we get
$\cos A + \cos B + \cos C + \cos D = 0$

34. $\cos(270° + \theta)\cos(90° - \theta) - \sin(270° - \theta)\cos\theta$
$= \sin\theta.\sin\theta + \cos\theta.\cos\theta = 1$

35. $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A)$
$= \cos A - \cos A + \cos A - \cos A = 0$

36. $\tan A + \cot(180° + A) + \cot(90° + A) + \cot(360° - A)$
$= \tan A + \cot A - \tan A - \cot A = 0$

37. $\sin 1° + \sin 2° + \ldots + \sin 359°$
$= (\sin 1° + \sin 359°) + (\sin 2° + \sin 358°) + \ldots + (\sin 179° + \sin 181°) + \sin 180°$
$= (\sin 1° - \sin 1°) + (\sin 2° - \sin 2°) + \ldots + (\sin 179° - \sin 179°) + \sin 180°$
$= 0$

38. $\sin 600° \cos 330° + \cos 120° \sin 150°$
$= -\sin 60° \cos 30° - \sin 30° \cos 60°$
$= -\{\sin(60° + 30°)\}$
$= -1$

39. Let $f(x) = 2\sin 3x + 3\cos 3x$
$\therefore \quad f\left(\dfrac{5\pi}{6}\right) = 2\sin\left(\dfrac{5\pi}{2}\right) + 3\cos\left(\dfrac{5\pi}{2}\right)$
$= 2\sin\left(2\pi + \dfrac{\pi}{2}\right) + 3\cos\left(2\pi + \dfrac{\pi}{2}\right)$
$= 2\sin\dfrac{\pi}{2} + 3\cos\dfrac{\pi}{2} = 2(1) + 3(0) = 2$

40. $\dfrac{1 - \tan 2° \cot 62°}{\tan 152° - \cot 88°} = \dfrac{1 - \tan 2° \cot 62°}{\tan(90° + 62°) - \cot(90° - 2°)}$
$= \dfrac{1 - \tan 2° \cot 62°}{-\cot 62° - \tan 2°} = \dfrac{\tan 62° - \tan 2°}{-(1 + \tan 2° \tan 62°)}$
$= -\tan(62° - 2°)$
$= -\tan 60° = -\sqrt{3}$

41. $\dfrac{\cos 12^\circ - \sin 12^\circ}{\cos 12^\circ + \sin 12^\circ} + \dfrac{\sin 147^\circ}{\cos 147^\circ}$

$= \dfrac{1 - \tan 12^\circ}{1 + \tan 12^\circ} + \tan 147^\circ$

$= \tan(45^\circ - 12^\circ) + \tan(180^\circ - 33^\circ)$

$= \tan 33^\circ - \tan 33^\circ = 0$

42. $\dfrac{\tan 160^\circ - \tan 110^\circ}{1 + (\tan 160^\circ)(\tan 110^\circ)}$

$= \dfrac{\tan(180^\circ - 160^\circ) - \cot(90^\circ - 110^\circ)}{1 + [\tan(180^\circ - 160^\circ)\cot(90^\circ - 110^\circ)]}$

$= \dfrac{-\tan 20^\circ + \cot 20^\circ}{1 + (-\tan 20^\circ)(-\cot 20^\circ)}$

$= \dfrac{-\lambda + \frac{1}{\lambda}}{1 + 1}$

$= \dfrac{1 - \lambda^2}{2\lambda}$

43. $\sin^2 17.5^\circ + \sin^2 72.5^\circ$

$= \sin^2 17.5^\circ + [\sin(90^\circ - 17.5^\circ)]^2$

$= \sin^2 17.5^\circ + \cos^2 17.5^\circ$

$= 1 = \tan^2 45^\circ$

44. $3\left\{\sin^4\left(\dfrac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right\}$

$-2\left\{\sin^6\left(\dfrac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right\}$

$= 3\{(-\cos\alpha)^4 + (-\sin\alpha)^4\} - 2(\cos^6\alpha + \sin^6\alpha)$

$= 3(1 - 2\sin^2\alpha\cos^2\alpha) - 2(1 - 3\sin^2\alpha\cos^2\alpha)$

$= 3 - 6\sin^2\alpha\cos^2\alpha - 2 + 6\sin^2\alpha\cos^2\alpha$

$= 3 - 2 = 1$

45. $\sin^2 5^\circ + \sin^2 10^\circ + \sin^2 15^\circ + \ldots + \sin^2 85^\circ + \sin^2 90^\circ$

Since, $\sin 90^\circ = 1$ or $\sin^2 90^\circ = 1$

Similarly, $\sin 45^\circ = \dfrac{1}{\sqrt{2}}$ or $\sin^2 45^\circ = \dfrac{1}{2}$ and the angles are in A.P. of 18 terms.

Also, $\sin^2 85^\circ = [\sin(90^\circ - 5^\circ)]^2 = \cos^2 5^\circ$

Therefore from the complementary rule, we have $\sin^2 5^\circ + \sin^2 85^\circ = \sin^2 5^\circ + \cos^2 5^\circ = 1$

$\therefore$ $\sin^2 5^\circ + \sin^2 10^\circ + \sin^2 15^\circ + \ldots + \sin^2 85^\circ + \sin^2 90^\circ$

$= (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) + 1 + \dfrac{1}{2}$

$= 9\dfrac{1}{2}$

46. $\cos 15^\circ = \sqrt{\dfrac{1 + \cos(2 \times 15^\circ)}{2}}$

$= \sqrt{\dfrac{1 + \cos 30^\circ}{2}}$

47. $\sin 15^\circ = \sin(45^\circ - 30^\circ) = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$ = irrational

$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$ = irrational

$\sin 15^\circ \cos 15^\circ = \dfrac{1}{2}(2\sin 15^\circ \cos 15^\circ)$

$= \dfrac{1}{2}\sin 30^\circ = \dfrac{1}{2}\cdot\dfrac{1}{2}$

$= \dfrac{1}{4}$ = rational

$\sin 15^\circ \cos 75^\circ = \sin 15^\circ \sin 15^\circ$

$= \sin^2 15^\circ$

$= \left(\dfrac{\sqrt{3} - 1}{2\sqrt{2}}\right)^2$

$= \dfrac{4 - 2\sqrt{3}}{8}$ = irrational

48. $\dfrac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$

$= \dfrac{1}{8}(3 - 4\cos 2\theta + 2\cos^2 2\theta - 1)$

$= \dfrac{1}{8}(2\cos^2 2\theta - 4\cos 2\theta + 2)$

$= \dfrac{1}{4}(\cos^2 2\theta - 2\cos 2\theta + 1)$

$= \dfrac{1}{4}(\cos 2\theta - 1)^2$

$= \dfrac{1}{4}(-2\sin^2\theta)^2$ ....[$\because \cos 2\theta = 1 - 2\sin^2\theta$]

$= \dfrac{1}{4}(4\sin^4\theta) = \sin^4\theta$

49. $\sec 2x - \tan 2x = \dfrac{1 - \sin 2x}{\cos 2x}$

$= \dfrac{(\cos x - \sin x)^2}{(\cos^2 x - \sin^2 x)}$

$= \dfrac{\cos x - \sin x}{\cos x + \sin x}$

$= \dfrac{1 - \tan x}{1 + \tan x}$

$$= \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\left(\frac{\pi}{4}\right)\tan x}$$

$$= \tan\left(\frac{\pi}{4} - x\right)$$

50. $\sqrt{3}$ cosec 20° − sec 20° $= \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$

$$= \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\left[\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right]}{\frac{2}{2}\sin 20^\circ \cos 20^\circ}$$

$$= \frac{4\cos(20^\circ + 30^\circ)}{\sin 40^\circ} = \frac{4\cos 50^\circ}{\sin 40^\circ}$$

$$= \frac{4\sin 40^\circ}{\sin 40^\circ}$$

$= 4$

51. $\tan(1^\circ) + \tan(89^\circ)$

$= \tan 1^\circ + \cot 1^\circ$ ….[$\because$ $\tan(90^\circ - \theta) = \cot\theta$]

$$= \frac{\tan^2 1^\circ + 1}{\tan 1^\circ}$$

$$= \frac{\sec^2 1^\circ}{\tan 1^\circ} = \frac{1}{\sin 1^\circ \cos 1^\circ} = \frac{2}{\sin 2^\circ}$$

52. $\cot 2\theta + \tan\theta = \frac{1}{\tan 2\theta} + \tan\theta$

$$= \frac{1 - \tan^2\theta}{2\tan\theta} + \tan\theta$$

$$= \frac{1 + \tan^2\theta}{2\tan\theta} = \frac{1}{\sin 2\theta}$$

$= \text{cosec } 2\theta$

Now, $\cot\frac{2x}{3} + \tan\frac{x}{3} = \text{cosec}\frac{kx}{3}$

$\Rightarrow \text{cosec}\frac{2x}{3} = \text{cosec}\frac{kx}{3}$

$\Rightarrow k = 2$

53. $2\sin^2\left[\left(\frac{\pi}{2}\right)\cos^2 x\right] = 1 - \cos(\pi\sin 2x)$

$\Rightarrow 2\sin^2\left[\left(\frac{\pi}{2}\right)\cos^2 x\right] = 2\sin^2\left[\frac{\pi\sin 2x}{2}\right]$

$\Rightarrow \cos^2 x = \sin 2x \Rightarrow \cos^2 x = 2\sin x\cos x$

$\Rightarrow \tan x = \frac{1}{2}$

Now, $\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{1 - \frac{1}{4}}{1 + \frac{1}{4}} = \frac{3}{5}$

54. $8\cos 2\theta + 8\sec 2\theta = 65$

$\Rightarrow 8\cos^2 2\theta + 8 = 65\cos 2\theta$

$\Rightarrow 8\cos^2 2\theta - 65\cos 2\theta + 8 = 0$

$\Rightarrow (\cos 2\theta - 8)(8\cos 2\theta - 1) = 0$

Since, $\cos 2\theta \in [-1, 1]$

$\therefore$ $\cos 2\theta = \frac{1}{8}$

Now, $4\cos 4\theta = 4(2\cos^2 2\theta - 1)$

$$= 4\left[2\left(\frac{1}{8}\right)^2 - 1\right] = -\frac{31}{8}$$

55. $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$

Put $\cos^2 x = t$

$\therefore$ $5\left(\frac{1-t}{t} - t\right) = 2(2t - 1) + 9$

$\Rightarrow 5(1 - t - t^2) = 4t^2 - 2t + 9t$

$\Rightarrow 9t^2 + 12t - 5 = 0$

$\Rightarrow 9t^2 + 15t - 3t - 5 = 0$

$\Rightarrow 3t(3t + 5) - 1(3t + 5) = 0$

$\Rightarrow (3t + 5)(3t - 1) = 0$

$\Rightarrow t = \frac{1}{3}$ or $t = \frac{-5}{3}$

But t cannot be negative

$\therefore$ $t = \frac{1}{3}$

$\therefore$ $\cos^2 x = \frac{1}{3}$

$\Rightarrow \cos 2x = 2\cos^2 x - 1$

$$= \frac{2}{3} - 1 = \frac{-1}{3}$$

$\Rightarrow \cos 4x = 2\cos^2 2x - 1$

$$= 2\left(\frac{-1}{3}\right)^2 - 1$$

$$= \frac{-7}{9}$$

56. $x + \frac{1}{x} = 2\cos\alpha$

Squaring on both sides, we get

$x^2 + \frac{1}{x^2} + 2 = 4\cos^2\alpha$

$\Rightarrow x^2 + \dfrac{1}{x^2} = 4\cos^2\alpha - 2$

$\Rightarrow x^2 + \dfrac{1}{x^2} = 2(2\cos^2\alpha - 1)$

$= 2\cos 2\alpha$

Similarly, $x^n + \dfrac{1}{x^n} = 2\cos n\alpha$

57. $\sin x + \cos x = \dfrac{1}{5}$

$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x\cos x = \dfrac{1}{25}$

$\Rightarrow \sin 2x = \dfrac{-24}{25} \Rightarrow \cos 2x = \dfrac{-7}{25}$

$\Rightarrow \tan 2x = \dfrac{24}{7}$

58. $3\tan A - 4 = 0$

$\Rightarrow \tan A = \dfrac{4}{3}$

$\Rightarrow \sin A = -\dfrac{4}{5}, \cos A = -\dfrac{3}{5}$

$\therefore$ $5\sin 2A + 3\sin A + 4\cos A$

$= 10\sin A\cos A + 3\sin A + 4\cos A$

$= 10\left(\dfrac{12}{25}\right) - \dfrac{12}{5} - \dfrac{12}{5} = 0$

59. $a\cos 2\theta + b\sin 2\theta = a\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right) + b\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$= a\left(\dfrac{1-\dfrac{b^2}{a^2}}{1+\dfrac{b^2}{a^2}}\right) + b\left(\dfrac{\dfrac{2b}{a}}{1+\dfrac{b^2}{a^2}}\right)$

$\ldots\left[\because \tan\theta = \dfrac{b}{a}\text{(given)}\right]$

$= a\left(\dfrac{a^2-b^2}{a^2+b^2}\right) + b\left(\dfrac{2ba}{a^2+b^2}\right)$

$= \dfrac{1}{(a^2+b^2)}(a^3 - ab^2 + 2ab^2) = \dfrac{a(a^2+b^2)}{a^2+b^2}$

$= a$

60. $a\cos 2\theta + b\sin 2\theta = c$

$\Rightarrow a\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right) + b\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = c$

$\Rightarrow a - a\tan^2\theta + 2b\tan\theta = c + c\tan^2\theta$

$\Rightarrow -(a+c)\tan^2\theta + 2b\tan\theta + (a-c) = 0$

$\therefore$ $\tan\alpha + \tan\beta = -\dfrac{2b}{-(c+a)} = \dfrac{2b}{c+a}$

61. $6\cos\theta + 8\sin\theta = 9$ ....(i)

$\Rightarrow 8\sin\theta = 9 - 6\cos\theta$

Squaring on both sides, we get

$64\sin^2\theta = 81 - 108\cos\theta + 36\cos^2\theta$

$\Rightarrow 64(1-\cos^2\theta) = 81 - 108\cos\theta + 36\cos^2\theta$

$\Rightarrow 100\cos^2\theta - 108\cos\theta + 17 = 0$

$\therefore$ $\cos\alpha\cos\beta = \dfrac{17}{100}$

From (i), $6\cos\theta = 9 - 8\sin\theta$

Squaring on both sides, we get

$36\cos^2\theta = 81 - 144\sin\theta + 64\sin^2\theta$

$\Rightarrow 36(1-\sin^2\theta) = 81 - 144\sin\theta + 64\sin^2\theta$

$\Rightarrow 100\sin^2\theta - 144\sin\theta + 45 = 0$

$\therefore$ $\sin\alpha\sin\beta = \dfrac{45}{100}$

Now, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \dfrac{17}{100} - \dfrac{45}{100} = \dfrac{-28}{100} = \dfrac{-14}{50}$

$\therefore$ $\sin(\alpha+\beta) = \sqrt{1-\cos^2(\alpha+\beta)}$

$= \sqrt{1-\left(\dfrac{-14}{50}\right)^2} = \dfrac{1}{50}\sqrt{2500-196}$

$= \dfrac{48}{50} = \dfrac{24}{25}$

62. $25\cos^2\alpha + 5\cos\alpha - 12 = 0$

$\Rightarrow \cos\alpha = \dfrac{-5\pm\sqrt{25+1200}}{50} = \dfrac{-5\pm 35}{50}$

$\Rightarrow \cos\alpha = \dfrac{-4}{5}$ $\ldots\left[\because \dfrac{\pi}{2} < \alpha < \pi \Rightarrow \cos\alpha < 0\right]$

$\Rightarrow \sin\alpha = \sqrt{1-\left(\dfrac{-4}{5}\right)^2} = \dfrac{3}{5}$

$\therefore$ $\sin 2\alpha = 2\sin\alpha\cos\alpha = \dfrac{-24}{25}$

63. $2\cos^2\theta - 2\sin^2\theta = 1$

$\Rightarrow 2\cos 2\theta = 1 \Rightarrow \cos 2\theta = \dfrac{1}{2} = \cos 60^\circ$

$\Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ$

64. $2\sin A\cos^3 A - 2\sin^3 A\cos A$

$= 2\sin A\cos A(\cos^2 A - \sin^2 A)$

$= 2\sin A\cos A\cos 2A$

$= \sin 2A\cos 2A$

$= \dfrac{1}{2}\sin 4A$

65. $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$

$=\frac{1}{4}\left[\left(2\sin^2\frac{\pi}{8}\right)^2+\left(2\sin^2\frac{3\pi}{8}\right)^2\right]$

$+\frac{1}{4}\left[\left(2\sin^2\frac{\pi}{8}\right)^2+\left(2\sin^2\frac{3\pi}{8}\right)^2\right]$

$=\frac{1}{4}\left[\left(1-\cos\frac{\pi}{4}\right)^2+\left(1-\cos\frac{3\pi}{4}\right)^2\right]$

$+\frac{1}{4}\left[\left(1-\cos\frac{\pi}{4}\right)^2+\left(1-\cos\frac{3\pi}{4}\right)^2\right]$

$=\frac{1}{4}\left[\left(1-\frac{1}{\sqrt{2}}\right)^2+\left(1+\frac{1}{\sqrt{2}}\right)^2\right]$

$+\frac{1}{4}\left[\left(1-\frac{1}{\sqrt{2}}\right)^2+\left(1+\frac{1}{\sqrt{2}}\right)^2\right]$

$=\frac{1}{4}(3)+\frac{1}{4}(3)=\frac{3}{2}$

66. $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x)$

$=3(\sin^2 x+\cos^2 x-2\sin x\cos x)^2+6(\sin^2 x+\cos^2 x+2\sin x\cos x)+4(\sin^6 x+\cos^6 x)$

$=3(1-\sin 2x)^2+6(1+\sin 2x)+4(1-3\sin^2 x\cos^2 x)$

$=3+3\sin^2 2x-6\sin 2x+6+6\sin 2x+4-3\sin^2 2x$

$=9+4+3\sin^2 2x-3\sin^2 2x=13$

67. $\sin 2\theta+\sin 2\phi=\frac{1}{2}$ …..(i)

and $\cos 2\theta+\cos 2\phi=\frac{3}{2}$ …..(ii)

Squaring and adding (i) and (ii), we get

$(\sin^2 2\theta+\cos^2 2\theta)+(\sin^2 2\phi+\cos^2 2\phi)+2(\sin 2\theta\sin 2\phi+\cos 2\theta\cos 2\phi)=\frac{1}{4}+\frac{9}{4}$

$\Rightarrow \cos 2\theta\cos 2\phi+\sin 2\theta\sin 2\phi=\frac{1}{4}$

$\Rightarrow \cos(2\theta-2\phi)=\frac{1}{4}$

$\Rightarrow 2\cos^2(\theta-\phi)-1=\frac{1}{4}$

$\Rightarrow \cos^2(\theta-\phi)=\frac{5}{8}$

68. $\cos\alpha\cos 2\alpha\cos 2^2\alpha\cos 2^3\alpha\ldots\cos 2^{n-1}\alpha$

$=\frac{2^n\sin\alpha\cos\alpha\cos 2\alpha\cos 2^2\alpha\ldots\cos 2^{n-1}\alpha}{2^n\sin\alpha}$

$=\frac{\sin\{2(2^{n-1}\alpha)\}}{2^n\sin\alpha}$

(using $2\sin\theta\cos\theta=\sin 2\theta$ again and again)

$=\frac{\sin 2^n\alpha}{2^n\sin\alpha}$

69. $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$

$=\frac{\sin 2^4\frac{2\pi}{15}}{2^4\sin\frac{2\pi}{15}}=\frac{\sin\frac{32\pi}{15}}{16\sin\frac{2\pi}{15}}$

$=\frac{\sin\left(2\pi+\frac{2\pi}{15}\right)}{16\sin\frac{2\pi}{15}}=\frac{\sin\frac{2\pi}{15}}{16\sin\frac{2\pi}{15}}=\frac{1}{16}$

70. $k=\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$

$=\cos\left(\frac{\pi}{2}-\frac{\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$

$=\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{\sin 2^3\frac{\pi}{9}}{2^3\sin\frac{\pi}{9}}$

$=\frac{\sin\frac{8\pi}{9}}{8\sin\frac{\pi}{9}}=\frac{\sin\left(\pi-\frac{\pi}{9}\right)}{8\sin\frac{\pi}{9}}=\frac{1}{8}$

71. $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$

$=\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\times 1\times\sin\left(\pi-\frac{5\pi}{14}\right)\sin\left(\pi-\frac{3\pi}{14}\right)\sin\left(\pi-\frac{\pi}{14}\right)$

$=\left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2$

$=\left(\cos\frac{6\pi}{14}\cos\frac{4\pi}{14}\cos\frac{2\pi}{14}\right)^2$ ….$\left[\because\ \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta\right]$

$$=\left(\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right)^2$$

$$=\left(-\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\right)^2$$

$$\ldots\left[\because\ \cos\frac{3\pi}{7}=\cos\left(\pi-\frac{4\pi}{7}\right)=-\cos\frac{4\pi}{7}\right]$$

$$=\left(-\frac{\sin\frac{2^3\pi}{7}}{2^3\sin\frac{\pi}{7}}\right)^2=\frac{1}{64}\left(-\frac{\sin\frac{8\pi}{7}}{\sin\frac{\pi}{7}}\right)^2$$

$$=\frac{1}{64}\quad\ldots\left[\because\sin\frac{8\pi}{7}=\sin\left(\pi+\frac{\pi}{7}\right)=-\sin\frac{\pi}{7}\right]$$

72. $\sin\left(\frac{31}{3}\pi\right)=\sin\left(10\pi+\frac{\pi}{3}\right)$

$=\sin\frac{\pi}{3}$

$=\frac{\sqrt{3}}{2}$

73. Since, $\tan\alpha+2\tan 2\alpha+2^2\tan 2^2\alpha+\ldots.+2^n\tan 2^n\alpha+2^{n+1}\cot 2^{n+1}\alpha=\cot\alpha\ \forall\ n\in N$

Here, $\alpha=\frac{\pi}{5}$

$\therefore\quad \tan\frac{\pi}{5}+2\tan\frac{2\pi}{5}+4\cot\frac{4\pi}{5}=\cot\frac{\pi}{5}$

74. $$\frac{\cot x-\tan x}{\cot 2x}=\frac{\cos^2 x-\sin^2 x}{\sin x\cos x}\times\frac{\sin 2x}{\cos 2x}$$

$$=\frac{2\cos 2x}{\sin 2x}\times\frac{\sin 2x}{\cos 2x}=2$$

75. $$\frac{\sec 8A-1}{\sec 4A-1}=\frac{1-\cos 8A}{\cos 8A}.\frac{\cos 4A}{1-\cos 4A}$$

$$=\frac{2\sin^2 4A}{\cos 8A}\cdot\frac{\cos 4A}{2\sin^2 2A}$$

$$=\frac{2\sin 4A\cos 4A}{\cos 8A}\cdot\frac{\sin 4A}{2\sin^2 2A}$$

$$=\tan 8A\ \frac{2\sin 2A\cos 2A}{2\sin^2 2A}$$

$$=\frac{\tan 8A}{\tan 2A}$$

76. $2\tan A=3\tan B$

$\Rightarrow\ \tan A=\frac{3}{2}\tan B=\frac{3}{2}t$ [Let tan B = t]

$\sin 2B=\frac{2t}{1+t^2}$, $\cos 2B=\frac{1-t^2}{1+t^2}$

$$\therefore\quad \frac{\sin 2B}{5-\cos 2B}=\frac{\left(\frac{2t}{1+t^2}\right)}{5-\left(\frac{1-t^2}{1+t^2}\right)}=\frac{2t}{4+6t^2}$$

$$=\frac{t}{2+3t^2}=\frac{\frac{3t}{2}-t}{1+\frac{3t^2}{2}}$$

$$=\tan(A-B)$$

77. $\cos 2\alpha=\frac{3\cos 2\beta-1}{3-\cos 2\beta}$

By componendo – dividendo, we get

$$\frac{\cos 2\alpha+1}{\cos 2\alpha-1}=\frac{3\cos 2\beta-1+3-\cos 2\beta}{3\cos 2\beta-1-(3-\cos 2\beta)}$$

$$\Rightarrow\frac{2\cos^2\alpha}{-2\sin^2\alpha}=\frac{2+2\cos 2\beta}{4\cos 2\beta-4}$$

$$\Rightarrow\frac{-\cos^2\alpha}{\sin^2\alpha}=\frac{1+\cos 2\beta}{2(\cos 2\beta-1)}=\frac{2\cos^2\beta}{-4\sin^2\beta}$$

$$\Rightarrow\frac{\sin^2\alpha}{\cos^2\alpha}=\frac{2\sin^2\beta}{\cos^2\beta}\Rightarrow\tan^2\alpha=2\tan^2\beta$$

$$\Rightarrow\tan\alpha=\sqrt{2}\tan\beta$$

78. Since, $\cos 3\theta=4\cos^3\theta-3\cos\theta$

$$\therefore\quad \cos 3\theta=4\left[\frac{1}{2^3}\left(a+\frac{1}{a}\right)^3\right]-3\left[\frac{1}{2}\left(a+\frac{1}{a}\right)\right]$$

$$\Rightarrow\cos 3\theta=\frac{1}{2}\left(a+\frac{1}{a}\right)\left[\left(a+\frac{1}{a}\right)^2-3\right]$$

$$\Rightarrow\cos 3\theta=\frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$$

79. $\cos^3\theta+\cos^3(120°-\theta)+\cos^3(120°+\theta)$

$=\frac{3}{4}\cos(3\theta)$

$\therefore\quad \cos^3 10°+\cos^3 110°+\cos^3 130°$

$=\cos^3(10°)+\cos^3(120°-10°)+\cos^3(120°+10°)$

$=\frac{3}{4}\cos(3\times 10°)=\frac{3}{4}\times\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{8}$

80. $\sin 6\theta = 2\sin 3\theta \cos 3\theta$
$= 2(3\sin\theta - 4\sin^3\theta)(4\cos^3\theta - 3\cos\theta)$
$= 24\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta) - 18\sin\theta\cos\theta - 32\sin^3\theta\cos^3\theta$
$= 6\sin\theta\cos\theta - 32\sin\theta\cos^3\theta\sin^2\theta$
$= 3\sin 2\theta - 32\sin\theta\cos^3\theta(1-\cos^2\theta)$
$= 32\cos^5\theta\sin\theta - 32\cos^3\theta\sin\theta + 3\sin 2\theta$
Given, $\sin 6\theta = 32\cos^5\theta\sin\theta - 32\cos^3\theta\sin\theta + 3x$
$\therefore$ On comparing, we get $x = \sin 2\theta$

81. $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

$= \dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + \dfrac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$

$= \dfrac{\left(1+\tan\frac{\theta}{2}\right)^2 + \left(1-\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}}$

$= 2\left(\dfrac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right)$

$= \dfrac{2}{\cos\theta}$
$= 2\sec\theta$

82. Given, $\tan x = \frac{3}{4}$

$\Rightarrow \cos x = -\frac{4}{5}$ ....$\left[\because \pi < x < \frac{3\pi}{2}\right]$

Since, $1 + \cos x = 2\cos^2\frac{x}{2}$

$\therefore$ $1 - \frac{4}{5} = 2\cos^2\frac{x}{2}$ $\therefore$ $\cos^2\frac{x}{2} = \frac{1}{10}$

$\therefore$ $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$ ....$\left[\because \pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}\right]$

83. $\sin A = \frac{4}{5}$

$\Rightarrow \tan A = -\frac{4}{3}$ ....[$\because$ 90° < A < 180°]

Now, $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$ (Let $\tan\frac{A}{2} = P$)

$\Rightarrow -\frac{4}{3} = \frac{2P}{1-P^2}$

$\Rightarrow 4P^2 - 6P - 4 = 0$

$\Rightarrow P = \frac{-1}{2}$ or $P = 2$

$P = \frac{-1}{2}$ is not possible

$\therefore$ $P = 2 \Rightarrow \tan\frac{A}{2} = 2$

84. Given, $\sin\frac{\theta}{2} = \sqrt{\frac{x-1}{2x}}$

$\therefore$ $\cos\frac{\theta}{2} = \sqrt{1-\sin^2\frac{\theta}{2}} = \sqrt{\frac{x+1}{2x}}$

and $\tan\frac{\theta}{2} = \frac{\sqrt{x-1}}{\sqrt{x+1}}$

Since, $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

$\therefore$ $\tan\theta = \sqrt{x^2-1}$

85. $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$= \frac{2y}{1+y^2}$ ....$\left[\text{Let } y = \tan\left(\frac{x}{2}\right)\right]$

$\therefore$ $\tan\frac{x}{2} = \operatorname{cosec} x - \sin x = \frac{1}{\sin x} - \sin x$

$\Rightarrow y = \frac{1+y^2}{2y} - \frac{2y}{1+y^2}$

$\Rightarrow 2y^2(1+y^2) = 1 + y^4 + 2y^2 - 4y^2$

$\Rightarrow 1 - y^4 - 4y^2 = 0 \Rightarrow y^4 + 4y^2 - 1 = 0$

$\Rightarrow y^2 = \frac{-4 \pm \sqrt{16+4}}{2} = \frac{-4 \pm \sqrt{20}}{2}$

$\Rightarrow \tan^2\left(\frac{x}{2}\right) = -2 \pm \sqrt{5}$

But, $\tan^2\left(\frac{x}{2}\right) \neq -2 - \sqrt{5}$

$\Rightarrow \tan^2\left(\frac{x}{2}\right) = -2 + \sqrt{5}$

86. $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$

By componendo – dividendo, we get

$\dfrac{\cos\theta + 1}{\cos\theta - 1} = \dfrac{\cos\alpha - \cos\beta + 1 - \cos\alpha\cos\beta}{\cos\alpha - \cos\beta - (1 - \cos\alpha\cos\beta)}$

$\Rightarrow \dfrac{\cos\theta+1}{\cos\theta-1} = \dfrac{(1+\cos\alpha)(1-\cos\beta)}{-(1-\cos\alpha)(1+\cos\beta)}$

$\Rightarrow \dfrac{1+\cos\theta}{1-\cos\theta} = \dfrac{1+\cos\alpha}{1-\cos\alpha} \times \dfrac{1-\cos\beta}{1+\cos\beta}$

$\Rightarrow \cot^2\dfrac{\theta}{2} = \cot^2\dfrac{\alpha}{2}\tan^2\dfrac{\beta}{2}$

$\Rightarrow \tan^2\dfrac{\theta}{2} = \tan^2\dfrac{\alpha}{2}\cot^2\dfrac{\beta}{2}$

$\Rightarrow \tan\dfrac{\theta}{2} = \pm\tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$

87. $\sqrt{4\cos^4\theta + \sin^2 2\theta} + 4\cot\theta\cos^2\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$

$= \sqrt{4\cos^4\theta + 4\sin^2\theta\cos^2\theta} + 4\cot\theta\left[\dfrac{1+\cos 2\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)}{2}\right]$

$= \sqrt{4\cos^2\theta(\cos^2\theta + \sin^2\theta)} + 2\cot\theta\left[1 + \cos\left(\dfrac{\pi}{2} - \theta\right)\right]$

$= |2\cos\theta| + 2\cot\theta + 2\cos\theta$

$= 2\cot\theta \qquad \ldots\left[\because \theta \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)\right]$

88. Put $\tan\left(\dfrac{\theta}{2}\right) = t$

$\therefore \quad (m+2)\left(\dfrac{2t}{1+t^2}\right) + (2m-1)\left(\dfrac{1-t^2}{1+t^2}\right) = 2m+1$

$\Rightarrow (2m+4)t + (2m-1)(1-t^2) = (2m+1)(1+t^2)$

$\Rightarrow 4mt^2 - (2m+4)t + 2 = 0$

$\Rightarrow 2mt^2 - mt - 2t + 1 = 0$

$\Rightarrow mt(2t-1) - 1(2t-1) = 0$

$\Rightarrow (2t-1)(mt-1) = 0$

$\Rightarrow t = \dfrac{1}{2}$ or $t = \dfrac{1}{m}$

If $t = \dfrac{1}{2}$, then

$\tan\theta = \dfrac{2t}{1-t^2} = \dfrac{1}{1-\dfrac{1}{4}} = \dfrac{4}{3}$

If $t = \dfrac{1}{m}$, then

$\tan\theta = \dfrac{\dfrac{2}{m}}{1-\left(\dfrac{1}{m^2}\right)} = \dfrac{2m}{m^2-1}$

89. $\cos\left(\dfrac{\alpha-\beta}{2}\right) = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)$

$\Rightarrow \cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2} + \sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2} = 2\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2} - 2\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}$

$\Rightarrow 3\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2} = \cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}$

$\Rightarrow \tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} = \dfrac{1}{3}$

90. Since, $\tan\dfrac{A}{2} = \dfrac{1-\cos A}{\sin A}$

Putting $\dfrac{A}{2} = \left(7\dfrac{1}{2}\right)^\circ$, we get

$\tan\left(7\dfrac{1}{2}\right)^\circ = \dfrac{1-\cos 15^\circ}{\sin 15^\circ}$

$= \dfrac{1 - \dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}}$

$= \dfrac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$

91. $\cot\dfrac{A}{2} = \dfrac{1+\cos A}{\sin A}$

Putting $A = \left(7\dfrac{1}{2}\right)^\circ$, we get

$\cot\left(7\dfrac{1}{2}\right)^\circ = \dfrac{1+\cos 15^\circ}{\sin 15^\circ}$

$= \dfrac{1 + \dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}}$

$= \dfrac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \sqrt{6} + \sqrt{2} + \sqrt{3} + \sqrt{4}$

92. Since, $\sin\left(22\frac{1}{2}\right)^\circ = \frac{1}{2}\sqrt{2-\sqrt{2}} = \cos\left(67\frac{1}{2}\right)^\circ$

and $\cos\left(22\frac{1}{2}\right)^\circ = \frac{1}{2}\sqrt{2+\sqrt{2}} = \sin\left(67\frac{1}{2}\right)^\circ$

$\therefore \left[1+\cos\left(22\frac{1}{2}\right)^\circ\right]\left[1+\cos\left(67\frac{1}{2}\right)^\circ\right]\left[1+\cos\left(112\frac{1}{2}\right)^\circ\right]\left[1+\cos\left(157\frac{1}{2}\right)^\circ\right]$

$= \left[1+\cos\left(22\frac{1}{2}\right)^\circ\right]\left[1+\cos\left(67\frac{1}{2}\right)^\circ\right]\left[1-\sin\left(22\frac{1}{2}\right)^\circ\right]\left[1-\sin\left(67\frac{1}{2}\right)^\circ\right]$

….[$\because \cos(90^\circ+\theta) = -\sin\theta$]

$= \left[1+\frac{1}{2}\sqrt{2+\sqrt{2}}\right]\left[1+\frac{1}{2}\sqrt{2-\sqrt{2}}\right]\left[1-\frac{1}{2}\sqrt{2-\sqrt{2}}\right]\left[1-\frac{1}{2}\sqrt{2+\sqrt{2}}\right]$

$= \left[1-\frac{1}{4}(2+\sqrt{2})\right]\left[1-\frac{1}{4}(2-\sqrt{2})\right]$

$= \frac{(4-2-\sqrt{2})(4-2+\sqrt{2})}{16}$

$= \frac{(2-\sqrt{2})(2+\sqrt{2})}{16} = \frac{4-2}{16} = \frac{1}{8}$

93. $\tan A = \frac{1-\cos B}{\sin B} = \frac{2\sin^2\left(\frac{B}{2}\right)}{2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)} = \tan\frac{B}{2}$

$\Rightarrow \tan 2A = \tan B$

94. Given, $\operatorname{cosec}\theta = \frac{p+q}{p-q} \Rightarrow \frac{1}{\sin\theta} = \frac{p+q}{p-q}$

By componendo – dividendo, we get

$\frac{1+\sin\theta}{1-\sin\theta} = \frac{p+q+p-q}{p+q-p+q}$

$\Rightarrow \left\{\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right\}^2 = \frac{p}{q}$

$\Rightarrow \left\{\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right\}^2 = \frac{p}{q} \Rightarrow \left\{\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right\}^2 = \frac{p}{q}$

$\Rightarrow \tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{p}{q} \Rightarrow \cot^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{q}{p}$

$\Rightarrow \cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \sqrt{\frac{q}{p}}$

95. $\frac{\sqrt{2}-\sin\alpha-\cos\alpha}{\sin\alpha-\cos\alpha}$

$= \frac{\sqrt{2}-\sqrt{2}\left\{\frac{1}{\sqrt{2}}\sin\alpha+\frac{1}{\sqrt{2}}\cos\alpha\right\}}{\sqrt{2}\left\{\frac{1}{\sqrt{2}}\sin\alpha-\frac{1}{\sqrt{2}}\cos\alpha\right\}}$

$= \frac{\sqrt{2}-\sqrt{2}\cos\left(\alpha-\frac{\pi}{4}\right)}{\sqrt{2}\sin\left(\alpha-\frac{\pi}{4}\right)}$

$= \frac{\sqrt{2}(1-\cos\theta)}{\sqrt{2}\sin\theta}$, where $\theta = \alpha-\frac{\pi}{4}$

$= \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$

$= \tan\frac{\theta}{2} = \tan\left(\frac{\alpha}{2}-\frac{\pi}{8}\right)$ ….$\left[\because \theta = \alpha-\frac{\pi}{4}\right]$

96. Putting $\theta = \phi = \frac{\pi}{4}$ in the given expression, we get

$\cos 2\left(\frac{\pi}{2}\right) - 4\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + 2\sin^2\left(\frac{\pi}{4}\right) = 0$

Put $\theta = \phi = \frac{\pi}{4}$ in option (A), then

$\cos 2\theta = \cos\frac{\pi}{2} = 0$

Hence, option (A) is the correct answer.

97. Given that $\sin\theta + \sin\phi = a$ …..(i)

and $\cos\theta + \cos\phi = b$ …..(ii)

Squaring (i) and (ii) and adding, we get

$2 + 2(\sin\theta\sin\phi + \cos\theta\cos\phi) = a^2 + b^2$

$\Rightarrow 2\cos(\theta-\phi) = a^2 + b^2 - 2$

$\Rightarrow \cos(\theta-\phi) = \frac{a^2+b^2-2}{2}$

$\Rightarrow \frac{1-\tan^2\frac{\theta-\phi}{2}}{1+\tan^2\frac{\theta-\phi}{2}} = \frac{a^2+b^2-2}{2}$

$\Rightarrow (a^2+b^2) + (a^2+b^2)\tan^2\frac{\theta-\phi}{2} - 2$

$- 2\tan^2\frac{\theta-\phi}{2} = 2 - 2\tan^2\frac{\theta-\phi}{2}$

$\Rightarrow \frac{4-a^2-b^2}{a^2+b^2} = \tan^2\frac{\theta-\phi}{2}$

$\Rightarrow \tan\left(\frac{\theta-\phi}{2}\right) = \sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

Trick : Putting $\theta = \frac{\pi}{4} = \phi$, we get $\tan\frac{\theta-\phi}{2} = 0$, which is given by option (B).

98. $\frac{1+\sin 2\alpha}{\cos(2\alpha-2\pi)\tan\left(\alpha-\frac{3\pi}{4}\right)}$

$-\frac{1}{4}\sin 2\alpha\left(\cot\frac{\alpha}{2}+\cot\left(\frac{3\pi}{2}+\frac{\alpha}{2}\right)\right)$

$= \frac{1+2\sin\alpha\cos\alpha}{\cos 2\alpha\left(\frac{\tan\alpha+1}{1-\tan\alpha}\right)}$

$-\frac{1}{4}(2\sin\alpha\cos\alpha)\left(\frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)$

$= \frac{(\cos\alpha+\sin\alpha)^2}{\cos 2\alpha\left(\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}\right)}$

$-\frac{1}{4}(2\sin\alpha\cos\alpha)\left(2\frac{\cos\alpha}{\sin\alpha}\right)$

$= 1 - \cos^2\alpha = \sin^2\alpha$

99. When $\cos 4\theta = \frac{1}{3}$, then $2\cos^2 2\theta - 1 = \frac{1}{3}$

$\Rightarrow 2\cos^2 2\theta = \frac{4}{3} \Rightarrow \cos^2 2\theta = \frac{2}{3}$

$\Rightarrow \cos 2\theta = \pm\sqrt{\frac{2}{3}}$ ....$\left[\begin{array}{l}\because\ \theta \in \left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right),\\ \therefore\ 2\theta \in \left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)\end{array}\right]$

$\therefore\ f\left(\frac{1}{3}\right) = f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$

$= \frac{2\cos^2\theta}{2\cos^2\theta-1} = \frac{1+\cos 2\theta}{\cos 2\theta}$

$\therefore\ f\left(\frac{1}{3}\right) = 1 \pm \sqrt{\frac{3}{2}}$ ....$\left[\because\ \cos 2\theta = \pm\sqrt{\frac{2}{3}}\right]$

100. $f_n(\theta) = \left(\tan\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)$

$(1+\sec 4\theta)\ldots.(1+\sec 2^n\theta)$

$= \left(\tan\frac{\theta}{2}\right)\left(\frac{1+\cos\theta}{\cos\theta}\right)(1+\sec 2\theta)$

$(1+\sec 4\theta)\ldots.(1+\sec 2^n\theta)$

$= \left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) \times \frac{2\cos^2\frac{\theta}{2}}{\cos\theta}(1+\sec 2\theta)$

$(1+\sec 4\theta)\ldots.(1+\sec 2^n\theta)$

$= \frac{\sin\theta}{\cos\theta}(1+\sec 2\theta)(1+\sec 4\theta)\ldots.(1+\sec 2^n\theta)$

....[$\because$ sin 2A = 2 sin A cos A]

$= \tan\theta\left(\frac{1+\cos 2\theta}{\cos 2\theta}\right)(1+\sec 4\theta)\ldots.(1+\sec 2^n\theta)$

$= \frac{\sin\theta}{\cos\theta}\left(\frac{2\cos^2\theta}{\cos 2\theta}\right)(1+\sec 4\theta)\ldots.(1+\sec 2^n\theta)$

$= \tan 2\theta\,(1+\sec 4\theta)\ldots.(1+\sec 2^n\theta)$

$= \tan 2^n\theta$

$\therefore\ f_2\left(\frac{\pi}{16}\right) = \tan\left(2^2\times\frac{\pi}{16}\right) = 1$

$f_3\left(\frac{\pi}{32}\right) = \tan\left(2^3\times\frac{\pi}{32}\right) = 1$

$f_4\left(\frac{\pi}{64}\right) = \tan\left(2^4\times\frac{\pi}{64}\right) = 1$

$f_5\left(\frac{\pi}{128}\right) = \tan\left(2^5\times\frac{\pi}{128}\right) = 1$

$\therefore$ option (D) is incorrect.

101. $\sum_{n=1}^{\infty}\sin\left(\frac{n!\pi}{720}\right) = \left(\sin\frac{1!\pi}{720}+\sin\frac{2!\pi}{720}+\ldots.+\frac{\sin 5!\pi}{720}\right)$

$+\sum_{n=6}^{\infty}\sin\frac{n!\pi}{720}$

$= \sin\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{30}\right)+\sin\left(\frac{\pi}{120}\right)+\sin\left(\frac{\pi}{360}\right)$

$+\sin\left(\frac{\pi}{720}\right)$

....$\left[\because\ \sum_{n=6}^{\infty}\sin\frac{n!\pi}{720} = 0\right]$

Evaluation Test

1. $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = \mathrm{k(say)}$

$\Rightarrow \cos\theta = \frac{k}{x}$, $\cos\left(\theta+\frac{2\pi}{3}\right) = \frac{k}{y}$

and $\cos\left(\theta+\frac{4\pi}{3}\right) = \frac{k}{z}$

$\therefore$ $\frac{k}{x}+\frac{k}{y}+\frac{k}{z}$

$= \cos\theta + \cos\left(\theta+\frac{2\pi}{3}\right) + \cos\left(\theta+\frac{4\pi}{3}\right)$

$= \cos\theta + \cos\left(\pi-\left(\frac{\pi}{3}-\theta\right)\right) + \cos\left(\pi+\left(\frac{\pi}{3}+\theta\right)\right)$

$= \cos\theta - \cos\left(\frac{\pi}{3}-\theta\right) - \cos\left(\frac{\pi}{3}+\theta\right)$

$= \cos\theta - \left[\cos\left(\frac{\pi}{3}-\theta\right) + \cos\left(\frac{\pi}{3}+\theta\right)\right]$

$= \cos\theta - 2\cos\frac{\pi}{3}\cos\theta$

$= \cos\theta - 2\times\frac{1}{2}\cos\theta$

$\therefore$ $\frac{k}{x}+\frac{k}{y}+\frac{k}{z} = 0$

$\Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$

2. $\tan 70° - \tan 20° - 2\tan 40°$

$= (\cot 20° - \tan 20°) - 2\tan 40°$

....[$\because \tan 70° = \tan(90° - 20°) = \cot 20°$]

$= 2\cot 40° - 2\tan 40°$

....[$\because \cot\theta - \tan\theta = 2\cot 2\theta$]

$= 2(\cot 40° - \tan 40°)$

$= 2(2\cot 80°) = 4\cot 80°$

$= 4\cot(90° - 10°) = 4\tan 10°$

3. Given, $\sqrt{x}+\frac{1}{\sqrt{x}} = 2\cos\theta$

Squaring on both sides, we get

$x+\frac{1}{x}+2 = 4\cos^2\theta$

$\Rightarrow x+\frac{1}{x} = 4\cos^2\theta - 2$

$\Rightarrow x+\frac{1}{x} = 2(2\cos^2\theta - 1) = 2\cos 2\theta$

Again, squaring on both sides, we get

$x^2+\frac{1}{x^2}+2 = 4\cos^2 2\theta$

$\Rightarrow x^2+\frac{1}{x^2} = 4\cos^2 2\theta - 2$

$\Rightarrow x^2+\frac{1}{x^2} = 2(2\cos^2 2\theta - 1)$

$\Rightarrow x^2+\frac{1}{x^2} = 2\cos 4\theta$ ....(i)

Cubing on both sides, we get

$\left(x^2+\frac{1}{x^2}\right)^3 = (2\cos 4\theta)^3$

$\Rightarrow x^6+\frac{1}{x^6}+3x^2\times\frac{1}{x^2}\left(x^2+\frac{1}{x^2}\right) = 8\cos^3 4\theta$

$\Rightarrow x^6+\frac{1}{x^6}+3(2\cos 4\theta) = 8\cos^3 4\theta$

….[From (i)]

$\Rightarrow x^6+\frac{1}{x^6} = 8\cos^3 4\theta - 6\cos 4\theta$

$= 2(4\cos^3 4\theta - 3\cos 4\theta)$

$= 2\cos 3(4\theta)$

….[$\because \cos 3A = 4\cos^3 A - 3\cos A$]

$= 2\cos 12\theta$

4. $\cos^3\theta + \cos^3(\theta + 120°) + \cos^3(\theta - 120°)$

$= \frac{\cos 3\theta + 3\cos\theta}{4} + \frac{\cos(3\theta+360°)+3\cos(\theta+120°)}{4} + \frac{\cos(3\theta-360°)+3\cos(\theta-120°)}{4}$

.... $\left[\begin{array}{l}\because \cos 3A = 4\cos^3 A - 3\cos A,\\ \therefore \cos^3 A = \frac{\cos 3A + 3\cos A}{4}\end{array}\right]$

$= \frac{\cos 3\theta}{4}+\frac{3\cos\theta}{4}+\frac{\cos 3\theta}{4}+\frac{3\cos(\theta+120°)}{4}+\frac{\cos 3\theta}{4}+\frac{3\cos(\theta-120°)}{4}$

$= \frac{3}{4}\cos 3\theta + \frac{3}{4}\{\cos(\theta+120°) + \cos(\theta-120°) + \cos\theta\}$

$= \frac{3}{4}\cos 3\theta + \frac{3}{4}\{2\cos\theta\cos 120° + \cos\theta\}$

$= \frac{3}{4}\cos 3\theta + \frac{3}{4}\{2\cos\theta\,(-\sin 30°) + \cos\theta\}$

$= \frac{3}{4}\cos 3\theta + \frac{3}{4}\left\{2\cos\theta\left(-\frac{1}{2}\right) + \cos\theta\right\}$

$= \frac{3}{4}\cos 3\theta + \frac{3}{4}(-\cos\theta + \cos\theta)$

$= \frac{3}{4}\cos 3\theta$

5. $\tan 2\alpha = \tan(\alpha + \alpha)$

$= \dfrac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{25}}$

$\therefore$ $\tan 2\alpha = \frac{5}{12}$

$\tan 4\alpha = \tan(2\alpha + 2\alpha)$

$= \dfrac{\frac{5}{12}+\frac{5}{12}}{1-\frac{25}{144}}$

$= \frac{120}{119}$

$\therefore$ $\tan(4\alpha - \beta) = \dfrac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}$

$= \dfrac{120\times 239 - 119}{119\times 239 + 120}$

$= \dfrac{(119+1)239 - 119}{119\times 239 + 120}$

$= \dfrac{119\times 239 + (239 - 119)}{119\times 239 + 120} = 1$

6. Given, $\sec(\theta + \phi)$, $\sec\theta$ and $\sec(\theta - \phi)$ are in A. P.

$\therefore$ $2\sec\theta = \sec(\theta + \phi) + \sec(\theta - \phi)$

$\Rightarrow \frac{2}{\cos\theta} = \frac{1}{\cos(\theta+\phi)} + \frac{1}{\cos(\theta-\phi)}$

$\Rightarrow \frac{2}{\cos\theta} = \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{\cos(\theta+\phi)\cos(\theta-\phi)}$

$\Rightarrow \frac{2}{\cos\theta} = \frac{2\cos\theta\cos\phi}{\cos^2\theta - \sin^2\phi}$

….[$\because \cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$]

$\Rightarrow \cos^2\theta\cos\phi = \cos^2\theta - \sin^2\phi$

$\Rightarrow \cos^2\theta - \cos^2\theta\cos\phi = \sin^2\phi$

$\Rightarrow \cos^2\theta\,(1 - \cos\phi) = (1 - \cos^2\phi)$

$\Rightarrow \cos^2\theta = 1 + \cos\phi$

$\Rightarrow \cos^2\theta = 2\cos^2\frac{\phi}{2}$

$\Rightarrow \cos\theta = \pm\sqrt{2}\cos\frac{\phi}{2}$

Comparing with $\cos\theta = k\cos\frac{\phi}{2}$, we get

$k = \pm\sqrt{2}$

7. $\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta$

$= \frac{1}{2}\{2\sin^2\theta + 3(2\sin\theta\cos\theta) + 5(2\cos^2\theta)\}$

$= \frac{1}{2}\{1 - \cos 2\theta + 3\sin 2\theta + 5(1 + \cos 2\theta)\}$

$= 3 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta$

Now,

$-\sqrt{4+\frac{9}{4}} \le 2\cos 2\theta + \frac{3}{2}\sin 2\theta \le \sqrt{4+\frac{9}{4}}$

$\Rightarrow -\frac{5}{2} \le 2\cos 2\theta + \frac{3}{2}\sin 2\theta \le \frac{5}{2}$

$\Rightarrow \frac{1}{2} \le 3 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta \le \frac{11}{2}$

$\Rightarrow \frac{1}{2} \le \sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta \le \frac{11}{2}$

$\Rightarrow \frac{2}{11} \le \frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta} \le 2$

Hence, the maximum value of the given expression is 2.

8. $\sin\alpha + \sin\beta = -\frac{21}{65}$, $\cos\alpha + \cos\beta = -\frac{27}{65}$

$\therefore$ $(\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2$

$= \left(\frac{-21}{65}\right)^2 + \left(\frac{-27}{65}\right)^2$

$\Rightarrow (\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta)$

$+ 2\sin\alpha\sin\beta + 2\cos\alpha\cos\beta = \frac{441}{(65)^2} + \frac{729}{(65)^2}$

$\Rightarrow 2 + 2\sin\alpha\sin\beta + 2\cos\alpha\cos\beta$

$= \frac{441}{(65)^2} + \frac{729}{(65)^2}$

$\Rightarrow 2 + 2[\cos(\alpha - \beta)] = \frac{1170}{(65)^2}$

$\Rightarrow 2[1 + \cos(\alpha - \beta)] = \frac{1170}{(65)^2}$

$\Rightarrow 2 \times 2 \cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1170}{(65)^2}$

$\therefore \quad \cos\left(\frac{\alpha-\beta}{2}\right) = -\frac{3\sqrt{130}}{130}$

$\ldots\left[\because \frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{3\pi}{2}\right]$

$= \frac{-3}{\sqrt{130}}$

9. $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$\Rightarrow \tan(\alpha+\beta) = \dfrac{\dfrac{1}{1+\frac{1}{2^x}} + \dfrac{1}{1+2^{x+1}}}{1-\left(\dfrac{1}{1+\frac{1}{2^x}}\right)\left(\dfrac{1}{1+2^{x+1}}\right)}$

$\Rightarrow \tan(\alpha+\beta) = \frac{2^x + 2.2^{x+x} + 2^x + 1}{1+2^x + 2.2^x + 2.2^{x+x} - 2^x}$

$\Rightarrow \tan(\alpha+\beta) = 1$

$\Rightarrow \tan(\alpha+\beta) = \tan\frac{\pi}{4}$

$\Rightarrow \alpha+\beta = \frac{\pi}{4}$

10. We have, $\frac{3\sin A}{\sin B} = \frac{2\cos B}{\cos A}$

$\Rightarrow \frac{3\sin A}{\cos A} = \frac{2\cos B\sin B}{\cos^2 A}$

$\Rightarrow \tan A = \frac{\sin 2B}{3\cos^2 A}$

$\Rightarrow \tan A = \frac{\sin 2B}{3\cos 2B} \times \frac{\cos 2B}{\cos^2 A}$

$\Rightarrow \tan A = \frac{\tan 2B}{3\cos^2 A}(2\cos^2 B - 1)$

$\Rightarrow \tan A = \frac{\tan 2B}{3\cos^2 A}(4 - 3\cos^2 A - 1)$

$\ldots[\because 2\cos^2 B = 4 - 3\cos^2 A \text{ (given)}]$

$\Rightarrow \tan A = \tan 2B \,\frac{\sin^2 A}{\cos^2 A}$

$\ldots[\because 1 - \cos^2 A = \sin^2 A]$

$\Rightarrow \tan A = \tan 2B \tan^2 A$

$\Rightarrow \tan A \tan 2B = 1$

$\Rightarrow \tan A = \cot 2B$

$\Rightarrow \tan A = \tan\left(\frac{\pi}{2} - 2B\right)$

$\Rightarrow A = \frac{\pi}{2} - 2B$

$\Rightarrow A + 2B = \frac{\pi}{2}$

11. $\frac{\sin^4 A}{a} + \frac{\cos^4 A}{b} = \frac{1}{a+b}$

$\Rightarrow \frac{(1-\cos 2A)^2}{4a} + \frac{(1+\cos 2A)^2}{4b} = \frac{1}{a+b}$

$\Rightarrow b(a+b)(1 - 2\cos 2A + \cos^2 2A)$
$+ a(a+b)(1 + 2\cos 2A + \cos^2 2A) = 4ab$

$\Rightarrow \{b(a+b) + a(a+b)\}\cos^2 2A$
$+ 2(a+b)(a-b)\cos 2A$
$+ a(a+b) + b(a+b) - 4ab = 0$

$\Rightarrow (a+b)^2\cos^2 2A + 2(a+b)(a-b)\cos 2A$
$+ (a-b)^2 = 0$

$\Rightarrow \{(a+b)\cos 2A + (a-b)\}^2 = 0$

$\Rightarrow \cos 2A = \frac{b-a}{b+a}$ ....(i)

$\therefore \quad \frac{\sin^8 A}{a^3} + \frac{\cos^8 A}{b^3} = \frac{(1-\cos 2A)^4}{16a^3} + \frac{(1+\cos 2A)^4}{16b^3}$

$= \frac{1}{16a^3}\left(1 - \frac{b-a}{b+a}\right)^4 + \frac{1}{16b^3}\left(1 + \frac{b-a}{b+a}\right)^4$

....[From (i)]

$= \frac{16a^4}{16a^3(b+a)^4} + \frac{16b^4}{16b^3(b+a)^4}$

$= \frac{1}{(b+a)^4}(a+b) = \frac{1}{(a+b)^3}$

12. $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$

$\Rightarrow \sin(\pi\cos\theta) = \sin\left(\frac{\pi}{2} + \pi\sin\theta\right)$

$\ldots\left[\because \cos\theta = \sin\left(\frac{\pi}{2} + \theta\right)\right]$

$\Rightarrow \pi\cos\theta = \frac{\pi}{2} + \pi\sin\theta$

$\Rightarrow \cos\theta - \sin\theta = \frac{1}{2}$ ....(i)

$\therefore \quad \cos\left(\theta + \frac{\pi}{4}\right) = \cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)$

$= \frac{1}{2\sqrt{2}}$ ....[From (i)]

Textbook Chapter No.

04 Factorization Formulae

Hints

Classical Thinking

1. $\cos 5° - \sin 25° = \sin (90 - 5)° - \sin 25°$
$= \sin 85° - \sin 25°$
$= 2 \cos 55° \sin 30°$
$= \cos 55°$

2. $\cos 57° + \sin 27° = \cos 57° + \cos (90° - 27°)$
$= \cos 57° + \cos 63°$
$= 2 \cos 60° \cos 3°$
$= \cos 3°$

3. $\cos 18° - \sin 18° = \cos 18° - \cos 72°$
$= 2 \sin 45° \sin 27°$
$= \sqrt{2} \sin 27°$

4. $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)$
$= 2 \sin \left(\frac{\frac{3\pi}{4}+x+\frac{3\pi}{4}-x}{2}\right) \times \sin\left(\frac{\frac{3\pi}{4}-x-\frac{3\pi}{4}-x}{2}\right)$
$= 2 \sin \left(\frac{3\pi}{4}\right)\sin(-x)$
$= -2 \sin \left(\frac{\pi}{2}+\frac{\pi}{4}\right)\sin x$
$= -2 \cos\left(\frac{\pi}{4}\right)\sin x$
$= -\sqrt{2} \sin x$

5. $(\sin 50° - \sin 70°) + \sin 10°$
$= 2 \cos 60° \sin (-10°) + \sin 10°$
$= -2.\frac{1}{2} \sin 10° + \sin 10°$
$= 0$

6. $\cos 52° + \cos 68° + \cos 172°$
$= (\cos 52° + \cos 172°) + \cos 68°$
$= 2 \cos 112° \cos 60° + \cos 68°$
$= \cos 112° + \cos 68°$
$= 2 \cos 90° \cos 22°$
$= 0$

7. $\{\sin (\beta + \gamma - \alpha) + \sin (\gamma + \alpha - \beta)\} + \{\sin (\alpha + \beta - \gamma) - \sin (\alpha + \beta + \gamma)\}$
$= 2 \sin \gamma \cos (\beta - \alpha) + 2 \sin (-\gamma) \cos (\alpha + \beta)$
$= 2 \sin \gamma [\cos (\beta - \alpha) - \cos (\alpha + \beta)]$
$= 2 \sin \gamma . 2 \sin \alpha \sin \beta$
$= 4 \sin \alpha \sin \beta \sin \gamma$

8. $\frac{\sin 3x - \sin x}{\cos 2x} = \frac{2 \cos 2x \sin x}{\cos 2x}$
$= 2 \sin x$

9. $\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \frac{2\sin 4x \cos x}{2\cos 4x \cos x}$
$= \tan 4x$

10. $\frac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \frac{2 \cos 6A \cos A}{2 \cos 6A \sin A}$
$= \cot A$

11. $\frac{\sin 70° + \cos 40°}{\cos 70° + \sin 40°} = \frac{\sin 70° + \sin 50°}{\cos 70° + \cos 50°}$
$= \frac{2 \sin 60° \cos 10°}{2 \cos 60° \cos 10°}$
$= \tan 60°$
$= \sqrt{3}$

12. $\frac{\sin 3A - \cos\left(\frac{\pi}{2}-A\right)}{\cos A + \cos(\pi + 3A)} = \frac{\sin 3A - \sin A}{\cos A - \cos 3A}$
$= \frac{2\cos 2A \sin A}{2 \sin 2A \sin A}$
$= \cot 2A$

13. $\frac{\sin 3\theta + \sin 5\theta + \sin 7\theta + \sin 9\theta}{\cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta}$
$= \frac{(\sin 3\theta + \sin 9\theta) + (\sin 5\theta + \sin 7\theta)}{(\cos 3\theta + \cos 9\theta) + (\cos 5\theta + \cos 7\theta)}$
$= \frac{2\sin 6\theta \cos 3\theta + 2\sin 6\theta \cos\theta}{2\cos 6\theta \cos 3\theta + 2\cos 6\theta \cos\theta}$
$= \frac{2\sin 6\theta(\cos 3\theta + \cos\theta)}{2\cos 6\theta(\cos 3\theta + \cos\theta)}$
$= \tan 6\theta$

14. $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{a+b}{a-b}$

By componendo and dividendo, we get

$\dfrac{\sin(x+y)+\sin(x-y)}{\sin(x+y)-\sin(x-y)} = \dfrac{(a+b)+(a-b)}{(a+b)-(a-b)}$

$\Rightarrow \dfrac{2\sin x\cos y}{2\cos x\sin y} = \dfrac{2a}{2b}$

$\Rightarrow \dfrac{\tan x}{\tan y} = \dfrac{a}{b}$

15. $2\sin 3x\cos 2x = \sin(3x+2x) + \sin(3x-2x)$
$= \sin 5x + \sin x$

16. $2\sin\dfrac{5\pi}{12}\cos\dfrac{\pi}{12} = \sin\left(\dfrac{5\pi}{12}+\dfrac{\pi}{12}\right) + \sin\left(\dfrac{5\pi}{12}-\dfrac{\pi}{12}\right)$

$= \sin\dfrac{\pi}{2} + \sin\dfrac{\pi}{3}$

$= 1 + \dfrac{\sqrt{3}}{2} = \dfrac{2+\sqrt{3}}{2}$

17. $\cos 75^\circ \cos 15^\circ$

$= \dfrac{1}{2}[2\cos 75^\circ \cos 15^\circ]$

$= \dfrac{1}{2}[\cos(75^\circ+15^\circ) + \cos(75^\circ-15^\circ)]$

$= \dfrac{1}{2}[\cos 90^\circ + \cos 60^\circ]$

$= \dfrac{1}{4}$

18. $\sin(45^\circ + A)\sin(45^\circ - A)$

$= \dfrac{1}{2}[2\sin(45^\circ + A)\sin(45^\circ - A)]$

$= \dfrac{1}{2}[\cos(45^\circ + A - 45^\circ + A)$
$- \cos(45^\circ + A + 45^\circ - A)]$

$= \dfrac{1}{2}[\cos 2A - \cos 90^\circ]$

$= \dfrac{1}{2}\cos 2A$

19. $4\sin\left(\dfrac{\pi}{3}+\theta\right)\sin\left(\dfrac{\pi}{3}-\theta\right)$

$= 2\left[2\sin\left(\dfrac{\pi}{3}+\theta\right)\sin\left(\dfrac{\pi}{3}-\theta\right)\right]$

$= 2\left[\cos\left(\dfrac{\pi}{3}+\theta-\dfrac{\pi}{3}+\theta\right) - \cos\left(\dfrac{\pi}{3}+\theta+\dfrac{\pi}{3}-\theta\right)\right]$

$= 2\left[\cos 2\theta - \cos\left(\dfrac{2\pi}{3}\right)\right]$

$= 2\cos 2\theta + 1$

20. $\sin 18^\circ \sin 70^\circ + \sin 16^\circ \sin 36^\circ$

$= \dfrac{1}{2}[2\sin 18^\circ \sin 70^\circ + 2\sin 16^\circ \sin 36^\circ]$

$= \dfrac{1}{2}[\cos 52^\circ - \cos 88^\circ + \cos 20^\circ - \cos 52^\circ]$

$= \dfrac{1}{2}[\cos 20^\circ - \cos 88^\circ]$

$= \dfrac{1}{2}[2\sin 54^\circ \sin 34^\circ]$

$= \sin 54^\circ \sin 34^\circ$

21. $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$

$= \dfrac{1}{2}.\dfrac{1}{2}(2\sin 10^\circ \sin 50^\circ)\sin 70^\circ$

$= \dfrac{1}{4}(\cos 40^\circ - \cos 60^\circ)\sin 70^\circ$

$= \dfrac{1}{8}(2\sin 70^\circ \cos 40^\circ - \sin 70^\circ)$

$= \dfrac{1}{8}(\sin 110^\circ + \sin 30^\circ - \sin 70^\circ)$

$= \dfrac{1}{8}\left(\sin 70^\circ + \dfrac{1}{2} - \sin 70^\circ\right)$

....[$\because \sin(180^\circ - A) = \sin A$]

$= \dfrac{1}{16}$

22. $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

$= \dfrac{1}{2}.\dfrac{1}{2}(2\cos 40^\circ \cos 20^\circ)\cos 80^\circ$

$= \dfrac{1}{4}(\cos 60^\circ + \cos 20^\circ)\cos 80^\circ$

$= \dfrac{1}{8}(\cos 80^\circ + 2\cos 20^\circ \cos 80^\circ)$

$= \dfrac{1}{8}(\cos 80^\circ + \cos 100^\circ + \cos 60^\circ)$

$= \dfrac{1}{16}$ ....[$\because \cos(180^\circ - A) = -\cos A$]

24. cosec A (sin B cos C + cos B sin C)
= cosec A sin (B + C)
= cosec A sin (180° – A)
= cosec A sin A
= 1

25. $\sin^2 \alpha + \sin^2 \beta - \sin^2 \gamma$
$= \sin^2 \alpha + \sin(\beta - \gamma)\sin(\beta + \gamma)$
$= \sin^2 \alpha + \sin(\pi - \alpha)\sin(\beta - \gamma)$
$= \sin \alpha\,[\sin \alpha + \sin(\beta - \gamma)]$
$= \sin \alpha\,[\sin(\beta + \gamma) + \sin(\beta - \gamma)]$
$= 2\sin \alpha \sin \beta \cos \gamma$

26. $\cos^2 A + \cos^2 B - \cos^2 C$
$= \frac{1}{2}(1 + \cos 2A) + \frac{1}{2}(1 + \cos 2B) - \frac{1}{2}(1 + \cos 2C)$
$= \frac{1}{2} + \frac{1}{2}(\cos 2A + \cos 2B - \cos 2C)$
$= \frac{1}{2} + \frac{1}{2}(1 - 4\sin A \sin B \cos C)$
$= 1 - 2\sin A \sin B \cos C$

27. $\tan(A + B) = \tan(180° - C)$
$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$
$\Rightarrow \tan A + \tan B = -\tan C(1 - \tan A \tan B)$
$\Rightarrow \tan A + \tan B = -\tan C + \tan A \tan B \tan C$
$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$

28. $\cos A = \cos B \cos C$
$\Rightarrow \cos(\pi - B - C) = \cos B \cos C$
$\Rightarrow -\cos(B + C) = \cos B \cos C$
$\Rightarrow -\cos B \cos C + \sin B \sin C = \cos B \cos C$
$\Rightarrow 2\cos B \cos C = \sin B \sin C$
$\Rightarrow \cot B \cot C = \frac{1}{2}$

Critical Thinking

1. $\cos 12° + \cos 84° + \cos 156° + \cos 132°$
$= (\cos 12° + \cos 132°) + (\cos 84° + \cos 156°)$
$= 2\cos 72° \cos 60° + 2\cos 120° \cos 36°$
$= 2\cos 72° \times \frac{1}{2} - 2 \times \frac{1}{2} \times \cos 36°$
$= \cos 72° - \cos 36°$
$= \frac{\sqrt{5} - 1}{4} - \frac{\sqrt{5} + 1}{4}$
$= \frac{-1}{2}$

2. $\cot 70° + 4\cos 70°$
$= \frac{\cos 70° + 4\sin 70° \cos 70°}{\sin 70°}$
$= \frac{\cos 70° + 2\sin 140°}{\sin 70°}$
$= \frac{\cos(90° - 20°) + 2\sin(180° - 40°)}{\sin 70°}$
$= \frac{\sin 20° + \sin 40° + \sin 40°}{\sin 70°}$
$= \frac{2\sin 30° \cos 10° + \sin 40°}{\sin 70°}$
$= \frac{\cos 10° + \sin 40°}{\sin 70°}$
$= \frac{\sin 80° + \sin 40°}{\sin 70°}$
$= \frac{2\sin 60° \cos 20°}{\sin 70°}$
$= \sqrt{3}$

3. $\cos 10x + \cos 8x + 3\cos 4x + 3\cos 2x$
$= (\cos 10x + \cos 8x) + 3(\cos 4x + \cos 2x)$
$= 2\cos 9x \cos x + 6\cos 3x \cos x$
$= 2\cos x(\cos 9x + 3\cos 3x)$
$= 2\cos x[\cos(3(3x)) + 3\cos 3x]$
$= 2\cos x(4\cos^3 3x - 3\cos 3x + 3\cos 3x)$
$= 8\cos^3 3x \cos x$

4. $1 + \cos 2x + \cos 4x + \cos 6x$
$= (1 + \cos 6x) + (\cos 2x + \cos 4x)$
$= 2\cos^2 3x + 2\cos 3x \cos x$
$= 2\cos 3x(\cos 3x + \cos x)$
$= 4\cos x \cos 2x \cos 3x$

5. $\frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$
$= \frac{(\cos 6x + \cos 4x) + 5(\cos 4x + \cos 2x) + 10(\cos 2x + 1)}{\cos 5x + 5\cos 3x + 10\cos x}$
$= \frac{2\cos x\cos 5x + 10\cos x\cos 3x + 10(2\cos^2 x - 1 + 1)}{\cos 5x + 5\cos 3x + 10\cos x}$
$= \frac{2\cos x(\cos 5x + 5\cos 3x + 10\cos x)}{\cos 5x + 5\cos 3x + 10\cos x}$
$= 2\cos x$

6. $\sin\theta + \sin 3\theta + \sin 2\theta = \sin\alpha$
$\Rightarrow 2\sin 2\theta \cos\theta + \sin 2\theta = \sin\alpha$
$\Rightarrow \sin 2\theta(2\cos\theta + 1) = \sin\alpha$ …..(i)
Also, $\cos\theta + \cos 3\theta + \cos 2\theta = \cos\alpha$
$\Rightarrow 2\cos 2\theta \cos\theta + \cos 2\theta = \cos\alpha$
$\Rightarrow \cos 2\theta(2\cos\theta + 1) = \cos\alpha$ …..(ii)

From (i) and (ii), we get

$\tan 2\theta = \tan \alpha$

$\Rightarrow 2\theta = \alpha$

$\Rightarrow \theta = \dfrac{\alpha}{2}$

7. $\cos x + \cos y + \cos \alpha = 0$

$\Rightarrow \cos x + \cos y = -\cos \alpha$

$\Rightarrow 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right) = -\cos\alpha$ ....(i)

Also, $\sin x + \sin y + \sin \alpha = 0$

$\Rightarrow \sin x + \sin y = -\sin \alpha$

$\Rightarrow 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right) = -\sin\alpha$ ....(ii)

Dividing (i) by (ii), we get

$$\frac{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} = \frac{\cos\alpha}{\sin\alpha}$$

$\Rightarrow \cot\left(\dfrac{x+y}{2}\right) = \cot\alpha$

8. $(\cos A + \cos B)^2 + (\sin A - \sin B)^2$

$= \left[2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)\right]^2 + \left[2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)\right]^2$

$= 4\cos^2\left(\dfrac{A+B}{2}\right)\left[\cos^2\left(\dfrac{A-B}{2}\right) + \sin^2\left(\dfrac{A-B}{2}\right)\right]$

$= 4\cos^2\left(\dfrac{A+B}{2}\right)$

9. $\cos^2\alpha + \cos^2(\alpha + 120°) + \cos^2(\alpha - 120°)$

$= \cos^2\alpha + \{\cos(\alpha + 120°) + \cos(\alpha - 120°)\}^2 - 2\cos(\alpha + 120°)\cos(\alpha - 120°)$

$= \cos^2\alpha + \{2\cos\alpha\cos 120°\}^2 - 2\{\cos^2\alpha - \sin^2 120°\}$

$= \cos^2\alpha + \cos^2\alpha - 2\cos^2\alpha + 2\sin^2 120°$

$= 2\sin^2 120°$

$= 2 \times \dfrac{3}{4} = \dfrac{3}{2}$

10. $\dfrac{\cos(A+B)}{\cos(A-B)} = \dfrac{\sin(C+D)}{\sin(C-D)}$

By componendo and dividendo, we get

$$\frac{\cos(A+B)+\cos(A-B)}{\cos(A+B)-\cos(A-B)} = \frac{\sin(C+D)+\sin(C-D)}{\sin(C+D)-\sin(C-D)}$$

$\Rightarrow \dfrac{2\cos A\cos B}{-2\sin A\sin B} = \dfrac{2\sin C\cos D}{2\cos C\sin D}$

$\Rightarrow -\cot A\cot B = \tan C\cot D$

$\Rightarrow \tan A\tan B\tan C + \tan D = 0$

11. $\dfrac{\sin^2 A - \sin^2 B}{\sin A\cos A - \sin B\cos B}$

$= \dfrac{2\sin(A+B)\sin(A-B)}{\sin 2A - \sin 2B}$

....[$\because \sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$]

$= \dfrac{2\sin(A+B)\sin(A-B)}{2\cos(A+B)\sin(A-B)}$

$= \tan(A+B)$

12. Since, $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \ldots\ldots + \cos\{\alpha + (n-1)\beta\}$

$= \dfrac{\cos\left\{\alpha + (n-1)\frac{\beta}{2}\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$

Here, $\alpha = \dfrac{\pi}{11}$ and $\beta = \dfrac{2\pi}{11}$

$\therefore$ $\cos\dfrac{\pi}{11} + \cos\dfrac{3\pi}{11} + \cos\dfrac{5\pi}{11} + \cos\dfrac{7\pi}{11} + \cos\dfrac{9\pi}{11}$

$= \dfrac{\cos\left(\frac{\pi}{11} + \frac{4\pi}{11}\right)\sin\left(\frac{5\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)}$

$= \dfrac{\cos\frac{5\pi}{11}\sin\frac{5\pi}{11}}{\sin\frac{\pi}{11}}$

$= \dfrac{1}{2}\dfrac{\sin\frac{10\pi}{11}}{\sin\frac{\pi}{11}}$ ....[$\because 2\sin\theta\cos\theta = \sin 2\theta$]

$= \dfrac{1}{2}$

13. $2\sin\dfrac{5A}{2}\sin\dfrac{A}{2} = \cos 2A - \cos 3A$

$= 2\cos^2 A - 1 - 4\cos^3 A + 3\cos A$

$= 2\left(\frac{9}{16}\right) - 1 - 4\left(\frac{27}{64}\right) + 3\left(\frac{3}{4}\right)$

$....\left[\because \cos A = \frac{3}{4}\right]$

$= \frac{9}{8} - 1 - \frac{27}{16} + \frac{9}{4}$

$= \frac{11}{16}$

14. $\sin\frac{\pi}{16}\sin\frac{3\pi}{16}\sin\frac{5\pi}{16}\sin\frac{7\pi}{16}$

$= \frac{1}{4}\left[2\sin\frac{\pi}{16}\sin\frac{3\pi}{16}.2\sin\frac{5\pi}{16}\sin\frac{7\pi}{16}\right]$

$= \frac{1}{4}\left[\left(\cos\frac{\pi}{8} - \cos\frac{\pi}{4}\right)\left(\cos\frac{\pi}{8} - \cos\frac{3\pi}{4}\right)\right]$

$= \frac{1}{4}\left[\left(\cos\frac{\pi}{8} - \frac{1}{\sqrt{2}}\right)\left(\cos\frac{\pi}{8} + \frac{1}{\sqrt{2}}\right)\right]$

$= \frac{1}{4}\left[\left(\cos^2\frac{\pi}{8} - \frac{1}{2}\right)\right]$

$= \frac{1}{8}\left[2\cos^2\frac{\pi}{8} - 1\right] = \frac{1}{8}\left[\cos\frac{\pi}{4}\right]$

$= \frac{1}{8} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{16}$

15. $\sin 20° \sin 40° \sin 60° \sin 80°$

$= \frac{\sqrt{3}}{2} \sin 20° \sin (60° - 20°) \sin (60° + 20°)$

$= \frac{\sqrt{3}}{2} . \frac{1}{4} \sin 60°$

$....\left[\because \sin\theta \sin(60° - \theta)\sin(60° + \theta) = \frac{1}{4}\sin 3\theta\right]$

$= \frac{\sqrt{3}}{8} . \frac{\sqrt{3}}{2}$

$= \frac{3}{16}$

16. $\tan 20° \tan 40° \tan 60° \tan 80°$

$= \tan 20° \tan (60° - 20°). \sqrt{3} . \tan(60° + 20°)$

$= \sqrt{3} \tan 20°. \tan (60° - 20°) . \tan (60° + 20°)$

$= \sqrt{3} \tan 3(20°)$

$….[\because \tan\theta \tan(60° - \theta) \tan(60° + \theta) = \tan 3\theta]$

$= \sqrt{3} . \sqrt{3} = 3$

17. $\frac{\tan 70^\circ - \tan 20^\circ}{\tan 50^\circ} = \frac{\frac{\sin 70^\circ}{\cos 70^\circ} - \frac{\sin 20^\circ}{\cos 20^\circ}}{\frac{\sin 50^\circ}{\cos 50^\circ}}$

$= \frac{\frac{\sin 70^\circ \cos 20^\circ - \cos 70^\circ \sin 20^\circ}{\cos 70^\circ \cos 20^\circ}}{\frac{\sin 50^\circ}{\cos 50^\circ}}$

$= \frac{\sin(70^\circ - 20^\circ)\cos 50^\circ}{\cos 70^\circ \cos 20^\circ \sin 50^\circ}$

$= \frac{2\sin 50^\circ \cos 50^\circ}{2\cos 70^\circ \cos 20^\circ \sin 50^\circ}$

$= \frac{2\cos 50^\circ}{\cos 90^\circ + \cos 50^\circ}$

$= \frac{2\cos 50^\circ}{0 + \cos 50^\circ} = 2$

18. $\tan 20° + 2 \tan 50° - \tan 70°$

$= \frac{\sin 20^\circ}{\cos 20^\circ} - \frac{\sin 70^\circ}{\cos 70^\circ} + 2 \tan 50°$

$= \frac{\sin 20^\circ \cos 70^\circ - \cos 20^\circ \sin 70^\circ}{\cos 20^\circ \cos 70^\circ} + 2 \tan 50°$

$= \frac{\sin(20^\circ - 70^\circ)}{\frac{1}{2}[\cos(70^\circ + 20^\circ) + \cos(70^\circ - 20^\circ)]} + 2 \tan 50°$

$= \frac{2\sin(-50^\circ)}{\cos 90^\circ + \cos 50^\circ} + 2 \tan 50°$

$= \frac{-2\sin 50^\circ}{0 + \cos 50^\circ} + 2 \tan 50°$

$= - 2 \tan 50° + 2 \tan 50°$

$= 0$

19. $\operatorname{cosec} 48° + \operatorname{cosec} 84° + \operatorname{cosec} 192° + \operatorname{cosec} 384°$

$= \frac{1}{\sin 48^\circ} + \frac{1}{\sin 84^\circ} + \frac{1}{-\sin 12^\circ} + \frac{1}{\sin 24^\circ}$

$= \left(\frac{1}{\sin 48^\circ} - \frac{1}{\sin 12^\circ}\right) + \left(\frac{1}{\sin 84^\circ} + \frac{1}{\sin 24^\circ}\right)$

$= -\frac{(\sin 48^\circ - \sin 12^\circ)}{\sin 48^\circ \sin 12^\circ} + \frac{(\sin 84^\circ + \sin 24^\circ)}{\sin 84^\circ \sin 24^\circ}$

$= -\frac{2\cos 30^\circ \sin 18^\circ}{\frac{1}{2}(\cos 36^\circ - \cos 60^\circ)} + \frac{2\sin 54^\circ \cos 30^\circ}{\frac{1}{2}(\cos 60^\circ - \cos 108^\circ)}$

$= \frac{4\cos 30° \sin 18°}{\cos 60° - \cos 36°} + \frac{4\sin 54° \cos 30°}{\cos 60° + \sin 18°}$

$= 4\cos 30°\left[\frac{\sin 18°}{\cos 60° - \cos 36°} + \frac{\sin 54°}{\cos 60° + \sin 18°}\right]$

$= 4\cos 30°\left[\frac{\sin 18°}{\cos 60° - \cos 36°} + \frac{\cos 36°}{\cos 60° + \sin 18°}\right]$

$= 4\cos 30°\left[\frac{\frac{\sqrt{5}-1}{4}}{\frac{1}{2} - \frac{\sqrt{5}+1}{4}} + \frac{\frac{\sqrt{5}+1}{4}}{\frac{1}{2} + \frac{\sqrt{5}-1}{4}}\right]$

$= 4\cos 30° (-1 + 1) = 0$

20. $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{7\pi}{7}$

$= \frac{1}{2\sin\frac{\pi}{7}}\left\{2\cos\frac{2\pi}{7}\sin\frac{\pi}{7} + 2\cos\frac{4\pi}{7}\sin\frac{\pi}{7}\right.$

$\left. + 2\cos\frac{6\pi}{7}\sin\frac{\pi}{7}\right\} + \cos\pi$

$= \frac{1}{2\sin\frac{\pi}{7}}\left[\sin\frac{3\pi}{7} - \sin\frac{\pi}{7} + \sin\frac{5\pi}{7} - \sin\frac{3\pi}{7}\right.$

$\left. + \sin\pi - \sin\frac{5\pi}{7}\right] + \cos\pi$

$= -\frac{1}{2} - 1$

$= -\frac{3}{2}$

21. $\cos^2(A - B) + \cos^2 B - 2\cos(A - B)\cos A\cos B$

$= \cos^2(A - B) + \cos^2 B$

$\quad - \cos(A - B)\{\cos(A - B) + \cos(A + B)\}$

$= \cos^2 B - \cos(A - B)\cos(A + B)$

$= \cos^2 B - (\cos^2 A - \sin^2 B)$

$= 1 - \cos^2 A$

Hence, it depends on A.

22. $\frac{A}{B} = \tan 6° \tan 42° \tan 66° \tan 78°$

$= \frac{(2\sin 66°\sin 6°)(2\sin 78°\sin 42°)}{(2\cos 66°\cos 6°)(2\cos 78°\cos 42°)}$

$= \frac{(\cos 60° - \cos 72°)(\cos 36° - \cos 120°)}{(\cos 60° + \cos 72°)(\cos 36° + \cos 120°)}$

$= \frac{(\cos 60° - \sin 18°)(\cos 36° + \sin 30°)}{(\cos 60° + \sin 18°)(\cos 36° - \sin 30°)}$

....[$\because \cos(90° + \theta) = -\sin\theta$]

$= \frac{\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right)}{\left(\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)}$

$= \frac{9-5}{5-1} = 1$

$\Rightarrow A = B$

23. $\sin 2A + \sin 2B - \sin 2C$

$= 2\sin A\cos A + 2\cos(B + C)\sin(B - C)$

$= 2\sin A\cos A - 2\cos A\sin(B - C)$

$= 2\cos A[\sin A - \sin(B - C)]$

$= 2\cos A[\sin(B + C) - \sin(B - C)]$

....[$\because \sin(B + C) = \sin A$]

$= 2\cos A(2\cos B\sin C)$

$= 4\cos A\cos B\sin C$

Trick : Check by assuming A = B = 45° and C = 90°

24. $\cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A + B)\cos(A - B) + (2\cos^2 C - 1)$

$= -1 - 2\cos C\cos(A - B) + 2\cos^2 C$

....[$\because \cos(A + B) = -\cos C$]

$= -1 - 2\cos C[\cos(A - B) + \cos(A + B)]$

$= -1 - 4\cos A\cos B\cos C$

25. $\cos 2x + \cos 2y - \cos 2z$

$= 2\cos(x + y)\cos(x - y) - 2\cos^2 z + 1$

$= 2\cos(\pi - z)\cos(x - y) - 2\cos^2 z + 1$

$= 1 - 2\cos z\{\cos(x - y) - \cos(x + y)\}$

$= 1 - 2\cos z . 2\sin x\sin y$

$= 1 - 4\sin x\sin y\cos z$

26. $\cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A + B)\cos(A - B) + \cos 2C$

$= 2\cos\left(\frac{3\pi}{2} - C\right)\cos(A - B) + \cos 2C$

....$\left[\because A + B = \frac{3\pi}{2} - C\right]$

$= -2\sin C\cos(A - B) + 1 - 2\sin^2 C$

$= 1 - 2\sin C\{\cos(A - B) + \sin C\}$

$= 1 - 2\sin C\{\cos(A - B) - \cos(A + B)\}$

$= 1 - 4\sin A\sin B\sin C$

Trick : Check by assuming $A = B = C = \frac{\pi}{2}$

27. $\frac{\cos A}{\sin B \sin C}+\frac{\cos B}{\sin C \sin A}+\frac{\cos C}{\sin A \sin B}$

$= \frac{2\sin A\cos A}{2\sin A\sin B\sin C}+\frac{2\sin B\cos B}{2\sin A\sin B\sin C}$

$+\frac{2\sin C\cos C}{2\sin A\sin B\sin C}$

$= \frac{\sin 2A+\sin 2B+\sin 2C}{2\sin A\sin B\sin C}$

$= \frac{4\sin A\sin B\sin C}{2\sin A\sin B\sin C}$

$= 2$

28. $\sin^2 A + \sin^2 B + \sin^2 C$
$= 1 - \cos^2 A + 1 - \cos^2 B + \sin^2 C$
$= 2 - \cos^2 A - (\cos^2 B - \sin^2 C)$
$= 2 - \cos^2 A - \cos(B + C)\cos(B - C)$
$= 2 - \cos A[\cos A - \cos(B - C)]$
$= 2 - \cos A[-\cos(B + C) - \cos(B - C)]$
$= 2 + \cos A.2\cos B\cos C$
$\therefore$ $\sin^2 A + \sin^2 B + \sin^2 C - 2\cos A\cos B\cos C = 2$

29. $\cos^2 A + \cos^2\left(A+\frac{\pi}{3}\right)+\cos^2\left(A-\frac{\pi}{3}\right)$

$= \frac{1}{2}(1+\cos 2A)+\frac{1}{2}\left\{1+\cos\left(2A+\frac{2\pi}{3}\right)\right\}$

$+\frac{1}{2}\left\{1+\cos\left(2A-\frac{2\pi}{3}\right)\right\}$

$= \frac{3}{2}+\frac{1}{2}\cos 2A$

$+\frac{1}{2}\left\{\cos\left(2A+\frac{2\pi}{3}\right)+\cos\left(2A-\frac{2\pi}{3}\right)\right\}$

$= \frac{3}{2}+\frac{1}{2}\cos 2A+\cos 2A\cos\frac{2\pi}{3}$

….[$\because$ $\cos(A + B) + \cos(A - B) = 2\cos A\cos B$]

$= \frac{3}{2}+\frac{1}{2}\cos 2A-\frac{1}{2}\cos 2A=\frac{3}{2}$

30. We have,

$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}$

$= \frac{1}{2}(1+\cos A)+\frac{1}{2}(1+\cos B)-\frac{1}{2}(1+\cos C)$

$= \frac{1}{2}+\frac{1}{2}(\cos A+\cos B-\cos C)$

$= \frac{1}{2}+\frac{1}{2}\left[4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}-1\right]$

$= 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

31. $\sin A + \sin B + \sin C$

$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$

$= 2\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\frac{A-B}{2}$

$+2\cos\frac{C}{2}\sin\left(\frac{\pi}{2}-\left(\frac{A+B}{2}\right)\right)$

$= 2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\cos\frac{C}{2}\cos\frac{A+B}{2}$

$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right]$

$= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right)$

$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

32. $\frac{\sin 2A+\sin 2B+\sin 2C}{\cos A+\cos B+\cos C-1}$

$= \frac{4\sin A\sin B\sin C}{1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}-1}$

$= \frac{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$

….$\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$

$= 8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

33. We have, $\alpha + \beta + \gamma = 2\pi$

$\Rightarrow \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=\pi$

$\Rightarrow \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)=\tan\pi=0$

$\Rightarrow \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=0$

$\Rightarrow \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

34. $\sum \dfrac{\cot A + \cot B}{\tan A + \tan B} = \sum \dfrac{\dfrac{\cos A}{\sin A} + \dfrac{\cos B}{\sin B}}{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}$

$= \sum \left(\dfrac{\sin B \cos A + \sin A \cos B}{\sin A.\sin B}\right)\left(\dfrac{\cos A.\cos B}{\sin A \cos B + \cos A.\sin B}\right)$

$= \sum \cot A \cot B$

$= \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

$\ldots\left[\begin{array}{l}\because A + B + C = \pi, \\ \therefore \cot A \cot B + \cot B \cot C + \cot C \cot A = 1\end{array}\right]$

Competitive Thinking

1. $\sin 47° + \sin 61° - (\sin 11° + \sin 25°)$

$= 2 \sin 54° \cos 7° - 2 \sin 18° \cos 7°$

$= 2 \cos 7° (\sin 54° - \sin 18°)$

$= 2 \cos 7° (2 \cos 36° \sin 18°)$

$= 4 \cos 7° . \dfrac{\sqrt{5}+1}{4} . \dfrac{\sqrt{5}-1}{4}$

$= \cos 7°$

2. $\cos A + \cos(240° + A) + \cos(240° - A)$

$= \cos A + 2 \cos 240° \cos A$

$= \cos A \{1 + 2 \cos (180° + 60°)\}$

$= \cos A \left\{1 + 2\left(-\dfrac{1}{2}\right)\right\}$

$= 0$

3. $\cos\dfrac{10\pi}{13} + \cos\dfrac{8\pi}{13} + \cos\dfrac{3\pi}{13} + \cos\dfrac{5\pi}{13}$

$= \left(\cos\dfrac{10\pi}{13} + \cos\dfrac{3\pi}{13}\right) + \left(\cos\dfrac{8\pi}{13} + \cos\dfrac{5\pi}{13}\right)$

$= 2\cos\left(\dfrac{13\pi}{2\times13}\right)\cdot\cos\left(\dfrac{7\pi}{2\times13}\right)$

$+ 2\cos\left(\dfrac{13\pi}{2\times13}\right)\cos\left(\dfrac{3\pi}{2\times13}\right)$

$= 2 \cos \dfrac{\pi}{2}\left(\cos\dfrac{7\pi}{26} + \cos\dfrac{3\pi}{26}\right)$

$= 0$ $\ldots\left[\because \cos\dfrac{\pi}{2} = 0\right]$

4. $2 \cos\dfrac{\pi}{13} \cos\dfrac{9\pi}{13} + \cos\dfrac{3\pi}{13} + \cos\dfrac{5\pi}{13}$

$= 2 \cos\dfrac{\pi}{13} \cos\dfrac{9\pi}{13} + 2 \cos\dfrac{4\pi}{13} \cos\dfrac{\pi}{13}$

$= 2 \cos\dfrac{\pi}{13}\left[\cos\dfrac{9\pi}{13} + \cos\dfrac{4\pi}{13}\right]$

$= 2 \cos\dfrac{\pi}{13}\left[2 \cos\dfrac{\pi}{2} \cos\dfrac{5\pi}{26}\right]$

$= 0$ $\ldots\left[\because \cos\dfrac{\pi}{2} = 0\right]$

5. $2 \cos x - \cos 3x - \cos 5x$

$= 2 \cos x (1 - \cos 4x)$

$= 2 \cos x \, 2 \sin^2 2x$

$= 4 \cos x \sin^2 2x$

$= 4 \cos x (2 \sin x \cos x)^2$

$= 16 \sin^2 x \cos^3 x$

6. $1 + \cos 10° + \cos 20° + \cos 30°$

$= 2 \cos^2 5° + 2 \cos 25° \cos 5°$

$= 2 \cos 5° (\cos 5° + \cos 25°)$

$= 2 \cos 5° (2 \cos 15° \cos 10°)$

$= 4 \cos 5° \cos 10° \cos 15°$

7. $1 + \cos 56° + \cos 58° - \cos 66°$

$= 2 \cos^2 28° + 2 \sin 62° \sin 4°$

$= 2 \cos^2 28° + 2 \cos 28° \cos 86°$

$\ldots[\because \sin (90° - \theta) = \cos \theta]$

$= 2 \cos 28° (\cos 28° + \cos 86°)$

$= 2 \cos 28°.2 \cos 57° \cos 29°$

$= 4 \cos 28° \cos 29° \sin 33°$

$\ldots[\because \cos (90° - \theta) = \sin \theta]$

8. $\dfrac{\sin 85° - \sin 35°}{\cos 65°} = \dfrac{2 \cos 60° \sin 25°}{\sin 25°}$

$= 2 \times \dfrac{1}{2}$

$= 1$

9. $\dfrac{\sin 55° - \cos 55°}{\sin 10°} = \dfrac{\sin 55° - \sin 35°}{\sin 10°}$

$= \dfrac{2 \cos 45° \sin 10°}{\sin 10°} = \sqrt{2}$

10. $\tan 9° - \tan 27° - \tan 63° + \tan 81°$

$= \tan 9° - \tan 27° - \cot 27° + \cot 9°$

$\ldots[\because \tan (90° - \theta) = \cot \theta]$

$= (\tan 9° + \cot 9°) - (\tan 27° + \cot 27°)$

$= \dfrac{1}{\sin 9^{\circ} \cos 9^{\circ}} - \dfrac{1}{\sin 27^{\circ} \cos 27^{\circ}}$

$= \dfrac{2}{\sin 18^{\circ}} - \dfrac{2}{\sin 54^{\circ}}$

…[$\because \sin 2\theta = 2 \sin \theta \cos \theta$]

$= 2\left\{\dfrac{\sin 54^{\circ} - \sin 18^{\circ}}{\sin 18^{\circ} \sin 54^{\circ}}\right\}$

$= 2.\dfrac{2 \cos 36^{\circ} \sin 18^{\circ}}{\sin 18^{\circ} \sin 54^{\circ}}$

$= \dfrac{4\cos 36°}{\cos 36°}$

$= 4$

11. $\dfrac{\sin A - \sin B}{\cos A + \cos B} = \dfrac{2.\cos\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)}{2.\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)}$

$= \tan\left(\dfrac{A-B}{2}\right)$

12. $\dfrac{\sin(B+A) + \cos(B-A)}{\sin(B-A) + \cos(B+A)}$

$= \dfrac{\sin(B+A) + \sin\{(90^{\circ} - (B-A)\}}{\sin(B-A) + \sin\{(90^{\circ} - (A+B)\}}$

$= \dfrac{2\sin(A+45^{\circ})\cos(45^{\circ} - B)}{2\sin(45^{\circ} - A)\cos(45^{\circ} - B)}$

$= \dfrac{\sin(A+45^{\circ})}{\sin(45^{\circ} - A)} = \dfrac{\cos A + \sin A}{\cos A - \sin A}$

13. $\sin 4A - \cos 2A = \cos 4A - \sin 2A$

$\Rightarrow \sin 4A + \sin 2A = \cos 4A + \cos 2A$

$\Rightarrow 2 \sin 3A \cos A = 2 \cos 3A \cos A$

$\Rightarrow \tan 3A = 1$

$\Rightarrow 3A = \dfrac{\pi}{4}$

$\Rightarrow 4A = \dfrac{\pi}{3}$

$\therefore$ $\tan 4A = \tan \dfrac{\pi}{3}$

$= \sqrt{3}$

14. $\sin x + \sin y = \dfrac{1}{2}$

$\Rightarrow 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right) = \dfrac{1}{2}$ …. (i)

$\cos x + \cos y = 1$

$\Rightarrow 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right) = 1$ ….(ii)

Dividing (i) by (ii), we get

$\tan\left(\dfrac{x+y}{2}\right) = \dfrac{1}{2}$

Now, $\tan(x+y) = \dfrac{2\tan\left(\frac{x+y}{2}\right)}{1-\tan^2\left(\frac{x+y}{2}\right)}$

$= \dfrac{2\left(\frac{1}{2}\right)}{1-\frac{1}{4}} = \dfrac{4}{3}$

15. $\cos x = 3 \cos y \Rightarrow \dfrac{\cos x}{\cos y} = \dfrac{3}{1}$

By componendo and dividendo, we get

$\dfrac{\cos x + \cos y}{\cos x - \cos y} = \dfrac{3+1}{3-1}$

$\Rightarrow \dfrac{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)} = \dfrac{4}{2}$

$\Rightarrow -\cot\left(\dfrac{x+y}{2}\right)\cot\left(\dfrac{x-y}{2}\right) = 2$

$\Rightarrow \cot\left(\dfrac{x+y}{2}\right)\cot\left(\dfrac{y-x}{2}\right) = 2$

$\Rightarrow 2\tan\left(\dfrac{y-x}{2}\right) = \cot\left(\dfrac{x+y}{2}\right)$

16. Given that, $\cos A = m \cos B$

$\Rightarrow \dfrac{m}{1} = \dfrac{\cos A}{\cos B}$

By componendo and dividendo, we get

$\dfrac{m+1}{m-1} = \dfrac{\cos A + \cos B}{\cos A - \cos B}$

$= \dfrac{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{B-A}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)}$

….[$\because \cos(B-A) = \cos(A-B)$]

$= \cot\left(\dfrac{A+B}{2}\right)\cot\left(\dfrac{B-A}{2}\right)$

$\Rightarrow \cot\left(\dfrac{A+B}{2}\right) = \dfrac{m+1}{m-1}\tan\left(\dfrac{B-A}{2}\right)$

17. $S = \sin\theta + \sin 2\theta + \sin 3\theta + \ldots. + \sin n\theta$

Since,

$\sin\theta + \sin(\theta + \beta) + \sin(\theta + 2\beta) + \ldots\ldots + \sin[\theta + (n-1)\beta]$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \sin\left[\theta + \left(\frac{(n-1)}{2}\right)\beta\right]$$

Here, $\beta = \theta$

$$\therefore \quad S = \frac{\sin\frac{n\theta}{2}.\sin\frac{\theta(n+1)}{2}}{\sin\frac{\theta}{2}}$$

18. Since, $\cos\alpha + \cos 2\alpha + \cos 3\alpha + \ldots. + \cos n\alpha$

$$= \frac{\cos\left(\frac{n+1}{2}\right)\alpha.\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

Here, $n = 3$ and $\alpha = \frac{2\pi}{7}$

$$\therefore \quad \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$$

$$= \frac{\cos\left(\frac{3+1}{2}\right)\left(\frac{2\pi}{7}\right)\sin\left(\frac{3\times2\pi}{2\times7}\right)}{\sin\left(\frac{2\pi}{7\times2}\right)}$$

$$= \frac{\cos\left(\frac{4\pi}{7}\right).\sin\left(\frac{3\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)}$$

Since, the values of $\cos\left(\frac{4\pi}{7}\right)$, $\sin\left(\frac{3\pi}{7}\right)$ and $\sin\left(\frac{\pi}{7}\right)$ are –ve, +ve and +ve respectively.

$\therefore$ option (C) is the correct answer.

19. $\cos A = \frac{3}{4} \Rightarrow \sin A = \frac{\sqrt{7}}{4}$

Now, $32 \sin\frac{A}{2}\cos\frac{5A}{2}$

$= 16 (\sin 3A - \sin 2A)$

$= 16 (3\sin A - 4\sin^3 A - 2\sin A\cos A)$

$= 16 \sin A(3 - 4\sin^2 A - 2\cos A)$

$= 16.\ \frac{\sqrt{7}}{4}\left(3 - 4.\frac{7}{16} - 2.\frac{3}{4}\right)$

$= -\sqrt{7}$

20. $\sin 12° \sin 48° \sin 54°$

$= \frac{1}{2}(\cos 36° - \cos 60°)\cos 36°$

$= \frac{1}{2}\left[\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right]\left[\frac{\sqrt{5}+1}{4}\right]$

$= \frac{1}{2}\left[\frac{\sqrt{5}-1}{4}\right]\left[\frac{\sqrt{5}+1}{4}\right]$

$= \frac{5-1}{32}$

$= \frac{1}{8}$

21. $\sin 12° \sin 24° \sin 48° \sin 84°$

$= \frac{1}{4}(2\sin 12° \sin 48°)(2\sin 24° \sin 84°)$

$= \frac{1}{2}(\cos 36° - \cos 60°)(\cos 60° - \cos 108°)$

$= \frac{1}{4}\left(\cos 36^{\circ} - \frac{1}{2}\right)\left(\frac{1}{2} + \sin 18^{\circ}\right)$

$= \frac{1}{4}\left\{\frac{1}{4}(\sqrt{5}+1) - \frac{1}{2}\right\}\left\{\frac{1}{2} + \frac{1}{4}(\sqrt{5}-1)\right\}$

$= \frac{1}{16}$

Consider, $\cos 20° \cos 40° \cos 60° \cos 80°$

$= \frac{1}{2}[\cos(60° - 20°)\cos 20° \cos(60° + 20°)]$

$= \frac{1}{2}\left[\frac{1}{4}\cos 3(20^{\circ})\right]$

....$\left[\because \cos\theta\cos(60° - \theta)\cos(60° + \theta) = \frac{1}{4}\cos 3\theta\right]$

$= \frac{1}{8}\cos 60° = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$

$\therefore$ option (A) is the correct answer.

22. $\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)$

$= \left(1 + \cos\frac{\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)$

….$[\because \cos(\pi - \theta) = -\cos\theta]$

$= \left(1 - \cos^2\frac{\pi}{8}\right)\left(1 - \cos^2\frac{3\pi}{8}\right)$

$= \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$

$= \frac{1}{4}\left(2\sin\frac{\pi}{8}.\sin\frac{3\pi}{8}\right)^2$

$= \frac{1}{4}\left(\cos\frac{\pi}{4} - \cos\frac{\pi}{2}\right)^2$

$= \frac{1}{8}$

23. $\frac{m}{n} = \frac{\tan(\theta+120^\circ)}{\tan(\theta-30^\circ)}$

$\Rightarrow \frac{m+n}{m-n} = \frac{\tan(\theta+120^\circ)+\tan(\theta-30^\circ)}{\tan(\theta+120^\circ)-\tan(\theta-30^\circ)}$

….[By componendo and dividendo]

$= \frac{\sin(\theta+120°)\cos(\theta-30°)+\cos(\theta+120°)\sin(\theta-30°)}{\sin(\theta+120°)\cos(\theta-30°)-\cos(\theta+120°)\sin(\theta-30°)}$

$= \frac{\sin(2\theta+90^\circ)}{\sin(150^\circ)} = \frac{\cos 2\theta}{\frac{1}{2}}$

$= 2\cos 2\theta$

24. $\cos^2 76° + \cos^2 16° - \cos 76° \cos 16°$

$= \frac{1}{2}[1+\cos 152°+1+\cos 32°-\cos 92°-\cos 60°]$

$= \frac{1}{2}[2 - \frac{1}{2} + \cos 152° + \cos 32° - \cos 92°]$

$= \frac{1}{2}\left[\frac{3}{2} + 2\cos 92^\circ\cos 60^\circ - \cos 92^\circ\right]$

$= \frac{1}{2}\left[\frac{3}{2} + \cos 92^\circ - \cos 92^\circ\right]$

$= \frac{3}{4}$

25. $\cos 2(\alpha+\beta) = 2\cos^2(\alpha+\beta) - 1$

and $2\sin^2\beta = 1 - \cos 2\beta$

Now, $2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$

$= 1 - \cos 2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + 2\cos^2(\alpha+\beta) - 1$

$= 2\cos(\alpha+\beta)[2\sin\alpha\sin\beta + \cos(\alpha+\beta)] - \cos 2\beta$

$= -\cos 2\beta + 2\cos(\alpha+\beta)\cos(\alpha-\beta)$

$= -\cos 2\beta + \cos 2\alpha + \cos 2\beta$

$= \cos 2\alpha$

26. In ΔABC, $A + B + C = \pi$

$\sin 2A + \sin 2B + \sin 2C$

$= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C$

$= 2\sin(\pi-C)\cos(A-B) + 2\sin C\cos\{\pi-(A+B)\}$

$= 2\sin C\cos(A-B) - 2\sin C\cos(A+B)$

$= 2\sin C\{\cos(A-B) - \cos(A+B)\}$

$= 2\sin C(2\sin A\sin B)$

$= 4\sin A\sin B\sin C$

27. $\cos A = \cos B \cos C$

$\Rightarrow \cos[\pi - (B+C)] = \cos B\cos C$

$\Rightarrow -\cos(B+C) = \cos B\cos C$

$\Rightarrow -[\cos B\cos C - \sin B\sin C] = \cos B\cos C$

$\Rightarrow \sin B\sin C = 2\cos B\cos C$

$\Rightarrow \tan B\tan C = 2$

28. $\cot(A+B) = \cot(\pi - C)$

$\Rightarrow \frac{\cot A\cot B - 1}{\cot A + \cot B} = -\cot C$

$\Rightarrow \cot A\cot B - 1 = -\cot A\cot C - \cot B\cot C$

$\Rightarrow \cot B\cot C + \cot C\cot A + \cot A\cot B = 1$

29. $\cot B + \cot C = \frac{\sin C\cos B + \sin B\cos C}{\sin B\sin C}$

$= \frac{\sin(B+C)}{\sin B\sin C}$

$= \frac{\sin(180^\circ - A)}{\sin B\sin C}$

$= \frac{\sin A}{\sin B\sin C}$

Similarly, $\cot C + \cot A = \frac{\sin B}{\sin C\sin A}$

and $\cot A + \cot B = \frac{\sin C}{\sin A\sin B}$

$\therefore$ $(\cot B + \cot C)(\cot C + \cot A)(\cot A + \cot B)$

$= \frac{\sin A}{\sin B\sin C}.\frac{\sin B}{\sin C\sin A}.\frac{\sin C}{\sin A\sin B}$

$= \text{cosec } A \text{ cosec } B \text{ cosec } C$

30. Since, $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

$\therefore$ $\frac{\tan A + \tan B + \tan C}{\tan A\tan B\tan C} = 1$

31. Given, $\angle A = \frac{\pi}{2}$

In Δ ABC, $\angle A + \angle B + \angle C = \pi$

$\therefore \quad \angle B + \angle C = \frac{\pi}{2}$

$\Rightarrow B = \frac{\pi}{2} - C$

$\Rightarrow \cos^2 B = \cos^2\left(\frac{\pi}{2} - C\right) = \sin^2 C$

$\therefore \quad \cos^2 B + \cos^2 C = \sin^2 C + \cos^2 C = 1$

32. $A + B + C = 180°$

$\Rightarrow \cot\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\left(90° - \frac{C}{2}\right)$

$\Rightarrow \dfrac{\cot\frac{A}{2}.\cot\frac{B}{2} - 1}{\cot\frac{B}{2} + \cot\frac{A}{2}} = \tan\frac{C}{2} = \dfrac{1}{\cot\frac{C}{2}}$

$\Rightarrow \left(\cot\frac{A}{2}\cot\frac{B}{2} - 1\right)\cot\frac{C}{2} = \cot\frac{B}{2} + \cot\frac{A}{2}$

$\Rightarrow \cot\frac{A}{2}\ \cot\frac{B}{2}\ \cot\frac{C}{2} = \cot\frac{C}{2} + \cot\frac{B}{2} + \cot\frac{A}{2}$

33. $A + B + C = \pi$

$\therefore \quad \tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)$

$\Rightarrow \dfrac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}.\tan\frac{B}{2}} = \cot\frac{C}{2}$

$\Rightarrow \dfrac{\frac{1}{3} + \frac{2}{3}}{1 - \frac{1}{3}.\frac{2}{3}} = \cot\frac{C}{2}$

$\Rightarrow \frac{9}{7} = \cot\frac{C}{2}$

$\Rightarrow \tan\frac{C}{2} = \frac{7}{9}$

34. For A = B = C = 60°, only option (C) satisfies the condition.

Evaluation Test

1. We have, $A + B + C = 180°$

$\Rightarrow \frac{A}{2} = 90° - \left(\frac{B+C}{2}\right)$

$\Rightarrow \cot\frac{A}{2} = \cot\left(90° - \left(\frac{B+C}{2}\right)\right)$

$\Rightarrow \cot\frac{A}{2} = \tan\left(\frac{B}{2} + \frac{C}{2}\right)$

$\Rightarrow \dfrac{1}{\tan\frac{A}{2}} = \dfrac{\tan\frac{B}{2} + \tan\frac{C}{2}}{1 - \tan\frac{B}{2}\tan\frac{C}{2}}$

$\Rightarrow 1 - \tan\frac{B}{2}\tan\frac{C}{2} = \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{A}{2}\tan\frac{C}{2}$

$\Rightarrow \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{A}{2}\tan\frac{C}{2} = 1$

i.e., $\sum\tan\frac{A}{2}\tan\frac{B}{2} = 1$

2. $\sin^2\alpha + \sin^2\beta - \sin^2\gamma$

$= \sin^2\alpha + \sin(\beta - \gamma)\sin(\beta + \gamma)$

….[$\because \sin^2 A - \sin^2 B = \sin(A + B)\sin(A - B)$]

$= \sin^2\alpha + \sin(\pi - \alpha)\sin(\beta + \gamma)$

….[$\because \alpha + \beta - \gamma = \pi$ (given)]

$= \sin^2\alpha + \sin\alpha\sin(\beta + \gamma)$

$= \sin\alpha\{\sin\alpha + \sin(\beta + \gamma)\}$

$= \sin\alpha\{\sin(\pi - (\beta - \gamma)) + \sin(\beta + \gamma)\}$

$= \sin\alpha\{\sin(\beta - \gamma) + \sin(\beta + \gamma)\}$

$= \sin\alpha(2\sin\beta\cos\gamma)$

$= 2\sin\alpha\sin\beta\cos\gamma$

3. $(\cos\theta + \cos 7\theta) + (\cos 3\theta + \cos 5\theta) = 0$

$\Rightarrow 2\cos 4\theta\cos 3\theta + 2\cos 4\theta\cos\theta = 0$

….$\left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right]$

$\Rightarrow 2\cos 4\theta(\cos 3\theta + \cos\theta) = 0$

$\Rightarrow 4\cos 4\theta\cos 2\theta\cos\theta = 0$

$\Rightarrow 4\dfrac{\sin 2^3\theta}{2^3\sin\theta} = 0$

….$\left[\because \cos A\cos 2A\cos 2^2A\cos 2^3A\ldots\cos 2^{n-1}A = \dfrac{\sin 2^n A}{2^n\sin A}\right]$

$\Rightarrow \sin 8\theta = 0 \Rightarrow 8\theta = n\pi$

$\Rightarrow \theta = \frac{n\pi}{8}$

4. Given, sin A + sin B = C
and cos A + cos B = D

$\therefore \quad \dfrac{\sin A + \sin B}{\cos A + \cos B} = \dfrac{C}{D}$

$\Rightarrow \dfrac{2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)}{2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)} = \dfrac{C}{D}$

$\Rightarrow \tan\left(\dfrac{A+B}{2}\right) = \dfrac{C}{D}$ ….(i)

Now, $\sin(A+B) = \dfrac{2\tan\left(\dfrac{A+B}{2}\right)}{1+\tan^2\left(\dfrac{A+B}{2}\right)}$

….$\left[\because \sin\theta = \dfrac{2\tan\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}}\right]$

$= \dfrac{2\dfrac{C}{D}}{1+\dfrac{C^2}{D^2}}$ ….[From (i)]

$= \dfrac{2CD}{C^2+D^2}$

5. $\dfrac{\sin\alpha - \sin\gamma}{\cos\gamma - \cos\alpha} = \dfrac{2\cos\dfrac{\alpha+\gamma}{2}\sin\dfrac{\alpha-\gamma}{2}}{2\sin\dfrac{\alpha+\gamma}{2}\sin\dfrac{\alpha-\gamma}{2}}$

$= \cot\dfrac{\alpha+\gamma}{2}$

But α, β, γ are in A.P. $\Rightarrow \dfrac{\alpha+\gamma}{2} = \beta$

$\therefore \quad \dfrac{\sin\alpha - \sin\gamma}{\cos\gamma - \cos\alpha} = \cot\beta$

6. $\sin^3 x \sin 3x$

$= \dfrac{1}{4}(3\sin x - \sin 3x)\sin 3x$

….$\left[\begin{array}{l}\because \sin 3A = 3\sin A - 4\sin^3 A \\ \Rightarrow \sin^3 A = \dfrac{1}{4}(3\sin A - \sin 3A)\end{array}\right]$

$= \dfrac{3}{8}(2\sin x \sin 3x) - \dfrac{1}{8}(2\sin^2 3x)$

$= \dfrac{3}{8}(\cos 2x - \cos 4x) - \dfrac{1}{8}(1 - \cos 6x)$

$= -\dfrac{1}{8} + \dfrac{3}{8}\cos 2x - \dfrac{3}{8}\cos 4x + \dfrac{1}{8}\cos 6x$ ....(i)

and $\sum_{m=0}^{n} c_m \cos mx$

$= c_0 + c_1\cos x + c_2\cos 2x + c_3\cos 3x + \ldots + c_n\cos nx$ ….(ii)

But, $\sin^3 x \sin 3x = \sum_{m=0}^{n} c_m \cos mx$

$\therefore$ from (i) and (ii), we get n = 6.

7. Given, $\sin B = \dfrac{1}{5}\sin(2A+B)$

$\Rightarrow \dfrac{\sin(2A+B)}{\sin B} = \dfrac{5}{1}$

By componendo and dividendo, we get

$\dfrac{\sin(2A+B)+\sin B}{\sin(2A+B)-\sin B} = \dfrac{5+1}{5-1}$

$\Rightarrow \dfrac{2\sin(A+B)\cos A}{2\cos(A+B)\sin A} = \dfrac{6}{4}$

$\Rightarrow \dfrac{\tan(A+B)}{\tan A} = \dfrac{3}{2}$

8. $\sin A + \sin 2A + \sin 3A = \cos A + \cos 2A + \cos 3A$

$\therefore$ (sin 3A + sin A) + sin 2A = (cos 3A + cos A) + cos 2A

$\therefore$ 2 sin 2A.cos A + sin 2A = 2 cos 2A.cos A + cos 2A

$\therefore$ sin 2A(2 cos A + 1) = cos 2A(2 cos A + 1)

$\therefore$ tan 2A = 1

Textbook Chapter No.

06 Straight Line

Hints

Classical Thinking

2. Gradient of the line which passes through $(1, 0)$ and $(-2, \sqrt{3})$ is $m = \frac{\sqrt{3}-0}{-2-1} = -\frac{1}{\sqrt{3}}$

$\Rightarrow \tan\theta = -\frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = 150°$

3. The required equation is
$y + 6 = \tan 45°(x - 4)$
$\Rightarrow x - y - 10 = 0$

4. The required equation passing through (0, 0) and having gradient $m = \frac{1}{0}$, is $y = \frac{1}{0}x$

$\Rightarrow x = 0$

5. Midpoint is (3, 4) and slope of AB $= \frac{6}{4}$

$\therefore$ Slope of perpendicular $= \frac{-1}{6/4} = \frac{-2}{3}$

$\therefore$ the required equation is $y - 4 = \frac{-2}{3}(x - 3)$

$\Rightarrow 2x + 3y = 18$

6. $m = \frac{-1}{\frac{b'-b}{a'-a}} = \frac{a'-a}{b-b'}$

Midpoint is $\left(\frac{a+a'}{2}, \frac{b+b'}{2}\right)$

$\therefore$ the required equation is

$y - \left(\frac{b+b'}{2}\right) = \frac{a'-a}{b-b'}\left[x - \left(\frac{a+a'}{2}\right)\right]$

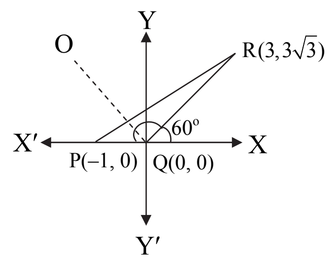
$\Rightarrow 2(b - b')y + 2(a - a')x = b^2 - b'^2 + a^2 - a'^2$

7. Midpoint $\equiv (4, -9)$ and slope $= \frac{-1}{\frac{3+1}{-1-5}} = \frac{3}{2}$

Hence, the required line is $y + 9 = \frac{3}{2}(x - 4)$

$\Rightarrow 3x - 2y = 30$

8.



Slope of QR $= \frac{3\sqrt{3}-0}{3-0} = \sqrt{3}$ i.e., $\theta = 60°$

Clearly, $\angle PQR = 120°$

OQ is the angle bisector of the angle PQR, so line OQ makes 120° with the positive direction of X-axis.

Therefore, equation of the bisector of $\angle PQR$ is $y = \tan 120° x \Rightarrow y = -\sqrt{3}x \Rightarrow \sqrt{3}x + y = 0$

9. $m = \frac{5-0}{-4-0} = \frac{5}{-4}$

$\therefore$ the required equation is $5x + 4y = 0$.

10. Equation of a line passing through the given points is $\frac{y-(-6)}{-6-10} = \frac{x-(-5)}{-5-3}$

$\Rightarrow \frac{y+6}{-16} = \frac{x+5}{-8} \Rightarrow 2x - y + 4 = 0$

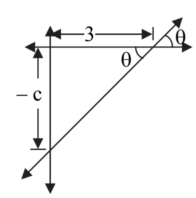
11. The point of intersection is (0, 0)
Thus, the equation of line passing through the points (0, 0) and (2, 2) is $y = x$.

12. Equation of line is $y = mx + c$
$\Rightarrow y = (\tan 135°)x - 5 \Rightarrow y = -x - 5$
$\Rightarrow x + y + 5 = 0$

13. From the figure, $m = \tan\theta = \frac{-c}{3}$

$\Rightarrow 3 = \frac{-c}{3}$

$\Rightarrow c = -9$



Hence, the required equation is $y = 3x - 9$

14. Here, intercept on X-axis is 3 and intercept on Y-axis is –2.
So, using double intercept form, the required equation of the line is $\frac{x}{3} - \frac{y}{2} = 1$.

15. Using double intercept form, we get
$$\frac{x}{2a\sec\theta} + \frac{y}{2a\,\text{cosec}\,\theta} = 1$$
$\Rightarrow x\cos\theta + y\sin\theta = 2a$

16. Intersection point on X-axis is $(2x_1, 0)$ and on Y-axis is $(0, 2y_1)$. Thus, equation of line passing through these points is $\frac{x}{x_1} + \frac{y}{y_1} = 2$.

17. Since, the given line passes through (2, –3) and (4, –5).
$\therefore$ $\frac{2}{a} - \frac{3}{b} = 1$ and $\frac{4}{a} - \frac{5}{b} = 1$
$\Rightarrow b = -1, a = -1$

20. The equation of line is $\frac{x}{a} + \frac{y}{a} = 1$.
$\Rightarrow x + y - a = 0$
$\therefore$ Slope $= -\dfrac{\text{coefficient of } x}{\text{coefficient of } y} = -1$

21. The required equation which passes through (1, 2) and its gradient m = 3, is $y - 2 = 3(x - 1)$.

22. The required equation which passes through (c, d) and its gradient $-\frac{a}{b}$, is
$y - d = -\frac{a}{b}(x - c)$
$\Rightarrow a(x - c) + b(y - d) = 0$

23. The required equation passing through (3, –4) and having gradient $\frac{4}{3}$ is $y + 4 = \frac{4}{3}(x - 3)$.

24. Equation of line perpendicular to
$ax + by + c = 0$ is $bx - ay + \lambda = 0$ ……(i)
It passes through (a, b).
$\therefore$ $ab - ab + \lambda = 0 \Rightarrow \lambda = 0$
Putting $\lambda = 0$ in (i), we get $bx - ay = 0$ which is the required equation.

25. Slope of perpendicular $= \frac{-y'}{2a}$
$\therefore$ the required equation is $y - y' = -\frac{y'}{2a}(x - x')$
$\Rightarrow xy' + 2ay - 2ay' - x'y' = 0$

26. $m_1 = \sqrt{3}$, $m_2 = 0$
$\therefore$ $\tan\theta = \left|\frac{\sqrt{3} - 0}{1 + 0}\right|$
$\Rightarrow \tan\theta = \sqrt{3}$
$\Rightarrow \theta = 60^\circ$

27. $\theta = \tan^{-1}\left|\frac{2 - \sqrt{3} - 2 - \sqrt{3}}{1 + 4 - 3}\right| = \tan^{-1}\left(\sqrt{3}\right)$
$\Rightarrow \theta = 60^\circ$

28. $\theta = \tan^{-1}\left|\frac{-\cot 30^\circ + \cot 60^\circ}{1 + \cot 30^\circ \cot 60^\circ}\right|$
$= \tan^{-1}\left|\frac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 30^\circ \tan 60^\circ}\right| = 30^\circ$

29. Equation of lines are $\frac{x}{a} - \frac{y}{b} = 1$ and $\frac{x}{b} - \frac{y}{a} = 1$
$\Rightarrow m_1 = \frac{b}{a}$ and $m_2 = \frac{a}{b}$
$\therefore$ $\theta = \tan^{-1}\left|\dfrac{\frac{b}{a} - \frac{a}{b}}{1 + \frac{b}{a}.\frac{a}{b}}\right|$
$= \tan^{-1}\frac{b^2 - a^2}{2ab}$

30. Let $L_1 \equiv 2x + 3y - 7 = 0$ and
$L_2 \equiv 2x + 3y - 5 = 0$
Here, slope of L_1 = slope of $L_2 = -\frac{2}{3}$
Hence, the lines are parallel.

31. Slope of given line is $\frac{1}{2}$
Thus, $\tan 45^\circ = \pm\dfrac{m - \frac{1}{2}}{1 + m.\frac{1}{2}} \Rightarrow m = 3$ or $\frac{-1}{3}$
Hence option (B) is correct.

33. Let $L_1 \equiv 2x + 5y - 7 = 0$ and $L_2 \equiv 2x - 5y - 9 = 0$,
so that $m_1 = -\frac{2}{5}$, $m_2 = \frac{2}{5}$
Lines are neither parallel nor perpendicular, also not coincident.
Hence, the lines are intersecting.

34. $\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} m_1 & m_2 & m_3 \\ -1 & -1 & -1 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\Rightarrow m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$

35. The lines are concurrent, if $\begin{vmatrix} 7 & -8 & 5 \\ 3 & -4 & 5 \\ 4 & 5 & k \end{vmatrix} = 0$

$\Rightarrow 7(-4k - 25) + 8(3k - 20) + 5(15 + 16) = 0$
$\Rightarrow k = -45$

36. consider $\begin{vmatrix} 15 & -18 & 1 \\ 12 & 10 & -3 \\ 6 & 66 & -11 \end{vmatrix}$

$= 15\,(-110 + 198) + 18\,(-132 + 18) + 1\,(792 - 60)$
$= 0$

37. $u = a_1x + b_1y + c_1 = 0$, $v = a_2x + b_2y + c_2 = 0$

let $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = c$

$\Rightarrow a_2 = \dfrac{a_1}{c}, b_2 = \dfrac{b_1}{c}, c_2 = \dfrac{c_1}{c}$

Given that, $u + kv = 0$
$\Rightarrow a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$
$\Rightarrow a_1x + b_1y + c_1 + k\left(\dfrac{a_1}{c}\right)x + k\left(\dfrac{b_1}{c}\right)y + k\left(\dfrac{c_1}{c}\right) = 0$

$\Rightarrow a_1x\left(1 + \dfrac{k}{c}\right) + b_1y\left(1 + \dfrac{k}{c}\right) + c_1\left(1 + \dfrac{k}{c}\right) = 0$

$\Rightarrow a_1x + b_1y + c_1 = 0 = u$

38. Required length $= \left|\dfrac{4(3) + 3(1) + 20}{5}\right| = 7$

39. Required distance $= \left|\dfrac{-2 - 3 - 5}{\sqrt{1+1}}\right| = \dfrac{10}{\sqrt{2}} = 5\sqrt{2}$

40. Required distance $= \left|\dfrac{-7}{\sqrt{12^2 + 5^2}}\right| = \dfrac{7}{13}$

41. Here, equation of line is $y = x \tan\alpha + c$, $c > 0$
Length of the perpendicular drawn on line from point $(a \cos\alpha, a \sin\alpha)$ is

$p = \left|\dfrac{-a\sin\alpha + a\cos\alpha\tan\alpha + c}{\sqrt{1+\tan^2\alpha}}\right| = \dfrac{c}{\sec\alpha} = c \cos\alpha$

42. $p = \left|\dfrac{ab}{\sqrt{a^2 + b^2}}\right|$

$\Rightarrow \dfrac{a^2 + b^2}{a^2b^2} = \dfrac{1}{p^2}$

$\Rightarrow \dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{p^2}$

43. Length of perpendicular is

$\left|\dfrac{\dfrac{b}{a} - \dfrac{a}{b} - 1}{\sqrt{\left(\dfrac{1}{a}\right)^2 + \left(-\dfrac{1}{b}\right)^2}}\right| = \left|\dfrac{b^2 - a^2 - ab}{\sqrt{a^2 + b^2}}\right|$

44. Straight line $y - y' = \dfrac{y'' - y'}{x'' - x'}(x - x')$

$\therefore$ Length of perpendicular

$= \left|\dfrac{x'(y'' - y') - y'(x'' - x')}{\sqrt{(x'' - x')^2 + (y'' - y')^2}}\right|$

$= \left|\dfrac{x'y'' - y'x''}{\sqrt{(x'' - x')^2 + (y'' - y')^2}}\right|$

45. Given lines are $5x + 3y - 7 = 0$ .....(i)
and $15x + 9y + 14 = 0$ or

$5x + 3y + \dfrac{14}{3} = 0$ ....(ii)

Lines (i) and (ii) are parallel.

$\therefore$ Required distance $= \left|\dfrac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right| = \dfrac{\left|-7 - \dfrac{14}{3}\right|}{\left|\sqrt{5^2 + 3^2}\right|}$

$= \left|\dfrac{-35}{3\sqrt{34}}\right| = \dfrac{35}{3\sqrt{34}}$

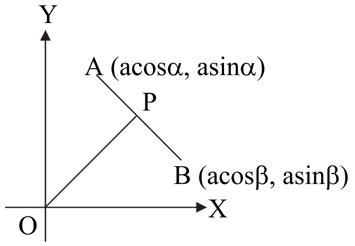
Critical Thinking

1. The four vertices on solving are A(–3, 3), B(1, 1), C(1, –1) and D(–2, –2).
m_1 = slope of AC = –1,
m_2 = slope of BD = 1

$\therefore$ $m_1m_2 = -1$
Hence, the angle between diagonals AC and BD is 90°.

2. Mid point of $(a \cos\alpha, a \sin\alpha)$ and $(a \cos\beta, a \sin\beta)$ is

$P\left(\dfrac{a(\cos\alpha + \cos\beta)}{2}, \dfrac{a(\sin\alpha + \sin\beta)}{2}\right)$



∴ Slope of line AB is
$\frac{a\sin\beta - a\sin\alpha}{a\cos\beta - a\cos\alpha} = \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = m_1$
and slope of OP is $\frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = m_2$
Now, $m_1 \times m_2 = \frac{\sin^2\beta - \sin^2\alpha}{\cos^2\beta - \cos^2\alpha} = -1$
Hence, the lines are perpendicular.

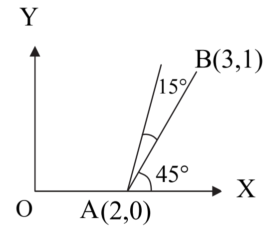
3. Slope $= \frac{8-2}{3-1} = 3$
The diagonal is $y - 2 = 3(x - 1)$
$\Rightarrow 3x - y - 1 = 0$

4. S = midpoint of QR $= \left(\frac{6+7}{2}, \frac{-1+3}{2}\right) = \left(\frac{13}{2}, 1\right)$
∴ 'm' of $PS = \frac{2-1}{2-\frac{13}{2}} = -\frac{2}{9}$
∴ The required equation is $y + 1 = \frac{-2}{9}(x-1)$
i.e., $2x + 9y + 7 = 0$

5. Point P(a, b) is on $3x + 2y = 13$
So, $3a + 2b = 13$ .....(i)
Point Q(b, a) is on $4x - y = 5$
So, $4b - a = 5$ .....(ii)
By solving (i) and (ii), we get
a = 3, b = 2
Now, equation of PQ is
$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
$\Rightarrow y - 2 = \frac{3-2}{2-3}(x-3) \Rightarrow y - 2 = -(x - 3)$
$\Rightarrow x + y = 5$

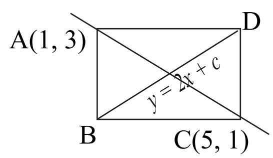
6. Here, slope of AB = 1
$\Rightarrow \tan\theta = m_1 = 1$
or $\theta = 45°$
∴ $\tan(45° + 15°) = \tan 60°$
[∵ It is rotated anticlockwise so the angle will be $45° + 15° = 60°$]

Thus, slope of new line is $\sqrt{3}$



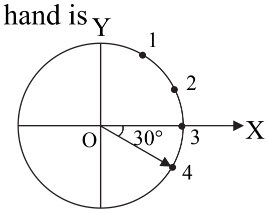
∴ The required equation of line passing through (2, 0) and m = $\sqrt{3}$ is $y = \sqrt{3}(x - 2)$
i.e., $y = \sqrt{3}x - 2\sqrt{3}$

7. Let ABCD be a rectangle.
Given, A (1, 3) and C (5, 1).



Intersecting point of diagonal of a rectangle is same or at midpoint.
∴ midpoint of AC is (3, 2).
Also, $y = 2x + c$ passes through (3, 2).
Hence, c = – 4

8. Since the hour, minute and second hands always pass through origin because one end of these hands is always at origin.
Now, at 4 O' clock, the hour hand makes 30° angle in fourth quadrant.
So, the equation of hour hand is
$y = mx \Rightarrow y = -\frac{1}{\sqrt{3}}x$
$\Rightarrow x + \sqrt{3}y = 0$



9. Let the co-ordinates of axes are A (a, 0) and B(0, b), but the point (–5, 4) divides the line AB in the ratio of 1 : 2.
∴ the co-ordinates of axes are $\left(\frac{-15}{2}, 0\right)$ and (0, 12).
Therefore, the equation of line passing through these coordinate axes is given by
$8x - 5y + 60 = 0$

10. Let the intercept be a and 2a, then the equation of line is $\frac{x}{a} + \frac{y}{2a} = 1$, but it also passes through (1, 2), therefore a = 2.
Hence, the required equation is $2x + y = 4$

11. Take two perpendicular lines as the coordinate axes. If a, b be the intercepts made by the moving line on the coordinate axes, then the equation of the line is

$\frac{x}{a}+\frac{y}{b}=1$ .....(i)

Let $\frac{1}{a}+\frac{1}{b}=\frac{1}{k}$

i.e., $\frac{k}{a}+\frac{k}{b}=1$ .....(ii)

The result (ii) shows that the straight line (i) passes through a fixed point (k, k).

12. Given, a + b = 14 ⇒ a = 14 – b

Hence, the equation of straight line is $\frac{x}{14-b}+\frac{y}{b}=1$

Also, it passes through (3, 4)

∴ $\frac{3}{14-b}+\frac{4}{b}=1$

$\Rightarrow b = 8$ or 7

Therefore, equations are $4x + 3y = 24$ and $x + y = 7$

13. Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$

The co-ordinates of the mid point of the intercept AB between the axes are $\left(\frac{a}{2},\frac{b}{2}\right)$

∴ $\frac{a}{2}=1,\frac{b}{2}=2 \Rightarrow a=2, b=4$

Hence, the equation of the line is

$\frac{x}{2}+\frac{y}{4}=1 \Rightarrow 2x+y=4$

14. A line perpendicular to the line $5x - y = 1$ is given by $x + 5y - \lambda = 0 = L$

In intercept form $\frac{x}{\lambda}+\frac{y}{\lambda/5}=1$

So, area of triangle is $\frac{1}{2}$ × (Multiplication of intercepts)

$\Rightarrow \frac{1}{2}(\lambda)\times\left(\frac{\lambda}{5}\right) = 5$

$\Rightarrow \lambda = \pm 5\sqrt{2}$

Hence, the equation of required line is $x + 5y = \pm 5\sqrt{2}$

15. Given form is $3x + 3y + 7 = 0$

$\Rightarrow \frac{3}{\sqrt{3^2+3^2}}x+\frac{3}{\sqrt{3^2+3^2}}y+\frac{7}{\sqrt{3^2+3^2}}=0$

$\Rightarrow \frac{3}{3\sqrt{2}}x+\frac{3}{3\sqrt{2}}y=\frac{-7}{3\sqrt{2}}$

∴ $p=\left|\frac{-7}{3\sqrt{2}}\right|=\frac{7}{3\sqrt{2}}$

16. Let p be the length of the perpendicular from the origin on the given line. Then its equation in normal form is

$x \cos30° + y \sin30° = p$ or $\sqrt{3}x+y=2p$

This meets the coordinate axes at $A\left(\frac{2p}{\sqrt{3}},0\right)$ and B(0, 2p)

∴ Area of $\Delta OAB=\frac{1}{2}\left(\frac{2p}{\sqrt{3}}\right)2p=\frac{2p^2}{\sqrt{3}}$

$\Rightarrow \frac{2p^2}{\sqrt{3}}=\frac{50}{\sqrt{3}} \Rightarrow p=\pm5$

Hence, the lines are $\sqrt{3}x+y\pm10=0$

17. The equation of line passing through A(– 5, –4) is $\frac{x+5}{\cos\theta}=\frac{y+4}{\sin\theta}$

Let $AB = r_1$, $AC = r_2$, $AD = r_3$

The co-ordinate of B is

$(r_1 \cos\theta - 5, r_1 \sin\theta - 4)$

which lies on $x + 3y + 2 = 0$

∴ $r_1=\frac{15}{\cos\theta+3\sin\theta}$

Similarly, $\frac{10}{AC}=2\cos\theta+\sin\theta$ and

$\frac{6}{AD}=\cos\theta-\sin\theta$

Putting in the given relation, we get $(2\cos\theta+3\sin\theta)^2=0$

$\Rightarrow \tan\theta=-\frac{2}{3}$

∴ The equation of line is $y+4=-\frac{2}{3}(x+5)$

$\Rightarrow 2x + 3y + 22 = 0$

18. Let the required line through the point (1,2) be inclined at an angle θ to the axis of X. Then its equation is $\frac{x-1}{\cos\theta}=\frac{y-2}{\sin\theta}=r$ .....(i)

The co-ordinates of a point on the line (i) are $(1 + r\cos\theta, 2 + r\sin\theta)$

If this point is at a distance $\frac{\sqrt{6}}{3}$ form (1, 2),

then $r = \frac{\sqrt{6}}{3}$

Therefore, the point is

$$\left(1+\frac{\sqrt{6}}{3}\cos\theta,\ \ 2+\frac{\sqrt{6}}{3}\sin\theta\right).$$

But this point lies on the line $x + y = 4$

$\Rightarrow \frac{\sqrt{6}}{3}(\cos\theta + \sin\theta) = 1$ or

$\sin\theta+\cos\theta=\frac{3}{\sqrt{6}}$

$\Rightarrow \frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=\frac{\sqrt{3}}{2}$

….(Dividing both sides by $\sqrt{2}$)

$\Rightarrow \sin(\theta+45^{\circ})=\sin 60^{\circ}$ or $\sin 120^{\circ}$

$\Rightarrow \theta = 15^{\circ}$ or 75°

19. The slope of line $x + y = 1$ is -1.

$\therefore$ It makes an angle of 135° with X-axis.

The equation of line passing through (1, 1) and making an angle of 135° is,

$$\frac{x-1}{\cos 135^{\circ}}=\frac{y-1}{\sin 135^{\circ}}=r$$

$$\Rightarrow \frac{x-1}{-1/\sqrt{2}}=\frac{y-1}{1/\sqrt{2}}=r$$

$\therefore$ Co-ordinates of any point on this line are $\left(1-\frac{r}{\sqrt{2}},1+\frac{r}{\sqrt{2}}\right)$

If this point lies on $2x - 3y = 4$, then

$$2\left(1-\frac{r}{\sqrt{2}}\right)-3\left(1+\frac{r}{\sqrt{2}}\right)=4$$

$\Rightarrow r = \sqrt{2}$

21. $\left(\frac{-2}{3a}\right)\left(\frac{-3}{4}\right)=-1$ or $a=\frac{-1}{2}$

22. $x\cos\theta - y\sin\theta = a(\cos^4\theta - \sin^4\theta) = a\cos 2\theta$

23. Mid point $\equiv\left(\frac{1+1}{2},\frac{3-7}{2}\right)=(1,-2)$

Therefore, required line is

$y + 2 = \frac{2}{3}(x - 1) \Rightarrow 2x - 3y = 8$

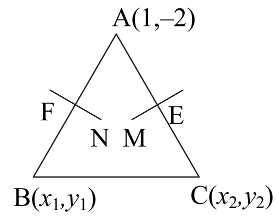
24. Given, line AB makes 0 intercepts on X–axis and Y – axis so, $(x_1, y_1) = (0, 0)$

Slope of perpendicular $= \frac{4}{3}$

$\therefore$ Equation is $y - 0 = \frac{4}{3}(x - 0)$

$\Rightarrow 4x - 3y = 0$

25. Let the equation of perpendicular bisector FN of AB is $x - y + 5 = 0$ ......(i)



The middle point F of AB is $\left(\frac{x_1+1}{2},\frac{y_1-2}{2}\right)$ Which lies on line (i).

$\therefore$ $x_1 - y_1 = -13$ …..(ii)

Also AB is perpendicular to FN. So the product of their slopes is -1.

i.e., $\frac{y_1+2}{x_1-1}\times 1=-1$ or $x_1 + y_1 = -1$ ….(iii)

On solving (ii) and (iii), we get B(–7, 6)

Similarly, C $\left(\frac{11}{5},\frac{2}{5}\right)$

Hence, the equation of BC is $14x + 23y - 40 = 0$

26. The equation of any line parallel to $2x + 6y + 7 = 0$ is $2x + 6y + k = 0$.

This meets the axes at $A\left(-\frac{k}{2},0\right)$ and $B\left(0,-\frac{k}{6}\right)$

Since, AB = 10

$\Rightarrow \sqrt{\frac{k^2}{4}+\frac{k^2}{36}}=10 \Rightarrow \sqrt{\frac{10k^2}{36}}=10$

$\Rightarrow 10k^2 = 3600 \Rightarrow k = \pm 6\sqrt{10}$

Hence, there are two lines given by $2x + 6y \pm 6\sqrt{10} = 0$

27. $\theta=\tan^{-1}\left|\frac{-\cot\alpha_1+\cot\alpha_2}{1+\cot\alpha_1\cot\alpha_2}\right|$

$=\tan^{-1}\left|\frac{\tan\alpha_1-\tan\alpha_2}{1+\tan\alpha_2\tan\alpha_1}\right| = \alpha_1 - \alpha_2$

28. Here, equation of AB is
$x + 4y - 4 = 0$ .....(i)
and equation of BC is
$2x + y - 22 = 0$ .....(ii)
Thus angle between (i) and (ii) is given by
$$\tan^{-1}\left(\frac{-\frac{1}{4}+2}{1+\left(-\frac{1}{4}\right)(-2)}\right) = \tan^{-1}\frac{7}{6}$$

29. $\frac{k-(2+\sqrt{3})}{1+k(2+\sqrt{3})} = \sqrt{3}$
$\Rightarrow k - 2 - \sqrt{3} = \sqrt{3} + 2k\sqrt{3} + 3k$
$\Rightarrow k = \frac{-2(1+\sqrt{3})}{2(1+\sqrt{3})} = -1$

30. Let θ be the acute angle which the line $y = mx + 4$ makes with the lines $y = 3x + 1$ and $2y = x + 3$.
Then,
$$\tan\theta = \left|\frac{m-3}{1+3m}\right| \text{ and } \tan\theta = \left|\frac{m-\frac{1}{2}}{1+\frac{m}{2}}\right|$$
$\Rightarrow \left|\frac{m-3}{1+3m}\right| = \left|\frac{2m-1}{m+2}\right|$
$\Rightarrow m^2 - m - 6 = \pm(6m^2 - m - 1)$
$\Rightarrow 5m^2 + 5 = 0$ or $7m^2 - 2m - 7 = 0$
$\Rightarrow 7m^2 - 2m - 7 = 0$
$\Rightarrow m = \frac{1 \pm 5\sqrt{2}}{7}$

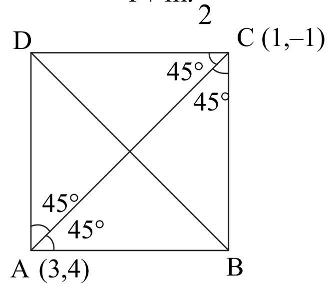
31. Any line through (1, –10) is given by
$y + 10 = m(x - 1)$
Since, it makes equal angle say 'α' with the given lines $7x - y + 3 = 0$ and $x + y - 3 = 0$
$\therefore \tan\alpha = \frac{m-7}{1+7m} = \frac{m-(-1)}{1+m(-1)}$
$\Rightarrow m = \frac{1}{3}$ or -3
Hence, the two possible equations of third side are $3x + y + 7 = 0$, $x - 3y - 31 = 0$.

32. Here,
Slope of I^{st} diagonal $= m_1 = \frac{2-0}{2-0} = 1$
$\Rightarrow \theta_1 = 45°$
Slope of II^{nd} diagonal $= m_2 = \frac{2-0}{1-1} = \infty$
$\Rightarrow \theta_2 = 90°$
$\Rightarrow \theta_2 - \theta_1 = 45° = \frac{\pi}{4}$

33. Intersection point of the line is $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$, which is satisfying all the equations given in options (A), (B) and (C).
Hence, (D) is correct.

34. Putting k = 1, 2, we get
$3x + 2y = 12$ .....(i)
$4x + 3y = 19$ .....(ii)
The given lines are not parallel.
Hence on solving them, we get
$x = -2, y = 9$
Therefore, the lines pass through (–2, 9)

35. Slope of AC = 5/2.
Let m be the slope of a line inclined at an angle of 45° to AC,
Then $\tan 45^\circ = \pm\frac{m-\frac{5}{2}}{1+m.\frac{5}{2}} \Rightarrow m = -\frac{7}{3}, \frac{3}{7}$



Thus, let the slope of AB or DC be $\frac{3}{7}$ and that of AD or BC be $-\frac{7}{3}$.
Then, equation of AB is $3x - 7y + 19 = 0$.
Also the equation of BC is $7x + 3y - 4 = 0$
On solving these equations, we get $B\left(-\frac{1}{2}, \frac{5}{2}\right)$
Now let the co-ordinates of the vertex D be (h, k). Since the middle points of AC and BD are same
$\therefore \frac{1}{2}\left(h - \frac{1}{2}\right) = \frac{1}{2}(3+1) \Rightarrow h = \frac{9}{2}$
$\Rightarrow \frac{1}{2}\left(k + \frac{5}{2}\right) = \frac{1}{2}(4-1)$
$\Rightarrow k = \frac{1}{2}$
Hence, $D = \left(\frac{9}{2}, \frac{1}{2}\right)$

36. $\begin{vmatrix} p-q & q-r & r-p \\ q-r & r-p & p-q \\ r-p & p-q & q-r \end{vmatrix} = \begin{vmatrix} 0 & q-r & r-p \\ 0 & r-p & p-q \\ 0 & p-q & q-r \end{vmatrix} = 0$

….[By $C_1 \rightarrow C_1 + C_2 + C_3$]

Hence, the lines are concurrent.

37. Given lines are $3x + 4y = 5$, $5x + 4y = 4$, $\lambda x + 4y = 6$. These lines meet at a point if the point of intersection of first two lines lies on the third line.

From $3x + 4y = 5$ and $5x + 4y = 4$

We get $x = \frac{-1}{2}$, $y = \frac{13}{8}$

This lies on $\lambda x + 4y = 6$, if $\lambda\left(-\frac{1}{2}\right) + 4\left(\frac{13}{8}\right) = 6$

$\Rightarrow \lambda = 1$

38. If the given lines are concurrent, then

$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$

[By $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$]

$\Rightarrow a(b-1)(c-1) - (b-1)(1-a) - (c-1)(1-a) = 0$

$\Rightarrow \frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$

….[Divide by $(1-a)(1-b)(1-c)$]

$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$

39. From option (B),

$\begin{vmatrix} 1 & 2 & -10 \\ 2 & 1 & 5 \\ 5 & 4 & 0 \end{vmatrix} = 1(0-20) - 2(-25) - 10(3) = 0$

Hence, option (B) is the correct answer.

40. From option (B), we get

$\begin{vmatrix} 3 & 4 & 6 \\ 6 & 5 & 9 \\ 3 & 3 & 5 \end{vmatrix} = 3(25-27) - 4(3) + 6(3) = 0$

41. The three lines are concurrent, if

$\begin{vmatrix} 1 & 2 & -9 \\ 3 & 5 & -5 \\ a & b & -1 \end{vmatrix} = 0$

$\Rightarrow 35a - 22b + 1 = 0$

which is true if the line $35x - 22y + 1 = 0$ passes through (a, b).

42. By the given condition of $a + b + c = 0$, the three lines reduce to

$x - y = \frac{p}{a}$ or $\frac{p}{b}$ or $\frac{p}{c}$ $(p \neq 0)$.

All these lines are parallel. Hence, they do not intersect in finite plane.

43. The point of intersection of the lines is (1, 1).

and slope of the line $2y - 3x + 2 = 0$ is $\frac{3}{2}$

Hence, the equation is $y - 1 = \frac{3}{2}(x - 1)$

$\Rightarrow 3x - 2y = 1$

44. The intersection point of lines $x - 2y = 1$ and $x + 3y = 2$ is $\left(\frac{7}{5}, \frac{1}{5}\right)$ and the slope of required line $= -\frac{3}{4}$

$\therefore$ Equation of required line is

$y - \frac{1}{5} = \frac{-3}{4}\left(x - \frac{7}{5}\right)$

$\Rightarrow \frac{3x}{4} + y = \frac{21}{20} + \frac{1}{5} \Rightarrow 3x + 4y = 5$

$\Rightarrow 3x + 4y - 5 = 0$

45. The point of intersection of $5x - 6y - 1 = 0$ and $3x + 2y + 5 = 0$ is $(-1, -1)$.

Now the line perpendicular to $3x - 5y + 11 = 0$ is $5x + 3y + k = 0$, but it passes through $(-1, -1)$

$\Rightarrow -5 - 3 + k = 0 \Rightarrow k = 8$

Hence, required line is $5x + 3y + 8 = 0$.

46. Equation of line passing through point of intersection of $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ is

$(x + 2y + 3) + k(3x + 4y + 7) = 0$

$\Rightarrow (1 + 3k)x + (2 + 4k)y + 3 + 7k = 0$ ....(i)

Slope of equation (i) is $m_1 = \frac{-(1+3k)}{2+4k}$

and slope of given line is $m_2 = \frac{-1}{-1} = 1$ ....(ii)

Since (i) and (ii) represent perpendicular lines.

$\therefore$ $m_1 m_2 = -1$

$\therefore$ $\frac{-(1+3k)}{(2+4k)} \times 1 = -1$

$\therefore$ equation of required line is

$(x + 2y + 3) - 1(3x + 4y + 7) = 0$

$\Rightarrow x + y + 2 = 0$

47. Equation of line passing through point of intersection of u = 0 and v = 0 is u + kv = 0

$\therefore$ $(x + 2y + 5) + k(3x + 4y + 1) = 0$

It is passing through (3, 2)

$\therefore$ $(3 + 2 \times 2 + 5) + k(3 \times 3 + 4 \times 2 + 1) = 0$

$\therefore$ $k = -\dfrac{2}{3}$

$\therefore$ equation of line will be

$(x + 2y + 5) - \dfrac{2}{3}(3x + 4y + 1) = 0$

$\Rightarrow 3x + 2y - 13 = 0$

48. Equation of line through the point of intersection of lines $2x + 3y + 1 = 0$ and $3x - 5y - 5 = 0$ is given by

$(2 + 3k)x + (3 - 5k)y + (1 - 5k) = 0$

Slope of line is given by

$\tan 45° = -\dfrac{(2+3k)}{3-5k}$

$\Rightarrow k = \dfrac{5}{2}$

$\therefore$ Equation of line is $19x - 19y - 23 = 0$

49. Required line should be

$(3x - y + 2) + \lambda(5x - 2y + 7) = 0$ .....(i)

$\Rightarrow (3 + 5\lambda)x - (2\lambda + 1)y + (2 + 7\lambda) = 0$

$\Rightarrow y = \dfrac{3+5\lambda}{2\lambda+1}x + \dfrac{2+7\lambda}{2\lambda+1}$ .....(ii)

As the equation (ii), has infinite slope,

$2\lambda + 1 = 0$

$\Rightarrow \lambda = -1/2$

Putting $\lambda = -1/2$ in equation (i) we have

$(3x - y + 2) + (-1/2)(5x - 2y + 7) = 0$

$\Rightarrow x = 3$

Alternate Method:

The point of intersection of $3x - y + 2 = 0$ and $5x - 2y + 7 = 0$ is (3, 11)

....[By solving equations simultaneously]

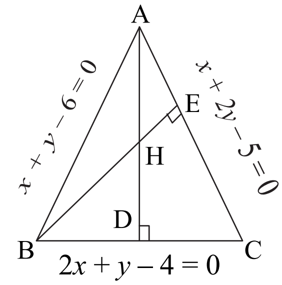
The required line has infinite slope (i.e. parallel to Y - axis) and passes through (3, 11).

$\Rightarrow x = 3$ is required equation.

50. Equation of AD is

$(x + y - 6) + k(x + 2y - 5) = 0$

$\Rightarrow (1 + k)x + (1 + 2k)y - (6 + 5k) = 0$ ....(i)



$\therefore$ Slope of AD = $m_1 = \dfrac{-(1+k)}{(1+2k)}$

and Slope of BC = $m_2 = -2$

$\therefore$ $m_1m_2 = -1$ ....[$\because$ AD $\perp$ BC]

$\therefore$ $k = -\dfrac{3}{4}$

$\therefore$ From (i), equation of AD is

$x - 2y = 9$ .....(ii)

Similarly, equation of BE is

$2x - y = -12$ .....(iii)

By solving equation (ii) and (iii), we get

$x = -11, y = -10$

$\therefore$ H $\equiv (-11, -10)$

51. Lengths of perpendicular from (0,0) on the given lines are each equal to 2.

52. $p_1.p_2 = \left(\dfrac{b\sqrt{a^2-b^2}\cos\theta + 0 - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right) \times \left(\dfrac{-b\sqrt{a^2-b^2}\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right)$

$= \dfrac{-[b^2(a^2-b^2)\cos^2\theta - a^2b^2]}{(b^2\cos^2\theta + a^2\sin^2\theta)}$

$= \dfrac{b^2[a^2 - a^2\cos^2\theta + b^2\cos^2\theta]}{b^2\cos^2\theta + a^2\sin^2\theta}$

$= \dfrac{b^2[a^2\sin^2\theta + b^2\cos^2\theta]}{b^2\cos^2\theta + a^2\sin^2\theta} = b^2$

53. Here, $p = \left|\dfrac{-k}{\sqrt{\sec^2\alpha + \text{cosec}^2\alpha}}\right|$

and $p' = \left|\dfrac{-k\cos 2\alpha}{\sqrt{\cos^2\alpha + \sin^2\alpha}}\right|$

$\therefore$ $4p^2 + p'^2 = \dfrac{4k^2}{\sec^2\alpha + \text{cosec}^2\alpha} + \dfrac{k^2(\cos^2\alpha - \sin^2\alpha)^2}{1}$

$= 4k^2 \sin^2\alpha \cos^2\alpha + k^2 (\cos^4\alpha + \sin^4\alpha)$
$\quad - 2k^2 \cos^2\alpha \sin^2\alpha$
$= k^2(\sin^2\alpha + \cos^2\alpha)^2$
$= k^2$

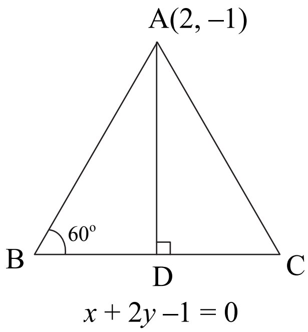
54. Let the point be (h, k), then h + k = 4.….(i) and

$$1 = \left|\frac{4h + 3k - 10}{\sqrt{4^2 + 3^2}}\right|$$

$\Rightarrow 4h + 3k = 15$ …..(ii)
and $4h + 3k = 5$ …..(iii)
On solving (i) and (ii), and (i) and (iii), we get the required points (3, 1) and (–7, 11).

55. $|AD| = \left|\frac{2-2-1}{\sqrt{1^2+2^2}}\right|$

$= \frac{1}{\sqrt{5}}$



$\tan 60° = \frac{AD}{BD}$

$\Rightarrow \sqrt{3} = \frac{1/\sqrt{5}}{BD}$

$\Rightarrow BD = \frac{1}{\sqrt{15}}$

$\therefore$ $BC = 2BD = 2/\sqrt{15}$

56. Equation of any line through (0, a) is
$y - a = m(x - 0)$ or $mx - y + a = 0$ …..(i)
If the length of perpendicular from (2a, 2a) to the line (i) is 'a', then $a = \pm\frac{m(2a) - 2a + a}{\sqrt{m^2+1}}$

$\Rightarrow m = 0, \frac{4}{3}$

Hence, the required equations of lines are
$y - a = 0$, $4x - 3y + 3a = 0$

57. The equation of lines passing through (1, 0) is given by $y = m(x - 1)$.

Its distance from origin is $\frac{\sqrt{3}}{2}$.

$\Rightarrow \left|\frac{-m}{\sqrt{1+m^2}}\right| = \frac{\sqrt{3}}{2} \Rightarrow m = \pm\sqrt{3}$

Hence, the lines are $\sqrt{3}x + y - \sqrt{3} = 0$ and $\sqrt{3}x - y - \sqrt{3} = 0$

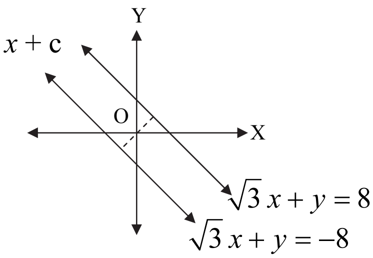
58. Point of intersection is (2, 3).
Therefore, the equation of line passing through (2, 3) is $y - 3 = m(x - 2)$
or $mx - y - (2m - 3) = 0$
According to the condition,

$$\left|\frac{3m - 2 - (2m - 3)}{\sqrt{1+m^2}}\right| = \frac{7}{5} \Rightarrow m = \frac{3}{4}, \frac{4}{3}$$

Hence, the equations are $3x - 4y + 6 = 0$ and $4x - 3y + 1 = 0$.

59. Slope $= -\sqrt{3}$

$\therefore$ Line is $y = -\sqrt{3}\,x + c$



$\Rightarrow \sqrt{3}\,x + y = c$

Now $\frac{c}{2} = |4|$

$\Rightarrow c = \pm 8$

$\Rightarrow x\sqrt{3} + y = \pm 8$

60. $d = \left|\frac{8-3}{\sqrt{(3)^2 + (4)^2}}\right| = 1$

61. If the given lines represent the same line, then the length of the perpendiculars from the origin to the lines are equal, so that

$$\frac{c}{\sqrt{1+m^2}} = \frac{p}{\sqrt{\cos^2\alpha + \sin^2\alpha}}$$

$\Rightarrow c = p\sqrt{1+m^2}$

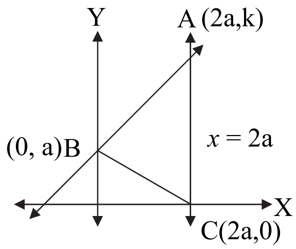
62. Lines $3x + 4y + 2 = 0$ and $3x + 4y + 5 = 0$ are on the same side of the origin. The distance between these lines is $d_1 = \left|\frac{2-5}{\sqrt{3^2+4^2}}\right| = \frac{3}{5}$.

Lines $3x + 4y + 2 = 0$ and $3x + 4y - 5 = 0$ are on the opposite sides of the origin. The distance between these lines is $d_2 = \left|\frac{2+5}{\sqrt{3^2+4^2}}\right| = \frac{7}{5}$.

Thus, $3x + 4y + 2 = 0$ divides the distance between $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ in the ratio $d_1 : d_2$ i.e., 3 : 7.

63. Since, the distance between the parallel lines $lx + my + n = 0$ and $lx + my + n' = 0$ is same as the distance between parallel lines $mx + y + n = 0$ and $mx + ly + n' = 0$. Therefore, the parallelogram is a rhombus. Since, the diagonals of a rhombus are at right angles, therefore the required angle is $\frac{\pi}{2}$.

64. Line AB will pass through (0, a) and (2a, k)

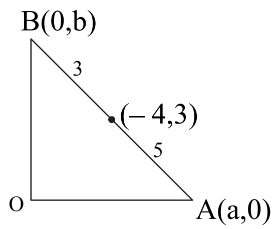


But as we are given AB = AC

$\Rightarrow k = \sqrt{4a^2 + (k-a)^2} \Rightarrow k = \dfrac{5a}{2}$

Hence, the required equation is $3x - 4y + 4a = 0$

65.



By the section formula, we get $a = -\dfrac{32}{3}$ and $b = \dfrac{24}{5}$

Hence, the required equation is given by

$\dfrac{x}{-(32/3)} + \dfrac{y}{(24/5)} = 1$

$\Rightarrow 9x - 20y + 96 = 0$

66. It is given that the lines $ax + 2y + 1 = 0$, $bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent, therefore $\begin{vmatrix} a & 2 & 1 \\ b & 3 & 1 \\ c & 4 & 1 \end{vmatrix} = 0$

$\Rightarrow -a + 2b - c = 0 \Rightarrow 2b = a + c$

$\Rightarrow$ a, b, c are in A. P.

67. The two lines will be identical if there exists some real number k such that

$b^3 - c^3 = k(b - c)$, $c^3 - a^3 = k(c - a)$,

$a^3 - b^3 = k(a - b)$

$\Rightarrow b - c = 0$ or $b^2 + c^2 + bc = k$

$\Rightarrow c - a = 0$ or $c^2 + a^2 + ac = k$

$\Rightarrow a - b = 0$ or $a^2 + b^2 + ab = k$

$\Rightarrow b = c, c = a, a = b$

or $b^2 + c^2 + bc = c^2 + a^2 + ca$

$\Rightarrow b^2 - a^2 = c(a - b)$

$\Rightarrow b = a$ or $a + b + c = 0$

68. $2p = \left| \dfrac{0 + 0 - 1}{\sqrt{\dfrac{1}{a^2} + \dfrac{1}{b^2}}} \right|$

$\Rightarrow \dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{4p^2} \Rightarrow \dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{2}{8p^2}$

$\Rightarrow a^2, 8p^2, b^2$ are in H.P.

Competitive Thinking

1. Since, the line makes an angle of measure 30° with Y-axis. Therefore, the line will make an angle of measure 60° or – 60° with X-axis.

$\therefore$ Slope of line = tan 60° or tan(–60°)

$= \sqrt{3}$ or $-\sqrt{3} = \pm\sqrt{3}$

2. Here, the straight line is parallel to X-axis. So, the slope of such a line = 0.

3. $m_1 = \dfrac{6+4}{-2-3} = \dfrac{10}{-5} = -2$ and $m_2 = \dfrac{-18-6}{9-(-3)} = -2$

Hence, the lines are parallel.

4. Midpoint of the line joining the points (4, –5) and (–2, 9) is $\left(\dfrac{4-2}{2}, \dfrac{-5+9}{2}\right)$ i.e., (1, 2)

$\therefore$ Inclination of straight line passing through point (–3, 6) and midpoint (1, 2) is

$m = \dfrac{2-6}{1+3} \Rightarrow \tan\theta = -1$

$\Rightarrow \theta = \dfrac{3\pi}{4}$

5. The required equation of line passing through (a, b) and having gradient $m = \dfrac{-b}{a}$, is

$(y - b) = \dfrac{-b}{a}(x - a)$

i.e. $\dfrac{x}{a} + \dfrac{y}{b} = 2$

6. The required equation of line passing through (–2, 3) and gradient $m = \dfrac{3}{4}$, is

$y - 3 = \dfrac{3}{4}[x - (-2)]$

i.e. $3x - 4y + 18 = 0$

7. Slope of line passing through (1, 0) and (–4, 1) = $\frac{1-0}{-4-1} = \frac{-1}{5}$

Slope of line perpendicular to the given line is m = 5

Equation of line passing through (–3, 5) and having slope 5 is

$y - 5 = 5(x + 3)$

$\Rightarrow 5x - y + 20 = 0$

8. Midpoint ≡ (2, 7)

Slope of perpendicular = – 6

∴ the required equation is $y - 7 = -6(x - 2)$

$\Rightarrow 6x + y - 19 = 0$

9. Midpoint of given line segment ≡ (2, –1)

Now, slope of the line segment = $\frac{-8}{8} = -1$

Slope of the required line segment is 1

∴ the required equation of line is $y + 1 = 1(x - 2)$

$\Rightarrow x - y = 3$

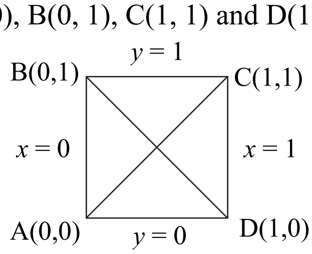
10. Midpoint ≡ (3, 2).

∴ the required equation is $y - 2 = 2(x - 3)$

$\Rightarrow 2x - y - 4 = 0$

11. The required diagonal passes through the midpoint of AB and is perpendicular to AB. So, its equation is $y - 2 = -3(x - 2)$ or $y + 3x - 8 = 0$.

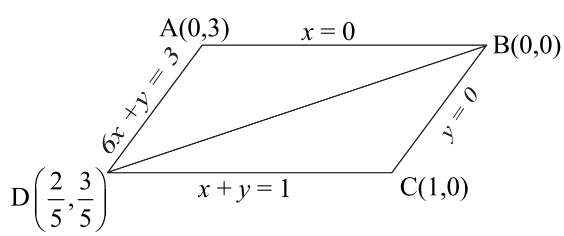
12. Co-ordinates of the vertices of the square are A(0, 0), B(0, 1), C(1, 1) and D(1, 0).



Now, the equation of AC is $y = x$ and of BD is

$y - 1 = -\frac{1}{1}(x - 0) \Rightarrow x + y = 1$

13.



From figure, diagonal BD is passing through origin, therefore its equation is given by

$$\left(y - \frac{3}{5}\right) = \frac{-(3/5)}{-(2/5)}\left(x - \frac{2}{5}\right)$$

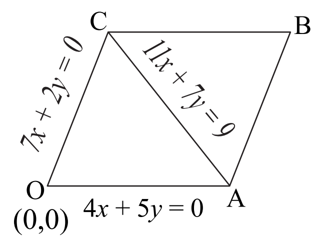
$\Rightarrow 3x - 2y = 0$

14. Since, the required line will be a line passing through A and B.

∴ $\frac{y-6}{6-(-4)} = \frac{x-1}{1-3}$

$\Rightarrow 10x - 10 = -2y + 12 \Rightarrow 5x + y - 11 = 0$

15. Since, equation of diagonal $11x + 7y = 9$ does not pass through origin, so it cannot be the equation of the diagonal OB. Thus, on solving the equation AC with the equations OA and OC, we get $A\left(\frac{5}{3}, -\frac{4}{3}\right)$ and $C\left(\frac{-2}{3}, \frac{7}{3}\right)$



Therefore, the midpoint of AC is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Hence, the equation of OB is $y = x$ i.e., $x - y = 0$.

16. Point of intersection of $x - y + 1 = 0$ and $7x - y - 5 = 0$ is (1, 2)

Equation of diagonal passing through (–1, –2) and (1, 2) is

$y + 2 = \frac{4}{2}(x + 1)$

$\Rightarrow 2x + 2 = y + 2$

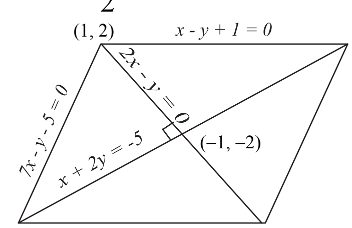
$\Rightarrow 2x - y = 0$

Equation of another diagonal passing through (–1, –2) and having slope $\frac{-1}{2}$ is

$y + 2 = \frac{-1}{2}(x + 1)$

$\Rightarrow 2y + 4 = -x - 1$

$\Rightarrow x + 2y = -5$



Point of intersection of $7x - y - 5 = 0$ and $x + 2y = -5$ is $\left(\frac{1}{3}, \frac{-8}{3}\right)$

∴ Answer is option (C)

17. Slope = $\dfrac{(2-1)}{1-\left(-\frac{1}{2}\right)} = \dfrac{1}{\left(\frac{3}{2}\right)} = \dfrac{2}{3}$

So, equation of the line is $y - 2 = \dfrac{2}{3}(x - 1)$

$\Rightarrow y = \dfrac{2}{3}x + \dfrac{4}{3}$

Putting $y = 0$, to find x-intercept, $\dfrac{2}{3}x + \dfrac{4}{3} = 0$

$\Rightarrow x = -2$

$\therefore$ x-intercept $= -2$

18. Let the equation be $\dfrac{x}{a} + \dfrac{y}{-a} = 1$.

$\Rightarrow x - y = a$ ….(i)

But, it passes through $(-3, 2)$

$\therefore$ $a = -3 - 2 = -5$

Putting the value of a in (i), we get

$x - y + 5 = 0$

19. Let the equation of the line be $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Given, $a = b$

So, equation of line is $x + y = a$

Since, this line passes through $(2, 4)$.

$\therefore$ $2 + 4 = a$

$\Rightarrow a = 6$

$\therefore$ the required equation of line is $x + y = 6$ i.e., $x + y - 6 = 0$

20. Here, $a + b = -1$

$\therefore$ required line is $\dfrac{x}{a} - \dfrac{y}{1+a} = 1$ .....(i)

Since, line (i) passes through $(4, 3)$.

$\therefore$ $\dfrac{4}{a} - \dfrac{3}{1+a} = 1$

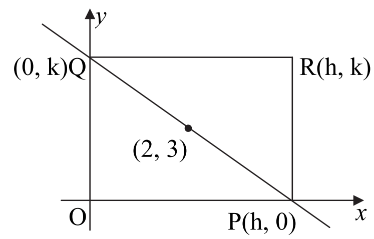
$\Rightarrow 4 + 4a - 3a = a + a^2$

$\Rightarrow a^2 = 4$

$\Rightarrow a = \pm 2$

$\therefore$ the required lines are $\dfrac{x}{2} - \dfrac{y}{3} = 1$ and $\dfrac{x}{-2} + \dfrac{y}{1} = 1$.

21. Let the equation of PQ be $\dfrac{x}{h} + \dfrac{y}{k} = 1$.



Since, the line passes through the fixed point $(2, 3)$.

$\Rightarrow \dfrac{2}{h} + \dfrac{3}{k} = 1$

$\Rightarrow$ Locus of R(h, k) is $\dfrac{2}{x} + \dfrac{3}{y} = 1$

$\Rightarrow 3x + 2y = xy$

22. Equation of the line has its intercepts on the X-axis and Y-axis in the ratio 2 : 1 i.e., 2a and a

$\therefore$ $\dfrac{x}{2a} + \dfrac{y}{a} = 1 \Rightarrow x + 2y = 2a$ .....(i)

Line (i) also passes through midpoint of $(3, -4)$ and $(5, 2)$ i.e., $(4, -1)$

$\therefore$ $4 + 2(-1) = 2a \Rightarrow a = 1$

Hence, the equation of required line is

$x + 2y = 2$

23. Let the points of the required line on X-axis and Y-axis be A(a, 0) and B(0, b) respectively.

Since, $\left(\dfrac{3}{2}, \dfrac{5}{2}\right)$ is midpoint of AB.

$\therefore$ $\dfrac{a+0}{2} = \dfrac{3}{2}$ and $\dfrac{0+b}{2} = \dfrac{5}{2} \Rightarrow a = 3$ and $b = 5$

$\therefore$ the equation of line is $\dfrac{x}{3} + \dfrac{y}{5} = 1$

$\Rightarrow 5x + 3y - 15 = 0$

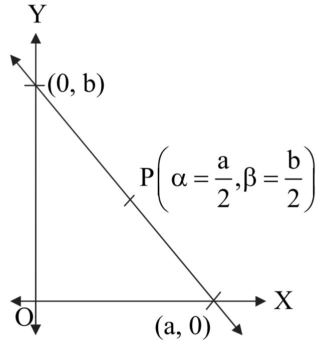
24. The required equation of line is

$\dfrac{x}{6} + \dfrac{y}{8} = 1 \Rightarrow 4x + 3y = 24$

25. Let $P\left(\alpha = \dfrac{a}{2}, \beta = \dfrac{b}{2}\right)$ be the midpoint of the line joining (a, 0) and (0, b).

$\therefore$ $\alpha = \dfrac{a}{2} \Rightarrow a = 2\alpha$ ….(i)

and $\beta = \dfrac{b}{2} \Rightarrow b = 2\beta$ ….(ii)



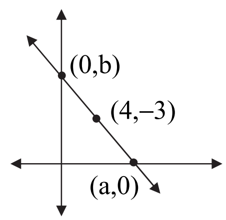
$\therefore$ Equation of a straight line cutting off intercepts a and b on X-axis and Y-axis respectively is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow \frac{x}{2\alpha} + \frac{y}{2\beta} = 1 \quad \text{....[From (i) and (ii)]}$$

$$\Rightarrow \frac{x}{\alpha} + \frac{y}{\beta} = 2$$

26. $\frac{a+0}{2} = 4 \Rightarrow a = 8$

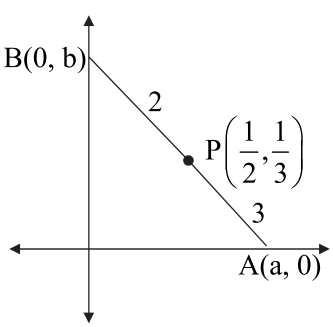


and $\frac{b+0}{2} = -3 \Rightarrow b = -6$

$\therefore$ the required equation of the line is $\frac{x}{8} + \frac{y}{-6} = 1$

$$\Rightarrow \frac{3x-4y}{24} = 1 \Rightarrow 3x - 4y = 24$$

27.



Point P divides AB in the ratio 2:3

$$\therefore \left(\frac{1}{2}, \frac{1}{3.}\right) = \left(\frac{0+2a}{5}, \frac{3b+0}{5}\right)$$

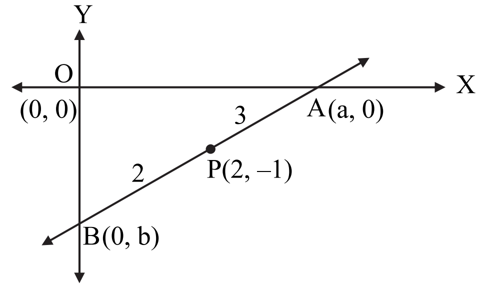
$$\Rightarrow a = \frac{5}{4} \text{ and } b = \frac{5}{9}$$

$\therefore$ equation of line AB is

$$\frac{x}{a} + \frac{y}{b} = 1$$

i.e., $4x + 9y = 5$

28.



Point P divides AB in the ratio 3:2

$$\therefore (2, -1) = \left(\frac{0+2a}{5}, \frac{3b+0}{5}\right)$$

$$\Rightarrow a = 5 \text{ and } b = \frac{-5}{3}$$

$\therefore$ Equation of the line AB is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow x - 3y - 5 = 0$$

29. Since, $px - qy = r$ intersects at X-axis and Y-axis.

$$\therefore a = \frac{r}{p} \text{ and } b = -\frac{r}{q}$$

$$\therefore a + b = \frac{r}{p} - \frac{r}{q} = r\left(\frac{q-p}{pq}\right)$$

30. Any line through the middle point M(1, 5) of the intercept AB may be taken as

$$\frac{x-1}{\cos\theta} = \frac{y-5}{\sin\theta} = r \quad \text{.....(i)}$$

Since, the points A and B are equidistant from M and on the opposite sides of it.

Therefore, if the co-ordinates of A are obtained by putting r = d in (i), then the co-ordinates of B are given by putting r = –d.

Now, the point A(1 + d cosθ, 5 + d sinθ) lies on the line $5x - y - 4 = 0$ and point B(1 – d cosθ, 5 – d sinθ) lies on the line $3x + 4y - 4 = 0$.

$$\therefore 5(1 + d\cos\theta) - (5 + d\sin\theta) - 4 = 0$$

and $3(1 - d\cos\theta) + 4(5 - d\sin\theta) - 4 = 0$

Eliminating 'd', we get $\frac{\cos\theta}{35} = \frac{\sin\theta}{83}$

Hence, the required line is $\frac{x-1}{35} = \frac{y-5}{83}$ or $83x - 35y + 92 = 0$.

31. Since, $m_1m_2 = (2)\left(-\frac{1}{2}\right) = -1$

$\therefore$ the lines are perpendicular.

32. Here, $m_1 = -1$, $m_2 = -\frac{1}{k}$.

For orthogonal lines,

$m_1m_2 = -1 \Rightarrow \frac{1}{k} = -1 \Rightarrow k = -1$

33. Here, $m_1 = \frac{-2}{3}$ and $m_2 = \frac{-1}{k}$

for perpendicular lines

$m_1m_2 = -1$

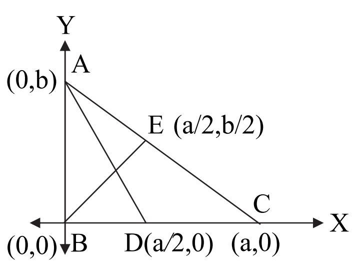
$\therefore$ $\frac{-2}{3} \times \frac{-1}{k} = -1$

$\Rightarrow k = \frac{-2}{3}$

34. $m_1m_2 = -1$

$\Rightarrow \left(\frac{k-3}{2-4}\right)(2) = -1 \Rightarrow 2k - 6 = 2 \Rightarrow k = 4$

35.



From figure,

$\left(\frac{b/2}{a/2}\right)\left(\frac{b}{-a/2}\right) = -1$

$\Rightarrow a^2 = 2b^2 \Rightarrow a = \pm\sqrt{2}b$

36. Since, the point (–4, 5) does not lie on the diagonal $7x - y + 8 = 0$, so point will lie on the other diagonal.

Also, diagonals are perpendicular.

$\therefore$ Slope of other diagonal = $\frac{-1}{7}$

$\therefore$ equation of the other diagonal is

$y - 5 = -\frac{1}{7}(x + 4) \Rightarrow 7y + x = 31$

37. The equation of lines in intercept form are

$\frac{x}{-8/a} + \frac{y}{-8/b} = 1$

$\frac{x}{-3} + \frac{y}{2} = 1$

According to the given condition,

$-\frac{8}{a} = -(-3)$ and $-\frac{8}{b} = -2$

$\Rightarrow a = -\frac{8}{3}$ and $b = 4$

38. The equation of a line perpendicular to $x - y = 0$ is $-x - y + c = 0$ ....(i)

Since, the line passes through (3, 2).

$\therefore$ $-3 - 2 + c = 0$

$\therefore$ $c = 5$

Putting c = 5 in (i), we get

$x + y = 5$

39. The equation of a line perpendicular to $x + y + 1 = 0$ is $x - y + \lambda = 0$.

Since, the line passes through the point (1, 2).

$\therefore$ $1 - 2 + \lambda = 0$

$\Rightarrow \lambda = 1$

Hence, required equation of line is

$y - x - 1 = 0$

40. Slope of $y = 3x - 1$ is 3

$\therefore$ Slope of line perpendicular to the above line is

$m = \frac{-1}{3}$

Equation of line passing through (1, 2) and having slope (m) = $\frac{-1}{3}$ is

$(y - 2) = \frac{-1}{3}(x - 1)$

$\Rightarrow 3y - 6 = -x + 1$

$\Rightarrow x + 3y - 7 = 0$

41. The required equation passing through (–1, 1) and having gradient $\frac{3}{2}$ is

$y - 1 = \frac{3}{2}(x + 1) \Rightarrow 2(y - 1) = 3(x + 1)$

42. $5x - 6y - 1 = 0$ ...(i)

$3x + 2y + 5 = 0$ ...(ii)

On solving (i) and (ii), we get $x = -1$, $y = -1$

Slope of line $3x - 5y + 11 = 0$ is $\frac{3}{5}$.

Slope of line perpendicular to above line = $\frac{-5}{3}$

$\therefore$ Equation of line passing through (–1, –1) and having slope $-\frac{5}{3}$ is

$(y + 1) = -\frac{5}{3}(x + 1)$

$\Rightarrow 3y + 3 = -5x - 5$

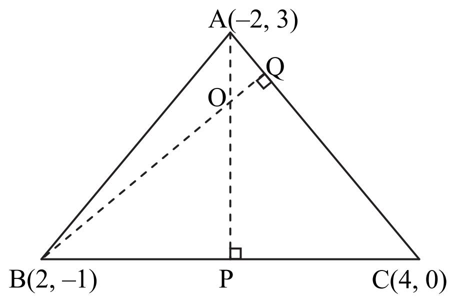
$\Rightarrow 5x + 3y + 8 = 0$

43. The given line is $bx - ay = ab$ ….(i)
It cuts X-axis at (a, 0).
The equation of a line perpendicular to (i) is $ax + by = k$.
Since, the line passes through (a, 0) $\Rightarrow k = a^2$
Hence, required equation of line is $ax + by = a^2$
i.e., $\frac{x}{b} + \frac{y}{a} = \frac{a}{b}$

44. The equation of a line passing through (2, 2) and perpendicular to $3x + y = 3$ is $y - 2 = \frac{1}{3}(x - 2)$ or $x - 3y + 4 = 0$.
Putting $x = 0$ in this equation, we get $y = \frac{4}{3}$

$\therefore$ y – intercept = $\frac{4}{3}$

45.



In ΔABC;

slope of BC = $\frac{0-(-1)}{4-2} = \frac{1}{2}$

slope of AC = $\frac{0-3}{4-(-2)} = \frac{-3}{6} = \frac{-1}{2}$

Since, AP $\perp$ BC and BQ $\perp$ AC,

$\therefore$ slope of AP = –2,
slope of BQ = 2

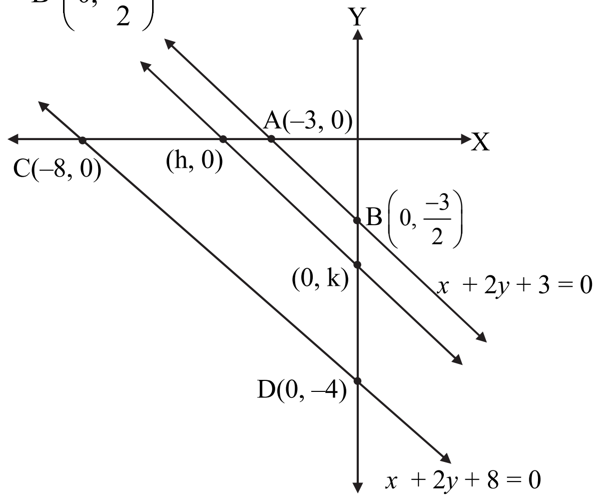
$\therefore$ Equation of AP is $2x + y + 1 = 0$ and equation of BQ is $2x - y - 5 = 0$
Solving the above equations, we get orthocentre, O = (1, –3)
Also, centroid of the triangle,

$C = \left(\frac{-2+2+4}{3}, \frac{3-1+0}{3}\right)$

i.e. $C = \left(\frac{4}{3}, \frac{2}{3}\right)$

$\therefore$ Equation of line OC is $11x - y - 14 = 0$

46. Line; $x + 2y + 3 = 0$
intersects the co-ordinate axes at A (–3, 0) and B $\left(0, \frac{-3}{2}\right)$



Line: $x + 2y + 8 = 0$ intersects the coordinate axes at C(–8, 0) and D(0, –4)
Since the required line divides the distance between the two lines in the ratio 1 : 2

$\therefore$ (h, 0) divides the distance between A(–3, 0) and C(–8, 0) in the ratio 1 : 2

$\therefore$ $(h, 0) = \left(\frac{1(-8)+2(-3)}{3}, 0\right)$

$\therefore$ $(h, 0) = \left(\frac{-14}{3}, 0\right)$

$\therefore$ the required equation of line passing through $\left(\frac{-14}{3}, 0\right)$ and having gradient $m = \frac{-1}{2}$, is

$(y - 0) = \frac{-1}{2}\left(x + \frac{14}{3}\right)$

$\therefore$ $-3x - 6y = 14$
Writing in normal form,

$\frac{-3x}{\sqrt{45}} - \frac{6y}{\sqrt{45}} = \frac{14}{\sqrt{45}}$

i.e. $x \cos a + y \sin a = \frac{14}{\sqrt{45}}$

where, $\cos a = \frac{-3}{\sqrt{45}}$, $\sin a = \frac{-6}{\sqrt{45}}$

$\therefore$ $a = \pi + \tan^{-1}\frac{\frac{-6}{\sqrt{45}}}{\frac{-3}{\sqrt{45}}}$

i.e. $a = \pi + \tan^{-1} 2$

47. Here, $m_1 = -\cot\alpha$, $m_2 = \tan\beta$

$\therefore$ $\tan\theta = \left|\dfrac{-\cot\alpha - \tan\beta}{1 - \cot\alpha\tan\beta}\right|$

$\therefore$ $\tan\theta = -\cot(\alpha - \beta)$

$\therefore$ $\theta = \dfrac{\pi}{2} - \beta + \alpha$

48. The lines are $bx + ay - ab = 0$ and $bx - ay - ab = 0$.
Hence, the required angle is

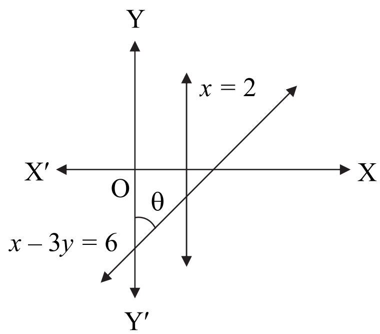
$$\theta = \tan^{-1}\left|\dfrac{\dfrac{-b}{a} - \dfrac{b}{a}}{1 - \dfrac{b^2}{a^2}}\right| = \tan^{-1}\left|\dfrac{2ab}{b^2 - a^2}\right|$$

$$= 2\tan^{-1}\dfrac{b}{a} \quad \ldots. \left[\because 2\tan^{-1}\dfrac{y}{x} = \tan^{-1}\left|\dfrac{2xy}{y^2 - x^2}\right|\right]$$

49. $\theta = 90^\circ - \tan^{-1}\left(\dfrac{1}{3}\right)$

$\Rightarrow \tan\theta = \cot\left[\tan^{-1}\left(\dfrac{1}{3}\right)\right] = 3$

$\Rightarrow \theta = \tan^{-1}(3)$



50. Given lines are $ax + by + c = 0$
and $x = \alpha t + \beta$, $y = \gamma t + \delta$
After eliminating t, we get
$\gamma x - \alpha y + \alpha\delta - \gamma\beta = 0$
For parallelism condition,
$\dfrac{a}{\gamma} = \dfrac{b}{-\alpha} \Rightarrow a\alpha + b\gamma = 0$

51. The given lines are perpendicular because $m_1m_2 = (2)\left(\dfrac{-1}{2}\right) = -1$

Hence, the angle between the two lines is 90°.

52. The slopes of the lines are $m_1 = \dfrac{-1}{2}$, $m_2 = 2$

$\therefore$ $m_1m_2 = -1$
So, the lines are perpendicular i.e., $\theta = 90^\circ$

53. Slopes of lines $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{b'} + \dfrac{y}{a'} = 1$

are $\dfrac{-b}{a}$ and $\dfrac{-a'}{b'}$ respectively

$\therefore$ Product of slopes is $\dfrac{a'b}{ab'}$

But $\dfrac{1}{ab'} + \dfrac{1}{ba'} = 0$
$\Rightarrow ab' = -a'b$
$\Rightarrow$ Product of slopes $= -1$
Hence option (C)

54. The equation of a straight line passing through $(3, -2)$ is
$y + 2 = m(x - 3)$ .....(i)
The slope of the line $\sqrt{3}\,x + y = 1$ is $-\sqrt{3}$

So, $\tan 60^\circ = \pm\dfrac{m - (-\sqrt{3})}{1 + m\,(-\sqrt{3})}$

On solving, we get
$m = 0$ or $\sqrt{3}$
Putting the values of m in (i), the required equation of lines are $y + 2 = 0$ and
$\sqrt{3}x - y - 2 - 3\sqrt{3} = 0$

55. Here the lines are $x - 3 = 0$, $y - 4 = 0$ and $4x - 3y + a = 0$.
These will be concurrent, if

$$\begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 4 & -3 & a \end{vmatrix} = 0 \Rightarrow a = 0$$

56. Lines are concurrent, if $\begin{vmatrix} 4 & 3 & -1 \\ 1 & -1 & 5 \\ b & 5 & -3 \end{vmatrix} = 0$

$\Rightarrow 4(3 - 25) - 3(-3 - 5b) - 1(5 + b) = 0$
$\Rightarrow -88 + 9 + 15b - 5 - b = 0$
$\Rightarrow -84 + 14b = 0$
$\Rightarrow b = 6$

57. Given lines are concurrent, if $\begin{vmatrix} 2 & 1 & -1 \\ a & 3 & -3 \\ 3 & 2 & -2 \end{vmatrix} = 0$

$$\Rightarrow -\begin{vmatrix} 2 & 1 & 1 \\ a & 3 & 3 \\ 3 & 2 & 2 \end{vmatrix} = 0$$

This is true for all values of a because C_2 and C_3 are identical.

58. Here, the given lines are
$ax + by + c = 0$
$bx + cy + a = 0$
$cx + ay + b = 0$
The lines will be concurrent, if $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$
$\Rightarrow a^3 + b^3 + c^3 - 3abc = 0$

59. Dividing both sides of relation $3a + 2b + 4c = 0$ by 4, we get $\frac{3}{4}a + \frac{1}{2}b + c = 0$, which shows that for all values of a, b and c each member of the set of lines $ax + by + c = 0$ passes through the point $\left(\frac{3}{4}, \frac{1}{2}\right)$

60. Since, lines $x + 3y - 9 = 0$, $4x + by - 2 = 0$, and $2x - y - 4 = 0$ are concurrent
$\begin{vmatrix} 1 & 3 & -9 \\ 4 & b & -2 \\ 2 & -1 & -4 \end{vmatrix} = 0$
∴ $1(-4b - 2) - 3(-16 + 4) - 9(-4 - 2b) = 0$
$\Rightarrow b = -5$
∴ the required line passes through $(-5, 0)$
Now, consider option (D) and $x + 3y - 9 = 0$, $4x - 5y - 2 = 0$
∴ $\begin{vmatrix} 1 & 3 & -9 \\ 4 & -5 & -2 \\ 1 & -4 & 5 \end{vmatrix} = 0$
∴ option (D) is correct

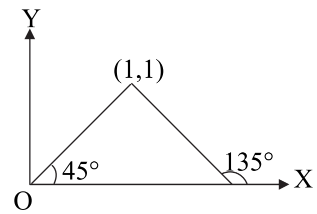
61. Required equation of line which is parallel to $x + 2y = 5$ is $x + 2y + k = 0$ ….(i)
Given equation of lines are
$x + y = 2$ ….(ii)
$x - y = 0$ ….(iii)
Adding (ii) and (iii), we get $2x = 2 \Rightarrow x = 1$
From (iii), we get $y = 1$
∴ Point of intersection is (1, 1).
Putting $x = 1$, $y = 1$ in (i), we get $k = -3$
∴ the required equation of line is $x + 2y = 3$.

62. Point of intersection is $y = -\frac{21}{5}$ and $x = \frac{23}{5}$
∴ $3x + 4y = \frac{3(23) + 4(-21)}{5} = \frac{69 - 84}{5} = -3$
Hence, required line is $3x + 4y + 3 = 0$

63. The lines passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$ is
$ax + 2by + 3b + \lambda(bx - 2ay - 3a) = 0$
$\Rightarrow (a + b\lambda)x + (2b - 2a\lambda)y + 3b - 3\lambda a = 0$ …(i)
Line (i) is parallel to X-axis,
$a + b\lambda = 0 \Rightarrow \lambda = \frac{-a}{b}$
Putting the value of λ in (i), we get
$ax + 2by + 3b - \frac{a}{b}(bx - 2ay - 3a) = 0$
$\Rightarrow y\left(2b + \frac{2a^2}{b}\right) + 3b + \frac{3a^2}{b} = 0$
$\Rightarrow y\left(\frac{2b^2 + 2a^2}{b}\right) = -\left(\frac{3b^2 + 3a^2}{b}\right)$
$\Rightarrow y = \frac{-3(a^2 + b^2)}{2(b^2 + a^2)} = \frac{-3}{2}$
So, it is 3/2 unit below X-axis.

64. Point of intersection of the lines is (3, –2)
Also, slope of perpendicular = $\frac{2}{7}$
Hence, the equation is $y + 2 = \frac{2}{7}(x - 3)$
$\Rightarrow 2x - 7y - 20 = 0$

65. Slopes of the lines are 1 and –1



Since, the point of intersection is (1, 1)
Hence, the required equations are
$y - 1 = \pm 1(x - 1)$

66. The point of intersection of the lines $3x + y + 1 = 0$ and $2x - y + 3 = 0$ are $\left(\frac{-4}{5}, \frac{7}{5}\right)$.
The equation of line which makes equal intercepts with the axes is $x + y = a$.
∴ $-\frac{4}{5} + \frac{7}{5} = a \Rightarrow a = \frac{3}{5}$
∴ the required equation of the line is
$x + y - \frac{3}{5} = 0$ i.e., $5x + 5y - 3 = 0$

67. $(a-2b)x + (a + 3b)y + 3a + 4b = 0$
or $a(x + y + 3) + b(-2x + 3y + 4) = 0$, which represents a family of straight lines through point of intersection of $x + y + 3 = 0$ and $-2x + 3y + 4 = 0$ i.e, $(-1, -2)$.

68. Required distance $= \left|\dfrac{3(3)-5(-4)-26}{\sqrt{9+16}}\right| = \dfrac{3}{5}$

69. Since, $L(p, q) = \dfrac{ap + bq + c}{\sqrt{a^2 + b^2}}$

and $L\left(\dfrac{2}{3}, \dfrac{1}{3}\right) + L\left(\dfrac{1}{3}, \dfrac{2}{3}\right) + L(2, 2) = 0$

$\therefore \quad \dfrac{a\left(\dfrac{2}{3}\right) + b\left(\dfrac{1}{3}\right) + c}{\sqrt{a^2 + b^2}} + \dfrac{a\left(\dfrac{1}{3}\right) + b\left(\dfrac{2}{3}\right) + c}{\sqrt{a^2 + b^2}} + \dfrac{a(2) + b(2) + c}{\sqrt{a^2 + b^2}} = 0$

$\therefore \quad \dfrac{3a + 3b + 3c}{\sqrt{a^2 + b^2}} = 0$

$\therefore \quad a + b + c = 0 \quad \ldots(i)$

Comparing equation (i) with $ax + by + c = 0$, we get
$a + b + c = ax + by + c$
i.e. $x = 1$ and $y = 1$

$\therefore$ The line $ax + by + c = 0$ passes through the point (1, 1).

70. The line is $4x - 3y - 12 = 0$.

$\therefore$ Required length $= \left|\dfrac{-12}{\sqrt{4^2 + (-3)^2}}\right| = \dfrac{12}{5} = 2\dfrac{2}{5}$

71. Given, equation of line is
$\dfrac{x \sin\alpha}{b} - \dfrac{y\cos\alpha}{a} - 1 = 0$

$\therefore$ perpendicular distance from origin

$= \left|\dfrac{0.\dfrac{\sin\alpha}{b} - \dfrac{0.\cos\alpha}{a} - 1}{\sqrt{\dfrac{\sin^2\alpha}{b^2} + \dfrac{\cos^2\alpha}{a^2}}}\right| = \dfrac{|ab|}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}$

72. Equation of the line is
$y - 0 = \left(\dfrac{3-0}{-5}\right)(x-5)$

$\Rightarrow 3x + 5y - 15 = 0$

$\therefore \quad d = \left|\dfrac{3(4)+5(4)-15}{\sqrt{3^2 + 5^2}}\right| = \dfrac{17}{\sqrt{34}} = \sqrt{\dfrac{17}{2}}$

73. Distance of (1, 1) from $3x + 4y + c = 0$ is
$d = \left|\dfrac{3(1) + 4(1) + c}{\sqrt{9+16}}\right|$

$\Rightarrow \pm 7 = \dfrac{7+c}{5}$

$\Rightarrow c = -42, 28$

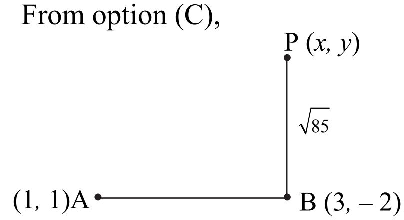
74. Let the point be (h, 0), then $a = \left|\dfrac{bh + 0 - ab}{\sqrt{a^2 + b^2}}\right|$

$\Rightarrow bh = \pm a\sqrt{a^2 + b^2} + ab$

$\Rightarrow h = \dfrac{a}{b}(b \pm \sqrt{a^2 + b^2})$

Hence, the points are $\left\{\dfrac{a}{b}(b \pm \sqrt{a^2+b^2}), 0\right\}$.

75. From option (C),



$BP = \sqrt{(5-3)^2 + (7+2)^2}$
$= \sqrt{4 + 81} = \sqrt{85}$

Hence, option (C) is correct.

76. $L_{12} \equiv x - 3y + 1 = 0$
$L_{23} \equiv 2x + y - 12 = 0$
$L_{13} \equiv 3x - 2y - 4 = 0$
Therefore, the required distances are

$D_3 = \left|\dfrac{4 - 3\times4 + 1}{\sqrt{10}}\right| = \dfrac{7}{\sqrt{10}}$

$D_1 = \left|\dfrac{4 + 1 - 12}{\sqrt{5}}\right| = \dfrac{7}{\sqrt{5}}$

$D_2 = \left|\dfrac{3\times5 - 2\times2 - 4}{\sqrt{9+4}}\right|$

$= \dfrac{7}{\sqrt{13}}$

77. Let p be the length of the perpendicular from the vertex (2, –1) to the base $x + y = 2$.

Then, $p = \left|\dfrac{2 - 1 - 2}{\sqrt{1^2 + 1^2}}\right| = \dfrac{1}{\sqrt{2}}$

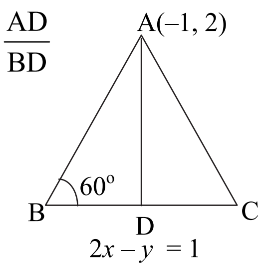
If 'a' is the length of the side of triangle, then
$p = a\sin 60^\circ$

$\Rightarrow \frac{1}{\sqrt{2}} = \frac{a\sqrt{3}}{2}$

$\Rightarrow a = \sqrt{\frac{2}{3}}$

78. $AD = \left| \frac{-2-2-1}{\sqrt{(2)^2+(-1)^2}} \right| = \left| \frac{-5}{\sqrt{5}} \right| = \sqrt{5}$

Since, $\tan 60° = \frac{AD}{BD}$



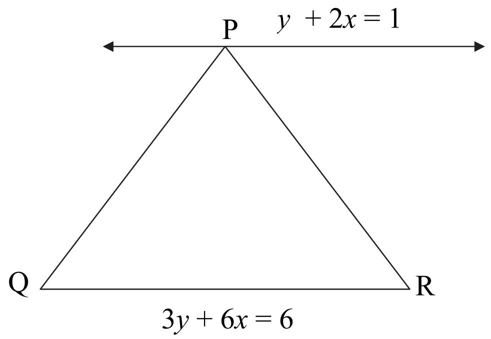
$\Rightarrow \sqrt{3} = \frac{\sqrt{5}}{BD}$

$\Rightarrow BD = \sqrt{\frac{5}{3}}$

$\therefore$ BC = 2BD

$= 2\sqrt{\frac{5}{3}} = \sqrt{\frac{20}{3}}$

79.



lines $y + 2x = 1$ and $3y + 6x = 6$ are parallel to each other

$\therefore$ $d = \left| \frac{-1+2}{\sqrt{4+1}} \right|$

$d = \frac{1}{\sqrt{5}}$

Side of equilateral triangle $= \frac{2}{\sqrt{3}} \times d$

$= \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{5}} = \frac{2}{\sqrt{15}}$

80. Point of intersection is (1, 2)
Therefore, the equation of line passing through (1, 2) is $(y - 2) = m(x - 1)$
or $mx - y + 2 - m = 0$
Since, the line is at distance of $\sqrt{5}$ from origin i.e. (0, 0),

$\left| \frac{(0)m-(0)+2-m}{\sqrt{m^2+1}} \right| = \sqrt{5}$

$\Rightarrow m = \frac{-1}{2}$

$\therefore$ Equation of the line is $x + 2y - 5 = 0$

81. Given b = 2a

$\therefore$ The equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$

$\Rightarrow \frac{x}{a} + \frac{y}{2a} = 1$

$\Rightarrow 2x + y = 2a$

Distance of the line from (0, 0) is

$d = \left| \frac{2(0)+1(0)-2a}{\sqrt{4+1}} \right|$

$\Rightarrow 1 = \left| \frac{-2a}{\sqrt{5}} \right|$

$\Rightarrow a = \pm \frac{\sqrt{5}}{2}$

$\therefore$ Equation of line is $2x + y = \pm\sqrt{5}$

82. Gradient of BC = –1 and its equation is $x + y + 4 = 0$. Therefore, the equation of line parallel to BC is $x + y + \lambda = 0$.
Also, it is $\frac{1}{2}$ unit distant from origin.

Thus, $\frac{\lambda}{\sqrt{2}} = \frac{1}{2} \Rightarrow \lambda = \frac{\sqrt{2}}{2}$

Hence, the required equation of line is $2x + 2y + \sqrt{2} = 0$

83. Equation of straight line parallel to $4x - 3y = 5$ is $4x - 3y = \lambda$
According to the given condition,

$\frac{4(-1)-3(-4)-\lambda}{\sqrt{16+9}} = \pm 1$

$\Rightarrow 8 - \lambda = \pm 5$

$\Rightarrow \lambda = 3, 13$

$\therefore$ the equation of one of the lines is $4x - 3y - 3 = 0$

84. Equation of AB: $4x - 3y - 17 = 0$
Equation of BC: $3x + 4y - 19 = 0$
If P(x, y) is a point on the bisector of $\angle$ABC then,

$\left| \frac{4x-3y-17}{\sqrt{(4)^2+(-3)^2}} \right| = \left| \frac{3x+4y-19}{\sqrt{(3)^2+(4)^2}} \right|$

$\therefore$ $7y = x + 2$ is the required equation of the angle bisector.

85. Let point (x_1, y_1) be at distance $\sqrt{5}$ from $x - 2y + 1 = 0$

$\therefore \quad \sqrt{5} = \left|\dfrac{x_1 - 2y_1 + 1}{\sqrt{1+4}}\right|$

$\therefore \quad x_1 - 2y_1 + 1 = \pm 5 \qquad \text{....(i)}$

Let point (x_2, y_2) be at distance $\sqrt{13}$ from $2x + 3y - 1 = 0$

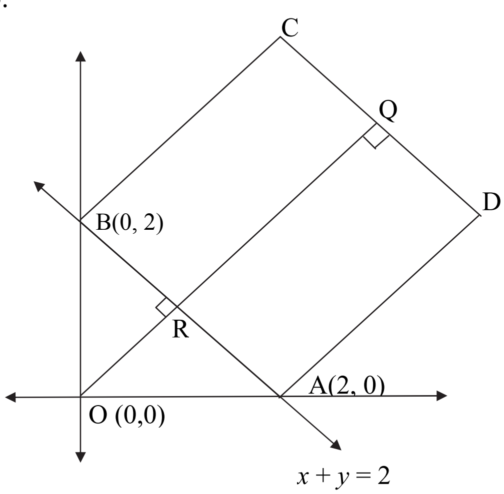
$\therefore \quad \sqrt{13} = \left|\dfrac{2x_2 + 3y_2 - 1}{\sqrt{4+9}}\right|$

$\therefore \quad 2x_2 + 3y_2 - 1 = \pm 13 \qquad \text{....(ii)}$

$\therefore$ Equation (i) and (ii) will give us total 4 points.

$\therefore$ answer is option (C)

86.



$l(\text{AB}) = l(\text{RQ}) = \sqrt{(2-0)^2 + (0-2)^2}$

$= \sqrt{8} = 2\sqrt{2}$

$l(\text{OR}) = \left|\dfrac{1(0) + 1(0) - 2}{\sqrt{1+1}}\right| = \left|\dfrac{-2}{\sqrt{2}}\right| = \sqrt{2}$

$\therefore$ Perpendicular distance (p) = OR + RQ

$= \sqrt{2} + 2\sqrt{2}$

$= 3\sqrt{2}$

87. Slope of given line $ax + by + c = 0$ is $-\dfrac{a}{b}$.

$\therefore \quad -\dfrac{a}{b} = \pm 1 \Rightarrow a = \pm b \qquad \text{....(i)}$

Distance of line $ax + by + c = 0$ from $(1, -2)$

$= \dfrac{|a - 2b + c|}{\sqrt{a^2 + b^2}}$

Distance of line $ax + by + c = 0$ from $(3, 4)$

$= \dfrac{|3a + 4b + c|}{\sqrt{a^2 + b^2}}$

According to the given condition,

$\dfrac{|a - 2b + c|}{\sqrt{a^2 + b^2}} = \dfrac{|3a + 4b + c|}{\sqrt{a^2 + b^2}}$

$\Rightarrow 3a + 4b + c = \pm(a - 2b + c)$

$\Rightarrow a + 3b = 0 \quad \text{(taking +ve)} \qquad \text{....(ii)}$

$\Rightarrow 2a + b + c = 0 \text{ (taking–ve)} \qquad \text{....(iii)}$

From, (i) and (ii), we get a = b = 0 which is not possible so taking (i) and (iii), (taking a = – b) we get

$a + c = 0 \Rightarrow c = -a$

$a : b : c = a : -a : -a = 1 : -1 : -1$

or $a = 1, b = -1, c = -1$

From (i) and (iii) (taking a = b), we get

$3a + c = 0 \Rightarrow c = -3a$

$a : b : c = a : a : -3a = 1 : 1 : -3$

$\therefore$ option (B) is the correct answer.

88. Here, the lines are $3x + 4y - 9 = 0$ and $6x + 8y - 15 = 0$ or $3x + 4y - \dfrac{15}{2} = 0$.

$\therefore$ Required distance $= \left|\dfrac{-9 - \left(\dfrac{-15}{2}\right)}{\sqrt{3^2 + 4^2}}\right| = \left|\dfrac{-3}{10}\right| = \dfrac{3}{10}$

89. Given equation of parallel lines are

$x - y + a = 0, x - y + b = 0$

$\therefore$ required distance $= \left|\dfrac{a - b}{\sqrt{(1)^2 + (-1)^2}}\right| = \dfrac{|a - b|}{\sqrt{2}}$

90. Line L passes through (13, 32).

$\therefore \quad \dfrac{13}{5} + \dfrac{32}{b} = 1$

$\Rightarrow b = -20$

So, equation of L is $\dfrac{x}{5} - \dfrac{y}{20} = 1 \Rightarrow 4x - y = 20$

Slope of L is $m_1 = 4$.

Slope of $\dfrac{x}{c} + \dfrac{y}{3} = 1$ is $m_2 = -\dfrac{3}{c}$

$\Rightarrow -\dfrac{3}{c} = 4$

$\Rightarrow c = -\dfrac{3}{4}$

Equation of line K is $-\dfrac{4x}{3} + \dfrac{y}{3} = 1$

$\Rightarrow 4x - y = -3$

Distance between L and K is $\left|\dfrac{20 + 3}{\sqrt{16 + 1}}\right| = \dfrac{23}{\sqrt{17}}$

91. Distance between lines $-x + y = 2$ and $x - y = 2$ is $\alpha = \left|\frac{2+2}{\sqrt{2}}\right| = 2\sqrt{2}$ ....(i)

Distance between lines $4x - 3y = 5$ and $6y - 8x = 1$ is

$$\beta = \left|\frac{5-\left(\frac{-1}{2}\right)}{5}\right| = \frac{11}{10} \quad \text{....(ii)}$$

From (i) and (ii), we get

$$\frac{\alpha}{\beta} = \frac{2\sqrt{2}}{11/10}$$

$\Rightarrow 20\sqrt{2}\beta = 11\alpha$

92. $(K+1)^2x + Ky - 2K^2 - 2 = 0$

$\therefore$ $(K^2 + 2K + 1)x + Ky - 2K^2 - 2 = 0$

$\therefore$ $K^2(x-2) + K(2x+y) + (x-2) = 0$

$\therefore$ $(K^2+1)(x-2) + K(2x+y) = 0$

$\therefore$ $x - 2 = 0$ i.e. $x = 2$

and $2x + y = 0$

$\therefore$ $2(2) + y = 0$

$\therefore$ $y = -4$

$\therefore$ The fixed point is $(2, -4)$

$\because$ The required line has slope 2 and passes through the point $(2, -4)$

$\therefore$ Equation of line;

$y - (-4) = 2(x-2)$

$\therefore$ $y + 4 = 2x - 4$

$\therefore$ $y = 2x - 8$

93. $ax + by + c = 0$ always passes through $(1, -2)$.

$\therefore$ $a - 2b + c = 0 \Rightarrow 2b = a + c$

Therefore, a, b and c are in A.P.

94. Since, l, m, n are in A.P.

$\therefore$ $2m = l + n$

Given equation of line is $lx + my = n = 0$

Consider, option (B),

If the point $(1, -2)$ satisfy the given equation.

$\therefore$ $l - 2m + n = 0 \Rightarrow 2m = l + n$

$\Rightarrow$ l, m, n are A.P.

95. a, b, c are in H. P., then $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$ .....(i)

Given, line is $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ .....(ii)

From (i) and (ii), we get

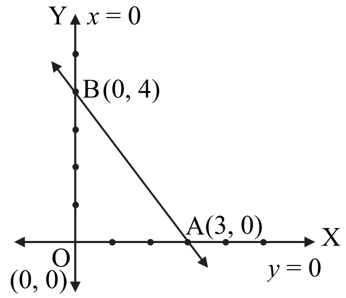
$$\frac{1}{a}(x-1) + \frac{1}{b}(y+2) = 0$$

Since, $a \neq 0$, $b \neq 0$

So, $(x-1) = 0$ and $(y+2) = 0$

$\Rightarrow x = 1$ and $y = -2$

96.



For a triangle with side lengths a, b and c and vertices at points opposite to these sides (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively, the incentre is given by,

$$(x_i, y_i) = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1by_2 + cy_3}{a+b+c}\right)$$

For the given triangle,

OA = 3 units

OB = 4 units

AB = $\sqrt{(4-0)^2 + (0-3)^2}$ = 5 units

$\therefore$ Incentre

$$= \left(\frac{3(0)+4(3)+5(0)}{3+4+5}, \frac{3(4)4(0)5(0)}{3+4+5}\right)$$

$$= \left(\frac{12}{12}, \frac{12}{12}\right)$$

$= (1, 1)$

$\therefore$ Incentre is $(1, 1)$

97. Two sides $x - 3y = 0$ and $3x + y = 0$ of the given triangle are perpendicular to each other. Therefore, its orthocentre is the point of intersection of $x - 3y = 0$ and $3x + y = 0$ i.e., $(0, 0)$.

98. The vertices of triangle are the intersection points of these given lines. The vertices of Δ are A(0, 4), B(1, 1), C(4, 0)

Now,

$$AB = \sqrt{(0-1)^2 + (4-1)^2} = \sqrt{10}$$

$$BC = \sqrt{(1-4)^2 + (1-0)^2} = \sqrt{10}$$

$$AC = \sqrt{(0-4)^2 + (4-0)^2} = 4\sqrt{2}$$

$\because$ AB = BC

$\therefore$ Δ is isosceles.

99. The point of intersection of the given lines are $(-1, 1)$, $(1, -1)$ and $(2/3, 2/3)$ which is the vertices of an isosceles triangle.

Evaluation Test

1. Given, $f(\alpha) = x \cos \alpha + y \sin \alpha - p(\alpha)$

$\therefore$ $f(\beta) = x \cos \beta + y \sin \beta - p(\beta)$

Since, both the lines are perpendicular to each other.

$\therefore$ $a_1a_2 + b_1b_2 = 0$

$\Rightarrow \cos \alpha \cos \beta + \sin \alpha \sin \beta = 0$

$\Rightarrow \cos (\alpha - \beta) = 0 \Rightarrow |\alpha - \beta| = \dfrac{\pi}{2}$

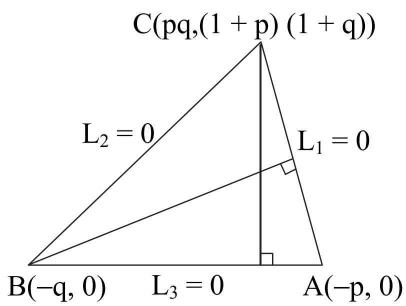
2. The equations of the sides of the triangle are

$L_1 \equiv \dfrac{x}{p} - \dfrac{y}{1+p} = -1,$

$L_2 \equiv \dfrac{x}{q} - \dfrac{y}{1+q} = -1,$

$L_3 \equiv y = 0$

The coordinates of vertices are A(–p, 0), B(–q, 0) and C(pq,(1 + p) (1 + q)).



The equation of the altitude through C is $x = pq$ and the equation of the altitude through B is $(1 + p) y + px + pq = 0$.

Solving these equations, we get

$x = pq$ and $y = -pq$

Let (h, k) be the coordinates of the orthocentre. Then,

$h = pq$ and $k = -pq \Rightarrow k = -h$

Hence, the locus of (h, k) is $y = -x$, which is a straight line.

3. The line $ax + by + c = 0$ meets the coordinate axes at $A\left(-\dfrac{c}{a}, 0\right)$ and $B\left(0, -\dfrac{c}{b}\right)$.

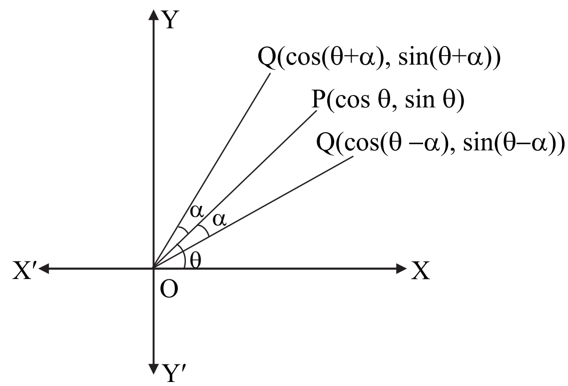
$\therefore$ Area of $\Delta OAB = \dfrac{1}{2} \times OA \times OB$

$$= \frac{1}{2} \times \left|-\frac{c}{a}\right|\left|-\frac{c}{b}\right| = \frac{c^2}{2ab}$$

This will be constant, if a, c, b are in G.P.

4. The clockwise rotation of the point $P(\cos \theta, \sin \theta)$ through an angle α takes it to the point $(\cos(\theta - \alpha), \sin(\theta - \alpha))$ and anticlockwise rotation through angle α takes P to the point $(\cos(\theta + \alpha), \sin(\theta + \alpha))$.

$\therefore$ options (A) and (B) are not correct.



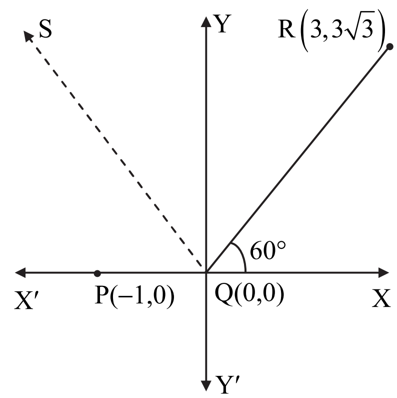
Slope of PQ $= \dfrac{\sin(\alpha - \theta) - \sin \theta}{\cos(\alpha - \theta) - \cos \theta}$

$$= -\frac{2\sin\left(\frac{\alpha}{2} - \theta\right)\cos\frac{\alpha}{2}}{2\sin\left(\frac{\alpha}{2} - \theta\right)\sin\frac{\alpha}{2}}$$

$$= -\cot\frac{\alpha}{2}$$

This shows that PQ is perpendicular to a line with slope $\tan\dfrac{\alpha}{2}$. Thus, Q can be obtained from P by taking its reflection in the line through origin with slope $\tan\dfrac{\alpha}{2}$.

5. Let QS be the bisector of $\angle PQR$.



Slope of QR $= \dfrac{3\sqrt{3} - 0}{3 - 0} = \sqrt{3}$

$\therefore$ $\angle XQR = 60°$

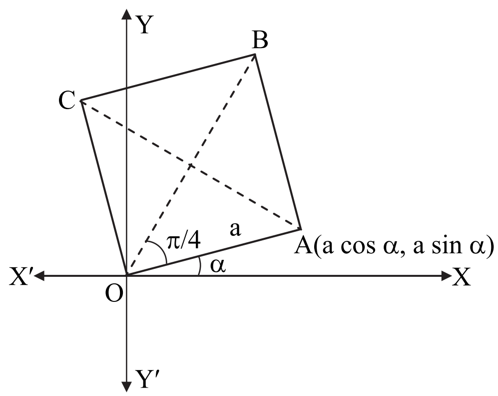
$\Rightarrow \angle PQR = 120°$

$\Rightarrow \angle PQS = \angle SQR = 60° \Rightarrow \angle XQS = 120°$

$\therefore$ Slope of QS = tan 120° = $-\sqrt{3}$

$\therefore$ the equation of QS is $y = -\sqrt{3}x$ i.e., $\sqrt{3}x + y = 0$

6.



Slope of OB = $\tan\left(\frac{\pi}{4}+\alpha\right)$

$\therefore$ Slope of AC = $-\cot\left(\frac{\pi}{4}+\alpha\right)$

$$= -\left(\frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha}\right)$$

$$= \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$$

$\therefore$ the equation of AC is

$$y - a\sin\alpha = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}(x - a\cos\alpha)$$

$\Rightarrow y(\sin\alpha + \cos\alpha) + x(\cos\alpha - \sin\alpha) = a$

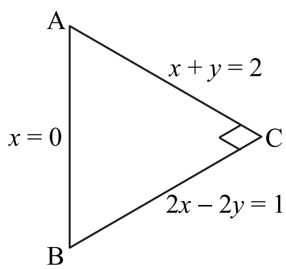
7. Equation of line L passing through (1, 1) and (2, 0) is

$$y - 1 = \frac{0-1}{2-1}(x - 1)$$

$\Rightarrow x + y = 2$ ….(i)

$\therefore$ Slope of L = -1

Also, slope of L′ = 1 ….[$\because$ L $\perp$ L′]



Equation of line L′ passing through $\left(\frac{1}{2}, 0\right)$ and having slope 1 is

$$y - 0 = 1\left(x - \frac{1}{2}\right)$$

$\Rightarrow 2x - 2y = 1$ ….(ii)

Equation of Y axis, $x = 0$ ….(iii)

From (i), (ii) and (iii),

the vertices of the triangle are A(0, 2), B$\left(0, -\frac{1}{2}\right)$ and C$\left(\frac{5}{4}, \frac{3}{4}\right)$.

$\therefore$ the area of the triangle is

$$\frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 0 & -\frac{1}{2} & 1 \\ \frac{5}{4} & \frac{3}{4} & 1 \end{vmatrix} = \frac{25}{16} \text{ square units}$$

8. By solving $3x + 4y = 9$, $y = mx + 1$, we get $x = \frac{5}{3+4m}$.

Now, x is an integer, if 3 + 4m = 1, – 1, 5, – 5

$\therefore$ $m = \frac{-2}{4}, \frac{-4}{4}, \frac{2}{4}, \frac{-8}{4}$.

Since, m = $\frac{-2}{4}, \frac{2}{4}$ do not give integral values of m.

$\therefore$ m has two integral values.

9. Given, the lines $ax + by + p = 0$ and $x\cos\alpha + y\sin\alpha - p = 0$ are inclined at an angle $\frac{\pi}{4}$.

$\therefore$ $$\tan\frac{\pi}{4} = \left|\frac{-\frac{a}{b} + \frac{\cos\alpha}{\sin\alpha}}{1 + \frac{a\cos\alpha}{b\sin\alpha}}\right|$$

$\Rightarrow$ a cos α + b sin α = – a sin α + b cos α

.....(i)

Also, the lines $ax + by + p = 0$,

$x\cos\alpha + y\sin\alpha - p = 0$ and $x\sin\alpha - y\cos\alpha = 0$ are concurrent.

$$\therefore \quad \begin{vmatrix} a & b & p \\ \cos\alpha & \sin\alpha & -p \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix} = 0$$

$\Rightarrow -$ ap cos α $-$ bp sin α $-$ p = 0

$\Rightarrow$ a cos α + b sin α = $-$ 1 ....(ii)

From (i) and (ii), we get

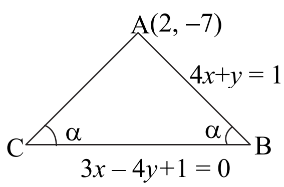
$-$ a sin α + b cos α = $-$ 1 ….(iii)

Squaring (ii) and (iii) and adding, we get

$(a \cos \alpha + b \sin \alpha)^2 + (- a \sin \alpha + b \cos \alpha)^2 = 2$

$\Rightarrow a^2 + b^2 = 2$

10. Slopes of AB and BC are -4 and $\frac{3}{4}$ respectively.



Let α be the angle between AB and BC.

Then, $\tan\alpha = \left|\dfrac{-4-\frac{3}{4}}{1-4\left(\frac{3}{4}\right)}\right| = \dfrac{19}{8}$ .....(i)

Since, AB = AC

$\Rightarrow \angle ABC = \angle ACB = \alpha$

$\therefore$ the line AC also makes an angle α with BC.

If m is the slope of the line AC, then its equation is $y + 7 = m(x - 2)$ .....(ii)

Now, $\tan\alpha = \pm\left[\dfrac{m-\frac{3}{4}}{1+m.\frac{3}{4}}\right]$

$\Rightarrow \dfrac{19}{8} = \pm\dfrac{4m-3}{4+3m}$ ….[From (i)]

$\Rightarrow m = -4$ or $-\dfrac{52}{89}$

But slope of AB is -4, so slope of AC is $-\dfrac{52}{89}$.

Therefore, the equation of line AC given by (ii) is $52x + 89y + 519 = 0$.

Textbook Chapter No.

07 Circle and Conics

Hints

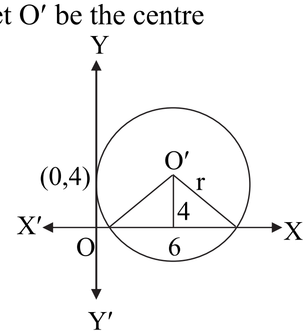
Classical Thinking

2. Required equation is $(x - a)^2 + (y - a)^2 = a^2$
$\Rightarrow x^2 + y^2 - 2ax - 2ay + a^2 = 0$

3. The equation of circle with centre (x_1, y_1) is
$(x - x_1)^2 + (y - y_1)^2 = r^2$
since, the circle touches both the axes,
$x_1 = y_1 = r$
$\therefore$ $(x - x_1)^2 + (y - x_1)^2 = x_1^2$
$\Rightarrow x^2 + y^2 - 2x_1(x + y) + x_1^2 = 0$

4. Since, the circle touches X-axis
$\therefore$ Radius = 2.
$\therefore$ the equation of the circle is
$(x - 1)^2 + (y - 2)^2 = 2^2$
$\Rightarrow x^2 + y^2 - 2x - 4y + 1 = 0$

5. Let O′ be the centre



Now, from the figure
Radius (r) $= \sqrt{(4)^2 + (3)^2} = 5$

6. Extremities of diameter are (5, 7) and (1, 4) Radius is half of the distance between them
$\therefore$ Radius $= \dfrac{1}{2}\sqrt{(4)^2 + (3)^2}$
$= \dfrac{5}{2}$

7. Using condition of point circle
Radius $= \sqrt{g^2 + f^2 - c} = 0$
$\Rightarrow g^2 + f^2 = c$

8. $(\text{Radius})^2 = g^2 + f^2 - c$
$\Rightarrow 121 = 81 + 36 - k \Rightarrow k = -4$

9. If c = 0; circle passes through origin.

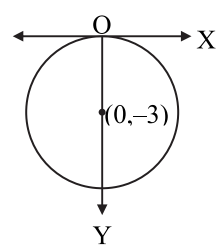
10. Intercept made by the circle on the X-axis
$= 2\sqrt{g^2 - c}$
$= 2\sqrt{9 - 9} = 0$
$\therefore$ Intercept cut on X-axis is zero.
Hence, circle touches X-axis.

11. Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches X-axis
$\therefore$ radius = ordinate of centre
$\Rightarrow \sqrt{g^2 + f^2 - c} = (-f)$
$\Rightarrow g^2 = c$

12. Required conditions are g = f = r and
$\sqrt{g^2 + f^2 - c} = r$
$\Rightarrow g = \sqrt{c} = f = r$

13. Both axis, as centre is (– 2, 2) and radius is 2.

14. Centre is (0, – 3) and radius $= \sqrt{0^2 + 9 - 0} = 3$



Hence, circle touches X-axis at the origin.

15. Centre (3, 4) of the given circle is satisfying only $x + y = 7$
$\therefore$ Correct answer is the option (C)

16. Here, the centre of circle (3, – 1) must lie on the line $x + 2by + 7 = 0$.
$\therefore$ $3 - 2b + 7 = 0$
$\Rightarrow b = 5$

17. Centre of the required circle is (– 4, – 5) and it passes through (2, 3)
$\therefore$ Radius $= \sqrt{(-4-2)^2 + (-5-3)^2}$
$= 10$
$\therefore$ Equation of the required circle is
$(x + 4)^2 + (y + 5)^2 = (10)^2$
$\Rightarrow x^2 + y^2 + 8x + 10y - 59 = 0$

18. The equation of circle in third quadrant touching the coordinate axes with centre (–a, –a) and radius 'a' is
$x^2+y^2+2ax+2ay+a^2= 0$ ….(i)
Since, line $3x - 4y + 8 = 0$ touches the circle
∴ perpendiular distance from centre of the circle to the line = radius
∴ $\left|\dfrac{3(-a)-4(-a)+8}{\sqrt{9+16}}\right| = a$
$\Rightarrow a = 2$
Substituting a = 2 in equation (i), we get
$x^2 + y^2 + 4x + 4y + 4 = 0$
This is the required equation of the circle

19. Radius of circle = $\left|\dfrac{2(1)-1(-3)-4}{\sqrt{4+1}}\right| = \dfrac{1}{\sqrt{5}}$
∴ Equation is $(x-1)^2 + (y+3)^2 = \left(\dfrac{1}{\sqrt{5}}\right)^2$
$\Rightarrow x^2 + y^2 - 2x + 6y + 10 = \dfrac{1}{5}$
$\Rightarrow 5x^2 + 5y^2 - 10x + 30y + 49 = 0$

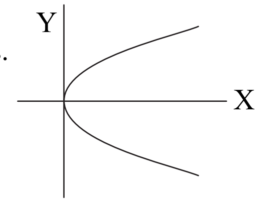
20. $4x^2 + 4y^2 = 9$
$\Rightarrow x^2 + y^2 = \dfrac{9}{4} \Rightarrow x^2 + y^2 = \left(\dfrac{3}{2}\right)^2$
∴ $x = \dfrac{3}{2}\cos\theta, y = \dfrac{3}{2}\sin\theta$

21. $(x-3)^2 + (y+4)^2 = 5^2$
Comparing with $(x-h)^2 + (y-k)^2 = r^2$, we get
h = 3, k = – 4, r = 5
∴ Parametric equations are
$x = 3 + 5\cos\theta, y = -4 + 5\sin\theta$

22. Given equation can be written as
$(x^2 + 2x + 1 - 1) + (y^2 - 4y + 4 - 4) - 4 = 0$
$\Rightarrow (x+1)^2 + (y-2)^2 = 3^2$
∴ h = – 1, k = 2 and r = 3
∴ Parametric form of equation are
$x = -1 + 3\cos\theta, y = 2 + 3\sin\theta$

23. $\dfrac{x+1}{2} = \cos\theta$ ….(i)
and $\dfrac{y-3}{2} = \sin\theta$ ….(ii)
Squaring (i) and (ii) and adding, we get
$\left(\dfrac{x+1}{2}\right)^2 + \left(\dfrac{y-3}{2}\right)^2 = 1$
$\Rightarrow (x+1)^2 + (y-3)^2 = 4,$
∴ centre is (– 1, 3)

24. Parabola $y^2 = x$ is symmetric about X-axis.



25. S ≡ (5,0)
Therefore, latus rectum = 4a = 20

26. Since, parabola $y^2 = 4ax$ passes through (– 3, 2)
∴ 4 = – 12a
$\Rightarrow 4a = -\dfrac{4}{3} = \dfrac{4}{3}$ ....[Taking positive sign]

27. Let the equation of parabola be $x^2 = -4ay$
Since, parabola passing through the point (– 4, – 2).
∴ $(-4)^2 = -4a(-2)$
$\Rightarrow a = 2$
∴ equation becomes $x^2 = -8y$ and latus rectum = 4a = 8

29. $y^2 = 4.\dfrac{1}{5}x \Rightarrow a = \dfrac{1}{5}$
co-ordinates of latus rectum are (a, 2a) and (a, – 2a)
i.e., $\left(\dfrac{1}{5},\dfrac{2}{5}\right)$ and $\left(\dfrac{1}{5},\dfrac{-2}{5}\right)$

30. $x^2 = -8y$
$\Rightarrow a = 2$
So, focus = (0, – 2)
∴ Ends of latus rectum = (4, – 2),(– 4, – 2).

31. $y^2 = 12x$
∴ a = 3
$\Rightarrow$ abscissa is 4 – 3 = 1 and $y^2 = 12$, $y = \pm 2\sqrt{3}$
Hence, points are $(1, 2\sqrt{3}), (1, -2\sqrt{3})$

32. We have, $y = 3x$
According to given condition,
$(3x)^2 = 36x$
$\Rightarrow x = 4$ and $y = 12$
∴ Required point is (4, 12)

34. $y^2 + 2y + x = 0$
$\Rightarrow y^2 + 2y + 1 = -x + 1$
$(y+1)^2 = -(x-1)$
Hence, vertex is (1, – 1), which lies in IV$^{\text{th}}$ quadrant.

35. Vertex = (2, 0)
$\Rightarrow$ focus is (2 + 2, 0) = (4, 0)

36. $y^2 = 4y - 4x$
$\Rightarrow y^2 - 4y + 4 = -4x + 4$
$\Rightarrow (y - 2)^2 = -4(x - 1)$
Comparing this equation with $Y^2 = -4aX$, we get
$a = 1$, $X = x - 1$ and $Y = y - 2$
Focus of the parabola is
$X = -a$, $Y = 0$
$\Rightarrow x - 1 = -1$, $y - 2 = 0$
$\Rightarrow x = 0$, $y = 2$
$\therefore$ focus = (0, 2)

37. Equation of parabola is $x^2 - 4x - 8y + 12 = 0$
$\Rightarrow x^2 - 4x + 4 = 8y - 8$
$\Rightarrow (x - 2)^2 = 8(y - 1) \Rightarrow X^2 = 8Y$
Comparing with $X^2 = 4aY$, we get a = 2
$\therefore$ Directrix is $Y = -a \Rightarrow y - 1 = -2 \Rightarrow y = -1$.

38. The parabola is $(x - 2)^2 = (3y - 6)$
Hence, axis is $x - 2 = 0$.

39. The given equation of parabola is $x^2 - 4x - 8y + 12 = 0$
$\Rightarrow x^2 - 4x = 8y - 12$
$\Rightarrow x^2 - 4x + 4 = 8y - 12 + 4$
$\Rightarrow (x - 2)^2 = 8(y - 1)$
Hence, the length of latus rectum = 4a = 8.

40. $x^2 + 5y = 0 \Rightarrow x^2 = -5y$
On comparing with $x^2 = -4ay$, we get
$a = \frac{5}{4}$
End points of latus rectum of the parabola are
$(\pm 2a, -a) = \left(\pm\frac{5}{2}, \frac{-5}{4}\right)$

41. Here, ae = 4 and $e = \frac{4}{5}$
$\Rightarrow a = 5$
Now, $b^2 = a^2(1 - e^2)$
$\Rightarrow b^2 = 25\left(1 - \frac{16}{25}\right) = 9$
$\therefore$ Equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

42. Since, $e^2 = 1 - \frac{b^2}{a^2}$
$\Rightarrow \left(\frac{2}{3}\right)^2 = 1 - \frac{b^2}{a^2}$
$\Rightarrow b^2 = \frac{5a^2}{9}$ ….(i)
Given length of latus rectum = 5
$\Rightarrow \frac{2b^2}{a} = 5$
$\Rightarrow b^2 = \frac{5a}{2}$ ….(ii)
From (i) and (ii), we get
$\Rightarrow a^2 = \frac{81}{4}$, $b^2 = \frac{45}{4}$
$\therefore$ Equation of ellipse is $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$

43. $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{28}{64}$
$\Rightarrow e^2 = \frac{36}{64} \Rightarrow e = \frac{3}{4}$

44. $\frac{x^2}{\frac{112}{16}} + \frac{y^2}{\frac{112}{7}} = 1$
$\therefore$ $e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{112}{16}.\frac{7}{112}} = \frac{3}{4}$

45. According to the condition, $\frac{2a}{e} = 3(2ae)$
$\Rightarrow e = \frac{1}{\sqrt{3}}$

46. Foci are ($\pm$ae, 0)
$\therefore$ According to the condition, 2ae = 2b
$\Rightarrow$ ae = b ….(i)
Also, $b^2 = a^2(1 - e^2)$
$\Rightarrow e^2 = (1 - e^2)$ ….[From (i)]
$\Rightarrow e = \frac{1}{\sqrt{2}}$

47. We have, ae = 1 and a = 2
$\Rightarrow e = \frac{1}{2}$
Also, $b^2 = a^2(1 - e^2)$
$\Rightarrow b = 2\sqrt{1 - \frac{1}{4}} = \sqrt{3}$
$\therefore$ Minor axis = 2b = $2\sqrt{3}$

49. $3x^2 + 4y^2 = 12$
$\Rightarrow \frac{x^2}{4} + \frac{y^2}{3} = 1$
$\therefore$ Latus rectum = $\frac{2b^2}{a} = 3$

50. Here, $a^2 = 36$, $b^2 = 49$
Since, $b > a$
$\therefore$ the length of the latus rectum
$= \frac{2a^2}{b} = 2 \times \frac{36}{7} = \frac{72}{7}$

51. We have, $e = \frac{1}{\sqrt{2}}$
$\therefore$ Latus rectum $= \frac{2b^2}{a} = \frac{2}{a} \times a^2\left(1 - e^2\right)$
$= 2a\left(1 - \frac{1}{2}\right) = a$
i.e., semi-major axis

52. We have, $2ae = 10 \Rightarrow a = \frac{10}{\frac{2 \times 5}{8}} = 8$
Also, $b^2 = a^2(1 - e^2)$
$\Rightarrow b = 8\sqrt{1 - \frac{25}{64}} = \sqrt{39}$
Now, Latus rectum $= \frac{2b^2}{a} = \frac{2 \times 39}{8} = \frac{39}{4}$

53. Focal distance of any point P (x,y) on the ellipse is equal to $SP = a + ex$.
Here, $x = a \cos \theta$
$\therefore$ $SP = a + ae \cos \theta$
$= a(1 + e \cos \theta)$

54. $4x^2 + 9y^2 - 16x - 54y + 61 = 0$
$\Rightarrow 4x^2 - 16x + 9y^2 - 54y = -61$
$\Rightarrow 4(x^2 - 4x + 4 - 4) + 9(y^2 - 6y + 9 - 9) = -61$
$\Rightarrow 4(x - 2)^2 + 9(y - 3)^2 = 36$
$\Rightarrow \frac{(x-2)^2}{9} + \frac{(y-3)^2}{4} = 1$
Hence, the centre is (2, 3)

55. Distance between foci is $4 = 2ae$
$\Rightarrow a^2 = \frac{9}{4}$
Also, $e^2 = 1 + \frac{b^2}{a^2}$
$\Rightarrow \frac{16}{9} - 1 = \frac{b^2}{a^2} \quad \Rightarrow b^2 = \frac{7}{4}$
Centre is (0, 4)
Hence, equation of hyperbola is
$\frac{x^2}{9} - \frac{4(y-4)^2}{7} = 1$

56. Let $S(1, -1)$ be the focus and $P \equiv (x, y)$ be any point on conic.
Now, $PS = e\,PM$
$\Rightarrow \sqrt{(x-1)^2 + (y+1)^2} = \sqrt{2}\left|\frac{x - y + 1}{\sqrt{1+1}}\right|$
$\Rightarrow \sqrt{x^2 - 2x + 1 + y^2 + 2y + 1} = |x - y + 1|$
Squaring both sides, we get
$4x - 4y - 2xy - 1 = 0$
$\Rightarrow 2xy - 4x + 4y + 1 = 0$

57. $2a = 8$, $2b = 6$
Difference of focal distances of any point of the hyperbola $= 2a = 8$

58. The equation of hyperbola is $9x^2 - 16y^2 = 144$
$\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$
$e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{16+9}}{4} = \frac{5}{4}$
Hence, foci are $(\pm ae, 0) \Rightarrow \left(\pm 4 \times \frac{5}{4}, 0\right)$
i.e., $(\pm 5, 0)$.

59. Since, $e > 1$ always for hyperbola and $\frac{2}{3} < 1$.

60. $\frac{x^2}{25} - \frac{y^2}{25} = 1$
$\therefore$ Eccentricity $= \sqrt{2}$ as $a = b$.

61. $\frac{y^2}{k^2} - \frac{x^2}{-k} = 1$
Also, $a^2 = b^2(e^2 - 1)$
$\Rightarrow -k = k^2(e^2 - 1) \Rightarrow -\frac{1}{k} = e^2 - 1$
$\Rightarrow e^2 = 1 - \frac{1}{k} \quad \Rightarrow e = \sqrt{1 - \frac{1}{k}}$

62. Vertices $(\pm 4, 0) \equiv (\pm a, 0)$
$\Rightarrow a = 4$
Foci $(\pm 6, 0) \equiv (\pm ae, 0)$
$\Rightarrow e = \frac{6}{4} = \frac{3}{2}$

63. The given equation of hyperbola is
$16x^2 - 9y^2 = 144 \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$
$\therefore$ L.R. $= \frac{2b^2}{a} = \frac{2 \times 16}{3} = \frac{32}{3}$

64. Length of latus rectum = $\dfrac{2b^2}{a} = \dfrac{2\times4}{\frac{2}{\sqrt{3}}} = 4\sqrt{3}$

66. Equation of hyperbola is
$x = 8 \sec\theta$, $y = 8\tan\theta$
$\Rightarrow \dfrac{x}{8} = \sec\theta, \dfrac{y}{8} = \tan\theta$
$\because$ $\sec^2\theta - \tan^2\theta = 1$
$\Rightarrow \dfrac{x^2}{8^2} - \dfrac{y^2}{8^2} = 1$
Here, a = 8, b = 8
Here, a = b
$\therefore$ it is rectangular hyperbola
$\Rightarrow$ e = $\sqrt{2}$
$\therefore$ Distance between directrices $= \dfrac{2a}{e}$
$= \dfrac{2\times8}{\sqrt{2}}$
$= 8\sqrt{2}$

67. The given equation of hyperbola is
$9x^2 - 36x - 16y^2 + 96y - 252 = 0$
Partially differentiating with respect to x, we get
$18x - 36 = 0$
$\Rightarrow x = 2$
Now partially differentiating with respect to y, we get
$-32y + 96 = 0$
$\Rightarrow -32y = -96 \Rightarrow y = 3$
$\therefore$ Centre $\equiv (2, 3)$

68. Given equation of hyperbola is
$5x^2 - 4y^2 + 20x + 8y = 4$
$\Rightarrow 5(x^2 + 4x + 4) - 4(y^2 - 2y + 1) = 4 + 20 - 4$
$\Rightarrow 5(x + 2)^2 - 4(y - 1)^2 = 20$
$\Rightarrow \dfrac{(x+2)^2}{4} - \dfrac{(y-1)^2}{5} = 1$
$\Rightarrow a^2 = 4, b^2 = 5$
$e = \dfrac{\sqrt{a^2+b^2}}{a} = \dfrac{\sqrt{4+5}}{2} = \dfrac{3}{2}$

69. $2x = e^{t} + e^{-t}$ and $2y = e^t - e^{-t}$
$\Rightarrow 4x^2 = e^{2t} + 2 + e^{-2t}$ ………. (i)
and $4y^2 = e^{2t} - 2 + e^{-2t}$ ………. (ii)
Subtracting (ii) from (i), we get
$4x^2 - 4y^2 = 4$
$\Rightarrow x^2 - y^2 = 1$
The equation represents hyperbola.

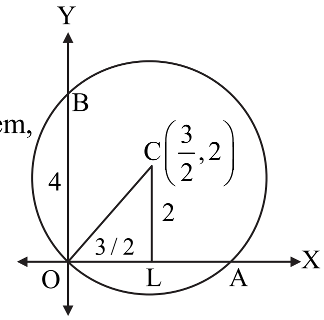
70. The equation is $(x - 0)^2 + (y - 0)^2 = a^2$

Critical Thinking

1. Radius = $\sqrt{\cos^2\theta + \sin^2\theta + 8} = 3$

2. The point of intersection of $3x + y - 14 = 0$ and $2x + 5y - 18 = 0$ is (4, 2).
Centre of the circle is (1, –2).
$\therefore$ radius $= \sqrt{(4-1)^2 + (2+2)^2} = 5$
$\therefore$ the equation of the circle is
$(x - 1)^2 + (y + 2)^2 = 5^2$
$\therefore$ $x^2 + y^2 - 2x + 4y - 20 = 0$.

3. Given, OA = 3 and
OB = 4



$\therefore$ OL = $\dfrac{3}{2}$ and CL = 2
By pythagoras theorem,
$OC^2 = OL^2 + LC^2$
$\therefore$ $OC^2 = \left(\dfrac{3}{2}\right)^2 + 2^2$
$= \dfrac{25}{4}$
$\therefore$ OC $= \dfrac{5}{2}$
The centre of the circle is $\left(\dfrac{3}{2}, 2\right)$ and
radius $= \dfrac{5}{2}$.
$\therefore$ the equation of the circle is
$\left(x - \dfrac{3}{2}\right)^2 + (y - 2)^2 = \left(\dfrac{5}{2}\right)^2$
$\therefore$ $x^2 + y^2 - 3x - 4y = 0$

4. Here, r = 10 (radius)
Centre will be the point of intersection of the diameters, i.e., (8, – 2).
Hence, required equation is
$(x - 8)^2 + (y + 2)^2 = 10^2$
$\Rightarrow x^2 + y^2 - 16x + 4y - 32 = 0$

5. Let its centre be (h, k), then
h – k = 1 ....(i)
Also, radius a = 3
$\therefore$ Equation of the circle is
$(x - h)^2 + (y - k)^2 = 9$
Also, it passes through (7, 3)
i.e., $(7 - h)^2 + (3 - k)^2 = 9$ ....(ii)
From (i) and (ii), we get
h = 4, k = 3
$\therefore$ Equation is $x^2 + y^2 - 8x - 6y + 16 = 0$

6. Centre is (– 4, 3)
Radius = Distance between centres – Radius of other circle = 5 – 1 = 4
Hence, equation of circle is $x^2+y^2+8x-6y+9=0$

7. Centre of the given circle is (0, – 1)
∴ the required circle passes through (0, – 1).
∴ $r = \sqrt{(0-1)^2+(-1+2)^2} = \sqrt{2}$
Hence, the required equation is
$(x-1)^2 + (y+2)^2 = (\sqrt{2})^2$
$\Rightarrow x^2 + y^2 - 2x + 4y + 3 = 0.$

8. Centre of the circle
$x^2 + y^2 - 4x - 6y - 3 = 0$ is C(2,3).
Since, it touches the Y-axis
∴ r = 2
Hence, required equation of the circle is
$(x-2)^2 + (y-3)^2 = 2^2$
$\Rightarrow x^2 + y^2 - 4x - 6y + 9 = 0$

9. Let centre of circle be (h, k).
Since it touches both axes, therefore h = k = a
Hence, equation can be $(x-a)^2 + (y-a)^2 = a^2$
But it also touches the line $3x + 4y = 4$
∴ $\left|\frac{3a+4a-4}{\sqrt{9+16}}\right| = a$
$\Rightarrow a = 2$
Hence, the required equation of circle is
$(x-2)^2 + (y-2)^2 = 2^2$
$\Rightarrow x^2 + y^2 - 4x - 4y + 4 = 0$

10. Let the centre of the required circle be (x_1, y_1).
Centre of given circle is (1, 2) and
$r = \sqrt{1+4+20} = 5$
∴ radii of both circles are same.
∴ Point of contact (5, 5) is the mid point of the line joining the centres of both circles.
∴ $\frac{x_1+1}{2} = 5$ and $\frac{y_1+2}{2} = 5$
$\Rightarrow x_1 = 9, y_1 = 8$
Hence, the required equation is
$(x-9)^2 + (y-8)^2 = 25$
$\Rightarrow x^2 + y^2 - 18x - 16y + 120 = 0$

11. Equation of circle concentric to given circle is
$x^2 + y^2 - 6x + 12y + k = 0$
Since, area of required circle = 2 (area of given circle)
$\Rightarrow \sqrt{9+36-k} = \sqrt{2}\ \sqrt{9+36-15}$
$\Rightarrow 45 - k = 60$
$\Rightarrow k = -15$
Hence, the required equation of circle is
$x^2 + y^2 - 6x + 12y - 15 = 0.$

12. We have the equation of circle
$x^2 + y^2 + 2gx + 2fy + c = 0$
But it passes through (0, 0) and (2, 1)
∴ c = 0
$5 + 4g + 2f = 0$ ….(i)
Also $\sqrt{g^2 + f^2 - c} = |g|$
$\Rightarrow f = 0$ ….[∵ c = 0]
∴ $g = -\frac{5}{4}$ ….[From (i)]
Hence, the equation will be $2x^2 + 2y^2 - 5x = 0$.

13. since, X–intercept = 2a
∴ $2\sqrt{g^2 - c} = 2a$ ….(i)
Also, Y-intercept = 2b
∴ $2\sqrt{f^2 - c} = 2b$ ….(ii)
On squaring (i) and (ii) and then subtracting (ii) from (i), we get
$g^2 - f^2 = a^2 - b^2$
Hence, the locus is
$x^2 - y^2 = a^2 - b^2$

14. Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.
Now on passing through the given points, we get three equations
$c = 0$ ….(i)
$a^2 + 2ga + c = 0$ ….(ii)
$b^2 + 2fb + c = 0$ ….(iii)
solving equations (i), (ii) and (iii), we get
$g = -\frac{a}{2}, f = -\frac{b}{2}$
Hence, the centre is $\left(\frac{a}{2}, \frac{b}{2}\right)$.

15. The equation of circle through points (0, 0), (1, 3) and (2, 4) is
$x^2 + y^2 - 10x = 0$
Point (k, 3) will be on the circle, if
$k^2 + 9 - 10k = 0$
$\Rightarrow k^2 - 10k + 9 = 0$
$\Rightarrow k^2 - 9k - k + 9 = 0$
$\Rightarrow (k-1)(k-9) = 0$
$\Rightarrow k = 1$ or $k = 9$

16. Given, $x = 2 + 3\cos\theta$
i.e, $x - 2 = 3\cos\theta$ ....(i)
and $y = 3 - 3\sin\theta$
i.e., $y - 3 = -3\sin\theta$ ....(ii)
Squaring and adding equation (i) and (ii), we get

$(x-2)^2 + (y-3)^2 = 9\cos^2\theta + 9\sin^2\theta$

$\therefore$ $(x-2)^2 + (y-3)^2 = 3^2$

$\therefore$ centre $\equiv (2, 3)$, radius = 3 units

17. $x = 2a\left(\dfrac{1-t^2}{1+t^2}\right)$ ….(i)

$y = \dfrac{4at}{1+t^2}$ ….(ii)

Squaring and adding (i) and (ii), we get

$x^2 + y^2 = 4a^2.\dfrac{(1-t^2)^2}{(1+t^2)^2} + \dfrac{16a^2t^2}{(1+t^2)^2}$

$= \dfrac{4a^2}{(1+t^2)^2}[1 - 2t^2 + t^4 + 4t^2]$

$= \dfrac{4a^2}{(1+t^2)^2}(1+t^2)^2$

$\therefore$ $x^2 + y^2 = (2a)^2$

$\therefore$ Radius = 2a

18. The point of intersection is

$x = a\cos\theta + b\sin\theta$

$y = a\sin\theta - b\cos\theta$

$\therefore$ $x^2 + y^2 = a^2 + b^2$

Hence, it is equation of a circle.

19. Since, the axis of parabola is Y-axis

$\therefore$ Equation of parabola $x^2 = 4ay$

Since, it passes through $(6, -3)$

$\therefore$ $36 = -12a$

$\Rightarrow a = -3$

$\therefore$ Equation of parabola is $x^2 = -12y$

20. Since, the axis of parabola is vertical and it's vertex is $(-1, -2)$.

$\therefore$ equation of parabola is

$(x+1)^2 = 4a(y+2)$

Also, it passes through $(3, 6)$

$\therefore$ $16 = 4a \times 8$

$\Rightarrow a = \dfrac{1}{2} \Rightarrow (x+1)^2 = 2(y+2)$

$\Rightarrow x^2 + 2x - 2y - 3 = 0$

21. a = VS

$= \sqrt{(2-2)^2 + (-3+1)^2}$

$= 2$

Since,

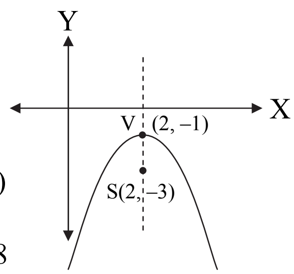
$(x-h)^2 = -4a(y-k)$

$\therefore$ $(x-2)^2 = -4 \times 2(y+1)$

$\Rightarrow (x-2)^2 = -8(y+1)$

$\Rightarrow x^2 + 4 - 4x = -8y - 8$

$\Rightarrow x^2 - 4x + 8y + 12 = 0$



22. Let any point on required parabola be P(x, y), then from definition of parabola, we get

PS = PM

$\Rightarrow \sqrt{(x+8)^2 + (y+2)^2} = \left|\dfrac{2x-y-9}{\sqrt{5}}\right|$

$\Rightarrow 5(x^2 + 16x + 64 + y^2 + 4y + 4)$

$= 4x^2 + y^2 + 81 - 4xy + 18y - 36x$

$\Rightarrow x^2 + 4y^2 + 4xy + 116x + 2y + 259 = 0$

23. Let (x, y) be any point on the required parabola

$\therefore$ by using definition of parabola, we get

$(x-a)^2 + (y-b)^2 = \left(\dfrac{bx + ay - ab}{\sqrt{a^2+b^2}}\right)^2$

$\Rightarrow (a^2 + b^2)(x^2 - 2ax + a^2 + y^2 - 2yb + b^2)$

$= b^2x^2 + a^2y^2 + a^2b^2 + 2abxy - 2a^2yb - 2ab^2x$

$\Rightarrow a^2x^2 - 2a^3x + a^4 + a^2b^2 + b^2y^2 - 2yb^3 + b^4$

$= 2abxy$

$\Rightarrow a^2x^2 - 2abxy + b^2y^2 - 2a^3x - 2b^3y$

$+ a^4 + a^2b^2 + b^4 = 0$

$\Rightarrow (ax - by)^2 - 2a^3x - 2b^3y + a^4 + a^2b^2 + b^4 = 0$

24. By using definition of parabola, we get

SP = PM

$\Rightarrow SP^2 = PM^2$

$\Rightarrow x^2 + y^2 = \left(\dfrac{x+y-4}{\sqrt{2}}\right)^2$

$\Rightarrow 2x^2 + 2y^2 = x^2 + y^2 + 16 + 2xy - 8y - 8x$

$\Rightarrow x^2 + y^2 - 2xy + 8x + 8y - 16 = 0$

25. $(13x-1)^2 + (13y-1)^2 = k(5x - 12y + 1)^2$

$\Rightarrow 169\left[\left(x - \dfrac{1}{13}\right)^2 + \left(y - \dfrac{1}{13}\right)^2\right] = k(5x - 12y + 1)^2$

Taking square root, we get

$\sqrt{\left(x-\dfrac{1}{13}\right)^2 + \left(y - \dfrac{1}{13}\right)^2} = \dfrac{\sqrt{k}(5x - 12y + 1)}{13}$

Now, condition for eccentricity is PS = ePM

$\Rightarrow PS = \sqrt{k}PM$

$\Rightarrow \sqrt{k} = 1 \Rightarrow k = 1$

26. $9x^2 - 6x + 36y + 9 = 0$

$\Rightarrow x^2 - \dfrac{2}{3}x + 4y + 1 = 0$

$\Rightarrow x^2 - \dfrac{2}{3}x + \dfrac{1}{9} + 4y + 1 - \dfrac{1}{9} = 0$

$\Rightarrow \left(x - \dfrac{1}{3}\right)^2 = -4\left(y + \dfrac{2}{9}\right)$.

Hence, the vertex is $\left(\dfrac{1}{3}, -\dfrac{2}{9}\right)$

27. $y^2 + 8x - 12y + 20 = 0$
$\Rightarrow y^2 - 12y = -8x - 20$
$\Rightarrow y^2 - 12y + 36 = -8x - 20 + 36$
$\Rightarrow (y-6)^2 = -8x + 16$
$\Rightarrow (y-6)^2 = -8(x-2)$
$\Rightarrow$ Vertex is (2, 6)

28. The given equation can be written as $(x-4)^2 = 1[y - (c-16)]$
Therefore, the vertex of the parabola is (4, c – 16).
The point lies on X-axis.
$\Rightarrow c - 16 = 0$
$\Rightarrow c = 16$

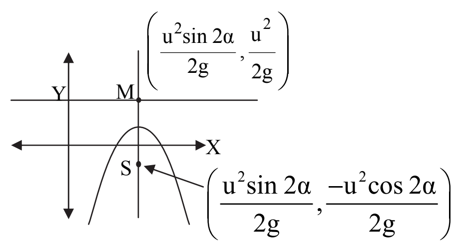
29. Given, equation can be written as
$y^2 = \frac{4k}{4}\left(x - \frac{8}{k}\right)$
The standard equation of parabola is
$y^2 = 4ax$
$\Rightarrow a = \frac{k}{4}$
$\therefore$ Equation of directrix is $X + \frac{k}{4} = 0$
$\Rightarrow x - \frac{8}{k} + \frac{k}{4} = 0$
But the given equation of directrix is $x - 1 = 0$.
Since, both equation are same
$\therefore$ $\frac{8}{k} - \frac{k}{4} = 1$
$\Rightarrow 32 - k^2 = 4k \Rightarrow k = -8, 4$

30. Since, $9y^2 - 16x - 12y - 57 = 0$
$\Rightarrow y^2 - \frac{16}{9}x - \frac{4}{3}y - \frac{57}{9} = 0$
$\Rightarrow y^2 - \frac{4}{3}y + \frac{4}{9} = \frac{16}{9}x + \frac{57}{9} + \frac{4}{9}$
$\Rightarrow \left(y - \frac{2}{3}\right)^2 = \frac{16}{9}\left(x + \frac{61}{16}\right)$
this equation can be written as
$Y^2 = 4\left(\frac{4}{9}\right)X$
Axis of the parabola is Y = 0
$\Rightarrow y - \frac{2}{3} = 0$
$\Rightarrow 3y = 2$

31. Distance between focus and directrix is
$= \left|\frac{3-4-2}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$
Hence, latus rectum is $3\sqrt{2}$
….[Since, latus rectum is two times the distance between focus and directrix].

32. Since, a = distance between tangent at vertex and latus rectum
$\therefore$ $a = \left|\frac{-8-(-12)}{\sqrt{1+1}}\right| = \frac{4}{\sqrt{2}}$
$\therefore$ Length of latus rectum = $4a = 4 \times \frac{4}{\sqrt{2}} = 8\sqrt{2}$

33. $y^2 + 2Ax + 2By + C = 0$
$\Rightarrow y^2 + 2By + B^2 = -2Ax - C + B^2$
$\Rightarrow (y+B)^2 = -2A\left(x + \frac{C}{2A} - \frac{B^2}{2A}\right)$
$\therefore$ focus $\equiv \left(\frac{-C+B^2}{2A} - \frac{A}{2}, -B\right)$
Equation of latus rectum is $x = -a$
$= \frac{-C+B^2}{2A} - \frac{A}{2} = \frac{B^2 - A^2 - C}{2A}$

34.


According to the figure, the length of latus rectum is
$2(SM) = 2 \times \frac{u^2}{2g}(1 + \cos 2\alpha) = \frac{2u^2\cos^2\alpha}{g}$

35. Given $t = -2$
$2y^2 = 7x$
$\Rightarrow y^2 = \frac{7}{2}x$
Comparing with $y^2 = 4ax$, we get
$4a = \frac{7}{2} \Rightarrow a = \frac{7}{8}$
$\therefore$ the point is
$P = (at^2, 2at) = \left(\frac{7}{8} \times 4, 2 \times \frac{7}{8} \times (-2)\right)$
$\therefore$ $P = \left(\frac{7}{2}, \frac{-7}{2}\right)$

36. Here, $\frac{y}{2}$ = t and $x - 2 = t^2$

$\Rightarrow (x-2) = \left(\frac{y}{2}\right)^2 \Rightarrow y^2 = 4(x-2)$

37. $x = 2 + t^2$ and $y = 2t + 1$

$\Rightarrow t^2 = (x-2)$ and $t^2 = \left(\frac{y-1}{2}\right)^2 = \frac{(y-1)^2}{4}$

$\Rightarrow \frac{(y-1)^2}{4} = (x-2) \Rightarrow (y-1)^2 = 4(x-2)$

$\therefore$ Vertex is (2, 1)

38. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since, it passes through (– 3, 1) and (2, – 2)

$\therefore$ $\frac{9}{a^2} + \frac{1}{b^2} = 1$ and $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{4}$

$\Rightarrow a^2 = \frac{32}{3}, b^2 = \frac{32}{5}$

Hence, required equation of ellipse is $3x^2 + 5y^2 = 32$

39. Let point P (x_1, y_1)

$\therefore$ $\sqrt{(x_1+2)^2 + y_1^2} = \frac{2}{3}\left(x_1 + \frac{9}{2}\right)$

$\Rightarrow (x_1+2)^2 + y_1^2 = \frac{4}{9}\left(x_1 + \frac{9}{2}\right)^2$

$\Rightarrow 9(x_1^2 + y_1^2 + 4x_1 + 4) = 4\left(x_1^2 + \frac{81}{4} + 9x_1\right)$

$\Rightarrow 5x_1^2 + 9y_1^2 = 45 \Rightarrow \frac{x_1^2}{9} + \frac{y_1^2}{5} = 1$

$\therefore$ Locus of (x_1, y_1) is $\frac{x^2}{9} + \frac{y^2}{5} = 1$, which is equation of an ellipse.

40. Foci = (3, – 3)

$\Rightarrow$ ae = 3 – 2 = 1

Vertex = (4, – 3)

$\Rightarrow$ a = 4 – 2 = 2

$\Rightarrow e = \frac{1}{2}$

$\Rightarrow b = 2\sqrt{\left(1 - \frac{1}{4}\right)} = \frac{2}{2}\sqrt{3} = \sqrt{3}$

Therefore, equation of ellipse with centre (2, – 3) is

$\frac{(x-2)^2}{4} + \frac{(y+3)^2}{3} = 1$

41. Vertices (± 5, 0) ≡ (± a, 0)

$\Rightarrow$ a = 5

Foci (± 4, 0) ≡ (± ae, 0)

$\Rightarrow e = \frac{4}{5}$

$\therefore$ $b = \sqrt{a^2(1-e^2)} = \sqrt{25 \times \frac{9}{25}} = 3$

Hence, equation is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

i.e., $9x^2 + 25y^2 = 225$

42. Foci (± 5, 0) ≡ (± ae, 0),

$\Rightarrow$ ae = 5 ….(i)

Equation of directrix is $x = \frac{a}{e}$

Given, $x = \frac{36}{5}$

$\Rightarrow \frac{a}{e} = \frac{36}{5}$ ….(ii)

$\Rightarrow$ a = 6 and $e = \frac{5}{6}$ ….[From (i) and (ii)]

$\therefore$ $b = a\sqrt{1-e^2} = 6\sqrt{1 - \frac{25}{36}} = \sqrt{11}$

Hence, equation is $\frac{x^2}{36} + \frac{y^2}{11} = 1$

43. Vertex (0,7) and directrix $y = 12$

$\therefore$ b = 7 and $\frac{b}{e} = 12$

$\Rightarrow e = \frac{7}{12}$

Also, $a = b\sqrt{1-e^2}$

$\Rightarrow a = 7\sqrt{\frac{95}{144}}$

$\Rightarrow a^2 = \frac{4655}{144}$

Hence, equation of ellipse is

$\frac{x^2}{4655/144} + \frac{y^2}{49} = 1$ i.e, $144x^2 + 95y^2 = 4655$

44. Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\because$ It passes through (– 3, 1)

$\therefore$ $\frac{9}{a^2} + \frac{1}{b^2} = 1$

$\Rightarrow 9 + \frac{a^2}{b^2} = a^2$ .....(i)

Given, eccentricity is $\sqrt{\frac{2}{5}}$

$\therefore \quad \frac{2}{5} = 1 - \frac{b^2}{a^2}$

$\Rightarrow \frac{b^2}{a^2} = \frac{3}{5}$ .....(ii)

From equation (i) and (ii), we get

$a^2 = \frac{32}{3}, b^2 = \frac{32}{5}$

Hence, required equation of ellipse is $3x^2 + 5y^2 = 32$

45. Given, $\frac{2b^2}{a} = 10 \Rightarrow b^2 = 5a$ and $2b = 2ae$

$\Rightarrow \frac{b}{a} = e$

Also, $b^2 = a^2(1 - e^2)$

$\Rightarrow e^2 = (1 - e^2)$ ....$\left[\because \ e = \frac{b}{a}\right]$

$\Rightarrow e = \frac{1}{\sqrt{2}}$

$\Rightarrow b = \frac{a}{\sqrt{2}}$ or $b = 5\sqrt{2}$, $a = 10$

Hence, equation of ellipse is

$\frac{x^2}{(10)^2} + \frac{y^2}{(5\sqrt{2})^2} = 1$

i.e., $x^2 + 2y^2 = 100$

46. We have, $2ae = 8$, $\frac{2a}{e} = 18$

$\Rightarrow ae \times \frac{a}{e} = 4 \times 9$

$\Rightarrow a = \sqrt{4 \times 9} = 6$ and $e = \frac{2}{3}$

Also, $b = a\sqrt{1 - e^2}$

$\Rightarrow b = 6\sqrt{1 - \frac{4}{9}} = \frac{6}{3}\sqrt{5} = 2\sqrt{5}$

Hence, the required equation is $\frac{x^2}{36} + \frac{y^2}{20} = 1$

i.e., $5x^2 + 9y^2 = 180$

47. $\frac{x^2}{2 - r} + \frac{y^2}{r - 5} + 1 = 0$

$\Rightarrow \frac{x^2}{r - 2} + \frac{y^2}{5 - r} = 1$

Hence, $r > 2$ and $r < 5$

$\Rightarrow 2 < r < 5$

48. $16x^2 + 25y^2 = 400$

$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$

$\Rightarrow e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

Therefore, directrices are $x = \pm \frac{5}{\frac{3}{5}}$ or $3x = \pm 25$

49. Given that, $\frac{a}{e} - ae = 8$ and $e = \frac{1}{2}$

$\Rightarrow a = \frac{8e}{\left(1 - e^2\right)}$

$= \frac{8.4}{2(3)} = \frac{16}{3}$

$\therefore \quad b = \frac{16}{3}\sqrt{\left(1 - \frac{1}{4}\right)}$

$= \frac{16}{3}\frac{\sqrt{3}}{2} = \frac{8\sqrt{3}}{3}$

Hence, the length of minor axis is $\frac{16\sqrt{3}}{3}$

50. $3x^2 + 4y^2 = 48$

$\Rightarrow \frac{x^2}{16} + \frac{y^2}{12} = 1$

$\therefore \quad a^2 = 16, b^2 = 12$

$\Rightarrow e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{12}{16}} = \frac{1}{2}$

$\therefore$ Distance between the foci $= 2ae$

$= 2 \times 4 \times \frac{1}{2}$

$= 4$

51. We have, $\frac{2b^2}{a} = 2ae$

$\Rightarrow b^2 = a^2e$

$\Rightarrow e = \frac{b^2}{a^2}$ ....(i)

Also, $e = \sqrt{1 - \frac{b^2}{a^2}}$

$\Rightarrow e^2 = 1 - e$ ....[From (i)]

$\Rightarrow e^2 + e - 1 = 0$

$\therefore \quad e = \frac{-1 \pm \sqrt{5}}{2}$

$\therefore \quad e = \frac{\sqrt{5} - 1}{2}$

52. $a = 6, b = 2\sqrt{5}$

Now, $b^2 = a^2(1 - e^2)$

$\Rightarrow \dfrac{20}{36} = (1 - e^2)$

$\Rightarrow e = \sqrt{\dfrac{16}{36}} = \dfrac{2}{3}$

Distance between directrix $= \dfrac{2a}{e} = \dfrac{2\times 6}{\frac{2}{3}} = 18$

53. Comparing $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we get

$a^2 = 16, b^2 = 9$

$\Rightarrow a = 4, b = 3$

Now, $b^2 = a^2(1 - e^2)$

$\Rightarrow \dfrac{9}{16} = 1 - e^2$

$\Rightarrow e^2 = 1 - \dfrac{9}{16} = \dfrac{7}{16}$

$\therefore$ $e = \dfrac{\sqrt{7}}{4}$

$\therefore$ $S = (ae, 0) \equiv (\sqrt{7}, 0)$

$\therefore$ radius of the circle $= \sqrt{\left(0-\sqrt{7}\right)^2 + \left(3-0\right)^2}$

$= \sqrt{16} = 4$

54. $x = 3(\cos t + \sin t)$, $y = 4(\cos t - \sin t)$

$\Rightarrow \dfrac{x}{3} = \cos t + \sin t$, $\dfrac{y}{4} = \cos t - \sin t$

$\Rightarrow \dfrac{x^2}{9} = 1 + \sin 2t$, $\dfrac{y^2}{16} = 1 - \sin 2t$

$\Rightarrow \dfrac{x^2}{9} + \dfrac{y^2}{16} = 2$

$\Rightarrow \dfrac{x^2}{18} + \dfrac{y^2}{32} = 1$ which is an ellipse.

55. $9x^2 + 4y^2 - 6x + 4y + 1 = 0$

$\Rightarrow 9\left(x^2 - \dfrac{2}{3}x + \dfrac{1}{9}\right) + 4\left(y^2 + y + \dfrac{1}{4}\right) = -1 + 1 + 1$

$\Rightarrow 9\left(x - \dfrac{1}{3}\right)^2 + 4\left(y + \dfrac{1}{2}\right)^2 = 1$

$\Rightarrow \dfrac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}} + \dfrac{\left(y+\frac{1}{2}\right)^2}{\frac{1}{4}} = 1$

Here, $a = \dfrac{1}{3}, b = \dfrac{1}{2}$

$\Rightarrow 2a = \dfrac{2}{3}, 2b = 1$

$\therefore$ Length of axes are $1, \dfrac{2}{3}$.

56. $3x^2 - 12x + 4y^2 - 8y = -4$

$\Rightarrow 3(x^2 - 4x + 4) + 4(y^2 - 2y + 1) = -4 + 12 + 4$

$\Rightarrow 3(x - 2)^2 + 4(y - 1)^2 = 12$

$\Rightarrow \dfrac{(x-2)^2}{4} + \dfrac{(y-1)^2}{3} = 1$

$\Rightarrow \dfrac{X^2}{4} + \dfrac{Y^2}{3} = 1$

$\therefore$ $e = \sqrt{1 - \dfrac{3}{4}} = \dfrac{1}{2}$

Foci are $x = \pm ae, y = 0$

$\Rightarrow x - 2 = \pm 1, y - 1 = 0$

$\Rightarrow x = 3$ or $1, y = 1$

i.e., (3, 1) and (1, 1)

57. $4(x^2 + 2x + 1) + 9(y^2 + 4y + 4) = 36$

$\Rightarrow 4(x + 1)^2 + 9(y + 2)^2 = 36$

$\Rightarrow \dfrac{(x+1)^2}{9} + \dfrac{(y+2)^2}{4} = 1$

Comparing with standard form, we get $a^2 = 9, b^2 = 4$

Now, condition for eccentricity is $b^2 = a^2(1 - e^2)$

$\Rightarrow 4 = 9(1 - e^2)$

$\Rightarrow e^2 = 1 - \dfrac{4}{9} = \dfrac{5}{9}$

$\Rightarrow e = \dfrac{\sqrt{5}}{3}$

58. Given, $2a = 6, 2b = 4$

i.e., $a = 3, b = 2$

Also, $e^2 = 1 - \dfrac{b^2}{a^2} = \dfrac{5}{9}$

$\Rightarrow e = \dfrac{\sqrt{5}}{3}$

$\therefore$ Distance between the pins $= 2ae = 2\sqrt{5}$ cm and length of string $= 2a + 2ae = 6 + 2\sqrt{5}$ cm

59. $2a = 7$ or $a = \frac{7}{2}$

Also (5, –2) satisfies $\frac{4}{49}(25) - \frac{51}{196}(4) = 1$

and $a^2 = \frac{49}{4}$

$\Rightarrow a = \frac{7}{2}$

∴ option (C) is correct answer

60. Centre (0, 0), vertex (4, 0) $\Rightarrow a = 4$ and focus (6, 0)

$\Rightarrow ae = 6$

$\Rightarrow e = \frac{3}{2}$

Also, $b^2 = a^2(e^2 - 1)$

$= 20$

Hence, required equation is $\frac{x^2}{16} - \frac{y^2}{20} = 1$

i.e., $5x^2 - 4y^2 = 80$

61. Given that, $e = \frac{3}{2}$

foci = $(\pm 2, 0) = (\pm ae, 0)$

$\Rightarrow ae = 2$

$\Rightarrow a = \frac{4}{3}$

$\Rightarrow a^2 = \frac{16}{9}$

Now, condition for eccentricity is

$b^2 = a^2(e^2 - 1)$

∴ $b^2 = \frac{16}{9}\left(\frac{9}{4} - 1\right) = \frac{16}{9}\left(\frac{5}{4}\right) = \frac{20}{9}$

Now, equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{9x^2}{16} - \frac{9y^2}{20} = 1 \Rightarrow \frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$

62. Given, $\frac{2b^2}{a} = 8$ and $\frac{3}{\sqrt{5}} = \sqrt{1 + \frac{b^2}{a^2}}$

$\Rightarrow \frac{4}{5} = \frac{b^2}{a^2}$

$\Rightarrow a = 5, b = 2\sqrt{5}$

Hence, the required equation of hyperbola is $\frac{x^2}{25} - \frac{y^2}{20} = 1$

i.e., $4x^2 - 5y^2 = 100$

63. Conjugate axis is 5 and distance between foci = 13

$\Rightarrow 2b = 5$ and $2ae = 13$

Also, $b^2 = a^2(e^2 - 1)$

$\Rightarrow \frac{25}{4} = \frac{(13)^2}{4e^2}(e^2 - 1)$

$\Rightarrow \frac{25}{4} = \frac{169}{4} - \frac{169}{4e^2}$

$\Rightarrow e = \frac{13}{12}$

$\Rightarrow a = 6, b = \frac{5}{2}$

Hence, the required equation of hyperbola is $\frac{x^2}{36} - \frac{y^2}{\frac{25}{4}} = 1$

i.e., $25x^2 - 144y^2 = 900$

64. Equation of hyperbola passes through (x_1, y_1)

$\Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \Rightarrow \frac{x_1^2}{a^2} - 1 = \frac{y_1^2}{b^2}$

$\Rightarrow \frac{x_1^2 - a^2}{a^2} = \frac{y_1^2}{b^2} \Rightarrow \frac{b^2}{a^2} = \frac{y_1^2}{x_1^2 - a^2}$

Now, $\frac{b^2}{a^2} = e^2 - 1$

$\Rightarrow \frac{y_1^2}{x_1^2 - a^2} = e^2 - 1$

$\Rightarrow e^2 = \frac{y_1^2 + (x_1^2 - a^2)}{x_1^2 - a^2}$

$\Rightarrow e = \sqrt{\frac{x_1^2 - a^2 + y_1^2}{x_1^2 - a^2}} = \sqrt{\frac{a^2 - x_1^2 - y_1^2}{a^2 - x_1^2}}$

65. Center of the hyperbola is midpoint of foci.

Hence, its center is (1, 5) also distance between foci is $2ae = 10$

$\Rightarrow a = 4$ ....$\left[\because \ e = \frac{5}{4}\right]$

$\Rightarrow a^2 = 16$

Now, $b^2 = a^2(e^2 - 1)$

$= a^2e^2 - a^2 = 25 - 16 \Rightarrow b^2 = 9$

Hence, equation of hyperbola is

$\frac{(x-1)^2}{16} - \frac{(y-5)^2}{9} = 1$

66. Let S(1, 1) be the focus and P(x, y) be a point on the hyperbola
Now, PS = ePM

$$\Rightarrow \sqrt{(x-1)^2+(y-1)^2} = \sqrt{3}\left|\frac{2x+y-1}{\sqrt{2^2+1^2}}\right|$$

Squaring both sides, we get

$$(x-1)^2 + (y-1)^2 = \frac{3}{5}(2x+y-1)^2$$

On simplification, the required equation is
$7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$

67. For ellipse, e < 1 and also e′ < 1
$\therefore$ $e^2 + e'^2 < 2$
For parabola, e = 1 and e′ = 1
$\therefore$ $e^2 + e'^2 = 2$
For hyperbola, e > 1 and e′ > 1
$\therefore$ $e^2 + e'^2 > 2$
Hence, it can be 3 in case of hyperbola.

68. $96x^2 - 16y^2 - 36x + 96y - 252 = 0$

$$\Rightarrow \frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

$$\Rightarrow \frac{X^2}{16} - \frac{Y^2}{9} = 1$$

$\therefore$ Vertices are (X = ± a, Y = 0)
i.e., ($x - 2 = \pm 4$, $y - 3 = 0$)
$\therefore$ The vertices of the hyperbola are (6, 3) and (– 2, 3)

69. $(x + 1)^2 - y^2 - 1 + 5 = 0$

$$\Rightarrow -\frac{(x+1)^2}{4} + \frac{y^2}{4} = 1$$

Equation of directrices of $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are $y = \pm\frac{b}{e}$

Here, b = 2, e = $\sqrt{1+1} = \sqrt{2}$

Hence, $y = \pm\frac{2}{\sqrt{2}}$

$\Rightarrow y = \pm\sqrt{2}$

70. Given, equation of hyperbola is
$9x^2 - 16y^2 + 72x - 32y - 16 = 0$
$\Rightarrow 9(x^2 + 8x) - 16(y^2 + 2y) - 16 = 0$
$\Rightarrow 9(x + 4)^2 - 16(y + 1)^2 = 144$

$$\Rightarrow \frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$$

$\therefore$ Latus rectum = $\frac{2b^2}{a} = 2 \times \frac{9}{4} = \frac{9}{2}$

71. Squaring and subtracting, we get
$a^2x^2 - b^2y^2 = a^2 - b^2$, which is the equation of hyperbola.

72. $2x = t + \frac{1}{t}$ and $2y = t - \frac{1}{t}$

$\Rightarrow 4x^2 = t^2 + 2 + \frac{1}{t^2}$ ....(i)

and $4y^2 = t^2 - 2 + \frac{1}{t^2}$ ....(ii)

Subtracting (ii) from (i), we get
$4x^2 - 4y^2 = 4 \Rightarrow x^2 - y^2 = 1$
The equation is of hyperbola.

73. Multiplying both, we get
$(bx)^2 - (ay)^2 = (ab)^2$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which is the standard equation of hyperbola.

74. $e^2 = 1 - \frac{b^2}{a^2} = \frac{4}{9}$

$\Rightarrow e = \frac{2}{3}$

and $e'^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{\frac{45}{4}}{9} = \frac{9}{4}$

$\Rightarrow e' = \frac{3}{2}$

$\therefore$ ee′ = 1

75. Centres of the circles are (0, 0), (– 3, 1) and (6, – 2) respectively. Line passing through any two points say (0, 0) and (– 3, 1) is $y = -\frac{1}{3}x$ point (6, –2) lies on it.
Hence, points are collinear.

Competitive Thinking

1. Radius = Distance from origin = $\sqrt{\alpha^2+\beta^2}$
$\therefore$ $(x - \alpha)^2 + (y - \beta)^2 = \alpha^2 + \beta^2$
$\Rightarrow x^2 + y^2 - 2\alpha x - 2\beta y = 0$

2. Centre (2, 2) and
r = $\sqrt{(4-2)^2+(5-2)^2}$
= $\sqrt{13}$
Hence, required equation is
$(x - 2)^2 + (y - 2)^2 = \left(\sqrt{13}\right)^2$
$\Rightarrow x^2 + y^2 - 4x - 4y - 5 = 0$

3. Let r be the radius of the circle.
Given, circumference = 10π

$\therefore$ $2\pi r = 10\pi$

$\Rightarrow r = 5$

$\therefore$ the equation of the circle is

$(x-2)^2 + (y+3)^2 = 5^2$

$\Rightarrow x^2 + y^2 - 4x + 6y - 12 = 0$

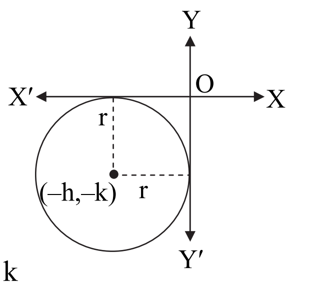
4. The centre of the circle which touches each axis in first quadrant at a distance 5, will be (5, 5) and radius will be 5.

$\therefore$ equation of the circle is

$(x-5)^2 + (y-5)^2 = (5)^2$

$\Rightarrow x^2 + y^2 - 10x - 10y + 25 = 0$

5. Since, circle touches the co-ordinate axes in III quadrant.

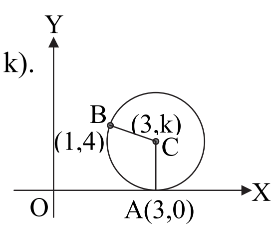


$\therefore$ Radius = – h = – k

Hence, h = k = – 5

$\therefore$ Equation of circle is $(x+5)^2 + (y+5)^2 = 25$

6. Since, the circle touches X-axis at (3, 0).



$\therefore$ centre of the circle is (3, k).

Now, $CA^2 = CB^2$

$\therefore$ $(3-3)^2 + (k-0)^2$
$= (3-1)^2 + (k-4)^2$

$\therefore$ $k^2 = 4 + k^2 - 8k + 16$

$\therefore$ $k = \dfrac{5}{2}$

$\therefore$ the required equation of circle is

$$(x-3)^2 + \left(y - \frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2$$

$\Rightarrow x^2 + y^2 - 6x - 5y + 9 = 0$

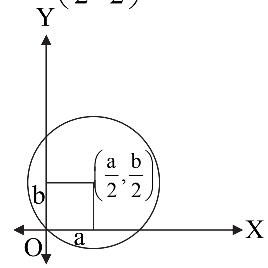
7. Since, the circle has centre at (1, 2) and line $x = y$ i.e. $x - y = 0$ as tangent,

$\therefore$ Radius of circle $= \left|\dfrac{(1)-(2)}{\sqrt{(1)^2 + (-1)^2}}\right|$

$= \dfrac{1}{\sqrt{2}}$

$\therefore$ Area of circle $= \pi \times \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{\pi}{2}$

8. Centre is $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ and radius $= \sqrt{\dfrac{a^2+b^2}{4}}$



Hence, equation of circle is

$x^2 + y^2 - ax - by = 0$

9. diameter $= \sqrt{[4-(-2)]^2 + [7-(-1)]^2}$

$= \sqrt{6^2 + 8^2}$

$= 10$

Centre $\equiv \left(\dfrac{4+(-2)}{2}, \dfrac{7+(-1)}{2}\right) \equiv (1, 3)$

$\therefore$ equation of the circle is

$(x-1)^2 + (y-3)^2 = (5)^2$

If this circle cuts X-axis, then

$(x-1)^2 + (0-3)^2 = 25$

$\Rightarrow (x-1)^2 = 16$

$\Rightarrow (x-1) = \pm 4$

$\Rightarrow x = 5, -3$

$\therefore$ the points on X-axis are A(5, 0) and B(–3, 0)

$\therefore$ AB $= \sqrt{(5+3)^2 + 0^2} = 8$

10. Since, the centre always lies on the diameter.
Solving $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$, the co-ordinates of the centre are (1, –1).
Given, circumference = 10π

$\therefore$ $2\pi r = 10\pi \Rightarrow r = 5$

$\therefore$ the equation of the circle is

$(x-1)^2 + (y+1)^2 = 5^2$

$\Rightarrow x^2 + y^2 - 2x + 2y - 23 = 0$

11. Centre of circle = Point of intersection of diameters = (1, – 1)
Now, area = 154

$\Rightarrow \pi r^2 = 154 \Rightarrow r = 7$

Hence, the equation of required circle is

$(x-1)^2 + (y+1)^2 = 7^2$

$\Rightarrow x^2 + y^2 - 2x + 2y = 47$

12. $3x - y - 4 = 0$ ….(i)

$x + 3y + 2 = 0$ ….(ii)

solving equation (i) and (ii), we get

$(x, y) = (1, -1)$

Since, $\pi r^2 = 154$

$\therefore$ r = 7

$\therefore$ Equation of required circle is
$(x-1)^2 + (y+1)^2 = 49$
$\Rightarrow x^2 + y^2 - 2x + 2y - 47 = 0$

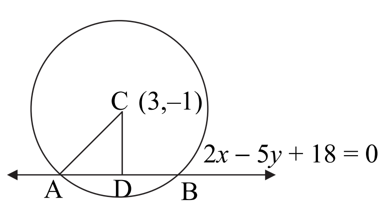
13. Since, the centre always lies on the diameter.
Solving $2x + 3y = 3$ and $16x - y = 4$, we get
co-ordinates of the centre $= \left(\frac{3}{10}, \frac{4}{5}\right)$.
The circle passes through (4, 6).

$\therefore$ $r^2 = \left(4-\frac{3}{10}\right)^2 + \left(6-\frac{4}{5}\right)^2$
$= \left(\frac{37}{10}\right)^2 + \left(\frac{26}{5}\right)^2 = \frac{4073}{100}$

$\therefore$ the equation of the circle is
$\left(x-\frac{3}{10}\right)^2 + \left(y-\frac{4}{5}\right)^2 = \frac{4073}{100}$
$\Rightarrow 100x^2 + 100y^2 - 60x - 160y = 4000$
$\Rightarrow 5(x^2 + y^2) - 3x - 8y = 200$

14. Let centre be (h, k). Then,
$\sqrt{(h-2)^2 + (k-3)^2} = \sqrt{(h-4)^2 + (k-5)^2}$
$\Rightarrow -4h + 4 - 6k + 9 = -8h + 16 - 10k + 25$
$\Rightarrow 4h + 4k - 28 = 0$
$\Rightarrow h + k - 7 = 0$ ….(i)
Since, centre lies on the given line.
$\therefore$ $k - 4h + 3 = 0$ ….(ii)
Solving (i) and (ii), we get
(h, k) = (2, 5)
$\therefore$ centre is (2, 5) and
radius $= \sqrt{(2-2)^2 + (5-3)^2} = 2$
$\therefore$ the required equation of the circle is
$(x-2)^2 + (y-5)^2 = (2)^2$
$\Rightarrow x^2 + y^2 - 4x - 10y + 25 = 0$

15.



Let AB be the chord cut off by the circle on the line $2x - 5y + 18 = 0$.
Let CD be the perpendicular drawn from centre (3, –1) to AB.

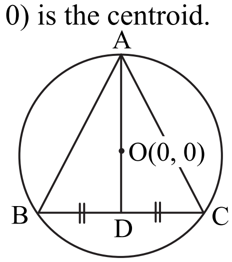
$\therefore$ $CD = \left|\frac{2(3) - 5(-1) + 18}{\sqrt{2^2 + (-5)^2}}\right| = \sqrt{29}$

and AD = 3

$\therefore$ $CA^2 = AD^2 + CD^2 = 3^2 + \left(\sqrt{29}\right)^2 = 38$

$\therefore$ the equation of the circle is
$(x-3)^2 + (y+1)^2 = 38$.

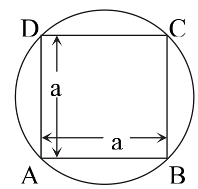
16. ΔABC is equilateral.
$\therefore$ O(0, 0) is the centroid.



O divides AD in the ratio 2 : 1

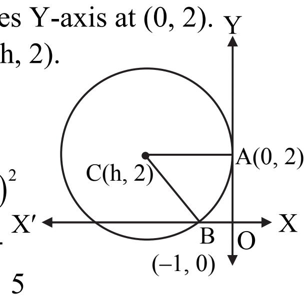
$\therefore$ $\frac{AO}{OD} = \frac{2}{1}$
$\Rightarrow \frac{AO}{AD - AO} = \frac{2}{1} \Rightarrow \frac{AO}{9 - AO} = \frac{2}{1}$
$\Rightarrow AO = 18 - 2\,AO \Rightarrow AO = 6$ units
$\therefore$ radius = 6 units
$\therefore$ equation of circle is $x^2 + y^2 = 36$.

17. According to the figure, A(0, 0), B(a,0), C(a, a) and D(0, a).
and centre is $\left(\frac{a}{2}, \frac{a}{2}\right)$.



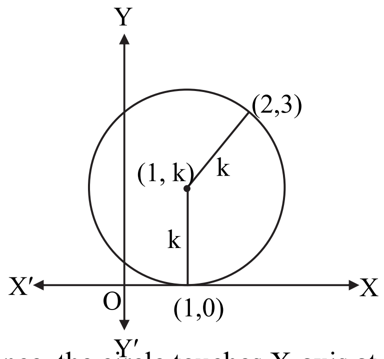
$\therefore$ the equation of the circle is
$\left(x-\frac{a}{2}\right)^2 + \left(y-\frac{a}{2}\right)^2 = \frac{a^2}{2}$
$\Rightarrow x^2 + y^2 - ax - ay = 0$

18. Since, the circle touches Y-axis at (0, 2).
$\therefore$ centre of the circle is (h, 2).
Now, $CA^2 = CB^2$



$\therefore$ $(h-0)^2 + (2-2)^2$
$= (h-(-1))^2 + (2-0)^2$
$\Rightarrow h^2 = h^2 + 2h + 1 + 4$
$\Rightarrow 2h + 5 = 0 \Rightarrow h = -\frac{5}{2}$
$\therefore$ equation of circle is
$\left(x+\frac{5}{2}\right)^2 + (y-2)^2 = \left(-\frac{5}{2}\right)^2$
$\Rightarrow x^2 + \frac{25}{4} + 5x + y^2 - 4y + 4 = \frac{25}{4}$
$\Rightarrow x^2 + y^2 + 5x - 4y + 4 = 0$
Point (–4, 0) satisfies this equation.
$\therefore$ option (D) is the correct answer.

19.



Since, the circle touches X-axis at (1, 0).

∴ centre of the circle is (1, k) and radius = k

∴ equation of the circle is $(2-1)^2 + (3-k)^2 = k^2$

$\Rightarrow 1 + k^2 - 6k + 9 = k^2$

$\Rightarrow k = \dfrac{5}{3}$

∴ diameter = $2k = \dfrac{10}{3}$

20. The equation of circle touching the coordinate axes with centre (a, a) and radius 'a' is

$x^2 + y^2 - 2ax - 2ay + a^2 = 0$ ...(i)

Since, line $3x - 4y - 12 = 0$ touches the circle

∴ perpendicular distance from centre of the circle to the line = radius

∴ $\left|\dfrac{3(a)-4(a)-12}{\sqrt{9+16}}\right| = a$

$\Rightarrow a = 3$

Substituting, a = 3 in equation (i), we get

$x^2 + y^2 - 6x - 6y + 9 = 0$

This is the required equation of the circle.

21. The given equation represents a circle having line segment joining (x_1, y_1) and (x_2, y_2) as a diameter.

∴ the coordinates of its centre are $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$.

22. By diameter form,

the required equation is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

∴ $(x+4)(x-12)+(y-3)(y+1)=0$

∴ $x^2+y^2-8x-2y-51=0$

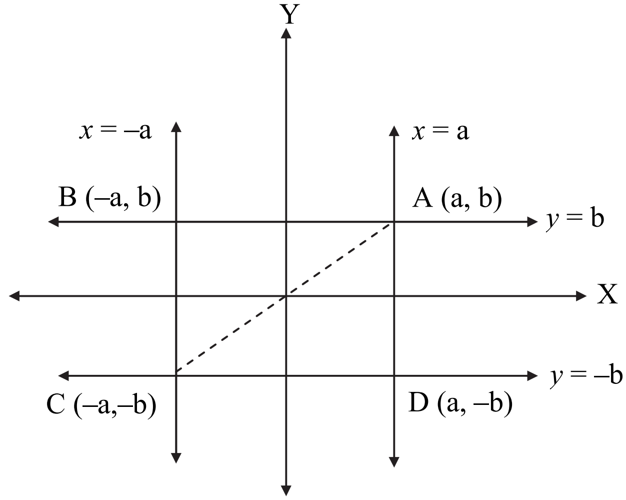
23. Circle whose diametric end points are (1, 0) and (0, 1) will be of smallest radius.

∴ By using diameter form, equation of circle is

$(x-1)(x-0)+(y-0)(y-1)=0$

$\Rightarrow x^2+y^2-x-y=0$

24.



Here, the diagonals AC and BD of rectangle ABCD are diameters of the circle passing through the vertices A, B, C and D.

Considering diagonal AC with end points A(a, b) and C (–a, –b), we get

Equation of circle in diameter form as,

$(x-a)(x-(-a)) + (y-b)(y-(-b)) = 0$

∴ $x^2 - a^2 - y^2 - b^2 = 0$

∴ $x^2 + y^2 = a^2 + b^2$

26. The given equation represents a circle, if coeff. of x^2 = coeff. of y^2 and coeff. of $xy = 0$

∴ a = 2 and b = 0

Also, it passes through origin.

∴ c = 0

27. Here, g = 2, f = 3 and c = 13

∴ $r = \sqrt{g^2+f^2-c}$

∴ $r = \sqrt{4+9-13} = 0$

∴ option (D) is the correct answer.

28. Consider option (A),

$x^2 + y^2 + 2ax + 2by + 2b^2 = 0$

Centre = (–a, –b)

∴ option (A) is the correct answer.

29. The given equation represents a circle, if coeff. of x^2 = coeff. of y^2

After solving the given equation , we get

$\dfrac{K}{3} = \dfrac{1}{4} \Rightarrow K = \dfrac{3}{4}$

30. The given equation represents a circle, if coeff. of $xy = 0$.

∴ $2k - 1 = 0 \Rightarrow k = \dfrac{1}{2}$

radius = $\sqrt{(-1)^2+(2)^2-3} = \sqrt{2}$

31. The given equation represents a circle, if coeff. of $xy = 0$.

$\therefore$ h = 0

and $\sqrt{(-3)^2 + (-1)^2 - k} = 2$

$\Rightarrow 10 - k = 4$

$\Rightarrow k = 6$

32. Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is concentric with $x^2 + y^2 - 2x + 4y + 20 = 0$.

$\therefore$ centre is (1, –2) and

radius = $\sqrt{(4-1)^2 + (-2+2)^2} = \sqrt{3^2 + 0^2} = 3$

Also, r = $\sqrt{g^2 + f^2 - c}$

$\therefore$ $3 = \sqrt{(-1)^2 + (2)^2 - c}$

$\therefore$ $9 = 1 + 4 - c$

$\therefore$ $c = -4$

33. Here, $g = \frac{-1}{4}$, f = 0 and c = 0

$\therefore$ centre $(-g, -f) = \left(\frac{1}{4}, 0\right)$

and $r = \sqrt{\frac{1}{16} + 0 - 0} = \frac{1}{4}$

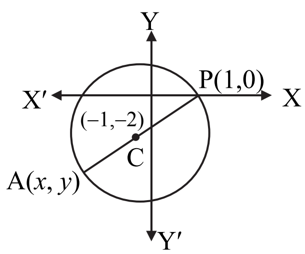
34. Let another end of the diameter be (x, y).
Centre of the given circle is (2, 3).
Since, centre is the midpoint of the diameter

$\therefore$ $2 = \frac{3 + x}{2}, 3 = \frac{4 + y}{2}$

$\Rightarrow x = 1, y = 2$

$\Rightarrow (x, y) = (1, 2)$

35. Let $A(x, y)$ be the required point.



given equation of circle is

$x^2 + y^2 + 2x + 4y - 3 = 0$

$\therefore$ Centre = (–1, –2)

Since, C is the midpoint of AP.

$\therefore$ A= (–3, –4)

36. Here, the centre of circle $(3, -1)$ must lie on the line $x + 2by + 7 = 0$.

$\therefore$ $3 - 2b + 7 = 0$

$\Rightarrow b = 5$

37. Given equation of circle is

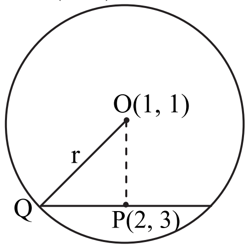
$x^2 + y^2 - 4x - 6y + 9 = 0$

$\Rightarrow x^2 - 4x + 4 + y^2 - 6y + 9 - 4 = 0$

$\Rightarrow (x - 2)^2 + (y - 3)^2 = 4$

$\therefore$ centre = (2, 3), radius = 2

The diameter of this circle is a chord of circle with centre O(1, 1).

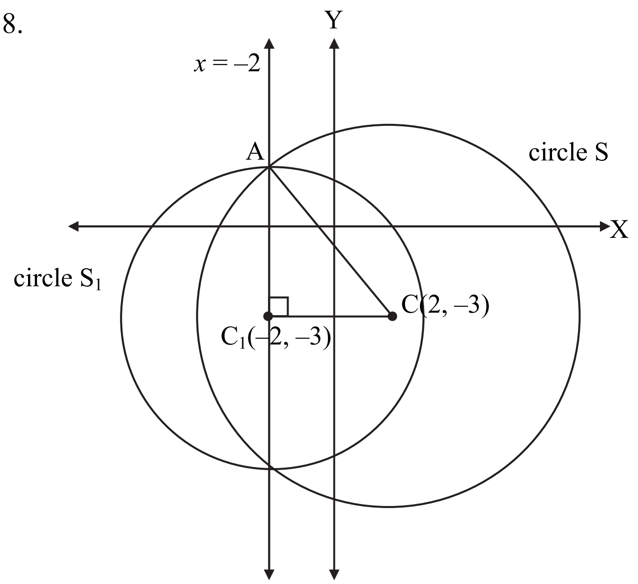


OP = $\sqrt{(3-1)^2 + (2-1)^2} = \sqrt{5}$

QP = 2

$\therefore$ $r^2 = \left(\sqrt{5}\right)^2 + 2^2 \Rightarrow r = 3$

38.



Given circle S_1, $x^2 + y^2 + 4x + 6y - 12 = 0$

$\therefore$ Its centre $C_1 = (-2, -3)$

radius, $r_1 = 5$

centre of circle S, is C = (2, –3)

From the figure, we have

Diameter, $x = -2$ of circle S_1 is the chord of circle S.

$\therefore$ In ΔCC_1A,

$C_1A = r = 5$ unit

$CC_1 = \sqrt{\left(-2-2\right)^2 + \left(3-3\right)^2} = 4$ unit

$\therefore$ $(CA)^2 = (CC_1)^2 + (C_1A)^2$

$= (4)^2 + (5)^2 = 16 + 25 = 41$

$\therefore$ CA = $\sqrt{41}$ unit

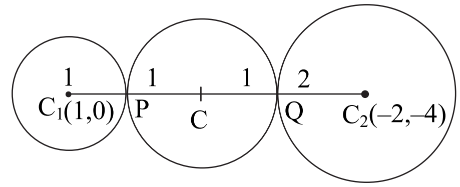
$\therefore$ Radius of circle S is $\sqrt{41}$ unit.

39. Consider option (A)
$x^2 + y^2 + 8x + 2y - 8 = 0$
Point (–1, 3) is common to both circle and lies on above circle also
Since, point (–1, 3) satisfies the equation of circle in option (A)
$\therefore$ correct answer is option (A)

40. $C_1 : x^2 + y^2 - 6x = 0$ ....(i)
$C_2 : x^2 + y^2 - 6y = 0$ ....(ii)
Solving (i) and (ii), we get
$x = y$ ....(iii)
Substituting (iii) in (i), we get
$y = 3$
$\therefore$ $x = 3$
Point on circle is P(3, 3) and
centre = $\left(\frac{3}{2}, \frac{3}{2}\right)$
$\therefore$ Radius $= \sqrt{\left(3-\frac{3}{2}\right)^2 + \left(3-\frac{3}{2}\right)^2}$
$= \frac{3}{\sqrt{2}}$
$\therefore$ equation of the circle is
$\left(x-\frac{3}{2}\right)^2 + \left(y-\frac{3}{2}\right)^2 = \frac{9}{2}$
$\Rightarrow x^2 + y^2 - 3x - 3y = 0$

41. Putting $y = x$ in $x^2 + y^2 - 2x = 0$, we get
$2x^2 - 2x = 0$
$\Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$
$\therefore$ $y = 0$ or $y = 1$
$\therefore$ Points of intersection are (0, 0) and (1, 1).
These are end points of a diameter of required circle.
$\therefore$ equation of required circle is
$(x-0)(x-1) + (y-0)(y-1) = 0$
$\Rightarrow x^2 + y^2 - x - y = 0$

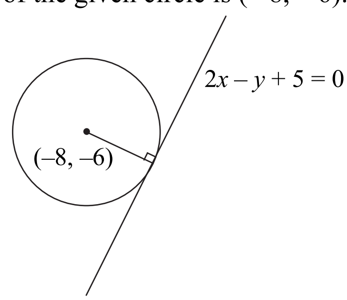
42. The centres of two circles are $C_1(1, 0)$ and $C_2(-2, -4)$ and their radii are 1 and 2 units respectively.
Let C be the centre of the required circle. Then, CP = CQ = 1.
$\therefore$ $CC_1 = 2$ and $CC_2 = 3$.
Clearly, C divides $C_1 C_2$ in the ratio 2 : 3. Therefore, coordinates of C are
$\left(\frac{-4+3}{2+3}, \frac{-8+0}{2+3}\right) = \left(-\frac{1}{5}, -\frac{8}{5}\right)$.



Hence, equation of the required circle is
$\left(x+\frac{1}{5}\right)^2 + \left(y+\frac{8}{5}\right)^2 = 1^2$
$\Rightarrow 5x^2 + 5y^2 + 2x + 16y + 8 = 0$

43. Let, P = (x, y)
$\therefore$ according to the given condition
$\frac{\sqrt{x^2+y^2-2x+4y-20}}{\sqrt{x^2+y^2-2x-8y+1}} = \frac{2}{1}$
$\Rightarrow \frac{x^2+y^2-2x+4y-20}{x^2+y^2-2x-8y+1} = 4$
$\Rightarrow x^2 + y^2 - 2x - 12y + 8 = 0$

44. Equation of the tangent at (1, 7) to $x^2 = y - 6$ is $2x - y + 5 = 0$
Centre of the given circle is (– 8, – 6).

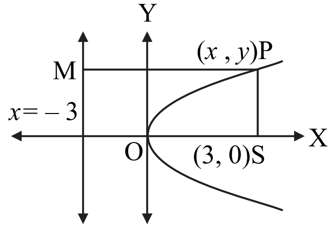


Perpendicular from the centre (– 8, – 6) to $2x - y + 5 = 0$ is equal to the radius of the circle.
$\Rightarrow \left|\frac{2(-8)-(-6)+5}{\sqrt{2^2+1^2}}\right| = \sqrt{8^2+6^2-c}$
$\Rightarrow \left|\frac{-5}{\sqrt{5}}\right| = \sqrt{100-c} \Rightarrow c = 95$

45. Let, $x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle.
This circle passes through (1, 0)
$\therefore$ $1 + 2g + c = 0$
i.e. $2g + c = -1$ …(i)
Also, this circle is orthogonal to the circle $x^2 + y^2 - 2x + 4y + 1 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$
$\therefore$ $2g + 4f + c = -1$ …(ii), and
$6g - f - c = 1$ …(iii)
Solving, (i), (ii) and (iii), we get
$g = f = 0$ and $c = -1$
$\therefore$ Centre = $(-g, -f) = (0, 0)$

46. $x^2 + y^2 - 4x - 6y - 12 = 0$
$C_1 = (2, 3)$, $r = \sqrt{2^2 + 3^2 + 12} = 5$
$x^2 + y^2 + 6x + 18y + 26 = 0$
$C_2 = (-3, -9)$, $r = \sqrt{9 + 81 - 26} = \sqrt{64} = 8$
$l(C_1C_2) = r_1 + r_2$
$\therefore$ The circles touch externally at a single point
$\therefore$ Number of common tangents is 3.

47.



$\because$ $SP^2 = PM^2$

$\Rightarrow (x-3)^2 + y^2 = \left|\frac{x+3}{\sqrt{1}}\right|^2$

$\Rightarrow x^2 + 9 - 6x + y^2 = x^2 + 9 + 6x$
$\Rightarrow y^2 = 12x$

48. $\sqrt{(x-1)^2 + (y+1)^2} = \left|\frac{x_1 + y_1 + 3}{\sqrt{1+1}}\right|$

$x^2 + y^2 - 10x - 2y - 2xy - 5 = 0$

49. Equation of parabola having vertex (p, q) and focus (p, b + q) is given by;
$(x - p)^2 = 4b(y - q)$
Given, vertex A = (1, 1) and focus S = (1, –1)
$\therefore$ p = 1, q = 1, b = –2
$\therefore$ Equation of parabola is;
$(x - 1)^2 = 4(-2)(y - 1)$
i.e. $x^2 - 2x + 8y - 7 = 0$
Only $\left(3, \frac{1}{2}\right)$ satisfies the above equation of parabola.

50. a = 4, vertex = (0, 0), focus = (0, – 4)

51. Given, equation is $x^2 = -8ay$
Here, A = 2a
Focus of parabola (0, – A) i.e., (0, – 2a)
Directrix y = A i.e., $y = 2a$

52. $y^2 - 4y - x + 3 = 0$
$\Rightarrow y^2 - 4y + 4 - x + 3 - 4 = 0$
$\Rightarrow (y-2)^2 - (x+1) = 0$
$\Rightarrow (y-2)^2 = (x+1)$
Comparing with $Y^2 = 4aX$, we have
$a = \frac{1}{4}$, $Y = y - 2$, $X = x + 1$
Focus of the parabola is Y = 0, X = a
$\Rightarrow y - 2 = 0, x + 1 = \frac{1}{4} \Rightarrow y = 2, x = \frac{-3}{4}$

$\therefore$ focus $= \left(\frac{-3}{4}, 2\right)$

53. $(y+1)^2 = -8(x+2)$
Comparing this equation with $Y^2 = -4aX$, we get
a = 2, X = x + 2 and Y = y + 1
Focus of the parabola is,
X = –a, Y = 0
$\Rightarrow x + 2 = -2, y + 1 = 0 \Rightarrow x = -4, y = -1$
$\therefore$ focus = (–4, –1)

54. Parabola is $y^2 = -4ax$ (left handed parabola).
$\therefore$ its focus is (–a, 0).
$\therefore$ option (B) is false.

56. Since, (a, –2a) $\equiv$ (2, –8)
$\therefore$ another end (a, 2a) $\equiv$ (2, 8)

57. Given $y^2 = 5x$
here (x_1, y_1) = (a, 2a) and (x_2, y_2) = (a, –2a)

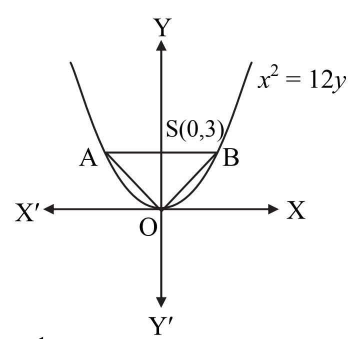
$\therefore$ $x_1x_2 = a^2 = \left(\frac{25}{16}\right)$

$\therefore$ $a = \frac{5}{4}$

$y_1 y_2 = -4a^2 = \frac{-25}{4}$

$\therefore$ $4x_1x_2 + y_1y_2 = 0$

58. $x^2 = 12y$
$\Rightarrow$ 4a = 12
$\Rightarrow$ a = 3



Area of triangle $= \frac{1}{2}$ (base) (height)

$= \frac{1}{2} \times AB \times OS = \frac{1}{2} \times 4a \times a$

$= \frac{1}{2}(12)(3) = 18$ sq. units

59. Equation of parabola is $y^2 = 12x$

$\therefore$ $a = 3$

Given $y = 6$

$\therefore$ substituting $y = 6$ in $y^2 = 12x$, we get

$36 = 12x$

$\Rightarrow x = 3$

Now, focal distance $= |x + a| = |3 + 3| = 6$

60. Given, $y = 3x$

Substituting $y = 3x$ in $y^2 = 18x$, we get

$(3x)^2 = 18x$

$\Rightarrow 9x^2 = 18x$

$\Rightarrow x = 2$ and $y = 6$

61. Vertex $= (0, 4)$, focus $= (0, 2)$

$\Rightarrow a = 2$

Hence, equation of parabola is

$(x - 0)^2 = -4 \times 2(y - 4)$

i.e., $x^2 + 8y = 32$

62. Given, vertex of parabola $(h, k) \equiv (1,1)$ and its focus $(a + h, k) \equiv (3, 1)$ or $a + h = 3$ or $a = 2$. The y-coordinates of vertex and focus are same, therefore axis of parabola is parallel to X-axis.Thus, equation of the parabola is $(y - k)^2 = 4a(x - h)$ or $(y - 1)^2 = 4 \times 2(x - 1)$ or $(y - 1)^2 = 8(x - 1)$

63. Directrix $= x + 5 = 0$

Focus is $(-3, 0)$

$\Rightarrow 2a = (5 - 3) = 2$

$\Rightarrow a = 1$

Vertex is $\left(\dfrac{-3+(-5)}{2}, 0\right) = (-4, 0)$

Therefore, equation is $(y - 0)^2 = 4(x + 4)$

i.e., $y^2 = 4(x + 4)$

64. Let $P(x, y)$ be any point on the parabola.

$\therefore$ $SP^2 = PM^2$

$\Rightarrow (x - 5)^2 + (y - 3)^2 = \left|\dfrac{3x - 4y + 1}{\sqrt{9+16}}\right|^2$

$\Rightarrow 25(x^2 + 25 - 10x + y^2 + 9 - 6y)$

$= 9x^2 + 16y^2 + 1 - 24xy + 6x - 8y$

$\Rightarrow 16x^2 + 9y^2 - 256x - 142y + 24xy + 849 = 0$

$\Rightarrow (4x + 3y)^2 - 256x - 142y + 849 = 0$

65. Equation will be of the form $y^2 = 4A(x - a)$, where $A = (a' - a)$ or $y^2 = 4(a' - a)(x - a)$.

66. Equation of parabola

$y^2 = 8x$

$\therefore$ $a = 2$

$P(2t^2, 4t)$ $A(1, 0)$

Mid point $= (x, y) = \left(\dfrac{2t^2+1}{2}, \dfrac{4t+0}{2}\right)$

$\Rightarrow \dfrac{2x-1}{2} = t^2$ and $\dfrac{y^2}{4} = t^2$

$\Rightarrow \dfrac{y^2}{4} = \dfrac{2x-1}{2} \Rightarrow y^2 = 4\left(x - \dfrac{1}{2}\right)$

67. Eccentricity of parabola is always 1 i.e., $e = 1$.

68. Since, vertex is the midpoint of focus and directrix.

$\therefore$ vertex $= \left(\dfrac{0+2}{2}, \dfrac{0+0}{2}\right) = (1, 0)$

69. $y^2 - 4y - x + 3 = 0$

$\Rightarrow y^2 - 4y + 4 - x + 3 - 4 = 0$

$\Rightarrow (y - 2)^2 - (x + 1) = 0 \Rightarrow (y - 2)^2 = (x + 1)$

Comparing with $(y - k)^2 = 4a(x - h)$, we get $h = -1, k = 2$

$\therefore$ vertex $= (-1, 2)$

70. $x^2 + 4x + 2y - 7 = 0$

$\Rightarrow x^2 + 4x + 4 = -2y + 7 + 4$

$\Rightarrow (x + 2)^2 = -2y + 11$

$\Rightarrow (x + 2)^2 = -2\left(y - \dfrac{11}{2}\right)$

Hence, vertex is $\left(-2, \dfrac{11}{2}\right)$

71. The given equation of parabola is

$y = 2x^2 + x$

$\Rightarrow x^2 + \dfrac{x}{2} = \dfrac{y}{2}$

$\Rightarrow \left(x + \dfrac{1}{4}\right)^2 = \dfrac{y}{2} + \dfrac{1}{16} \Rightarrow \left(x + \dfrac{1}{4}\right)^2 = \dfrac{1}{2}\left(y + \dfrac{1}{8}\right)$

Let $X^2 = \dfrac{1}{2}Y$ ....(i)

Here $A = \dfrac{1}{8}$, focus of (i) is $\left(0, \dfrac{1}{8}\right)$

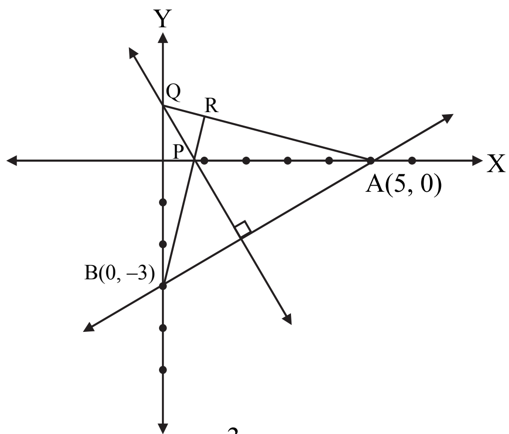
i.e., $X = 0$, $Y = \dfrac{1}{8}$

$\Rightarrow x + \dfrac{1}{4} = 0$, $y + \dfrac{1}{8} = \dfrac{1}{8} \Rightarrow x = -\dfrac{1}{4}$, $y = 0$

$\therefore$ focus of given parabola is $\left(-\dfrac{1}{4}, 0\right)$.

72. Given equation of conic
$x^2 - 6x + 4y + 1 = 0$
$\Rightarrow x^2 - 6x + 9 + 4y + 1 - 9 = 0$
$\Rightarrow (x-3)^2 + 4(y-2) = 0$
$\Rightarrow (x-3)^2 = -4(y-2)$
Comparing with $X^2 = -4aY$, we get
$a = 1$, $X = x - 3$, $Y = y - 2$
Focus of the parabola is $X = 0$, $Y = -a$
$\Rightarrow x - 3 = 0, y - 2 = -1 \Rightarrow x = 3, y = 1$
$\therefore$ Focus = (3, 1)

73.



Slope of AB = $\frac{-3}{5}$

$\therefore$ Slope of PQ = $\frac{5}{3}$

$\therefore$ Equation of PQ: $y = \frac{5}{3}x + c$

i.e. $5x - 3y = -3c$

At P, $y = 0 \Rightarrow x = \frac{-3c}{5}$

At Q, $x = 0 \Rightarrow y = c$

$\therefore$ P = $\left(\frac{-3c}{5}, 0\right)$ and Q = (0, c)

$\therefore$ Equation of AQ: $\frac{y}{-c} = \frac{x-5}{5}$

i.e. $c = \frac{-5y}{x-5}$ ...(i)

Equation of BP: $\frac{y+3}{3} = \frac{x}{\frac{3c}{5}}$

i.e. $c = \frac{5x}{y+3}$ ...(ii)

Since, lines AQ and BP intersect,

$\therefore$ $\frac{-5y}{x-5} = \frac{5x}{y+3}$

$\therefore$ $x^2 + y^2 - 5x + 3y = 0$ is the required locus of R.

74. Given equation of parabola
$x^2 - 2x + 3y - 2 = 0$
$\Rightarrow x^2 - 2x + 1 = -3y + 2 + 1$
$\Rightarrow (x-1)^2 = -3(y-1)$
$\therefore$ vertex = (h, k) = (1, 1),
focus = (h, k + b) = $\left(1, \frac{1}{4}\right)$
$\therefore$ distance between focus and vertex
$= \sqrt{0 + \left(1 - \frac{1}{4}\right)^2} = \sqrt{\left(\frac{3}{4}\right)^2}$
$= \frac{3}{4}$

75. $x^2 + 4x + 2y = 0$
$\Rightarrow x^2 + 4x + 4 = -2y + 4$
$\Rightarrow (x+2)^2 = -2(y-2)$
$\therefore$ Equation of directrix is $y - 2 = -\frac{1}{2}$
$\Rightarrow y = \frac{3}{2}$
$\Rightarrow 2y = 3$

76. Given, equation of parabola is $x^2 + 8y - 2x = 7$
$\Rightarrow x^2 - 2x + 8y - 7 = 0$
$\Rightarrow x^2 - 2x + 1 + 8y - 7 - 1 = 0$
$\Rightarrow (x-1)^2 + 8y = 8$
$\Rightarrow (x-1)^2 = -8(y-1)$
$\Rightarrow (x-1)^2 = -4 \times 2(y-1)$
Here, a = 2
$\therefore$ Equation of directrix is $y - 1 = 2$ i.e., $y = 3$

77. $y^2 + 6y - 2x = -5$
$\Rightarrow y^2 + 6y + 9 = 2x - 5 + 9$
$\Rightarrow (y+3)^2 = 2(x+2)$
$\therefore$ vertex = (–2, –3)
Here, $a = \frac{1}{2}$
$\therefore$ Equation of directix is $x + 2 = -\frac{1}{2}$
$\Rightarrow 2x + 5 = 0$

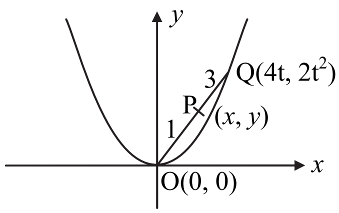
78. Given, $x^2 = 4y$ ….(i)
$y^2 = 4x$ ….(ii)
$\Rightarrow \frac{x^4}{16} = 4x$ ….[From (i) and (ii)]
$\Rightarrow x^4 = 64x$
$\Rightarrow x = 0, 4$
Substituting the values of x in (ii), we get
$y = 0, 4$
$\therefore$ Other point is (4, 4).

79. $x^2 = 8y$

$\Rightarrow$ a = 4

By internal division formula

$P(x, y) = \left(t, \frac{t^2}{2}\right)$



$\therefore \quad x = t,\ y = \frac{t^2}{2}$

$\Rightarrow x^2 = 2y$

80. $y^2 = -16x$

$\Rightarrow$ a = – 4

$t = \frac{1}{2}$ … (given)

$\therefore \quad x = at^2 = (-4) \times \left(\frac{1}{2}\right)^2$

$\Rightarrow x = -1$

$y = 2at = 2 \times (-4) \times \left(\frac{1}{2}\right)$

$\Rightarrow y = -4$

$\therefore$ The cartesian co-ordinates are (– 1, – 4).

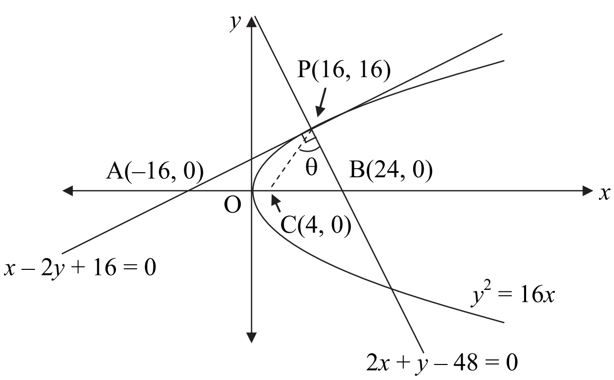
81. Equation of the tangent at P(16, 16) is $x - 2y + 16 = 0$

Equation of the normal at P(16, 16) is $2x + y - 48 = 0$

Tangent and normal intersect the axis of parabola at A(–16, 0) and B(24, 0) respectively.

AB is the diameter of the circle.

Centre of the circle is (4, 0).



$m_{PC} = \frac{16 - 0}{16 - 4} = \frac{16}{12} = \frac{4}{3}$

$m_{PB} = \frac{16 - 0}{16 - 24} = \frac{16}{-8} = -2$

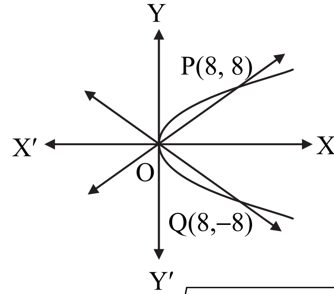
$\tan\theta = \left|\frac{\frac{4}{3} - (-2)}{1 + \left(\frac{4}{3}\right)(-2)}\right| = 2$

82. Given, $y = \pm x$ ….(i)

$y^2 = 8x$ ….(ii)

Solving (i) and (ii),

the point of intersection are P(8, 8) and Q(8, –8)



$\therefore$ Length of PQ = $\sqrt{(8-8)^2 + (8+8)^2}$

= 16

83. Given parabolas are $y^2 = 4ax$ ....(i)

$x^2 = 4ay$ ....(ii)

Putting the value of y from (ii) in (i), we get

$\frac{x^4}{16a^2} = 4ax$

$\Rightarrow x(x^3 - 64a^3) = 0$

$\Rightarrow x = 0, 4a$

From (ii), $y = 0, 4a$

Let A ≡ (0, 0), B ≡ (4a, 4a)

Since, the given line $2bx + 3cy + 4d = 0$ passes through A and B.

$\therefore \quad d = 0$ and $8ab + 12ac = 0$

$\Rightarrow 2b + 3c = 0$ ….[$\because a \neq 0$]

$\therefore \quad d^2 + (2b + 3c)^2 = 0$

84. $\frac{x^2}{36} + \frac{y^2}{16} = 1$

Here $a^2 = 36, b^2 = 16$

$b^2 = a^2(1 - e^2)$

$\Rightarrow 16 = 36(1 - e^2)$

$\Rightarrow e = \frac{\sqrt{5}}{3} = \frac{2\sqrt{5}}{6}$

85. $4x^2 + y^2 - 8x + 4y - 8 = 0$

$\Rightarrow 4(x^2 - 2x) + y^2 + 4y = 8$

$\Rightarrow 4(x^2 - 2x + 1) + y^2 + 4y + 4 = 16$

$\Rightarrow 4(x - 1)^2 + (y + 2)^2 = 16$

$\Rightarrow \frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$

which is an ellipse with $a^2 = 4$ and $b^2 = 16$

$\therefore \quad e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{4}{16}} = \sqrt{\frac{12}{16}}$

$\therefore \quad e = \frac{\sqrt{3}}{2}$

86. $x^2 + 2y^2 - 2x + 3y + 2 = 0$

$\therefore$ $(x^2 - 2x + 1) + 2\left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{1}{8}$

$\therefore$ $(x-1)^2 + 2\left(y + \frac{3}{4}\right)^2 = \frac{1}{8}$

$\therefore$ $\frac{(x-1)^2}{\frac{1}{8}} + \frac{\left(y + \frac{3}{4}\right)^2}{\frac{1}{16}} = 1$

$\therefore$ Comparing this with $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, we get

$a^2 = \frac{1}{8}$, $b^2 = \frac{1}{16}$

$\therefore$ $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{\frac{1}{16}}{\frac{1}{8}}} = 1 - \frac{8}{16}$

$\therefore$ $e = \frac{1}{\sqrt{2}}$

87. Here, given that $2b = 10$, $2a = 8$

$\Rightarrow b = 5, a = 4$

Hence, the required equation is $\frac{x^2}{16} + \frac{y^2}{25} = 1$

88. Given, centre (0,0), focus (0,3), b = 5

Focus (0,3)

$\Rightarrow be = 3$

$\Rightarrow e = \frac{3}{5}$

Also, $a = b\sqrt{1 - e^2} = 5\sqrt{1 - \frac{9}{25}} = 4$

Hence, the required equation is $\frac{x^2}{16} + \frac{y^2}{25} = 1$

89. Given, foci = $(\pm 2, 0) = (\pm ae, 0)$

$\Rightarrow ae = 2$

and $e = \frac{1}{2}$

$\therefore$ $a = 4$

Now, $b^2 = a^2(1 - e^2)$

$\Rightarrow b^2 = 16\left(1 - \frac{1}{4}\right)$

$\Rightarrow b^2 = 12$

Hence, equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$

$\Rightarrow 3x^2 + 4y^2 = 48$

90. We have, $ae = \pm\sqrt{5}$

$\Rightarrow a = \pm\sqrt{5}\left(\frac{3}{\sqrt{5}}\right)$ ....$\left[\because e = \frac{\sqrt{5}}{3}\right]$

$\Rightarrow a = \pm 3$

$\Rightarrow a^2 = 9$

Now, $b^2 = a^2(1 - e^2) = 9\left(1 - \frac{5}{9}\right) = 4$

Hence, equation of ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$\Rightarrow 4x^2 + 9y^2 = 36$

91. $b^2 = a^2(1 - e^2) = a^2\left(1 - \frac{2}{5}\right) = \frac{3a^2}{5}$

Let the equation of ellipse be

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{9}{a^2} + \frac{5}{3a^2} = 1$

$\Rightarrow a^2 = \frac{32}{3}$

$\therefore$ $b^2 = \frac{32}{5}$

$\therefore$ the required equation of ellipse is

$3x^2 + 5y^2 - 32 = 0$

92. Since point (–3, 1) satisfies equations in options (C) and (D) writing them in standard form, we have,

For option (C):

$\frac{x^2}{\frac{32}{3}} + \frac{y^2}{\frac{32}{5}} = 1$, here $a^2 > b^2$

For option (D):

$\frac{x^2}{\frac{48}{5}} + \frac{y^2}{16} = 1$, here, $a^2 < b^2$

Since, the ellipse has its major axis along Y-axis,

$\therefore$ $a^2 < b^2$

$\therefore$ Option (D) is correct

93. Since, directrix is parallel to Y-axis, hence axes of the ellipse are parallel to X-axis.

Let the equation of the ellipse be

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$

Now, $e^2 = 1 - \frac{b^2}{a^2}$

$\Rightarrow \frac{b^2}{a^2} = 1 - e^2 = 1 - \frac{1}{4} \Rightarrow \frac{b^2}{a^2} = \frac{3}{4}$

Also, one of the directrices is $x = 4$

$\Rightarrow \frac{a}{e} = 4 \Rightarrow a = 4e = 4 \times \frac{1}{2} = 2$;

$b^2 = \frac{3}{4}a^2 = \frac{3}{4} \times 4 = 3$

$\therefore$ Required equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$\Rightarrow 3x^2 + 4y^2 = 12$

95. Sum of focal distances of a point in an ellipse is always equal to length of major axis of that ellipse.

96. Equation of the curve is $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$

Here, a = 5, b = 4

$\therefore$ $PF_1 + PF_2 = 2a = 2 \times 5 = 10$

97. The equation of the ellipse is

$16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$

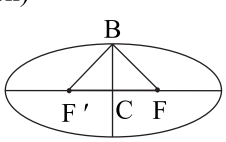
Here, $a^2 = 25$, $b^2 = 16$

$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{16}{25}} \Rightarrow e = \frac{3}{5}$

Hence, the foci are $(\pm ae, 0) = (\pm 3, 0)$

98. Since, $\angle FBF' = \frac{\pi}{2}$ (Given)

$\therefore$ $\angle FBC = \angle F'BC = \frac{\pi}{4}$



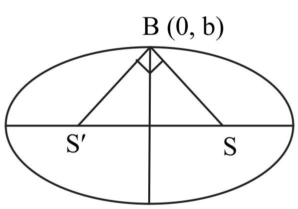
Now, CB = CF

$\Rightarrow b = ae \Rightarrow b^2 = a^2e^2$

$\Rightarrow a^2(1 - e^2) = a^2e^2 \Rightarrow 1 - e^2 = e^2$

$\Rightarrow 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$

99.

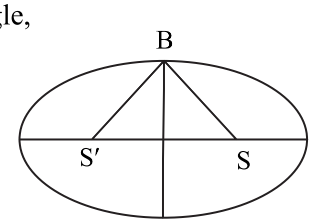


S = (ae, 0), S′ = (–ae, 0) and b = ae

Now, $a^2 e^2 = a^2 (1 - e^2)$

$\Rightarrow e = \frac{1}{\sqrt{2}}$

100. Since, ΔSBS′ is an isosceles right angled triangle,



$\therefore$ $\angle SBS' = \frac{\pi}{2}$, $\angle BSS' = \angle BS'S = \frac{\pi}{4}$

$\therefore$ $e = \frac{1}{\sqrt{2}}$

101. Distance between the foci = 2ae = 16 and

$e = \frac{1}{2}$

$\therefore$ Length of the major axis of the ellipse

$= 2a = \frac{2ae}{e} = \frac{16}{\frac{1}{2}} = 32$

102. According to the given condition,

$\sqrt{1 - \frac{25}{169}} = \sqrt{1 - \frac{b^2}{a^2}}$

$\Rightarrow \frac{144}{169} = 1 - \frac{b^2}{a^2}$

$\Rightarrow \frac{b^2}{a^2} = \frac{25}{169}$

$\Rightarrow \frac{b}{a} = \frac{5}{13}$ ....[$\because$ a > 0, b > 0]

$\Rightarrow \frac{a}{b} = \frac{13}{5}$

103. $5x^2 + 9y^2 = 45$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{5} = 1$

$\therefore$ Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 5}{3} = \frac{10}{3}$

104. Given, ellipse is $\frac{x^2}{\left(\frac{1}{3}\right)^2} + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$

Here, b > a

$\therefore$ Latus rectum $= \frac{2a^2}{b} = \frac{2 \times \frac{1}{9}}{\frac{1}{2}} = \frac{4}{9}$

105. Latus rectum = $\frac{1}{3}$ (major axis)

$\Rightarrow \frac{2b^2}{a} = \frac{2a}{3}$

$\Rightarrow a^2 = 3b^2$

$\Rightarrow a^2 = 3a^2(1 - e^2)$

$\Rightarrow e = \sqrt{\frac{2}{3}}$

106. Given, $\frac{2b^2}{a} = b$

$\Rightarrow \frac{b}{a} = \frac{1}{2}$

$\Rightarrow \frac{b^2}{a^2} = \frac{1}{4}$

Hence, $e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{3}}{2}$

107. $x^2 = 9\cos^2\theta$

$y^2 = 16\sin^2\theta$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1$

$\therefore$ $e = \frac{\sqrt{7}}{4}$

Distance between foci = 2be = $2\sqrt{7}$

108. Let (x, y) be any point on ellipse.

Then, by focus-directrix property of ellipse,

$\sqrt{(x+1)^2 + (y-1)^2} = \frac{1}{2}\left|\frac{x-y+3}{\sqrt{1+1}}\right|$

$\therefore$ $8(x^2 + 2x + 1 + y^2 - 2y + 1) = x^2 + y^2 + 9 - 2xy - 6y + 6x$

$\therefore$ $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$

109. solving $x + y - 3 = 0$

$x - y + 1 = 0$, we get

$(x, y) = (1, 2)$

110. Here, $a^2 = 9$, $b^2 = 25$

Since, b > a

$\therefore$ $e = \sqrt{\frac{b^2 - a^2}{b^2}} = \sqrt{\frac{25-9}{25}} = \frac{4}{5}$

111. $\frac{(x-1)^2}{2} + \left(y + \frac{3}{4}\right)^2 = \frac{1}{16}$

$\Rightarrow \frac{(x-1)^2}{2\left(\frac{1}{16}\right)} + \frac{\left(y+\frac{3}{4}\right)^2}{\left(\frac{1}{16}\right)} = 1$

$\Rightarrow \frac{(x-1)^2}{\left(\frac{1}{8}\right)} + \frac{\left(y+\frac{3}{4}\right)^2}{\left(\frac{1}{16}\right)} = 1$

$\therefore$ $a^2 = \frac{1}{8}$, $b^2 = \frac{1}{16}$

$b^2 = a^2(1 - e^2)$

$\Rightarrow \frac{1}{16} = \frac{1}{8}(1 - e^2)$

$\Rightarrow e^2 = 1 - \frac{1}{2}$

$\Rightarrow e = \frac{1}{\sqrt{2}}$

112. $4x^2 + y^2 - 8x + 4y - 8 = 0$

$\Rightarrow 4(x^2 - 2x) + y^2 + 4y = 8$

$\Rightarrow 4(x^2 - 2x + 1) + y^2 + 4y + 4 = 16$

$\Rightarrow 4(x - 1)^2 + (y + 2)^2 = 16$

$\Rightarrow \frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$

Comparing with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

h = 1, k = –2

$\therefore$ centre of the ellipse = (1, –2)

113. Given equation of ellipse is

$25x^2 + 9y^2 - 150x - 90y + 225 = 0$

$\Rightarrow 25(x-3)^2 + 9(y-5)^2 = 225$

$\Rightarrow \frac{(x-3)^2}{9} + \frac{(y-5)^2}{25} = 1$

Since, b > a

$\therefore$ $e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

114. $x^2 + 2y^2 - 2x + 3y + 2 = 0$

$\Rightarrow (x^2 - 2x + 1) + 2\left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = -2 + 1 + \frac{9}{8}$

$\Rightarrow (x - 1)^2 + 2\left(y + \frac{3}{4}\right)^2 = \frac{1}{8}$

$\Rightarrow \frac{(x-1)^2}{\frac{1}{8}} + \frac{\left(y+\frac{3}{4}\right)^2}{\frac{1}{16}} = 1,$

which is an ellipse with $a^2 = \frac{1}{8}$ and $b^2 = \frac{1}{16}$

Also, $b^2 = a^2(1-e^2)$

$\Rightarrow \frac{1}{16} = \frac{1}{8}(1-e^2)$

$\Rightarrow e^2 = 1 - \frac{1}{2}$

$\Rightarrow e = \frac{1}{\sqrt{2}}$

115. $25x^2 + 4y^2 + 100x - 4y + 100 = 0$

$\Rightarrow 25(x^2 + 4x) + 4(y^2 - y) = -100$

$\Rightarrow 25(x^2 + 4x + 4) + 4\left(y^2 - y + \frac{1}{4}\right) = -100 + 100 + 1$

$\Rightarrow 25(x+2)^2 + 4\left(y - \frac{1}{2}\right)^2 = 1$

$\Rightarrow \frac{(x+2)^2}{\frac{1}{25}} + \frac{\left(y-\frac{1}{2}\right)^2}{\frac{1}{4}} = 1$

Here $a^2 = \frac{1}{25}$, $b^2 = \frac{1}{4}$, $h = -2$, $k = \frac{1}{2}$

$\therefore$ $e = \frac{\sqrt{b^2 - a^2}}{b} = \frac{\sqrt{\frac{1}{4} - \frac{1}{25}}}{\frac{1}{2}} = \frac{\sqrt{21}}{5}$

$\therefore$ Foci $= (h, k \pm be)$

$= \left(-2, \frac{1}{2} \pm \frac{\sqrt{21}}{10}\right) = \left(-2, \frac{5 \pm \sqrt{21}}{10}\right)$

116. $5x^2 + y^2 + y = 8$

$\Rightarrow 5x^2 + y^2 + y + \frac{1}{4} = 8 + \frac{1}{4}$

$\Rightarrow 5x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{33}{4}$

$\Rightarrow \frac{x^2}{\left(\frac{33}{20}\right)} + \frac{\left(y + \frac{1}{2}\right)^2}{\left(\frac{33}{4}\right)} = 1$

The equation represent an ellipse.

117. Here, $2a = 10$m and $2ae = 8$m

$\therefore$ $e = \frac{4}{5}$, $a = 5$m

Now, $b^2 = a^2(1 - e^2) = 9$

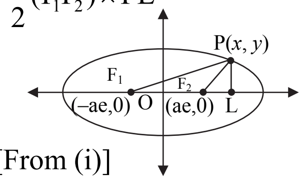
$\Rightarrow b = 3$

Thus, required area $= \pi ab = 15\pi$ sq. metre.

118. We have, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore$ $y = \frac{b}{a}\sqrt{a^2 - x^2}$ ....(i)

Now, area of Δ $PF_1F_2 = \frac{1}{2}(F_1F_2) \times PL$



$= \frac{1}{2}(2ae) \times y$

$= ae.\frac{b}{a}\sqrt{a^2 - x^2}$ ....[From (i)]

$\therefore$ $A = eb\sqrt{a^2 - x^2}$, which is maximum when $x = 0$.

Thus, the maximum value of A is abe.

119. Vertices $= (0, \pm 15)$, foci $= (0, \pm 20)$

$\therefore$ $b = 15$ and $be = 20$ $\Rightarrow e = \frac{4}{3}$

$a^2 = b^2(e^2 - 1)$

$= 15^2\left(\frac{16}{9} - 1\right)$

$= 175$

$\therefore$ The equation of hyperbola is

$\frac{-x^2}{175} + \frac{y^2}{225} = 1$

$\Rightarrow \frac{y^2}{225} - \frac{x^2}{175} = 1$

120. Given, $ae = 2$, $e = 2$

$\therefore$ $a = 1$

Now, $b^2 = a^2(e^2 - 1)$

$\Rightarrow b^2 = 1(4 - 1)$

$\Rightarrow b^2 = 3$

$\therefore$ the equation of hyperbola is

$\frac{x^2}{1} - \frac{y^2}{3} = 1$

$\Rightarrow 3x^2 - y^2 = 3$

121. Given: $ae = 8$, $e = \sqrt{2}$

$\therefore$ $a = 4\sqrt{2}$

Now, $b^2 = a^2(e^2 - 1)$

$\Rightarrow b^2 = 32(2 - 1)$

$\Rightarrow b^2 = 32$

$\therefore$ the equation of hyperbola is $\frac{x^2}{32} - \frac{y^2}{32} = 1$

$\therefore$ $x^2 - y^2 = 32$

122. Given, ae = 8 and $\frac{2b^2}{a} = 24$

$\Rightarrow b^2 = 12a$

Now, $b^2 = a^2(e^2 - 1)$

$\Rightarrow 12a = a^2e^2 - a^2$

$\Rightarrow 12a = 64 - a^2$

$\Rightarrow a^2 + 12a - 64 = 0$

$\Rightarrow a = 4$ ....[$\because$ a > 0]

$\therefore$ $b^2 = 12(4) = 48$

$\therefore$ the equation of hyperbola is

$\frac{x^2}{16} - \frac{y^2}{48} = 1 \Rightarrow 3x^2 - y^2 = 48$

123. $16x^2 - 9y^2 - 64x + 18y - 90 = 0$

$\therefore$ $16(x^2 - 4x + 4) - 9(y^2 - 2y + 1) = 145$

$\therefore$ $\frac{(x-2)^2}{\frac{145}{16}} - \frac{(y-1)^2}{\frac{145}{9}} = 1,$

Comparing with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$, we get

$X = x - 2$, $Y = y - 1$

$a^2 = \frac{145}{16}$, $b^2 = \frac{145}{9}$

$\therefore$ $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{9}} = \frac{\pm 5}{3}$

Focus of the hyperbola, $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$ is,

$(X = \pm ae, Y = 0)$

i.e. $x - 2 = \pm\left(\frac{\sqrt{145}}{4} \times \frac{5}{3}\right)$, $y - 1 = 0$

i.e. $x - 2 = \pm \frac{5\sqrt{145}}{12}$, $y = 1$

i.e. $x = \frac{24 \pm 5\sqrt{145}}{12}$, $y = 1$

$\therefore$ Focus = $\left(\frac{24 \pm 5\sqrt{145}}{12}, 1\right)$

124. $\frac{x^2}{36} - \frac{y^2}{k^2} = 1$ is a hyperbola $\Rightarrow k^2 > 0$

Now, $\frac{y^2}{k^2} = \frac{x^2}{36} - 1 = \frac{x^2 - 36}{36}$

$\Rightarrow k^2 = \frac{36y^2}{x^2 - 36} > 0 \Rightarrow x^2 - 36 > 0$

$\Rightarrow x^2 > 36$

This is true only for point (10, 4).

$\therefore$ (10, 4) lies on given hyperbola.

125. $\frac{x^2}{12 - k} + \frac{y^2}{8 - k} = 1$

$\Rightarrow \frac{x^2}{12 - k} - \frac{y^2}{k - 8} = 1$

$\therefore$ 12 > k and k > 8

$\Rightarrow 8 < k < 12$

$\therefore$ the given equation represents a hyperbola, if $8 < k < 12$.

126. Consider,

$x^2 - y^2 + 3x - 2y - 43 = 0$

Comparing with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 1, h = 0, b = -1, g = \frac{3}{2}, f = -1, c = -43$

$\therefore$ $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$\therefore$ $\Delta = 43 + 0 - 1 + \frac{9}{4} - 0 = \frac{177}{4}$

$\therefore$ $\Delta \neq 0$

Also, $ab - h^2 = -1$

$\therefore$ $ab - h^2 < 0$

$\therefore$ $x^2 - y^2 + 3x - 2y - 43 = 0$ is the equation of hyperbola.

127. The given equation can be written as

$\frac{x^2}{\frac{32}{3}} - \frac{y^2}{8} = 1$

$\Rightarrow \frac{x^2}{\left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^2} - \frac{y^2}{(2\sqrt{2})^2} = 1$

$\Rightarrow a = \frac{4\sqrt{2}}{\sqrt{3}}$

$\therefore$ Length of transverse axis of a hyperbola

$= 2a = 2 \times \frac{4\sqrt{2}}{\sqrt{3}} = \frac{8\sqrt{2}}{\sqrt{3}}$

128. $\frac{x^2}{9} - \frac{y^2}{4} = 1$

$\Rightarrow a^2 = 9, b^2 = 4$

$e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{9 + 4}}{3} = \frac{\sqrt{13}}{3}$

directrix of hyperbola is $x = \frac{a}{e}$

$\Rightarrow x = \frac{3}{\frac{\sqrt{13}}{3}} \Rightarrow x = \frac{9}{\sqrt{13}}$

129. $x^2 - 4y^2 = 1$

$\Rightarrow \dfrac{x^2}{(1)^2} - \dfrac{y^2}{\left(\dfrac{1}{2}\right)^2} = 1$

$\Rightarrow a^2 = 1,\ b^2 = \left(\dfrac{1}{2}\right)^2$

Also, $b^2 = a^2(e^2 - 1)$

$\Rightarrow \dfrac{1}{4} + 1 = e^2 \Rightarrow e = \dfrac{\sqrt{5}}{2}$

130. Here a = b, so it is a rectangular hyperbola.
Hence, eccentricity $e = \sqrt{2}$

131. Let the equation of hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Since, hyperbola passes through the points (3, 0), $(3\sqrt{2}, 2)$

$\therefore \quad \dfrac{9}{a^2} - 0 = 1$ and $\dfrac{18}{a^2} - \dfrac{4}{b^2} = 1$

$\Rightarrow a^2 = 9$ and $\dfrac{4}{b^2} = \dfrac{18}{a^2} - 1$

$\Rightarrow a^2 = 9$ and $\dfrac{4}{b^2} = \dfrac{18}{9} - 1$

$\Rightarrow a^2 = 9$ and $b^2 = 4$

Now, $e = \sqrt{1 + \dfrac{b^2}{a^2}} = \sqrt{1 + \dfrac{4}{9}} = \dfrac{\sqrt{13}}{3}$

132. Given length of LR = 8

$\Rightarrow \dfrac{2b^2}{a} = 8$

Also, $2b = \dfrac{1}{2} 2ae$

$\Rightarrow 4b^2 = a^2e^2 \Rightarrow 4a^2(e^2 - 1) = a^2e^2$

$\Rightarrow 4e^2 - e^2 = 4 \Rightarrow e = \dfrac{2}{\sqrt{3}}$

133. Eccentricity of hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$e = \sqrt{1 + \dfrac{b^2}{a^2}}$

$\Rightarrow e^2 = \dfrac{a^2 + b^2}{a^2}$ ….(i)

Eccentricity of hyperbola $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$ is

$e_1 = \sqrt{1 + \dfrac{a^2}{b^2}}$

$\Rightarrow e_1^2 = \dfrac{b^2 + a^2}{b^2}$ …. (ii)

From (i) and (ii), we get

$\dfrac{1}{e_1^2} + \dfrac{1}{e^2} = 1$

134. The hyperbola is $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

$\therefore$ Difference of focal distance = 2a = 8

135. Since, distance between directrices = $\dfrac{2a}{e}$ and eccentricity of rectangular hyperbola = $\sqrt{2}$.

$\therefore$ Distance between directrices = $\dfrac{2a}{\sqrt{2}}$

Given, $\dfrac{2a}{\sqrt{2}} = 10$

$\Rightarrow 2a = 10\sqrt{2}$

Now, distance between foci = 2ae

$= (10\sqrt{2})(\sqrt{2})$

$= 20$

136. Given, $x = a\left(t + \dfrac{1}{t}\right)$

$\Rightarrow \dfrac{x}{a} = t + \dfrac{1}{t}$ ....(i)

and $y = b\left(t - \dfrac{1}{t}\right)$

$\Rightarrow \dfrac{y}{b} = t - \dfrac{1}{t}$ ....(ii)

Squaring and subtracting equation (ii) from (i), we get

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = t^2 + 2 + \dfrac{1}{t^2} - t^2 + 2 - \dfrac{1}{t^2}$

$\Rightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 4$

which represents a hyperbola.

137. $x^2 - 2x - 4y^2 + 16y - 40 = 0$

$\Rightarrow (x^2 - 2x) - 4(y^2 - 4y) - 40 = 0$

$\Rightarrow (x - 1)^2 - 1 - 4[(y - 2)^2 - 4] - 40 = 0$

$\Rightarrow (x - 1)^2 - 4(y - 2)^2 = 25$

$\Rightarrow \dfrac{(x-1)^2}{25} - \dfrac{(y-2)^2}{\dfrac{25}{4}} = 1$, which is a hyperbola.

138. Given equation of lines are
$\sqrt{3}x - y - 4\sqrt{3}k = 0$
$\Rightarrow \sqrt{3}x - y = 4\sqrt{3}k$ ….(i)
and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$
$\Rightarrow k(\sqrt{3}x + y) = 4\sqrt{3}$
$\Rightarrow \sqrt{3}x + y = \dfrac{4\sqrt{3}}{k}$ ....(ii)
Multiplying equation (i) and (ii), we get
$(\sqrt{3}x - y)(\sqrt{3}x + y) = 4\sqrt{3}k \times \dfrac{4\sqrt{3}}{k}$
$\Rightarrow 3x^2 - y^2 = 48$
$\Rightarrow \dfrac{x^2}{(48/3)} - \dfrac{y^2}{48} = 1$, which is a hyperbola.

139. Given equation of hyperbola is
$x^2 - y^2 + 1 = 0$
$\Rightarrow x^2 - y^2 = -1$
$\Rightarrow \dfrac{x^2}{1} - \dfrac{y^2}{1} = -1$
$\therefore$ $a^2 = 1, b^2 = 1$
$e = \sqrt{\dfrac{a^2 + b^2}{b^2}} = \sqrt{\dfrac{1+1}{1}} = \sqrt{2}$
$\therefore$ foci $\equiv (0, \pm be) \equiv (0, \pm\sqrt{2})$, and
centre $\equiv (0, 0)$
Centre of circle $\equiv (0, 0)$, radius of circle $= \sqrt{2}$
$\therefore$ Equation of circle is $x^2 + y^2 = 2$.

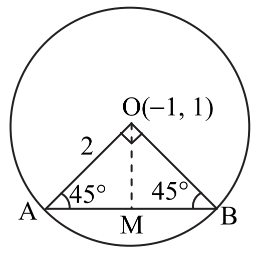
140. Hyperbola is $\dfrac{x^2}{144} - \dfrac{y^2}{81} = \dfrac{1}{25}$
$a = \sqrt{\dfrac{144}{25}}$,
$b = \sqrt{\dfrac{81}{25}}$
$\therefore$ $e_1 = \sqrt{1 + \dfrac{81}{144}} = \sqrt{\dfrac{225}{144}} = \dfrac{15}{12} = \dfrac{5}{4}$
$\therefore$ foci $= (ae_1, 0) = \left(\dfrac{12}{5} \times \dfrac{5}{4}, 0\right) = (3, 0)$
$\therefore$ focus of ellipse $= (4e, 0)$
Since, focus of ellipse and hyperbola is same
$\therefore$ $(4e, 0) = (3, 0)$
$\Rightarrow e = \dfrac{3}{4}$
Hence, $b^2 = 16\left(1 - \dfrac{9}{16}\right) = 7$

141. Eccentricity of ellipse $x^2 + 9y^2 = 9$ i.e. $\dfrac{x^2}{9} + \dfrac{y^2}{1} = 1$
$(e_1)^2 = 1 - \dfrac{1}{9} = \dfrac{8}{9}$
Now, eccentricity of hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is,
$(e_2)^2 = 1 + \dfrac{b^2}{a^2} = \dfrac{9}{8}$
$\therefore$ $\dfrac{b^2}{a^2} = \dfrac{9}{8} - 1$ $\therefore$ $\dfrac{a^2}{b^2} = \dfrac{8}{1}$

142. The equation of the ellipse is $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$
Let e be its eccentricity.
Then, $e = \sqrt{1 - \dfrac{1}{4}} = \dfrac{\sqrt{3}}{2}$
The foci of the ellipse are $S\left(\sqrt{3}, 0\right)$ and $S'\left(-\sqrt{3}, 0\right)$.
Eccentricity of the hyperbola $= \dfrac{1}{e} = \dfrac{2}{\sqrt{3}}$
$\therefore$ $b^2 = a^2\left(\dfrac{4}{3} - 1\right) = \dfrac{a^2}{3}$
The hyperbola passes through $S\left(\sqrt{3}, 0\right)$.
$\therefore$ $\dfrac{3}{a^2} - 0 = 1 \Rightarrow a^2 = 3 \Rightarrow a = \sqrt{3}$
$\therefore$ the co-ordinates of the foci of hyperbola are $(\pm 2, 0)$.

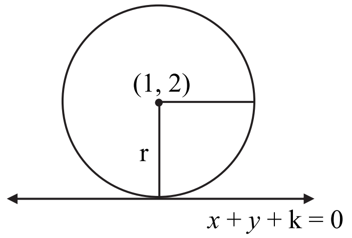
143. $x - 3y = 1$ ….(i)
and $x^2 - 4y^2 = 1$ ….(ii)
On solving (i) and (ii), we get
$A(1,0)$ and $B\left(-\dfrac{13}{5}, -\dfrac{6}{5}\right)$
These are the points of intersection of the straight line and hyperbola.
$\therefore$ Length of straight line intercepted by the hyperbola
$= \sqrt{\left(-\dfrac{13}{5} - 1\right)^2 + \left(-\dfrac{6}{5}\right)^2}$
$= \sqrt{\left(-\dfrac{18}{5}\right)^2 + \left(-\dfrac{6}{5}\right)^2} = \sqrt{\dfrac{324 + 36}{25}}$
$= \sqrt{\dfrac{360}{25}} = \dfrac{6}{5}\sqrt{10}$ units

144. Given equation of circle
$x^2 + y^2 + 2x - 2y - 2 = 0$
$\Rightarrow (x + 1)^2 + (y - 1)^2 = 4$
$\therefore$ centre = (–1, 1), and
radius = 2



$\sin 45° = \dfrac{OM}{OA}$
$\Rightarrow OM = \sqrt{2}$
$\therefore$ locus of the mid-points of the chord is
$(x + 1)^2 + (y - 1)^2 = \left(\sqrt{2}\right)^2$
$\Rightarrow x^2 + 2x + 1 + y^2 - 2y + 1 = 2$
$\Rightarrow x^2 + y^2 + 2x - 2y = 0$

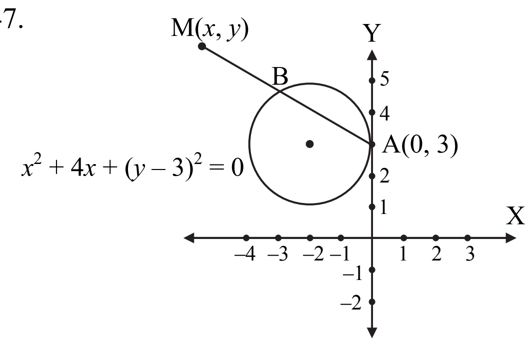
145.



Since, $x + y + k = 0$ touches the given circle.
$\therefore \left|\dfrac{1(1) + 1(2) + k}{\sqrt{1+1}}\right|$ = radius
$\Rightarrow \dfrac{3 + k}{\sqrt{2}} = \pm\sqrt{2} \Rightarrow k = -1$ or $k = -5$

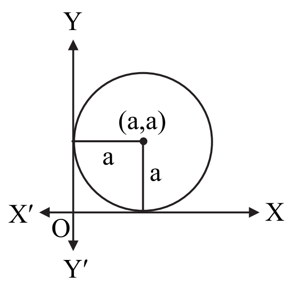
146. The diameter of the circle is perpendicular distance between the parallel lines (tangents)
Now, $3x - 4y + 4 = 0$ ….(i)
$6x - 8y - 7 = 0$
i.e., $3x - 4y - \dfrac{7}{2} = 0$ ….(ii)
Since, equation (i) and (ii) are parallel to each other.
$\therefore$ diameter $= \left|\dfrac{4 - \left(-\dfrac{7}{2}\right)}{\sqrt{(3)^2 + (-4)^2}}\right| = \dfrac{15}{2 \times 5} = \dfrac{3}{2}$
$\therefore$ radius $= \dfrac{3}{4}$

147.



Let M = (x, y)
Since, AM = 2AB
$\therefore \dfrac{AB}{AM} = \dfrac{1}{2}$
$\therefore$ B is the mid point of seg AM.
$\therefore B = \left(\dfrac{x}{2}, \dfrac{y+3}{2}\right)$
Since, B lies on circle $x^2 + 4x + (y - 3)^2 = 0$
$\therefore \left(\dfrac{x}{2}\right)^2 + 4\left(\dfrac{x}{2}\right) + \left(\dfrac{y+3}{2} - 3\right)^2 = 0$
i.e. $\dfrac{x^2}{4} + 2x + \dfrac{y - 6x + 9}{4} = 0$
i.e. $x^2 + y^2 + 8x - 6y + 9 = 0$
is the required locus of M.

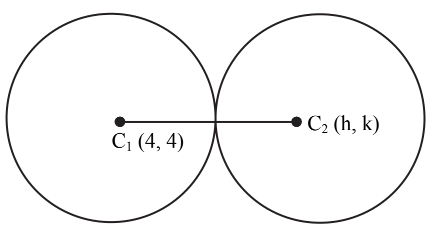
148. For given circle, $x^2 + y^2 - 2ax - 2ay + a^2 = 0$
Centre = (a, a)
Also, radius $= \sqrt{a^2 + a^2 - a^2} = a$



The above circle touches $x = 0$.

149. Given equation is $x^2 + y^2 - 6x + 2y = 0$.
Centre = (3, –1)
Since, diameter is passing through origin and (3, –1).
$\therefore$ option (A) is the correct answer.

150.



Given, $x^2 + y^2 - 8x - 8y - 4 = 0$
$\therefore (x^2 - 8x + 16) + (y^2 - 8y + 16) = 16 + 16 + 4$
$\Rightarrow (x - 4)^2 + (y - 4)^2 = 36$
Equation of circle touching X – axis
$(x - h)^2 + (y - k)^2 = k^2$
Since, both circle touches externally
$\therefore$ distance between their centre $= r_1 + r_2$
$\sqrt{(4 - h)^2 + (4 - k)^2} = 6 + k$

$\Rightarrow (4-h)^2 + (4-k)^2 = (6+k)^2$

$\Rightarrow (4-h)^2 = 36 + 12k + k^2 - 16 + 8k - k^2$

$\Rightarrow (4-h)^2 = 20k + 20$

This is equation of parabola

$\therefore$ answer is option (D)

151. Equations of tangents to the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ with centre A(1, 2),

At, B(1, 7) is $y = 7$

At, C(4, –2) is $3x - 4y - 20 = 0$

These tangents intersect each other at D(16, 7).

$\therefore$ Area of quadrilateral BACD

= Area of ΔABD + Area of ΔACD

$= \frac{1}{2}$(AB)(BD) + $\frac{1}{2}$(AC)(CD)

...[$\because \angle ABD = \angle ACD = 90°$]

$= \frac{1}{2} \times (5) \times (15) + \frac{1}{2} \times (5) \times (15)$

= 75 sq. units

152. $AB = \sqrt{(3+3)^2 + (3-5)^2}$

$= \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

Centroid divides orthocentre and circumcentre in the ratio 2 : 1.

AB : BC = 2 : 1

$AC = \frac{3}{2}$ AB

$= \frac{3}{2}\left(2\sqrt{10}\right) = 3\sqrt{10}$

radius $= \frac{1}{2}$ AC $= \frac{1}{2}\left(3\sqrt{10}\right) = 3\sqrt{\frac{5}{2}}$

153. Given, $x + y = 0$ ....(i)

$x^2 + y^2 + 4y = 0$ ....(ii)

Solving (i) and (ii), we get

$x = 0, y = 0; x = 2, y = -2$

$\therefore$ parabola passes through (0, 0) and (2, –2). These points are satisfied by the parabola $y^2 = 2x$.

154. Given equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{25} = 1$

$\Rightarrow$ a = 12, b = 5

Now, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{25}{144}} = \frac{\sqrt{119}}{12}$

$\therefore$ focus = (ae, 0) = $\left(\sqrt{119}, 0\right)$

$\therefore$ Radius $= \sqrt{\left(0-\sqrt{119}\right)^2 + \left(\sqrt{2}-0\right)^2} = 11$

155. $\frac{x^2}{16} + \frac{y^2}{12} = 1$

$\therefore$ centre of ellipse = (0, 0)

$\therefore$ centre of circle = (0, 0)

$y = 2x + \sqrt{76}$ …(i)

$2y + x = 8$ …(ii)

On solving (i) and (ii), we get

$x = \frac{8-2\sqrt{76}}{5}, \; y = \frac{16+\sqrt{76}}{5}$

Since (x, y) lies on the circle,

$\therefore$ radius $= \sqrt{(x-0)^2 + (y-0)^2}$

$= \sqrt{\left(\frac{8-2\sqrt{76}}{5}\right)^2 + \left(\frac{16+\sqrt{76}}{5}\right)^2}$

$= 2\sqrt{7}$

$\therefore$ Equation of circle is $x^2 + y^2 = \left(2\sqrt{7}\right)^2$

$\Rightarrow x^2 + y^2 = 28$

157. Midpoint of (4, 0) and (0, 4) is (2, 2).

Distance between (2, 2) and centre (0, 0)

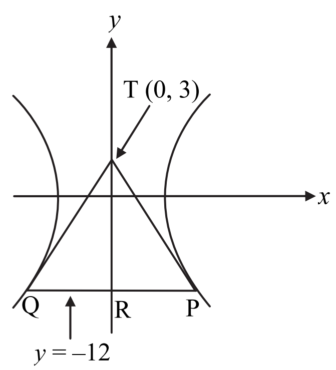
$= \sqrt{2^2 + 2^2} = 2\sqrt{2}$

158. PQ is a chord of contact.

Equation of PQ is

$\frac{xx_1}{9} - \frac{yy_1}{36} = 1$

$\Rightarrow 0 - \frac{3y}{36} = 1 \Rightarrow y = -12$



Substituting $y = -12$ in $4x^2 - y^2 = 36$, we get $x = \pm 3\sqrt{5}$

$P\left(3\sqrt{5}, -12\right), Q\left(-3\sqrt{5}, -12\right)$, T (0, 3)

$PQ = 6\sqrt{5}$, TR = 15

Area of ΔPTQ $= \frac{1}{2} \times PQ \times TR$

$= \frac{1}{2} \times 6\sqrt{5} \times 15$

$= 45\sqrt{5}$ sq.units

Evaluation Test

1. Since the triangle is equilateral.

$\therefore$ centroid of the triangle is same as the circumcentre

and radius of the circumcircle $= \frac{2}{3}$ (median)

$= \frac{2}{3}(3a) = 2a$

Hence, the equation of the circumcircle whose centre is at (0, 0) and radius 2a is $x^2 + y^2 = (2a)^2$

$\Rightarrow x^2 + y^2 = 4a^2$

2. Let the other end be $(t, 3-t)$.

$\therefore$ the equation of the circle in diameter form is

$(x-1)(x-t) + (y-1)(y-3+t) = 0$

$\Rightarrow x^2 + y^2 - (1+t)x - (4-t)y + 3 = 0$

$\therefore$ the centre (h, k) is given by

$h = \frac{1+t}{2}, k = \frac{4-t}{2}$

$\Rightarrow 2h + 2k = 5$

Hence, the locus is $2x + 2y = 5$.

3. We have, $x^2 - 8x + 12 = 0$

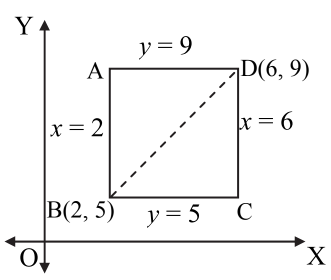
$\Rightarrow (x-6)(x-2) = 0$

$\Rightarrow x = 2, 6$

and $y^2 - 14y + 45 = 0$

$\Rightarrow (y-5)(y-9) = 0$

$\Rightarrow y = 5, 9$



Since, centre of circle is inscribed in square.

$\therefore$ BD is the diameter of circle

$\therefore$ centre = (h, k) = $\left(\frac{2+6}{2}, \frac{5+9}{2}\right)$ = (4, 7)

4. Let $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$.

According to the given condition,

$x_1 + x_2 = -2a$, $x_1x_2 = -b^2$

$y_1 + y_2 = -2p$, $y_1y_2 = -q^2$

The equation of the circle with A (x_1, y_1) and B (x_2, y_2) as the end points of diameter is

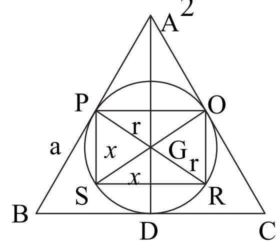
$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\Rightarrow x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 = 0$

$\Rightarrow x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

5. Let p be the altitude, then

$p = a\sin 60^\circ = \frac{a}{2}\sqrt{3}$.



Since, the triangle is equilateral, therefore centroid, orthocentre, circumcentre and incentre all coincide.

$\therefore$ radius of the inscribed circle $= \frac{p}{3} = \frac{a}{2\sqrt{3}} = r$

Let x be the side of the square inscribed, then angle in a semicircle being a right angle,

$x^2 + x^2 = (2r)^2 = 4r^2$

$\Rightarrow 2x^2 = \frac{4a^2}{12} = \frac{a^2}{3}$

$\therefore$ required area $= x^2 = \frac{a^2}{6}$

6. Given, parabola $y = x^2$ ....(i)

Straight line $y = 2x - 4$ ....(ii)

From (i) and (ii), $x^2 - 2x + 4 = 0$

Let $f(x) = x^2 - 2x + 4$

$\therefore$ $f'(x) = 2x - 2$

For least distance, $f'(x) = 0$

$\Rightarrow 2x - 2 = 0 \Rightarrow x = 1$

$\therefore$ $y = (1)^2 = 1$ ….[From (i)]

So the point least distance from the line is (1, 1).

7. For parabola, $y^2 = 8x$

$\Rightarrow 4a = 8 \Rightarrow a = 2$

vertex of $y^2 = 8x$ is $O \equiv (0, 0)$

Now, end points of latus rectum are

$L(a, 2a); L'(a, -2a) \Rightarrow L(2, 4); L'(2, -4)$

$\therefore$ the circle passes through the points (0,0), (2,4) and (2,–4).

All the three points are satisfied by the option (C).

$\therefore$ Option (C) is the correct answer.

8. Eccentricity of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $e=\sqrt{\frac{a^2+b^2}{a^2}}$

Eccentricity of conjugate hyperbola is $e'=\sqrt{\frac{a^2+b^2}{b^2}}$

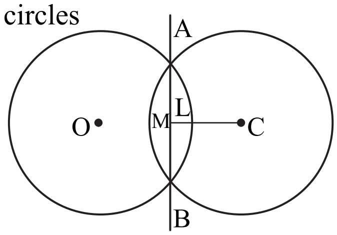
The given equation of hyperbola can be written as

$\frac{x^2}{1}-\frac{y^2}{\frac{1}{3}}=1$

Here, $a^2=1, b^2=\frac{1}{3}$

$\therefore$ $e'=\sqrt{\frac{1+\frac{1}{3}}{\frac{1}{3}}}=\sqrt{4}=2$

9. Let AB be the line of intersection of the two circles



$x^2+y^2=25$ ....(i)

$x^2+y^2-8x+7=0$ ....(ii)

Solving (i) and (ii), we get coordinates of A and B.

Subtracting (i) from (ii), we get

$-8x+32=0 \Rightarrow x=4$

From (i), we get $16+y^2=25$

$\Rightarrow y^2=9 \Rightarrow y=\pm 3$

Thus coordinates of A and B are (4, 3) and (4, –3).

$\therefore$ equation of L is $\frac{y-3}{3+3}=\frac{x-4}{4-4} \Rightarrow x-4=0$

Also coordinates of centre C of second circle is (4, 0).

Hence, CM = Length of perpendicular from C to the line $L=\left|\frac{4-4}{\sqrt{1}}\right|=0$

10. Here $a^2 = 16, b^2 = 9$

But $b^2 = a^2(1 - e^2)$

$\therefore$ $9 = 16(1 - e^2)$

$\Rightarrow e^2 = 1-\frac{9}{16}=\frac{7}{16} \Rightarrow e=\frac{\sqrt{7}}{4}$

Now, foci are (ae, 0), (–ae, 0)

i.e., $\left(\sqrt{7},0\right)$, $\left(-\sqrt{7},0\right)$

Centre of the circle is (0, 3)

$\therefore$ radius of the circle is

$=\sqrt{\left(\sqrt{7}-0\right)^2+\left(0-3\right)^2}=\sqrt{7+9}=4$

11. The equation of the ellipse is $\frac{x^2}{4}+\frac{y^2}{1}=1$.

Let e be its eccentricity.

Then, $e=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$

The foci of the ellipse are $S\left(\sqrt{3},0\right)$ and $S'\left(-\sqrt{3},0\right)$.

Eccentricity of the hyperbola $=\frac{1}{e}=\frac{2}{\sqrt{3}}$

$\therefore$ $b^2=a^2\left(\frac{4}{3}-1\right)=\frac{a^2}{3}$ ....(i)

The hyperbola passes through $S\left(\sqrt{3},0\right)$.

$\therefore$ $\frac{3}{a^2}-0=1 \Rightarrow a^2=3$

Putting $a^2 = 3$ in (i), we get

$b^2 = 1$

Hence, the equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{1}=1$ i.e., $x^2-3y^2=3$.

12. Semi minor axis = b = 2

Semi major axis = a = 4

Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\Rightarrow \frac{x^2}{16}+\frac{y^2}{4}=1$

$\Rightarrow x^2+4y^2=16$

13. Given equation of circle is

$x^2+y^2-4x-8y-5=0$

$\therefore$ Centre = (2, 4) and radius = $\sqrt{4+16+5}=5$

the circle is intersecting the line $3x-4y=m$ at two distinct points.

$\therefore$ length of perpendicular from centre on the line < radius

$\Rightarrow \left|\frac{6-16-m}{5}\right|<5$

$\Rightarrow |10+m|<25$

$\Rightarrow -25<m+10<25 \Rightarrow -35<m<15$

14. Let P = (1, 0) , Q(–1, 0) and A = (x, y)

Now, $\frac{AP}{AQ} = \frac{BP}{BQ} = \frac{CP}{CQ} = \frac{1}{3}$

$\Rightarrow \frac{AP}{AQ} = \frac{1}{3}$

$\Rightarrow 3AP = AQ \Rightarrow 9AP^2 = AQ^2$

$\Rightarrow 9(x-1)^2 + 9y^2 = (x+1)^2 + y^2$

$\Rightarrow 9x^2 - 18x + 9 + 9y^2 = x^2 + 2x + 1 + y^2$

$\Rightarrow 8x^2 - 20x + 8y^2 + 8 = 0$

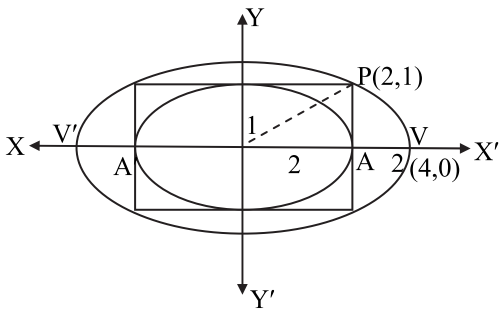
$\Rightarrow x^2 + y^2 - \frac{5}{2}x + 1 = 0$ ....(i)

Since, points A, B and C lies on the circle

$\therefore$ Circumcentre of ABC = Centre of Circle (i)

$= \left(\frac{5}{4}, 0\right)$

15.



Given equation of ellipse is

$x^2 + 4y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = 1 \Rightarrow a = 2, b = 1$

$\Rightarrow$ P(2, 1)

Let the required equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since, the ellipse passes through (4, 0).

$\therefore$ a = 4

Also, it is passes through P(2, 1).

$\therefore \frac{4}{16} + \frac{1}{b^2} = 1$

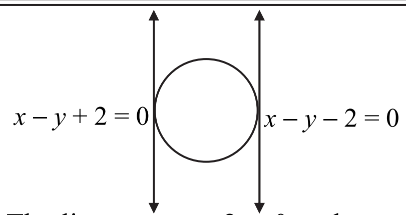
$\Rightarrow \frac{1}{b^2} = 1 - \frac{1}{4}$

$\Rightarrow b^2 = \frac{4}{3}$

$\therefore$ equation of ellipse becomes $\frac{x^2}{16} + \frac{3y^2}{4} = 1$

$\Rightarrow x^2 + 12y^2 = 16$

16.



The lines $x - y - 2 = 0$ and $x - y + 2 = 0$ are parallel, and tangent to the circle.

Distance between them = diameter of the circle

$= \frac{2-(-2)}{\sqrt{1^2+1^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

Let (h, k) be the centre of the circle.

Since, $x + y = 0$ is the diameter.

$\therefore$ h + k = 0

$\Rightarrow$ h = –k ....(i)

Now, perpendicular drawn from (h, k) to the $x - y - 2 = 0$ is equal to radius.

$\therefore \left|\frac{h-k-2}{\sqrt{2}}\right| = \sqrt{2}$

$\therefore \left|\frac{-k-k-2}{\sqrt{2}}\right| = \sqrt{2}$ ....[From (i)]

$\Rightarrow$ 2k + 2 = 2 k = 0

$\therefore$ h = 0 ....[From (i)]

$\therefore$ required equation of circle is

$(x-0)^2 + (y-0)^2 = \left(\sqrt{2}\right)^2$

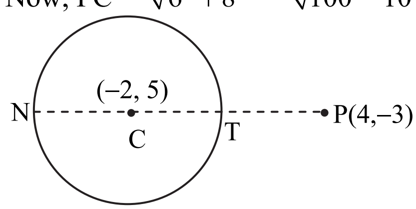
$\Rightarrow x^2 + y^2 = 2$

17. Centre of the given circle = C(–2, 5)

Radius of the circle CN = CT $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{2^2 + 5^2 + 7} = \sqrt{36} = 6$

Now, PC = $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$



We join the external point, (4, –3) to the centre of the circle (–2, 5). Then PT is the minimum distance, from external point P to the circle and PN is the maximum distance.

Minimum distance = PT = PC – CT = 10 – 6 = 4

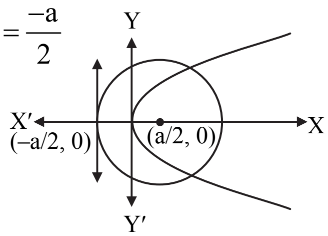
Maximum distance = PN = PC + CN = 10 + 6 = 16

So, sum of minimum and maximum distance = 16 + 4 = 20

18. General equation of circle
$f(x, y) = (x - h)^2 + (y - k)^2 - a^2 = 0$
$f(0, \lambda) = h^2 + (\lambda - k)^2 - a^2 = 0$
$\Rightarrow \lambda^2 - 2k\lambda + k^2 + h^2 - a^2 = 0$
Equation has equal roots (1, 1)
$\Rightarrow$ Sum of roots $= -\dfrac{b}{a} \Rightarrow 2 = 2k \Rightarrow k = 1$
Also, $f(\lambda, 0) = (\lambda - h)^2 + (0 - k)^2 - a^2 = 0$
$\lambda^2 - 2h\lambda + h^2 + k^2 - a^2 = 0$
Equation has roots $\lambda = \dfrac{1}{2}$ and 2
Sum of roots $= -\dfrac{b}{a} \Rightarrow \dfrac{5}{2} = 2h \Rightarrow h = \dfrac{5}{4}$
$\therefore$ Centre $(h, k) = \left(\dfrac{5}{4}, 1\right)$

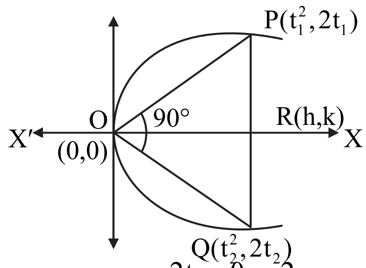
19. Here, $y = x^2 - 4x + 3$
$\Rightarrow y + 1 = x^2 - 4x + 4 \Rightarrow y + 1 = (x - 2)^2$
So, the vertex is (2, –1).
and for the circle, $x^2 + (y - 3)^2 = 9$
Centre = (0, 3)
Distance between vertex and centre
$= \sqrt{2^2 + (-4)^2} = 2\sqrt{5}$

20. Given parabola is $y^2 = 2ax$
$\therefore$ Focus (a/2, 0) and directrix is given by $x = -a/2$
Since, circle touches the directrix.
$\therefore$ Radius of circle = distance from the point (a/2, 0) to the line $x = \dfrac{-a}{2}$



$\therefore$ Radius $= \dfrac{\left|\dfrac{a}{2} + \dfrac{a}{2}\right|}{\sqrt{1}} = a$
$\therefore$ Equation of circle is
$\left(x - \dfrac{a}{2}\right)^2 + y^2 = a^2$ ....(i)
Also, $y^2 = 2ax$ ....(ii)
Solving (i) and (ii), we get
$x = \dfrac{a}{2}, -\dfrac{3a}{2}$
Putting these values in $y^2 = 2ax$ we get
$y = \pm a$ and $x = -3a/2$ gives imaginary values of y.
$\therefore$ Required points are $(a/2, \pm a)$.

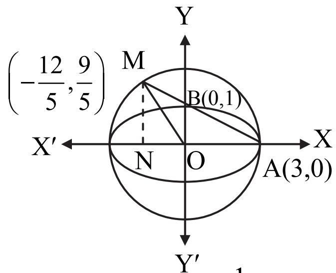
21. Let the coordinates of P and Q are $(t_1^2, 2t_1), (t_2^2, 2t_2)$ on parabola $y^2 = 4x$.



Slope of OP $= \dfrac{2t_1 - 0}{t_1^2 - 0} = \dfrac{2}{t_1}$
Slope of OQ $= \dfrac{2t_2 - 0}{t_2^2 - 0} = \dfrac{2}{t_2}$
Slope of OP × slope of OQ = –1
$\Rightarrow \dfrac{2}{t_1} \cdot \dfrac{2}{t_2} = -1 \Rightarrow t_1 t_2 = -4$ ….(i)
Let the coordinates of mid point of PQ are (h, k)
$\therefore$ $t_1^2 + t_2^2 = 2h$ ….(ii)
$t_1 + t_2 = k$ ….(iii)
Now, $(t_1 + t_2)^2 = t_1^2 + t_2^2 + 2t_1t_2$
$\Rightarrow k^2 = 2h + 2(-4)$ ….[From (i), (ii) and (iii)]
$\Rightarrow y^2 = 2x - 8$, which is required locus.

22. According to the given condition,
$\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 8$ ….(i)
Squaring on both sides, we get
$(x - 2)^2 + y^2 = 64 + (x + 2)^2 + y^2 - 2 \times 8\sqrt{(x+2)^2 + y^2}$
$\Rightarrow -4x = 64 + 4x - 16\sqrt{(x+2)^2 + y^2}$
$\Rightarrow -x - 8 = -2\sqrt{(x+2)^2 + y^2}$
Put $y = 3$ in (i), we get
$-x - 8 = -2\sqrt{(x+2)^2 + 9}$
Squaring on both sides, we get
$x^2 + 64 + 16x = 4(x + 2)^2 + 36$
$3x^2 = 12$
$\Rightarrow x^2 = 4$
$\Rightarrow x = \pm 2$

23. Equation of auxiliary circle is $x^2 + y^2 = 9$ ….(i)
Equation of AB i.e., AM is $\dfrac{x}{3} + \dfrac{y}{1} = 1$ ….(ii)
Solving (i) and (ii), we get $M\left(-\dfrac{12}{5}, \dfrac{9}{5}\right)$



Now, area of $\Delta AOM = \frac{1}{2} \times OA \times MN$

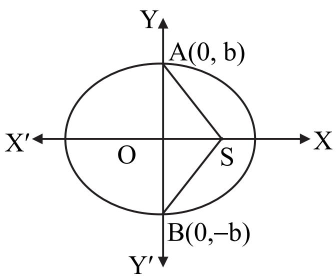
$= \frac{1}{2} \times 3 \times \frac{9}{5} = \frac{27}{10}$ sq. unit.

24. In the given figure, S is focus whose coordinates are (ae, 0).

$\because$ ΔABS is an equilateral triangle.

$\therefore$ Area of $\Delta ABS = \frac{1}{2} \times AB \times OS = \frac{\sqrt{3}}{4}(\text{side})^2$

$\Rightarrow \frac{1}{2} \times 2b \times ae = \frac{\sqrt{3}}{4}(2b)^2$



$\Rightarrow ae = \sqrt{3}\,b$ ....(i)

Also, $b^2 = a^2(1 - e^2)$

$\Rightarrow \left(\frac{ae}{\sqrt{3}}\right)^2 = a^2(1 - e^2)$ ....[From (i)]

$\Rightarrow e^2 = 3 - 3e^2$

$\Rightarrow e = \frac{\sqrt{3}}{2}$

25. The auxiliary circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$

Area of this circle $= \pi a^2$

$\therefore$ $\pi a^2 = 2 \times \pi ab$

$\Rightarrow a = 2b$

Eccentricity of ellipse $= \sqrt{1 - \frac{b^2}{a^2}}$

$= \sqrt{1 - \frac{b^2}{4b^2}} = \frac{\sqrt{3}}{2}$

26. Locus of the point P, if A and B are fixed and PA + PB = constant, is an ellipse.

We have, PA + PB = 4, which is a constant.

$\therefore$ Locus of the point P is an ellipse.

27. Locus of the point P, if A and B are fixed and $\angle APB = \frac{\pi}{2}$, is a circle with diameter AB.

But, we have $PA^2 + PB^2$ = constant.

$\therefore$ Locus of the point P is a circle.

28. $y = 7x - 25$ ....(i)

and $x^2 + y^2 = 25$

$\therefore$ $x^2 + (7x - 25)^2 = 25$

$\Rightarrow x^2 + 49x^2 + 625 - 350x = 25$

$\Rightarrow 50x^2 - 350x + 600 = 0$

$\Rightarrow x^2 - 7x + 12 = 0 \Rightarrow x = 3, 4$

Substituting $x = 3,4$ in (i), we get

$y = 21 - 25 \Rightarrow y = -4$, $y = 28 - 25 \Rightarrow y = 3$

Let $A \equiv (3, -4)$, $B \equiv (4, 3)$

Using distance formula, we get

$AB = \sqrt{(3-4)^2 + (-4-3)^2}$

$= \sqrt{1 + 49} = \sqrt{50} = 5\sqrt{2}$

Textbook Chapter No.

01 Sets, Relations and Functions

Hints

Classical Thinking

1. Adding 1 to even integers give odd integers.

6. $A \cup B = A$ if every element of B is contained in A i.e $B \subset A$

9. There is no real number which is both rational as well as irrational.

11. $A - B = \{x: x \in A \text{ and } x \notin B\} = A \cap B'$

12. $A - (B \cup C) = (A - B) \cap (A - C)$

15. $A = \{1, 2, 3, 4, 5, \ldots\}, B = \{2, 4, 6, 8, \ldots\}$
$\therefore$ $A \cap B = \{2, 4, 6, 8 \ldots.\}$

16. $B = \{2, 4, 6, 8, \ldots\}, C = \{1, 3, 5, 7, \ldots\}$
$B \cap C = \phi$

17. $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \ldots..\}$
$B = \{4, 8, 12, 16, 20, \ldots.\}$
$A \cup B = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20 \ldots.\}$.

19. $B \cup C = \{1, 3, 4, 5, 6, 7, 8, 9\}$
$A \cap B = \{5, 7\}, A \cap C = \{4, 8\}$
$A \cap (B \cup C) = \{4, 5, 7, 8\}$
$(A \cap B) \cup (A \cap C) = \{4, 5, 7, 8\}$

20. $B \cap C = \{\ \}$
$A \cup (B \cap C) = \{2, 4, 5, 7, 8\}$
$(A \cup B) \cap (A \cup C) = \{2, 4, 5, 7, 8\}$

21. $A = \{2, 4, 6, 8, 10, \ldots\}, B = \{5, 10, 15, 20, \ldots\}$
$C = \{10, 20, 30, 40, \ldots\}$
and $(A \cap B) = \{10, 20, 30, \ldots\}$
$\therefore$ $(A \cap B) \cap C = \{10, 20, 30, \ldots\}$

22. $(A \cap B)' = A' \cup B'$

23. Since A and B are disjoint,
$\therefore$ $A \cap B = \phi$
$\therefore$ $n(A \cap B) = 0$
Now $n(A \cup B) = n(A) + n(B) - n(A \cup B)$
$= n(A) + n(B) - 0$
$= n(A) + n(B)$.

24. Since A, B, C are disjoint sets.
$\therefore$ $n\,(A \cup B \cup C) = n(A) + n(B) + n(C) = 21$

25. $n(A) = 25, n(B) = 20$ and $n(A \cup B) = 35$
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$35 = 25 + 20 - n(A \cap B)$
$\Rightarrow n(A \cap B) = 10$

26. $n(U) = 100$
A = Students who play cricket, $n(A) = 60$
B = Students who play volleyball, $n(B) = 50$
$A \cap B$ = Students who play both the games, $n(A \cap B) = 28$
$\therefore$ Number of students who play atleast one game $= n(A \cup B) = n(A) + n(B) - n(A \cap B) = 82$

27. T = Set of members who like tea, $n(T) = 11$
C = Set of members who like coffee, $n(C) = 14$
$\therefore$ $n(T \cup C) = 20$
$T \cap C'$ = Set of members who like only tea and not coffee.
$\therefore$ $n(T \cup C') = n(T) - n(T \cap C)$
$T \cap C$ = Set of members who like both tea and coffee
$\therefore$ $n(T \cap C) = n(T) + n(C) - n(T \cup C) = 5$
$\therefore$ $n(T \cap C) = 5$
$\therefore$ $n(T \cup C') = n(T) - n(T \cap C) = 11 - 5 = 6$

28. $A \times B = \{(1, 0), (1, 1), (2, 0), (2, 1)\}$

30. $n(A \times A \times B) = n(A).\ n(A).\ n(B) = 3 \times 3 \times 4 = 36$

32. $A = \{2, 3\}, B = \{2, 4\}, C = \{4, 5\}$
$\therefore$ $(B \cap C) = \{4\}$
$\therefore$ $A \times (B \cap C) = \{2, 3\} \times \{4\} = \{(2, 4), (3, 4)\}$

33. $A \cap B = \{3\}$ and $A = \{1, 2, 3\}$

35. $A - B = \{1\}, B - C = \{4\}$
$\therefore$ $(A - B) \times (B - C) = \{(1, 4)\}$

36. Since (a, 2), (b, 3), (c, 2), (d, 3), (e, 2) are elements of $A \times B$
$\therefore$ $a, b, c, d, e \in A$ and $2, 3 \in B$

38. $A = \{a, b\}, B = \{1, 2, 3\}$
$\therefore$ $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
$B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, c)\}$
$\therefore$ $(A \times B) \cap (B \times A) = \phi$

39. $(Y \times A) \cap (Y \times B) = Y \times (A \cap B) = Y \times \phi = \phi$

40. Dom (R) = $\{1, 2, 3\}$

42. Since $x \nless x$ therefore R is not reflexive. Also $x < y$ does not imply that $y < x$, So R is not symmetric. Let x Ry and y Rz. Then, x, y and $y < z \Rightarrow x < z$ i.e., x Rz. Hence, R is transitive.

44. $f(x) = x^2 - 3x + 2 \Rightarrow f(-1) = (-1)^2 - 3(-1) + 2$

45. $f(x) = x^2 - 3x + 2$
$f(a + h) = (a + h)^2 - 3(a + h) + 2$
$= a^2 + (2a - 3)h - 3a + 2 + h^2$

46. $f(x) = x^2 + \frac{1}{x}$
$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^2 + \frac{1}{\left(\frac{1}{x}\right)} = \frac{1}{x^2} + x$

47. $f(x) = x^2 - 6x + 9, 0 \le x \le 4$
$f(3) = (3)^2 - 6(3) + 9 = 0$

48. $f(x) = x^2 - 6x + 5, 0 \le x \le 4$
f(8) does not exist (since $x = 8$ does not belong to the domain of f).

49. $f(x) = ax + 6 \Rightarrow f(1) = a(1) + 6 = a + 6$
$f(1) = 11 \Rightarrow 11 = a + 6 \Rightarrow a = 5$

50. $f(a + 1) - f(a - 1)$
$= 4(a + 1) - (a + 1)^2 - [4(a - 1) - (a - 1)^2]$
$= 4(2 - a)$

52. $\frac{3x^2 + 7x - 1}{3} = x^2 + \frac{7}{3}x - \frac{1}{3}$ is a polynomial function.

55. As f (b) is not defined, f is not a function.

56. $y = 2x - 3 \Rightarrow x = \frac{y+3}{2}$
$\Rightarrow f^{-1}(y) = \frac{y+3}{2} \Rightarrow f^{-1}(x) = \frac{x+3}{2}$

57. $g[f(x)] = 5[f(x)] - 6 = 5x^2 - 6$

58. Since $f(x) = 3x - 1$, $g(x) = x^2 + 1$
$\therefore$ $f[g(x)] = 3[g(x)] - 1 = 3[x^2 + 1] - 1 = 3x^2 + 2$

59. $f(f(x)) = f(x^2 + 1) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 2$

60. $(fog)(x) = f[g(x)] = f(x^3 + 1) = (x^3 + 1)^2$

61. $f\left(f\left(\frac{1}{x}\right)\right) = f\left(1 - \frac{1}{1/x}\right) = f(1 - x) = \frac{x}{x-1}$

62. $f(x) = \frac{x-1}{x+1}$
$\Rightarrow f\left(\frac{1}{f(x)}\right) = f\left(\frac{x+1}{x-1}\right) = \frac{\frac{x+1}{x-1} - 1}{\frac{x+1}{x-1} + 1} = \frac{1}{x}$

63. $f[g(x)] = \frac{3[g(x)] + 4}{5[g(x)] - 7} = \frac{3\left[\frac{7x+4}{5x-3}\right] + 4}{5\left[\frac{7x+4}{5x-3}\right] - 7} = x$

65. $-1 \le 5x \le \Rightarrow \frac{-1}{5} \le x \le \frac{1}{5}$
Hence, domain is $\left[\frac{-1}{5}, \frac{1}{5}\right]$.

66. For Dom(f), $5x - 7 > 0 \Rightarrow x > \frac{7}{5}$
Hence, $D_f = \left(\frac{7}{5}, \infty\right)$

67. $f(x) = \frac{(x-2)(x-1)}{(x-2)(x+3)}$
Hence, domain is $\{x : x \in R, x \ne 2, x \ne -3\}$.

68. For $x = -3, 3$, $|x^2 - 9| = 0$
Therefore, $\log|x^2 - 9|$ does not exist at $x = -3, 3$.
Hence, domain of function is $R - \{-3, 3\}$

69. $\log\left\{\frac{5x - x^2}{6}\right\} \ge 0 \Rightarrow \frac{5x - x^2}{6} \ge 1$
$\Rightarrow x^2 - 5x + 6 \le 0$ or $(x - 2)(x - 3) \le 0$.
Hence, $2 \le x \le 3$.

Critical Thinking

1. $\subset$ is a relation between two sets and 0 is not a set.
2. There is no real number x such that $x^2 + 1 = 0$.
3. Q is not a null set because Q = {0}
4. Number of proper subsets of A $= 2^n - 1$
$= 2^5 - 1$ ....[$\because$ o(A) = 5]
$= 32 - 1 = 31$
5. A – B is the set of those elements of A which are not common with B.
6. A – B = A iff A and B have no element in common.
7. A – B and B – A are always disjoint and hence A – B = B – A only if either of these is ϕ i.e., if $A \subset B$ and $B \subset A$ i.e., if A = B.
8. A – B, B – A and $A \cap B$ are pairwise disjoint and their union is $A \cup B$.

11. $A \cup B = \{1, 2, 3, 4, 6\}$
$\Rightarrow (A \cup B)' = \{5, 7, 8\}$

13. $A = \{3, 4\}$, $B = \{-3, 4\}$, $A \cap B = \{4\}$

14. $B = \{-3, 4\}$, $C = \{3, 5\}$, $B \cup C = \{-3, 3, 4, 5\}$

15. $C = \{1, 3, 5, 7,\ldots\}$, $D = \{2, 3, 5, 7, 11,\ldots\}$
$C \cap D = \{3, 5, 7, 11,\ldots.\}$

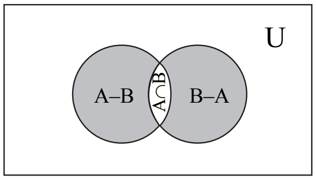
16. $A = \{2, 4, 6, 8, 10,\ldots\}$, $B = \{5, 10, 15, 20,\ldots\}$,
$C = \{10, 20, 30, 40,\ldots\}$
and $(B \cup C) = \{5, 10, 15, 20,\ldots\}$
$\therefore$ $A \cap (B \cup C) = \{10, 20, 30,\ldots\}$

17. $n(A) = n(X) - n(A') = 19$
$n(B) = n(X) - n(B') = 14$
$n(A \cap B) = n(X) - n(A \cap B)' = 5$
$\therefore$ $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 28$

18. For any $(a, b) \in A \times B$, $a \in A$ and $b \in B$.
Now (a, b) will belong to $B \times A$ only if $a \in B$ and $b \in A$ and that can happen only if $A \cap B \neq \phi$. But, in this case $A \cap B = \phi$.
$\therefore$ $(A \times B) \cap (B \times A) = \phi$

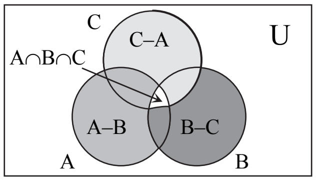
19. $A \cap (A \cup B)' = A \cap (A' \cap B')$, ….[$\because$ $(A \cup B)' = A' \cap B'$]
$= (A \cap A') \cap B'$ ….[by associative law]
$= \phi \cap B'$, ….[$\because$ $A \cap A' = \phi$]
$= \phi$.

20. From Venn-Euler's diagram,



$\therefore$ $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$

21. From Venn-Euler's Diagram,



Clearly, $\{(A - B) \cup (B - C) \cup (C - A)\}' = A \cap B \cap C$.

22. $A = \{4, 5\}$, $B = \{-6, -7\}$, $C = \{-7, 10\}$
$(B \cap C) = \{-7\} \Rightarrow A \cap (B \cap C) = \phi$

23. $A = \{x/6x^2 + x - 15 = 0\}$
$\therefore$ $6x^2 + x - 15 = 0$
$\Rightarrow (3x + 5)(2x - 3) = 0$
$\therefore$ $x = -\dfrac{5}{3}$ or $x = \dfrac{3}{2}$

$\Rightarrow A = \left\{-\dfrac{5}{3}, \dfrac{3}{2}\right\}$

Similarly, $B = \left\{1, \dfrac{3}{2}\right\}$ and $C = \left\{-1, \dfrac{3}{2}\right\}$

$A \cap B \cap C = \left\{\dfrac{3}{2}\right\}$

24. Let $A \equiv$ set of persons who take tea and
$B \equiv$ set of persons who take coffee
$n(A \cup B) = 50$, $n(A) = 35$, $n(B) = 25$
$\therefore$ $n(A \cap B) = 10$
Hence, $n(A - B) = n(A) - n(A \cap B)$
$= 35 - 10 = 25$

25. P = Set of children who like pizza
B = Set of children who like burger
$n(P) = 62$, $n(B) = 47$, $n(P \cap B) = 36$
$(P \cap B')$ = Set of children who like pizza but not burger
$\therefore$ $n(P \cap B') = n(P) - n(P \cap B) = 62 - 36 = 26$.

26. A = Set representing no. of consumers using Brand A, $n(A) = 15$
B = Set representing no. of consumers using Brand B, $n(B) = 20$
$A \cap B$ = Set representing no. of consumers using both the brands, $n(A \cap B) = 5$
$A \cup B$ = Set representing no. of consumers using atleast one brand.
$\therefore$ $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30$

27. Minimum value of $x = 100 - (30+20+25+15)$
$= 100 - 90 = 10$.

28. U = Universal set of all adults
M = Set of all males, F = Set of all females
V = Set of all vegetarians
Total number of adults = 20
Total number of males = 8
$\therefore$ Total number of females = 20 – 8 = 12
Total number of vegetarian = 9
Total number of male vegetarian = 5
$\therefore$ Total number of female vegetarian = 9 – 5 = 4
$\therefore$ Total number of female non-vegetarian
= 12 – 4 = 8

29. C = Set of students who play chess
T = Set of students who play table tennis
M = Set of students who play carrom
$\therefore$ $n(X) = 120$, $n(C) = 46$, $n(T) = 30$, $n(M) = 40$
$n(C \cap T) = 14$, $n(T \cap M) = 10$, $n(C \cap M) = 8$,
$n(C \cup T \cup M)' = 30$
$\therefore$ $n(C \cup T \cup M) = n(X) - n(C \cup T \cup M)' = 90$

$(C \cap T \cap M)$ = Set of students who play chess, table tennis and carrom.

$\therefore$ $n(C \cup T \cup M)$

$= n(C) + n(T) + n(M) - n(C \cap T) - n(T \cap M) - n(C \cap M) + n(C \cap T \cap M)$

$\therefore$ $90 = 46 + 30 + 40 - 14 - 10 - 8 + n(C \cap T \cap M)$

$\therefore$ $n(C \cap T \cap M) = 6$

30. Since, $y = e^x$, $y = e^{-x}$ will meet, when $e^x = e^{-x}$

$\Rightarrow e^{2x} = 1$,

$\therefore$ $x = 0, y = 1$

$\therefore$ A and B meet on (0, 1)

$\therefore$ $A \cap B \neq \phi$

31. Let A denote the set of Americans, who like cheese and let B denote the set of Americans, who like apples.

Let Population of Americans be 100.

Then $n(A) = 63$, $n(B) = 76$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 63 + 76 - n(A \cap B)$

$\therefore$ $n(A \cup B) + n(A \cap B) = 139$

$\Rightarrow n(A \cap B) = 139 - n(A \cup B)$

But, $n(A \cup B) \leq 100$

$\therefore$ $-n(A \cup B) \geq -100$

$\therefore$ $139 - n(A \cup B) \geq 139 - 100 = 39$

$\therefore$ $n(A \cap B) \geq 39$ i.e., $39 \leq n(A \cap B)$

Again, $A \cap B \subseteq A$, $A \cap B \subseteq B$

$\therefore$ $n(A \cap B) \leq n(A) = 63$ and

$n(A \cap B) \leq n(B) = 76$

$\therefore$ $n(A \cap B) \leq 63$

Then, $39 \leq n(A \cap B) \leq 63 \Rightarrow 39 \leq x \leq 63$

32. Since, $8^n - 7n - 1 = (7 + 1)^n - 7n - 1$

$= 7^n + {}^nC_1 7^{n-1} + {}^nC_2 7^{n-2} + \ldots + {}^nC_{n-1}7 + {}^nC_n - 7n - 1$

$= {}^nC_2 7^2 + {}^nC_3 7^3 + \ldots + {}^nC_n 7^n$,

$({}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}$ etc.)

$= 49[{}^nC_2 + {}^nC_3(7) + \ldots + {}^nC_n 7^{n-2}]$

$\therefore$ $8^n - 7n - 1$ is a multiple of 49 for $n \geq 2$

For $n = 1$, $8^n - 7n - 1 = 8 - 7 - 1 = 0$

For $n = 2$, $8^n - 7n - 1 = 64 - 14 - 1 = 49$

$\therefore$ $8^n - 7n - 1$ is a multiple of 49 for $n \in N$.

$\therefore$ X contains elements which are multiples of 49 and clearly Y contains all multiples of 49.

$\therefore$ $X \subseteq Y$

33. The given set is a cartesian product containing 6 elements. Only $A \times (B \cup C)$ contains 6 elements.

34. Here $1, 2, 3 \in A$ & $3, 5 \in B$

$\therefore$ $A \times B = \{1, 2, 3\} \times \{3, 5\}$

$\therefore$ The remaining elements are : (1, 5), (2, 3), (3, 5)

35. Clearly, A is the set of all first elements in ordered pairs in $A \times B$ and B is the set of all second elements in $A \times B$.

36. $(1, 4), (2, 6), (3, 6) \in A \times B$

$\Rightarrow \{1, 2, 3\} \subset A$ and $\{4, 6\} \subset B$

$\therefore$ A has 3 elements and B has 2 elements.

37. Number of relations on the set A = Number of subsets of $(A \times A) = 2^{n^2}$, $[\because n(A \times A) = n^2]$.

38. $n(A \times A) = n(A).\ n(A) = 3^2 = 9$

So, the total number of subsets of $A \times A$ is 2^9 and a subset of $A \times A$ is a relation over the set A.

39. Since, $(-1, 0) \in A \times A$ and $(0, 1) \in A \times A$

$\therefore$ $(-1, 0) \in A \times A \Rightarrow -1, 0 \in A$

and $(0, 1) \in A \times A \Rightarrow 0, 1 \in A$

$\therefore$ $\{-1, 0, 1\} \in A$

41. $R_2 \subseteq A \times B$, so it is a relation from A to B.

42. Number of relations from A to B = $2^{o(A).o(B)}$

43. Since, $R = \{(x, y)|\ x, y \in Z, x^2 + y^2 \leq 4\}$

$\therefore$ $R = \{(-2, 0), (-1, 0), (-1, 1), (0\ -1)\ (0, 1), (0, 2), (0, -2)\ (1, 0), (1, 1), (2, 0)\}$

Hence, Domain of $R = \{-2, -1, 0, 1, 2\}$.

46. Since R is an equivalence relation on set A, therefore $(a, a) \in R$ for all $a \in A$. Hence, R has at least n ordered pairs.

47. Let $(a, b) \in R$

Then, $(a, b) \in R \Rightarrow (b, a) \in R^{-1}$

$\Rightarrow (b, a) \in R$ ....$[\because R = R^{-1}]$

So, R is symmetric.

48. For any $a \in N$, we find that $a|a$, therefore R is reflexive but R is not symmetric, because aRb does not imply that bRa.

49. The relation is not symmetric, because $A \subseteq B$ does not imply that $B \subset A$. But it is anti-symmetric because $A \subset B$ and $B \subset A \Rightarrow A = B$

50. The given relation is not reflexive and transitive but it is symmetric, because $x^2 + y^2 = 1 \Rightarrow y^2 + x^2 = 1$.

51. R is a relation from {11, 12, 13} to {8, 10, 12} defined by $y = x - 3 \Rightarrow x - y = 3$

$\therefore$ R = {11, 8},{13, 10}.

Hence, $R^{-1} = \{(8, 11), (10, 13)\}$

52. We have, R = {(1, 3); (1, 5); (2, 3); (2, 5); (3, 5); (4, 5)}

R^{-1} = {(3, 1), (5, 1), (3, 2), (5, 2); (5, 3); (5, 4)}

Hence, RoR^{-1} = {(3, 3); (3, 5); (5, 3); (5, 5)}

53. $f(x) = f(x + 1)$
$\therefore$ $x^2 - 2x + 3 = (x + 1)^2 - 2(x + 1) + 3$
$\therefore$ $x^2 - 2x = x^2 + 2x + 1 - 2x - 2 \Rightarrow x = 1/2$

54. $f(x) = ax^2 + bx + 2$
$\therefore$ $f(1) = a(1)^2 + b(1) + 2 = a + b + 2$
But $f(1) = 3 \Rightarrow 3 = a + b + 2 \Rightarrow a + b = 1$ ….(i)
and $f(4) = a(4)^2 + b(4) + 2 = 16a + 4b + 2$
But $f(4) = 42 \Rightarrow 42 = 16a + 4b + 2$
$\therefore$ $40 = 16a + 4b \Rightarrow 4a + b = 10$ ….(ii)
By solving, (i) & (ii) a = 3 and b = – 2

55. $f(x) = x + \dfrac{1}{x} \Rightarrow f(x^3) = x^3 + \dfrac{1}{x^3}$
$\therefore$ $[f(x)]^3 = \left(x + \dfrac{1}{x}\right)^3 = \left(x^3 + \dfrac{1}{x^3}\right) + 3\left(x + \dfrac{1}{x}\right)$
$\therefore$ $[f(x)]^3 = f(x^3) + 3f(x)$
$\therefore$ $[f(x)]^3 = f(x^3) + 3f\left(\dfrac{1}{x}\right) \Rightarrow \lambda = 3$

56. $a.f(x) + b.f\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 5$
On replacing x by $\dfrac{1}{x}$, $b.f(x) + a.f\left(\dfrac{1}{x}\right) = x - 5$
Solving two equations,
$f(x) = \dfrac{1}{a^2 - b^2}\left(\dfrac{a}{x} - bx\right) - \dfrac{5}{a + b}$
$\therefore$ $f(2) = \dfrac{3(2b - 3a)}{2(a^2 - b^2)}$

58. As f (a) is not unique,
$\therefore$ f is not a function.

60. $[x] = I$ (Integers only).

61. Let $f(x) = x^2 + \sin^2 x$
Here, $f(-x) = f(x)$
$\therefore$ $f(x)$ is an even function.

62. If $f(x) = \sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}$, then
$f(-x) = \sqrt{1 - x + x^2} - \sqrt{1 + x + x^2}$
$\therefore$ $f(-x) = -f(x)$
So, $f(x)$ is an odd function.

63. The general expression for the function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in R$ is $f(x) = [f(1)]^x = a^x$ for all $x, y \in R$. [$\because f(1) = a$]

64. $f^{-1}(y) = \{x \in R: y = f(x)\}$
$\Rightarrow f^{-1}(2) = \{x \in R: 2 = f(x)\}$
$= \{x \in R: x^2 - 3x + 4 = 2\}$
$= \{x \in R: x^2 - 3x + 2 = 0\} = \{1, 2\}$

65. $f(f(x)) = \dfrac{1}{1 - f(x)} = \dfrac{1}{1 - \dfrac{1}{1 - x}} = \dfrac{x - 1}{x}$
$\therefore$ $f[f(f(x))] = f\left(\dfrac{x - 1}{x}\right) = \dfrac{x}{x - x + 1} = x$

66. $f(x) = \dfrac{x + 3}{4x - 5}$
$\therefore$ $f(t) = \dfrac{t + 3}{4t - 5} = \dfrac{\left(\dfrac{3 + 5x}{4x - 1}\right) + 3}{4\left(\dfrac{3 + 5x}{4x - 1}\right) - 5} = x$

67. (gof) (1) = g (f (1)) = g (4) = 8,
(gof) (2) = g (f (2)) = g (5) = 7
and (gof) (3) = g (f (3)) = g (6) = 9

68. $f(x) = \dfrac{x - 1}{x + 1} \Rightarrow \dfrac{f(x) + 1}{f(x) - 1} = \dfrac{2x}{-2}$
$\Rightarrow x = \dfrac{1 + f(x)}{1 - f(x)}$
$\therefore$ $f(\alpha x) = \dfrac{\alpha x - 1}{\alpha x + 1} = \dfrac{(\alpha + 1)f(x) + \alpha - 1}{(\alpha - 1)f(x) + \alpha + 1}$

69. Given, $(gof)\left(-\dfrac{5}{3}\right) - (fog)\left(-\dfrac{5}{3}\right)$
$= g\left\{f\left(\dfrac{-5}{3}\right)\right\} - f\left\{g\left(\dfrac{-5}{3}\right)\right\}$
$= g(-2) - f\left(\dfrac{5}{3}\right) = 2 - 1 = 1$

70. $f(-1) = f(1) = 1$
$\therefore$ function is many-one function.
$\therefore$ f is neither one-one nor onto.

71. $f'(x) = \dfrac{1}{(1 + x)^2} > 0 \ \forall x \in [0, \infty)$
and range $\in [0,1)$
$\Rightarrow$ function is one-one but not onto.

72. $f(x) = \dfrac{x - 2}{x - 3}, x \neq 3$
Let $y = f(x) \Rightarrow y = \dfrac{x - 2}{x - 3} \Rightarrow x = \dfrac{2 - 3y}{1 - y}$
$\Rightarrow y \neq 1 \Rightarrow$ Range of $f(x)$ is $R - \{1\}$
So, f is onto
For one-one, let $f(x_1) = f(x_2)$
$\Rightarrow \dfrac{x_1 - 2}{x_1 - 3} = \dfrac{x_2 - 2}{x_2 - 3} \Rightarrow x_1 = x_2$
Hence, f is one-one.

73. Let $f(x_1) = f(x_2) \Rightarrow [x_1] = [x_2] \not\Rightarrow x_1 = x_2$
{For example, if $x_1 = 1.4$, $x_2 = 1.5$, then $[1.4] = [1.5] = 1$}
$\therefore$ f is not one-one.
Also, f is not onto as its range I (set of integers) is a proper subset of its co-domain R.

74. Let $x_1, x_2 \in R$, then $f(x_1) = \cos x_1$, & $f(x_2) = \cos x_2$, Now $f(x_1) = f(x_2)$
$\Rightarrow \cos x_1 = \cos x_2 \Rightarrow x_1 = 2n\pi \pm x_2$
$\Rightarrow x_1 \neq x_2$,
$\therefore$ it is not one-one.
Again the value of f-image of x lies in between -1 to 1
$\Rightarrow f[R] = \{f(x) : -1 \leq f(x) \leq 1)\}$
So other numbers of co-domain (besides -1 and 1) is not f-image. $f[R] \in R$, so it is also not onto. So this mapping is neither one-one nor onto.

75. $x^2 - 6x + 7 = (x - 3)^2 - 2$
Here, minimum value is -2 and maximum ∞.
Hence, range of function is $[-2, \infty)$.

76. For domain, take $\frac{x}{1+x} \geq 0$
$\therefore$ $D_f = (-\infty, -1) \cup [0, \infty)$

77. $1 + x \geq 0$
$\Rightarrow x \geq -1;\ 1 - x \geq 0$
$\Rightarrow x \leq 1, x \neq 0$
Hence, domain is $[-1, 1] - \{0\}$.

78. $y = \sin^{-1}\left[\log_3\left(\frac{x}{3}\right)\right]$
$\therefore$ $-1 \leq \log_3\left(\frac{x}{3}\right) \leq 1$ $\therefore$ $\frac{1}{3} \leq \frac{x}{3} \leq 3$
$\therefore$ $1 \leq x \leq 9$
$\therefore$ $x \in [1, 9]$

79. $f(x)$ is defined, if
$x^2 - 5x + 6 \geq 0$ and $2x + 8 - x^2 \geq 0$
$\Rightarrow (x - 2)(x - 3) \geq 0$ and $(x - 4)(x + 2) \leq 0$
$\therefore$ $x \in (-\infty, 2] \cup [3, \infty)$ and $x \in [-2, 4]$
$\therefore$ $x \in [-2, 2] \cup [3, 4]$

80. Domain of $f(x) = R - \{3\}$,
and for Range : $x \neq 3 \Rightarrow x < 3$ or $x > 3$
Now, $x < 3 \Rightarrow x - 3 < 0 \Rightarrow |x - 3| = -(x - 3)$
$\Rightarrow f(x) = \frac{-(x-3)}{x-3} = -1$
Similarly, for $x > 3$, $f(x) = 1$
$\therefore$ Range (f) = $\{1, -1\}$.

81. $\text{Dom}(f) = R - \left\{-\frac{2}{3}\right\}$
For Range(f), let $y = f(x) = \frac{1}{3x+2}$
$\therefore$ $3x + 2 = \frac{1}{y} \Rightarrow x = \frac{1}{3}\left(2 - \frac{1}{y}\right)$
x is real if $y \neq 0$. Hence, $R_f = R - \{0\}$

82. $f(x)$ is defined for all $x \in R - \{0\}$.
So, dom(f) = $R - \{0\}$
Let $y = \frac{1+x^2}{x^2}$
$\Rightarrow x = \pm\sqrt{\frac{1}{y-1}}$
For x to be real, $y - 1 > 0 \Rightarrow y \in (1, \infty)$

83. $f(x)$ is defined for all $x \in R$. So, dom(f) = R.
Let $y = f(x) \Rightarrow y = \frac{x}{1+x^2}$
$\therefore$ $x = \frac{1 \pm \sqrt{1 - 4y^2}}{2y}$
For x to be real, $1 - 4y^2 \geq 0$ and $y \neq 0$
$\Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2}$ and $y \neq 0$

84. $f(x)$ is defined for $x^2 + x - 6 \neq 0$, i.e., $x \neq -3, 2$
$\therefore$ Dom(f) = $R - \{-3, 2\}$
Let $y = \frac{x^2 - 3x + 2}{x^2 + x - 6} = \frac{x-1}{x+3}$
$\Rightarrow x = \frac{3y+1}{y-1}$
x is real for $y - 1 \neq 0$, i.e., $y \neq 1$
Hence, range(f) = $R - \{1\}$

85. Here, $f(x) = \sqrt{x^2 + x + 1}$
$\Rightarrow y^2 = x^2 + x + 1$
$\Rightarrow x^2 + x + (1 - y^2) = 0$
$\Rightarrow x = \frac{-1 \pm \sqrt{1 - 4(1 - y^2)}}{2}$
$\Rightarrow x = \frac{-1 \pm \sqrt{4y^2 - 3}}{2}$
For x real, $4y^2 - 3 \geq 0$
$\therefore$ $y \geq \pm\frac{\sqrt{3}}{2}$
$\therefore$ $R_f = \left[\frac{\sqrt{3}}{2}, \infty\right)$

Competitive Thinking

4. $x^2 = 16 \Rightarrow x = \pm 4$
and $2x = 6 \Rightarrow x = 3$
There is no value of x which satisfies both the above equations. Thus, $A = \phi$.

5. $|2x + 3| < 7 \Rightarrow -7 < 2x + 3 < 7$
$\Rightarrow -10 < 2x < 4 \Rightarrow -5 < x < 2 \Rightarrow 0 < x + 5 < 7$

6. **Case I:** $0 \le x < 9$
$2\left(3 - \sqrt{x}\right) + \sqrt{x}\left(\sqrt{x} - 6\right) + 6 = 0$
$\Rightarrow \left(\sqrt{x}\right)^2 - 8\sqrt{x} + 12 = 0$
$\Rightarrow \sqrt{x} = 6, 2$
$\Rightarrow x = 36, 4 \Rightarrow x = 4$
Case II: $x \ge 9$
$2\left(\sqrt{x} - 3\right) + \sqrt{x}\left(\sqrt{x} - 6\right) + 6 = 0$
$\Rightarrow \left(\sqrt{x}\right)^2 - 4\sqrt{x} = 0$
$\Rightarrow \sqrt{x} = 0, 4$
$\Rightarrow x = 0, 16 \Rightarrow x = 16$
$\Rightarrow$ S contains exactly two elements.

7. The number of non- empty subsets $= 2^n - 1$
$= 2^4 - 1$ ....[$\because$ n = 4]
$= 15$.

8. Power set is the set of all subsets.
$n(A) = 5 \Rightarrow n(P(A)) = 2^5 = 32$

9. $A = \{4, 8, 12, 16, 20, 24, \ldots\}$
$B = \{6, 12, 18, 24, 30, \ldots\}$
$\therefore$ $A \subset B = \{12, 24, \ldots\} = \{x : x \text{ is a multiple of } 12\}$.

10. Given set is $\{(a, b) : 2a^2 + 3b^2 = 35, a, b \in Z\}$
We can see that, $2(\pm 2)^2 + 3(\pm 3)^2 = 35$
and $2(\pm 4)^2 + 3(\pm 1)^2 = 35$
$\therefore$ (2, 3), (2, –3), (–2, –3), (–2, 3), (4, 1), (4, –1), (–4, –1), (–4, 1) are 8 elements of the set.
$\therefore$ n = 8.

11. Let, $S = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\}$
$\because$ $A = \{(x, y) : x, y \in S, x \ne y\}$
$\therefore$ $A = \{(a_1, a_1), (a_1, a_2), (a_1, a_3), \ldots (a_1, a_{10}),$
$(a_2, a_1), (a_2, a_2), (a_2, a_3), \ldots (a_2, a_{10}),$
.
.
.
$(a_{10}, a_1), (a_{10}, a_2), (a_{10}, a_3), \ldots (a_{10}, a_{10})\}$
$\because$ $x \ne y$, removing groups with same elements i.e. $(a_1, a_1), (a_2, a_2) \ldots (a_{10}, a_{10})$, we get
$n(A) = 9 \times 10 = 90$

12. n (S) = 10
$\therefore$ Number of subsets of S which do not contain the element 6
= number of subsets containing the remaining nine elements
$= 2^9 = 512$

13. Since $2^m - 2^n = 56 = 8 \times 7 = 2^3 \times 7$
$\Rightarrow 2^n(2^{m-n} - 1) = 2^3 \times 7$
$\therefore$ $n = 3$ and $2^{m-n} = 8 = 2^3$
$\Rightarrow m - n = 3 \Rightarrow m - 3 = 3 \Rightarrow m = 6$
$\therefore$ m = 6, n = 3

14. $O(S) = O\left(\bigcup\limits_{i=1}^{30} A_i\right) = \dfrac{1}{10}(5 \times 30) = 15$
Since, element in the union S belongs to 10 of A_i's
Also, $O(S) = O\left(\bigcup\limits_{j=1}^{n} B_j\right) = \dfrac{3n}{9} = \dfrac{n}{3}$
$\therefore$ $\dfrac{n}{3} = 15 \Rightarrow n = 45$

15. $A - (A - B) = A \cap (A \cap B^c)^c = A \cap (A^c \cup B)$
$= \phi \cup (A \cap B) = A \cap B$

16. $A = B \cap C$, $B = C \cap A$
$\Rightarrow$ A, B are equivalent sets.
….[$\because$ A and B are interchangeable in both equations]

18. $A \cap X = B \cap X = \phi$
$\therefore$ A and X, B and X are disjoint sets
Also, $A \cup X = B \cup X \Rightarrow A = B$

19. If $A \subseteq B$, then $A \cup B = B$
$\therefore$ $n(A \cup B) = n(B) = 6$

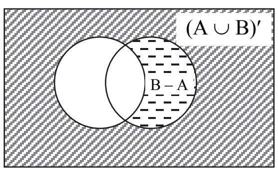
20. $n[(A \cap B)' \cap A]$
$= n[(A' \cup B') \cap A]$
….[By DeMorgan's law]
$= n(A' \cap A) \cup n(B' \cap A)$
….[By distributive law]
$= n(A) - n(A \cap B) = 8 - 2 = 6$

21. $A = \{x \mid x \text{ is a root of } x^2 - 1 = 0\}$
$= \{x \mid x \text{ is a root of } (x - 1)(x + 1) = 0\}$
$\Rightarrow x = \pm 1$
$B = \{x \mid x \text{ is a root of } x^2 - 2x + 1 = 0\}$
$= \{x \mid x \text{ is a root of } (x - 1)^2 = 0\}$
$\Rightarrow x = 1$
$\Rightarrow A \cup B = A$

22. Since, $4^n - 3n - 1 = (3+1)^n - 3n - 1$
$= 3^n + {}^nC_1 \cdot 3^{n-1} + {}^nC_2 \cdot 3^{n-2} + \ldots + {}^nC_{n-1} \cdot 3 + {}^nC_n - 3n - 1$
$= {}^nC_2 3^2 + {}^nC_3 \cdot 3^3 + \ldots + {}^nC_n \cdot 3^3$
$\ldots\left[{}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}, \text{etc.}\right]$
$= 9\left[{}^nC_2 = {}^nC_3(3) + \ldots + {}^nC_n \cdot 3^{n-1}\right]$
$\therefore$ $4^2 - 3n - 1$ is a multiple of 9 for $n \geq 2$.
for $n = 1$, $4^n - 3n - 1 = 4 - 3 - 1 = 0$,
for $n = 2$, $4^n - 3n - 1 = 16 - 6 - 1 = 9$,
$\therefore$ $4^n - 3n - 1$ is a multiple of 9 for all $n \in N$.
$\therefore$ X contains elements, which are multiples of 9, and clearly Y contains all multiples of 9.
$\therefore$ $X \subseteq Y$ i.e. $X \cap Y = X$

23. $N_5 \cap N_7 = N_{35}$,
[$\because$ 5 and 7 are relatively prime numbers].

24.



$\therefore$ $(B - A) \cap (A \cup B)' = \phi$

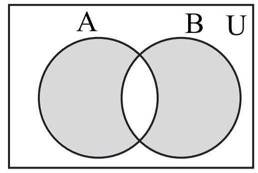
25. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$= 12 + 9 - 4 = 17$
Now, $n((A \cup B)^c) = n(U) - n(A \cup B)$
$= 20 - 17 = 3$

26. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$= 200 + 300 - 100$
$= 400$
$\therefore$ $n(A' \cap B') = n(U) - n(A \cup B)$
$= 700 - 400 = 300$

27. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$= 3 + 6 - n(A \cap B)$
Since, maximum number of elements in $A \cap B = 3$
$\therefore$ Minimum number of elements in
$A \cup B = 9 - 3 = 6$

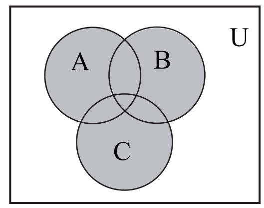
28. $(A \cup B)^C \cup (A^C \cap B)$
$= (A^C \cap B^C) \cup (A^C \cap B)$
$= A^C \cap (B^C \cup B)$
$= A^C \cap U = A^C$

29.

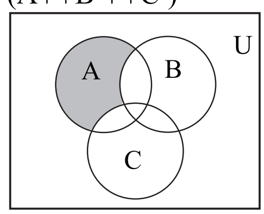


Since, $A - B = A - (A \cap B)$
and $B - A = B - (A \cap B)$
Option (D) is the correct answer.

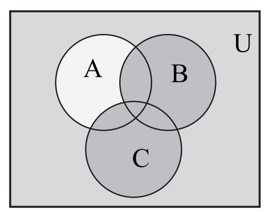
30. i. $A \cup B \cup C$



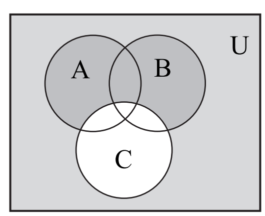
ii. $(A \cap B^c \cap C^c)$



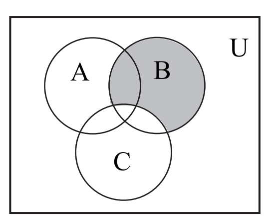
iii. $(A \cap B^c \cap C^c)^c$



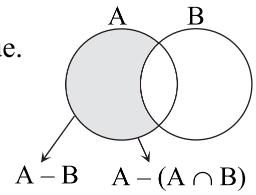
iv. C^c



v. $(A \cup B \cup C) \cap (A \cap B^c \cap C^c)^c \cap C^c = B \cap C^c$



31. $A - B = A - (A \cap B)$ is correct.
$A = (A \cap B) \cup (A - B)$ is correct.
(3) is false.
$\therefore$ (1) and (2) are true.



32. $n(A - B) = n(A) - n(A \cap B) = 25 - 10 = 15$

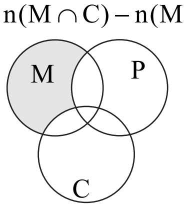
33. $n(X) = 60, n(C) = 25, n(T) = 20, n(C \cap T) = 10$
$\therefore$ $n(C \cup T) = n(C) + n(T) - n(C \cap T)$
$= 25 + 20 - 10 = 35$
$\therefore$ $n(C \cup T)' = n(X) - n(C \cup T) = 60 - 35 = 25$

34. $n(X \cap Y) = 12$ and these are 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200

35. Let number of newspapers be x. If every students reads one newspaper, the number of students would be $x(60) = 60x$
Since, every students reads 5 newspapers
$\therefore$ Numbers of students $= \dfrac{x \times 60}{5} = 300, x = 25$

36. E : English speakers
H : Hindi speakers
$n(H \cup E) = n(H) + n(E) - n(H \cap E)$
$= 50 + 20 - 10 = 60$

37. n(M alone)
$= n(M) - n(M \cap C) - n(M \cap P) + n(M \cap P \cap C)$



$= 100 - 28 - 30 + 18 = 60$

38. $n(C) = 224, n(H) = 240, n(B) = 336$
$n(H \cap B) = 64, n(B \cap C) = 80$
$n(H \cap C) = 40, n(C \cap H \cap B) = 24$
$n(C^c \cap H^c \cap B^c) = n(C \cup H \cup B)^c]$
$= n(U) - n(C \cup H \cup B)$
$= 800 - [n(C) + n(H) + n(B) - n(H \cap C)$
$- n(H \cap B) - n(C \cap B) + n(C \cap H \cap B)]$
$= 800 - [224 + 240 + 336 - 64 - 80 - 40 + 24]$
$= 800 - 640 = 160$

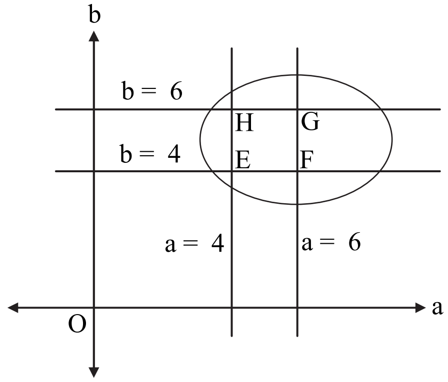
39. Given $n(N) = 12, n(P) = 16, n(H) = 18$,
$n(N \cup P \cup H) = 30$ and $n(N \cap P \cap H) = 0$
From, $n(N \cup P \cup H)$
$= n(N) + n(P) + n(H) - n(N \cap P) - n(P \cap H)$
$- n(N \cap H) + n(N \cap P \cap H)$
$\therefore$ $n(N \cap P) + n(P \cap H) + n(N \cap H) = 16$
Now, number of pupils taking two subjects
$= n(N \cap P) + n(P \cap H) + n(N \cap H)$
$- 3n(N \cap P \cap H)$
$= 16 - 0 = 16.$

40. $n(S \cup P \cup D) = 265, n(S) = 200, n(D) = 110$,
$n(P) = 55, n(S \cap D) = 60, n(S \cap P) = 30$,
$n(S \cap D \cap P) = 10$,
$n(S \cup P \cup D) = n(S) + n(D) + n(P) - n(S \cap D)$
$- n(D \cap P) - n(P \cap S) + n(S \cap D \cap P)$
$\therefore$ $265 = 200 + 110 + 55 - 60 - 30$
$- n(P \cap D) + 10$
$\therefore$ $n(P \cap D) = 285 - 265 = 20$
$\therefore$ $n(P \cap D) - n(P \cap D \cap S) = 20 - 10 = 10$

41. $n(A) = 40\%$ of $10{,}000 = 4{,}000$
$n(B) = 20\%$ of $10{,}000 = 2{,}000$
$n(C) = 10\%$ of $10{,}000 = 1{,}000$
$n(A \cap B) = 5\%$ of $10{,}000 = 500$
$n(B \cap C) = 3\%$ of $10{,}000 = 300$
$n(C \cap A) = 4\%$ of $10{,}000 = 400$
$n(A \cap B \cap C) = 2\%$ of $10{,}000 = 200$
We want to find,
$n(A \cap B^c \cap C^c) = n[A \cap (B \cup C)^c]$
$= n(A) - n[A \cap (B \cup C)]$
$= n(A) - n[(A \cap B) \cup (A \cap C)]$
$= n(A) - [n(A \cap B) + n(A \cap C)$
$- n(A \cap B \cap C)]$
$= 4000 - [500 + 400 - 200]$
$= 4000 - 700 = 3300.$

42. Since, $y = e^x$ and $y = x$ do not meet for any $x \in R$
$\therefore$ $A \cap B = \phi$

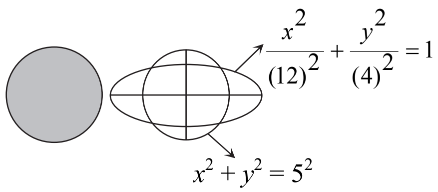
43. $|a - 5| < 1$ and $|b - 5| < 1$
$\Rightarrow 4 < a < 6$ and $4 < b < 6$
$4(a - 6)^2 + 9(b - 5)^2 \le 36$
$\Rightarrow \dfrac{(a-6)^2}{9} + \dfrac{(b-5)^2}{4} \le 1$



Set A represents square EFGH and Set B represents an ellipse.

$\Rightarrow A \subset B$

44. A = Set of all values $(x, y) : x^2 + y^2 = 25 = 5^2$



$B = \frac{x^2}{144} + \frac{y^2}{16} = 1$ i.e., $\frac{x^2}{(12)^2} + \frac{y^2}{(4)^2} = 1$

Clearly, $A \cap B$ consists of four points.

45. Let the original set contains (2n + 1) elements, then subsets of this set containing more than n elements, i.e., subsets containing (n + 1) elements, (n + 2) elements, ……. (2n + 1) elements.

$\therefore$ Required number of subsets

$= {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + ... + {}^{2n+1}C_{2n} + {}^{2n+1}C_{2n+1}$

$= {}^{2n+1}C_{n} + {}^{2n+1}C_{n-1} + ... + {}^{2n+1}C_{1} + {}^{2n+1}C_{0}$

$= {}^{2n+1}C_{0} + {}^{2n+1}C_{1} + {}^{2n+1}C_{2} + ... + {}^{2n+1}C_{n-1} + {}^{2n+1}C_{n}$

$= \frac{1}{2}\left[(1+1)^{2n+1}\right] = \frac{1}{2}[2^{2n+1}] = 2^{2n}$.

46. Since, $4^n - 3n - 1 = (3 + 1)^n - 3n - 1$

$= 3^n + {}^nC_1 3^{n-1} + {}^nC_2 3^{n-2} + ….. + {}^nC_{n-1}3 + {}^nC_n - 3n - 1$

$= {}^nC_2 3^2 + {}^nC_3.3^3 + …… + {}^nC_n 3^n$ ....$[{}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}$ etc]

$= 9[{}^nC_2 + {}^nC_3(3) + …… + {}^nC_n 3^{n-1}]$

$\therefore$ $4^n - 3n - 1$ is a multiple of 9 for $n \geq 2$.

For n = 1, $4^n - 3n - 1 = 4 - 3 - 1 = 0$,

For n = 2, $4^n - 3n - 1 = 16 - 6 - 1 = 9$

$\therefore$ $4^n - 3n - 1$ is a multiple of 9 for all $n \in N$

$\therefore$ X contains elements, which are multiples of 9, and clearly Y contains all multiples of 9.

$\therefore$ $X \subseteq Y$ i.e., $X \cup Y = Y$.

47. $R = A \times B$.

48. $A - B = \{a\}$, $B \cap C = \{c, d\}$

$\therefore$ $(A - B) \times (B \cap C) = \{a\} \times \{c, d\}$
$= \{(a, c), (a, d)\}$

49. $R \times (P^c \cup Q^c)^c = R \times [(P^c)^c \cap (Q^c)^c]$

$= R \times (P \cap Q) = (R \times P) \cap (R \times Q)$

50. A = {2, 4, 6}; B = {2, 3, 5}

$\therefore$ $A \times B$ contains $3 \times 3 = 9$ elements.

Hence, number of relations from A to B = 2^9.

51. $n((A \times B) \cap (B \times A)) = n^2 = 99^2$.

52. n(A) = 4, n(B) = 2

$\therefore$ $n(A \times B) = 4 \times 2 = 8$

Required numbers $= {}^8C_3 + {}^8C_4 + …. + {}^8C_8$

$= 2^8 - ({}^8C_0 + {}^8C_1 + {}^8C_2)$

$= 256 - 37$

$= 219$

53. In option (D), ordered pair $(a, d) \notin A \times B$. Thus it is not a relation from A to B.

54. A relation is equivalence if it is reflexive, symmetric and transitive.

56. Total number of reflexive relations in a set with n elements = 2^n.

Therefore, total number of reflexive relation set with 4 elements = 2^4.

57. $x^2 - 4x^2 + 3x^2 = 0$

$\therefore$ $xRx \Rightarrow$ Reflexive

58. Given A = {2, 4, 6, 8};

R = {(2, 4)(4, 2) (4, 6) (6, 4)}

$(a, b) \in R \Rightarrow (b, a) \in R$ and also $R^{-1} = R$.

Hence, R is symmetric.

59. For any $a \in N$, we find a|a, therefore R is reflexive.

But, R is not symmetric, because aRb does not imply that bRa.

60. Here, (3, 3), (6, 6), (9, 9), (12, 12), [Reflexive]; (3, 6), (6, 12), (3, 12), [Transitive].

Hence, reflexive and transitive only.

61. since $(5, 5) \notin S$.

$\therefore$ The relation S is not reflexive

It is symmetric and transitive.

62.

Equivalence classes	Product
(1, 11)	1
(3, 13)	3
(5, 15)	5
(7, 17)	7
(9, 19)	9
(10, 20)	0
(12, 21) (2, 12) (2, 21)	2
(4, 14) (4, 22) (14, 22)	4
(16, 23) (6, 16) (6, 23)	6
(8, 18) (8, 24) (18, 24)	8

$\therefore$ There are 10 different equivalence classes.

63. Given A = {1, 2, 3, 4}
R = {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)}
$(2, 3) \in R$ but $(3, 2) \notin R$.
Hence, R is not symmetric.
R is not reflexive as $(1, 1) \notin R$.
R is not a function as $(2, 4) \in R$ and $(2, 3) \in R$.
R is not transitive as $(1, 3) \in R$ and $(3, 1) \in R$ but $(1, 1) \notin R$.

64. For any $a \in R$, we have $a \geq a$. Therefore the relation R is reflexive, but it is not symmetric, as $(2, 1) \in R$ but $(1, 2) \notin R$. The relation R is transitive also, because $(a, b) \in R$, $(b, c) \in R$ imply that $a \geq b$ and $b \geq c$ which is turn imply that $a \geq c$.

65. $|a - a| = 0 < 1$
$\therefore$ a Ra $\forall$ $a \in R$
$\therefore$ R is reflexive.
Again, $a R b \Rightarrow |a - b| \leq 1 \Rightarrow |b - a| \leq 1 \Rightarrow bRa$
$\therefore$ R is symmetric,
Again, $1R\frac{1}{2}$ and $\frac{1}{2}R1$ but $\frac{1}{2} \neq 1$
$\therefore$ R is not anti-symmetric.
Further, 1 R 2 and 2 R 3 but $1 \not R 3$,
$[\because |1 - 3| = 2 > 1]$
$\therefore$ R is not transitive.

66. $x\rho y$, $y\rho z \Rightarrow 2x + y = 41$ and $2y + z = 41$ which do not imply $2x + z = 41$

67. for option D, $x > |x|$ is not true hence not reflexive
Take $x = 2$, $y = -1$, clearly $x > |y|$ but $y > |x|$ does not hold, hence not symmetric
Now, Let $x > |y|$ and $y > |z| \Rightarrow x, y > 0$.
$\therefore$ Rewriting, $x > |y|$ and $y > |z| \Rightarrow x > |z|$
Hence transitive.

68. r = {(a, b)| a,b $\in$ R and $a - b + \sqrt{3}$ is an irrational no.}
Here, r is reflexive as aRa = $a - a + \sqrt{3} = \sqrt{3}$ which is an irrational no.
$\sqrt{3}$ r1 = $\sqrt{3} - 1 + \sqrt{3} = 2\sqrt{3} - 1$, which is an irrational number.
But 1r$\sqrt{3}$ = $1 - \sqrt{3} + \sqrt{3} = 1$ which is not an irrational number.
$\therefore$ $\sqrt{3}$ r 1 $\Rightarrow \frac{1}{\sqrt{3}}$
$\therefore$ r is not symmetric.
Also, r is not transitive.
Since, $\sqrt{3}$ r 1 and 1 r $2\sqrt{3}$ $\not\Rightarrow$ $\sqrt{3}$ r $2\sqrt{3}$
$\therefore$ Option (B) is the correct answer.

69. On the set R;
$x\rho y \Leftrightarrow x - y = 0$ or $x - y \in Q'$
$\because$ $x - x = 0 \Rightarrow x\rho x$ (Reflexive)
if $x - y = 0 \Rightarrow y - x = 0$ or $x - y \in Q^c$
$\Rightarrow y - x \in Q'$ (Symmetric)
Take $x = 1 + \sqrt{2}$; $y = \sqrt{2} + \sqrt{3}$; $z = \sqrt{2} + 2$
$x - y = 1 - \sqrt{3} \in Q'$ and $y - z = \sqrt{3} - 2 \in Q'$
Here $x\rho y$ and $y\rho z$ but x is not related to z.
$\therefore$ Not transitive

70. Since, G. C. D. of a and a is 'a'
$\therefore$ if $a \neq 2$, then G. C. D. $\neq 2$
$\therefore$ R is not reflexive.
Let aRb
$\therefore$ G. C. D. of a, b = 2
i.e., (a, b) = 2
$\Rightarrow$ (b, a) = 2 $\Rightarrow$ G. C. D. of b, a = 2
$\therefore$ R is symmetric.
Again, let aRb and bRc
$\therefore$ $\left.\begin{array}{l}\text{G. C. D. of a, b} = 2 \\ \text{and G. C. D. of b, c} = 2\end{array}\right\} \not\Rightarrow$ G. C. D of (a, c) = 2
$\therefore$ R is not transitive.

71. Here, R = {$(x, y) \in W \times W$: the words x and y have at least one letter in common}
R is reflexive as the words x and x have all letters in common.
Hence, R is reflexive.
Also, if $(x, y) \in R$ i.e., x and y have a common letter, then y and x also have a letter in common
$\therefore$ R is symmetric.

R is not transitive as $(x, y) \in$ R and $(y, z) \in$ R need not imply $(x, z) \in$ R

For example, let x = CANE, y = NEST and z = WITH

then $(x, y) \in$ R and $(y, z) \in$ R, but $(x, z) \notin$ R

$\therefore$ R is reflexive and symmetric but not transitive.

72. For reflexive, $\theta = \phi$, so $\sec^2\theta - \tan^2\theta = 1$,

$\therefore$ R is reflexive.

For symmetric, $\sec^2\theta - \tan^2\phi = 1$

so, $(1 + \tan^2\theta) - (\sec^2\phi - 1) \Rightarrow \sec^2\phi - \tan^2\theta = 1$

$\therefore$ R is symmetric

For transitive, Let $\sec^2\theta - \tan^2\phi = 1$ ....(i)

and $\sec^2\phi - \tan^2\gamma = 1$

$\therefore$ $1 + \tan^2\phi - \tan^2\gamma = 1$

$\Rightarrow \sec^2\theta - \tan^2\gamma = 1$ ....[From (i)]

$\therefore$ R is transitive.

73. $f(x) = \sqrt{x} \Rightarrow \dfrac{f(25)}{f(16)+f(1)} = \dfrac{\sqrt{25}}{\sqrt{16}+\sqrt{1}} = \dfrac{5}{5} = 1$

74. $f(x) = 2x, \quad x > 3$

$= x^2, \quad 1 < x \leq 3$

$= 3x, \quad x \leq 1$

$\therefore$ as, $x = -1$, $f(x) = f(-1) = 3(-1) = -3$

as, $x = 2$, $f(x) = f(2) = (2)^2 = 4$

as, $x = 4$, $f(x) = f(4) = 2(4) = 8$

$\therefore$ $f(-1) + f(2) + f(4) = 9$

75. Since, $f(x)\, f(y) = f(xy)$

$\therefore$ $f(1).f(2) = f(2)$

$\therefore$ $f(1).4 = 4$

$\therefore$ $f(1) = 1$ ...(i)

Also, $f(2).f\left(\dfrac{1}{2}\right) = f(1)$

$\therefore$ $4 \times f\left(\dfrac{1}{2}\right) = 1$ ...[From (i)]

$\therefore$ $f\left(\dfrac{1}{2}\right) = \dfrac{1}{4}$

76. Given, $f(x) = \cos(\log x) \Rightarrow f(y) = \cos(\log y)$

Then, $f(x).f(y) - \dfrac{1}{2}\left[f\left(\dfrac{x}{y}\right) + f(xy)\right]$

$= \cos(\log x)\cos(\log y) - \dfrac{1}{2}\left[\cos\left(\log\dfrac{x}{y}\right) + \cos(\log xy)\right]$

$= \cos(\log x)\cos(\log y) - \dfrac{1}{2}[2\cos(\log x)\cos(\log y)] = 0$

77. $f(x + 1) - f(x) = 8x + 3$

$\Rightarrow [b(x + 1)^2 + c(x + 1) + d] - (bx^2 + cx + d) = 8x + 3$

$\Rightarrow (2b)x + (b + c) = 8x + 3$

$\Rightarrow 2b = 8,\ b + c = 3$

$\Rightarrow b = 4,\ c = -1$

78. $f(x + y) + f(x - y)$

$= \dfrac{1}{2}\left[a^{x+y} + a^{-x-y} + a^{x-y} + a^{-x+y}\right]$

$= \dfrac{1}{2}\left[a^x(a^y + a^{-y}) + a^{-x}(a^y + a^{-y})\right]$

$= \dfrac{1}{2}(a^x + a^{-x})\,(a^y + a^{-y}) = 2f(x)f(y)$

79. $f(x) = \log\left[\dfrac{1+x}{1-x}\right]$

$\therefore$ $f\left(\dfrac{2x}{1+x^2}\right) = \log\left[\dfrac{1+\dfrac{2x}{1+x^2}}{1-\dfrac{2x}{1+x^2}}\right] = \log\left[\dfrac{x^2+1+2x}{x^2+1-2x}\right]$

$= \log\left[\dfrac{1+x}{1-x}\right]^2 = 2\log\left[\dfrac{1+x}{1-x}\right] = 2f(x)$

80. $e^{f(x)} = \dfrac{10+x}{10-x},\ x \in (-10, 10)$

$\Rightarrow f(x) = \log\left(\dfrac{10+x}{10-x}\right)$

$\Rightarrow f\left(\dfrac{200x}{100+x^2}\right) = \log\left[\dfrac{10+\dfrac{200x}{100+x^2}}{10-\dfrac{200x}{100+x^2}}\right]$

$= \log\left[\dfrac{10(10+x)}{10(10-x)}\right]^2$

$= 2\log\left(\dfrac{10+x}{10-x}\right) = 2f(x)$

$\therefore$ $f(x) = \dfrac{1}{2}f\left(\dfrac{200x}{100+x^2}\right) \Rightarrow k = \dfrac{1}{2} = 0.5$

81. $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

$f(x) = \cos(9x) + \cos(-10x)$

$[\because \pi = 3.14,$

$\therefore$ $[9.85] = 9$ and $[-9.85] = -10]$

$= \cos(9x) + \cos(10x)$

$= 2\cos\left(\dfrac{19x}{2}\right)\cos\left(\dfrac{x}{2}\right)$

$\therefore \quad f\left(\frac{\pi}{2}\right) = 2\cos\left(\frac{19\pi}{4}\right)\cos\left(\frac{\pi}{4}\right);$

$\therefore \quad f\left(\frac{\pi}{2}\right) = 2\times\frac{-1}{\sqrt{2}}\times\frac{1}{\sqrt{2}} = -1$

83. $f(x) = f(-x) \Rightarrow f(0 + x) = f(0 - x)$ is symmetrical about $x = 0$.

$\therefore$ $f(2 + x) = f(2 - x)$ is symmetrical about $x = 2$.

84. $f = \{(1, 1), (2, 4), (0, -2), (-1, -5)\}$ be a linear function from Z to Z. The function satisfies the above points, if $f(x) = 3x - 2$

85. Here, $f(x) = \log\frac{1+x}{1-x}$

and $f(-x) = \log\left(\frac{1-x}{1+x}\right) = \log\left(\frac{1+x}{1-x}\right)^{-1}$

$= -\log\left(\frac{1+x}{1-x}\right) = -f(x) = f(-x)$

$\Rightarrow f(x)$ is an odd function.

86. Since, $f(x)$ is even.

$\therefore \quad f(-x) = f(x)$

$\therefore \quad \frac{a^{-x}-1}{(-x)^n\left(a^{-x}+1\right)} = \frac{a^x-1}{x^n\left(a^x+1\right)}$

$\Rightarrow \frac{1-a^x}{(-1)^n x^n\left(1+a^x\right)} = \frac{a^x-1}{x^n\left(a^x+1\right)}$

$\Rightarrow \frac{-1}{(-1)^n} = 1 \Rightarrow -1 = (-1)^n$

$\therefore \quad n = -\frac{1}{3}$ can satisfy the equation.

87. $f(-x) = \sec\left[\log\left(-x+\sqrt{1+(-x)^2}\right)\right]$

$= \sec\left[\log\left(-x+\sqrt{1+x^2}\right)\right]$

$= \sec\left[\log\left(\sqrt{1+x^2}-x\right)\right]$

$= \sec\left[\log\left(\frac{1+x^2-x^2}{\sqrt{1+x^2}+x}\right)\right]$

$= \sec\left[\log\left(\frac{1}{\sqrt{1+x^2}+x}\right)\right]$

$= \sec\left[-\log\left(\sqrt{1+x^2}+x\right)\right]$

$= \sec\left[\log\left(\sqrt{1+x^2}+x\right)\right]$

$\therefore \quad f(x)$ is an even function.

88. $f(x) = \sin\left(\log\left(x+\sqrt{1+x^2}\right)\right)$

$\Rightarrow f(-x) = \sin[\log(-x+\sqrt{1+x^2})]$

$\Rightarrow f(-x) = \sin\log\left((\sqrt{1+x^2}-x)\frac{(\sqrt{1+x^2}+x)}{(\sqrt{1+x^2}+x)}\right)$

$\Rightarrow f(-x) = \sin\log\left[\frac{1}{(x+\sqrt{1+(x^2)})}\right]$

$\Rightarrow f(-x) = \sin\left[-\log(x+\sqrt{1+x^2})\right]$

$\Rightarrow f(-x) = -\sin\left[\log(x+\sqrt{1+x^2})\right]$

$\Rightarrow f(-x) = -f(x)$

$\therefore \quad f(x)$ is odd function.

89. $f(x) + 2f\left(\frac{1}{x}\right) = 3x \quad \ldots. \text{(i)}$

$\therefore \quad f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x} \quad \ldots. \text{(ii)}$

From (i) and (ii), we get

$3f(x) = \frac{6}{x} - 3x$

$\Rightarrow f(x) = \frac{2}{x} - x \Rightarrow f(-x) = -\frac{2}{x} + x$

Since, $f(x) = f(-x)$

$\therefore \quad \frac{2}{x} - x = -\frac{2}{x} + x$

$\Rightarrow \frac{4}{x} = 2x \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

$\therefore$ option (B) is the correct answer.

90. Given expression $= \sum_{i=0}^{98}\left[\frac{2}{3}+\frac{i}{99}\right]$

$= \sum_{i=0}^{32}\left[\frac{2}{3}+\frac{i}{99}\right] + \sum_{i=33}^{98}\left[\frac{2}{3}+\frac{i}{99}\right]$

$= 0 + \sum_{i=33}^{98}\left[\frac{2}{3}+\frac{i}{99}\right]$

$\left[\because \frac{2}{3} \le \frac{2}{3}+\frac{i}{99} < 1 \text{ for } i = 0, 1, 2, \ldots, 32\right]$

$= 66$

[$\because$ each term in the summation is one or more but less than 2 when i = 33, 34, 35, ….,98]

91. $f(x) = \frac{1}{2}(1 + \cos 2x) + \frac{1}{2}\left[1+\cos\left(\frac{2\pi}{3}+2x\right)\right] - \frac{2\cos x\cos\left(\frac{\pi}{3}+x\right)}{2}$

$= 1 + \frac{1}{2}\left[\cos 2x+\cos\left(\frac{2\pi}{3}+2x\right) - \cos\left(2x+\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{3}\right)\right]$

$= \frac{3}{4}+\frac{1}{2}\left[\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right) - \cos\left(2x+\frac{\pi}{3}\right)\right]$

$= \frac{3}{4}+\frac{1}{2}\left[\cos 2x-2\sin\left(2x+\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right)\right]$

$= \frac{3}{4}+\frac{1}{2}\left[\cos 2x-2\sin\left(\frac{\pi}{2}+2x\right).\frac{1}{2}\right]$

$= \frac{3}{4}+\frac{1}{2}[\cos 2x-\cos 2x] = \frac{3}{4}$

92. $y = \frac{x+2}{x-1} \Rightarrow x = \frac{3}{y-1}+1 = \frac{y+2}{y-1} = f(y).$

93. Function given by $f(x) = ax + b$

$f^{-1}(x) = \frac{x-b}{a}$

So, $g(y) = y - 3$

94. $f(x) = |x|$

$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, the function $f^{-1}(x)$ does not exist.

95. Let $f(x) = y \Rightarrow x = f^{-1}(y)$

Now, $y = 3x - 5$

$\Rightarrow x = \frac{y+5}{3}$

$\Rightarrow f^{-1}(y) = x = \frac{y+5}{3}$

$\therefore$ $f^{-1}(x) = \frac{x+5}{3}$

Also f is one-one and onto, so f^{-1} exists and is given by $f^{-1}(x) = \frac{x+5}{3}$.

96. Let $f(x) = y \Rightarrow x = f^{-1}(y)$

Now, $y = 2x + 6$

$\Rightarrow 2x = y - 6$

$\Rightarrow x = \frac{y}{2} - 3$

$\Rightarrow f^{-1}(y) = \frac{y}{2} - 3$

$\Rightarrow f^{-1}(x) = \frac{x}{2} - 3$

97. Let $f(x) = y \Rightarrow x = f^{-1}(y)$

Now, $y = x^3 + 5$

$\Rightarrow y - 5 = x^3$

$\Rightarrow x = (y-5)^{\frac{1}{3}}$

$\Rightarrow f^{-1}(y) = (y-5)^{\frac{1}{3}}$

$\Rightarrow f^{-1}(x) = (x-5)^{\frac{1}{3}}$

98. Let $f(x) = y \Rightarrow x = f^{-1}(y)$. Now,

$y = \frac{2x-1}{x+5}$, $(x \neq -5)$

$xy + 5y = 2x - 1 \Rightarrow 5y + 1 = 2x - xy$

$\Rightarrow x(2 - y) = 5y + 1 \Rightarrow x = \frac{5y+1}{2-y}$

$\Rightarrow f^{-1}(y) = \frac{5y+1}{2-y}$

$\therefore$ $f^{-1}(x) = \frac{5x+1}{2-x}$, $x \neq 2$

99. We have, $f(x) = \frac{5x}{4x+5}$, $x \in R - \left\{\frac{5}{4}\right\}$

Let $f(x) = y$

$\Rightarrow x = f^{-1}(y)$

$y = \frac{5x}{4x+5}$

$\Rightarrow 4xy + 5y = 5x$

$\Rightarrow 5y = 5x - 4xy = x(5 - 4y)$

$\Rightarrow x = \frac{5y}{5-4y}$

$g(y) = f^{-1}(y) = \frac{5y}{5-4y}$, $x \in R - \left\{\frac{5}{4}\right\}$

100. Given, $f(x) = 2^{x(x-1)}$

$\Rightarrow x(x-1) = \log_2 f(x)$

$\Rightarrow x^2 - x - \log_2 f(x) = 0$

$\Rightarrow x = \dfrac{1 \pm \sqrt{1 + 4\log_2 f(x)}}{2}$

Only $x = \dfrac{1 + \sqrt{1 + 4\log_2 f(x)}}{2}$ lies in the domain

$\therefore \quad f^{-1}(x) = \dfrac{1}{2}[1 + \sqrt{1 + 4\log_2 x}]$.

101. Let $y = f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} + 2$

$\therefore \quad y - 2 = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

$\Rightarrow (y-2)\, e^{2x} + y - 2 = e^{2x} - 1$

$\Rightarrow e^{2x} = \dfrac{1-y}{y-3} = \dfrac{y-1}{3-y}$

$\Rightarrow 2x = \log_e \left(\dfrac{y-1}{3-y}\right)$

$\Rightarrow x = \dfrac{1}{2} \log_e \left(\dfrac{y-1}{3-y}\right)$

$\Rightarrow f^{-1}(y) = \dfrac{1}{2} \log_e \left(\dfrac{y-1}{3-y}\right)$

$\Rightarrow f^{-1}(x) = \log_e \left(\dfrac{x-1}{3-x}\right)^{\frac{1}{2}}$

102. Let $y = f(x) = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$

$\therefore \quad y = \dfrac{10^{2x} - 1}{10^{2x} + 1}$

$\Rightarrow 10^{2x} = \dfrac{1+y}{1-y}$

$\Rightarrow 2x = \log_{10} \dfrac{1+y}{1-y}$

$\Rightarrow x = \dfrac{1}{2} \log_{10} \dfrac{1+y}{1-y}$

$\Rightarrow f^{-1}(y) = \dfrac{1}{2} \log_{10} \dfrac{1+y}{1-y}$

$\Rightarrow f^{-1}(x) = \dfrac{1}{2} \log_{10} \dfrac{1+x}{1-x}$

103. Let $y = f(x) = \dfrac{16^x - 16^{-x}}{16^x + 16^{-x}}$

$\therefore \quad y = \dfrac{16^{2x} - 1}{16^{2x} + 1}$

$\Rightarrow 16^{2x} = \dfrac{1+y}{1-y}$

$\Rightarrow 2x = \log_{16} \dfrac{1+y}{1-y}$

$\Rightarrow x = \dfrac{1}{2} \log_{16} \dfrac{1+y}{1-y} \Rightarrow f^{-1}(y) = \dfrac{1}{2} \log_{16} \dfrac{1+y}{1-y}$

$\Rightarrow f^{-1}(x) = \dfrac{1}{2} \log_{16} \dfrac{1+x}{1-x}$

104. $f(g(-1)) = f(-3 -4) = f(-7) = 5 - 49 = -44$

105. $f(2) = \dfrac{2}{2^2 + 1} = \dfrac{2}{5}$

$\therefore \quad f(f(2)) = f\left(\dfrac{2}{5}\right) = \dfrac{\frac{2}{5}}{\left(\frac{2}{5}\right)^2 + 1} = \dfrac{10}{29}$

106. Here, $f(2) = \dfrac{2+1}{2-1} = 3$

$\therefore \quad f(f(2)) = f(3) = \dfrac{3+1}{3-1} = \dfrac{4}{2} = 2$

$\therefore \quad f(f(f(2))) = f(2) = \dfrac{2+1}{2-1} = 3$

107. $(fog)(x) = f(g(x)) = f(x^2) = \sin x^2$

108. $f(x) = \sin x + \cos x,\ g(x) = x^2$

$\therefore \quad fog(x) = \sin x^2 + \cos x^2$

109. $f[f(\cos 2\theta)] = f\left[\dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}\right] = f(\tan^2\theta)$

$= \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} = \cos 2\theta$

110. Here, $f\left(\dfrac{1}{2}\right) = \left(25 - \dfrac{1}{16}\right)^{\frac{1}{4}} = \left(\dfrac{399}{16}\right)^{\frac{1}{4}}$

$\Rightarrow f\left[f\left(\dfrac{1}{2}\right)\right] = f\left(\left(\dfrac{399}{16}\right)^{\frac{1}{4}}\right)$

$= \left(25 - \dfrac{399}{16}\right)^{\frac{1}{4}} = \left(\dfrac{1}{16}\right)^{\frac{1}{4}} = \dfrac{1}{2}$

111. $f(x) = \dfrac{x-1}{x+1}$

$\Rightarrow \dfrac{f(x)+1}{f(x)-1} = \dfrac{2x}{-2}$

$\Rightarrow x = \dfrac{f(x)+1}{1-f(x)}$

$\therefore \quad f(2x) = \dfrac{2x-1}{2x+1} = \dfrac{2\left[\dfrac{f(x)+1}{1-f(x)}\right]-1}{2\left[\dfrac{f(x)+1}{1-f(x)}\right]+1} = \dfrac{3f(x)+1}{f(x)+3}$

112. $f(x) = \dfrac{\alpha x}{x+1}$;

$f(f(x)) = f\left(\dfrac{\alpha x}{x+1}\right) = \dfrac{\alpha\left(\dfrac{\alpha x}{x+1}\right)}{\dfrac{\alpha x}{x+1}+1}$

But $f(f(x)) = x$

$\therefore \quad \dfrac{\alpha^2 x}{\alpha x + x + 1} = x$

L.H.S, Put $\alpha = -1$,

$\therefore \quad \dfrac{(-1)^2 x}{(-1)x + x + 1} = \dfrac{x}{-x+x+1} = x;$

$\therefore \quad \alpha = -1$

113. Given, $f(x) = a\,x + b$, $g(x) = cx + d$

and $f(g(x)) = g(f(x))$

$\Rightarrow f(c\,x + d) = g(a\,x + b)$

$\Rightarrow a(c\,x + d) + b = c(a\,x + b) + d$

$\Rightarrow ad + b = cb + d$

$\Rightarrow f(d) = g(b)$

114. $fog\,(x) = f\,[g(x)]$

$= f\left(x^{\frac{1}{3}}\right) = 8\left(x^{\frac{1}{3}}\right)^3$

$= 8x$

115. $(fog)\,(x) = f(g(x)) = f\left(\dfrac{x-1}{2}\right)$

$= 2\left(\dfrac{x-1}{2}\right) + 1 = x$

$\Rightarrow (fog)\,(x) = x \Rightarrow x = (fog)^{-1}(x)$

Hence, $(fog)^{-1}\left(\dfrac{1}{x}\right) = \dfrac{1}{x}$

116. $f(x) = \sin^2 x + \sin^2\left(x + \dfrac{\pi}{3}\right) + \cos x \cos\left(x + \dfrac{\pi}{3}\right)$

$= \sin^2 x + \left[\sin\left(x + \dfrac{\pi}{3}\right)\right]^2$

$+ \cos x\left[\cos x \cos\dfrac{\pi}{3} - \sin x \sin\dfrac{\pi}{3}\right]$

$= \sin^2 x + \left[\sin x \cos\dfrac{\pi}{3} + \cos x \sin\dfrac{\pi}{3}\right]^2$

$+ \cos x\left[\dfrac{1}{2}\cos x - \dfrac{\sqrt{3}}{2}\sin x\right]$

$= \sin^2 x + \left[\dfrac{\sin x}{2} + \dfrac{\sqrt{3}}{2}\cos x\right]^2$

$+ \dfrac{\cos^2 x}{2} - \dfrac{\sqrt{3}}{2}\sin x \cos x$

$= \sin^2 x + \dfrac{\sin^2 x}{4} + \dfrac{3}{4}\cos^2 x + \dfrac{\cos^2 x}{2}$

$+ \dfrac{\sqrt{3}}{2}\sin x \cos x - \dfrac{\sqrt{3}}{2}\sin x \cos x$

$= \dfrac{5}{4}(\sin^2 x + \cos^2 x) = \dfrac{5}{4}$

$(gof)(x) = g[f(x)] = g\left(\dfrac{5}{4}\right) = 1$

117. As $x - [x] \in [0, 1)$, $\forall\, x \in R$

$\therefore \quad 0 \le x - [x] < 1$, $\forall\, x \in R$

$\Rightarrow 1 \le 1 + x - [x] < 2$, $\forall\, x \in R$

$\Rightarrow 1 \le g(x) < 2$, $\forall\, x \in R$

Hence, $f(g(x)) = 1\ \forall\, x \in R$

118. $(gof)(e) + (fog)(\pi) = g(f(e)) + f(g(\pi))$

$= g(1) + f(0)$

$= -1 + 0$

$= -1$

119. $g(f(x)) = g(|x|) = [|x|]$

and $f(g(x)) = f([x]) = |[x]|$

When $x \ge 0$, $[|x|] = [x] = |[x]|$

$\Rightarrow f(g(x)) = g(f(x))$

When $x < 0$, $[x] \le x < 0$

$\Rightarrow |[x]| \ge |x|$

$\Rightarrow |[x]| \ge |x| \ge [|x|]$ ....[$\because$ $[t] \le t$ for all t]

$\Rightarrow f(g(x)) \ge g(f(x))$

$\therefore \quad g(f(x)) \le f(g(x))$ for all $x \in R$

120. $(\text{hofog})(x) = (\text{hof})(\text{g}(x))$
$= (\text{hof})\left(\sqrt{x^2+1}\right)$
$= \text{h}\left(\text{f}\left(\sqrt{x^2+1}\right)\right)$
$= \text{h}\left[\left(\sqrt{x^2+1}\right)^2 - 1\right]$
$= \text{h}(x^2 + 1 - 1)$
$= \text{h}(x^2) = \begin{cases} 0 & , \text{if } x = 0 \\ x^2 & , \text{if } x \neq 0 \end{cases}$

121. Given,
$$\text{f}(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos\left(x + \frac{\pi}{3}\right)\cos x$$
$$= \frac{1}{2}\left\{1 - \cos 2x + 1 - \cos\left(2x + \frac{2\pi}{3}\right) + \cos\left(2x + \frac{2\pi}{3}\right) + \cos\frac{\pi}{3}\right\}$$
$$= \frac{1}{2}\left[\frac{5}{2} - \left\{\cos 2x + \cos\left(2x + \frac{2\pi}{3}\right)\right\} + \cos\left(2x + \frac{\pi}{3}\right)\right]$$
$$= \frac{1}{2}\left[\frac{5}{2} - 2\cos\left(2x + \frac{\pi}{3}\right)\cos\frac{\pi}{3} + \cos\left(2x + \frac{\pi}{3}\right)\right]$$
$$= \frac{5}{4}$$
∴ $\text{gof}(x) = \text{g}[\text{f}(x)] = \text{g}\left(\frac{5}{4}\right)$
$= 1$ $\quad \ldots\left[\because \text{g}\left(\frac{5}{4}\right) = 1\right]$
Hence, $\text{gof}(x)$ is a constant function.

122. $(\text{gof})(x) = \sin x^2 \Rightarrow (\text{gogof})(x) = \sin(\sin x^2)$
$\Rightarrow (\text{fogogof})(x) = (\sin(\sin x^2))^2 = \sin^2(\sin x^2)$
Now, $\sin^2(\sin x^2) = \sin(\sin x^2)$
$\Rightarrow \sin(\sin x^2) = 0, 1$
$\Rightarrow \sin x^2 = \text{n}\pi, (4\text{n} + 1)\frac{\pi}{2}$ $\text{n} \in \text{I}$
$\Rightarrow \sin x^2 = 0 \Rightarrow x^2 = \text{n}\pi$
$\Rightarrow x = \pm\sqrt{\text{n}\pi}$ $\text{n} \in \text{W}$

123. $|x| = -x,$ $\quad$ if $x < 0$
$= x,$ $\quad$ if $x \geq 0$
Now, $(\text{fog})(x) = \text{f}[\text{g}(x)]$
$= |\text{g}(x)| + \text{g}(x)$
$= ||x| - x| + |x| - x$
When, $x < 0$
$(\text{fog})(x) = |-x - x| + (-x) - x$
$= -2x - 2x = -4x$

124. Let $\text{f}(x)$ be periodic with period T.
Then, $\text{f}(x + \text{T}) = \text{f}(x)$ for all $x \in \text{R}$
$\Rightarrow x + \text{T} - [x + \text{T}] = x - [x]$. for all $x \in \text{R}$
$\Rightarrow x + \text{T} - x = [x + \text{T}] - [x]$
$\Rightarrow [x + \text{T}] - [x] = \text{T}$ for all $x \in \text{R}$
$\Rightarrow \text{T} = 1, 2, 3, 4, \ldots\ldots.$
The smallest value of T satisfying
$\text{f}(x + \text{T}) = \text{f}(x)$ for all $x \in \text{R}$ is 1.
Hence, $\text{f}(x) = x - [x]$ has period 1.

125. $\text{g}(x)$ is neither injective nor surjective
$(\text{gof})(x) = \left(\text{e}^x\right)^2 = \text{e}^{2x}$
This is an injective function.

126. At $x = 0$, $\text{f}(x)$ is not defined.

128. $\text{f}(x) = \text{f}(y)$
$\Rightarrow x + 2 = y + 2 \Rightarrow x = y$
∴ Function f is one-one

129. Total number of distinct functions from $\text{A} \to \text{A} = \text{n}^\text{n} = 6^6$
Number of bijections = n! = 6!
∴ Number of functions which are not bijections $= 6^6 - 6!$

130. Number of bijective function from a set of 10 elements to itself is $^{10}\text{P}_{10}$.
So, required number = 10!

131. The total number of injective functions from a set A containing 3 elements to a set B containing 4 elements is equal to the total number of arrangements of 4 by taking 3 at a time i.e., $^4\text{P}_3 = 24$.

132. Number of injective mapping = $^5\text{P}_4 = 120$

133. $|x|$ is not one-one; x^2 is not one-one;
$x^2 + 1$ is not one-one. But $2x - 5$ is one-one because $\text{f}(x) = \text{f}(y) \Rightarrow 2x - 5 = 2y - 5 \Rightarrow x = y$
Now, $\text{f}(x) = 2x - 5$ is onto.
∴ $\text{f}(x) = 2x - 5$ is bijective.

134. Let $x_1, x_2 \in \text{R}$ such that $\text{f}(x_1) = \text{f}(x_2)$
$\Rightarrow x_1 = \pm x_2$
∴ $\text{f}(x_1) = \text{f}(x_2)$ does not impty that $x_1 = x_2$
∴ f is not one-one.
Consider an element 2 in the co-domain R.
There doesnot exist any x is domain R such that $\text{f}(x) = 2$.
∴ f is not onto.

135. $\text{f}'(x) = 2 + \cos x > 0$. So, $\text{f}(x)$ is strictly monotonic increasing so, $\text{f}(x)$ is one-to-one and onto.

136. Let $x, y \in N$ such that $f(x) = f(y)$

Then, $f(x) = f(y) \Rightarrow x^2 + x + 1 = y^2 + y + 1$

$\Rightarrow (x - y)(x + y + 1) = 0$

$\Rightarrow x = y$ or $x = (-y - 1) \notin N$

$\therefore$ f is one-one.

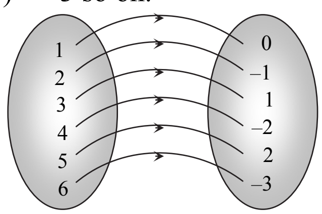
Again, since for each $y \in N$, there exist $x \in N$

$\therefore$ f is onto.

137. We have $f(x) = (x - 1)(x - 2)(x - 3)$ and $f(1) = f(2) = f(3) = 0 \Rightarrow f(x)$ is not one-one. For each $y \in R$, there exists $x \in R$ such that $f(x) = y$. Therefore f is onto. Hence, $f : R \to R$ is onto but not one-one.

138. $f : N \to I$

$f(1) = 0$, $f(2) = -1$, $f(3) = 1$, $f(4) = -2$, $f(5) = 2$ and $f(6) = -3$ so on.



In this type of function every element of set A has unique image in set B and there is no element left in set B. Hence f is one-one and onto function.

139. $f : N \to N$

$$f(n) = \begin{cases} \dfrac{n+1}{2} & \text{if n is odd} \\ \dfrac{n}{2} & \text{if n is even} \end{cases}$$

Now for n = 1, $f(1) = \dfrac{1+1}{2} = 1$

and if n = 2, $f(2) = \dfrac{2}{2} = 1$

$\therefore$ $f(1) = f(2)$, But $1 \neq 2$.

$\therefore$ $f(x)$ is not one-one.

$f(x) = \dfrac{n+1}{2}$ if n is odd

if $y = \dfrac{n+1}{2}$ then $n = 2y - 1$, $\forall y$

Also, $f(x) = \dfrac{n}{2}$ if n is even i.e., $y = \dfrac{n}{2}$

or $n = 2y \ \forall y$

$\therefore$ $f(x)$ is onto.

140. $\sigma : N \to Z$

$\sigma(1) = 0$, $\sigma(2) = 1$, $\sigma(3) = -1$, $\sigma(4) = 2$, $\sigma(5) = -2$, $\sigma(6) = 3$, $\sigma(7) = -3$

$\therefore$ σ is one-one and onto.

141. Function $f : R \to R$ is defined by $f(x) = e^x$. Let $x_1, x_2 \in R$ and $f(x_1) = f(x_2)$ or $e^{x_1} = e^{x_2}$ or $x_1 = x_2$. Therefore f is one-one. Let $f(x) = e^x = y$. Taking log on both sides, we get $x = \log y$. We know that negative real numbers have no pre-image or the function is not onto and zero is not the image of any real number. Therefore function f is into.

142. Here, $(f - g)(x) = f(x) - g(x)$

$\therefore$ $(f - g)(x) = \begin{cases} x - 0 = x, & \text{if } x \text{ is rational} \\ 0 - x = -x, & \text{if } x \text{ is irrational} \end{cases}$

Let $k = f - g$

Let x, y be any two distinct real numbers.

Then, $x \neq y$

$\Rightarrow -x \neq -y$

Now, $x \neq y$

$\Rightarrow k(x) \neq k(y) \Rightarrow (f - g)(x) \neq (f - g)(y)$

$\Rightarrow f - g$ is one-one.

Let y be any real number

If y is a rational number, then

$k(y) = y$

$\Rightarrow (f - g)(y) = y$

If y is an irrational number , then

$k(-y) = y$

$\Rightarrow (f - g)(-y) = y$

Thus, every $y \in R$ (co-domain) has its pre-image in R (domain)

$\therefore$ $f - g : R \to R$ is onto.

Hence, $f - g$ is one-one and onto.

143. Let $x, y \in R$ be such that

$f(x) = f(y)$

$\Rightarrow x^3 + 5x + 1 = y^3 + 5y + 1$

$\Rightarrow (x^3 - y^3) + 5(x - y) = 0$

$\Rightarrow (x - y)(x^2 + xy + y^2 + 5) = 0$

$\Rightarrow (x - y)\left[\left(x + \dfrac{y}{2}\right)^2 + \dfrac{3y^2}{4} + 5\right] = 0$

$\Rightarrow x = y$ and $\left(x + \dfrac{y}{2}\right)^2 + \dfrac{3y^2}{4} + 5 \neq 0$

$\therefore$ $f : R \to R$ is one-one

Let y be an arbitrary element in R (co-domain).

Then, $f(x) = y$ i.e., $x^3 + 5x + 1 = y$ has at least one real root, say β in R

$\therefore$ $\beta^3 + 5\beta + 1 = y$

$\Rightarrow f(\beta) = y$

Thus, for each $y \in R$ there exists $\beta \in R$ such that $f(\beta) = y$

$\therefore$ $f : R \to R$ is onto

Hence, $f : R \to R$ is one-one onto.

144. Since, f(x) and g(x) has same domain and co-domain A and B and $f(1) = (1)^2 - 1 = 0$

$g(1) = 2\left|1 - \frac{1}{2}\right| - 1 = 2 \times \frac{1}{2} - 1 = 0$

$f(1) = 0 = g(1)$, $f(0) = 0 = g(0)$

$f(-1) = 2 = g(-1)$, $f(2) = 2 = g(2)$

$A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$

$\therefore$ By definition, the two function are equal f = g

145. $-\sqrt{1+(-\sqrt{3})^2} \le (\sin x - \sqrt{3}\cos x) \le \sqrt{1+(-\sqrt{3})^2}$

$\therefore$ $-2 \le (\sin x - \sqrt{3}\ \cos x) \le 2$

$\therefore$ $-2 + 1 \le (\sin x - \sqrt{3}\ \cos x + 1) \le 2 + 1$

$\therefore$ $-1 \le (\sin x - \sqrt{3}\cos x + 1) \le 3$

i.e., range = $[-1, 3]$

$\therefore$ For f to be onto S = $[-1, 3]$.

146. Given, $f(x) = \sin x$

$\therefore$ $f : R \to R$ is neither one-one nor onto as $R_f = [-1, 1]$.

$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

is both one-one and onto.

$f : [0, \pi] \to [-1, 1]$

is neither one-one nor onto as $R_f = [0, 1]$.

$f : \left[0, \frac{\pi}{2}\right] \to [-1, 1]$ is one-one but not onto as $R_f = [0, 1]$.

147. Let $f(x) = \frac{x^2-1}{x^2+1} = \frac{x^2+1-2}{x^2+1} = 1 - \frac{2}{x^2+1}$

$\because$ $x^2 + 1 > 1$;

$\therefore$ $\frac{2}{x^2+1} \le 2$

So $1 - \frac{2}{x^2+1} \ge 1 - 2$;

$\therefore$ $-1 \le f(x) < 1$

Thus, f(x) has the minimum value equal to -1.

148. The quantity under root is positive, when $-1-\sqrt{3} \le x \le -1+\sqrt{3}$.

149. The function $f(x) = \sqrt{\log(x^2 - 6x + 6)}$ is defined, when $\log(x^2 - 6x + 6) \ge 0$

$\Rightarrow x^2 - 6x + x \ge 1 \ \Rightarrow (x - 5)(x - 1) \ge 0$

This inequality holds, if $x \le 1$ or $x \ge 5$. Hence, the domain of the function will be $(-\infty, 1] \cup [5, \infty)$.

150. Here, $x + 3 > 0$ and $x^2 + 3x + 2 \ne 0$

$\therefore$ $x > -3$ and $(x + 1)(x + 2) \ne 0$, i.e., $x \ne -1, -2$.

$\therefore$ Domain = $(-3, \infty) - \{-1, -2\}$.

151. $D_f = D_g \cap D_h$

where $g(x) = \frac{1}{\log_{10}(1-x)}$ and $h(x) = \sqrt{2+x}$

Now, $D_g = \{x \in R : 1 - x > 0, \log_{10}(1 - x) \ne 0\}$

$= \{x \in R : x < 1, 1 - x \ne 1\}$

$= \{x \in R : x < 1, x \ne 0\}$

and $D_h = \{x \in R : x + 2 \ge 0\}$

$= \{x \in R : x \ge -2\}$

$\therefore$ $D_f = [(-\infty, 1) - \{0\}] \cap [-2, \infty)$

$= [-2, 1) - \{0\}$

152. $f(x) = \log|\log x|$, f(x) is defined if $|\log x| > 0$ and $x > 0$ i.e., if $x > 0$ and $x \ne 1$ ($\because$ $|\log x| > 0$ if $x \ne 1$)

$\Rightarrow x \in (0, 1) \cup (1, \infty)$.

153. $\frac{1-|x|}{2-|x|} \ge 0$

$\Rightarrow \frac{|x|-1}{|x|-2} \ge 0$

$\Rightarrow |x| \le 1$ as $|x| > 2$

$\Rightarrow x \in (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$

154. $f(x) = \sqrt{\log\frac{1}{|\sin x|}}$

$\Rightarrow \sin x \ne 0 \Rightarrow x \ne n\pi + (-1)^n 0$

$\Rightarrow x \ne n\pi$. Domain of $f(x) = R - \{n\pi, n \in I\}$.

155. f(x) is to be defined when $x^2 - 1 > 0$

$\Rightarrow x^2 > 1, \Rightarrow x < -1$ or $x > 1$ and $3 + x > 0$

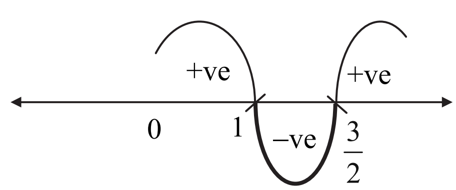
$\therefore$ $x > -3$ and $x \ne -2$

$\therefore$ $D_f = (-3, -2) \cup (-2, -1) \cup (1, \infty)$.

156. $f(x) = e^{\sqrt{5x-3-2x^2}}$

$\Rightarrow 5x - 3 - 2x^2 \ge 0$

$\Rightarrow (x - 1)\left(x - \frac{3}{2}\right) \le 0$



$\therefore$ $D_f = \left[1, \frac{3}{2}\right]$

157. To define f(x), $9 - x^2 > 0 \Rightarrow |x| < 3$
$\Rightarrow -3 < x < 3$, ...(i)
and $-1 \le (x - 3) \le 1$
$\Rightarrow 2 \le x \le 4$ .....(ii)
From (i) and (ii), $2 \le x < 3$ i.e., [2, 3).

158. $-1 \le 1 + 3x + 2x^2 \le 1$
Case I : $2x^2 + 3x + 1 \ge -1$; $2x^2 + 3x + 2 \ge 0$
$x = \dfrac{-3 \pm \sqrt{9-16}}{6} = \dfrac{-3 \pm i\sqrt{7}}{6}$ (imaginary).
Case II : $2x^2 + 3x + 1 \le 1$
$\Rightarrow 2x^2 + 3x \le 0 \Rightarrow 2x\left(x + \dfrac{3}{2}\right) \le 0$
$\Rightarrow \dfrac{-3}{2} \le x \le 0 \Rightarrow x \in \left[-\dfrac{3}{2}, 0\right]$
In case I, we get imaginary value hence, rejected
$\therefore$ Domain of function = $\left[\dfrac{-3}{2}, 0\right]$.

159. $f(x) = \sqrt{\cos^{-1}\left(\dfrac{1-|x|}{2}\right)}$
$\therefore$ $-1 \le \dfrac{1-|x|}{2} \le 1$
$\Rightarrow -2 - 1 \le -|x| \le 2 - 1$
$\Rightarrow -3 \le |x| \le 1$
$\Rightarrow -1 \le |x| \le 3$
$\Rightarrow x \in [-3, 3]$

160. $-1 \le \log_2(x^2 + 5x + 8) \le 1$
$\Rightarrow \dfrac{1}{2} \le (x^2 + 5x + 8) \le 2$
$\Rightarrow x^2 + 5x + \dfrac{15}{2} \ge 0$
$\Rightarrow x^2 + 2\left(\dfrac{5}{2}\right)x + \left(\dfrac{5}{2}\right)^2 - \left(\dfrac{5}{2}\right)^2 + \dfrac{15}{2} \ge 0$
$\Rightarrow \left(x + \dfrac{5}{2}\right)^2 + \dfrac{5}{4} \ge 0$ and $x^2 + 5x + 6 \le 0$
$\Rightarrow (x + 3)(x + 2) \le 0 \Rightarrow x \in [-3, -2]$

161. $f(x) = \sqrt{9 - x^2}$
f(0) = 3, f(3) = 0
$\therefore$ $0 \le f(x) \le 3$
$\therefore$ $x \in [0, 3]$

162. $f(x) = \dfrac{x+2}{|x+2|}$
$f(x) = \begin{cases} -1, & x < -2 \\ 1, & x > -2 \end{cases}$
$\therefore$ Range of f(x) is $\{-1, 1\}$.

163. Let $\dfrac{x^2 + 34x - 71}{x^2 + 2x - 7} = y$
$\Rightarrow x^2(1 - y) + 2(17 - y)x + (7y - 71) = 0$
For real value of x, $b^2 - 4ac \ge 0$
$\Rightarrow y^2 - 14y + 45 \ge 0 \Rightarrow y \ge 9, y \le 5$.

164. Dom (f) = R – {2}
For Range (f), let $y = f(x) = \dfrac{x^2 - 4}{x - 2}$
$\therefore$ $y = \dfrac{(x-2)(x+2)}{(x-2)}$
$\therefore$ $y = (x + 2)$
Since, Dom (f) = R – {2}
$\therefore$ $x \ne 2$
$\therefore$ $y \ne (2 + 2)$ i.e. $y \ne 4$
$\therefore$ Range (f) = R – {4}

165. Let $y = \dfrac{x^2 - x + 4}{x^2 + x + 4}$
$\Rightarrow (y - 1)x^2 + (y + 1)x + 4y - 4 = 0$
For real value of x, $b^2 - 4ac \ge 0$
$\Rightarrow (y + 1)^2 - 4(y - 1)(4y - 4) \ge 0$
$\Rightarrow -15y^2 + 34y - 15 \ge 0$
$\Rightarrow 15y^2 - 34y + 15 \ge 0$
$\Rightarrow \left(y - \dfrac{3}{5}\right)\left(y - \dfrac{5}{3}\right) \le 0$
$\Rightarrow \dfrac{3}{5} \le y \le \dfrac{5}{3}$

166. Since maximum and minimum values of cos – sin x are $\sqrt{2}$ and $-\sqrt{2}$ respectively, therefore range of f(x) is $[-\sqrt{2}, \sqrt{2}]$.

167. $\cos 2x + 7 = a(2 - \sin x) \Rightarrow a = \dfrac{\cos 2x + 7}{2 - \sin x}$
$\Rightarrow a = \dfrac{1 - 2\sin^2 x + 7}{2 - \sin x} = \dfrac{2(4 - \sin^2 x)}{2 - \sin x}$
$\Rightarrow a = 2(2 + \sin x)$
$\therefore$ $a \in [2, 6]$ ....[$\because -1 \le \sin x \le 1$]

168. Let $y = \log_e(3x^2 + 4)$

$\Rightarrow 3x^2 + 4 = e^y \Rightarrow x^2 = \dfrac{e^y - 4}{3}$

Since, $x^2 \geq 0$

$\therefore \quad \dfrac{e^y - 4}{3} \geq 0 \Rightarrow e^y - 4 \geq 0 \Rightarrow y \geq \log_e 4$

$\Rightarrow y \geq 2 \log_e 2$

So, range = $[2 \log_e 2, \infty)$

169. Let $y = \log_e \sqrt{4 - x^2} \Rightarrow e^y = \sqrt{4 - x^2}$

$\Rightarrow e^{2y} = 4 - x^2 \Rightarrow x^2 = 4 - e^{2y} \Rightarrow x = \sqrt{4 - e^{2y}}$

$\therefore \quad 4 - e^{2y} \geq 0$

$\Rightarrow e^{2y} \leq 4 \Rightarrow 2y \leq \log_e 4$

$\Rightarrow y \leq \dfrac{1}{2} \log_e 4 \Rightarrow y \leq \log_e 2$

$\therefore \quad y \in (-\infty, \log_e 2]$

170. $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$

$\therefore \quad f(\theta) = 2\sec^2\theta \geq 2$

$\therefore$ range of f is $[2, \infty)$.

171. $f(x) = \tan\sqrt{\dfrac{\pi^2}{9} - x^2}$

$f(x)$ is real valued function when $\dfrac{\pi^2}{9} - x^2 \geq 0$

$\Rightarrow x^2 \leq \dfrac{\pi^2}{9} \Rightarrow x \in \left[-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right]$ = Domain of $f(x)$

When domain is in closed interval, we use differentiation method.

$f'(x) = \sec^2\sqrt{\dfrac{\pi^2}{9} - x^2} \cdot \dfrac{1}{2\sqrt{\dfrac{\pi^2}{9} - x^2}}(-2x)$

When $f'(x) = 0$, $x = 0$

Finding values of $f(x)$ when $x = 0, -\dfrac{\pi}{3}, \dfrac{\pi}{3}$

[End points of domain]

$\therefore \quad f(0) = \tan\sqrt{\dfrac{\pi^2}{9}} = \sqrt{3}$ and $f\left(-\dfrac{\pi}{3}\right) = f\left(\dfrac{\pi}{3}\right) = 0$

$\therefore$ Range of function = $\left[0, \sqrt{3}\right]$

[Taking least value and greatest value for range]

Evaluation Test

1. Given, $g(x) = x^2 + x - 2$

and $\dfrac{1}{2}(gof)(x) = 2x^2 - 5x + 2$

$\Rightarrow g(f(x)) = 4x^2 - 10x + 4$

$\Rightarrow (f(x))^2 + f(x) - 2 = 4x^2 - 10x + 4$

$\Rightarrow (f(x))^2 + f(x) - (4x^2 - 10x + 6) = 0$

$\Rightarrow f(x) = \dfrac{-1 \pm \sqrt{1 + 16x^2 - 40x + 24}}{2}$

$= \dfrac{-1 \pm (4x - 5)}{2} = 2x - 3, -2x + 2$

2. $fog(x) = x^3 - \dfrac{1}{x^3}$

Since, $\left(x - \dfrac{1}{x}\right)^3 = x^3 - \dfrac{1}{x^3} - 3x.\dfrac{1}{x}\left(x - \dfrac{1}{x}\right)$

$\therefore \quad x^3 - \dfrac{1}{x^3} = \left(x - \dfrac{1}{x}\right)^3 + 3\left(x - \dfrac{1}{x}\right)$

Given, $f(g(x)) = x^3 - \dfrac{1}{x^3}$

$\therefore \quad f(g(x)) = \left(x - \dfrac{1}{x}\right)^3 + 3\left(x - \dfrac{1}{x}\right)$

$\therefore \quad f\left(x - \dfrac{1}{x}\right) = \left(x - \dfrac{1}{x}\right)^3 + 3\left(x - \dfrac{1}{x}\right)$

Put $x - \dfrac{1}{x} = t$

$\therefore \quad f(t) = t^3 + 3t \qquad \therefore \quad f(x) = x^3 + 3x$

$\therefore \quad f'(x) = 3x^2 + 3$

3. Given, $f(x + y) = f(x) + f(y) \ \forall\ x, y \in R$

$\Rightarrow f(2) = f(1+1) = f(1) + f(1) = 2f(1)$

$f(3) = f(2 + 1) = f(2) + f(1)$

$= 2f(1) + f(1) = 3f(1)$

Continuing in this way, we get

$f(r) = rf(1) \in N$

$\therefore \quad \sum_{r=1}^{n} f(r) = \sum_{r=1}^{n} rf(1) = f(1) \sum_{r=1}^{n} r$

$= 7(1 + 2 + 3 + \ldots + n)$

$= \dfrac{7n(n+1)}{2}$

4. Given, $f(x) = x^2 - 3$

$\therefore$ $f(-1) = (-1)^2 - 3 = -2$

$\Rightarrow (fof)(-1) = f(-2) = (-2)^2 - 3 = 1$

$\Rightarrow (fofof)(-1) = f(1) = 1^2 - 3 = -2$ ....(i)

Similarly, $(fofof)(0) = 33$ ....(ii)

and $(fofof)(1) = -2$ ....(iii)

From (i), (ii) and (iii), we get

$(fofof)(-1) + (fofof)(0) + (fofof)(1)$

$= -2 + 33 - 2 = 29 = f\left(4\sqrt{2}\right)$

5. $x = \left(\sqrt{3}+1\right)^5$

$= {}^5C_0\left(\sqrt{3}\right)^5 + {}^5C_1\left(\sqrt{3}\right)^4 + {}^5C_2\left(\sqrt{3}\right)^3 + {}^5C_3\left(\sqrt{3}\right)^2 + {}^5C_4\left(\sqrt{3}\right)^1 + {}^5C_5$

$= \left(\sqrt{3}\right)^5 + 5(9) + 10\left(3\sqrt{3}\right) + 30 + 5\sqrt{3} + 1$

$= 76 + 44\sqrt{3} = 152.20$

$\therefore$ $[x] = [152.20] = 152$

6. $f(x + y) = f(x) + f(y)$ .....(i)

Putting $x = y = 1$ in (i), we get

$f(2) = 2f(1)$

$\Rightarrow f(2) = 2(5)$ ....[$\because$ f(1) = 5(given)]

Putting $x = 2$ and $y = 1$ in (i), we get

$f(3) = f(2) + f(1) = 3(5)$

Similarly, $f(4) = 4(5)$

$f(5) = 5(5)$

...

...

...

$f(100) = 100(5) = 500$

7. For any $x \in (0, 2)$, $x \neq x + 1$

$\therefore$ $(x, x) \notin S$.

$\therefore$ S is not a reflexive relation.

So, S is not an equivalence relation.

$T = \{(x, y): x - y \text{ is an integer}\}$

For any $x \in R$,

$x - x = 0$, which is an integer.

$\Rightarrow (x, x) \in T$

$\therefore$ T is reflexive on R.

Let $(x, y) \in T$. Then,

$x - y$ is an integer

$\Rightarrow y - x$ is an integer

$\Rightarrow (y, x) \in T$

$\therefore$ T is symmetric on R.

Let $(x, y) \in T$ and $(y, z) \in T$.

Then, $x - y$ is an integer and $y - z$ is an integer

$\Rightarrow x - z$ is an integer

$\Rightarrow (x, z) \in T$

$\therefore$ T is transitive on R.

So, T is an equivalence relation on R.

8. Here, $(0, 3) \in R$, because $0 = 0 \times 3$

But, $(3, 0) \notin R$,

because $3 \neq$ (any rational number) $\times 0$

So, R is not a symmetric relation and hence it is not an equivalence relation.

$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right) : \text{m, n, p and q are integers such that n, q} \neq 0 \text{ and qm} = \text{pn}\right\}$

Let m, n $\in$ Z such that n $\neq$ 0. Then,

$mn = nm \Rightarrow \frac{m}{n} = \frac{m}{n} \Rightarrow \left(\frac{m}{n}, \frac{m}{n}\right) \in S$

So, S is reflexive.

Let $\left(\frac{m}{n}, \frac{p}{q}\right) \in S$. Then,

$qm = pn \Rightarrow np = mq$

$\Rightarrow \left(\frac{p}{q}, \frac{m}{n}\right) \in S$

So, S is symmetric.

Let $\left(\frac{m}{n}, \frac{p}{q}\right) \in S$ and $\left(\frac{p}{q}, \frac{r}{s}\right) \in S$.

Then, $qm = pn$ and $sp = rq$

$\Rightarrow (qm)(sp) = (pn)(rq)$

$\Rightarrow sm = rn \Rightarrow \left(\frac{m}{n}, \frac{r}{s}\right) \in S$

So, S is transitive.

Hence, S is an equivalence relation.

9. $\theta \in P$

$\Rightarrow \sin\theta - \cos\theta = \sqrt{2}\cos\theta$

$\Rightarrow (\sin\theta - \cos\theta)^2 = 2\cos^2\theta$

$\Rightarrow \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta = 2\cos^2\theta$

$\Rightarrow \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = 2\sin^2\theta$

$\Rightarrow (\cos\theta + \sin\theta)^2 = 2\sin^2\theta$

$\Rightarrow \cos\theta + \sin\theta = \sqrt{2}\sin\theta$

$\Rightarrow \theta \in Q$

$\therefore$ P = Q

10. f(x) is defined for

$$-1 \le \frac{8.3^{x-2}}{1-3^{2(x-1)}} \le 1$$

$$\Rightarrow -1 \le \frac{(3^2-1)(3^{x-2})}{1-3^{2x-2}} \le 1$$

$$\Rightarrow -1 \le \frac{3^x-3^{x-2}}{1-3^{2x-2}} \le 1$$

$$\Rightarrow \frac{3^x-3^{x-2}}{1-3^{2x-2}} + 1 \ge 0 \text{ and } \frac{3^x-3^{x-2}}{1-3^{2x-2}} - 1 \le 0$$

$$\Rightarrow \frac{1+3^x-3^{x-2}-3^{2x-2}}{1-3^{2x-2}} \ge 0$$

$$\text{and } \frac{3^x-3^{x-2}-1+3^{2x-2}}{1-3^{2x-2}} \le 0$$

$$\Rightarrow \frac{(3^x+1)(3^{x-2}-1)}{(3^x.3^{x-2}-1)} \ge 0 \text{ and } \frac{(3^x-1)(3^{x-2}+1)}{(3^{2x-2}-1)} \ge 0$$

$$\Rightarrow \frac{(3^{x-2}-1)}{(3^x.3^{x-2}-1)} \ge 0 \text{ and } \frac{(3^x-1)}{(3^{2x-2}-1)} \ge 0$$

$$\Rightarrow \frac{(3^x-3^2)}{(3^{2x}-3^2)} \ge 0 \text{ and } \frac{(3^x-1)}{(3^{2x}-3^2)} \ge 0$$

$$\Rightarrow \frac{(3^x-3^2)}{(3^x-3)} \ge 0 \text{ and } \frac{(3^x-1)}{(3^x-3)} \ge 0$$

$\Rightarrow x \in (-\infty, 1] \cup [2, \infty)$ and $x \in (-\infty, 0] \cup (1, \infty)$

$\Rightarrow x \in (-\infty, 0] \cup [2, \infty)$

11. $f(x - y) = f(x)\, f(y) - f(a - x)\, f(a + y)$

Putting $x = 0$ and $y = 0$, we get

$f(0) = \{f(0)\}^2 - \{f(a)\}^2$

$\Rightarrow 1 = 1 - \{f(a)\}^2 \Rightarrow f(a) = 0$

Now, $f(2a - x) = f(a - (x - a))$

$= f(a)\, f(x-a) - f(a-a)\, f(a+x-a)$

$= f(a)\, f(x - a) - f(0)\, f(x)$

$= f(a)\, f(x - a) - f(x)$

....[$\because$ f(0) = 1 (given)]

$= -f(x)$

12. $n[(A \times B) \cap (B \times A)] = n[(A \cap B) \times (B \cap A)]$

$= n(A \cap B) \times n(B \cap A) = 3 \times 3 = 9$

13. Given, $2\cos^2\theta + \sin\theta \le 2$ and $\dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}$

$\Rightarrow 2 - 2\sin^2\theta + \sin\theta \le 2$

$\Rightarrow 2\sin^2\theta - \sin\theta \ge 0$

$\Rightarrow \sin\theta\,(2\sin\theta - 1) \ge 0$

$\Rightarrow \sin\theta \ge 0$ and $2\sin\theta - 1 \ge 0$ or

$\sin\theta \le 0$ and $2\sin\theta - 1 \le 0$

Case I:

$\sin\theta \ge 0$ and $2\sin\theta - 1 \ge 0$

$\Rightarrow \sin\theta \ge 0$ and $\sin\theta \ge \dfrac{1}{2}$

$\Rightarrow \dfrac{\pi}{6} \le \theta \le \dfrac{5\pi}{6}$

$\therefore$ $A \cap B = \left\{\theta : \dfrac{\pi}{2} \le \theta \le \dfrac{5\pi}{6}\right\}$

....$\left[\because B = \left\{\theta : \dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}\right\}\right]$

Case II:

$\sin\theta \le 0$ and $2\sin\theta - 1 \le 0$

$\Rightarrow \sin\theta \le 0$ and $\sin\theta \le \dfrac{1}{2}$

$\Rightarrow \pi \le \theta \le 2\pi$

$\therefore$ $A \cap B = \left\{\theta : \pi \le \theta \le \dfrac{3\pi}{2}\right\}$

....$\left[\because B = \left\{\theta : \dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}\right\}\right]$

From Case I and II, we get

$$A \cap B = \left\{\theta : \frac{\pi}{2} \le \theta \le \frac{5\pi}{6}\right\} \cup \left\{\theta : \pi \le \theta \le \frac{3\pi}{2}\right\}$$

14. Given, $f(x) = \sqrt{\log_{10}\left\{\dfrac{\log_{10} x}{2(3-\log_{10} x)}\right\}}$

Now, f(x) is defined, if

$$\log_{10}\left\{\frac{\log_{10} x}{2(3-\log_{10} x)}\right\} \ge 0,\ \frac{\log_{10} x}{2(3-\log_{10} x)} > 0$$

and $x > 0$

$$\Rightarrow \frac{\log_{10} x}{2(3-\log_{10} x)} \ge 10^0 = 1,\ \frac{\log_{10} x}{(3-\log_{10} x)} > 0$$

and $x > 0$

$$\Rightarrow \frac{3(\log_{10} x - 2)}{2(\log_{10} x - 3)} \le 0,\ \frac{\log_{10} x}{\log_{10} x - 3} < 0 \text{ and } x > 0$$

$\Rightarrow 2 \le \log_{10}x < 3,\ 0 < \log_{10}x < 3$ and $x > 0$

$\Rightarrow 10^2 \le x < 10^3,\ 10^0 < x < 10^3$ and $x > 0$

$\Rightarrow 10^2 \le x < 10^3$

$\Rightarrow x \in [10^2, 10^3)$

Textbook Chapter No.

04 Sequence and Series

Hints

Classical Thinking

1. $a = 21, d = 16 - 21 = -5$
$t_n = a + (n-1)d$
$\therefore$ $t_{15} = 21 + (15-1)(-5) = 21 - 70 = -49$

2. $a = \sqrt{3}$, $d = \sqrt{12} - \sqrt{3} = \sqrt{3}$
$\therefore$ $t_{10} = \sqrt{3} + 9\sqrt{3} = 10\sqrt{3} = \sqrt{300}$

3. Given series
$\left(3 - \frac{1}{n}\right) + \left(3 - \frac{2}{n}\right) + \left(3 - \frac{3}{n}\right) + ...$ (A.P.)
Therefore, common difference
$d = \left(3 - \frac{2}{n}\right) - \left(3 - \frac{1}{n}\right) = -\frac{1}{n}$ and first term
$a = \left(3 - \frac{1}{n}\right)$
Now, p^{th} term of the series $= a + (p-1)d$
$= \left(3 - \frac{1}{n}\right) + (p-1)\left(-\frac{1}{n}\right)$
$= 3 - \frac{1}{n} + \frac{1}{n} - \frac{p}{n} = \left(3 - \frac{p}{n}\right)$

4. $d - c = e - d$
$\Rightarrow 2d = e + c$
$\Rightarrow 2d - 2c = e + c - 2c$
$\Rightarrow 2(d - c) = e - c$

5. a, b, c are in A.P., dividing by bc we get
$\frac{a}{bc}, \frac{1}{c}, \frac{1}{b}$ are in A.P.

6. $a = 3, d = 3$
Let there be n terms.
$\therefore$ $3 + (n-1)3 = 111$
$\Rightarrow n = 37$

7. $a = 72, d = -2$
Let n^{th} term be 40.
$\therefore$ $t_n = a + (n-1)d$
$\therefore$ $40 = 72 + (n-1)(-2)$
$\Rightarrow n = 17$

8. $d = -1 + 2i$, $t_4 = t_3 + d = 6 - 2i + (-1 + 2i) = 5$ which is purely real.

9. Given that, 9^{th} term $= a + (9-1)d = 0$
$\Rightarrow a + 8d = 0$
Now, ratio of 29^{th} and 19^{th} terms
$= \frac{a+28d}{a+18d} = \frac{(a+8d)+20d}{(a+8d)+10d} = \frac{20d}{10d} = \frac{2}{1}$

10. Let the first term and common difference of an A.P. be A and D respectively.
Now, p^{th} term $= A + (p-1)D = a$
q^{th} term $= A + (q-1)D = b$
and r^{th} term $= A + (r-1)D = c$
$\therefore$ $a(q-r) + b(r-p) + c(p-q)$
$= a\left\{\frac{b-c}{D}\right\} + b\left\{\frac{c-a}{D}\right\} + c\left\{\frac{a-b}{D}\right\}$
$= \frac{1}{D}(ab - ac + bc - ab + ca - bc) = 0$

11. $S_n = 3(4^n - 1)$
$\therefore$ $S_{n-1} = 3(4^{n-1} - 1)$
$\therefore$ $t_n = S_n - S_{n-1} = 3(4^n - 1) - 3(4^{n-1} - 1) = 9(4^{n-1})$

12. Required sum $= 1 + 3 + 5 + ….$ upto n terms
$= \frac{n}{2}[2 \times 1 + (n-1)2]$
$= n^2$

13. Given that first term $a = 10$, last term $l = 50$ and sum $S = 300$
$\therefore$ $S = \frac{n}{2}(a + l) \Rightarrow 300 = \frac{n}{2}(10 + 50) \Rightarrow n = 10$

14. $S_n = \frac{n}{2}[2a + (n-1)d]$
$\Rightarrow 406 = \frac{n}{2}[6 + (n-1)4]$
$\Rightarrow 812 = n[6 + 4n - 4] \Rightarrow 812 = 2n + 4n^2$
$\Rightarrow 406 = 2n^2 + n \Rightarrow 2n^2 + n - 406 = 0$
$\Rightarrow n = \frac{-1 \pm \sqrt{1 + 4.2.406}}{2.2} = \frac{-1 \pm \sqrt{3249}}{4}$
$= \frac{-1 \pm 57}{4}$
Taking (+) sign, $n = \frac{-1+57}{4} = 14$

15. $S_n = 3n^2 - n$

$\Rightarrow 3n^2 - n = \frac{n}{2}[2a + (n-1)6]$

$\Rightarrow a = 2$

16. $S_n = \frac{n}{2}[2a + (n-1)d]$

$\therefore$ $S_{16} = \frac{16}{2}[2(4) + (15)d]$

$\Rightarrow 784 = 8(8 + 15d)$

$\Rightarrow 8 + 15d = \frac{784}{8}$

$\Rightarrow 15d = 90$

$\Rightarrow d = 6$

17. $t_7 = 40 \Rightarrow a + 6d = 40$

$S_{13} = \frac{13}{2}[2a + (13-1)d] = 13(a + 6d) = 520$

18. $t_4 = a + 3d = 4$ and

$S_7 = \frac{7}{2}[2a + (7-1)d] = \frac{7}{2}[2a + 6d]$

$= 7(a + 3d)$

$= 7(4) = 28$

19. The terms of given sequence are in A.P. with $a = 1$, $d = 5$ and $S_n = 148$

$\therefore$ $\frac{n}{2}[2a + (n-1)d] = 148 \Rightarrow n = 8$

Now, $x = n^{th}$ term $\Rightarrow x = a + (n-1)d = 36$

20. $(x + 1) + (x + 4) + \ldots. + (x + 28) = 155$

Let n be the number of terms in the A.P. on L.H.S. Then,

$x + 28 = (x + 1) + (n-1)3 \Rightarrow n = 10$

$\therefore$ $(x + 1) + (x + 4) + \ldots.. + (x + 28) = 155$

$\Rightarrow \frac{10}{2}[(x + 1) + (x + 28)] = 155$

$\Rightarrow x = 1$

21. $S_5 = \frac{1}{4}(S_{10} - S_5) \Rightarrow 5S_5 = S_{10}$

$\therefore$ $5 \times \frac{5}{2}(2 \times 2 + 4d) = \frac{10}{2}(2 \times 2 + 9d)$

$\Rightarrow d = -6$

22. Here, $\frac{1}{3}$, A_1, A_2, $\frac{1}{24}$ will be in A.P.,

then $A_1 - \frac{1}{3} = \frac{1}{24} - A_2$

$\Rightarrow A_1 + A_2 = \frac{3}{8}$ ......(i)

Now, A_1 is a arithmetic mean of $\frac{1}{3}$ and A_2.

$\therefore$ $2A_1 = \frac{1}{3} + A_2 \Rightarrow 2A_1 - A_2 = \frac{1}{3}$ ......(ii)

From (i) and (ii), we get $A_1 = \frac{17}{72}$ and $A_2 = \frac{5}{36}$

23. Let the two numbers be a and b and let $A_1, A_2, \ldots..,A_n$ be the n A.M.'s between them. Then a, $A_1, A_2, \ldots., A_n$, b are in A.P. and let d be the common difference.

Now, $T_{n+2} = b = a + (n + 2 - 1)d$

$\Rightarrow d = \frac{b-a}{n+1}$

Also, $A_1 + A_2 + \ldots..+ A_n = S_{n+1} - a$

$= \frac{1}{2}(n+1)\left[2a + (n+1-1)\frac{(b-a)}{(n+1)}\right] - a$

$= \frac{n}{2}[2a + (b-a)] = \frac{n}{2}(a+b) = n\left(\frac{a+b}{2}\right)$

$\therefore$ S = Na

24. Let the three numbers be a + d, a, a – d. therefore, a + d + a + a – d = 33

$\Rightarrow a = 11$

and $a(a + d)(a - d) = 792$

$\Rightarrow 11(121 - d^2) = 792 \Rightarrow d = 7$

The required numbers are 4, 11, 18.

Hence, the smallest number is 4.

25. $t_n = ar^{n-1} = 1.(2)^{n-1} = 2^{n-1}$

26. Given sequence is $\sqrt{2}, \sqrt{10}, \sqrt{50}$........

Common ratio $r = \sqrt{5}$, first term $a = \sqrt{2}$, then 7^{th} term

$t_7 = \sqrt{2}(\sqrt{5})^{7-1} = \sqrt{2}(\sqrt{5})^6 = \sqrt{2}(5)^3 = 125\sqrt{2}$

28. $t_n = ar^{n-1} = 1\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n-1}$

29. $a = 3, r = \frac{\left(\frac{-3}{2}\right)}{3} = \frac{-1}{2}$

$\Rightarrow t_n = ar^{n-1} = 3\left(\frac{-1}{2}\right)^{n-1}$

30. Let r be common ratio of G.P.

$\Rightarrow t_3 = r^2, t_5 = r^4$

$\therefore$ $t_3 + t_5 = 90 \Rightarrow r^2 + r^4 = 90$

$\Rightarrow r^2 = 9$

$\Rightarrow r = \pm 3$

31. Accordingly, $ar^9 = 9$ and $ar^3 = 4$

$r^3 = \frac{3}{2}$ and $a = \frac{8}{3}$

$\therefore$ 7^{th} term i.e., $ar^6 = \frac{8}{3}\left(\frac{3}{2}\right)^2 = 6$

Trick : 7^{th} term is equidistant from 10^{th} and 4^{th} so it will be $\sqrt{9\times 4} = 6$.

33. $t_3 = ar^{3-1} = ar^2 = 20$ and
$t_7 = ar^{7-1} = ar^6 = 320$
Solving, $a = 5$ and $r = 2$

34. a, 8, b are in G.P. and $a \neq b$

$\Rightarrow \frac{8}{a} = \frac{b}{8} \Rightarrow ab = 64$

and a, b, -8 are in A.P.

$\Rightarrow b - a = -8 - b$

$\therefore$ $b = \left(\frac{a-8}{2}\right)$

Solving, $a = 16$ and $b = 4$

35. $a = 5, r = 3$

$S_n = \frac{a(r^n - 1)}{r-1} = \frac{5(3^n - 1)}{2}$

36. $a = 3$ and $r = \frac{12}{3} = 4 > 1$

$\therefore$ $S_n = a\left[\frac{r^n - 1}{r-1}\right] = 3\left[\frac{4^n - 1}{4-1}\right] = 4^n - 1$

38. $a = 1, r = 3$

$S_n = \frac{a(r^n - 1)}{r-1}$

$3280 = \frac{3^n - 1}{2}$

$6561 = 3^n$

$\Rightarrow 3^8 = 3^n \Rightarrow n = 8$

39. Let n be the number of terms needed.
For G.P. $2, 2^2, 2^3, \ldots$, $a = 2$, $r = 2$ and $S_n = 30$

$S_n = \frac{a\left(r^n - 1\right)}{r-1} \Rightarrow 30 = \frac{2\left(2^n - 1\right)}{2-1} \Rightarrow n = 4$

40. $S_8 = 82\ (S_4)$
Let the G.P. be $a + ar + a\,r^2 + \ldots$, then

$\frac{a\left(r^8 - 1\right)}{\left(r-1\right)} = 82\left\{\frac{a\left(r^4 - 1\right)}{r-1}\right\}$

$(r^4 - 1)(r^4 + 1) = 82(r^4 - 1)$

$\Rightarrow r^4 + 1 = 82$

$r^4 = 81$

$\Rightarrow r = 3$

41. $S_n = \frac{a\left(r^n - 1\right)}{r-1}$, $r = 2$

$\therefore$ $S_8 = \frac{a\left(2^8 - 1\right)}{2-1} \Rightarrow a(2^8 - 1) = 510 \Rightarrow a = 2$

$\therefore$ $t_3 = 2(2)^{3-1} = 2(2)^2 = 8$

42. $S_n = 2 + 22 + 222 + \ldots$ n terms

$= 2\,[1 + 11 + 111 + \ldots$ n terms]

$= \frac{2}{9}[(10 - 1) + (100 - 1) + (1000 - 1)$ $+ \ldots$ n terms]

$= \frac{2}{9}\left[10\left(\frac{10^n - 1}{10-1}\right) - n\right] = \frac{2}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

$= \frac{2}{81}[10(10^n - 1) - 9n]$

43. $S_n = 0.9 + 0.99 + 0.999 + \ldots$ n terms

$= 1 - 0.1 + 1 - 0.01 + 1 - 0.001$ $+ \ldots$ n terms

$= 1 + 1 + 1 + \ldots$ n terms $- [0.1 + 0.01 + 0.001 + \ldots$ n terms]

$= n - 0.1\left[\frac{1-(0.1)^n}{1-0.1}\right]$

$= n - \frac{1}{9}[1 - (0.1)^n]$

$= \frac{9n - [1 - (0.1)^n]}{9}$

44. $a = 2$, $S_\infty = 6$

Now, $S_\infty = \frac{a}{1-r}$

$\Rightarrow 6 = \frac{2}{1-r}$

$\Rightarrow 1 - r = \frac{1}{3}$

$\Rightarrow r = 1 - \frac{1}{3}$

$\Rightarrow r = \frac{2}{3}$

45. According to condition, $\frac{3/4}{1-r} = \frac{4}{3}$

$\Rightarrow r = \frac{7}{16}$

46. $\frac{g_1}{p} = \frac{q}{g_2} \Rightarrow g_1g_2 = pq$

47. Let 1,a, b, 64
$\Rightarrow a^2 = b$ and $b^2 = 64a$
$\Rightarrow a = 4$ and $b = 16$

48. Let the numbers be a, ar, ar^2
Sum = 70 $\Rightarrow a(1 + r + r^2) = 70$
It is given that 4a, 5ar, $4ar^2$ are in A.P.
$\therefore$ $2(5ar) = 4a + 4ar^2 \Rightarrow r = 2$ or $r = \frac{1}{2}$
Substituting values of r, a = 10 and a = 40
$\therefore$ The numbers are 10, 20, 40 or 40, 20, 10

49. Let numbers are $\frac{a}{r}$, a, ar
According to given conditions,
$\frac{a}{r} . a . ar = 216$
$\Rightarrow a = 6$
And, sum of product pairwise = 156
$\Rightarrow \frac{a}{r} . a + \frac{a}{r} . ar + a . ar = 156$
$\Rightarrow r = 3$
Hence, numbers are 2, 6, 18.
Trick : Since 2 × 6 × 18 = 216 (as given) and no other option gives the value.

50. Considering corresponding A.P.
$a + 6d = 10$ and $a + 11d = 25 \Rightarrow d = 3, a = -8$
$\Rightarrow t_{20} = a + 19d = -8 + 57 = 49$
Hence, 20^{th} term of the corresponding H.P. is $\frac{1}{49}$.

53. $\because$ H < G < A

54. (A.M.) (H.M.) = $(G.M)^2$
$\Rightarrow 9. 36 = (G.M)^2 \Rightarrow$ G.M. = 18

56. $G^2 = AH$
$\Rightarrow 144 = 25H$
$\Rightarrow H = 5.76$

58. Let $S = 1 + 3x + 5x^2 + 7x^3 + ….$
Then, $xS = 1x + 3x^2 + 5x^3 + ….$
$S - xS = 1 + 2x + 2x^2 + 2x^3 + ….$ to ∞
$\therefore$ $S(1 - x) = 1 + 2x + 2x^2 + 2x^3 + …$ to ∞
$$= 1 + \frac{2x}{1-x} = \frac{1-x+2x}{1-x}$$
$\therefore$ $S = \frac{1+x}{(1-x)^2}$

59. Here a = 3, d = 2 and r = r
Now $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} \quad (|r| < 1)$
$\therefore$ $S_\infty = \frac{3}{1-r} + \frac{2r}{(1-r)^2}$ $\therefore$ $\frac{44}{9} = \frac{3-r}{(1-r)^2}$
$\therefore$ $44r^2 - 79r + 17 = 0$
$\therefore$ $r = \frac{1}{4}$ or $\frac{17}{11}$
But, $r \neq \frac{17}{11}$
$\therefore$ $r = \frac{1}{4}$

60. $\sum_{r=1}^{n}(2r+5) = 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 5 = \frac{2(n)(n+1)}{2} + 5n$
$= n(n + 6)$

61. $(2^2 - 1^2) + (4^2 - 3^2) + (6^2 - 5^2) + …..$
$= (2^2 + 4^2 + 6^2 + …) - (1^2 + 3^2 + 5^2 + …)$
$= \sum_{r=1}^{n}(2r)^2 - \sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}4r - \sum_{r=1}^{n}1$
$= 4\left[\frac{n(n+1)}{2}\right] - n = n(2n + 1)$

62. $\frac{n(n+1)(2n+1)}{6} = 1015$
$\therefore$ $n(n + 1)(2n + 1) = 6090$
$\Rightarrow n(n + 1)(2n + 1) = 14 \times 15 \times 29$
$\Rightarrow n = 14$

63. $(31)^2 + (32)^2 + (33)^2 + ….. + (60)^2$
$= [(1)^2 + (2)^2 + (3)^2 + …. + (60)^2]$
$- [(1)^2 + (2)^2 + (3)^2 + ….. + (30)^2]$
$= \sum_{r=1}^{60} r^2 - \sum_{r=1}^{30} r^2 = 64355$

64. The first factors of the terms of the given series is 1, 2, 3, 4, …., n and second factors of the terms of the given series is
2, 3, 4, …….(n + 1)
$\therefore$ n^{th} term of the given series
$= n(n + 1) = n^2 + n$
Hence, sum =
$$\Sigma n^2 + \Sigma n = \frac{1}{6}n(n+1)(2n+1) + \frac{n}{2}(n+1)$$
$$= \frac{1}{6}n(n+1)(2n+1+3)$$
$$= \frac{1}{3}n(n+1)(n+2)$$

65. $1^3 + 2^3 + 3^3 + \ldots + 25^3 = \sum_{r=1}^{25} r^3$

$= \frac{(25)^2(25+1)^2}{4}$

$= 105625$

66. $2(1)^2 + 3(2)^2 + 4(3)^2 + \ldots$ upto 10 terms

$= \sum_{r=1}^{10}(r+1)r^2 = \sum_{r=1}^{10} r^3 + \sum_{r=1}^{10} r^2 = 3410$

67. $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$= \log_e (1 + x)$

$\therefore$ $1 + x = e^y \Rightarrow x = e^y - 1$

68. Sum of given series $= y + \frac{y^2}{2} + \frac{y^3}{3} + \ldots,$

where $y = x^2$.

$\therefore$ Sum of given series $= -\log(1 - y)$

$= -\log_e(1 - x^2)$

69. $\log_e 3 - \frac{\log_e 3^2}{2^2} + \frac{\log_3 3^3}{3^2} - \frac{\log_e 3^4}{4^2} + \ldots$

$= \log_e 3\left\{1 - \frac{2}{2^2} + \frac{3}{3^2} - \frac{4}{4^2} + \ldots\right\}$

$= \log_e 3\left\{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right\}$

$= \log_e 3 \log_e (1 + 1) = \log_e 3 \log_e 2$

Critical Thinking

1. $\frac{1}{1+\sqrt{x}}, \frac{1}{1-x}, \frac{1}{1-\sqrt{x}}, \ldots$

i.e., $\frac{1-\sqrt{x}}{1-x}, \frac{1}{1-x}, \frac{1+\sqrt{x}}{1-x}, \ldots,$

which is an A.P. with $d = \frac{\sqrt{x}}{1-x}$

$\therefore$ The fourth term $= t_3 + d = \frac{1+\sqrt{x}}{1-x} + \frac{\sqrt{x}}{1-x}$

$= \frac{1+2\sqrt{x}}{1-x}$

2. Here, $T_n = 3n - 1$, putting n = 1, 2, 3, 4, 5 we get first five terms, 2, 5, 8, 11, 14
Hence, sum is 2 + 5 + 8 + 11 + 14 = 40.

3. Given series 3.8 + 6.11 + 9.14 + 12.17 + …..
First factors are 3, 6, 9, 12 whose n^{th} term is 3n and second factors are 8, 11, 14, 17
$t_n = [8 + (n - 1)3] = (3n + 5)$
Hence n^{th} term of given series = 3n(3n + 5).

4. Required number n is the number of terms in the series 105 + 112 + 119 + …. + 994

$\therefore$ 994 = n^{th} term of the above A.P.

$\Rightarrow 994 = 105 + (n - 1) \times 7$

$\Rightarrow n = \frac{994 - 98}{7}$

$\Rightarrow n = 128$

5. Given sequence is in A.P.

$\therefore$ $a = 8 - 6i$, $d = -1 + 2i$

$\therefore$ $t_n = a + (n - 1)d = (9 - n) + i(2n - 8)$

For purely imaginary term, $9 - n = 0$

$\Rightarrow n = 9$

6. First term = a, d = b – a and last term = c
If the no. of terms is n, then

$t_n = c = a + (n - 1)(b - a) \Rightarrow \frac{c - a}{b - a} = n - 1$

Solving, $n = \frac{b + c - 2a}{b - a}$

7. d = b – a and if the number of terms is n, then
$2a = a + (n - 1)(b - a)$

$\Rightarrow \frac{a}{b-a} + 1 = n \Rightarrow n = \frac{b}{b-a}$

8. a, b, c are in A.P.

$\Rightarrow b - a = c - b \Rightarrow \frac{b-a}{c-b} = 1$

9. If D is the common difference of the A.P. a, b, c, d, e, then b = a + D, c = a + 2D, d = a + 3D, e = a + 4D

$\therefore$ $a - 4b + 6c - 4d + e$

$= a - 4(a + D) + 6(a + 2D) - 4(a + 3D) + a + 4D = 0$

10. Suppose that $\angle A = x°$, then $\angle B = x + 10°$, $\angle C = x + 20°$ and $\angle D = x + 30°$
So, we know that $\angle A + \angle B + \angle C + \angle D = 2\pi$
Putting these values, we get
$(x°) + (x° + 10°) + (x° + 20°) + (x° + 30°) = 360°$
$\Rightarrow x = 75°$
Hence, the angles of the quadrilateral are 75°, 85°, 95°, 105°.

11. As we know $T_n = S_n - S_{n-1}$
$= (2n^2 + 5n) - \{2(n - 1)^2 + 5(n - 1)\}$
$= 2n^2 + 5n - 2n^2 + 4n - 2 - 5n + 5$
$= 4n + 3$

12. $t_n = S_n - S_{n-1}$

$= \left\{nP + \frac{n(n-1)}{2}Q\right\} - \left\{(n-1)P + \frac{(n-1)(n-2)}{2}Q\right\}$

$= P + (n-1)Q$

$\therefore$ Common difference $= t_n - t_{n-1}$

$= [P + (n-1)Q] - [P + (n-2)Q] = Q$

13. $d = \frac{1}{3} - \frac{1}{2} = \frac{-1}{6}$

$\therefore$ $S_9 = \frac{9}{2}\left\{2\times\frac{1}{2} + (9-1)\left(\frac{-1}{6}\right)\right\} = -\frac{3}{2}$

14. Required sum $= 10 + 13 + 16 + \ldots + 97$

$= \frac{n}{2}(10 + 97)$ …. (i)

Here, $97 = 10 + (n-1)3 \Rightarrow n = 30$

$\therefore$ From (i), $S_n = \frac{30}{2}(10 + 97) = 1605$

15. The smallest 3 digit no. divisible by 7 is 105 and greatest is 994.

Given sequence is in A.P. with $d = 7$

$\therefore$ $994 = 105 + (n-1)7 \Rightarrow n = 128$

$\therefore$ $S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{128}{2}[2(105) + (128-1)7] = 70336$

16. According to the given condition

$\frac{15}{2}[10 + 14 \times d] = 390 \Rightarrow d = 3$

Hence, middle term i.e., 8^{th} term is given by

$5 + 7 \times 3 = 26$

17. $l = a + (n-1)d$ and

$S_n = \frac{n}{2}(a + l)$

Eliminating a, we get

$S_n = \frac{n}{2}\{l - (n-1)d + l\} = \frac{n}{2}\{2l - (n-1)d\}$

18. Suppose work is completed in n days

$\frac{n}{2}[2 \times 150 + (n-1)(-4)] = n(152 - 2n)$

Had no worker dropped from work, total no. of workers who would have worked all the n days is $150(n-8)$

$\therefore$ $n(152 - 2n) = 150(n-8) \Rightarrow n = 25$

19. $d = -2$, sum $= -5$

$\therefore$ $-5 = \frac{5}{2}\{2a + 4(-2)\} \Rightarrow a = 3$

Hence, the actual sum (when $d = 2$) is

$\frac{5}{2}\{2\times3 + (5-1)\times2\} = \frac{5}{2}(6+8) = 35$

21. Here $a = S_1 = 6$

$S_7 = 105 \Rightarrow \frac{7}{2}[2 \times 6 + (7-1)d] = 105 \Rightarrow d = 3$

$\therefore$ $\frac{S_n}{S_{n-3}} = \frac{\frac{n}{2}\{2\times6 + (n-1)3\}}{\frac{(n-3)}{2}\{2\times6 + (n-4)3\}} = \frac{n+3}{n-3}$

22. $S_{2n} = 3S_n$

$\therefore$ $\frac{2n}{2}[2a + (2n-1)d] = \frac{3n}{2}[2a + (n-1)d]$

$\Rightarrow$ $2a = (n+1)d$

$\therefore$ $\frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}[2a + (3n-1)d]}{\frac{n}{2}[2a + (n-1)d]} = 6$

23. Let S_n and S'_n be the sum of n terms of two A.P.'s and t_{11} and t'_{11} be the respective 11^{th} terms, then

$\frac{S_n}{S'_n} = \frac{\frac{n}{2}[2a + (n-1)d]}{\frac{n}{2}[2a' + (n-1)d']} = \frac{7n+1}{4n+27}$

$\Rightarrow$ $\frac{a + \frac{(n-1)}{2}d}{a' + \frac{(n-1)}{2}d'} = \frac{7n+1}{4n+27}$

Now put $n = 21$,

we get $\frac{a+10d}{a'+10d'} = \frac{t_{11}}{t'_{11}} = \frac{148}{111} = \frac{4}{3}$

24. a, b, c, are in A.P $\Rightarrow 2b = a + c$

Also, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. $\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$

$\therefore$ $\frac{2}{\frac{a+c}{2}} = \frac{a+c}{ac} \Rightarrow a = c$ and $b = a$

25. $(a + 2b - c)(2b + c - a)(c + a - b)$

$= (a + a + c - c)(a + c + c - a)(2b - b)$

$= 4abc$

($\because$ a, b, c are in A.P., $\therefore$ $2b = a + c$).

26. The sum of n arithmetic mean between a and b $= \frac{n}{2}(a+b)$

27. $\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$
$\Rightarrow a^{n+1} - ab^n + b^{n+1} - ba^n = 0$
$\Rightarrow (a - b)(a^n - b^n) = 0$
If $a^n - b^n = 0$. Then $\left(\frac{a}{b}\right)^n = 1 = \left(\frac{a}{b}\right)^0$
Hence, n = 0

28. The resulting progression will have n + 2 terms with 2 as the first term and 38 as the last term.
Therefore, the sum of the progression
$= \frac{n+2}{2}(2+38)$
$= 20(n + 2)$
By hypothesis, $20(n + 2) = 200$
$\Rightarrow n = 8$

29. As, log 2, $\log(2^n - 1)$ and $\log(2^n + 3)$ are in A.P.
Therefore,
$2\log(2^n - 1) = \log 2 + \log(2^n + 3)$
$2^{2n} - 4.2^n - 5 = 0$
$\Rightarrow (2^n - 5)(2^n + 1) = 0$
As 2^n cannot be negative, hence $2^n - 5 = 0$
$\Rightarrow 2^n = 5$ or $n = \log_2 5$

30. The given numbers are in A.P.
$\therefore$ $2\log_9(3^{1-x} + 2) = \log_3(4.3^x - 1) + 1$
$\Rightarrow 2\log_{3^2}(3^{1-x} + 2) = \log_3(4.3^x - 1) + \log_3 3$
$\Rightarrow \frac{2}{2}\log_3(3^{1-x} + 2) = \log_3[3(4.3^x - 1)]$
$\Rightarrow 3^{1-x} + 2 = 3(4.3^x - 1)$
$\Rightarrow \frac{3}{y} + 2 = 12y - 3$, where $y = 3^x$
$\Rightarrow 12y^2 - 5y - 3 = 0$
$\therefore$ $y = \frac{-1}{3}$ or $\frac{3}{4} \Rightarrow 3^x = \frac{-1}{3}$ or $3^x = \frac{3}{4}$
$\therefore$ $x = \log_3\left(\frac{3}{4}\right) \Rightarrow x = 1 - \log_3 4$

31. Let the three numbers be $a - d$, a, $a + d$
We get $a - d + a + a + d = 15$
$\Rightarrow a = 5$
and $(a - d)^2 + a^2 + (a + d)^2 = 83$
$\Rightarrow a^2 + d^2 - 2ad + a^2 + a^2 + d^2 + 2ad = 83$
$\Rightarrow 2(a^2 + d^2) + a^2 = 83$
Putting a = 5
$\Rightarrow 2(25 + d^2) + 25 = 83$
$\Rightarrow 2d^2 = 8$
$\Rightarrow d = 2$
Thus, numbers are 3, 5, 7.
Trick :
Since $3 + 5 + 7 = 15$ and $3^2 + 5^2 + 7^2 = 83$

32. $\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{8}\right) + \ldots$
$\therefore$ $t_n = 1 - \left(n^{th} \text{ term of G.P. } \frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$
$= 1 - \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$
$= 1 - \frac{1}{2^n}$

33. $t_3 = 4 \Rightarrow ar^2 = 4$
$\therefore$ $a \times ar \times ar^2 \times ar^3 \times ar^4 = (ar^2)^5 = 4^5$

34. $t_3 = ar^{3-1} = ar^2 = 36$ and $t_6 = ar^{6-1} = ar^5 = 972$
Solving, a = 4 and r = 3
$\therefore$ $t_8 = ar^7 = 4(3)^7 = 8748$

35. $t_n = ar^{n-1}$ and $r = 2$
$\therefore$ $t_n = a(2)^{n-1} \Rightarrow t_9 = a(2)^8$
$\therefore$ $a(2)^8 = 128 \Rightarrow a = \frac{128}{256} = \frac{1}{2}$

36. $ab^2 = a(ac)$ and $cb^2 = c(ac)$
$\therefore$ $ab^2 - cb^2 = a^2c - ac^2$
$\Rightarrow a(b^2 + c^2) = c(a^2 + b^2)$

37. $a + ar = -4$ and $ar^4 = 4ar^2 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$
Substituting $r = \pm 2$, we get $a = \frac{-4}{3}$ and $a = 4$
$\therefore$ Required G.P. is $\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, \ldots$
or 4, – 8, 16, – 32, ….

38. The common ratio of the G.P. is x^{n+4}
$\therefore$ 8^{th} term $= x^{52} = x^{-4}(x^{n+4})^7$
$\Rightarrow 7n = 28$
$\Rightarrow n = 4$

39. Let $AR^{p-1} = a$,
$AR^{q-1} = b$,
$AR^{r-1} = c$
So
$a^{q-r}b^{r-p}c^{p-q} = (AR^{p-1})^{q-r}(AR^{q-1})^{r-p}(AR^{r-1})^{p-q}$
$= A^{(q-r+r-p+p-q)}R^{(pq-pr-q+r+qr-pq-r+p+pr-rq-p+q)}$
$= A^0R^0 = 1$

40. $a = \frac{5}{2}$, $r = \frac{1}{2} < 1$

$\therefore \quad S_n = \frac{a\left(1-r^n\right)}{1-r} = 5\left[\frac{2^n-1}{2^n}\right]$

41. Given series is a G.P. with $a = \sqrt{2}$ and $r = \sqrt{3}$

$\therefore \quad S_{10} = \frac{\sqrt{2}\left(\left(\sqrt{3}\right)^{10}-1\right)}{\sqrt{3}-1} = \frac{\sqrt{2}(243-1)}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$

$= 121\sqrt{6} + 121\sqrt{2}$

$= 121\left(\sqrt{6}+\sqrt{2}\right)$

42. $\left(1-\frac{1}{2}\right) + \left(1-\frac{1}{4}\right) + \left(1-\frac{1}{8}\right) + \ldots.$

$\therefore \quad S_n = n - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots.\text{upto n terms}\right)$

$= n - \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = n - \left(1-\frac{1}{2^n}\right)$

$= n - 1 + 2^{-n}$

43. $t_4 = 24$ and $t_9 = 768$

$\therefore \quad t_4 = ar^3 \Rightarrow ar^3 = 24$

and $t_9 = ar^8 \Rightarrow ar^8 = 768$

Solving, $a = 3$ and $r = 2 > 1$

$\therefore \quad S_{10} = \frac{a\left[r^{10}-1\right]}{r-1} = \frac{3\left[2^{10}-1\right]}{2-1} = 3(2^{10} - 1)$

44. $S_{10} = 244\ S_5$

$\Rightarrow (1 - r^{10}) = 244\ (1 - r^5)$

$\Rightarrow r^{10} - 244\ r^5 + 243 = 0$

$\Rightarrow r^5 = 243$ or $r^5 = 1$

$\Rightarrow r = 3$ or $r = 1$

45. $a_1 = 3$, $a_n = 96$

$\Rightarrow a_1r^{n-1} = 96$

$\Rightarrow r^{n-1} = 32$

$r^{n-1} = 2^5$

$2^{n-1} = 2^5$

$\Rightarrow n - 1 = 5$

$\Rightarrow n = 6$

46. $a = 7$ and $ar^{n-1} = 448$

Now, $S_n = \frac{a(r^n-1)}{r-1} = 889$

$\Rightarrow \frac{(ar^{n-1}.r-a)}{r-1} = 889 \Rightarrow \frac{448r-7}{r-1} = 889$

$\Rightarrow r = 2$

47. Given that $\frac{a(r^n-1)}{r-1} = 255\ (\because\ r > 1)$ ….(i)

$ar^{n-1} = 128$ ….(ii)

and common ratio $r = 2$ ….(iii)

From (i), (ii) and (iii), we get

$a(2)^{n-1} = 128$ ….(iv)

and $\frac{a(2^n-1)}{2-1} = 255$ ….(v)

Dividing (v) by (iv), we get

$\frac{2^n-1}{2^{n-1}} = \frac{255}{128}$

$\Rightarrow 2 - 2^{-n+1} = \frac{255}{128}$

$\Rightarrow 2\left(1-2^{-n}\right) = \frac{255}{128}$

$\Rightarrow 2^{-n} = 2^{-8}$

$\Rightarrow n = 8$

Putting $n = 8$ in equation (iv),

we get $a.2^7 = 128 = 2^7$ or $a = 1$

48. $\frac{S_3}{S_6-S_3} = \frac{125}{27} \Rightarrow \frac{S_3}{S_6} = \frac{125}{152}$

$\therefore \quad \frac{a(1-r^3)}{a(1-r^6)} = \frac{125}{152} \Rightarrow \frac{1}{1+r^3} = \frac{125}{152}$

$\Rightarrow r^3 = \frac{27}{125} \Rightarrow r = \frac{3}{5}$

49. $r = \frac{t_2}{t_1} = \frac{b}{a}$; last term = c

$\Rightarrow ar^{n-1} = c$

$\Rightarrow \frac{ar^n}{r} = c$

$\Rightarrow ar^n = cr$

$\therefore \quad S_n = \frac{a(1-r^n)}{1-r} = \frac{a-ar^n}{1-r} = \frac{a-cr}{1-r} = \frac{a-c\left(\frac{b}{a}\right)}{1-\frac{b}{a}}$

50. We have

$1 + a + a^2 + \ldots.. + a^x = (1 + a)(1 + a^2)(1 + a^4)$

$\Rightarrow \frac{(1-a^{x+1})}{(1-a)} = (1 + a)(1 + a^2) + (1 + a^4)$

$\Rightarrow (1 - a^{x+1}) = (1 - a)(1 + a)(1 + a^2)(1 + a^4)$

$\Rightarrow (1 - a^{x+1}) = (1 - a^8)$

$\Rightarrow x + 1 = 8$

$\Rightarrow x = 7$

51. $S_n = 4 + 44 + 444 + \ldots\ldots$ to n terms

$= \frac{4}{9}\left[(10-1)+(10^2-1)+(10^3-1)+\ldots\ldots+(10^n-1)\right]$

$= \frac{4}{9}\left\{(10+10^2+10^3+\ldots.+10^n)-(1+1+1+\ldots\ldots\text{n times})\right\}$

$= \frac{4}{9}\left\{\frac{10(10^n-1)}{10-1}-n\right\} = \frac{4}{9}\left\{\frac{10}{9}(10^n-1)-n\right\}$

52. $1 + (1 + x) + (1 + x + x^2) + \ldots.$
$+ (1 + x + x^2 + \ldots. + x^{n-1})$

$= \frac{1-x}{1-x}+\frac{1-x^2}{1-x}+\frac{1-x^3}{1-x}+\ldots\ldots+\frac{1-x^n}{1-x}$

$= \frac{1}{1-x}[(1 + 1 + \ldots.\text{ n times})$
$- (x + x^2 + \ldots + x^n)]$

$= \frac{1}{1-x}\left[n-\frac{x(1-x^n)}{1-x}\right]$

$= \frac{n}{1-x}-\frac{x(1-x^n)}{(1-x)^2}$

53. Infinite series $9 - 3 + 1 - \frac{1}{3} + \ldots\ldots..\infty$ is a G.P. with $a = 9$, $r = \frac{-1}{3}$

$\therefore$ $S_\infty = \frac{a}{1-r} = \frac{9}{1+\left(\frac{1}{3}\right)} = \frac{9\times3}{4} = \frac{27}{4}$

54. Let the G.P. be $a + ar + ar^2 + \ldots.$, $|r| < 1$,

then $ar = 2$ and $\frac{a}{1-r} = 8$

$\therefore$ $\frac{ar(1-r)}{a} = \frac{2}{8} \Rightarrow r = \frac{1}{2}$ and $a = 4$

55. $\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+\ldots.$ upto ∞

$= \left(\frac{1}{7}+\frac{1}{7^3}+\frac{1}{7^5}+\ldots.\right)+2\left(\frac{1}{7^2}+\frac{1}{7^4}+\frac{1}{7^6}+\ldots.\right)$

$= \frac{\frac{1}{7}}{1-\frac{1}{7^2}}+\frac{2\left(\frac{1}{7^2}\right)}{1-\frac{1}{7^2}}$

$= \frac{3}{16}$

56. $2.3\overline{45} = 2.3 + 0.045 + 0.00045 + \ldots.$

$= \frac{23}{10}+\frac{45}{1000}+\frac{45}{100000}+\ldots.$

From 2^{nd} term onwards, the terms are in G.P.

$\therefore$ $s_\infty = \frac{a}{1-r} = \frac{\frac{45}{1000}}{1-\frac{1}{100}} = \frac{1}{22}$

$\therefore$ $2.3\overline{45} = \frac{23}{10}+\frac{1}{22} = \frac{129}{55}$

Alternate Method:

$2.3\overline{45} = 2 + \frac{345-3}{990} = 2 + \frac{342}{990} = \frac{129}{55}$

57. $4^{1/3}.\, 4^{1/9}.4^{1/27}\ldots\ldots.\infty$

$\therefore$ $S = 4^{1/3 + 1/9 + 1/27\ldots\ldots\infty}$

$\Rightarrow S = 4^{\left(\frac{1/3}{1-1/3}\right)} = 4^{\frac{1/3}{2/3}}$

$\Rightarrow S = 4^{1/2}$

$\Rightarrow S = 2$

58. $5 = \frac{x}{1-r} \Rightarrow 5 - 5r = x \Rightarrow r = 1 - \frac{x}{5}$

As $|r| < 1$ i.e., $\left|1-\frac{x}{5}\right| < 1$

$\therefore$ $-1 < 1 - \frac{x}{5} < 1$

$\therefore$ $-5 < 5 - x < 5$

$\therefore$ $-10 < -x < 0$

$\therefore$ $10 > x > 0$

$\therefore$ $0 < x < 10$

59. $A = 1 + r^z + r^{2z} + r^{3z} + \ldots.. \infty$

$A = 1 + [r^z + r^{2z} + r^{3z} + \ldots.. \infty]$

We know that sum of infinite G.P. is

$S_\infty = \frac{a}{1-r}\ (-1 < r < 1)$

Therefore, $A = 1 + \left[\frac{r^z}{1-r^z}\right]$

$\Rightarrow A = \frac{1-r^z+r^z}{1-r^z} \Rightarrow A = \frac{1}{1-r^z}$

$\Rightarrow 1 - r^z = \frac{1}{A} \Rightarrow r^z = \frac{A-1}{A}$

Hence, $r = \left[\frac{A-1}{A}\right]^{\frac{1}{z}}$

60. We have, $x = \sum_{n=0}^{\infty} a^n = \dfrac{1}{1-a} \Rightarrow a = \dfrac{x-1}{x}$

$y = \sum_{n=0}^{\infty} b^n = \dfrac{1}{1-b} \Rightarrow b = \dfrac{y-1}{y}$

$z = \sum_{n=0}^{\infty} a^n b^n = \dfrac{1}{1-ab} \Rightarrow ab = \dfrac{z-1}{z}$

$\therefore \quad \dfrac{x-1}{x} \cdot \dfrac{y-1}{y} = \dfrac{z-1}{z}$

$\Rightarrow xy + z = zx + yz$

61. $a = 3, r = 3$

G.M. $= (3.3^2.3^3 \ldots\ldots 3^n)^{1/n}$

$= (3^{1+2+3\ldots\ldots+n})^{1/n} = \left(3^{\frac{n(n+1)}{2}}\right)^{1/n} = 3^{\frac{(n+1)}{2}}$

62. As given, $G = \sqrt{xy}$

$\therefore \quad \dfrac{1}{G^2 - x^2} + \dfrac{1}{G^2 - y^2} = \dfrac{1}{xy - x^2} + \dfrac{1}{xy - y^2}$

$= \dfrac{1}{x-y}\left\{-\dfrac{1}{x} + \dfrac{1}{y}\right\} = \dfrac{1}{xy} = \dfrac{1}{G^2}$

63. a, g_1, g_2, b are in G.P. $\Rightarrow \dfrac{g_1}{a} = \dfrac{g_2}{g_1} = \dfrac{b}{g_2}$

$\therefore \quad \dfrac{g_1}{a} = \dfrac{g_2}{g_1}$ and $\dfrac{g_2}{g_1} = \dfrac{b}{g_2} \Rightarrow a = \dfrac{g_1^2}{g_2}$ and $b = \dfrac{g_2^2}{g_1}$

$\therefore \quad \dfrac{g_1^2}{g_2} + \dfrac{g_2^2}{g_1} = a + b$

64. G.M $= b = \sqrt{ac}$

$\dfrac{1}{b-a} + \dfrac{1}{b-c} = \dfrac{1}{\sqrt{ac} - a} + \dfrac{1}{\sqrt{ac} - c}$

$= \dfrac{1}{\sqrt{a}\left[\sqrt{c} - \sqrt{a}\right]} + \dfrac{1}{\sqrt{c}\left[\sqrt{a} - \sqrt{c}\right]}$

$= \dfrac{1}{\sqrt{a}\left[\sqrt{c} - \sqrt{a}\right]} - \dfrac{1}{\sqrt{c}\left[\sqrt{c} - \sqrt{a}\right]}$

$= \dfrac{1}{\sqrt{ac}} = \dfrac{1}{b}$

65. Let the 9 terms of a G.P. be

$\dfrac{a}{r^4}, \dfrac{a}{r^3}, \dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2, ar^3, ar^4$

Given, fifth term $a = 2$

Hence, product of 9 terms is $a^9 = (2)^9 = 512$

66. Let the terms of given G.P. be $\dfrac{a}{r}$, a, ar

then product $= \dfrac{a}{r} \times a \times ar = 1000$

$\dfrac{a}{r}$, $a + 6$, $ar + 7$ are in A.P.

$\therefore \quad 2(a + 6) = \dfrac{a}{r} + ar + 7 \quad \therefore \quad 25 = \dfrac{10}{r} + 10r$

$\therefore \quad 2r^2 - 5r + 2 = 0$

$\therefore \quad (2r - 1)(r - 2) = 0$

$\therefore \quad r = 2, \dfrac{1}{2}$

Hence, the G.P is 5, 10, 20,…. or 20, 10, 5.…

67. Suppose that x to be added then numbers 13, 15, 19 so that new numbers $x + 13$, $15 + x$, $19 + x$ will be in H.P.

$\Rightarrow (15 + x) = \dfrac{2(x+13)(19+x)}{x+13+x+19}$

$\Rightarrow x^2 + 31x + 240 = x^2 + 32x + 247 \Rightarrow x = -7$

68. a, b, c are in H.P. $\Rightarrow \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P.

$\therefore \quad \dfrac{1}{b} - \dfrac{1}{a} = \dfrac{1}{c} - \dfrac{1}{b} \Rightarrow \dfrac{a-b}{b-c} = \dfrac{a}{c}$

69. 7^{th} term of corresponding A.P. is $\dfrac{1}{8}$ and 8^{th} term will be $\dfrac{1}{7}$

$\Rightarrow a + 6d = \dfrac{1}{8}$ and $a + 7d = \dfrac{1}{7}$

Solving these, we get $d = \dfrac{1}{56}$ and $a = \dfrac{1}{56}$

Therefore, 15^{th} term of this A.P.

$= \dfrac{1}{56} + 14 \times \dfrac{1}{56} = \dfrac{15}{56}$

Hence, the required 15^{th} term of the H.P. is $\dfrac{56}{15}$.

70. Let a be the first term and d be the common difference of the corresponding A.P.

p^{th} term of A.P. $(T_p) = a + (p - 1)d$

$= \dfrac{1}{q}$ …..(i)

q^{th} term of A.P. $(T_q) = a + (q - 1)d$

$= \dfrac{1}{p}$ …..(ii)

From (i) – (ii), $(p-q)d = \frac{1}{q} - \frac{1}{p} = \frac{p-q}{pq}$

$\Rightarrow d = \frac{1}{pq}$

From (i),

$a + (p-1)\frac{1}{pq} = \frac{1}{q} \Rightarrow a = \frac{1}{pq}$

$\therefore$ $T_{pq} = a + (pq-1)d$

$= \frac{1}{pq} + (pq-1)\frac{1}{pq} = 1$

Therefore, pq^{th} term is 1.

71. $\text{H.M.} = \dfrac{2\left(\frac{a^2}{1-a^2b^2}\right)}{\frac{a}{1-ab} + \frac{a}{1+ab}} = \frac{2a^2}{2a} = a$

72. $c = \frac{2ab}{a+b} \Rightarrow \frac{c}{a} = \frac{2b}{a+b}$ and $\frac{c}{b} = \frac{2a}{a+b}$

$\therefore$ $\frac{c}{a} + \frac{c}{b} = \frac{2b}{a+b} + \frac{2a}{a+b} = 2$

73. $H = \frac{2ab}{a+b}$

$\Rightarrow H - a = \frac{2ab}{a+b} - a = \frac{ab-a^2}{a+b}$

and $H - b = \frac{2ab}{a+b} - b = \frac{ab-b^2}{a+b}$

$\therefore$ $\frac{1}{H-a} + \frac{1}{H-b} = \frac{a+b}{ab-a^2} + \frac{a+b}{ab-b^2}$

$= \frac{(a+b)}{(b-a)}\left[\frac{(b-a)}{ab}\right]$

$= \frac{1}{a} + \frac{1}{b}$

74. $H = \frac{2ab}{a+b} \Rightarrow \frac{H}{a} = \frac{2b}{a+b}$

$\therefore$ $\frac{H+a}{H-a} = \frac{3b+a}{b-a}$

Similarly, $\frac{H+b}{H-b} = \frac{3a+b}{a-b} = -\frac{3a+b}{b-a}$

$\therefore$ $\frac{H+a}{H-a} + \frac{H+b}{H-b} = \frac{2b-2a}{b-a} = 2$

75. a, b, c are in H.P.

$\therefore$ $b = \frac{2ac}{a+c}$

Also b, c, d are in H.P. $\Rightarrow c = \frac{2bd}{b+d}$

Multiplying we get, $bc = \frac{4abcd}{(a+c)(b+d)}$

$\therefore$ $ab + bc + cd + ad = 4ad$

$\Rightarrow ab + bc + cd = 3ad$

76. We know that A > G > H

Where A is arithmetic mean, G is geometric mean and H is harmonic mean, then A > G

$\Rightarrow \frac{a+b}{2} > \sqrt{ab}$ or $(a+b) > 2\sqrt{ab}$

77. Let the numbers be a and b, then

$4 = \frac{2ab}{a+b} \Rightarrow a + b = \frac{ab}{2}$

$A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Also, $2A + G^2 = 27$

$\therefore$ $a + b + ab = 27 \Rightarrow \frac{ab}{2} + ab = 27 \Rightarrow ab = 18$

and hence a + b = 9.

Only option A satisfies this condition.

78. As given, $2b = a + c \Rightarrow 3^{2b} = 3^{a+c}$

or $(3^b)^2 = 3^a.3^c$ i.e $3^a, 3^b, 3^c$ are in G.P.

79. $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\Rightarrow 2^{(b-a)x} = 2^{(c-b)x}$

$\Rightarrow (b-a)x = (c-b)x$

$\Rightarrow (b-a) = (c-b)\ \forall x, x \neq 0$

$\therefore$ $2^{ax+1}, 2^{bx+1}, 2^{cx+1}$ is a G.P., $\forall x \neq 0$

80. a, b, c are in A.P. $\Rightarrow 2b = a + c$

Now,

$(10^{bx+10})^2 = (10^{ax+10}.\ 10^{cx+10})$

$\Rightarrow 10^{2(bx+10)} = 10^{ax+cx+20}$

$\Rightarrow 2(bx + 10) = ax + cx + 20, \forall x$

$\Rightarrow 2b = a + c$ i.e. a, b, c are in A.P.

Hence, these are in G.P. $\forall x$

Alternate Method :

As we know if a, b, c are in A.P., then x^{an+r}, x^{bn+r}, x^{cn+r} are in G.P. for every n and r.

81. $\because$ a, b, c are in G.P. $\Rightarrow b^2 = ac$ …..(i)
Let $a^x = b^y = c^z = k$
$\Rightarrow a = k^{1/x}, b = k^{1/y}, c = k^{1/z}$
Putting these values in (i),
$k^{2/y} = k^{1/x}. k^{1/z} = k^{\frac{1}{x}+\frac{1}{z}}$ *i.e.*, $\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$
$\therefore$ $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P. or x, y, z are in H.P.

82. Here, $\frac{\log x}{\log a}, \frac{\log x}{\log b}, \frac{\log x}{\log c}$ are in H.P.
$\Rightarrow \frac{\log a}{\log x}, \frac{\log b}{\log x}, \frac{\log c}{\log x}$ are in A.P.
$\Rightarrow \log_x a, \log_x b, \log_x c$ are in A.P.
$\Rightarrow$ a, b, c are in G.P.

83. Clearly, $x = \frac{1}{1-a}, y = \frac{1}{1-b}, z = \frac{1}{1-c}$
Since a, b, c are in A.P.
$\Rightarrow 1 - a, 1 - b, 1 - c$ are also in A.P.
$\Rightarrow \frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c}$ are in H.P.
$\therefore$ x, y, z are in H.P.

84. Given that $\frac{a}{b} = \frac{9}{1}$ or $a = 9b$
Here, $H = \frac{2ab}{a+b}$ and $G = \sqrt{ab}$
$\Rightarrow H : G = \frac{2ab}{a+b} : \sqrt{ab} = \frac{2.9b^2}{10b} : 3b = \frac{3}{5}$
Hence, G : H = 5 : 3

85. Given that $\frac{\text{H.M.}}{\text{G.M.}} = \frac{12}{13}$
$\Rightarrow \frac{\frac{2ab}{a+b}}{\sqrt{ab}} = \frac{12}{13}$ or $\frac{a+b}{2\sqrt{ab}} = \frac{13}{12}$
$\Rightarrow \frac{(a+b)+2\sqrt{ab}}{(a+b)-2\sqrt{ab}} = \frac{13+12}{13-12} = \frac{25}{1}$
$\Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{5}{1}$
$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})} = \frac{5+1}{5-1}$
$\Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{6}{4}$
$\Rightarrow \left(\frac{a}{b}\right)^{1/2} = \frac{6}{4} \Rightarrow a : b = 9 : 4$

86. Let $S = 1 + 2.2 + 3.2^2 + 4.2^3 + …. + 100.2^{99}$
$2S = 1.2 + 2.2^2 + 3.2^3 + …. + 100.2^{100}$
$S - 2S = 1 + (1.2 + 1.2^2 + 1.2^3$
$+ ….$ upto 99 terms$) - 100\cdot2^{100}$
$\therefore$ $S = -1 - \frac{2(2^{99}-1)}{2-1} + 100.2^{100}$
$= -1 - 2^{100} + 2 + 100.2^{100}$
$= 1 + 99 \times 2^{100}$

87. Given series, let
$S_n = 1 + \frac{2}{5} + \frac{3}{5^2} + \frac{4}{5^3} + ... + \frac{n}{5^{n-1}}$
$\frac{1}{5}S_n = \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + ... + \frac{n}{5^n}$
Subtracting,
$\left(1-\frac{1}{5}\right)S_n = 1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3}$
$+$ ……..+ upto n terms $- \frac{n}{5^n}$
$\Rightarrow \frac{4}{5}S_n = \frac{1-\frac{1}{5^n}}{\frac{4}{5}} - \frac{n}{5^n}$
$\Rightarrow S_n = \frac{25}{16} - \frac{4n+5}{16\times5^{n-1}}$

88. Let $S_n = 1 + 2x + 3x^2 + 4x^3 + …. + nx^{n-1}$…. (i)
$xS_n = x + 2x^2 + 3x^3 + ….. + nx^n$ ……(ii)
Subtracting (ii) from (i), we get
$(1 - x) S_n = 1 + x + x^2 + x^3 +$…..to n terms $- nx^n$
$= \frac{(1-x^n)}{1-x} - nx^n$
$\Rightarrow S_n = \frac{(1-x^n) - nx^n(1-x)}{(1-x)^2}$
$= \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$

89. Let $S = 2 + 4 + 7 + 11 + 16 + …. + t_n$
$S = 2 + 4 + 7 + 11 + 16 + ….. t_{n-1} + t_n$
Subtracting, we get
$0 = 2 + \{2 + 3 + 4 + ….. + (t_n - t_{n-1})\} - t_n$
$\Rightarrow t_n = 1 + \{1 + 2 + 3 + 4 + …..$ upto n terms)
$\Rightarrow t_n = 1 + \frac{1}{2}n(n+1)$
$= \frac{2+n^2+n}{2} = \frac{n^2+n+2}{2}$

90. We have $\sqrt{2} + \sqrt{8} + \sqrt{18} + \sqrt{32} + \ldots\ldots$
$= 1\sqrt{2} + 2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + \ldots..$
$= \sqrt{2}\,[1 + 2 + 3 + 4 + \ldots.$ upto 24 terms$]$
$= \sqrt{2} \times \dfrac{24\times 25}{2}$
$= 300\sqrt{2}$

91. $t_r = \dfrac{1+2+3+\ldots..+r}{r} = \dfrac{\frac{r(r+1)}{2}}{r} = \dfrac{r+1}{2}$

$\therefore\ S_n = \sum\limits_{r=1}^{n} t_r = \sum\limits_{r=1}^{n}\dfrac{r+1}{2} = \dfrac{1}{2}\left[\sum\limits_{r=1}^{n} r + \sum\limits_{r=1}^{n} 1\right]$
$= \dfrac{1}{2}\left[\dfrac{n(n+1)}{2} + n\right]$
$= \dfrac{1}{4}(n^2 + 3n) = \dfrac{n(n+3)}{4}$

93. $S_n = \sum\limits_{r=1}^{n}(4r-3)(4r-1)$
$= \sum\limits_{r=1}^{n}(16r^2 - 16r + 3)$
$= \dfrac{16n(n+1)(2n+1)}{6} - \dfrac{16n(n+1)}{2} + 3n$
$= n\left(\dfrac{16n^2-7}{3}\right)$

94. $\sum n^2 = 330 + \sum n$
$\Rightarrow \dfrac{n(n+1)(2n+1)}{6} = 330 + \dfrac{n(n+1)}{2}$
$\Rightarrow \dfrac{n(n+1)}{2}\left[\dfrac{2n+1}{3} - 1\right] = 330$
$\Rightarrow \dfrac{n(n+1)}{2}\cdot\dfrac{2(n-1)}{3} = 330$
$\Rightarrow n(n+1)(n-1) = 990 \Rightarrow n = 10$

95. $t_r = \dfrac{1^3 + 2^3 + \ldots. + r^3}{(r+1)^2} = \dfrac{r^2}{4}$

$\therefore\ S_n = \sum\limits_{r=1}^{n} t_r = \sum\limits_{r=1}^{n}\dfrac{r^2}{4} = \dfrac{1}{4}\sum\limits_{r=1}^{n} r^2$
$= \dfrac{1}{4}\,\dfrac{n(n+1)(2n+1)}{6}$
$= \dfrac{n(n+1)(2n+1)}{24}$

96. $1^3 + 3^3 + 5^3 + \ldots. + 21^3$
$= (1^3 + 2^3 + 3^3 + 4^3 + \ldots. + 21^3)$
$\quad - (2^3 + 4^3 + 6^3 + \ldots.. + 20^3)$
$= \sum\limits_{r=1}^{21} r^3 - 8\sum\limits_{r=1}^{10} r^3$
$= \dfrac{(21)^2(21+1)^2}{4} - \dfrac{8\times 10^2(10+1)^2}{4}$
$= 29161$

97. $\sum\limits_{n=1}^{20}(n^3) - \sum\limits_{n=1}^{10}(n^3) = \left[\dfrac{n(n+1)}{2}\right]^2_{n=20} - \left[\dfrac{n(n+1)}{2}\right]^2_{n=10}$
$\Rightarrow \left[\dfrac{20\times 21}{2}\right]^2 - \left[\dfrac{10\times 11}{2}\right]^2$
$= 44100 - 3025$
$= 41075$

98. Here $T_n = \dfrac{1^3 + 2^3 + 3^3 + \ldots\ldots.. + n^3}{1 + 3 + 5 + \ldots\ldots..\text{upto n terms}}$
$= \dfrac{\Sigma n^3}{\frac{n}{2}[2 + (n-1)2]}$
$= \dfrac{1}{4}\,\dfrac{n^2(n+1)^2}{n^2}$
$= \dfrac{1}{4}(n^2 + 2n + 1)$

99. Given fraction $\dfrac{\sum\limits_{n=1}^{12} n^3}{\sum\limits_{n=1}^{12} n^2} = \dfrac{\left\{\dfrac{12(12+1)}{2}\right\}^2}{\dfrac{12(12+1)(2\times 12+1)}{6}}$
$= \dfrac{12\times 13}{4} \times \dfrac{6}{25} = \dfrac{234}{25}$

100. $S_1 = \dfrac{n(n+1)}{2}$, $S_2 = \dfrac{n(n+1)(2n+1)}{6}$
$S_3 = \left(n\left(\dfrac{n+1}{2}\right)\right)^2$
For, $S_3(1 + 8S_1) = \dfrac{n^2(n+1)^2}{4}\left(1 + \dfrac{8n(n+1)}{2}\right)$
$= \left(\dfrac{n(n+1)(2n+1)}{6}\right)^2 \times 9$
$= 9S_2^2$

101. $\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots$
which is the expansion of e^{-1}

102. $e^{-x} = (1-x) + \frac{x^2}{2!}\left(1-\frac{x}{3}\right) + \frac{x^4}{4!}\left(1-\frac{x}{5}\right)+\ldots$

$\therefore$ $e^{-1} = (1-1) + \frac{1}{2!}\left(1-\frac{1}{3}\right)+\frac{1}{4!}\left(1-\frac{1}{5}\right)+\ldots$

$= \frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\ldots$

103. Let $t_n = \frac{1}{(n+1)!}$

$S_n = \frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots$

$= \left[1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots\right]-\left[1+\frac{1}{1!}\right]$

$= e-(1+1) = e-2$

104. Given ratio $= \dfrac{\frac{1}{2}\left(e+\frac{1}{e}\right)-1}{\frac{1}{2}\left(e-\frac{1}{e}\right)} = \dfrac{(e-1)^2}{(e-1)(e+1)}$

$= \frac{e-1}{e+1}$

105. $\frac{1}{1\times 2} + \frac{1}{3\times 4} + \frac{1}{5\times 6} + \ldots \infty$

$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{5}-\frac{1}{6}\right) + \ldots$ to ∞

$= \log 2$

106. $T_n = \frac{3^n-1}{3^n} = 1-\left(\frac{1}{3}\right)^n$

$S_n = n - \sum_{n=1}^{n}\left(\frac{1}{3}\right)^n = n - \dfrac{\frac{1}{3}\left[1-\left(\frac{1}{3}\right)^n\right]}{\left(1-\frac{1}{3}\right)}$

$= n-\frac{1}{2}(1-3^{-n}) = n+\frac{1}{2}(3^{-n}-1)$

107. The series is

$\frac{2}{1!}+\frac{(2+5)}{2!}+\frac{(2+5+8)}{3!}+\frac{(2+5+8+11)}{4!}+\ldots$

Hence, $T_n = \frac{(2+5+8+\ldots\ldots n \text{ terms})}{n!}$

$= \dfrac{\frac{n}{2}[2.2+(n-1)3]}{n!}$

$T_n = \frac{n(3n+1)}{2(n)!}$

108. Let $S = i-2-3i+4+5i+\ldots\ldots+100i^{100}$

$\Rightarrow S = i+2i^2+3i^3+4i^4+5i^5+\ldots\ldots+100i^{100}$

$\Rightarrow iS = i^2+2i^3+3i^4+4i^5+\ldots+99i^{100}+100i^{101}$

$\therefore$ $S-iS = [i+i^2+i^3+i^4+\ldots+i^{100}]-100i^{101}$

$\Rightarrow S(1-i) = 0-100i^{101} = -100i$

$\therefore$ $S = \frac{-100i}{1-i} = -50i(1+i) = -50(i-1)$

$= 50(1-i)$

109. Here, $T_r = \frac{1}{r(r+1)}$, $r = 1, 2, \ldots n$

$\Rightarrow T_r = \frac{1}{r}-\frac{1}{r+1}$

$\therefore$ Required sum $= \sum_{r=1}^{n} T_r$

$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$= 1-\frac{1}{n+1} = \frac{n}{n+1}$

110. sin A, cos A and tan A are in G.P.

$\therefore$ $\cos^2 A = \sin A \tan A = \frac{\sin^2 A}{\cos A}$

$\Rightarrow \cos^3 A = \sin^2 A$

$\Rightarrow \cos^3 A = 1-\cos^2 A$

$\Rightarrow \cos^3 A+\cos^2 A = 1$

111. $\cos^4\theta \sec^2\alpha, \frac{1}{2}$ and $\sin^4\theta \operatorname{cosec}^2\alpha$ are in A.P.

$\therefore$ $1 = \cos^4\theta\sec^2\alpha+\sin^4\theta\operatorname{cosec}^2\alpha$

$\Rightarrow \cos^4\theta\sin^2\alpha+\sin^4\theta\cos^2\alpha = \sin^2\alpha\cos^2\alpha$

$\Rightarrow (1-\sin^2\theta)\cos^2\theta\sin^2\alpha+\sin^4\theta(1-\sin^2\alpha)$
$= \sin^2\alpha(1-\sin^2\alpha)$

$\Rightarrow \cos^2\theta\sin^2\alpha+\sin^4\theta-\sin^2\theta\sin^2\alpha(\cos^2\theta+\sin^2\theta)$
$= \sin^2\alpha-\sin^4\alpha$

$\Rightarrow \sin^4\theta+\sin^4\alpha-\sin^2\theta\sin^2\alpha-\sin^2\alpha(1-\cos^2\theta)$
$= 0$

$\Rightarrow \sin^4\theta+\sin^4\alpha-2\sin^2\theta\sin^2\alpha = 0$

$\Rightarrow (\sin^2\theta-\sin^2\alpha)^2 = 0$

$\Rightarrow \sin^2\theta = \sin^2\alpha$ and $\cos^2\theta = \cos^2\alpha$

$\therefore$ $\cos^8\theta\sec^6\alpha+\sin^8\theta\operatorname{cosec}^6\alpha$
$= \cos^2\theta+\sin^2\theta = 1$

$\Rightarrow \cos^8\theta\sec^6\alpha$, $\frac{1}{2}$ and $\sin^8\theta\operatorname{cosec}^6\alpha$ are in A.P.

112. $(0.05)^{\log_{\sqrt{20}}(0.1+0.01+.....)} = \left(\frac{1}{20}\right)^{2\log_{20}\left(\frac{0.1}{1-0.1}\right)}$

$= 20^{-2\log_{20}(1/9)} = 20^{2\log_{20}9}$

$= 20^{\log_{20}9^2} = 9^2 = 81$

113. If $\log_{ax}x$, $\log_{bx}x$, $\log_{cx}x$ are in H.P

Then $\frac{1}{\log_{ax}x}, \frac{1}{\log_{bx}x}, \frac{1}{\log_{cx}x}$ are in A.P.

i.e., $\log_x ax$, $\log_x bx$, $\log_x cx$ are in A.P.

$\therefore$ $2\log_x bx = \log_x ax + \log_x cx$

$\therefore$ $\log_x b^2x^2 = \log_x ac.\,x^2$

$\therefore$ $b^2x^2 = ac.\,x^2$

$\therefore$ $b^2 = ac$

$\therefore$ a, b, c, are in G.P

114. A.M. $\geq$ G.M.

$\therefore$ $\frac{27^{\cos x} + 81^{\sin x}}{2} \geq \sqrt{27^{\cos x} \times 81^{\sin x}}$

$\Rightarrow 27^{\cos x} + 81^{\sin x} \geq 2\sqrt{3^{3\cos x + 4\sin x}}$

$\Rightarrow 27^{\cos x} + 81^{\sin x} \geq 2 \times \sqrt{3^{-5}}$

....[$\because -5 \leq 3\cos x + 4\sin x \leq 5$]

$\Rightarrow 27^{\cos x} + 81^{\sin x} \geq \frac{2}{9\sqrt{3}}$

Hence, the minimum value of $27^{\cos x} + 81^{\sin x}$ is $\frac{2}{9\sqrt{3}}$.

115. Since, $\tan\frac{2\pi}{18}$, x, $\tan\frac{7\pi}{18}$ are in A.P. and $\tan\frac{2\pi}{18}$, y, $\tan\frac{5\pi}{18}$ are in A.P.

$\therefore$ $2x = \tan\frac{2\pi}{18} + \tan\frac{7\pi}{18}$ and $2y = \tan\frac{2\pi}{18} + \tan\frac{5\pi}{18}$

$\Rightarrow 2x = \tan 20° + \tan 70°$ and

$2y = \tan 20° + \tan 50°$

$\Rightarrow 2x = \frac{\sin(20°+70°)}{\cos 20° \cos 70°}$ and $2y = \frac{\sin(20°+50°)}{\cos 20° \cos 50°}$

$\Rightarrow 2x = \frac{\sin 90°}{\sin 20° \cos 20°}$ and $2y = \frac{\sin 70°}{\cos 20° \cos 50°}$

$\Rightarrow x = \frac{1}{2\sin 20° \cos 20°}$ and $2y = \frac{1}{\cos 50°}$

$\Rightarrow x = \frac{1}{\sin 40°}$ and $2y = \frac{1}{\sin 40°}$

$\Rightarrow x = 2y \Rightarrow \frac{x}{y} = 2$

116. Given, $\cos(\theta - \alpha)$, $\cos\theta$ and $\cos(\theta + \alpha)$ are in H.P.

$\Rightarrow \frac{1}{\cos(\theta-\alpha)}, \frac{1}{\cos\theta}, \frac{1}{\cos(\theta+\alpha)}$ will be in A.P.

$\Rightarrow \frac{2}{\cos\theta} = \frac{1}{\cos(\theta-\alpha)} + \frac{1}{\cos(\theta+\alpha)}$

$= \frac{\cos(\alpha+\theta)+\cos(\theta-\alpha)}{\cos^2\theta - \sin^2\alpha}$

$\Rightarrow \frac{2}{\cos\theta} = \frac{2\cos\theta\cos\alpha}{\cos^2\theta - \sin^2\alpha}$

$\Rightarrow \cos^2\theta - \sin^2\alpha = \cos^2\theta\cos\alpha$

$\Rightarrow \cos^2\theta\,(1 - \cos\alpha) = \sin^2\alpha$

$\Rightarrow \cos^2\theta\left(2\sin^2\frac{\alpha}{2}\right) = 4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}$

$\Rightarrow \cos^2\theta\sec^2\frac{\alpha}{2} = 2$

$\Rightarrow \cos\theta\sec\frac{\alpha}{2} = \pm\sqrt{2}$

117. $\sin\theta = \sqrt{\sin\phi\cos\phi}$

$\Rightarrow \sin\phi\cos\phi = \sin^2\theta$

$\Rightarrow \sin 2\phi = 2\sin^2\theta$

$\Rightarrow 1 - 2\sin^2\theta = 1 - \sin 2\phi$

$\Rightarrow \cos 2\theta = 1 - \cos\left(\frac{\pi}{2} - 2\phi\right)$

$= 2\sin^2\left(\frac{\pi}{4} - \phi\right)$

118. $\cos 2B = \frac{\cos(A+C)}{\cos(A-C)} = \frac{\cos A\cos C - \sin A\sin C}{\cos A\cos C + \sin A\sin C}$

$\Rightarrow \frac{1-\tan^2 B}{1+\tan^2 B} = \frac{1-\tan A\tan C}{1+\tan A\tan C}$

$\Rightarrow 1 + \tan^2 B - \tan A\tan C - \tan A\tan C\tan^2 B$

$= 1 - \tan^2 B + \tan A\tan C - \tan A\tan C\tan^2 B$

$\Rightarrow 2\tan^2 B = 2\tan A\tan C$

$\Rightarrow \tan^2 B = \tan A\tan C$

$\therefore$ tan A, tan B, tan C are in G.P.

Competitive Thinking

1. The given sequence is an A.P.

a = 10, d = –3

$t_{30} = 10 + (30 - 1)(-3)$

$= -77$

2. Given series $27 + 9 + 5\frac{2}{5} + 3\frac{6}{7} + \ldots$

$= 27 + \frac{27}{3} + \frac{27}{5} + \frac{27}{7} + \ldots + \frac{27}{2n-1} + \ldots$

Hence, n^{th} term of given series $t_n = \frac{27}{2n-1}$

So, $t_9 = \frac{27}{2\times9-1} = \frac{27}{17} = 1\frac{10}{17}$

3. Since, a, 9, 3a – b and 3a + b are in A.P.

$\therefore$ $9 - a = (3a + b) - (3a - b)$

$\Rightarrow 9 - a = 2b \Rightarrow a + 2b = 9$ ....(i)

Also, $9 - a = (3a - b) - 9$

$\Rightarrow 4a - b = 18$ ....(ii)

Eliminating b from (i) and (ii), we get

$4a - 18 = (9 - a)/2$

$\Rightarrow 8a - 36 = 9 - a \Rightarrow 9a = 45 \Rightarrow a = 5$

So, first 2 terms of the A.P. are 5 and 9

So, a = 5, d = 4

$\therefore$ 2011^{th} term $= a + 2010d$

$= 5 + 2010 \times 4$

$= 8045$

4. According to the given condition,

$100(a + 99d) = 50(a + 49d)$

$\Rightarrow 2a + 198d = a + 49d$

$\Rightarrow a + 149d = 0$

$\therefore$ $T_{150} = a + 149d = 0$

5. a + 1, 2a + 1, 4a – 1 are in AP.

$\therefore$ $2a + 1 - (a + 1) = 4a - 1 - (2a + 1)$

$\Rightarrow a = 2a - 2$

$\Rightarrow a = 2$

6. It is not possible to express a + b + 4c – 4d + e in terms of a.

7. Required ratio is $\frac{44}{99} = \frac{4}{9}$

8. Given series $63 + 65 + 67 + 69 + \ldots$ .... (i)

and $3 + 10 + 17 + 24 + \ldots$ .... (ii)

Now from (i), m^{th} term = (2m + 61) and m^{th} term of (ii) series = (7m – 4)

According to the given condition,

$7m - 4 = 2m + 61$

$\Rightarrow 5m = 65 \Rightarrow m = 13$

9. According to the given condition,

$p\{a+(p-1)d\} = q\{a+(q-1)d\}$

$\Rightarrow a(p - q) + (p^2 - q^2)d + (q - p)d = 0$

$\Rightarrow (p - q)\{a + (p + q - 1)d\} = 0$

$\Rightarrow a + (p + q - 1)d = 0$ ....[$\because p \neq q$]

$\Rightarrow t_{p+q} = 0$

10. Given that, $t_p = a + (p - 1)d = q$ .... (i)

and $t_q = a + (q - 1)d = p$ .... (ii)

From (i) and (ii), we get $d = -\frac{(p-q)}{(p-q)} = -1$

Putting the value of d in equation (i), we get

$a = p + q - 1$

$t_r = a + (r - 1)d = (p + q - 1) + (r - 1)(-1)$

$= p + q - r$

11. $t_m = a + (m - 1)d = \frac{1}{n}$ and

$t_n = a + (n - 1)d = \frac{1}{m}$

On solving, $a = \frac{1}{mn}$ and $d = \frac{1}{mn}$

$\therefore$ $t_{mn} = a + (mn - 1)d = \frac{1}{mn} + (mn - 1)\frac{1}{mn} = 1$

12. Let the first term be a and common difference be d.

The last 3 terms are T_{23}, T_{22} and T_{21}.

According to the given condition,

$T_{21} + T_{22} + T_{23} = 261$

$\Rightarrow (a + 20d) + (a + 21d) + (a + 22d) = 261$

$\Rightarrow 3a + 63d = 261$ ....(i)

Also, sum of 3 middle terms = 141

$\Rightarrow T_{11} + T_{12} + T_{13} = 141$

$\Rightarrow (a + 10d) + (a + 11d) + (a + 12d) = 141$

$\Rightarrow 3a + 33d = 141$ ....(ii)

Solving (i) and (ii), we get a = 3

13. $164 = (3m^2 + 5m) - \{3(m - 1)^2 + 5(m - 1)\}$

$= (3m^2 + 5m) - 3m^2 + 6m - 3 - 5m + 5$

$\Rightarrow 164 = 6m + 2 \Rightarrow m = 27$

14. a = 3, d = 2

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$= \frac{n}{2}[6 + (n - 1)2]$

$= n(n + 2)$

15. Series 108 + 117 + ... + 999 is an A.P.

Here, $a = 108$, $d = 9$, $t_n = l = 999$

Before 108, there are 11 multiples of 9 (and 108 is 12^{th} multiple. 999 is 111^{th} multiple of 9).

$\Rightarrow$ From 108 to 999 there are 100 terms.

$\Rightarrow$ Required sum

$= \frac{100}{2}(108 + 999)$ ...$\left[\because S_n = \frac{n}{2}(a + l)\right]$

$= 50 \times 1107 = 55350$

16. The series of all natural numbers is
3, 6, 9, 12, ... 99
Here $n = \frac{99}{3} = 33$, a = 3, d = 3
$l = 99$
$\therefore$ $S_{33} = \frac{33}{2}\{3 + 99\}$
$= \frac{33}{2} \times 102$
$= 33 \times 51 = 1683$

17. Series, 2 + 5 + 8 + 11 + …..
a = 2,d = 3 and let number of terms be n,
then sum of A.P. $= \frac{n}{2}\{2a + (n - 1)d\}$
$\Rightarrow 60100 = \frac{n}{2}\{2 \times 2 + (n - 1)3\}$
$\Rightarrow 120200 = n(3n + 1)$
$\Rightarrow 3n^2 + n - 120200 = 0$
$\Rightarrow (n - 200)(3n + 601) = 0$
Hence, n = 200

18. 12, 19, …, 96 is the series of numbers which are of two digits and leave remainder 5 when divided by 7.
Here, a = 12, d = 7
Last term (l) = 96
$S_{13} = \frac{13}{2}[12+96]$
$= \frac{13}{2}\times108$
$= 702$

19. For given series,
a = 1
d = 2
$\therefore$ $a_n = a + (n - 1)d$
$\therefore$ $2001 = 1 + (n - 1)(2)$
$\therefore$ $n = 1001$
$\therefore$ $S_{1001} = \frac{1001}{2}[2(1) + (1001 - 1) \times (2)]$
$\therefore$ $S_{1001} = (1001)^2$

20. k^{th} term = 5k + 1
$\therefore$ 1^{st} term = a = 6
2^{nd} term = 11
3^{rd} term = 16
$\therefore$ d = 5
$\therefore$ $S_{100} = \frac{100}{2}[2 \times 6 + (100 - 1) \times 5]$
$\therefore$ $S_{100} = 50\ (507)$

21. Here, a = ₹ 200, d = ₹ 40
Saving in first two months = ₹ 400
Remained saving = 200 + 240 + 280 + …. upto n terms
$\Rightarrow \frac{n}{2}[400 + (n - 1)40] = 11040 - 400$
$\Rightarrow 200n + 20n^2 - 20n = 10640$
$\Rightarrow 20n^2 + 180n - 10640 = 0$
$\Rightarrow n^2 + 9n - 532 = 0$
$\Rightarrow (n + 28)(n - 19) = 0$
$\Rightarrow n = 19$
$\therefore$ Number of months = 19 + 2 = 21

22. According to the given condition,
$4500 = 150 \times 10 + \{148 + 146 + \ldots \text{ upto n terms}\}$
$= 1500 + \frac{n}{2}\{296 + (n - 1)(-2)\}$
$\Rightarrow n^2 - 149n + 3000 = 0$
$\Rightarrow (n - 24)(n - 125) = 0$
$\Rightarrow n = 24$ ….[$\because n \neq 125$]
So, total time taken = 10 + 24 = 34 min.

23. Let the number of sides of the polygon be n.
Then the sum of interior angles of the polygon
$= (2n - 4)\frac{\pi}{2} = (n - 2)\pi$
Since, the angles are in A.P.and a = 120°,d = 5 therefore,
$\frac{n}{2}[2 \times 120 + (n - 1)5] = (n - 2)180$
$\Rightarrow n^2 - 25n + 144 = 0$
$\Rightarrow (n - 9)(n - 16) = 0$
$\Rightarrow n = 9, 16$
But n = 16 gives,
$T_{16} = a + 15d$
$= 120° + 15.5°$
= 195° which is impossible, as interior angle cannot be greater than 180°. Hence, n = 9.

24. We have $\frac{S_{n_1}}{S_{n_2}} = \frac{2n+3}{6n+5}$
$\Rightarrow \frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{2n+3}{6n+5}$

$$\Rightarrow \frac{2\left[a_1 + \left(\frac{n-1}{2}\right)d_1\right]}{2\left[a_2 + \left(\frac{n-1}{2}\right)d_2\right]} = \frac{2n+3}{6n+5}$$

$$\Rightarrow \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{2n+3}{6n+5}$$

Put n = 25 then $\frac{a_1 + 12d_1}{a_2 + 12d_2} = \frac{2(25)+3}{6(25)+5}$

$$\Rightarrow \frac{t_{13_1}}{t_{13_2}} = \frac{53}{155}$$

25. Let the first term be a and common difference be d.

Given, $\frac{a_1 + a_2 + ... + a_p}{a_1 + a_2 + ... + a_q} = \frac{p^2}{q^2}$

$$\Rightarrow \frac{pa + d[1+2+...+(p-1)]}{qa + d[1+2+...+(q-1)]} = \frac{p^2}{q^2}$$

$$\Rightarrow \frac{pa + \frac{p(p-1)}{2}d}{qa + \frac{q(q-1)}{2}d} = \frac{p^2}{q^2} \Rightarrow \frac{a + \left(\frac{p-1}{2}\right)d}{a + \left(\frac{q-1}{2}\right)d} = \frac{p}{q}$$

We have to find, $\frac{a_6}{a_{21}} = \frac{a+5d}{a+20d}$

Put $\frac{p-1}{2} = 5$ and $\frac{q-1}{2} = 20$

$\Rightarrow p = 11$ and $\Rightarrow q = 41$

$\therefore$ $\frac{a+5d}{a+20d} = \frac{11}{41}$

26. Acording to the given condition,

$\frac{n}{2}\{2a + (n-1)d\} = \frac{m}{2}\{2a + (m-1)d\}$

$\Rightarrow 2a(m-n) + d(m^2 - m - n^2 + n) = 0$

$\Rightarrow (m-n)\{2a + d(m+n-1)\} = 0$

$\Rightarrow 2a + (m+n-1)d = 0$ ....[$\because m \neq n$]

$\therefore$ $S_{m+n} = \frac{m+n}{2}\{2a + (m+n-1)d\}$

$= \frac{m+n}{2}\{0\} = 0$

27. Given that $S_n = nA + n^2B$

Putting n = 1, 2, 3, …… we get

$S_1 = A + B$, $S_2 = 2A + 4B$, $S_3 = 3A + 9B$

....

....

Therefore,

$T_1 = S_1 = A + B$,

$T_2 = S_2 - S_1 = A + 3B$,

$T_3 = S_3 - S_2 = A + 5B$,

....

....

Hence, the sequence is

(A + B), (A + 3B), (A + 5B)…….

Here, a = A + B and common difference d = 2B

28. $S_1 = a_2 + a_4 + a_6 + a_8 + …. + a_{100}$

$S_2 = a_1 + a_3 + a_5 + a_7 + …. + a_{99}$

$\therefore$ $S_1 - S_2 = (a_2 - a_1) + (a_4 - a_3) + … + (a_{100} - a_{99})$

$= d + d + … + d = 50d \Rightarrow d = \frac{S_1 - S_2}{50}$

29. As given $a_2 - a_1 = a_3 - a_2 = … = a_n - a_{n-1} = d$

Where d is the common difference of the given A.P.

Also $a_n = a_1 + (n-1)d$

Then by rationalising each term,

$$\frac{1}{\sqrt{a_2} + \sqrt{a_1}} + \frac{1}{\sqrt{a_3} + \sqrt{a_2}} + ... + \frac{1}{\sqrt{a_n} + \sqrt{a_{n-1}}}$$

$$= \frac{\sqrt{a_2} - \sqrt{a_1}}{a_2 - a_1} + \frac{\sqrt{a_3} - \sqrt{a_2}}{a_3 - a_2} + ... + \frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{a_n - a_{n-1}}$$

$$= \frac{1}{d}(\sqrt{a_2} - \sqrt{a_1} + \sqrt{a_3} - \sqrt{a_2} + ... + \sqrt{a_n} - \sqrt{a_{n-1}})$$

$$= \frac{1}{d}(\sqrt{a_n} - \sqrt{a_1}) = \frac{1}{d}\left(\frac{a_n - a_1}{\sqrt{a_n} + \sqrt{a_1}}\right)$$

$$= \frac{1}{d}\left\{\frac{(n-1)d}{\sqrt{a_n} + \sqrt{a_1}}\right\} = \frac{n-1}{\sqrt{a_n} + \sqrt{a_1}}$$

30. $\frac{1}{S_1S_2} + \frac{1}{S_2S_3} + ... + \frac{1}{S_{100}S_{101}} = \frac{1}{6}$

$$\Rightarrow \frac{1}{d}\left[\frac{S_2 - S_1}{S_1S_2} + \frac{S_3 - S_2}{S_2S_3} + ... + \frac{S_{101} - S_{100}}{S_{100}S_{101}}\right] = \frac{1}{6}$$

….[$\because S_2 - S_1 = S_3 - S_2 = … = d$]

$$\Rightarrow \frac{1}{d}\left[\frac{1}{S_1} - \frac{1}{S_2} + \frac{1}{S_2} - \frac{1}{S_3} + ... + \frac{1}{S_{100}} - \frac{1}{S_{101}}\right] = \frac{1}{6}$$

$\Rightarrow \frac{1}{d}\left[\frac{1}{S_1}-\frac{1}{S_{101}}\right]=\frac{1}{6} \Rightarrow \frac{1}{d}\left[\frac{1}{S_1}-\frac{1}{S_1+100d}\right]=\frac{1}{6}$

$\Rightarrow \frac{1}{d}\left[\frac{100d}{S_1.(S_1+100d)}\right]=\frac{1}{6}$

$\Rightarrow S_1.(S_1 + 100d) = 600$ ….(i)

Given, $S_1 + S_{101} = 50$

$\Rightarrow S_1 + (S_1 + 100d) = 50 \Rightarrow 2S_1 + 100d = 50$

$\Rightarrow S_1 + 50d = 25$

$\Rightarrow S_1 = 25 - 50d$ ….(ii)

Putting (ii) in (i), we get

$(25 - 50d).(25 + 50d) = 600$

$\Rightarrow 625 - 2500\,d^2 = 600$

$\Rightarrow d^2 = \frac{1}{100} \Rightarrow d = \pm\frac{1}{10}$

$\therefore$ $|S_1 - S_{101}| = |S_1 - (S_1 + 100d)|$

$= |-100d| = 100\,|d|$ ….[$\because$ $|xy| = |x|.|y|$]

$\therefore$ $|S_1 - S_{101}| = 10$ ….[$\because$ $d = \pm 1/10$]

31. $a_1, a_2, a_3, \ldots, a_{n+1}$ are in A.P. and common difference = d

Let $S=\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots\ldots\ldots+\frac{1}{a_na_{n+1}}$

$\Rightarrow S=\frac{1}{d}\left\{\frac{d}{a_1a_2}+\frac{d}{a_2a_3}+\ldots\ldots+\frac{d}{a_n\,a_{n+1}}\right\}$

$\Rightarrow S=\frac{1}{d}\left\{\frac{a_2-a_1}{a_1a_2}+\frac{a_3-a_2}{a_2a_3}+\ldots\ldots+\frac{a_{n+1}-a_n}{a_n\,a_{n+1}}\right\}$

$\Rightarrow S=\frac{1}{d}\left\{\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_2}-\frac{1}{a_3}+\ldots\ldots.+\frac{1}{a_n}-\frac{1}{a_{n+1}}\right\}$

$\Rightarrow S=\frac{1}{d}\left\{\frac{1}{a_1}-\frac{1}{a_{n+1}}\right\}=\frac{1}{d}\left\{\frac{a_{n+1}-a_1}{a_1a_{n+1}}\right\}$

$\Rightarrow S=\frac{1}{d}\left(\frac{nd}{a_1a_{n+1}}\right)=\frac{n}{a_1a_{n+1}}$

Trick: Check for n = 2.

32. Since, $a_1 = 0$

$\therefore$ $a_2 = d, a_3 = 2d, \ldots$

$\therefore$ $\left(\frac{a_3}{a_2}+\frac{a_4}{a_3}+\ldots+\frac{a_n}{a_{n-1}}\right)-a_2\left(\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_{n-2}}\right)$

$=\left(\frac{2d}{d}+\frac{3d}{2d}+\ldots+\frac{(n-1)d}{(n-2)d}\right)$

$-d\left(\frac{1}{d}+\frac{1}{2d}+\ldots+\frac{1}{(n-3)d}\right)$

$=\left(\frac{2}{1}+\frac{3}{2}+\ldots+\frac{n-1}{n-2}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-3}\right)$

$=\left[(1+1)+\left(1+\frac{1}{2}\right)+\ldots+\left(\frac{n-1}{n-2}\right)\right]$

$-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-3}\right)$

$= (n-3) + \frac{n-1}{n-2} = (n-3) + 1 + \frac{1}{n-2}$

$= (n-2) + \frac{1}{n-2}$

33. As given

$d = a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1}$

$\therefore$ $\sin d\,\{\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \ldots.. + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n\}$

$=\frac{\sin(a_2-a_1)}{\sin a_1.\sin a_2}+\ldots\ldots+\frac{\sin(a_n-a_{n-1})}{\sin a_{n-1}\sin a_n}$

$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + \ldots. + (\cot a_{n-1} - \cot a_n)$

$= \cot a_1 - \cot a_n$

34. $\log_3 2$, $\log_3(2^x - 5)$ and $\log_3\left(2^x-\frac{7}{2}\right)$ are in A.P.

$\Rightarrow 2\log_3(2^x - 5) = \log_3\left[(2)\left(2^x-\frac{7}{2}\right)\right]$

$\Rightarrow (2^x - 5)^2 = 2^{x+1} - 7$

$\Rightarrow 2^{2x} - 12.2^x + 32 = 0$

$\Rightarrow x = 2,3$

But $x = 2$ does not hold, hence $x = 3$

35. $2\tan^{-1}y = \tan^{-1}x + \tan^{-1}z$

$\Rightarrow \tan^{-1}\left(\frac{2y}{1-y^2}\right)=\tan^{-1}\left(\frac{x+z}{1-xz}\right)$

$\Rightarrow \frac{2y}{1-y^2}=\frac{x+z}{1-xz}$

But $2y = x + z$ …[$\because$ x, y, z are in A.P.]

$\therefore$ $1 - y^2 = 1 - xz$

$\Rightarrow y^2 = xz$

$\because$ x, y, z are both in G.P. and A.P.,

$\therefore$ $x = y = z$

36. Since, a,b,c are in A.P., we get

$b - c = -d$, ...(i)

$c - a = 2d$, ...(ii)

$a - b = -d$ ...(iii)

Also, since x, y, z are in G.P., we get

$y^2 = x.z$ ...(iv)

Now, $x^{b-c}.y^{c-a}.z^{a-b}$

$= x^{-d}.y^{2d}.z^{-d}$ ...[From (i), (ii), (iii)]

$= x^{-d}.(x.z)^d.z^{-d}$

$= 1$

37. If a, b, c are in A.P., then 2b = a + c

So, $\dfrac{(a-c)^2}{(b^2-ac)} = \dfrac{(a-c)^2}{\left\{\left(\dfrac{a+c}{2}\right)^2 - ac\right\}}$

$= \dfrac{4(a-c)^2}{[a^2+c^2+2ac-4ac]}$

$= \dfrac{4(a-c)^2}{(a-c)^2} = 4$

Trick: Put a = 1, b = 2, c = 3, then the required value is $\dfrac{4}{1} = 4$.

38. Let a – d, a, a + d be the roots of the equation $x^3 - 12x^2 + 39x - 28 = 0$

Then, (a – d) + a + (a + d) = 12 and (a – d)a(a + d) = 28

$\Rightarrow 3a = 12$ and $a(a^2 - d^2) = 28$

$\Rightarrow a = 4$ and $a(a^2 - d^2) = 28$

$\Rightarrow 16 - d^2 = 7$

$\Rightarrow d = \pm 3$

39. Arithmetic mean of ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ i.e. (n + 1) terms

$= \dfrac{{}^nC_0 + {}^nC_1 + {}^nC_2 + ... {}^nC_n}{n+1}$

$= \dfrac{2^n}{n+1}$

40. For set a to 2b,

2b is the $(n + 2)^{th}$ term

$\therefore$ 2b = a + (n + 1)d

$\Rightarrow d = \dfrac{2b-a}{n+1}$

$\therefore$ m^{th} mean $= a + md = a + m\left(\dfrac{2b-a}{n+1}\right)$ ...(i)

For set 2a to b,

b is the $(n + 2)^{th}$ term

$\therefore$ b = 2a + (n + 1)d

$\Rightarrow d = \dfrac{b-2a}{n+1}$

$\therefore$ m^{th} mean $= 2a + md = 2a + m\left(\dfrac{b-2a}{n+1}\right)$ ...(ii)

$\therefore$ From (i) and (ii)

$a + m\left(\dfrac{2b-a}{n+1}\right) = 2a + m\left(\dfrac{b-2a}{n+1}\right)$

$\Rightarrow \dfrac{a}{b} = \dfrac{m}{n+1-m}$

41. The given sequence is a G.P.

$a = 3, r = \dfrac{1}{3}$

$t_6 = 3\left(\dfrac{1}{3}\right)^{6-1}$

$= 3\left(\dfrac{1}{3}\right)^5$

$= \dfrac{1}{81}$

42. $r = \dfrac{1}{3}\sqrt{\dfrac{20}{3}.\dfrac{9}{10}} = \dfrac{\sqrt{60}}{10} = \sqrt{\dfrac{6}{10}} = \sqrt{\dfrac{3}{5}}$

$\therefore$ $t_5 = ar^4 = \left(\dfrac{10}{9}\right)\left(\dfrac{3}{5}\right)^2 = \dfrac{10}{9}.\dfrac{9}{25} = \dfrac{2}{5}$

43. Given that x, $2x + 2$, $3x + 3$ are in G.P.

Therefore,

$(2x + 2)^2 = x(3x + 3)$

$\Rightarrow x^2 + 5x + 4 = 0$

$\Rightarrow (x + 4)(x + 1) = 0$

$\Rightarrow x = -1, -4$

Now, first term: $a = x$

and second term: $ar = 2(x + 1)$

$\Rightarrow r = \dfrac{2(x+1)}{x}$

then 4^{th} term $= ar^3 = x\left[\dfrac{2(x+1)}{x}\right]^3 = \dfrac{8}{x^2}(x + 1)^3$

Putting, $x = -4$

We get, $t_4 = \dfrac{8}{16}(-3)^3 = -\dfrac{27}{2} = -13.5$

44. Let the first four terms be a, –ar, ar^2, $-ar^3$, where r > 0, a > 0

According to the given conditions,

$a - ar = 12$ and $ar^2 - ar^3 = 48$

By solving, we get r = 2 (r > 0)

So, a = –12

45. $t_5 = ar^4 = \dfrac{1}{3}$ …..(i)

and $t_9 = ar^8 = \dfrac{16}{243}$ …..(ii)

Solving (i) and (ii), we get $r = \dfrac{2}{3}$ and $a = \dfrac{27}{16}$

Now 4^{th} term $= ar^3 = \dfrac{3^3}{2^4}.\dfrac{2^3}{3^3} = \dfrac{1}{2}$

46. $t_n = t_{n+1} + t_{n+2}$
$\Rightarrow a r^{n-1} = a r^n + a r^{n+1}$
$\Rightarrow r^{n-1} = r^n (1 + r)$
$\Rightarrow r^2 + r = 1$
$\Rightarrow r^2 + r - 1 = 0$
$\therefore$ $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= \frac{-1 \pm \sqrt{1+4}}{2}$
$= \frac{-1 \pm \sqrt{5}}{2}$

47. Let first term and common ratio of G.P. are respectively a and r, then under condition,
$t_n = t_{n-1} + t_{n-2}$
$\Rightarrow ar^{n-1} = ar^{n-2} + ar^{n-3}$
$\Rightarrow ar^{n-1} = ar^{n-1} r^{-1} + ar^{n-1} r^{-2}$
$\Rightarrow 1 = \frac{1}{r} + \frac{1}{r^2}$
$\Rightarrow r^2 - r - 1 = 0$
$\Rightarrow r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2}$
Taking only (+) sign ($\because r > 1$)

48. Let the G.P. be a, ar, ar^2, ar^3, ar^4, …
$t_2 + t_5 = ar + ar^4 = 216$
$\frac{t_4}{t_6} = \frac{ar^3}{ar^5} = \frac{1}{4}$
$\Rightarrow r^2 = 4 \Rightarrow r = \pm 2$
For r = 2,
$a(2 + 2^4) = 216$
$\Rightarrow a(18) = 216$
$\Rightarrow a = \frac{216}{18} = 12$
For r = –2,
$a(-2 + 2^4) = 216$
$\Rightarrow a(14) = 216$
$\Rightarrow a = \frac{216}{14} = \frac{108}{7}$
$\therefore$ a = 12

49. Let first term of G.P.= A and
common ratio = r
We know that n^{th} term of G.P. = Ar^{n-1}
Now $t_4 = a = Ar^3$, $t_7 = b = Ar^6$ and $t_{10} = c = Ar^9$
Relation $b^2 = ac$ is true because
$b^2 = (Ar^6)^2 = A^2r^{12}$ and $ac = (Ar^3)(Ar^9) = A^2r^{12}$

Alternate method : As we know, if p, q, r in A.P., then p^{th}, q^{th}, r^{th} terms of a G.P. are always in G.P., therefore, a, b, c will be in G.P. i.e. $b^2 = ac$.

50. The given series is a G.P. with a = i, r = –i
$\therefore$ $S_{100} = \frac{i(1-i^{100})}{1+i}$
$= \frac{i(1-(i^2)^{50})}{1+i}$
$= \frac{i(1-1)}{1+i} = 0$

51. $\frac{ar^n - a}{r-1} = 364$
$\Rightarrow \frac{ar^{n-1}.r - a}{r-1} = 364$ …..(i)
$\Rightarrow \frac{3 \times 243 - a}{2} = 364$
$\Rightarrow$ a = 1
Now, putting this in (i), n = 6

52. $\therefore$ n^{th} term of series = $ar^{n-1} = a(3)^{n-1}$
= 486 …..(i)
and sum of n terms of series.
$S_n = \frac{a(3^n - 1)}{3-1} = 728$ ($\because r > 1$) ......(ii)
From (i), $a\left(\frac{3^n}{3}\right) = 486$ or $a.3^n = 3 \times 486$
= 1458
From (ii), $a.3^n - a = 728 \times 2$
or $a.3^n - a = 1456$
$\therefore$ 1458 – a = 1456
$\Rightarrow$ a = 2

53. $t_2 = ar = 24$
$t_5 = ar^4 = 3$
$\frac{t_5}{t_2} = \frac{1}{8} = r^3$
$\Rightarrow r = \frac{1}{2}$ & a = 48
$S_6 = \frac{a\left(1-r^6\right)}{1-r}$
$= \frac{48\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = \frac{189}{2}$

54. $9 + 99 + 999 + ...10$ terms

$= (10 - 1) + (100 - 1) + (1000 - 1) + \ldots 10$ terms

$= (10 + 100 + 1000 + \ldots 10 \text{ terms}) - (1 + 1 + 1 + \ldots 10 \text{ terms})$

$= (10 + 10^2 + 10^3 + \ldots 10 \text{ terms}) - (10)$

$= \dfrac{10\left(10^{10}-1\right)}{10-1} - 1$

$= \dfrac{10\left(10^{10}-1\right)}{9} - 10$

$= \dfrac{10\left(10^{10}-1\right)-90}{9}$

$= \dfrac{100}{9}\left(10^9-1\right)$

55. Series $3 + 33 + 333 + \ldots\ldots + n$ terms

Given series can be written as,

$= \dfrac{1}{3}[9 + 99 + 999 + \ldots + n \text{ terms}]$

$= \dfrac{1}{3}[(10-1) + (10^2-1) + (10^3-1) + \ldots + n \text{ terms}]$

$= \dfrac{1}{3}[10 + 10^2 + \ldots + 10^n] - \dfrac{1}{3}[1 + 1 + 1 + \ldots + n \text{ terms}]$

$= \dfrac{1}{3}\cdot\dfrac{10(10^n-1)}{10-1} - \dfrac{1}{3}.n$

$= \dfrac{1}{3}\left[\dfrac{10^{n+1}-10}{9} - n\right]$

$= \dfrac{1}{3}\left[\dfrac{10^{n+1}-9n-10}{9}\right]$

$= \dfrac{1}{27}[10^{n+1}-9n-10]$

56. Let the G.P. be $a, ar, ar^2, ar^3, \ldots, ar^{48}, ar^{49}$

i.e., $a_1 = a$, $a_2 = ar$, $a_3 = ar^2$, …, $a_{49} = ar^{48}$ and $a_{50} = ar^{49}$

$\therefore \quad \dfrac{a_1 - a_3 + a_5 - \ldots + a_{49}}{a_2 - a_4 + a_6 - \ldots + a_{50}}$

$= \dfrac{a - ar^2 + ar^4 - \ldots + ar^{48}}{ar - ar^3 + ar^5 - \ldots + ar^{49}}$

$= \dfrac{\dfrac{a\left(1-(-r^2)^{25}\right)}{1-(-r^2)}}{\dfrac{ar\left(1-(-r^2)^{25}\right)}{1-(-r^2)}} = \dfrac{1}{r} = \dfrac{a}{ar} = \dfrac{a_1}{a_2}$

57. Since $n^m + 1$ divides $1 + n + n^2 + \ldots + n^{127}$

Therefore, $\dfrac{1+n+n^2+\ldots+n^{127}}{n^m+1}$ is an integer

$\Rightarrow \dfrac{1-n^{128}}{1-n} \times \dfrac{1}{n^m+1}$ is an integer

$\Rightarrow \dfrac{(1-n^{64})(1+n^{64})}{(1-n)(n^m+1)}$

is an integer, when largest $m = 64$.

58. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}, \dfrac{1}{\sqrt{2}(\sqrt{2}-1)}, \dfrac{1}{2}, \ldots\ldots$

Common ratio of the series $= \dfrac{1}{\sqrt{2}(\sqrt{2}+1)}$

$\therefore \quad \text{sum} = \dfrac{a}{1-r} = \left(\dfrac{\sqrt{2}+1}{\sqrt{2}-1}\right)\Big/\left(1-\dfrac{1}{\sqrt{2}(\sqrt{2}+1)}\right)$

$= \dfrac{(\sqrt{2}+1)}{(\sqrt{2}-1)}\cdot\dfrac{\sqrt{2}(\sqrt{2}+1)}{(1+\sqrt{2})}$

$= \sqrt{2}(\sqrt{2}+1)^2$

59. Clearly it is a infinite G.P. whose common ratio is 0.24.

$\therefore \quad S_\infty = \dfrac{a}{1-r} = \dfrac{5.05}{1-0.24} = 6.64474$

60. $(32)(32)^{1/6}(32)^{1/36}\ldots\infty = (32)^{1+\frac{1}{6}+\frac{1}{36}+\ldots\infty}$

$= (32)^{\frac{1}{1-(1/6)}} = (32)^{\frac{1}{5/6}} = (32)^{\frac{6}{5}}$

$= 2^6 = 64$

61. According to the given condition,

$\dfrac{a}{1-r} = \dfrac{4}{3}$

$\Rightarrow \dfrac{3}{4}\left(\dfrac{1}{1-r}\right) = \dfrac{4}{3}$

$\Rightarrow r = 1 - \dfrac{9}{16} = \dfrac{7}{16}$

62. According to the given condition,

$4 = \dfrac{a}{1-r}$

$\Rightarrow 4 \Rightarrow a = 4 - 4r$

$\Rightarrow 4r = 4 - a$

Only option (D) satisfies this condition.

63. $3 + 3\alpha + 3\alpha^2 + 3\alpha^3 + \ldots\ldots \infty = \dfrac{45}{8}$

$\Rightarrow 3\left[\dfrac{1}{1-\alpha}\right] = \dfrac{45}{8} \Rightarrow 8 = 15(1-\alpha) \Rightarrow \alpha = \dfrac{7}{15}$

64. $y = x - x^2 + x^3 - x^4 + \ldots\ldots \infty$
Then $xy = x^2 - x^3 + x^4 - \ldots\ldots\infty$
Adding, $y + xy = x + 0 + 0 \ldots\ldots + 0$
$\Rightarrow x - xy = y$
$\Rightarrow x(1 - y) = y$
$\Rightarrow x = \dfrac{y}{1-y}$
Alternate method:
$y = \dfrac{x}{1-(-x)} \Rightarrow y = \dfrac{x}{1+x}$
$\Rightarrow y + yx = x \Rightarrow x = \dfrac{y}{1-y}$

65. Common ratio (r) $= \dfrac{2}{x}$
For sum to be finite $r < 1 \Rightarrow \dfrac{2}{x} < 1$
$\Rightarrow 2 < x$
$\Rightarrow x > 2$

66. We have $\dfrac{a}{1-r} = x$
and $\dfrac{a^2}{1-r^2} = \dfrac{a}{1-r} \cdot \dfrac{a}{1+r} = y$
$\Rightarrow y = x \cdot \dfrac{a}{1+r} = x\dfrac{x(1-r)}{1+r}$
$\Rightarrow \dfrac{y}{x^2} = \dfrac{1-r}{1+r} \Rightarrow \dfrac{x^2}{y} = \dfrac{1+r}{1-r}$
$\Rightarrow \dfrac{x^2}{y}(1 - r) = 1 + r$
$\Rightarrow r\left[1+\dfrac{x^2}{y}\right] = -1 + \dfrac{x^2}{y}$
$\Rightarrow r = \dfrac{x^2 - y}{x^2 + y}$

67. Let r be the common ratio of the G.P. Then
$S = \dfrac{a}{1-r} \Rightarrow r = 1 - \dfrac{a}{S}$
Now S_n = Sum of n terms
$= a\left(\dfrac{1-r^n}{1-r}\right)$
$= \dfrac{a}{1-r}(1 - r^n)$
$= S\left[1-\left(1-\dfrac{a}{S}\right)^n\right]$

68. Since the series are in G.P., therefore
$x = \dfrac{1}{1-a}$ and $y = \dfrac{1}{1-b}$
$\therefore$ $a = \dfrac{x-1}{x}$, $b = \dfrac{y-1}{y}$
$\therefore$ $1 + ab + a^2b^2 + \ldots\ldots \infty$
$= \dfrac{1}{1-ab} = \dfrac{1}{1-\dfrac{x-1}{x}\cdot\dfrac{y-1}{y}} = \dfrac{xy}{x+y-1}$

69. $1-\cos\alpha = \dfrac{1}{2-\sqrt{2}} = 1+\dfrac{1}{\sqrt{2}}$ $\ldots\left[\because \dfrac{a}{1-r} = 2-\sqrt{2}\right]$
$\Rightarrow \cos\alpha = -\dfrac{1}{\sqrt{2}} = \cos\dfrac{3\pi}{4}$
$\Rightarrow \alpha = \dfrac{3\pi}{4}$

70. $1 + \sin x + \sin^2 x + \ldots.$ upto $\infty = 4 + 2\sqrt{3}$
$\Rightarrow \dfrac{1}{1-\sin x} = 4+2\sqrt{3}$
$\Rightarrow 1-\sin x = \dfrac{1}{2(2+\sqrt{3})}$
$\Rightarrow \sin x = \dfrac{4+2\sqrt{3}-1}{2(2+\sqrt{3})}$
$\Rightarrow \sin x = \dfrac{\sqrt{3}}{2} \Rightarrow x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

71. $0.2\dot{3}\dot{4} = \dfrac{234-2}{990} = \dfrac{232}{990}$

72. $0.4\dot{2}\dot{3} = \dfrac{423-4}{990} = \dfrac{419}{990}$

73. $0.14189189189\ldots.$
$= 0.14 + 0.00189 + 0.00000189 + \ldots\ldots$
$= \dfrac{14}{100} + 189\left[\dfrac{1}{10^5} + \dfrac{1}{10^8} + \ldots.\infty\right]$
$= \dfrac{7}{50} + 189\left[\dfrac{\frac{1}{10^5}}{1-\left(\frac{1}{10^3}\right)}\right]$
$= \dfrac{7}{50} + 189\left[\dfrac{1}{10^5} \times \dfrac{10^3}{999}\right]$
$= \dfrac{7}{50} + \dfrac{189}{999\times100} = \dfrac{7}{50} + \dfrac{7}{3700}$
$= \dfrac{7}{50} + \dfrac{7}{25\times148} = \dfrac{21}{148}$

Alternate Method:

$0.14\dot{1}\dot{8}\dot{9}$

$= \frac{14189-14}{99900} = \frac{14175}{99900} = \frac{21}{148}$

74. Let α and β be the roots of equation
$x^2 - 18x + 9 = 0$
$\therefore$ G.M. of α and $\beta = \sqrt{\alpha\beta} = \sqrt{9} = 3$ $[\because \alpha\beta = 9]$

75. Let G_1, G_2, G_3, G_4, G_5 be the G.M.'s are inserted between 486 and $\frac{2}{3}$. So total terms are 7.
$t_n = ar^{n-1}$
$\Rightarrow \frac{2}{3} = 486(r)^6 \Rightarrow r = \frac{1}{3}$
Hence, 4^{th} G.M. will be, $t_5 = ar^4$
$= 486\left(\frac{1}{3}\right)^4$
$= 6$

76. Let a – d, a, a + d be three numbers in A.P.
$\therefore$ $a + d + a + a - d = 15$
$\Rightarrow a = 5$
$a - d + 1, a + 4, a + d + 19$ are in G.P.
$\Rightarrow 6 - d, 9, 24 + d$ are in G.P.
$\therefore$ $81 = (6 - d)(24 + d)$
$\Rightarrow 81 = 144 + 6d - 24d - d^2$
$\Rightarrow d^2 + 18d - 63 = 0$
$\therefore$ $d = 3, -21$
$\therefore$ the numbers are 2, 5, 8 and 26, 5, –16

77. x, y, z are in G.P., then $y^2 = x.z$
Now $a^x = b^y = c^z = m$
$\Rightarrow x \log_e a = y \log_e b = z \log_e c = \log_e m$
$\Rightarrow x = \log_a m, y = \log_b m, z = \log_c m$
Again as x, y, z are in G.P., so $\frac{y}{x} = \frac{z}{y}$
$\Rightarrow \frac{\log_b m}{\log_a m} = \frac{\log_c m}{\log_b m}$
$\Rightarrow \log_b a = \log_c b$

78. Let $a^{1/x} = b^{1/y} = c^{1/z}$
$\Rightarrow a = k^x, b = k^y, c = k^z$
Now, a, b, c are in G.P.
$\Rightarrow b^2 = ac$
$\Rightarrow k^{2y} = k^x.k^z = k^{x+z}$
$\Rightarrow 2y = x + z$
$\Rightarrow x, y$, z are in A.P.

79. Since, a, b, c are in G.P.
$\therefore$ $b^2 = ac$
$\Rightarrow \log_e b^2 = \log_e ac$
$\Rightarrow \log_e a - 2\log_e b + \log_e c = 0$
Given, $(\log_e a)x^2 - (2\log_e b)x + \log_e c = 0$
Since, 1 satisfies this equation.
Therefore, 1 is one root and other root say β.
$\therefore$ $1.\beta = \frac{\log_e c}{\log_e a}$
$\therefore$ $\beta = \log_a c$

80. $\frac{x^{n+1} + y^{n+1}}{x^n + y^n} = \sqrt{xy} \Rightarrow x^{n+1} + y^{n+1} = \sqrt{xy}\,(x^n + y^n)$
$\Rightarrow x^{n+\frac{1}{2}}\left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right) = y^{n+\frac{1}{2}}\left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right)$
$\Rightarrow \left(\frac{x}{y}\right)^{n+\frac{1}{2}} = 1 \Rightarrow n = -\frac{1}{2}$

81. a + d, a + 4d, a + 8d, are in G.P.
$\Rightarrow (a + 4d)^2 = (a + d)(a + 8d)$
$\Rightarrow 8d^2 = ad \Rightarrow \frac{a}{d} = 8$
$\therefore$ common ratio $= \frac{a+4d}{a+d}$
$= \frac{8+4}{8+1} = \frac{4}{3}$

82. Series, 2, $2\frac{1}{2}$, $3\frac{1}{3}$, …… are in H.P.
$\Rightarrow \frac{1}{2}, \frac{2}{5}, \frac{3}{10}$,….. will be in A.P.
Now, first term $a = \frac{1}{2}$ and
common difference $d = -\frac{1}{10}$
So, 5^{th} term of the A.P.
$= \frac{1}{2} + (5 - 1)\left(-\frac{1}{10}\right) = \frac{1}{10}$
Hence, 5^{th} term of the H.P. is 10.

83. Here, 5^{th} term of the corresponding A.P. $= a + 4d = 45$ …..(i)
and 11^{th} term of the corresponding A.P. $= a + 10d = 69$ …..(ii)
From (i) and (ii), we get $a = 29$, $d = 4$
Therefore, 16^{th} term of the corresponding A.P.
$= a + 15d = 29 + 15 \times 4 = 89$
Hence, 16^{th} term of the H.P. is $\frac{1}{89}$.

84. Since $a_1, a_2, a_3, \ldots\ldots a_n$ are in H.P

Therefore $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots. \frac{1}{a_n}$ will be in A.P.

Which gives, $\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \ldots\ldots$

$$= \frac{1}{a_n} - \frac{1}{a_{n-1}} = d$$

$$\Rightarrow \frac{a_1 - a_2}{a_1 a_2} = \frac{a_2 - a_3}{a_2 a_3} = \ldots\ldots = \frac{a_{n-1} - a_n}{a_{n-1} a_n} = d$$

$\Rightarrow a_1 - a_2 = da_1a_2,\ a_2 - a_3 = da_2a_3$

and $a_{n-1} - a_n = da_{n-1}a_n$

Adding these, we get

$d(a_1a_2 + a_2a_3 + \ldots\ldots + a_{n-1}a_n)$

$= (a_1 + a_2 + \ldots.. + a_{n-1}) - (a_2 + a_3 + \ldots.. + a_n)$

$= a_1 - a_n$ .....(i)

Also n^{th} term of this A.P. is given by

$$\frac{1}{a_n} = \frac{1}{a_1} + (n-1)d \Rightarrow d = \frac{a_1 - a_n}{a_1 a_n (n-1)}$$

Substituting this value of d in (i)

$$(a_1 - a_n) = \frac{a_1 - a_n}{a_1 a_n (n-1)}(a_1a_2 + a_2a_3 + \ldots.. + a_{n-1}a_n)$$

$(a_1a_2 + a_2a_3 + \ldots\ldots + a_{n-1}a_n) = a_1a_n(n-1)$

85. We know that, $x_n = \frac{(n+1)ab}{na+b}$

$\therefore$ Sixth H.M. i.e. $x_6 = \frac{7.3.\left(\frac{6}{13}\right)}{\left(6.3 + \frac{6}{13}\right)}$

$$= \frac{126}{240}$$

$$= \frac{63}{120}$$

86. Let roots be α, β then

$\alpha + \beta = -\frac{b}{a} = 10$

$\alpha\beta = \frac{c}{a} = 11$

H.M. $= \frac{2\alpha\beta}{\alpha+\beta} = \frac{11 \times 2}{10} = \frac{11}{5}$

87. a,b,c are in H.P. $\Rightarrow b = \frac{2ac}{a+c}$

By inspection, we get (A) False (B) False (C) False

88. Since, a, b, c are in H.P.

$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow b = \frac{2ac}{a+c}$

Consider option (B),

$$\frac{1}{ca} = \frac{2\left(\frac{1}{bc}.\frac{1}{ab}\right)}{\frac{1}{bc} + \frac{1}{ab}} = \frac{\left(\frac{2}{ab^2c}\right)}{\frac{a+c}{abc}}$$

$$= \frac{2(abc)}{ab^2c(a+c)} = \frac{2}{b(a+c)}$$

$$\Rightarrow ca = \frac{b(a+c)}{2}$$

$$\Rightarrow b = \frac{2ac}{a+c}$$

$\therefore \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in H.P.

89. As given $H = \frac{2pq}{p+q}$

$$\therefore \frac{H}{p} + \frac{H}{q} = \frac{2q}{p+q} + \frac{2p}{p+q} = \frac{2(p+q)}{p+q} = 2$$

90. Let, the distance of school from home = d and time taken are t_1 and t_2 .

$\therefore t_1 = \frac{d}{x}$ and $t_2 = \frac{d}{y}$

Avg. velocity $= \frac{\text{Total distance}}{\text{Total time}}$

$= \frac{2d}{\left(\frac{d}{x} + \frac{d}{y}\right)} = \frac{2xy}{x+y}$, which is the H.M. of x and y.

91. If x, y, z are in H.P., then $y = \frac{2xz}{x+z}$

$\therefore \log_e(x + z) + \log_e(x - 2y + z)$

$= \log_e\{(x + z)(x - 2y + z)\}$

$= \log_e\left[(x+z)\left(x + z - \frac{4xz}{x+z}\right)\right]$

$= \log_e[(x + z)^2 - 4xz]$

$= \log_e(x - z)^2$

$= 2\log_e(x - z)$

92. We have $\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{2ab}{a+b}$

$\Rightarrow a^{n+2} + ab^{n+1} + ba^{n+1} + b^{n+2} = 2a^{n+1}b + 2b^{n+1}a$

$\Rightarrow a^{n+1}(a - b) = b^{n+1}(a - b)$

or $\left(\frac{a}{b}\right)^{n+1} = (1) = \left(\frac{a}{b}\right)^0$

Hence, n = −1

93. $f(x) = x + \frac{1}{2} = \frac{2x+1}{2}$

$f(2x) = \frac{4x+1}{2}$

$f(4x) = \frac{8x+1}{2}$

$f(x)$, $f(2x)$, $f(4x)$ are in H.P.

$\therefore$ $f(2x) = \frac{2f(x)f(4x)}{f(x)+f(4x)}$

$\Rightarrow x = 0, \frac{1}{4}$

At $x = 0$, terms are equal, so only solution is $x = \frac{1}{4}$

94. Given $x_1.x_2.x_3 \ldots\ldots x_n = 1$

$\because$ A.M. $\geq$ G.M.

$\therefore$ $\left(\frac{x_1 + x_2 + x_3 + \ldots\ldots + x_n}{n}\right) \geq (x_1.x_2.x_3\ldots\ldots x_n)^{\frac{1}{n}}$

$= (1)^{\frac{1}{n}}$

$= 1$

$\therefore$ $x_1 + x_2 + x_3 + \ldots.. + x_n \geq n$

$\therefore$ $x_1 + x_2 + x_3 + \ldots.. + x_n$ can never be less than n.

95. A.M. $\geq$ G.M.

$\Rightarrow \frac{a_1 + a_2 + \ldots + a_{n-1} + 2a_n}{n}$

$\geq (a_1.a_2,\ldots a_{n-1}2a_n)^{\frac{1}{n}} \geq (2c)^{\frac{1}{n}}$

$\therefore$ Minimum value of

$a_1 + a_2 + \ldots.+ a_{n-1} + 2a_n = n(2c)^{\frac{1}{n}}$

96. Since, p, q, r are in G.P.

$\therefore$ $q^2 = pr$

Also, $\tan^{-1} p$, $\tan^{-1} q$, $\tan^{-1} r$ are in A.P.

$\Rightarrow \tan^{-1} p + \tan^{-1} r = 2\tan^{-1} q$

$\Rightarrow p + r = 2q$

$\Rightarrow$ p, q, r are in A.P.

Here, p, q, r are both in A.P. and G.P., which is possible only, if p = q = r.

97. $\frac{x+y}{2} = 3$

$\Rightarrow x + y = 6$

$xy = 1^2 = 1$

$x^2 + y^2 = (x + y)^2 - 2xy$

$= 36 - 2$

$= 34$

98. Given that A.M. = 8 and G.M. = 5, if α, β are roots of quadratic equation, then quadratic equation is

$x^2 - x(\alpha + \beta) + \alpha\beta = 0$

A.M. $= \frac{\alpha+\beta}{2} = 8 \Rightarrow \alpha + \beta = 16$

and G.M. $= \sqrt{\alpha\beta} = 5 \Rightarrow \alpha\beta = 25$

So the required quadratic equation will be

$x^2 - 16x + 25 = 0$.

99. $\frac{x+y}{2} = 9$

$\Rightarrow x + y = 18$

$xy = 4^2 = 16$

x and y are the roots.

$\therefore$ The equation is

$x^2 - 18x + 16 = 0$

100. $GM = \sqrt{ab} = 2$

$\Rightarrow ab = 4$

$HM = \frac{2ab}{a+b} = \frac{-8}{5}$

$\Rightarrow \frac{8}{a+b} = \frac{-8}{5}$

$\Rightarrow a + b = -5$

$\Rightarrow 2a + 2b = -10$

$ab = 4 \Rightarrow (2a)(2b) = 16$

$\therefore$ The required quadratic equation is

$x^2 - (2a + 2b)x + (2a)(2b) = 0$

$\Rightarrow x^2 + 10x + 16 = 0$

101. Sum of the roots of $x^2 - 2ax + b^2 = 0$ is 2a,

Therefore, A = A.M. of the roots = a.

Product of the roots of $x^2 - 2bx + a^2 = 0$ is a^2

Therefore, G.M. of the roots is G = a

Thus, A = G

102. We have, $\tan n\theta = \tan m\theta$

$\Rightarrow n\theta = N\pi + (m\theta)$

$\Rightarrow \theta = \frac{N\pi}{n-m}$, putting N = 1,2,3……, we get

$\frac{\pi}{n-m}, \frac{2\pi}{n-m}, \frac{3\pi}{n-m}$……. which are in A.P.

Since, common difference, $d = \frac{\pi}{n-m}$.

103. Let three numbers a, b and c in G.P., then $b^2 = ac$

$\Rightarrow 2\log_e b = \log_e a + \log_e c$ or

$\log_e b = \frac{\log_e a + \log_e c}{2}$

Thus, their logarithms are in A.P.

104. $225 = 3^2 \times 5^2 = d\,(225) = 3 \times 3 = 9$
$1125 = 3^2 \times 5^3 = d\,(1125) = 3 \times 4 = 12$
$640 = 2^7 \times 5 = d\,(640) = 8 \times 2 = 16$
9, 12, 16 are in G.P

105. If $\frac{x+y}{2}, y, \frac{y+z}{2}$ are in H.P., then

$$y = \frac{2\left(\frac{x+y}{2}.\frac{y+z}{2}\right)}{\frac{x+y}{2}+\frac{y+z}{2}}$$

$$= \frac{\frac{2}{4}(x+y)(y+z)}{\frac{1}{2}(x+2y+z)}$$

$$y = \frac{xy+xz+y^2+yz}{x+2y+z}$$

$\Rightarrow xy + 2y^2 + yz = xy + xz + y^2 + yz$
$\Rightarrow y^2 = xz$
Thus, x, y, z will be in G.P.

106. $(y - x)$, $2(y - a)$, $(y - z)$ are in H.P.

$\Rightarrow \frac{1}{y-x}, \frac{1}{2(y-a)}, \frac{1}{y-z}$ are in A.P.

$$\Rightarrow \frac{1}{2(y-a)} - \frac{1}{(y-x)} = \frac{1}{(y-z)} - \frac{1}{2(y-a)}$$

$$\Rightarrow \frac{y-x-2y+2a}{y-x} = \frac{2y-2a-y+z}{y-z}$$

$$\Rightarrow \frac{-x-y+2a}{(y-x)} = \frac{y+z-2a}{(y-z)}$$

$$\Rightarrow \frac{(x-a)+(y-a)}{(x-a)-(y-a)} = \frac{(y-a)+(z-a)}{(y-a)-(z-a)}$$

$$\Rightarrow \frac{(x-a)}{(y-a)} = \frac{(y-a)}{(z-a)}$$

$\Rightarrow (x - a), (y - a), (z - a)$ are in G. P.

107. x, l, z are in A.P., then $2 = x + z$ ......(i)
and $4 = xz$ ......(ii)
Divide (ii) by (i), we get

$$\frac{x\,.\,z}{x+z} = \frac{4}{2} \text{ or } \frac{2xz}{x+z} = 4$$

Hence, x, 4, z will be in H.P.

108. Given, a, b, c are in G.P.
$\Rightarrow \log_x a, \log_x b \ \log_x c$ are in A.P.

$\Rightarrow \frac{\log a}{\log x}, \frac{\log b}{\log x}, \frac{\log c}{\log x}$ are in A.P.

$\Rightarrow \frac{\log x}{\log a}, \frac{\log x}{\log b}, \frac{\log x}{\log c}$ are in H.P.

i.e., $\log_a x, \log_b x, \log_c x$ are in H.P.

109. x, y, z are in G.P.
Hence, $y^2 = xz$
$\therefore$ $2 \log y = \log x + \log z$
$\Rightarrow 2 (\log y + 1) = (1 + \log x) + (1 + \log z)$
$\Rightarrow 1 + \log x, 1 + \log y, 1 + \log z$ are in A.P.

$\Rightarrow \frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z}$ are is H.P.

110. Since, b^2, a^2, c^2 are in A.P.
$\therefore$ $a^2 - b^2 = c^2 - a^2$
$\Rightarrow (a - b)(a + b) = (c - a)(c + a)$

$$\Rightarrow \frac{1}{b+c} - \frac{1}{a+b} = \frac{1}{c+a} - \frac{1}{b+c}$$

$\Rightarrow \frac{1}{a+b}, \frac{1}{b+c}, \frac{1}{c+a}$ are in A.P.

$\therefore$ $(a + b), (b + c), (c + a)$ are in H.P.

111. Given, a, b, c are in A.P.
$\Rightarrow 2b = a + c \Rightarrow b - c = a - b$
Also, a^2, b^2, c^2 are in H.P.

$$\frac{1}{b^2} - \frac{1}{a^2} = \frac{1}{c^2} - \frac{1}{b^2}$$

$$\Rightarrow \frac{a^2-b^2}{a^2b^2} = \frac{b^2-c^2}{b^2c^2}$$

$\Rightarrow (a - b)\,[c^2(a + b) - a^2(b + c)] = 0$
....[$\because$ $(b - c) = (a - b)$]
$\Rightarrow a = b$ or $c^2a + c^2b - a^2b - a^2c = 0$
$\Rightarrow c^2a + c^2b - a^2b - a^2c = 0$
$\Rightarrow ac(c - a) = b(a^2 - c^2)$
$\Rightarrow ac = -b(c + a)$
$\Rightarrow -\,ac = b.2b$

$$\Rightarrow b^2 = -\left(\frac{a}{2}\right)c$$

$\therefore$ $-\frac{a}{2}$, b, c are in G.P.

112. $x + y + z = 15$, if 9, x, y, z, a are in A.P.

Sum $= 9 + 15 + a = \frac{5}{2}(9+a)$

$\Rightarrow 24 + a = \frac{5}{2}(9+a)$

$\Rightarrow 48 + 2a = 45 + 5a$
$\Rightarrow 3a = 3$
$\Rightarrow a = 1$

and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}$, if 9, x, y, z, a are in H.P.

Sum $= \frac{1}{9} + \frac{5}{3} + \frac{1}{a} = \frac{5}{2}\left[\frac{1}{9} + \frac{1}{a}\right] \Rightarrow a = 1$

113. Given, a, b, c are in G.P.

$\therefore$ $b^2 = ac$

$$x = \frac{a+b}{2},\ y = \frac{b+c}{2}$$

$$\therefore \quad \frac{a}{x}+\frac{c}{y} = \frac{2a}{a+b}+\frac{2c}{b+c}$$

$$= \frac{2(ab+bc+2ca)}{ab+ac+b^2+bc}$$

$$= \frac{2(ab+bc+2ca)}{(ab+ac+ac+bc)}$$

$$= 2 \quad \text{....}[\because b^2 = ac]$$

114. Given that a, A_1, A_2, b are in A.P.

Therefore, $A_1 = \frac{a+A_2}{2},\ A_2 = \frac{A_1+b}{2}$

$\Rightarrow A_1 + A_2 = \frac{1}{2}(a + b + A_1 + A_2)$

$\Rightarrow \frac{1}{2}(A_1 + A_2) = \frac{1}{2}(a + b)$ or

$A_1 + A_2 = a + b$

and a, G_1, G_2, b are in G.P.

Therefore, $G_1^2 = aG_2, G_2^2 = bG_1$

$\Rightarrow G_1^2G_2^2 = abG_1G_2 \Rightarrow G_1G_2 = ab$

Hence, $\frac{A_1+A_2}{G_1G_2} = \frac{a+b}{ab}$

Trick : Let a = 1, b = 2,

then $A_1 + A_2 = 1 + 2 = 3$

and $G_1 . G_2 = 2 \times 1 = 2$

$\therefore$ $\frac{A_1+A_2}{G_1G_2} = \frac{3}{2}$, which is given by (A).

115. Given numbers a and 2.

A.M. = $\frac{a+2}{2}$ and G.M. = $\sqrt{2a}$

According to the given condition,

A.M. − G.M. = 1

$\Rightarrow \frac{a+2}{2} - \sqrt{2a} = 1$

$\Rightarrow \frac{a}{2}+1-1 = \sqrt{2a}$

$\Rightarrow a = 2\sqrt{2a} \Rightarrow a^2 = 8a$

$\Rightarrow a(a - 8) = 0$

$\Rightarrow a = 0$ or 8

Since, $a \neq 0$

$\therefore$ a = 8

116. Let the two numbers be x, y.

$\therefore$ $x - y = 48$ ....(i)

and $\frac{x+y}{2} - \sqrt{xy} = 18$

$\Rightarrow x + y - 2\sqrt{xy} = 36$

$\Rightarrow 48 + y + y - 2\sqrt{(48+y)y} = 36$ ....[From (i)]

$\Rightarrow 12 + 2y = 2\sqrt{y(48+y)}$

$\Rightarrow 6 + y = \sqrt{y(48+y)}$

$\Rightarrow 36 + y^2 + 12y = 48y + y^2$

$\Rightarrow 36y = 36 \Rightarrow y = 1$

$\therefore$ $x = 48 + 1 = 49$

117. Since, H_1, H_2 are two harmonic means between a and b.

$\therefore$ $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P.

We know that 2A = a + b and $G^2 = ab$

$\therefore$ $2 \times \frac{1}{H_1} = \frac{1}{a}+\frac{1}{H_2}$

Similarly, $2 \times \frac{1}{H_2} = \frac{1}{b}+\frac{1}{H_1}$

On adding and solving we get,

$$2\left(\frac{1}{H_1}+\frac{1}{H_2}\right) - \left(\frac{1}{H_1}+\frac{1}{H_2}\right) = \frac{1}{a}+\frac{1}{b}$$

$$\frac{1}{H_1}+\frac{1}{H_2} = \frac{a+b}{ab} = \frac{2A}{G^2}$$

118. Let a and b be two numbers.

Sum of n A.M.'s = n × single A.M.

$\Rightarrow A_1 + A_2 = 2 \times \left(\frac{a+b}{2}\right) = a + b$

Product of n G.M.'s = (Single G.M.)^n

$\Rightarrow G_1 . G_2 = \left(\sqrt{ab}\right)^2 = ab$

$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P.

$\Rightarrow \frac{1}{H_1}+\frac{1}{H_2} = \frac{1}{a}+\frac{1}{b} = \frac{a+b}{ab}$

$\Rightarrow \frac{H_1H_2}{H_1+H_2} = \frac{G_1G_2}{A_1+A_2}$

$\Rightarrow \frac{G_1G_2}{H_1H_2} \times \frac{H_1+H_2}{A_1+A_2} = 1$

119. Given, $\sqrt{ab} = 10$

$\Rightarrow ab = 100$ and $\frac{2ab}{a+b} = 8$

$\Rightarrow a + b = 25$

$\therefore$ $a = 5, b = 20$

120. Let the positive numbers be a_1 and a_2.

$a_1, A, a_2, \ldots\ldots$ are in A.P. then $A = \frac{a_1 + a_2}{2}$

Also, $a_1, G, a_2, \ldots\ldots$ are in G.P.

$\therefore$ $G = \sqrt{a_1 a_2}$

$\frac{1}{a_1}, \frac{1}{H}, \frac{1}{a_2}, \ldots\ldots$ are in H.P.

$\therefore$ $\frac{2}{H} = \frac{1}{a_1} + \frac{1}{a_2} \Rightarrow H = \frac{2a_1a_2}{a_1 + a_2} \Rightarrow H = \frac{G^2}{A}$

121. Given A.M. = 2(G.M.) or $\frac{1}{2}(a+b) = 2\sqrt{ab}$

or $\frac{a+b}{2\sqrt{ab}} = \frac{2}{1}$

$\Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{2+1}{2-1} = \frac{3}{1}$

$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{3}{1}$

$\Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1} \Rightarrow \frac{a}{b} = \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2$

$\Rightarrow \frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$ or $a : b = (2+\sqrt{3}) : (2-\sqrt{3})$.

122. We have H.M. $= \frac{2ab}{a+b}$ and G.M. $= \sqrt{ab}$

So $\frac{\text{H.M.}}{\text{G.M.}} = \frac{4}{5} \Rightarrow \frac{2ab/(a+b)}{\sqrt{ab}} = \frac{4}{5}$

$\Rightarrow \frac{2\sqrt{ab}}{(a+b)} = \frac{4}{5} \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{5}{4}$

$\Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{5+4}{5-4}$

$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{9}{1} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{3}{1}$

$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})} = \frac{3+1}{3-1}$

$\Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{4}{2} \Rightarrow \left(\frac{a}{b}\right) = 2^2 = 4$

$\Rightarrow a : b = 4 : 1$ or $b : a = 1 : 4$

123. According to the given condition,

$$\frac{\frac{x+y}{2}}{\sqrt{xy}} = \frac{p}{q}$$

$$\Rightarrow \frac{x+y}{2\left(\sqrt{xy}\right)} = \frac{p}{q} \qquad \text{....(i)}$$

$$\Rightarrow \frac{x^2+y^2+2xy}{4xy} = \frac{p^2}{q^2}$$

$$\Rightarrow \frac{x^2+y^2+2xy-4xy}{4xy} = \frac{p^2-q^2}{q^2}$$

$$\Rightarrow \frac{(x-y)^2}{4xy} = \frac{p^2-q^2}{q^2}$$

$$\Rightarrow \frac{x-y}{2\sqrt{xy}} = \frac{\sqrt{p^2-q^2}}{q} \qquad \text{....(ii)}$$

Dividing (ii) by (i), we get

$$\frac{x+y}{x-y} = \frac{p}{\sqrt{p^2-q^2}} \Rightarrow \frac{x}{y} = \frac{p+\sqrt{p^2-q^2}}{p-\sqrt{p^2-q^2}}$$

124. $\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \ldots + \frac{1}{t_{2003}}$

$= 4\left[\frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \ldots + \frac{1}{(2005).(2006)}\right]$

$= 4\left[\frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \ldots + \frac{1}{2005} - \frac{1}{2006}\right]$

$= 4\left[\frac{1}{3} - \frac{1}{2006}\right]$

$= 4.\frac{2003}{3(2006)}$

$= \frac{4006}{3009}$

125. It is an arithmetico-geometric series

$\therefore$ $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} = \frac{1}{1-\frac{1}{2}} + \frac{2 \times \frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}$

$= 2 + 4$

$= 6$

126. $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

Here, $a = 1$, $r = \frac{1}{5}$, $d = 3$

$\therefore$ $S_\infty = \frac{1}{1-\frac{1}{5}} + \frac{3 \times \frac{1}{5}}{\left(1-\frac{1}{5}\right)^2} = \frac{35}{16}$

127. Let $S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \text{.... to } \infty$

$\Rightarrow (S-1) = \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \text{.... to } \infty$ ....(i)

$\Rightarrow (S-1)\,\frac{1}{3} = \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + \text{.... to } \infty$ ....(ii)

Subtracting (ii) from (i), we get

$\frac{2}{3}(S-1) = \frac{2}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \text{.... to } \infty$

$\Rightarrow \frac{2}{3}(S-1) = \frac{2}{3} + \frac{\frac{4}{3^2}}{1-\frac{1}{3}}$

$\Rightarrow \frac{2}{3}(S-1) = \frac{2}{3} + \frac{2}{3} \Rightarrow S = 3$

128. $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{....}\infty = \frac{\pi^4}{90}$

$\therefore \quad \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \text{....}\infty + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{....}\infty\right) = \frac{\pi^4}{90}$

$\therefore \quad \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{.... }\infty + \frac{1}{16} \times \frac{\pi^4}{90} = \frac{\pi^4}{90}$

$\therefore \quad \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{....} + \infty$

$= \frac{\pi^4}{90} - \frac{1}{16}\left(\frac{\pi^4}{90}\right)$

$= \frac{15}{16}\left(\frac{\pi^4}{90}\right) = \frac{\pi^4}{96}$

129. The sequence can be written as log a, (2 log a – log b), (3 log a – 2 log b), …. which are in A.P. having common difference as log a – log b.

130. $\sum_{k=1}^{n} k(k+2) = \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k$

$= \frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$

$= \frac{n(n+1)}{6}\left[2n+1+6\right]$

$= \frac{n(n+1)(2n+7)}{6}$

131. Given series $\frac{1}{1} + \frac{1+2}{2} + \frac{1+2+3}{3} + \ldots\ldots$

So, n^{th} term of series is given by

$t_n = \frac{1+2+3+\text{.......}+n}{n}$

$= \frac{\frac{1}{2}n(n+1)}{n} = \frac{n+1}{2}$

132. Here $t_n = \frac{n(n+1)}{2}$

$\therefore \quad S_n = \frac{1}{2}\left(\Sigma n^2 + \Sigma n\right) = \frac{n(n+1)(n+2)}{6}$

133. $t_n = \frac{(2n+1)}{\frac{n(n+1)(2n+1)}{6}}$

$= \frac{6}{n(n+1)}$

$S_n = \sum(t_n)$

$= \sum 6\left[\frac{1}{n} - \frac{1}{n+1}\right]$

$= 6\left[1 - \frac{1}{n+1}\right]$

$S_n = \frac{6n}{n+1}$

134. General term $t_n = \frac{1^2 + 2^2 + \text{....} + n^2}{1 + 2 + \text{....} + n}$

$\Rightarrow t_n = \frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)}{2}} = \frac{1}{3}.(2n+1)$

$\therefore \quad \sum t_n = \frac{2}{3}\sum n + \frac{1}{3}n$

$= \frac{2}{3}.\frac{n(n+1)}{2} + \frac{1}{3}n$

$= \frac{1}{3}n.(n+1) + \frac{1}{3}n$

$= \frac{n(n+2)}{3}$

135. Mean, $\bar{x} = \frac{1.(1) + 2.(2) + 3.(3) + \ldots + n.(n)}{1 + 2 + 3 + \ldots + n}$

$= \frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)}{2}}$

$= \frac{2n+1}{3}$

136. $S = 3.6 + 4.7 + \ldots$ upto $n - 2$ terms

$= (1.4 + 2.5 + 3.6 + 4.7 + \ldots$ upto n terms$) - 14$

$= \Sigma n(n + 3) - 14$

$= \frac{1}{6}(2n^3 + 12n^2 + 10n) - 14$

$= \left(\frac{2n^3 + 12n^2 + 10n - 84}{6}\right)$,

where n = 3, 4, 5 …..

Trick : $S_1 = 18$, $S_2 = 46$

Now put in options $(n - 2) = 1,2$ i.e. $n = 3,4$

Option (B) gives the values.

137. $\left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + \ldots$

$= \frac{1}{5^2}[8^2 + 12^2 + 16^2 + \ldots]$

$= \frac{1}{5^2}[(4 \times 2)^2 + (4 \times 3)^2 + (4 \times 4)^2 + \ldots]$

$= \frac{4^2}{5^2}[2^2 + 3^2 + 4^2 + \ldots + 11^2]$

$= \frac{4^2}{5^2}[1^2 + 2^2 + \ldots 11^2 - 1^2]$

$= \frac{4^2}{5^2}\left[\frac{11(12)(23)}{6} - 1\right] = \frac{16}{5}m$

$\therefore$ $m = \frac{1}{5}\left[\frac{3036 - 6}{6}\right] = \frac{3030}{30}$

$\therefore$ $m = 101$

138. $\sum_{k=0}^{12} a_{4k+1} = 416$

$\Rightarrow a_1 + a_5 + a_9 + \ldots + a_{49} = 416$

$\Rightarrow \frac{13}{2}(2a_1 + 48d) = 416$

$\Rightarrow a_1 + 24d = 32$ …(i)

$a_9 + a_{43} = 66$

$\Rightarrow 2a_1 + 50d = 66$

$\Rightarrow a_1 + 25d = 33$ …(ii)

Solving (i) and (ii), we get

$a_1 = 8$ and $d = 1$

$a_1^2 + a_2^2 + \ldots + a_{17}^2 = 140m$

$\Rightarrow 8^2 + 9^2 + \ldots + 24^2 = 140m$

$\Rightarrow (1^2 + 2^2 + \ldots + 24^2)$

$- (1^2 + 2^2 + \ldots + 7^2) = 140m$

$\Rightarrow \frac{24 \times 25 \times 49}{6} - \frac{7 \times 8 \times 15}{6} = 140m$

$\Rightarrow 4900 - 140 = 140m$

$\Rightarrow 140m = 4760$

$\Rightarrow m = 34$

139. $S_n = cn^2$

$S_{n-1} = c(n - 1)^2 = cn^2 + c - 2cn$

$T_n = 2cn - c$

$T_n^2 = (2cn - c)^2 = 4c^2n^2 + c^2 - 4c^2n$

$\therefore$ Required sum $= \sum T_n^2$

$= \frac{4c^2.n(n+1)(2n+1)}{6} + nc^2 - 2c^2n(n + 1)$

$= \frac{2c^2n(n+1)(2n+1) + 3nc^2 - 6c^2n(n+1)}{3}$

$= \frac{nc^2(4n^2 + 6n + 2 + 3 - 6n - 6)}{3}$

$= \frac{nc^2(4n^2 - 1)}{3}$

140. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j$

$= \sum_{i=1}^{n}\left(\frac{i(i+1)}{2}\right)$

$= \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right]$

$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

$= \frac{1}{4}n(n+1)\left[\frac{2n+1}{3} + 1\right]$

$= \frac{n(n+1)}{4}\left[\frac{2n+1+3}{3}\right] = \frac{n(n+1)(n+2)}{6}$

141. We have $S = 2 + 4 + 7 + 11 + 16 + \ldots + t_n$

Again, $S = 2 + 4 + 7 + 11 + \ldots + t_{n-1} + t_n$

Subtracting, we get

$0 = 2 + \{2 + 3 + 4 + 5 + \ldots + (t_n - t_{n-1})\} - t_n$

$t_n = 2 + \frac{1}{2}(n - 1)\{(4 + (n - 2)1\}$

$= \frac{1}{2}(n^2 + n + 2)$

Now,

$S = \Sigma t_n = \frac{1}{2}\Sigma(n^2 + n + 2)$

$= \frac{1}{2}(\Sigma n^2 + \Sigma n + 2\Sigma 1)$

$= \frac{1}{2}\{\frac{1}{6}n(n + 1)(2n + 1) + \frac{1}{2}n(n + 1) + 2n\}$

$= \frac{n}{12}\{(n + 1)(2n + 1 + 3) + 12\}$

$= \frac{n}{6}\{(n + 1)(n + 2) + 6\} = \frac{n}{6}(n^2 + 3n + 8)$

142. Sum of cubes of ‘n’ natural number
$= \dfrac{n^2(n+1)^2}{4}$
$= \dfrac{15^2(16)^2}{4} = 14{,}400$

143. $t_n = n(n+1)(n+2) = n(n^2 + 3n + 2)$
$= n^3 + 3n^2 + 2n$

$\therefore$ $S_n = \Sigma(n^3) + \Sigma(3n^2) + \Sigma(2n)$

$S_n = \left[\dfrac{n(n+1)}{2}\right]^2 + \dfrac{3.n(n+1)(2n+1)}{6} + \dfrac{2.n(n+1)}{2}$

$S_n = \dfrac{1}{4}n(n+1)(n+2)(n+3)$

144. Here, t_n of the A.P. 1, 2, 3, ….. = n
and t_n of the A.P. 3, 5, 7, ….. = 2n + 1
$\therefore$ t_n of given series $= n(2n+1)^2 = 4n^3 + 4n^2 + n$
Hence,

$S = \sum_1^{20} t_n = 4\sum_1^{20} n^3 + 4\sum_1^{20} n^2 + \sum_1^{20} n$

$= 4.\dfrac{1}{4}20^2.21^2 + 4.\dfrac{1}{6}20.21.41 + \dfrac{1}{2}20.21$

$= 188090$

145. $1^3 + 3^3 + 5^3 + 7^3 + ... = \sum(2n-1)^3$
$= \sum(8n^3 - 3.4n^2 + 3.2n - 1)$
$= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$
$= 2n^4 + 4n^3 + 2n^2 - 2n(2n^2 + 3n + 1) + 3n^2 + 3n - n$
$= 2n^4 + 4n^3 + 2n^2 - 4n^3 - 6n^2 - 2n + 3n^2 + 3n - n$
$= 2n^4 - n^2 = n^2(2n^2 - 1)$

146. $(n^2 - 1^2) + 2(n^2 - 2^2) + \ldots\ldots$
$= n^2(1 + 2 + 3 + \ldots.) - (1.1^2 + 2.2^2 + 3.3^2 + \ldots.)$
$= n^2\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^3$
$= \dfrac{n^3(n+1)}{2} - \left[\dfrac{n(n+1)}{2}\right]^2$
$= \dfrac{n^2(n+1)}{2}\left(\dfrac{n-1}{2}\right) = \dfrac{1}{4}n^2(n^2-1)$

147. $S_n = 1(1!) + 2(2!) + 3(3!) + \ldots.. + n(n!)$
$= (2-1)(1!) + (3-1)(2!) + (4-1)(3!) + \ldots.. + [(n+1)-1](n!)$
$= (2.1! - 1!) + (3.2! - 2!) + (4.3! - 3!) + \ldots. + [(n+1)! - (n!)]$
$= (2! - 1!) + (3.2! - 2!) + (4.3! - 3!) + \ldots + [(n+1)(n!) - (n!)]$
$= (n+1)! - 1!$

148. $1^2 - 2^2 + 3^2 - 4^2 + ... + 11^2$
$= (1^2 - 2^2) + (3^2 - 4^2) + ... + (9^2 - 10^2) + 11^2$
Now, $a^2 - b^2 = (a-b)(a+b)$
$\therefore$ $1^2 - 2^2 + 3^2 - 4^2 + ... + 11^2$
$= (1-2)(1+2) + (3-4)(3+4) + ... + (9-10)(9+10) + 11^2$
$= (-1)[1 + 2 + 3 + ... + 9 + 10] + 11^2$
$= (-1).\dfrac{10\times11}{2} + 11^2 = 66$

149. G.M. of $1, 2, 2^2, 2^3, \ldots., 2^n$
Here, no. of terms = (n + 1)

$\therefore$ G.M. $= (1.2.2^2.2^3.....2^n)^{\frac{1}{(n+1)}}$

$= \left(2^{0+1+2+....+n}\right)^{\frac{1}{(n+1)}} = \left[2^{\frac{n(n+1)}{2}}\right]^{1/(n+1)}$

$\therefore$ G.M. $= 2^{\frac{n}{2}}$

150. $1 + 3 + 7 + \ldots.. + t_n$
$= 2 - 1 + 2^2 - 1 + 2^3 - 1 + \ldots\ldots. + 2^n - 1$
$= (2 + 2^2 + \ldots\ldots + 2^n) - n = 2^{n+1} - 2 - n$

151. Let n^{th} term of series is t_n, then
$S_n = 12 + 16 + 24 + 40 + \ldots. + t_n$
Again $S_n = 12 + 16 + 24 + \ldots\ldots + t_n$
On subtraction
$0 = (12 + 4 + 8 + 16 + \ldots. + \text{upto n terms}) - t_n$
$\Rightarrow t_n = 12 + [4 + 8 + 16 + \ldots..+ \text{upto (n– 1) terms}]$
$= 12 + \dfrac{4(2^{n-1}-1)}{2-1} = 2^{n+1} + 8$
On putting n = 1, 2, 3 …..
$t_1 = 2^2 + 8$, $t_2 = 2^3 + 8$, $t_3 = 2^4 + 8$ …. etc.
$S_n = t_1 + t_2 + t_3 + \ldots. + t_n$
$= (2^2 + 2^3 + 2^4 + \ldots..\text{ upto n terms}) + (8 + 8 + 8 + \ldots.\text{ upto n terms})$
$= \dfrac{2^2(2^n-1)}{2-1} + 8n = 4(2^n - 1) + 8n$

152. $(1 + 2) + (1 + 2 + 2^2) + \ldots$ upto n terms

$\therefore \quad T_n = 1 + 2 + 2^2 + \ldots + 2^n$

$\therefore \quad T_n = \dfrac{1(2^{n+1}-1)}{2-1} = 2^{n+1} - 1$

$\therefore \quad S_n = \sum T_n = \sum (2^{n+1}-1)$

$\therefore \quad S_n = \sum 2^{n+1} - \sum 1$

$= 2^2 + 2^3 + 2^4 + \ldots + 2^n - (n)$

$= 2^{n+2} - 4 - n$

153. $S = \dfrac{1}{2}\left(\dfrac{1}{2}+\dfrac{1}{3}\right) - \dfrac{1}{4}\left(\dfrac{1}{2^2}+\dfrac{1}{3^2}\right) + \dfrac{1}{6}\left(\dfrac{1}{2^3}+\dfrac{1}{3^3}\right) - \ldots$

$= \dfrac{1}{2}\left(\dfrac{1}{2} - \dfrac{1}{2}\cdot\dfrac{1}{2^2} + \dfrac{1}{3}\cdot\dfrac{1}{2^3} - \ldots\right)$

$+ \dfrac{1}{2}\left(\dfrac{1}{3} - \dfrac{1}{2}\cdot\dfrac{1}{3^2} + \dfrac{1}{3}\cdot\dfrac{1}{3^3} - \ldots\right)$

$= \dfrac{1}{2}\left[\dfrac{1}{2} - \dfrac{1}{2}\left(\dfrac{1}{2}\right)^2 + \dfrac{1}{3}\left(\dfrac{1}{2}\right)^3 - \ldots\right]$

$+ \dfrac{1}{2}\left[\dfrac{1}{3} + \dfrac{1}{2}\left(\dfrac{1}{3}\right)^2 + \dfrac{1}{3}\left(\dfrac{1}{3}\right)^3 - \ldots\right]$

$= \dfrac{1}{2}\log\left(1+\dfrac{1}{2}\right) + \dfrac{1}{2}\log\left(1+\dfrac{1}{3}\right)$

$= \dfrac{1}{2}\left[\log\dfrac{3}{2} + \log\dfrac{4}{3}\right]$

$= \dfrac{1}{2}\log\left(\dfrac{3}{2}\times\dfrac{4}{3}\right)$

$= \dfrac{1}{2}\log 2$

154. Let, $S = 2 + 7 + 14 + 23 + 34 + \ldots + t_n +$ ....(i)

and $S = 2 + 7 + 14 + \ldots + t_{n-1} + t_n +$ ....(ii)

From (i) and (ii), we get

$0 = 2 + [5 + 7 + 9 + 11 \ldots + t_n - t_{n-1}] - t_n$

$\Rightarrow \quad t_n = 2 + \left[\dfrac{n-1}{2}\{2\times 5 + (n-2)2\}\right]$

$\Rightarrow \quad t_n = 2 + (n-1)(n+3)$

Now,

put n = 99

$\Rightarrow \quad t_{99} = 2 + 98 \times 102 = 9998$

155. $A = 1^2 + 2.2^2 + 3^2 + 2.4^2 + \ldots + 2.20^2$

$= (1^2 + 2^2 + 3^2 + \ldots + 20^2)$

$+ (2^2 + 4^2 + \ldots + 20^2)$

$= (1^2 + 2^2 + 3^2 + \ldots + 20^2)$

$+ 4(1^2 + 2^2 + \ldots + 10^2)$

$= \dfrac{20\times 21\times 41}{6} + 4\left(\dfrac{10\times 11\times 21}{6}\right)$

$= 2870 + 4(385)$

$= 4410$

$B = 1^2 + 2.2^2 + 3^2 + 2.4^2 + \ldots + 2.40^2$

$= (1^2 + 2^2 + 3^2 + \ldots + 40^2)$

$+ (2^2 + 4^2 + \ldots + 40^2)$

$= (1^2 + 2^2 + 3^2 + \ldots + 40^2)$

$+ 4(1^2 + 2^2 + \ldots + 20^2)$

$= \dfrac{40\times 41\times 81}{6} + 4\left(\dfrac{20\times 21\times 41}{6}\right)$

$= 22140 + 4(2870)$

$= 33620$

$B - 2A = 33620 - 2(4410) = 24800$

$\Rightarrow 100\lambda = 24800 \Rightarrow \lambda = 248$

156. Since, $\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P.

$\therefore \quad \dfrac{\cos\theta}{\sin\theta} = \dfrac{\tan\theta}{\cos\theta} = \dfrac{\sin\theta}{\cos^2\theta}$

$\Rightarrow \cos^3\theta = \sin^2\theta$ ....(i)

$\therefore \quad \cot^6\theta - \cot^2\theta = \dfrac{\cos^6\theta}{\sin^6\theta} - \dfrac{\cos^2\theta}{\sin^2\theta}$

$= \dfrac{\cos^6\theta}{\cos^9\theta} - \dfrac{\cos^2\theta}{\cos^3\theta}$ ....[From (i)]

$= \dfrac{1}{\cos^3\theta} - \dfrac{1}{\cos\theta} = \dfrac{1-\cos^2\theta}{\cos^3\theta} = \dfrac{\sin^2\theta}{\cos^3\theta}$

$= 1$ ....[From (i)]

157. Let $1 - \cos\theta = x$

$\therefore$ the given series $= 1 + 2x + 3x^2 + 4x^3 + \ldots \infty$

$= (1-x)^{-2}$

$= (1 - 1 + \cos\theta)^{-2} = \sec^2\theta$

$= 1 + \tan^2\theta = 1 + \dfrac{3}{2} = \dfrac{5}{2}$ ....$\left[\because \tan\theta = \sqrt{\dfrac{3}{2}}\right]$

158. cos (θ – α), cos θ, cos (θ + α) are in HP

$\therefore \quad \frac{1}{\cos(\theta-\alpha)}, \frac{1}{\cos\theta}, \frac{1}{\cos(\theta+\alpha)}$ are in AP

$\therefore \quad \frac{1}{\cos(\theta-\alpha)} + \frac{1}{\cos(\theta+\alpha)} = \frac{2}{\cos\theta}$

$\Rightarrow \frac{\cos(\theta+\alpha)+\cos(\theta-\alpha)}{\cos(\theta+\alpha)\cos(\theta-\alpha)} = \frac{2}{\cos\theta}$

$\Rightarrow 2\cos\theta \cos\alpha.\cos\theta = 2\cos(\theta+\alpha)\cos(\theta-\alpha)$

$\Rightarrow \cos^2\theta \cos\alpha = \cos^2\theta - \sin^2\alpha$

$\Rightarrow \cos^2\theta (\cos\alpha - 1) = -(1 - \cos^2\alpha)$

$\Rightarrow \cos^2\theta - 1 + \cos\alpha$

159. $\sum_{k=1}^{n} f(a+k) = f(a+1) + f(a+2) + f(a+3) + \dots f(a+n)$

$= f(a)\, f(1) + f(a)\, f(2) + f(a)\, f(3) + \dots f(a)\, f(n)$ ...[$\because f(x+y) = f(x)\, f(y)$]

$\therefore \quad \sum_{k=1}^{n} f(a+k) = f(a)[f(1) + f(2) + f(3) + \dots f(n)]$ ...(i)

$\because \quad f(1) = 2$

$\therefore \quad f(2) = f(1+1) = f(1).f(1) = 4$

$\therefore \quad f(3) = f(2+1) = f(2).f(1) = 8$

and so on.

$\therefore$ substituting above values in (i), we get

$\sum_{k=1}^{n} f(a+k) = f(a)[2 + 4 + 8 + \dots f(n)]$

$= f(a).2(2^n - 1)$

$\therefore \quad f(a).2(2^n - 1) = 16.(2^n - 1)$

$\therefore \quad f(a) = 8$

Since, f(3) = 8

$\therefore \quad a = 3$

Evaluation Test

1. $a_1, a_2, a_3, \dots, a_n$ are in A.P. with common difference = 5

i.e., $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \dots = a_n - a_{n-1} = 5$

$\therefore \quad \tan^{-1}\left(\frac{5}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{5}{1+a_2a_3}\right) + \dots + \tan^{-1}\left(\frac{5}{1+a_{n-1}a_n}\right)$

$= \tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right) + \dots + \tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right)$

$= \tan^{-1}(a_2) - \tan^{-1}(a_1) + \tan^{-1}(a_3) - \tan^{-1}(a_2) + \dots + \tan^{-1}(a_n) - \tan^{-1}(a_{n-1})$

$= \tan^{-1}(a_n) - \tan^{-1}(a_1) = \tan^{-1}\left(\frac{a_n-a_1}{1+a_1a_n}\right)$

$= \tan^{-1}\left(\frac{(n-1)5}{1+a_1a_n}\right)$ ....[$\because a_n = a_1 + (n-1)5$]

$= \tan^{-1}\left(\frac{5n-5}{1+a_1a_n}\right)$

2. Since, $a_1, a_2, a_3, \dots$ are in H.P.

$\therefore \quad \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots$ are in A.P.

$\therefore \quad \frac{1}{25} = \frac{1}{5} + 19d$

$\Rightarrow d = \frac{1}{19}\left(\frac{-4}{25}\right) = -\frac{4}{19\times25}$

Since, $a_n < 0$

$\therefore \quad \frac{1}{5} - \frac{4}{19\times25}\times(n-1) < 0$

$\Rightarrow \frac{19\times5}{4} < n - 1$

$\Rightarrow n > 24.75$

3. p, q, r are positive and are in A.P.

$\therefore \quad q = \frac{p+r}{2}$ ....(i)

Since, the roots of $px^2 + qx + r = 0$ are real

$\therefore \quad q^2 \ge 4pr \Rightarrow \left[\frac{p+r}{2}\right]^2 \ge 4pr$ ....[From (i)]

$\Rightarrow p^2 + r^2 - 14pr \ge 0$

$$\Rightarrow \left(\frac{r}{p}\right)^2 - 14\left(\frac{r}{p}\right) + 1 \geq 0$$

....[$\because$ p > 0 and p ≠ 0]

$$\Rightarrow \left(\frac{r}{p} - 7\right)^2 - 48 \geq 0$$

$$\Rightarrow \left(\frac{r}{p} - 7\right)^2 - (4\sqrt{3})^2 \geq 0$$

$$\Rightarrow \left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$$

4. Let A be the first term and R be the common ratio of the G.P. Then,
$a = AR^{p-1}$
$\Rightarrow \log a = \log A + (p-1)\log R$ ....(i)
$b = AR^{q-1}$
$\Rightarrow \log b = \log A + (q-1)\log R$ ....(ii)
$c = AR^{r-1}$
$\Rightarrow \log c = \log A + (r-1)\log R$ ....(iii)
Multiplying (i), (ii) and (iii) by (q – r), (r – p) and (p – q) respectively and adding, we get
$(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$
Expanding along first row, we get
$\Delta = (q-r)\log a + (r-p)\log b + (p-q)\log c$
$\Rightarrow \Delta = 0$

5. Let x_1, x_2, x_3 be a, ar, ar^2 and y_1, y_2, y_3 be b, br, br^2.
$\therefore$ A, B, C are (a,b), (ar, br), (ar^2, br^2) resp.
Now Slope of AB = $\frac{b}{a}$ = slope of BC.
Hence, the points are collinear.
i.e., lie on a straight line

Alternate method:

Given $x_2 = rx_1, x_3 = r^2x_1, y_2 = ry_1, y_3 = r^2y_1$

$$\text{Area of } \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ rx_1 & ry_1 & 1 \\ r^2x_1 & r^2y_1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}x_1y_1\begin{vmatrix} 1 & 1 & 1 \\ r & r & 1 \\ r^2 & r^2 & 1 \end{vmatrix} = 0$$

i.e., lie on a straight line.

6. We have,
Length of a side of S_n
= Length of a diagonal of S_{n+1}
$\Rightarrow$ Length of a side of S_n
= $\sqrt{2}$ (Length of a side of S_{n+1})

$$\Rightarrow \frac{\text{Length of a side of } S_{n+1}}{\text{Length of a side of } S_n} = \frac{1}{\sqrt{2}} \text{ for all } n \geq 1$$

$\therefore$ sides of $S_1, S_2, ..., S_n$ form a G.P. with common ratio $\frac{1}{\sqrt{2}}$ and first term 10.

$$\therefore \text{ length of the side of } S_n = 10\left(\frac{1}{\sqrt{2}}\right)^{n-1}$$

$$= \frac{10}{2^{\frac{n-1}{2}}}$$

$$\text{Now, area of } S_n = (\text{side})^2 = \left(\frac{10}{2^{\frac{n-1}{2}}}\right)^2$$

$$= \frac{100}{2^{n-1}}$$

But, area of $S_n < 1$

$$\Rightarrow \frac{100}{2^{n-1}} < 1$$

$\Rightarrow 2^{n-1} > 100$

The above inequality is satisfied if $n - 1 \geq 7$ i.e., $n \geq 8$

7. We have, $T_p = a + (p-1)d = \frac{1}{q}$ ....(i)

and $T_q = a + (q-1)d = \frac{1}{p}$ ....(ii)

From (i) and (ii), we get $a = \frac{1}{pq}$ and $d = \frac{1}{pq}$

$$\therefore \text{ sum of } (pq)^{th} \text{ terms } = \frac{pq}{2}\left[\frac{2}{pq} + (pq-1)\frac{1}{pq}\right]$$

$$= \frac{pq}{2}.\frac{2}{pq}\left[1 + \frac{1}{2}(pq-1)\right]$$

$$= \frac{2 + pq - 1}{2} = \frac{pq+1}{2}$$

8. $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$,

$= \begin{vmatrix} -1 & x+2 & x+a \\ -1 & x+3 & x+b \\ -1 & x+4 & x+c \end{vmatrix}$

$= (-1) \begin{vmatrix} 1 & x+2 & x+a \\ 1 & x+3 & x+b \\ 1 & x+4 & x+c \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$= (-1) \begin{vmatrix} 1 & x+2 & x+a \\ 0 & 1 & b-a \\ 0 & 2 & c-a \end{vmatrix}$

$= (-1)(c + a - 2b) = 0$

....[$\because$ a, b, c are in A.P. $\Rightarrow$ 2b = a + c]

9. Let $S = 1 + \frac{1.3}{6} + \frac{1.3.5}{6.8} + ...\infty$

$\therefore$ $\frac{S}{4} = \frac{1}{4} + \frac{1.3}{4.6} + \frac{1.3.5}{4.6.8} + ...\infty$

Multiplying on both sides by $\frac{1}{2}$, we get

$\frac{S}{8} = \frac{1}{2.4} + \frac{1.3}{2.4.6} + \frac{1.3.5}{2.4.6.8} + ... \infty$

$\therefore$ $\frac{1}{2} - \frac{S}{8} = \frac{1}{2} - \left[\frac{1}{2.4} + \frac{1.3}{2.4.6} + \frac{1.3.5}{2.4.6.8} + ...\infty\right]$

$\Rightarrow \frac{1}{2} - \frac{S}{8} = \frac{1}{2} - \frac{1}{2}.\frac{1}{4} - \frac{1}{2}.\frac{1.3}{4.6} - \frac{1}{2}.\frac{1.3.5}{4.6.8} - ....\infty$

$$\Rightarrow \frac{1}{2} - \frac{S}{8} = 1 - \frac{1}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2} - \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4} - ....\infty$$

$\Rightarrow \frac{1}{2} - \frac{S}{8} = (1-1)^{\frac{1}{2}} = 0$

$\therefore$ $\frac{S}{8} = \frac{1}{2} \Rightarrow S = 4$

10. Let the first installment be a and common difference of A.P. be d.

Given, 3600 = sum of 40 terms

$= \frac{40}{2}[2a + (40 - 1)d]$

$\therefore$ 3600 = 20(2a + 39d)

$\therefore$ 180 = 2a + 39d ....(i)

After 30 installments one third of the debt is unpaid

$\therefore$ $\frac{3600}{3}$ = 1200 is unpaid and 2400 is paid.

Now, $2400 = \frac{30}{2}[2a + (30 - 1)d]$

$\therefore$ 160 = 2a + 29d ....(ii)

Subtracting (ii) from (i), we get 20 = 10d

$\therefore$ d = 2

From (i), 180 = 2a + 39(2)

$\therefore$ 2a = 180 – 78 = 102

$\therefore$ a = 51

$\therefore$ value of the 8^{th} installment

= a + (8 – 1) d = 51 + 7(2) = ₹ 65

Textbook Chapter No.

11 Probability

Hints

Classical Thinking

5. Here, P(A) = 1

∴ $P(\overline{A}) = 1 - P(A) = 0$

6. Here, n(S) = 2 × 2 × 2 × 2 = 16

A: Event of getting all heads

⇒ A = {(HHHH)}

∴ n (A) = 1

$\Rightarrow P(A) = \frac{1}{16}$

7. Here, n(S) = 52

There is one queen of club and one king of heart

∴ Favourable ways = 1 + 1 = 2

∴ Required Probability = $\frac{2}{52} = \frac{1}{26}$

8. Required probability = $\frac{12}{52} = \frac{3}{13}$.

9. Total number of outcomes = 36

Favourable number of outcomes = 6

i.e., {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}

∴ Required probability = $\frac{6}{36} = \frac{1}{6}$

10. Required probability = $\frac{3}{36} = \frac{1}{12}$

11. Required probability = $\frac{5}{25} = \frac{1}{5}$

12. Odd and perfect square (< 10) are 1, 9.

Hence, required probability = $\frac{2}{10} = \frac{1}{5}$

13. Since there are one A, two I and one O, hence the required probability = $\frac{1+2+1}{11} = \frac{4}{11}$

14. Two fruits out of 6 can be chosen in ${}^6C_2 = 15$ ways.

One mango and one apple can be chosen in ${}^3C_1 \times {}^3C_1 = 9$ ways

∴ Probability = $\frac{9}{15} = \frac{3}{5}$

15. Three persons can be chosen out of 8 in ${}^8C_3 = 56$ ways.

The number of girls is more than that of the boys if either 3 girls are chosen or two girls and one boy is chosen. This can be done in ${}^3C_3 + {}^3C_2 \times {}^5C_1$ ways

= 1 + 3 × 5 = 16 ways.

∴ Required probability = $\frac{16}{56} = \frac{2}{7}$

16. Number of tickets, numbered such that it is divisible by 20 are $\frac{10000}{20} = 500$

Hence, required probability = $\frac{500}{10000} = \frac{1}{20}$.

17. In a non-leap year, we have 365 days i.e., 52 weeks and one day. So, we may have any day of seven days.

18. Total no. of ways = 3! = 6

Favourable ways = 1

⇒ Probability = $\frac{1}{6}$

19. Probability of keeping at least one letter in wrong envelope = $1 - \frac{1}{n!}$

∴ option (B) is the correct answer.

20. Sample space when six dice are thrown = 6^6

All dice show the same face means we are getting same number on all six dice which can be any one of the six numbers 1, 2, …, 6.

∴ No. of ways of selecting a number is ${}^6C_1$.

∴ Required probability = $\frac{{}^6C_1}{6^6} = \frac{1}{6^5}$

21. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

∴ $\frac{5}{8} = \frac{1}{4} + \frac{1}{2} - P(A \cap B)$

∴ $P(A \cap B) = \frac{1}{8}$

22. Since, events are mutually exclusive, therefore $P(A \cap B) = 0$ i.e., $P(A \cup B) = P(A) + P(B)$

$\Rightarrow 0.7 = 0.4 + x \Rightarrow x = \frac{3}{10}$

23. $P(A \text{ or } B) = P(A \cup B)$
$= P(A) + P(B) - P(A \cap B)$
$= 0.25 + 0.5 - 0.15 = 0.6$

24. $P(A) = P(A \cap B) + P(A \cup B) - P(B)$
$= \frac{1}{3} + \frac{5}{6} - \frac{2}{3} = \frac{3}{6} = \frac{1}{2}$

25. $P(A) = 0.28$, $P(B) = 0.55$, $P(A \cap B) = 0.14$
$P(A' \cap B') = P[(A \cup B)'] = 1 - P(A \cup B)$
$= 1 - [P(A) + P(B) - P(A \cap B)]$
$= 1 - (0.28 + 0.55 - 0.14) = 0.31$

26. Here, $P(A \cup B) = 0.6$ and $P(A \cap B) = 0.3$
$\therefore$ $P(A) + P(B) = P(A \cup B) + P(A \cap B) = 0.9$
$\therefore$ $P(A') + P(B') = 1 - P(A) + 1 - P(B)$
$= 2 - 0.9 = 1.1$

27. Probability of getting either first class or second class or third class = P(A)
$= \frac{2}{7} + \frac{3}{5} + \frac{1}{10}$
$= \frac{69}{70}$
Probability of failing = $P(A') = 1 - P(A) = \frac{1}{70}$

28. There are 4 kings, 13 hearts and a king of hearts is common to the two blocks.
$\therefore$ Required probability = $\frac{4+13-1}{52} = \frac{16}{52}$

29. Total number of ways = {HH, HT, TH, TT}
$\therefore$ P (head on first toss) = $\frac{2}{4} = \frac{1}{2} = P(A)$,
P (head on second toss) = $\frac{2}{4} = \frac{1}{2} = P(B)$
and P (head on both toss) = $\frac{1}{4} = P(A \cap B)$
Hence, required probability is,
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

31. If A and B are independent, A′ and B′ are also independent.

33. $P(A/B) = \frac{P(A \cap B)}{P(B)}$
Since, A and B are mutually exclusive.
So, $P(A \cap B) = 0$.
Hence, $P(A/B) = \frac{0}{P(B)} = 0$

34. $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.5}{0.6} = \frac{5}{6}$

35. $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{(3/8) + (5/8) - (3/4)}{(5/8)}$
$= \frac{2}{5}$

36. $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

37. $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{1 - P(B')} = \frac{0.15}{1 - 0.10}$
$= \frac{1}{6}$

38. $P\left(\frac{\overline{A}}{\overline{B}}\right) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{P(\overline{A \cup B})}{P(\overline{B})} = \frac{1 - P(A \cup B)}{P(\overline{B})}$

39. Let E_1 be the event that man will be selected and E_2 be the event that woman will be selected. Then
$P(E_1) = \frac{1}{2}$, So $P(\overline{E_1}) = 1 - \frac{1}{2} = \frac{1}{2}$ and
$P(E_2) = \frac{1}{3}$, So $P(\overline{E_2}) = 1 - \frac{1}{3} = \frac{2}{3}$
Clearly, E_1 and E_2 are independent events.
$\therefore$ $P(\overline{E_1} \cap \overline{E_2}) = P(\overline{E_1}) \times P(\overline{E_2})$
$= \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

40. Let A be the event of selecting bag X, B be the event of selecting bag Y and E be the event of drawing a white ball, then
$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(E/A) = \frac{2}{5}$
and $P(E/B) = \frac{4}{6} = \frac{2}{3}$
$\therefore$ $P(E) = P(A)\, P(E/A) + P(B)\, P(E/B)$
$= \frac{1}{2} . \frac{2}{5} + \frac{1}{2} . \frac{2}{3} = \frac{8}{15}$

41. Required probability = $\frac{3}{5}$
....$\left[\because \text{The probability of the occurrence } = \frac{b}{a+b}\right]$

42. Required probability = $\frac{6}{6+5} = \frac{6}{11}$

....$\left[\because \text{ The probability of the occurrence } = \frac{a}{a+b}\right]$

43. Here, $P(A) = \frac{3}{7}$, $P(B) = \frac{7}{12}$

$\therefore$ $P(A') = \frac{4}{7}$ and $P(B') = \frac{5}{12}$

$\therefore$ P(Problem will be considered solved even if one person solves it)

$= 1 - [P(A')\cdot P(B')] = 1 - \frac{5}{21} = \frac{16}{21}$

Critical Thinking

1. Here, $n(S) = 2 \times 2 = 4$
A: Event of getting 2 heads or 2 tails
$\therefore$ A = {(H H), (T T)}
$\Rightarrow n(A) = 2$
$\Rightarrow P(A) = \frac{2}{4} = \frac{1}{2}$

2. One card can be selected from a pack in $^{52}C_1$ ways.
$\therefore$ $n(S) = {}^{52}C_1 = 52$
A: Event of getting a red queen
$\therefore$ P(A) = P(diamond queen or heart queen)
$= \frac{^2C_1}{^{52}C_1}$

3. Favourable ways
= {29, 92, 38, 83, 47, 74, 56, 65}
Hence, required probability = $\frac{8}{100} = \frac{2}{25}$

4. Two digits, one from each set can be selected in $9 \times 9 = 81$ ways.
Favourable outcomes are (1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2) and (9, 1).
$\therefore$ $n(S) = 81$
and $n(A) = 9$
$\therefore$ $P(A) = \frac{9}{81} = \frac{1}{9}$

5. When six dice are thrown, the total number of outcomes is 6^6.They can show different number in $^6P_6 = 6!$ ways
$\therefore$ Required probability = $\frac{6!}{6^6} = \frac{5!}{6^5} = \frac{5}{324}$

6. The sum 2 can be found in one way i.e., {(1, 1)}
The sum 8 can be found in five ways i.e., {(6, 2), (5, 3), (4, 4), (3, 5), (2, 6)}. Similarly, the sum twelve can be found in one way i.e., {(6, 6)}.
Hence, required probability = $\frac{7}{36}$.

7. Between 1 and 100, there are 25 prime numbers.
$\therefore$ $n(S) = 98$ and $n(A) = 25$
$\therefore$ $P(A) = \frac{25}{98}$

8. Total cases = 4
So, probability of correct answer = $\frac{1}{4}$

9. In a leap year, there are 366 days in which 52 weeks and two days. The combination of 2 days may be: Sun – Mon, Mon – Tue, Tue – Wed, Wed – Thu, Thu – Fri, Fri – Sat, Sat – Sun.
$\therefore$ P(53 Sun) = $\frac{2}{7}$

10. When a coin is tossed, there are two outcomes and when a dice is rolled, there are six possible outcomes.
Hence, there are 8 (2 corresponding to head and six corresponding to tail at first toss) sample points in the sample space.
Sample space is {HH, HT, T1, T2, T3, T4, T5, T6}.

11. It six does not appear on either dice then, there are only five possible outcomes associated with one dice, the number of sample points is 5×5.

12. Since, the total '13' can't be found.

13. Probabilities of H_1, H_2 and H_3 winning a race must be in the ratio 4 : 2 : 1 (due to given condition) and should also add up to 1.

14. Here, $n(S) = {}^6C_2 = 15$
If both are vowels, then they are selected in $^2C_2$ ways = 1.
$\therefore$ Required probability = $\frac{1}{15}$

15. Here, $n(S) = {}^{10}C_2$
A: Event that the watches selected are defective
$\therefore$ $n(A) = {}^2C_2 = 1$
$\therefore$ $P(A) = \frac{1}{^{10}C_2} = \frac{1}{45}$

16. Total no. of ways in which 2 socks can be drawn out of 9 is ${}^9C_2$. The two socks match if either they are both black or they are both blue. So, two matching socks can be drawn in ${}^5C_2 + {}^4C_2$ ways.

$\therefore$ Required probability $= \dfrac{{}^5C_2 + {}^4C_2}{{}^9C_2}$

$= \dfrac{10+6}{36} = \dfrac{4}{9}$

17. Ace is not drawn in 26 cards.
It means 26 cards are drawn from 48 cards.

$\therefore$ Required Probability $= \dfrac{{}^{48}C_{26}}{{}^{52}C_{26}}$

18. $n(S) = {}^{16}C_{11}$
A: Event that the team has exactly four bowlers.

$\therefore$ $n(A) = {}^6C_4 \,.\, {}^{10}C_7$

$\Rightarrow P(A) = \dfrac{{}^6C_4 . {}^{10}C_7}{{}^{16}C_{11}} = \dfrac{75}{182}$

19. We have to select exactly 2 children

$\therefore$ selection contain 2 children out of 4 children and remaining 2 person can be selected from 2 women and 4 men
i.e., $4C_2 \times 6C_2$ ways

$\therefore$ Total favourable ways $= 6 \times 15 = 90$

$\therefore$ Required probability $= \dfrac{90}{210} = \dfrac{3}{7}$

20. A committee of 4 can be formed in ${}^{25}C_4$ ways
A: Event that the committee contains at least 3 doctors

$\therefore$ $n(A) = {}^4C_3.{}^{21}C_1 + {}^4C_4 = 85$

$\therefore$ $P(A) = \dfrac{85}{{}^{25}C_4} = \dfrac{85}{12650} = \dfrac{17}{2530}$

21. Since, cards are drawn with replacement.

$\therefore$ Total no. of ways $= 52 \times 52$.
Now, we can choose one suit out of four in ${}^4C_1$ ways and two cards in 13×13 ways.

$\therefore$ Required Probability $= \dfrac{{}^4C_1 \times 13 \times 13}{52 \times 52} = \dfrac{1}{4}$

22. Besides ground floor, there are 7 floors. Since a person can leave the cabin at any of the seven floors, total no. of ways in which each of the five persons can leave the cabin at any of the 7 floors $= 7^5$
Five persons can leave the cabin at five different floors in ${}^7C_5 \times 5!$ ways

Hence, required probability $= \dfrac{{}^7C_5 \times 5!}{7^5}$

23. Here, $n(S) = 2 \times 2 \times 2 = 8$
If A is the event that there is no tail, then
$A = \{(HHH)\}$
$\Rightarrow n(A) = 1$

$\Rightarrow P(A) = \dfrac{1}{8}$

$\therefore$ $P(A') = 1 - P(A) = 1 - \dfrac{1}{8} = \dfrac{7}{8}$

24. Required probability

$= \left(1-\dfrac{1}{3}\right)\left(1-\dfrac{1}{4}\right)\left(1-\dfrac{1}{5}\right) = \dfrac{2}{3}.\dfrac{3}{4}.\dfrac{4}{5} = \dfrac{2}{5}$

25. Out of 30 numbers from 1 to 30, three numbers can be chosen in ${}^{30}C_3$ ways.
Three consecutive numbers can be chosen in one of the following ways:
$\{(1, 2, 3), (2, 3, 4), \ldots, (28, 29, 30)\} = 28$ ways

$\therefore$ Probability that numbers are consecutive

$= \dfrac{28}{{}^{30}C_3} = \dfrac{1}{145}$

Hence, required probability $= 1 - \dfrac{1}{145} = \dfrac{144}{145}$

26. Total no. of ways = 7!
Arrangement of boys and girls in alternate seats is B G B G B G B
Boys can occupy seat in 4! ways and girls in 3! ways.

$\therefore$ Required Probability $= \dfrac{3! \times 4!}{7!} = \dfrac{1}{35}$

27. Two 3s, one 6 and one 8 can be dialled in $\dfrac{4!}{2!} = 12$ ways of which only one is the correct way of dialling.

$\therefore$ Required probability $= \dfrac{1}{12}$

28. As $\{(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6)\}$ are only favourable outcomes

$\Rightarrow$ Required probability $= \dfrac{6}{216}$

29. Since there are 3 A′s and 2 N's.

Total no. of arrangements $= \dfrac{10!}{3!2!}$

Hence, the number of arrangements in which ANAND occurs without any split = 6!

$\therefore$ Required probability $= \dfrac{6!3!2!}{10!} = \dfrac{1}{420}$

30. 15 places are occupied. This includes the owner's car also. 14 cars are parked in 24 places of which 22 places are available (excluding the neighbouring places) and so the required probability $\frac{{}^{22}C_{14}}{{}^{24}C_{14}} = \frac{15}{92}$

31. Three numbers can be chosen out of 10 numbers in ${}^{10}C_3$ ways.
The product of two numbers, out of the three chosen numbers, will be equal to the third number, if the numbers are chosen in one of the following ways:
{(2, 3, 6), (2, 4, 8), (2, 5, 10)} = 3 ways
Hence, required probability = $\frac{3}{{}^{10}C_3} = \frac{1}{40}$

32. 4 cards can drop out of 52 in ${}^{52}C_4$ ways. They can be one from each suit in
${}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = (13\times13\times13\times13)$ ways.
∴ Required probability = $\frac{13\times13\times13\times13}{{}^{52}C_4}$
$= \frac{13\times13\times13\times13\times4!}{52\times51\times50\times49}$
$= \frac{2197}{20825}$

33. $0.7 = 0.4 + x - 0.4x$
$\Rightarrow x = \frac{1}{2}$

34. Since, we have
$P(A\cup B) + P(A \cap B) = P(A) + P(B)$
$= P(A) + \frac{P(A)}{2}$
$\Rightarrow \frac{7}{8} = \frac{3P(A)}{2}$
$\Rightarrow P(A) = \frac{7}{12}$

35. Since, $A \cup B = S$.
∴ $P(A \cup B) = P(S) = 1$
∴ $1 = P(A) + 2P(A)$ [$\because P(A \cup B) = P(A) + P(B)$]
$\Rightarrow 3(P(A)) = 1$
$\Rightarrow P(A) = \frac{1}{3}$
∴ $P(B) = \frac{2}{3}$

36. A: Event of obtaining an even sum and B: Event of obtaining a sum less than five. Since A, B are not mutually exclusive,
∴ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= \frac{18}{36} + \frac{6}{36} - \frac{4}{36} = \frac{5}{9}$
[$\because$ there are 18 ways to get an even sum i.e {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)} and there are 6 ways to get a sum < 5 i.e., {(1, 3), (3, 1), (2, 2), (1, 2), (2, 1), (1, 1)} and 4 ways to get an even sum < 5 i.e., {(1, 3), (3, 1), (2, 2), (1, 1)}]

37. Here, A = {4, 5, 6}
$\Rightarrow P(A) = \frac{3}{6} = \frac{1}{2}$
and B = {4, 3, 2, 1}
$\Rightarrow P(B) = \frac{4}{6} = \frac{2}{3}$
∴ $A \cap B = \{4\}$
$\Rightarrow P(A \cap B) = \frac{1}{6}$
∴ $P(A \cup B) = \frac{1}{2} + \frac{2}{3} - \frac{1}{6} = 1$

38. A is independent of itself, if
$P(A \cap A) = P(A).P(A)$
$\Rightarrow P(A) = P(A)^2$
$\Rightarrow P(A) = 0, 1$

39. We have $P(A + B) = P(A) + P(B) - P(AB)$
$\Rightarrow \frac{5}{6} = \frac{1}{2} + P(B) - \frac{1}{3} \Rightarrow P(B) = \frac{4}{6} = \frac{2}{3}$
Thus, $P(A).P(B) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3} = P(AB)$
Hence, events A and B are independent.

40. Let $P(A) = \frac{20}{100} = \frac{1}{5}$, $P(B) = \frac{10}{100} = \frac{1}{10}$
Since, events are independent and we have to find $P(A \cup B) = P(A) + P(B) - P(A).P(B)$
$= \frac{1}{5} + \frac{1}{10} - \frac{1}{5} \times \frac{1}{10}$
$= \frac{3}{10} - \frac{1}{50} = \frac{14}{50} \times 100 = 28\%$

41. In a leap year, there are 366 days in which 52 weeks and two days. The combination of 2 days may be: Sun-Mon, Mon-Tue, Tue-Wed, Wed-Thu, Thu-Fri, Fri-Sat, Sat-Sun.

$\therefore$ P(53 fri) = $\frac{2}{7}$; P(53 Sat) = $\frac{2}{7}$

There is one combination in common i.e., (Fri-Sat)

$\therefore$ P(53 Fri and 53 Sat) = $\frac{1}{7}$

$\therefore$ P(53 Fri or 53 Sat) = P(53 Fri) + P(53 Sat) − P(53 Fri and Sat)

$$= \frac{2}{7} + \frac{2}{7} - \frac{1}{7} = \frac{3}{7}$$

42. Here, P(A) = P(B) = 2 P(C),
and P(A) + P(B) + P(C) = 1

$\Rightarrow$ P(C) = $\frac{1}{5}$ and P(A) = P(B) = $\frac{2}{5}$

Hence, $P(A \cup B) = P(A) + P(B) = \frac{2}{5} + \frac{2}{5} = \frac{4}{5}$

43. For both to be boys, the probability

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

44. We have to consider order for IIT

$\therefore$ Required probability = $\frac{10}{20} \times \frac{9}{19} \times \frac{10}{18} = \frac{5}{38}$

45. In the word 'MULTIPLE' there are 3 vowels, out of total of 8, 1 vowel can be chosen in ${}^3C_1$ ways. In the word 'CHOICE' there are 3 vowels, out of the total of 6, 1 vowel can be chosen in ${}^3C_1$ ways.

$\therefore$ Required probability = $\frac{{}^3C_1}{8} \times \frac{{}^3C_1}{6} = \frac{3}{16}$

46. A total of 7 and a total of 9 cannot occur simultaneously.

$\therefore$ P(total of 7 or 9)

= P(total of 7) + P(total of 9) = $\frac{6}{36} + \frac{4}{36} = \frac{5}{18}$

(A total of 7 and a total of 9 cannot occur simultaneously)

47. $\frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{3}{10} = \frac{7}{20}$

48. P(G) = $\frac{25}{80}$, P(R) = $\frac{10}{80}$, P(I) = $\frac{20}{80}$

Since events are independent,

$\therefore$ P(selecting rich and intelligent girls)

= P(G)·P(R)·P(I) = $\frac{5}{512}$

49. $P(A' \cup B') = P[(A \cap B)']$

$$= 1 - P(A \cap B) = 1 - \frac{1}{4} = \frac{3}{4}$$

50. $P(A' \cap B') = \frac{1}{3}$

$\Rightarrow P[(A \cup B)'] = \frac{1}{3}$

$\therefore$ $1 - P(A \cup B) = \frac{1}{3}$

$\Rightarrow P(A \cup B) = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore$ $P(A) + P(B) - P(A \cap B) = \frac{2}{3}$

$\therefore$ $p + 2p - \frac{1}{2} = \frac{2}{3}$

$\Rightarrow 3p = \frac{2}{3} + \frac{1}{2} = \frac{7}{6} \Rightarrow p = \frac{7}{18}$

51. Required Probability

$= P[(A \cap B') \cup (A' \cap B)]$

$= P(A \cap B') + P(A' \cap B)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= P(A) + P(B) - 2P(A \cap B)$

52. P(neither A nor B)

$= 1 - P(\text{either A or B}) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - 0.25 - 0.50 + 0.14 = 0.39$

53. M: Event that student passed in Mathematics.
E: Event that student passed in Electronics

$\therefore$ n(M) = 30, n(E) = 20, $n(M \cap E)$ = 10, n(S) = 80.

$\therefore$ P(M) = $\frac{30}{80}$, P(E) = $\frac{20}{80}$, $P(M \cap E) = \frac{10}{80}$

$\therefore$ $P(M \cup E) = P(M) + P(E) - P(M \cap E)$

$$= \frac{30}{80} + \frac{20}{80} - \frac{10}{80} = \frac{1}{2}$$

$\therefore$ P(Student has passed in none of the subject)

$= P[(M \cup E)'] = 1 - P(M \cup E) = 1 - \frac{1}{2} = \frac{1}{2}$

54. P(neither E_1 nor E_2 occurs) = $P\left(E_1' \cap E_2'\right)$

$$= P\left(E_1'\right) P\left(E_2'\right)$$

$$= (1 - p_1)(1 - p_2)$$

55. $P(M) = \frac{1}{4} \Rightarrow P(M') = \frac{3}{4}$

and $P(W) = \frac{1}{3} \Rightarrow P(W') = \frac{2}{3}$

Both events are independent so that probability that no one will be alive is

$P(W' \cap M') = P(W')\ P(M') = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

56. Here, $P(A) = p$

$\Rightarrow P(\overline{A}) = 1 - p$

and $P(B) = q \Rightarrow P(\overline{B}) = 1 - q$

Probability that one person is alive is the sum of two cases A dies B lives and A lives B dies

$= p(1 - q) + q(1 - p) = p + q - 2pq$

57. Here, $P(A) = 0.6$; $P(B) = 0.9$

$\therefore$ Required pobability

$= P(A) \cdot P(\overline{B}) + P(B) \cdot P(\overline{A}) = (0.6)\ (0.1) + (0.9)\ (0.4)$

$= 0.06 + 0.36 = 42$

58. Since, E and F are independent

$\therefore$ $P(E \cap F) = P(E)\ P(F)$

$\Rightarrow P(E)\ P(F) = \frac{1}{12}$

Now, E and F are independent

$\therefore$ E′ and F′ are also independent

$\therefore$ $P(E' \cap F') = P(E') \cdot P(F') = \frac{1}{2}$

$\therefore$ $[1 - P(E)] \cdot [1 - P(F)] = \frac{1}{2}$

$\therefore$ $1 - P(E) - P(F) + P(E) \cdot P(F) = \frac{1}{2}$

$\therefore$ $1 - P(E) - P(F) + \frac{1}{12} = \frac{1}{2}$

$\Rightarrow P(E) + P(F) = \frac{7}{12}$

Solving, $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{3}$

59. $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{P(A).P(B)}{P(B)} = P(A)$.

60. Since, $A \subseteq B \Rightarrow A \cap B = B \cap A = A$

Hence, $P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$

61. $P\left(\frac{\overline{B}}{\overline{A}}\right) = \frac{1 - P(A \cup B)}{P(\overline{A})} = \frac{1 - \frac{23}{60}}{1 - \frac{1}{3}} = \frac{37}{60} \times \frac{3}{2} = \frac{37}{40}$

62. A: Brown hair

$\Rightarrow P(A) = \frac{40}{100}$

B: Brown eyes

$\Rightarrow P(B) = \frac{25}{100}$

$\therefore$ $P(A \cap B) = \frac{15}{100}$

$\therefore$ $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{15}{100}}{\frac{40}{100}} = \frac{3}{8}$

63. $P(A \cap B) = P(A) \cdot P(B/A) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

Since, $P(A \cap B) = P(B)\ P(A/B)$

$\therefore$ $\frac{1}{8} = P(B) \times \frac{1}{4}$

$\Rightarrow P(B) = \frac{1}{2}$

$\therefore$ $P(A) \cdot P(B) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} = P(A \cap B)$

$\therefore$ A and B are independent

64. It is based on Baye's theorem.

Probability of picked bag A, i.e., $P(A) = \frac{1}{2}$

Probability of picked bag B, i.e., $P(B) = \frac{1}{2}$

Probability of green ball picked from bag A

$= P(A).P\left(\frac{G}{A}\right) = \frac{1}{2} \times \frac{4}{7} = \frac{2}{7}$

Probability of green ball picked from bag B

$= P(B).P\left(\frac{G}{B}\right) = \frac{1}{2} \times \frac{3}{7} = \frac{3}{14}$

$\therefore$ Total probability of green ball $= \frac{2}{7} + \frac{3}{14} = \frac{1}{2}$

$\therefore$ Probability of fact that green ball is drawn from bag B

$$= \frac{P(B)P\left(\frac{G}{B}\right)}{P(A)P\left(\frac{G}{A}\right) + P(B)P\left(\frac{G}{B}\right)} = \frac{\frac{1}{2} \times \frac{3}{7}}{\frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{3}{7}} = \frac{3}{7}$$

65. Consider the following events :

$A \to$ Ball drawn is black;

$E_1 \to$ Bag I is chosen;

$E_2 \to$ Bag II is chosen and

$E_3 \to$ Bag III is chosen.

Then $P(E_1) = (E_2) = P(E_3) = \frac{1}{3}$, $P\left(\frac{A}{E_1}\right) = \frac{3}{5}$

$P\left(\frac{A}{E_2}\right) = \frac{1}{5}$, $P\left(\frac{A}{E_3}\right) = \frac{7}{10}$

$\therefore$ Required probability $= P\left(\frac{E_3}{A}\right)$

$$= \frac{P(E_3)P\left(\frac{A}{E_3}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)}$$

$$= \frac{7}{15}$$

66. Let E denote the event that a six occurs and A is the event that the man reports that it is a '6', we have

$P(E) = \frac{1}{6}$, $P(E') = \frac{5}{6}$, $P(A/E) = \frac{3}{4}$ and

$P(A/E') = \frac{1}{4}$

$\therefore$ From Baye's theorem,

$$P(E/A) = \frac{P(E).P\left(\frac{A}{E}\right)}{P(E).P\left(\frac{A}{E}\right) + P(E').P\left(\frac{A}{E'}\right)}$$

$$= \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4}} = \frac{3}{8}$$

67. We define the following events :

A_1: He knows the answer.

A_2 : He does not know the answer.

E : He gets the correct answer.

Then $P(A_1) = \frac{9}{10}$, $P(A_2) = 1 - \frac{9}{10} = \frac{1}{10}$,

$P\left(\frac{E}{A_1}\right) = 1$ and $P\left(\frac{E}{A_2}\right) = \frac{1}{4}$

$\therefore$ Required probability is

$$P\left(\frac{A_2}{E}\right) = \frac{P(A_2)P\left(\frac{E}{A_2}\right)}{P(A_1)P\left(\frac{E}{A_1}\right) + P(A_2)P\left(\frac{E}{A_2}\right)} = \frac{1}{37}$$

68. We define the following events :

A_1 : Selecting a pair of consecutive letter from the word LONDON.

A_2 : Selecting a pair of consecutive letters from the word CLIFTON.

E : Selecting a pair of letters 'ON'.

Then $P(A_1 \cap E) = \frac{2}{5}$; as there are 5 pairs of consecutive letters out of which 2 are ON.

$P(A_2 \cap E) = \frac{1}{6}$; as there are 6 pairs of consecutive letters of which one is ON.

$\therefore$ The required probability is

$$P\left(\frac{A_1}{E}\right) = \frac{P(A_1 \cap E)}{P(A_1 \cap E) + P(A_2 \cap E)} = \frac{\frac{2}{5}}{\frac{2}{5} + \frac{1}{6}}$$

$$= \frac{12}{17}$$

69. Required probability $= \frac{5}{5+3} = \frac{5}{8}$

$$\ldots\left[\begin{array}{l} \because \text{If odds in favours of an event are a : b,} \\ \text{then the probability of non} - \text{occurrence} \\ \text{of that event is } \frac{b}{a+b} \end{array}\right]$$

70. Required probability $= \frac{4}{4+5} = \frac{4}{9}$

71. Let p be the probability of the other event. Then the probability of the first event is $\frac{2}{3}$p.

$$\therefore \quad \frac{p}{p + \frac{2}{3}p} = \frac{3}{3+2}$$

$\therefore$ odds in favour of the other are 3 : 2

72. Probabilities of winning the race by three horses are $\frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$.

Hence, required probability $= \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

73. Required probability $= \frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{6}{8} = \frac{37}{56}$

74. Probability of the card being a spade or an ace $= \frac{16}{52} = \frac{4}{13}$. Hence, odds in favour is 4 : 9.

So, the odds against his winning is 9: 4

75. We have ratio of the ships A, B and C for arriving safely are 2 : 5, 3 : 7 and 6 : 11 respectively.

$\therefore$ The probability of ship A for arriving safely $= \frac{2}{2+5} = \frac{2}{7}$

Similarly, for B $= \frac{3}{3+7} = \frac{3}{10}$ and for C $= \frac{6}{6+11} = \frac{6}{17}$

$\therefore$ Probability of all the ships for arriving safely $= \frac{2}{7} \times \frac{3}{10} \times \frac{6}{17} = \frac{18}{595}$.

76. Let A and B be two given events. The odds against A are 5:2, therefore P(A) $= \frac{2}{7}$.

And the odds in favour of B are 6:5, therefore P(B) $= \frac{6}{11}$

$\therefore$ The required probability $= 1 - P(\overline{A})P(\overline{B})$

$= 1 - \left(1 - \frac{2}{7}\right)\left(1 - \frac{6}{11}\right) = \frac{52}{77}$

Competitive Thinking

1. $n(S) = 36$

$E = \{(1, 4), (4, 1), (2, 3), (3, 2)\}$

$\therefore$ $P(E) = \frac{4}{36} = \frac{1}{9}$

2. Required probability $= \frac{26}{36} = \frac{13}{18}$

3. Required probability $= \frac{4}{36} = \frac{1}{9}$

4. Total number of ways = 36

and Favourable number of cases are

$\{(1, 4), (2, 3), (3, 2), (4, 1), (1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\} = 9$

Hence, the required probability $= \frac{9}{36} = \frac{1}{4}$.

5. Required probability $= \frac{15}{36} = \frac{5}{12}$

6. Prime numbers are {2, 3, 5, 7, 11}.

Hence, required probability

$= \frac{1+2+4+6+2}{36} = \frac{15}{36} = \frac{5}{12}$

7. $n(S) = 36$

A: Event that product of numbers is even

$n(A) = 27$

$P(A) = \frac{27}{36} = \frac{3}{4}$

8.

	9	10	11	12
Ways	↓ 4	↓ 3	↓ 2	↓ 1

Hence, required probability $= \frac{10}{36} = \frac{5}{18}$

10. $n(S) = 6$

$P(T) = P(R) = \frac{1}{6}$

$\therefore$ $P(T \text{ or } R) = P(T) + P(R)$

$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

11. Total number of ways $= 2^n$

If head comes odd times, then favourable ways $= 2^{n-1}$.

$\therefore$ Required probability $= \frac{2^{n-1}}{2^n} = \frac{1}{2}$.

12. For m sided die, which is thrown n times, the probability that the number on the top is increasing is given by $\frac{{}^mC_n}{m^n}$

Here 6-faced die is thrown three times.

$\therefore$ Required probability $= \frac{{}^6C_3}{6^3} = \frac{5}{54}$

13. 3 coins are tossed

$\therefore$ S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

A: Event of getting 2 heads

$\Rightarrow$ A = {HHT, HTH, THH}

$\therefore$ $n(A) = 3 \Rightarrow P(A) = \frac{3}{8}$

14. $n(S) = 8$

$P(2 \text{ tails}) = \frac{3}{8}$

$P(3 \text{ tails}) = \frac{1}{8}$

P(at least 2 tails) = P(2 tails) + P(3 tails)

$= \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$

15. Three dice can be thrown in $6 \times 6 \times 6 = 216$ ways. A total 17 can be obtained as {(5, 6, 6), (6, 5, 6), (6, 6, 5)}. A total 18 can be obtained as (6, 6, 6).

Hence, the required probability $= \frac{4}{216} = \frac{1}{54}$

16. Required combinations are {(2, 2, 1), (1, 2, 2), (2, 1, 2), (1, 3, 1,), (3, 1, 1), (1, 1, 3)}

$\therefore$ Required probability $= \frac{6}{4^3} = \frac{6}{64} = \frac{3}{32}$

17. Required probability $= \frac{2}{10} = \frac{1}{5}$

18. $n(S) = {}^4C_2$

P(no black ball) = P(red ball)

$= \frac{{}^2C_2}{{}^4C_2} = \frac{1}{6}$

19. 3 batteries can be selected from 10 batteries in ${}^{10}C_3$ ways.

3 dead batteries can be selected from 4 dead batteries in ${}^4C_3$ ways.

$\therefore$ Probability that the all 3 selected batteries are dead $= \frac{{}^4C_3}{{}^{10}C_3} = \frac{4 \times 3 \times 2}{10 \times 9 \times 8} = \frac{1}{30}$

20. $n(S) = {}^{10}C_4$

A: Event of getting 2 red balls

$n(A) = {}^4C_2 \cdot {}^6C_2$

$\therefore$ $P(A) = \frac{{}^4C_2 \cdot {}^6C_2}{{}^{10}C_4} = \frac{9}{21}$

21. STATISTICS $\Rightarrow$ SSS TTT A II C

ASSISTANT $\Rightarrow$ SSS TT AA I N

$\therefore$ S, T, A and I are the common letters.

$\therefore$ Probability of choosing S $= \frac{{}^3C_1}{10} \times \frac{{}^3C_1}{9} = \frac{1}{10}$

Probability of choosing T $= \frac{{}^3C_1}{10} \times \frac{{}^2C_1}{9} = \frac{1}{15}$

Probability of choosing A $= \frac{{}^1C_1}{10} \times \frac{{}^2C_1}{9} = \frac{1}{45}$

Probability of choosing I $= \frac{{}^2C_1}{10} \times \frac{{}^1C_1}{9} = \frac{1}{45}$

$\therefore$ Required probability $= \frac{1}{10} + \frac{1}{15} + \frac{1}{45} + \frac{1}{45}$

$= \frac{19}{90}$

22. $n(S) = {}^{12}C_3$

P(not of same colour) = 1 − P (Same colour)

$= 1 - [P(\text{red ball}) + P(\text{black ball}) + P(\text{white ball})]$

$= 1 - \left[\frac{{}^5C_3}{{}^{12}C_3} + \frac{{}^3C_3}{{}^{12}C_3} + \frac{{}^4C_3}{{}^{12}C_3}\right]$

$= 1 - \left(\frac{60 + 6 + 24}{1320}\right)$

$= \frac{41}{44}$

23. Total rusted items = 3 + 5 = 8;

unrusted nails = 3.

$\therefore$ Required probability $= \frac{3+8}{6+10} = \frac{11}{16}$.

24. If both integers are even, then product is even.

If both integers are odd, then product is odd.

If one integer is odd and other is even, then product is even.

$\therefore$ Required probability $= \frac{2}{3}$.

25. Number which are cubes

$1^3 = 1, 2^3 = 8, 3^3 = 27, 4^3 = 64$

$\therefore$ Required probability $= \frac{4}{100} = \frac{1}{25}$

26. S = {−18, −16, −14, …., 20}

$n(S) = 20$

A : no. divisible by both 4 and 6

A = {−12, 0, 12}

$P(A) = \frac{n(A)}{n(S)} = \frac{3}{20}$

27. In a non leap year, there are 365 days which has 52 weeks and 1 day.

$\therefore$ P(53 Sundays) = $\frac{1}{7}$

28. Here, n(S) = 36
Also, n(F), where F is the set of favourable cases.
F = {(6, 1), (5, 2), (4, 3)}
where 1st number in ordered pair gives the number of black die and 2nd number gives the number on white die.

$\therefore$ required probability = $\frac{3}{36} = \frac{1}{12}$

29. Here, $n(S) = {}^{52}C_1 \times {}^{51}C_1 = 52 \times 51$
A: Event that both cards chosen are Ace.

$\therefore$ $n(A) = {}^{4}C_1 \times {}^{3}C_1 = 12$

$\therefore$ $P(A) = \frac{12}{52 \times 51} = \frac{1}{221}$

30. There are 8 even numbers from 1 to 17

$\therefore$ Probability of selecting 1 even number = $\frac{8}{17}$

Remaining number of tickets = 16
There are 7 even numbers in the remaining tickets.

$\therefore$ Probability of selecting second even number $= \frac{7}{16}$

$\therefore$ Required probability = $\frac{8}{17} \times \frac{7}{16} = \frac{7}{34}$

31. Required probability = $\dfrac{\frac{10!}{2!}}{\frac{11!}{2!2!}} = \frac{2}{11}$

32. HULULULU $\Rightarrow$ contains 4U, 3L, 1H
Consider 3L together i.e. we have to arrange 6 units which contains 4U.
Hence number of possible arrangements $= \frac{6!}{4!} = 6 \times 5 = 30$

Number of ways of arranging all letters of given word = $\frac{8!}{3!\,4!} = \frac{8 \times 7 \times 6 \times 5}{3 \times 2}$
$= 8 \times 7 \times 5$

Hence required probability = $\frac{30}{8 \times 7 \times 5}$
$= \frac{6}{8 \times 7} = \frac{3}{28}$

33. Let E be the event that the numbers are divisible by 4.

$\therefore$ E = {4, 8, 12, 16, 20, 24}

$\therefore$ n(E) = 6

$\therefore$ $n(\overline{E}) = 20$

$\therefore$ Required probability = $P(\overline{E}) = \frac{20}{26} = \frac{10}{13}$

34. P (at least 1H) = 1 – P (No head)
$= 1 - P\text{ (four tail)} = 1 - \frac{1}{16} = \frac{15}{16}$

35. Required probability is 1 – P (no die show up 1)
$= 1 - \left(\frac{5}{6}\right)^3 = \frac{216 - 125}{216} = \frac{91}{216}$

36. We have $P\left(\overline{A}\right) = 0.05 \Rightarrow P(A) = 0.95$
Hence, the probability that the event will take place in 4 consecutive occasions
$= \{P(A)\}^4 = (0.95)^4 = 0.81450625$

37. Probability that A does not solve the problem
$= 1 - \frac{1}{2} = \frac{1}{2}$
Probability that B does not solve the problem
$= 1 - \frac{1}{3} = \frac{2}{3}$
Probability that C does not solve the problem
$= 1 - \frac{1}{5} = \frac{4}{5}$
Probability that at least one of them solve problem = 1 – no one solves the problem
$= 1 - \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)$
$= 1 - \frac{4}{15} = \frac{11}{15}$

38. The probability of A, B, and C not finishing the game is, $1 - \frac{1}{2} = \frac{1}{2}$, $1 - \frac{1}{3} = \frac{2}{3}$ and $1 - \frac{1}{4} = \frac{3}{4}$ respectively.

$\therefore$ The probability that the game is not finished by any one of them = $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$

$\therefore$ The probability that the game is finished $= 1 - \frac{1}{4} = \frac{3}{4}$

39. Total balls = $5 + x$
Two balls are drawn.
$\therefore$ $n(S) = {}^{5+x}C_2$

Given, probability of red balls drawn = $\frac{5}{14}$

$\therefore$ $\frac{5}{14} = \frac{{}^5C_2}{{}^{5+x}C_2}$

$\Rightarrow \frac{5}{14} = \frac{5!}{3!2!} \times \frac{(3+x)!\,2!}{(5+x)!}$

$\Rightarrow \frac{5}{14} = \frac{20}{1} \times \frac{1}{(5+x)(4+x)}$

$\Rightarrow (5 + x)(4 + x) = \frac{20 \times 14}{5}$

$\Rightarrow (5 + x)(4 + x) = 56 \Rightarrow x = 3$

40. Number of ways in which two faulty machines may be detected (depending upon the test done to identify the faulty machines) $= {}^4C_2 = 6$
and Number of favourable cases = 1
[When faulty machines are identified in the first and the second test]

Hence, required probability = $\frac{1}{6}$.

41. Favorable number of cases = ${}^{20}C_1 = 20$

Sample space = ${}^{62}C_1 = 62$

$\therefore$ Required probability = $\frac{20}{62} = \frac{10}{31}$

42. The number of ways to arrange 7 white and 3 black balls in a row $= \frac{10!}{7!.3!} = \frac{10.9.8}{1.2.3} = 120$

Numbers of blank places between 7 balls are 6. There is 1 place before first ball and 1 place after last ball. Hence, total number of places are 8.
Hence, 3 black balls are arranged on these 8 places so that no two black balls are together in number of ways

$= {}^8C_3 = \frac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56$

So required probability = $\frac{56}{120} = \frac{7}{15}$.

43. Since, we have
$P(A + B) = P(A) + P(B) - P(AB)$
$\Rightarrow 0.7 = 0.4 + P(B) - 0.2$
$\Rightarrow P(B) = 0.5$.

44. $0.8 = 0.3 + x - 0.3x \Rightarrow x = \frac{5}{7}$.

45. Since events are mutually exclusive, therefore $P(A \cap B) = 0$ i.e., $P(A \cup B) = P(A) + P(B)$

$\Rightarrow 0.7 = 0.4 + x \Rightarrow x = \frac{3}{10}$

46. Since, $P(A + B + C)$
$= P(A) + P(B) + P(C)$
$= \frac{2}{3} + \frac{1}{4} + \frac{1}{6} = \frac{13}{12}$, which is greater than 1.
Hence, the statement is wrong.

48. If $P(A) = P(B)$
As this gives,
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
or
$P(A) = 2P(A) - P(A)$
$\Rightarrow P(A \cup B) = P(A \cap B)$

49. A: Student who know lesson I
B: Student who know lesson II
$P(A) = 0.6$, $P(B) = 0.4$, $P(A \cap B) = 0.2$
Required probability $= 1 - P(A \cup B)$
$= 1 - [P(A) + P(B) - P(A \cap B)]$
$= 1 - (0.6 + 0.4 - 0.2)$
$= 0.2$
$= \frac{1}{5}$

50. Set of even numbers that can come up on die $= \{2, 4, 6\}$
$\therefore$ Probability of it being either 2 or 4
$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

51. Here, $P(A) = \frac{3}{6} = \frac{1}{2}$, $P(B) = \frac{4}{6} = \frac{2}{3}$

and $P(A \cap B)$ = Probability of getting a number greater than 3 and less than 5

= Probability of getting 4 = $\frac{1}{6}$

$\therefore$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{2}{3} - \frac{1}{6} = 1$

52. $n(S) = {}^{10}C_3$

A: event that minimum of chosen numbers is 3

B: event that maximum of chosen number is 7.

$P(A) = \frac{{}^7C_2}{{}^{10}C_3}$, $P(B) = \frac{{}^6C_2}{{}^{10}C_3}$, $P(A \cap B) = \frac{{}^3C_1}{{}^{10}C_3}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{{}^7C_2}{{}^{10}C_3} + \frac{{}^6C_2}{{}^{10}C_3} - \frac{{}^3C_1}{{}^{10}C_3}$

$= \frac{33}{120}$

$= \frac{11}{40}$

53. Let R_1 be the event that the first ball drawn is red,
B_1 be the event that the first ball drawn is black,
R_2 be the event that the second ball drawn is red.
Required probability

$= P(R_1) . P\left(\frac{R_2}{R_1}\right) + P(B_1) . P\left(\frac{R_2}{B_1}\right)$

$= \frac{4}{10} \times \frac{6}{12} + \frac{6}{10} \times \frac{4}{12}$

$= \frac{2}{5}$

54. Given, $P(A \cup B) = 0.6$ and $P(A \cap B) = 0.2$

We know that, if A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 0.6 = 1 - P(\overline{A}) + 1 - P(\overline{B}) - 0.2$

$\Rightarrow P(\overline{A}) + P(\overline{B}) = 2 - 0.8 = 1.2$

55. Given $P(A \cup B) = \frac{3}{5}$ and $P(A \cap B) = \frac{1}{5}$

We know $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \quad \frac{3}{5} = 1 - P(\overline{A}) + 1 - P(\overline{B}) - \frac{1}{5}$

$\therefore \quad 2 - \frac{4}{5} = P(\overline{A}) + P(\overline{B})$

$\Rightarrow P(\overline{A}) + P(\overline{B}) = \frac{6}{5}$.

56. $P(A \cap B') = P(A) - P(A \cap B)$

$= \frac{4}{5} - \frac{1}{2} = \frac{3}{10}$

57. $P(A \cap B') = P(A) - P(A \cap B)$

$= 0.7 - 0.3 = 0.4 = \frac{2}{5}$

58. $P(\overline{A} \cap B) = P(B) - P(A \cap B) = y - z$.

59. $P(\overline{A} \cap \overline{B}) = P(A \cup B)'$

$= 1 - P(A \cup B)$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= 1 - 0.25 - 0.50 + 0.14 = 0.39$

60. $P(A' \cap B') = 1 - P(A \cup B)$

$\Rightarrow P(A \cup B) = \frac{2}{3}$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \frac{2}{3} = x + x - \frac{1}{3} \Rightarrow x = \frac{1}{2}$

61. Since A and B are mutually exclusive,
$P(A \cup B) = P(A) + P(B)$

$= \frac{3}{5} + \frac{1}{5} = \frac{4}{5} = 0.8$

62. Probability of getting head $= \frac{1}{2}$

Probability of die showing 3 $= \frac{1}{6}$

Since both events are independent, the required probability $= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

63. When two dice are thrown simultaneously,
$n(S) = 36$
A: Event that both the numbers on top are prime number

$\therefore$ A = {(2, 2), (2, 3), (2, 5), (3, 2), (3, 3), (3, 5), (5, 2), (5, 3), (5, 5)}

$\therefore$ $n(A) = 9$

$\therefore$ $P(A) = \frac{9}{36} = \frac{1}{4}$

When two coins are tossed simultaneously, $n(S) = 4$

B : Event that we get one head and one tail

$\therefore$ $n(B) = 2$

$\therefore$ $P(B) = \frac{2}{4} = \frac{1}{2}$

Since both the events are independent of each other,

$\therefore$ Required probabiity $= P(A) . P(B) = \frac{1}{8}$

64. $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

Since A and B are mutually exclusive, so

$P(A \cup B) = P(A) + P(B)$

Hence, required probability $= 1 - (0.5 + 0.3)$

$= 0.2.$

65. Consider option (B)

$P(A' \cap B') = [1 - P(A)]\,[1 - P(B)]$

$\Rightarrow P(A' \cap B') = P(A') \cdot P(B')$

$\therefore$ A and B are independent events.

66. P(neither A nor B) $= P\left(\overline{A} \cap \overline{B}\right)$

$= P(\overline{A}).P(\overline{B}) = 0.6 \times 0.5$

$= 0.3$

67. $P\left(A' \cap B'\right) = 1 - P\left(A \cup B\right)$

$= 1 - \left[P(A) + P(B) - P\left(A \cap B\right)\right]$

$= 1 - 1 = 0$

68. Here, $P(X) = 0.3$; $P(Y) = 0.2$

Now $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

Since, these are independent events

$\therefore$ $P(X \cap Y) = P(X).P(Y)$

Thus, required probability

$= 0.3 + 0.2 - 0.06 = 0.44$

69. Let A be the event that a man will live 10 more years.

$\therefore$ $P(A) = \frac{1}{4}$

Let B be the event that his wife will live 10 more years.

$\therefore$ $P(B) = \frac{1}{3}$

$\therefore$ Required probability $= P(A' \cap B')$

$= P(A')\,P(B')$

$= \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

70. $P(A) = \frac{3}{8}$ and $P(B) = \frac{1}{2}$

$\therefore$ $P(A)\,P(B) = \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16}$

and $P(A \cap B) = \frac{2}{8} = \frac{1}{4} \neq P(A).P(B)$

$\therefore$ A and B are dependent.

71. Since, A and B are independent events

$\therefore$ $P(A \cap B) = P(A).P(B)$

$= [1 - P(\overline{A})]\,[1 - P(\overline{B})]$

$= [1 - 2/3]\,[1 - 2/7]$

$= \frac{1}{3} \cdot \frac{5}{7} = \frac{5}{21}$

72. $P\left(A \cup \overline{B}\right) = 0.8$ and $P(B) = \frac{2}{7} \Rightarrow P\left(\overline{B}\right) = \frac{5}{7}$

$\Rightarrow P(A) + P\left(\overline{B}\right) - P\left(A \cap \overline{B}\right) = 0.8$

$\Rightarrow P(A) + \frac{5}{7} - \frac{5}{7}P(A) = 0.8$

$\Rightarrow \frac{2}{7}P(A) = \frac{3}{35} \Rightarrow P(A) = 0.3$

73. Since $\overline{E_1} \cap \overline{E_2} = \overline{E_1 \cup E_2}$

and $(E_1 \cup E_2) \cap (\overline{E_1 \cup E_2}) = \phi$

$\therefore$ $P\{(E_1 \cup E_2) \cap (\overline{E_1} \cap \overline{E_2})\} = P(\phi) = 0 < \frac{1}{4}$

74. $P\left(\overline{A \cup B}\right) = \frac{1}{6}$

$\Rightarrow 1 - P(A \cup B) = \frac{1}{6}$

$\Rightarrow P(A \cup B) = \frac{5}{6}$

$\Rightarrow P(A) + P(B) - P(A \cap B) = \frac{5}{6}$

$\Rightarrow \frac{3}{4} + P(B) - \frac{1}{4} = \frac{5}{6} \Rightarrow P(B) = \frac{1}{3}$

Clearly, $P(A \cap B) = \frac{1}{4} = \frac{3}{4} \times \frac{1}{3} = P(A)\,P(B)$

So, A and B are independent.

Also, $P(A) \neq P(B)$. So, A and B are not equally likely.

75. $P(A \cap B) = \frac{1}{6}$ and $P\left(\overline{A} \cap \overline{B}\right) = \frac{1}{3}$

$\Rightarrow P(A)\,P(B) = \frac{1}{6}$ and $P(\overline{A})P(\overline{B}) = \frac{1}{3}$

$\Rightarrow xy = \frac{1}{6}$ and $(1 - x)(1 - y) = \frac{1}{3}$,

where $P(A) = x$, $P(B) = y$

$\Rightarrow xy = \frac{1}{6}$ and $1 - x - y + \frac{1}{6} = \frac{1}{3}$

$\Rightarrow xy = \frac{1}{6}$ and $x + y = \frac{5}{6}$

$\Rightarrow x = \frac{1}{2}$ and $y = \frac{1}{3}$ or $x = \frac{1}{3}$ and $y = \frac{1}{2}$

76. Let $A_i(i = 1, 2)$ denote the event that i^{th} plane hits the target.
Clearly, A_1 and A_2 are independent events.
Required probability $= P(\overline{A_1} \cap A_2)$
$= P(\overline{A_1})P(A_2)$
$= (1 - 0.3)(0.2) = 0.14$

77. Total number of defective items $= \frac{2}{100} \times 2500 + \frac{3}{100} \times 3500 + \frac{5}{100} \times 4000$
$= 355$
Number of defective items from machine C $= \frac{5}{100} \times 4000 = 200$
$\therefore$ Required probability $= \frac{200}{355} = \frac{40}{71}$

78. $P[(A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$
$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$
$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$

79. $P(B \cap C) = P(B) - \left[P\left(A \cap B \cap \overline{C}\right) + P\left(\overline{A} \cap B \cap \overline{C}\right)\right]$
$= \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$

80. Let A_1 – student passes in Test - I
A_2 – student passes in Test - II
A_3 – student passes in Test – III
A – student is successful
$A - (A_1 \cap A_2 \cap A'_3) \cup (A_1 \cap A'_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3)$
$\therefore$ $P(A) = P(A_1) \,.\, P(A_2) \,.\, P(A'_3) + P(A_1) \,.\, P(A'_2) \,.\, P(A_3) + P(A_1) \,.\, P(A_2) \,.\, P(A_3)$
$\therefore$ $\frac{1}{2} = p \,.\, q. \frac{1}{2} + p \,.\, (1 - q) \,.\, \frac{1}{2} + p \,.\, q \,.\, \frac{1}{2}$
$\therefore$ $p + pq = 1 \quad \Rightarrow p\,(1 + q) = 1$

81. Probability of first card to be a king $= \frac{4}{52}$
and probability of also second to be a king $= \frac{3}{51}$
Hence, required probability $= \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$.

82. Required probability = P(Diamond).P(king)
$= \frac{13}{52} \cdot \frac{4}{52} = \frac{1}{52}$

83. Second white ball can draw in two ways.
i. First is white and second is white
Probability $= \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$
ii. First is black and second is white
Probability $= \frac{3}{7} \times \frac{4}{6} = \frac{2}{7}$
Hence, required probability $= \frac{2}{7} + \frac{2}{7} = \frac{4}{7}$.

84. The sample space is [LWW, WLW]
$\therefore$ P(LWW) + P(WLW)
= Probability that in 5 match series, it is India's second win
= P(L)P(W)P(W) + P(W)P(L)P(W)
$= \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$

85. Here, $P(A) = \frac{3}{4}$, $P(B) = \frac{4}{5}$
$\therefore$ $P(\overline{A}) = \frac{1}{4}$ and $P(\overline{B}) = \frac{1}{5}$
$\therefore$ Required probability
$= P(A).P(\overline{B}) + P(\overline{A}).P(B) = \frac{7}{20}$.

86. $P(A) = \frac{4}{5}$, $P(A') = \frac{1}{5}$
$P(B) = \frac{3}{5}$, $P(B') = \frac{2}{5}$
$\therefore$ P(both are false) $= P(A') \,.\, P(B')$
$= \frac{1}{5} \cdot \frac{2}{5}$
$= \frac{2}{25}$
$\therefore$ P (atleast one of them is true)
= 1 – P (both are false)
$= 1 - \frac{2}{25} = \frac{23}{25}$

87. Consider the following events:
A = 'X' speaks truth, B = 'Y' speaks truth.
Then, $P(A) = \frac{60}{100} = \frac{3}{5}$ and $P(B) = \frac{50}{100} = \frac{1}{2}$

Required probability = $P\left((A\cap\overline{B})\cup(\overline{A}\cap B)\right)$

$= P(A\cap\overline{B}) + P(\overline{A}\cap B)$

$= \frac{3}{5}\times\frac{1}{2} + \frac{2}{5}\times\frac{1}{2} = \frac{1}{2}$

88. Consider the following events:
X = 'A' speaks truth, Y = 'B' speaks truth

Then, $P(X) = \frac{70}{100} = \frac{7}{10}$ and $P(Y) = \frac{80}{100} = \frac{4}{5}$

Required probability $= P[(X\cap\overline{Y})\cup(\overline{X}\cap Y)]$

$= \frac{7}{10}\times\frac{1}{5}+\frac{3}{10}\times\frac{4}{5}$

$= \frac{19}{50} = 0.38$

89. Consider the following events:
A = family who owns a car,
B = family who owns a house
Required probability = $P(A\cup B) - P(A\cap B)$

$= \frac{60+30-20}{100} - \frac{20}{100} = \frac{70-20}{100} = 0.5$

90. The probability of husband is not selected

$= 1 - \frac{1}{7} = \frac{6}{7}$

The probability that wife is not selected

$= 1 - \frac{1}{5} = \frac{4}{5}$

The probability that only husband selected

$= \frac{1}{7}\times\frac{4}{5} = \frac{4}{35}$

The probability that only wife selected

$= \frac{1}{5}\times\frac{6}{7} = \frac{6}{35}$

Hence, required probability $= \frac{6}{35} + \frac{4}{35} = \frac{10}{35}$

$= \frac{2}{7}$

91. The probability of students not solving the problem are $1-\frac{1}{3}=\frac{2}{3}$, $1-\frac{1}{4}=\frac{3}{4}$ and $1-\frac{1}{5}=\frac{4}{5}$

Therefore, the probability that the problem is not solved by any one of them $= \frac{2}{3}\times\frac{3}{4}\times\frac{4}{5} = \frac{2}{5}$

Hence, the probability that problem is solved $= 1-\frac{2}{5}=\frac{3}{5}$.

92. i. This question can also be solved by one student

ii. This question can be solved by two students simultaneously

ii. This question can be solved by three students all together.

We have, $P(A)= \frac{1}{2}$, $P(B)= \frac{1}{4}$, $P(C)= \frac{1}{6}$

$\therefore$ $P(A\cup B\cup C) = P(A) + P(B) + P(C)$
$-[P(A).P(B) + P(B).P(C) + P(C).P(A)] +$
$[P(A).P(B).P(C)]$

$= \frac{1}{2}+\frac{1}{4}+\frac{1}{6} - \left[\frac{1}{2}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{6}+\frac{1}{6}\times\frac{1}{2}\right] + \left[\frac{1}{2}\times\frac{1}{4}\times\frac{1}{6}\right]$

$= \frac{33}{48}$

93. $P\left(\frac{B}{A}\right) = \frac{P(A\cap B)}{P(A)} = \frac{\left(\frac{1}{10}\right)}{\left(\frac{1}{4}\right)} = \frac{2}{5}$.

94. For S and T as independent events,

$P(S/T) = P(S)$. Thus, $P(S/T) = 0.3$.

95. $P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{7}{10}}{\frac{17}{20}} = \frac{7}{10}\times\frac{20}{17} = \frac{14}{17}$

96. $P(A\cap B) = P(A)\,P(B/A)$

$\therefore$ $P(A\cap B) = \frac{1}{4}\times\frac{2}{3} = \frac{1}{6}$

Now, $P(A/B) = \frac{P(A\cap B)}{P(B)}$

$\Rightarrow \frac{1}{2} = \frac{1}{6}\times\frac{1}{P(B)}$

$\Rightarrow P(B) = \frac{1}{3}$

97. $P(B / (A \cup B^c)] = \dfrac{P(B \cap (A \cup B^c))}{P(A \cup B^c)}$

$= \dfrac{P(A \cap B)}{P(A) + P(B^c) - P(A \cap B^c)}$

$= \dfrac{P(A) - P(A \cap B^c)}{P(A) + P(B^c) - P(A \cap B^c)}$

$= \dfrac{0.7 - 0.5}{0.8} = \dfrac{1}{4}$

98. S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

$\therefore$ $n(E) = 4$, $n(F) = 4$ and $n(E \cap F) = 3$

$\therefore$ $P\left(\dfrac{E}{F}\right) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{\frac{3}{8}}{\frac{4}{8}} = \dfrac{3}{4}$

99. Event that at least one of them is a boy $\rightarrow$ A,
Event that other is girl $\rightarrow$ B,
So, required probability

$P(B/A) = \dfrac{P(B \cap A)}{P(A)}$

Now, total cases are 3 (BG, BB, GG)

$\therefore$ $\dfrac{P(B \cap A)}{P(A)} = \dfrac{\frac{1}{3}}{\frac{2}{3}} = \dfrac{1}{2}$

....[$\because$ $B \cap A = \{BG\}$ and $A = \{BG, BB\}$]

100. Consider the following events:
A = Sum of the digits on the selected tickets is 8.
B = Product of the digits on the selected ticket is zero.
There are 14 tickets having product of digits appearing on them as zero. The numbers on such tickets are 00, 01, 02, 03, 04, 05, 06, 07, 08, 09, 10, 20, 30, 40.

$\therefore$ $P(B) = \dfrac{14}{50}$ and $P(A \cap B) = \dfrac{1}{50}$

$\therefore$ Required probability $= P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$= \dfrac{1}{14}$

101. M: student studying maths
S: student studying science

$\therefore$ $P(M \cap S) = 40\% = 0.4$
$P(M) = 60\% = 0.6$
Probability of student studying science given the student is already studying maths
$= P(S/M) = P(M \cap S) / P(M)$

$= \dfrac{0.4}{0.6} = \dfrac{2}{3}$

102. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cap B)$

$\{\because P(A \cap B) = P(A \cup B)\}$

$\Rightarrow 2\,P(A \cap B) = P(A) + P(B)$

$\Rightarrow 2\,P(A). \dfrac{P(A \cap B)}{P(A)} = P(A) + P(B)$

$\Rightarrow 2\,P(A). P\left(\dfrac{B}{A}\right) = P(A) + P(B)$

103. We know that $P(A / B) = \dfrac{P(A \cap B)}{P(B)}$

Also we know that $P(A \cup B) \leq 1$

$\Rightarrow P(A) + P(B) - P(A \cap B) \leq 1$

$\Rightarrow P(A \cap B) \geq P(A) + P(B) - 1$

$\Rightarrow \dfrac{P(A \cap B)}{P(B)} \geq \dfrac{P(A) + P(B) - 1}{P(B)}$

$\Rightarrow P(A / B) \geq \dfrac{P(A) + P(B) - 1}{P(B)}$

104. $P(E \cap F) = P(E).P(F)$

Now, $P(E \cap F^c) = P(E) - P(E \cap F)$
$= P(E)[1 - P(F)]$
$= P(E).P(F^c)$

and $P(E^c \cap F^c) = 1 - P(E \cup F)$
$= 1 - [P(E) + P(F) - P(E \cap F)$
$= [1 - P(E)][1 - P(F)] = P(E^c)P(F^c)$

Also $P\left(\dfrac{E}{F}\right) = P(E)$ and $P\left(\dfrac{E^c}{F^c}\right) = P(E^c)$

$\Rightarrow P\left(\dfrac{E}{F}\right) + P\left(\dfrac{E^c}{F^c}\right) = 1.$

105. $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} \Rightarrow \frac{1}{2} = \frac{P(A \cap B)}{1/4}$

$\Rightarrow P(A \cap B) = \frac{1}{8}$

Hence, events A and B are not mutually exclusive.

$\therefore$ Statement II is incorrect.

Now, $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(B) = \frac{1}{2}$

$\ldots.\left[\because\ P(A \cap B) = \frac{1}{8} = P(A).P(B)\right]$

$\therefore$ events A and B are independent events.

$\therefore$ $P\left(\frac{A^c}{B^c}\right) = \frac{P(A^c \cap B^c)}{P(B^c)} = \frac{P(A^c)P(B^c)}{P(B^c)}$

$= \frac{3}{4}.\frac{1}{2}.\frac{2}{1} = \frac{3}{4}$

Hence, statement I is correct.

Again $P\left(\frac{A}{B}\right) + P\left(\frac{A}{B^c}\right) = \frac{1}{4} + \frac{P(A \cap B^c)}{P(B^c)}$

$= \frac{1}{4} + \frac{P(A) - P(A \cap B)}{P(B^c)}$

$= \frac{1}{4} + \frac{\frac{1}{4} - \frac{1}{8}}{\frac{1}{2}}$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Hence, statement III is incorrect.

106. Consider the following events:
S = person is smoker,
NS = person is non smoker,
D = death due to lung cancer

$P(D) = P(S) \cdot P\left(\frac{D}{S}\right) + P(NS) \cdot P\left(\frac{D}{NS}\right)$

$\Rightarrow 0.006 = \frac{20}{100} \times P\left(\frac{D}{S}\right) + \frac{80}{100} \times \frac{1}{10} \times P\left(\frac{D}{S}\right)$

$\Rightarrow P\left(\frac{D}{S}\right) = \frac{1000 \times 0.006}{280} = \frac{6}{280} = \frac{3}{140}$

107. Let E denote the event that a five occurs and A be the event that the man reports it as '6'.

Then, $P(E) = \frac{1}{6}$, $P(E') = \frac{5}{6}$

$P(A/E) = \frac{2}{3}$, $P(A/E') = \frac{1}{3}$

From Baye's theorem,

$P(E/A) = \frac{P(E) \cdot P(A/E)}{P(E) \cdot P(A/E) + P(E') \cdot P(A/E')}$

$= \frac{\frac{1}{6} \times \frac{2}{3}}{\frac{1}{6} \times \frac{2}{3} + \frac{5}{6} \times \frac{1}{3}}$

$= \frac{2}{7}$

108. Let E_1 be the event that the ball is drawn from bag A, E_2 the event that it is drawn from bag B and E that the ball is red. We have to find $P(E_2/E)$.

Since both the bags are equally likely to be selected, we have $P(E_1) = P(E_2) = \frac{1}{2}$

Also $P(E/E_1) = \frac{3}{5}$, $P(E/E_2) = \frac{5}{9}$

Hence by Baye's theorem, we have

$P(E_2/E) = \frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2)}$

$= \frac{\frac{1}{2}.\frac{5}{9}}{\frac{1}{2}.\frac{3}{5} + \frac{1}{2}.\frac{5}{9}} = \frac{25}{52}$

109. Let A be the event of selecting bag X, B be the event of selecting bag Y and E be the event of drawing a white ball, the P(A) = 1/2, P(B) = 1/2, P(E/A) = 2/5, P(E/B) = 4/6 = 2/3

$\therefore$ P(E) = P(A) P(E/A) + P(B)P(E/B)

$= \frac{1}{2} \cdot \frac{2}{5} + \frac{1}{2} \cdot \frac{2}{3} = \frac{8}{15}$

110. K = He knows the answers, NK = He randomly ticks the answers, C = He is correct

$$P\left(\frac{K}{C}\right) = \frac{P(K).P\left(\frac{C}{K}\right)}{P(K).P\left(\frac{C}{K}\right) + P(NK).P\left(\frac{C}{NK}\right)}$$

$$= \frac{p \times 1}{p \times 1 + (1-p) \times \frac{1}{5}} = \frac{5p}{4p+1}$$

111. Consider the following events:

$E_1 \rightarrow$ He knows the answer, $E_2 \rightarrow$ He guesses the answer

$A \rightarrow$ He gets the correct answer.

We have,

$P(E_1) = \frac{90}{100} = \frac{9}{10}$, $P(E_2) = \frac{1}{10}$,

$P(A/E_1) = 1$, $P(A/E_2) = \frac{1}{4}$

$\therefore$ Required probability = $P(E_2/A)$

$$= \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{10} \times \frac{1}{4}}{\frac{9}{10} \times 1 + \frac{1}{10} \times \frac{1}{4}} = \frac{1}{37}$$

112. Required probability

$$= \frac{\frac{1}{7} \times \frac{7}{9}}{\frac{1}{7} \times \frac{7}{9} + \frac{3}{7} \times \frac{8}{9} + \frac{2}{7} \times \frac{5}{9} + \frac{1}{7} \times \frac{8}{9}} = \frac{1}{7}$$

113. Required probability $= \frac{(21)!2!}{(22)!} = \frac{1}{11} = \frac{1}{1+10}$

$\therefore$ Odds against = 10 : 1.

114. Required probability $= \frac{0.1}{0.1 + 0.32}$

$$= \frac{0.1}{0.42} = \frac{5}{21}$$

115. Probability [Person A will die in 30 years]

$$= \frac{8}{8+5}$$

$\therefore$ $P(A) = \frac{8}{13} \Rightarrow P(\bar{A}) = \frac{5}{13}$

Similarly, $P(B) = \frac{4}{7} \Rightarrow P(\bar{B}) = \frac{3}{7}$

There are two ways in which one person is alive after 30 years. $\bar{A}B$ and $A\bar{B}$ are independent events.

So, required probability

$= P(\bar{A}).P(B) + P(A).P(\bar{B})$

$= \frac{5}{13} \times \frac{4}{7} + \frac{8}{13} \times \frac{3}{7} = \frac{44}{91}$

116. The probability of solving the question by these three students are $\frac{1}{3}, \frac{2}{7}$ and $\frac{3}{8}$ respectively.

$\therefore$ $P(A) = \frac{1}{3}$; $P(B) = \frac{2}{7}$; $P(C) = \frac{3}{8}$

Then, probability of question solved by only one student = $P(A\bar{B}\bar{C}$ or $\bar{A}B\bar{C}$ or $\bar{A}\bar{B}C)$

$= P(A)\, P(\bar{B}) P(\bar{C}) + P(\bar{A}) P(B)\, P(\bar{C})$

$+ P(\bar{A})\, P(\bar{B}) P(C)$

$$= \frac{1}{3}.\frac{5}{7}.\frac{5}{8} + \frac{2}{3}.\frac{2}{7}.\frac{5}{8} + \frac{2}{3}.\frac{5}{7}.\frac{3}{8}$$

$$= \frac{25+20+30}{168} = \frac{25}{56}$$

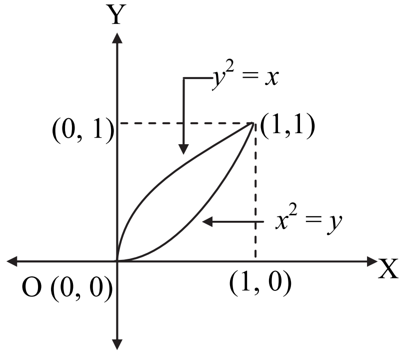
117. The quadratic equation $ax^2 + bx + c = 0$ has real roots when, $\Delta = b^2 - 4ac \geq 0$

Since a, b, c are chosen from the numbers 2, 3, 5.

6 different equations having distinct coefficients can be formed. Of these, only two equations having b = 5 will have real roots.

$\therefore$ Required probability $= \frac{2}{6} = \frac{1}{3}$

118.



A is an event of (x, y)which satisfies $y^2 \le x$

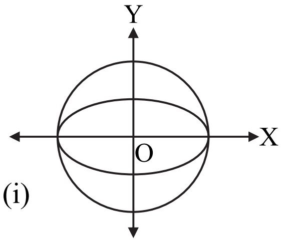
$\therefore$ $P(A) = \int_0^1 y\,dx = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}$

B is an event of (x, y) which satisfies $x^2 \le y$

$P(B) = \int_0^1 y\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$

$\therefore$ $P(A \cap B) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

119. Let, $\frac{x}{a^2} + \frac{y^2}{b^2} = 1$ be the given ellipse with eccentricity, $e = \frac{2\sqrt{2}}{3}$



$\therefore$ $e^2 = 1 - \frac{b^2}{a^2}$

$\therefore$ $b^2 = a^2 (1 - e^2)$ …(i)

Area of ellipse = πab

$= \pi a. \sqrt{a^2(1 - e^2)}$ …[From (i)]

$= \pi a^2 \sqrt{1 - e^2}$

$= \pi a^2 \sqrt{1 - \frac{8}{9}} = \frac{\pi a^2}{3}$

Also, radius of the circle = a

$\therefore$ Area of circle = πa^2

$\therefore$ Probability that point inside the circle lies outside the ellipse $= 1 - \frac{\frac{\pi a^2}{3}}{\pi a^2} = 1 - \frac{1}{3} = \frac{2}{3}$

Evaluation Test

1. Out of 30 numbers from 1 to 30, three numbers can be chosen in ${}^{30}C_3$ ways.

So, total number of elementary events = ${}^{30}C_3$.

Three consecutive numbers can be chosen in one of the following ways:

(1, 2, 3), (2, 3, 4),….,(28, 29, 30).

$\therefore$ Number of elementary events in which three numbers are consecutive is 28.

$\therefore$ Probability that the numbers are consecutive $= \frac{28}{{}^{30}C_3} = \frac{1}{145}$

$\therefore$ required probability $= 1 - \frac{1}{145} = \frac{144}{145}$

2. We have,

$P(E) + P(F) - 2P(E \cap F) = \frac{11}{25}$ and

$P(\overline{E} \cap \overline{F}) = \frac{2}{25}$

$\Rightarrow P(E) + P(F) - 2P(E)P(F) = \frac{11}{25}$ and

$P(\overline{E})P(\overline{F}) = \frac{2}{25}$

$\Rightarrow x + y - 2xy = \frac{11}{25}$ and $1 - x - y + xy = \frac{2}{25}$,

Where, P(E) = x and P(F) = y

$\Rightarrow x + y + 2 - 2x - 2y = \frac{11}{25} + 2 \times \frac{2}{25}$

....[On eliminating xy]

$\Rightarrow x + y = \frac{7}{5} \Rightarrow y = \frac{7}{5} - x$

Substituting $y = \frac{7}{5} - x$ in $1 - x - y + xy = \frac{2}{25}$,

we get

$1 - \frac{7}{5} + x\left(\frac{7}{5} - x\right) = \frac{2}{25}$

$\Rightarrow 25x^2 - 35x + 12 = 0$

$\Rightarrow x = \frac{3}{5}, \frac{4}{5}$

When $x = \frac{3}{5}, y = \frac{4}{5}$ and $y = \frac{3}{5}$ for $x = \frac{4}{5}$

Hence, $P(E) = \frac{3}{5}$, $P(F) = \frac{4}{5}$ or $P(E) = \frac{4}{5}$, $P(F) = \frac{3}{5}$

3. Let A denote the event that each American man is seated adjacent to his wife and B denote the event that Indian man is seated adjacent to his wife. Then,
required probability = P(B/A)

$$= \frac{\text{Number of ways in which Indian man sits adjacent to his wife when each man is sited adjacents to his wife}}{\text{Number of ways in which each American man is seated adjacent to his wife}}$$

$$= \frac{(2!)^5 \times (5-1)!}{(2!)^4\,(6-1)!} = \frac{2}{5}$$

4. We have 13 denominations Ace, 2, 3, 4, …., 10, J, Q, K. For selecting exactly one pair, we select first any 3 denominations, 2 cards from 1 and one each from the other two
Thus, favourable ways = ${}^{13}C_3 . 3 . {}^4C_2 . {}^4C_1 . {}^4C_1$
Total ways = ${}^{52}C_4$

∴ required probability = $\frac{13.12.11.3.6.4.4.24}{6.52.51.50.49}$

$$= \frac{6336}{20825} = 0.3042 = 0.3$$

5. Let event A that minimum of the chosen number is 3 and B be the event that maximum of the chosen number is

∴ P(A) = P (choosing 3 and two other numbers from 4 to 10)

$$= \frac{{}^7C_2}{{}^{10}C_3} = \frac{7\times6\times3}{10\times9\times8} = \frac{7}{40}$$

P(B) = P(choosing 7 and choosing two other numbers from 1 to 6)

$$= \frac{{}^6C_2}{{}^{10}C_3} = \frac{6\times5\times3}{10\times9\times7} = \frac{1}{8}$$

$P(A \cap B)$ = P (choosing 3 and 7 and one other from 4 to 6)

$$= \frac{3}{{}^{10}C_3} = \frac{3\times3\times2}{10\times9\times8} = \frac{1}{40}$$

∴ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{7}{40} + \frac{1}{8} - \frac{1}{40} = \frac{11}{40}$$

6. In the 22nd century there are 25 leap years viz. 2100, 2104, …., 2196 and 75 non-leap years.
Consider the following events:
E_1 = Selecting a leap year from 22nd century
E_2 = Selecting a non-leap year from 22nd century
A = There are 53 Sundays in a year of 22nd century
We have,

$P(E_1) = \frac{25}{100}$, $P(E_2) = \frac{75}{100}$

$P(A/E_1) = \frac{2}{7}$ and $P(A/E_2) = \frac{1}{7}$

∴ Required probability = P(A)

$$= P\big((A\cap E_1)\cup(A\cap E_2)\big)$$
$$= P(A \cap E_1) + P(A \cap E_2)$$
$$= P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$$
$$= \frac{25}{100}\times\frac{2}{7} + \frac{75}{100}\times\frac{1}{7}$$
$$= \frac{5}{28}$$

7. We know that the probability of occurrence of an event is always less than or equal to 1 and it is given that $P(A \cup B \cup C) \geq 0.75$

∴ $0.75 \leq P(A \cup B \cup C) \leq 1$

$\Rightarrow 0.75 \leq P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \leq 1$

$\Rightarrow 0.75 \leq 0.3 + 0.4 + 0.8 - 0.08 - P(B \cap C) - 0.28 + 0.09 \leq 1$

$\Rightarrow 0.75 \leq 1.23 - P(B \cap C) \leq 1$

$\Rightarrow -0.48 \leq -P(B \cap C) \leq -0.23$

$\Rightarrow 0.23 \leq P(B \cap C) \leq 0.48$

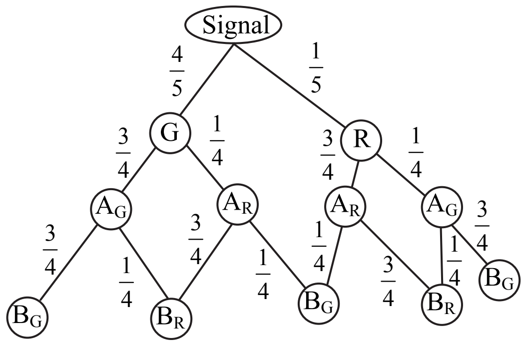
8. From the tree diagram,
$P(B_G)$

$$= \frac{4}{5}\times\frac{3}{4}\times\frac{3}{4} + \frac{4}{5}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{5}\times\frac{3}{4}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4}\times\frac{3}{4}$$
$$= \frac{23}{40}$$

$$P\left(\frac{B_G}{G}\right) = \frac{3}{4}\times\frac{3}{4}+\frac{1}{4}\times\frac{1}{4} = \frac{5}{8}$$

$\therefore \quad P(B_G \cap G) = P(G)\,P\left(\frac{B_G}{G}\right)$

$$\Rightarrow P(B_G \cap G) = \frac{4}{5}\times\frac{5}{8} = \frac{1}{2}$$



$\therefore \quad$ Required probability $= P\left(\frac{G}{B_G}\right) = \frac{P(B_G \cap G)}{P(B_G)}$

$$= \frac{\frac{1}{2}}{\frac{23}{40}} = \frac{20}{23}$$

Textbook Chapter No.

01 Mathematical Logic

Hints

Classical Thinking

1. 'Bombay is the capital of India' is a statement. The other options are exclamatory and interrogative sentences.

2. 'Two plus two is four' is a statement. The other options are imperative sentences.

3. Even though $2 = 3$ is false, it is a statement in logic with truth value F.

5. $\sim$q: Ram studies on holiday, 'and' is expressed by '$\wedge$' symbol
$\therefore$ Symbolic form is $p \wedge \sim q$.

6. p: There are clouds in the sky, $\sim$q: It is not raining, 'and' is expressed by '$\wedge$' symbol.
$\therefore$ $p \wedge \sim q$

7. $\sim$p: The sun has not set, $\sim$q: The moon has not risen, 'or' is expressed by '$\vee$' symbol.
$\therefore$ $\sim p \vee \sim q$

8. $\sim$p: Rohit is short, 'or' is expressed by '$\vee$' symbol, 'and' is expressed by '$\wedge$' symbol.

9. p: Candidates are present,
q: Voters are ready to vote
r: Ballot papers $\Rightarrow \sim r$: no Ballot papers
'and' and 'but' are represented by '$\wedge$' symbol.

10. $\sim$p: She is not beautiful, '$\vee$' indicates 'or'.

11. $\sim$p: Ram is not lazy, $\sim$q: Ram does not fail in the examination, '$\vee$' indicates 'or'.

15. "Implies" is expressed as '$\rightarrow$'.
$\therefore$ symbolic form is $p \rightarrow q$

16. ($\sim$d: Driver is not drunk) implies ($\sim$a: He cannot meet with an accident).

17. "if and only if" is expressed as '$\leftrightarrow$'
$\therefore$ symbolic form is $a \leftrightarrow b$.

19. p: A, B,C, are distinct points
q: Points are collinear
r: Points form a triangle
$\therefore$ p implies (q or r) i.e. $p \rightarrow (q \vee r)$

20. '$m \rightarrow n$' means 'If m then n',
$\therefore$ option (C) is correct.

23. Let p : x^2 is not even,
q : x is not even
Converse of $p \rightarrow q$ is $q \rightarrow p$
i.e., If x is not even then x^2 is not even

24. Converse of $p \rightarrow q$ is $q \rightarrow p$.

25. Let p : $x > y$
q : $x + a > y + a$
Converse of $p \rightarrow q$ is $q \rightarrow p$
i.e., If $x + a > y + a$, then $x > y$

26. Let p: You access the internet
q: You have to pay the charges
Given statement is written symbolically as, $p \rightarrow q$
Inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$
i.e. If you do not access the internet then you do not have to pay the charges.

27. Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

28. $\sim$p: Sita does not get promotion and '$\leftrightarrow$' symbol indicates 'if and only if'.

33. r: It is raining, c: I will go to college.
The given statement is $r \rightarrow c \equiv \sim c \rightarrow \sim r$

36.

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

37.

p	q	$\sim q$	$p \wedge q$	$p \rightarrow \sim q$	$(p \wedge q) \wedge (p \rightarrow \sim q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F

38.

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$	$p \rightarrow \sim(p \wedge \sim q)$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

39.

p	q	$p \to q$	~p	~q	$\sim p \to \sim q$	$(p \to q) \leftrightarrow (\sim p \to \sim q)$
T	T	T	F	F	T	T
T	F	F	F	T	T	F
F	T	T	T	F	F	F
F	F	T	T	T	T	T

40. Option (C) is a true statement, since, $x = 3 \in N$ satisfies $x + 5 = 8$.

41. Option (D) is the required true statement since $x = 6 \in W$ satisfies $x^2 - 4 = 32$

43. p: Manoj has the job, q: he is not happy
Symbolic form is $p \wedge q$.
Its dual is $p \vee q$.
$\therefore$ Manoj has the job or he is not happy.

44. $\sim(p \wedge q) \equiv \sim p \vee \sim q$

45. $\sim[p \vee (\sim q)] \equiv \sim p \wedge \sim(\sim q) \equiv \sim p \wedge q$

46. p : I like Mathematics
q : I like English.
$\sim (p \wedge q) \equiv \sim p \vee \sim q$
$\therefore$ Option (D) is correct.

47. We know that,
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$\therefore$ $\sim(p \leftrightarrow q) \equiv \sim[(p \to q) \wedge (q \to p)]$
$\equiv \sim (p \to q) \vee \sim(q \to p)$
….[By Demorgan's Law]
$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$
….[$\because \sim(p \to q) = p \wedge \sim q$]

48. p : It is Sunday
q : It is a holiday
$\therefore$ Symbolic form $p \to q$
$\sim (p \to q) \equiv p \wedge \sim q$
i.e. It is Sunday, but it is not a holiday

49. Given statement is '$\forall x \in N, x + 5 > 4$'
$\therefore$ $\sim [\forall x \in N, , x + 5 > 4]$
$\equiv \exists x \in N$, such that $x + 5 \le 4$
i.e., there exists a natural number x, for which $x + 5 \le 4$

51. Current will flow in the circuit if switch p and q are closed or switch r is closed.
It is represented by
$(p \wedge q) \vee r$.
$\therefore$ option (A) is correct.

Critical Thinking

1. 'Incorrect statement' means a statement in logic with truth value false.
Options (A) and (C) are not statements in logic.
Option (D) has truth value True.
Option (B) is a statement in logic with truth value false.

2. p: One being lucky,
q: One should stop working
$\therefore$ Symbolic form: $(p \vee \sim p) \wedge \sim q$

3. p: Physics is interesting.
q: Physics is difficult.
$\therefore$ Symbolic form: $\sim (\sim p \vee q)$

4. p: Intelligent persons are polite.
q: Intelligent persons are helpful.
$\therefore$ Symbolic form: $\sim (\sim p \wedge \sim q)$

5. $\sim p \wedge (q \vee \sim r)$ and $(p \to q) \wedge r$
$\therefore$ $\sim T \wedge (T \vee \sim F)$ and $(T \to T) \wedge F$
$\Rightarrow F \wedge (T \vee T)$ and $(T \wedge F)$
$\Rightarrow F \wedge T$ and $T \wedge F \Rightarrow F$ and F

6. $(\sim p \vee q) \leftrightarrow \sim(p \wedge q)$ and $\sim p \leftrightarrow (p \to \sim q)$
$\therefore$ $(\sim F \vee F) \leftrightarrow \sim(F \wedge F)$ and $\sim F \leftrightarrow (F \to \sim F)$
$\Rightarrow (T \vee F) \leftrightarrow \sim F$ and $T \leftrightarrow (F \to T)$
$\Rightarrow T \leftrightarrow T$ and $T \leftrightarrow T$
$\Rightarrow$ T and T

7. $(p \to q) \leftrightarrow (\sim q \to \sim p)$ and $(\sim p \vee q) \wedge (\sim q \vee p)$
$\therefore$ $(T \to F) \leftrightarrow (\sim F \to \sim T)$ and $(\sim T \vee F) \wedge (\sim F \vee T)$
$\Rightarrow F \leftrightarrow (T \to F)$ and $(F \vee F) \wedge (T \vee T)$
$\Rightarrow F \leftrightarrow F$ and $F \wedge T \Rightarrow$ T and F

8. $p \wedge q \equiv F \wedge T \equiv F$
$p \vee \sim q \equiv F \vee \sim T \equiv F \vee F \equiv F$
$q \to p \equiv T \to F \equiv F$
$p \to q \equiv F \to T \equiv T$

9. $\sim p \to \sim q \equiv \sim F \to \sim T \equiv T \to F \equiv F$
$p \to (q \wedge p) \equiv F \to (T \wedge F) \equiv F \to F \equiv T$
$p \to \sim q \equiv F \to \sim T \equiv F \to F \equiv T$
$q \to \sim p \equiv T \to \sim F \equiv T \to T \equiv T$

10. Consider option (C)
$(p \vee q) \wedge (p \vee r) \equiv (T \vee T) \wedge (T \vee F)$
$\equiv T \wedge T$
$\equiv T$
$\therefore$ option (C) is correct.

11.

p	q	~q	~q ∨ p	p ↔ q	p → q	q → p
T	T	F	T	T	T	T
T	F	T	T	F	F	T
F	T	F	F	F	T	F
F	F	T	T	T	T	T

Alternate Method:

~ q ∨ p: F

∴ ~ q is F, p is F

i.e., q is T, p is F

∴ p → q ≡ F → T ≡ T

12. p: Seema solves a problem

q: She is happy

i. p → q ii. ~p → ~q

iii. ~q → ~p iv. q → p

(i) and (iii) have the same meaning,

(ii) and (iv) have the same meaning.

13. i. b → r

ii. ~b → ~r

iii. r → b

iv. ~r → ~b

(i) and (iv) are the same and (ii) and (iii) are the same.

14. p ∧ (p → q)

≡ p ∧ (~p ∨ q) ….[Conditional law]

≡ (p ∧ ~p) ∨ (p ∧ q) ….[Distributive law]

≡ F ∨ (p ∧ q) ….[Complement law]

≡ p ∧ q ….[Identity law]

15. ~[p → (p ∨ ~q)] ≡ ~[~p ∨ (p ∨ ~q)]

....[∵ p → q ≡ ~p ∨ q]

≡ p ∧ ~[p ∨ (~q)]

≡ p ∧ [~p ∧ ~(~q)]

≡ p ∧ (~p ∧ q)

16. (~q) → (~p) is contrapositive of p → q and hence both are logically equivalent of each other.

17.

p	~p	~(~p)	~(~p) ↔ p
T	F	T	T
F	T	F	T

All the entries in the last column of the above truth table is T.

∴ ~(~p) ↔ p is a tautology.

18.

p	q	r	~ p	~ q	~p ∧~q	q ∧ r	(~p ∧ ~q) ∧ (q ∧ r)
T	T	T	F	F	F	T	F
T	T	F	F	F	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	T	F	F	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	F
F	F	T	T	T	T	F	F
F	F	F	T	T	T	F	F

∴ Given statement is contradiction.

19. Consider option (C)

p	q	~q	p→q	~(p→q)	p∧~q	~(p→q)↔(p∧~q)
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

∴ option (C) is correct.

20. consider option (B)

P	q	~ q	p ∧ ~q	p → q	(p ∧ ~ q) ∧ (p → q)
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F

∴ option (B) is correct.

21. Consider option (B)

p	q	~p	~q	p∧q	~p→ ~q	(p∧q)↔(~p→~q)
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	F	F	T
F	F	T	T	F	T	F

∴ option (B) is correct.

22. Since, $x = 4, 5, 7, 9$ satisfies $x + 1 \leq 10$

∴ option (B) is correct.

23. Option (A) is the true statement since square of every natural number is positive.

24. Option (C) is false, since for every natural number the statement $x - 1 \geq 0$ is always true.

25. Dual of (p ∨ q) ∨ s is (p ∧ q) ∧ s.

27. Negation of (p ∨ q) ∧ (~q ∧ r) is

~[(p ∨ q) ∧ (~q ∧ r)]

≡ ~(p ∨ q) ∨ ~(~q ∧ r)

≡ (~p ∧ ~q) ∨ [~(~q) ∨ ~r]

≡ (~p ∧ ~q) ∨ (q ∨ ~r)

28. $\sim[p \vee (\sim q \wedge \sim p)]$
$\equiv \sim p \wedge \sim(\sim q \wedge \sim p)$ ….[By De Morgan's law]
$\equiv \sim p \wedge [\sim(\sim q) \vee \sim(\sim p)]$
$\equiv \sim p \wedge (q \vee p)$
$\equiv (\sim p \wedge q) \vee (\sim p \wedge p)$
….[Distributive property]
$\equiv (\sim p \wedge q) \vee F$ ….[Complement law]
$\equiv \sim p \wedge q$ ….[Identity law]

29. $\sim[p \to (p \vee \sim q)] \equiv p \wedge \sim[p \vee (\sim q)]$
$\equiv p \wedge (\sim p \wedge q)$

30. $\sim [\exists\, x \in R$, such that $x^2 + 3 > 0]$
$= \forall\, x \in R, x^2 + 3 \leq 0$

31. p: Saral Mart does not reduce the prices.
q: I will not shop there any more.
Symbolic form is $p \to q$
$\sim (p \to q) \equiv p \wedge \sim q$
i.e. Saral Mart does not reduce the prices and still I will shop there.

36. The symbolic form of circuit is
$(p \wedge q) \vee (\sim p \wedge q) \equiv (p \vee \sim p) \wedge q$
$\equiv T \wedge q$
$\equiv q$

37. The symbolic form of circuit is
$[(\sim p \wedge \sim q) \vee p \vee q] \wedge r$
$\equiv [\sim(p \vee q) \vee (p \vee q)] \wedge r$
$\equiv T \wedge r$
$\equiv r$

Competitive Thinking

2. Man is not rich : $\sim q$
Man is not happy : $\sim p$
$\therefore$ The symbolic representation of the given statement is $\sim q \to \sim p$.

3. $\sim p$: Ram is not rich
$\sim q$: Ram is not successful
$\sim r$: Ram is not talented
$\therefore$ The symbolic form of the given statement is $\sim p \wedge \sim q \wedge \sim r$.

4. "Not a correct statement" means it is a statement whose truth value is false.
Option (A) is not a statement.
Options (C) and (D) are statements with truth value true.
'$\sqrt{3}$ is a prime' is false statement.
Hence, option (B) is correct.

5. The symbol $p \wedge q$ means
Mathematics is interesting and Mathematics is difficult.

6. p : roses are red
q : The sun is a star
$(\sim p) \vee q$: roses are not red or the sun is a star.

7. $\sim p$: Boys are not playing
The symbol '$\vee$' means 'or'.

8. Consider option (C),
$(p \wedge \sim q) \to q \equiv (T \wedge \sim T) \to T$
$\equiv (T \wedge F) \to T$
$\equiv F \to T$
$\equiv T$
$\therefore$ option (C) is correct.

9.

p	q	~p	~p ∨ q	p → (~p ∨ q)
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\therefore$ From the table $p \to (\sim p \vee q)$ is false when p is true and q is false.

10. Since, $(p \wedge \sim q) \to (\sim p \vee r) \equiv F$
$\Rightarrow p \wedge \sim q \equiv T$ and $\sim p \vee r \equiv F$
$\Rightarrow p \equiv T, \sim q \equiv T$ and $\sim p \equiv F, r \equiv F$
$\Rightarrow p \equiv T, q \equiv F, r \equiv F$
$\therefore$ The truth values of p, q and r are T, F, F respectively.

11. Since, both the given statements p and q have truth values T,
$\therefore$ $p \to q \equiv T \to T \equiv T$, and
$p \leftrightarrow q \equiv T \leftrightarrow T \equiv T$

12. Contrapositive of $(p \vee q) \to r$ is
$\sim r \to \sim(p \vee q)$ i.e. $\sim r \to \sim p \wedge \sim q$

13. Given $p \to q$
Its contrapositive is $\sim q \to \sim p$
and its converse is $\sim p \to \sim q$

14. Let p : Ram secures 100 marks in maths
q : Ram will get a mobile
Converse of $p \to q$ is $q \to p$
i.e., If Ram will get a mobile, then he secures 100 marks in maths.

15. Inverse of $q \to p$ is $\sim q \to \sim p$
i.e., If a triangle is not equiangular then it is not equilateral.

16. Let p : It is raining
q : I will not come
Contrapositive of $p \to q$ is $\sim q \to \sim p$
i.e., If I will come, then it is not raining.

17. Let p = x is a prime number, q = x is odd.
Contrapositive of $p \to q$ is $\sim q \to \sim p$

18. p: The weather is fine.
q: My friends will come and we will go for a picnic.
∴ Statement is $p \to q$
Contrapositive of $p \to q$ is $\sim q \to \sim p$
i.e., if my friends do not come or we do not go for a picnic then weather will not be fine.

19. Let p : x is prime number
q : x is odd
∴ Statement is $p \to q$
Converse of $p \to q$ is $q \to p$
Contrapositive of $q \to p$ is $\sim p \to \sim q$.

21.

1	2	3	4	5	6
p	q	$\sim p$	$\sim p \wedge q$	$q \to p$	$\sim(q \to p)$
T	T	F	F	T	F
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	F	T	F

The entries in the columns 4 and 6 are identical.
∴ $\sim p \wedge q \equiv \sim(q \to p)$

22. Consider option (B)
$(p \vee q) \wedge \sim p \equiv (p \wedge \sim p) \vee (q \wedge \sim p)$
$\equiv F \vee (q \wedge \sim p)$
$\equiv q \wedge \sim p$
$\equiv \sim p \wedge q$

23. $(p \wedge q) \vee (\sim q \wedge p) \equiv (p \wedge q) \vee (p \wedge \sim q)$
$\equiv p \wedge (q \vee \sim q)$
$\equiv p \wedge T \equiv p$

24. $\sim (p \vee q) \vee (\sim p \wedge q)$
$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$
$\equiv \sim p \wedge (\sim q \vee q)$
$\equiv \sim p \wedge T$
$\equiv \sim p$

25. $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$
$\equiv [(p \vee q) \wedge (\sim q \vee q)] \vee (\sim p \wedge q)$
$\equiv [(p \vee q) \wedge T] \vee (\sim p \wedge q)]$
$\equiv (p \vee q) \vee (\sim p \wedge q)$
$\equiv (p \vee q \vee \sim p) \wedge (p \vee q \vee q)$
$\equiv (T \vee q) \wedge (p \vee q) \equiv T \wedge (p \vee q)$
$\equiv p \vee q$

26.

1	2	3	4	5	6
p	q	$q \to p$	$p \to (q \to p)$	$q \vee p$	$p \to (q \vee p)$
T	T	T	T	T	T
T	F	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	T

The entries in the columns 4 and 6 are identical.
∴ $p \to (q \to p) \equiv p \to (p \vee q)$

27. $p \to (\sim q) \equiv \sim p \vee \sim q$
$\equiv \sim q \vee \sim p$

28. $\sim(p \vee q) \equiv \sim p \vee \sim q$ is not true as it contradicts De Morgan's law.
∴ option (D) is not true.

29. $p \wedge (\sim p \wedge q) \equiv (p \wedge \sim p) \wedge q$
$\equiv F \wedge q$
$\equiv F$

30.

p	$\sim p$	$p \to \sim p$	$\sim p \to p$	$(p \to \sim p) \wedge (\sim p \to p)$
T	F	F	T	F
F	T	T	F	F

31.

p	q	$\sim p$	$\sim q$	$(p \wedge \sim q)$	$(\sim p \wedge q)$	$(p \wedge \sim q) \wedge (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	F	F

∴ Given statement is contradiction.

32. Since, $p \vee \sim p \equiv T$
∴ $(\sim q \wedge p) \vee (p \vee \sim p) \equiv (\sim q \wedge p) \vee T \equiv T$
∴ $(\sim q \wedge p) \vee (p \vee \sim p)$ is a tautology.

33. Consider option (C)

A	B	$A \to B$	$A \wedge (A \to B)$	$[A \wedge (A \to B)] \to B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

∴ option (C) is correct.

34. Consider option (C)

p	q	$q \to p$	$\sim p$	$\sim p \leftrightarrow q$	$(q \to p) \vee (\sim p \leftrightarrow q)$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	T	T
F	F	T	T	F	T

35. $p \to q$ is logically equivalent to $\sim q \to \sim p$

$\therefore$ $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is tautology
But, it is given contradiction.
Hence, it is false statement.

36.

1	2	3	4	5	6
p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$	$p \leftrightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

The entries in the columns 5 and 6 are identical.

$\therefore$ $\sim(p \leftrightarrow \sim q) \equiv p \leftrightarrow q$

37.

p	q	$\sim p$	$p \to q$	$\sim p \to q$	$(\sim p \to q) \to q$	$(p \to q) \to [(\sim p \to q) \to q]$
T	T	F	T	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	F	T	T	F	T	T

38. Option (C) is the correct answer since there exists a real number $x = 0$, such that $x^2 = 0$. Zero is neither positive nor negative.

39. Dual of $\sim p \wedge (q \vee c) = \sim p \vee (q \wedge t)$

40. Negation of $q \vee \sim(p \wedge r)$ is

$\sim[q \vee \sim(p \wedge r)] \equiv \sim q \wedge \sim(\sim(p \wedge r))$
$\equiv \sim q \wedge (p \wedge r)$

41. $\sim[(p \vee \sim q) \wedge q] \equiv \sim(p \vee \sim q) \vee \sim q$ ....[De Morgan's Law]
$\equiv (\sim p) \wedge [\sim(\sim q)] \vee \sim q$
$\equiv (\sim p \wedge q) \vee \sim q$

42. p : A is rich, q : A is silly

$\therefore$ $\sim(p \wedge q) \equiv \sim p \vee \sim q$

43. $\sim(p \wedge q) \equiv \sim p \vee \sim q$

44. p: 72 is divisible by 2.
q: 72 is divisible by 3.

$\therefore$ Statement is $p \wedge q$
$\sim (p \wedge q) \equiv \sim p \vee \sim q$

45. $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
i.e., 7 is greater than 4 and Paris is not in France.

46. $\sim[\sim s \vee (\sim r \wedge s)]$
$\equiv \sim(\sim s) \wedge \sim(\sim r \wedge s)$ ....[De Morgan's Law]
$\equiv s \wedge (r \vee \sim s)$
$\equiv (s \wedge r) \vee (s \wedge \sim s)$ ....[Distributive property]
$\equiv (s \wedge r) \vee F$ ....[Complement law]
$\equiv s \wedge r$ ....[Identity law]

47. $p \to q \equiv \sim p \vee q$

$\therefore$ $\sim (p \to q) \equiv p \wedge \sim q$

48. Since, $p \to q \equiv \sim p \vee q$

$\therefore$ $\sim p \to q \equiv p \vee q$

$\therefore$ $\sim(\sim p \to q) \equiv \sim(p \vee q)$
$\equiv \sim p \wedge \sim q$

49. $\sim[p \to (\sim p \vee q)] \equiv p \wedge \sim (\sim p \vee q)$
$\equiv p \wedge (p \wedge \sim q)$
$\equiv (p \wedge p) \wedge \sim q$
$\equiv p \wedge \sim q$

50. Since, $p \to q \equiv \sim p \vee q$

$\therefore$ $\sim[(p \wedge q) \to (\sim p \vee r)]$
$\equiv \sim[\sim(p \wedge q) \vee (\sim p \vee r)]$
$\equiv \sim[(\sim p \vee \sim q) \vee (\sim p \vee r)]$
$\equiv \sim(\sim p \vee \sim q) \wedge \sim(\sim p \vee r)$
$\equiv (p \wedge q) \wedge (p \wedge \sim r)$

52. Let p : 2 is prime, q : 3 is odd

$\therefore$ Symbolic form $p \to q$

$\therefore$ $\sim(p \to q) \equiv p \wedge \sim q$
i.e., 2 is prime and 3 is not odd.

53. p: Hema gets admission in good college.
q: Hema gets above 95% marks.

$\therefore$ Statement is $p \to q$
$\sim (p \to q) \equiv p \wedge \sim q$

54. Given statement is
$\exists\, x \in S$, such that $x > 0$

$\therefore$ $\sim (\exists\, x \in S$, such that $x > 0)$
$\equiv \forall\, x \in S, x \leq 0$
i.e., Every rational number $x \in S$ satisfies $x \leq 0$.

55. The current will flow through the circuit if p, q, r are closed or p, q′, r are closed.

$\therefore$ option (C) is the correct answer.

56. Let p : switch s_1 is closed.
q: switch s_2 is closed.
~p : switch s_1 is open
~q : switch s_2 is open
The current can flow in the circuit iff either s_1' and s_2 are closed or s_1 and s_2' are closed.
It is represented by $(\sim p \wedge q) \vee (p \wedge \sim q)$.

58. The symbolic form of the given circuit is
$(p \vee \sim p) \wedge q \equiv T \wedge q$
$\equiv q$

59. Symbolic form of the circuit is
$(p \wedge \sim q) \vee (\sim p \wedge q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$
$\equiv \sim (p \leftrightarrow q)$

Evaluation Test

1. $x + 3 = 10$ is an open sentence.
∴ It is not a statement.
∴ option (C) is correct.

2. Since $p \rightarrow q$ is false, when p is true and q is false.
$p \rightarrow (q \vee r)$ is false,
∴ p is true and $q \vee r$ is false
$\Rightarrow$ p is true and both q and r are false.

3. Since, contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.
∴ contrapositive of $(\sim p \wedge q) \rightarrow \sim r$ is
$\sim(\sim r) \rightarrow \sim(\sim p \wedge q) \equiv r \rightarrow (p \vee \sim q)$

4. ~p: Rohit is short.
The given statement can be written symbolically as $p \vee (\sim p \wedge q)$.

5. Let p: $\sqrt{x}$ is a complex number
q: x is a negative number
∴ Logical statement is $p \rightarrow q$
∴ converse of $p \rightarrow q$ is $q \rightarrow p$
∴ option (B) is correct.

6. Consider option (C)

p	q	r	~q	$p \wedge \sim q$	$(p \wedge \sim q) \rightarrow r$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

∴ $(p \wedge \sim q) \rightarrow r$ is a contingency
∴ option (C) is correct.

7. Consider option (A)

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge (\sim(p \vee q))$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

∴ $(p \wedge q) \wedge (\sim(p \vee q))$ is a contradiction.
∴ option (A) is correct.

8.

1	2	3	4	5	6
p	q	~q	$p \leftrightarrow q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$
T	T	F	T	F	T
T	F	T	F	T	F
F	T	F	F	T	F
F	F	T	T	F	T

The entries in the columns 4 and 6 are identical.
∴ $\sim(p \leftrightarrow \sim q) \equiv p \leftrightarrow q$
∴ statement-I is true.
Also, all the entries in the last column of the above truth table are not T.
∴ $\sim(p \leftrightarrow \sim q)$ is not a tautology.
∴ statement-2 is false.
∴ option (B) is correct.

9. Consider option (C)
$(p \vee q) \wedge (p \vee r) \equiv (T \vee T) \wedge (T \vee F)$
$\equiv T \wedge T \equiv T$
∴ option (C) is correct.

11. The statement "Suman is brilliant and dishonest iff suman is rich" can be expressed as $Q \leftrightarrow (P \wedge \sim R)$
The negation of this statement is
$\sim(Q \leftrightarrow (P \wedge \sim R))$

12. $(\sim q) \rightarrow (\sim p)$ is contrapositive of $p \rightarrow q$.
∴ $p \rightarrow q \equiv (\sim q) \rightarrow (\sim p)$
∴ option (D) is true.

13. $(\sim p \wedge \sim q) \vee (p \wedge q) \vee (\sim p \wedge q)$
$\equiv \sim p \wedge (\sim q \vee q) \vee (p \wedge q)$
$\equiv (\sim p \wedge T) \vee (p \wedge q)$
$\equiv \sim p \vee (p \wedge q)$
$\equiv (\sim p \vee p) \wedge (\sim p \vee q)$
$\equiv T \wedge (\sim p \vee q)$
$\equiv \sim p \vee q$
∴ option (B) is correct.

14. Since, inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.
∴ inverse of $(p \wedge \sim q) \rightarrow r$
is $\sim(p \wedge \sim q) \rightarrow \sim r$
i.e., $\sim p \vee q \rightarrow \sim r$

Textbook Chapter No.

02 Matrices

Hints

Classical Thinking

1. $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $R_2 \to R_2 - 2R_1$,

$\begin{bmatrix} 1 & -1 \\ 2-2(1) & 3-2(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0-2(1) & 1-2(0) \end{bmatrix} A$

$\begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$

2. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 5 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_2$,

$A \sim \begin{bmatrix} 3 & -2 & 5 \\ 1 & 2 & -1 \end{bmatrix}$

Applying $C_1 \to C_1 + 2C_3$,

$A \sim \begin{bmatrix} 13 & -2 & 5 \\ -1 & 2 & -1 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$,

$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 5 & -2 \end{bmatrix}$

Applying $R_3 \to R_3 - \left(\frac{5}{3}\right)R_2$,

$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$

which is an upper triangular matrix.

4. If $|A| \neq 0$, then A^{-1} exists

$\therefore$ $|A|$ is non zero

5. M_{11} = minor of $a_{11} = |a_{22}| = a_{22}$

….[By leaving first row and first column]

6. The minor of element $a_{21} = M_{21} = -1$

….[By leaving R_2 and C_1]

7. $M_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}$ ….[By leaving R_3 and C_1]

$= -8$

8. $M_{23} = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3$

9. $A_{12} = (-1)^{1+2} M_{12} = (-1)^3(-3) = 3$

10. $A_{21} = (-1)^3 M_{21} = -(3) = -3$

11. $A_{32} = (-1)^{3+2}.M_{32} = (-1)^5 \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2$

12. $A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -3 - 1 = -4$

$A_{32} = -(-3 - 2) = -(-5) = 5$

$A_{33} = 1 - 2 = -1$

$\therefore$ Co-factors are $-4, 5, -1$

13. Matrix of co-factors

$= \left[A_{ij}\right]_{2\times 2} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 2 & -(-3) \\ -5 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$\therefore$ adj $A = \left[A_{ij}\right]^T_{2\times 2} = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}^T = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

14. Let $A = \begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix}$

Matrix of co-factors is:

$\left[A_{ij}\right]_{2\times 2} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$

$\therefore$ adj $A = \left[A_{ij}\right]^T_{2\times 2} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$

15. Matrix of co-factors is :

$$\left[A_{ij}\right]_{3\times3} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

$\therefore$ $\text{adj } A = \left[A_{ij}\right]^T_{3\times3} = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}^T$

$\therefore$ $\text{adj } A = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

16. Matrix of co-factors is :

$$\left[A_{ij}\right]_{3\times3} = \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}$$

$$\text{adj} A = [A_{ij}]^T_{3\times3} = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

17. $|\text{ adj (adj A)}| = |A| = 12 - 10 = 2$

18. $|A| = a^3$

$|A|\,|\text{adj } A| = |A\text{ (adj A)}| = |A|\, I$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3 = (a^3)^3 = a^9$$

19. $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$

$\therefore$ A^{-1} does not exist.

20. Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}$

$\therefore$ $|A| = \begin{vmatrix} 1 & 3 \\ 3 & 10 \end{vmatrix} = 1$

$\text{adj } A = \begin{bmatrix} 10 & -3 \\ -3 & 1 \end{bmatrix}$

$A^{-1} = \dfrac{1}{|A|}\text{adj } A = \begin{bmatrix} 10 & -3 \\ -3 & 1 \end{bmatrix}$

21. The multiplicative inverse of $A = A^{-1}$

$|A| = \begin{vmatrix} 2 & 1 \\ 7 & 4 \end{vmatrix} = 1 \neq 0$

$\text{adj } A = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{1}{|A|}.\text{adj} A = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$

22. Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \Rightarrow |A| = 14 \neq 0$

$\text{adj } A = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} \frac{4}{14} & \frac{2}{14} \\ \frac{-1}{14} & \frac{3}{14} \end{bmatrix}$

23. The inverse of the given diagonal matrix is

$A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & b \end{bmatrix}$

24. $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 4 \neq 0$

$\text{adj } A = \begin{bmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{bmatrix}^T = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

25. Let $A = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow |A| = 1 \neq 0$

$\text{adj } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$

26. The inverse of the given diagonal matrix is,

$A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

27. $|A| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1$

$\text{adj } A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\therefore \quad A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

28. Given, $A^{-1} = \dfrac{1}{k} \text{adj } A$

$\therefore \quad k = |A|$

$\therefore \quad |A| = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix}$

$= 3(2 + 1) - 2(1 - 0) + 4(1 - 0)$
$= 9 - 2 + 4 = 11$

$\Rightarrow k = 11$

29. $AB = AC$

$\Rightarrow A^{-1} AB = A^{-1} AC$

$\Rightarrow IB = IC$

$\Rightarrow B = C$

$\therefore$ For $B = C$, A^{-1} must exist

$\Rightarrow$ A is non-singular

30. Consider option (B),

A^{-1} is a matrix and $|A|^{-1}$ is a number.

$\therefore$ option (B) is not true.

32. $(A^2 - 4A) A^{-1} = A \cdot A \cdot A^{-1} - 4 A \cdot A^{-1}$

$= A - 4I$

$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

33. The given system of equations can be written in matrix form $AX = B$, where

$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\therefore$ Now, $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$,

$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\Rightarrow x + 2y = 3$, and ….(i)

$-y = -2$ ….(ii)

$\Rightarrow y = 2$

putting $y = 2$ in (i), we get

$x + 2(2) = 3$

$\Rightarrow x = -1$

Alternate method:

$AX = B \Rightarrow X = A^{-1} B$

$|A| = -1 \neq 0$

$A^{-1} = \dfrac{1}{-1} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$X = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\therefore \quad x = -1, y = 2$

34. $AX = B$

$\therefore \quad \begin{bmatrix} 3 & -4 & 2 \\ 2 & 3 & 5 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$

$R_2 - 5R_3 \Rightarrow \begin{bmatrix} 3 & -4 & 2 \\ -3 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$

$R_1 - 2R_3 \Rightarrow \begin{bmatrix} 1 & -4 & 0 \\ -3 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}$

$\Rightarrow x - 4y = -5$, and ….(i)

$-3x + 3y = -3$ ….(ii)

Solving (i) and (ii), we get $x = 3$

35. $\begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_1 \to R_1 - R_3$,

$\begin{bmatrix} a-1 & 0 & 0 \\ 1 & a & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \quad (a - 1)x + 0 + 0 = 0$

$\Rightarrow a - 1 = 0$

$\Rightarrow a = 1$

36. $|A| = -\frac{1}{2} \neq 0$

$\text{adj } A = \begin{bmatrix} \frac{i}{2} & 0 \\ 0 & i \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{\frac{-1}{2}} \begin{bmatrix} \frac{i}{2} & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & -2i \end{bmatrix}$

37. $\because$ adj (AB) = adj (B) adj (A)

Critical Thinking

1. $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Applying $C_2 \to C_2 - 3C_1$ and $C_3 \to C_3 + 2C_1$,

$\begin{bmatrix} 1 & 3-3 & -2+2 \\ -3 & 0+9 & -5-6 \\ 2 & 5-6 & 0+4 \end{bmatrix} = A \begin{bmatrix} 1 & 0-3 & 0+2 \\ 0 & 1-0 & 0+0 \\ 0 & 0-0 & 1+0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3 & 9 & -11 \\ 2 & -1 & 4 \end{bmatrix} = A \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

Applying $C_2 \to C_2 + 2C_1$,

$\therefore$ $A \sim \begin{bmatrix} 2 & 1 & 3 \\ 2 & 6 & 3 \\ 3 & 4 & 2 \end{bmatrix}$

Applying $R_1 \to R_1 + R_3$,

$A \sim \begin{bmatrix} 5 & 5 & 5 \\ 2 & 6 & 3 \\ 3 & 4 & 2 \end{bmatrix}$

3. $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

$= 1(4-3) + 3[-(4-1)] + 2(6-2) = 0$

and $|A| = 1(4-3) - 2(6-6) + 1(3-4) = 0$

$\therefore$ $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} = |A|$

4. Matrix of co-factors,

$\left[A_{ij}\right]_{3\times3} = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}$

$\therefore$ $\text{adj } N = [A_{ij}]^T_{3\times3} = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = N$

5. $AB = \begin{bmatrix} 34 & 39 \\ 82 & 94 \end{bmatrix} \Rightarrow \text{adj } (AB) = \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

6. A is a 2×2 matrix

$\therefore$ $|\text{adj}A| = |A| = 10$

7. A is a 3×3 Matrix

$\therefore$ $|\text{adj } A| = |A|^2 = (12)^2 = 144$

8. $A(\text{Adj } A) = |A| . (I_n)$

$\therefore$ $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix}$

$\Rightarrow |A| = 10$

9. $A(\text{adj } A) = |A|\, I$

$\therefore$ $|A\,(\text{adj } A)| = |A|^n$ (If A is of order $n \times n$)

$\Rightarrow |A|\,|\text{adj } A| = |A|^n$

$\Rightarrow |\text{Adj } A| = |A|^{n-1}$

Since, A is singular

$\therefore$ $|A| = 0$

$\Rightarrow |\text{Adj } A| = 0$

Hence, adj A is a singular matrix.

10. A is a Singular matrix.

$\therefore$ $|A| = 0$ and $A.(\text{adj } A) = |A|.\, I = 0.I = 0$

$\Rightarrow$ A (adj A) is a zero matrix.

11. $\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\therefore$ $\text{adj } (\text{adj } A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

12. $|A| = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 1 \neq 0$

$\text{adj } A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{adj } A) = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

13. $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & 7 & 8 \\ 6 & 7 & -8 \end{vmatrix} = -16 \neq 0$

$\text{adj } A = \begin{bmatrix} -112 & 96 & 0 \\ 15 & -14 & -1 \\ 1 & -2 & 1 \end{bmatrix}^T$

$= \begin{bmatrix} -112 & 15 & 1 \\ 96 & -14 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

$\therefore \quad A^{-1} = -\frac{1}{16}\begin{bmatrix} -112 & 15 & 1 \\ 96 & -14 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

14. $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

The inverse of the given diagonal matrix is

$D^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

$\therefore \quad D^{-1} = \text{diag}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right]$

15. If AC = B, then $A = BC^{-1}$

$\therefore \quad A = \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & -16 \\ 6 & -30 \end{bmatrix}$

16. If AB = C, then $B^{-1} A^{-1} = C^{-1}$

$\therefore \quad A^{-1} = BC^{-1}$

Here, $A\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$\therefore \quad A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$

17. If AB = C, then $B^{-1} A^{-1} = C^{-1}$

$\therefore \quad A^{-1} = BC^{-1}$

$\because \quad A\begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} -1 & 3 \\ 6 & -16 \end{bmatrix}$

18. If XAY = I, then $A = X^{-1} Y^{-1} = (YX)^{-1}$

Here, $YX = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -11 & -7 \end{bmatrix}$

$\therefore \quad A = \begin{bmatrix} 8 & 5 \\ -11 & -7 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 7 & 5 \\ -11 & -8 \end{bmatrix}$

19. $(BA)^{-1} = C$

$\Rightarrow A^{-1}B^{-1} = C \Rightarrow A^{-1} = CB$

$\therefore \quad A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 6 & 4 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -3 & -5 & -5 \\ 0 & 9 & 2 \\ 2 & 14 & 6 \end{bmatrix}$

20. Since, $PQ = -5I_3$

$\therefore \quad (PQ)^{-1} = -\frac{1}{5} I_3$

21. $|A| = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7 \neq 0$

$\text{adj} A = \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj} A = \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{7} A$

$\therefore \quad \alpha = \frac{1}{7}$

22. $|A| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 5 \\ 2 & 6 & 7 \end{vmatrix} = -34$

Co-factor of element a_{23} of A = A_{23}

$\therefore$ $A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 2$

$\therefore$ Element a_{32} of $A^{-1} = \frac{A_{23}}{|A|} = \frac{2}{-34} = \frac{-1}{17}$

23. $A^2 - 3A - 7I = 0$

$\Rightarrow A - 3I - 7A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{7}(A - 3I)$

$\therefore$ $A^{-1} = \frac{1}{7}\left\{\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right\}$

$= \begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ -\frac{1}{7} & -\frac{5}{7} \end{bmatrix}$

24. $A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow x = 0$

$\therefore$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^2 = I$

$\therefore$ $A^{-1}\,A.A = A^{-1}\,I$

$\Rightarrow I.A = A^{-1}\,I$

$\Rightarrow A^{-1} = A$

25. $AB = 3I$

$\Rightarrow A^{-1}\,AB = 3\,A^{-1}\,I$

$\Rightarrow B = 3A^{-1}$

$\therefore$ $A^{-1} = \frac{1}{3}B$

26. $A^2 - A + 2I = 0$

$\Rightarrow A.A - A + 2I = 0$

$\Rightarrow A^{-1}.A.A - A^{-1}.A + 2\,A^{-1}.I = 0$

$\Rightarrow A - I + 2\,A^{-1} = 0$

$\Rightarrow 2\,A^{-1} = I - A$

$\Rightarrow A^{-1} = \frac{1}{2}\,(I - A)$

27. $A^2 + mA + nI = 0$

$\Rightarrow A.A + mA + nI = 0$

$\Rightarrow A^{-1}.A.A + mA^{-1}\,.A + nA^{-1}.I = 0$

$\Rightarrow A + mI + nA^{-1} = 0$

$\Rightarrow nA^{-1} = -A - mI$

$\Rightarrow A^{-1} = \frac{-1}{n}(A + mI)$

28. $4A^3 + 2A^2 + 7A + I = 0$

$\Rightarrow 4A^{-1}A^3 + 2A^{-1}A^2 + 7A^{-1}A + A^{-1}I = 0$

$\Rightarrow 4A^2 + 2A + 7I + A^{-1} = 0$

$\Rightarrow A^{-1} = -(4A^2 + 2A + 7I)$

29. The given system of equations can be written in the matrix form as AX = B, where

$A = \begin{bmatrix} 5 & -7 \\ 7 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & -7 \\ 7 & -5 \end{vmatrix} = 24 \neq 0$

$AX = B$

$\therefore$ $\begin{bmatrix} 5 & -7 \\ 7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$R_1 \to R_1 - \frac{5}{7}R_2$

$\Rightarrow \begin{bmatrix} 0 & \frac{-24}{7} \\ 7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-1}{7} \\ 3 \end{bmatrix}$

$\therefore$ $\frac{-24}{7}y = \frac{-1}{7} \Rightarrow y = \frac{1}{24}$

$7x - 5y = 3 \Rightarrow x = \frac{11}{24}$

Alternate method:

the given system of equations has a unique solution which is given by $X = A^{-1}B$.

$\text{adj } A = \begin{bmatrix} -5 & 7 \\ -7 & 5 \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{24}\begin{bmatrix} -5 & 7 \\ -7 & 5 \end{bmatrix}$

$\therefore$ $X = A^{-1}B = \frac{1}{24}\begin{bmatrix} -5 & 7 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 11 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{11}{24} \\ \frac{1}{24} \end{bmatrix}$

$\Rightarrow x = \frac{11}{24},\ y = \frac{1}{24}$

30. $AX = B$

$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

$R_1 \to 2R_1 + R_3$

$$\begin{bmatrix} 5 & 0 & 5 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 4 \end{bmatrix}$$

$R_1 \to R_1 - 5R_2$

$$\begin{bmatrix} -5 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$\therefore$ $-5x_1 = 5 \Rightarrow x_1 = -1$

$2x_1 + x_3 = 1 \Rightarrow x_3 = 3$

$3x_1 + 2x_2 + x_3 = 4 \Rightarrow x_2 = 2$

$\therefore$ $X = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

31. $X = A^{-1} D$

$\Rightarrow AX = D$

$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix}$$

$R_1 \to R_1 + R_2$, $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 1 \\ 6 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 16 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 8 \end{bmatrix}$$

$\therefore$ $3x = 8 \Rightarrow x = \frac{8}{3}$

$3x - z = 8 \Rightarrow z = 0$

$2x + y + z = 5 \Rightarrow y = \frac{-1}{3}$

$\therefore$ $X = \begin{bmatrix} \frac{8}{3} \\ \frac{-1}{3} \\ 0 \end{bmatrix}$

32. $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1+\omega^n+\omega^{2n} & \omega^n & \omega^{2n} \\ 1+\omega^n+\omega^{2n} & 1 & \omega^n \\ 1+\omega^n+\omega^{2n} & \omega^{2n} & 1 \end{vmatrix} = \begin{vmatrix} 0 & \omega^n & \omega^{2n} \\ 0 & 1 & \omega^n \\ 0 & \omega^{2n} & 1 \end{vmatrix}$$

….[$\because$ $1 + \omega^n + \omega^{2n} = 0$, if n is not multiple of 3]

$\therefore$ $\Delta = 0$

33. $|A| = \begin{vmatrix} 1 & \sin\alpha \\ -\sin\alpha & -1 \end{vmatrix} = -1 + \sin^2\alpha \neq 0$

$\text{adj } A = \begin{bmatrix} -1 & -\sin\alpha \\ \sin\alpha & 1 \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-1+\sin^2\alpha}\begin{bmatrix} -1 & -\sin\alpha \\ \sin\alpha & 1 \end{bmatrix}$

$= \frac{1}{\cos^2\alpha}\begin{bmatrix} 1 & \sin\alpha \\ -\sin\alpha & -1 \end{bmatrix} = \sec^2\alpha\begin{bmatrix} 1 & \sin\alpha \\ -\sin\alpha & -1 \end{bmatrix}$

Competitive Thinking

1. $|A| = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}$

$= 4 - 4 = 0$

$|B| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$

$\therefore$ A^{-1} and B^{-1} does not exist

2. The matrix is not invertible if $\begin{vmatrix} 1 & a & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{vmatrix} = 0$

$\Rightarrow 1(2 - 5) - a(1 - 10) + 2(1 - 4) = 0$

$\Rightarrow -3 + 9a - 6 = 0$

$\Rightarrow a = 1$

3. $|A| = k^2 + 1$, which can be never zero.

Hence matrix A is invertible for all real k.

4. The given matrix will be invertible, if

$\begin{vmatrix} \lambda & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{vmatrix} \neq 0$

$\Rightarrow \lambda(0 - 1) + 1(-6 + 1) + 4(-3) \neq 0$

$\Rightarrow -\lambda - 5 - 12 \neq 0$

$\Rightarrow \lambda \neq -17$

5. Let $A = \begin{bmatrix} \alpha & 14 & -1 \\ 2 & 3 & 1 \\ 6 & 2 & 3 \end{bmatrix} = 0$

Since, A^{-1} doesnot exist

$\therefore$ $|A| = 0$

$\Rightarrow \begin{vmatrix} \alpha & 14 & -1 \\ 2 & 3 & 1 \\ 6 & 2 & 3 \end{vmatrix}$

$\Rightarrow \alpha(9-2) - 14(6-6) - 1(4-18) = 0$

$\Rightarrow 7\alpha = -14$

$\Rightarrow \alpha = -2$

6. $a_{11} = 1, a_{12} = 1, a_{13} = 0$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$

$\therefore$ $a_{11}.A_{21} + a_{12}.A_{22} + a_{13}.A_{23} = 1 \times -1 + 1 \times 1 + 0 \times -1$
$= 0$

7. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$

$a_{31}\, A_{31} + a_{32}\, A_{32} + a_{33}\, A_{33}$
$= 2(10-3) + 4[-(5-3)] + 7(1-2)$
$= 14 - 8 - 7 = -1$

8. Co-factor matrix of $X = \begin{bmatrix} t & -z \\ y & -x \end{bmatrix}$

$\therefore$ Transpose of adj X = co-factor matrix of X
$= \begin{bmatrix} t & -z \\ y & -x \end{bmatrix}$

9. Matrix of co-factors is

$\left[A_{ij}\right]_{3\times3} = \begin{bmatrix} 2 & -5 & 32 \\ 0 & 1 & -6 \\ 0 & 0 & 2 \end{bmatrix}$

$\Rightarrow \text{adj } A = \left[A_{ij}\right]^T_{3\times3} = \begin{bmatrix} 2 & 0 & 0 \\ -5 & 1 & 0 \\ 32 & -6 & 2 \end{bmatrix}$

10. $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$3A^2 = 3\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$= 3\begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix} = \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix}$

$3A^2 + 12A = \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix} + \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$

$= \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

$\therefore$ $\text{adj}(3A^2 + 12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$

$B = \text{adj } A = \begin{bmatrix} 3 & 1 & 1 \\ -6 & -2 & 3 \\ -4 & -3 & 2 \end{bmatrix}$

$\therefore$ $\text{adj } B = \begin{bmatrix} 5 & -5 & 5 \\ 0 & 10 & -15 \\ 10 & 5 & 0 \end{bmatrix} = 5A$

$\therefore$ adj B = C ….[$\because$ C = 5A(given)]

$\Rightarrow |\text{adj } B| = |C|$

$\Rightarrow \dfrac{|\text{adj} B|}{|C|} = 1$

12. $A(\text{adj } A) = |A|.I_n$
Where, n = order of the matrix

$\therefore$ $A(\text{adj } A) = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

13. $|A| = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Since, $A(\text{adj } A) = |A|.I$

$\therefore$ $A(\text{adj } A) = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

14. Since, $A(\text{adj } A) = |A|.\, I$

$\Rightarrow \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = (\cos^2\alpha + \sin^2\alpha)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow k = 1$

15. Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $kI = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$

$\Rightarrow \text{adj}(kI) = \begin{bmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & k^2 \end{bmatrix} = k^2 I$

16. $\text{adj}(\lambda X) = \lambda^{3-1}$ (adj X)

....[$\because$ adj(kA) = k^{n-1} (adj A)]

$= \lambda^2$ adj X

17. Given, A is a singular matrix.

$\therefore$ $|A| = 0$

Since, $|\text{adj} A| = |A|^{n-1}$

$\Rightarrow |\text{adj} A| = 0$

$\Rightarrow$ adj A is also singular.

18. $|\text{Adj } A| = |A|^{n-1} = d^{n-1}$

19. $|A| = \begin{vmatrix} 4 & 2 \\ 3 & 4 \end{vmatrix} = 16 - 6 = 10$

$\because$ $|\text{ adj } A\ | = |A|^{n-1}$

where n $\Rightarrow$ order of matrix.

$\therefore$ $|\text{ adj } A| = |A| = 10$

20. $A\ (\text{adj } A) = |A|\ I_n$

$\Rightarrow \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = |A|\ I_n$

$\Rightarrow 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|\ I_n$

$\Rightarrow 10\ I_n = |A|\ I_n$

$\Rightarrow |A| = 10$

21. Since, $A(\text{Adj } A) = |A|\ I$

$\therefore$ $|A| = 10$

$|\text{Adj } A| = |A|^{n-1}$

$\therefore$ $|\text{Adj } A| = |A|^{3-1} = |A|^2 = 10^2 = 100$

22. $\text{adj } P = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$

$|\text{adj } P| = |P|^2$ ....[$\because$ $|\text{adj } A| = |A|^{n-1}$]

$\Rightarrow \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix} = |P|^2$

$\Rightarrow |P|^2 = 1(-4) - 4(-1) + 4(1)$

$\Rightarrow |P|^2 = 4 \Rightarrow |P| = \pm 2$

23. $|\text{adj } A| = |A|^{n-1} = |A|^{2-1} = |A|$

$\text{Adj}(\text{adj } A) = |A|^{n-2} A = |A|^0 A = A$

$\therefore$ option (B) is the correct answer.

24. adj AB – (adj B) (adj A)

= (adj B) (adj A) – (adj B) (adj A)

....[$\because$adj AB = (adj B) (adj A)]

= O

25. Since, $AA^{-1} = I$

$\therefore$ $\begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 2x & 0 \\ 0 & 2x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By equality of matrices,

$x = \dfrac{1}{2}$

26. Since, $AA^{-1} = I$

$\therefore$ $\begin{bmatrix} x & -2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} \frac{7}{34} & \frac{1}{17} \\ \frac{-3}{34} & \frac{2}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \frac{7x+6}{34} & \frac{x-4}{17} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By equality of matrices,

$\dfrac{x-4}{17} = 0 \Rightarrow x - 4 = 0$

$\Rightarrow x = 4$

27. $10\ A^{-1} = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$ ....[$\because$ B = A^{-1}]

$\Rightarrow 10\ A^{-1} A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} A$

$\Rightarrow 10\ I = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ -5+\alpha & 5+\alpha & -5+\alpha \\ 0 & 0 & 10 \end{bmatrix}$

$\therefore$ $-5 + \alpha = 0 \Rightarrow \alpha = 5$

28. $|A| = \begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix} = -2$

adj $A = \begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$

$A^{-1} = -\frac{1}{2}\begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$

29. $|A| = \begin{vmatrix} 2 & -3 \\ -4 & 2 \end{vmatrix} = -8$

adj $A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

$A^{-1} = -\frac{1}{8}\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

30. $|U| = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = 1 \neq 0$

adj $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$\therefore$ $U^{-1} = \frac{1}{|U|}(\text{adj } U) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$= U^T$

31. $|A| = \begin{vmatrix} a & c \\ d & b \end{vmatrix} = ab - cd$

adj $A = \begin{bmatrix} b & -c \\ -d & a \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{ab - cd}\begin{bmatrix} b & -c \\ -d & a \end{bmatrix}$

32. The inverse of diagonal matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is

$\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

$\therefore$ The inverse of the given diagonal matrix is

$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

33. If $B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $B^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} \frac{2}{k} & 0 & 0 \\ 0 & \frac{3}{l} & 0 \\ 0 & 0 & \frac{4}{m} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

$\Rightarrow \frac{2}{k} = \frac{1}{2} \Rightarrow k = 4,$

$\frac{3}{l} = \frac{1}{3} \Rightarrow l = 9$ and

$\frac{4}{m} = \frac{1}{4} \Rightarrow m = 16$

$\therefore$ $k + l + m = 4 + 9 + 16 = 29$

34. $|A| = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0$

adj $A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$

35. $|A| = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1 \neq 0$

adj $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{|A|}\text{ (adj A)}$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A^T$$

36. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

$|A| = -3 \neq 0$

$\text{adj A} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\text{ adj A}$

$$= \frac{-1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

37. $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = -2 \neq 0$

$\text{adj A} = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

38. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \Rightarrow |A| = 1 \neq 0$

$\text{adj A} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$

$\therefore \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$

39. $|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 1 \neq 0$

$\text{adj A} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

$A^2 = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$

$A^3 = A^2.\,A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = A^{-1}$

40. $A = [a_{ij}]_{2\times2} \Rightarrow A = \begin{bmatrix} -1 & 3 \\ 3 & 0 \end{bmatrix}$

$\therefore \quad |A| = -9$

$\text{adj A} = \begin{bmatrix} 0 & -3 \\ -3 & -1 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{-1}{9}\begin{bmatrix} 0 & -3 \\ -3 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$

41. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \Rightarrow |A| = -2 \neq 0$

Now, co-factor of element a_{32} of A = A_{32}

$\therefore \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2$

$\therefore$ Element a_{23} of $A^{-1} = \frac{A_{32}}{|A|} = \frac{2}{-2} = -1$

Alternate method:

$|A| = -2 \neq 0$

$\text{adj A} = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 2 \\ 1 & -2 & -3 \end{bmatrix}$

$\therefore \quad A^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & -1 & -1 \\ -\frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix}$

$\therefore$ Element a_{23} of $A^{-1} = -1$.

42. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 7$

$\therefore$ Element a_{13} of $A^{-1} = \dfrac{A_{31}}{|A|} = 7$

43. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$

$\therefore$ sum of all the diagonal entries $= \dfrac{1}{2} + 3 + \dfrac{1}{2} = 4$

44. $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$Ax = I \Rightarrow A^{-1}Ax = A^{-1} I$

$\Rightarrow x = A^{-1}$

$|A| = -5$

$\therefore$ $A^{-1} = \dfrac{-1}{5}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$

45. $A\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -16 \\ 6 & -30 \end{bmatrix}$

46. $|A| = \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3$, adj $A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{1}{3}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$

$(A^{-1})^3 = \dfrac{1}{27}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}^3$

$= \dfrac{1}{27}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$

$= \dfrac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$

47. $|A| = \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} = -6 \neq 0$

$A^{-1} = \dfrac{1}{|A|}\text{adj}\,A = \lambda(\text{adj } A)$ ….[Given]

$\therefore$ $\lambda = \dfrac{1}{|A|} = -\dfrac{1}{6}$

48. $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{vmatrix} = 6 \neq 0$

adj $A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$

$\therefore$ $A^2 + cA + dI$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} + \begin{bmatrix} c & 0 & 0 \\ 0 & c & c \\ 0 & -2c & 4c \end{bmatrix} + \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$

$= \begin{bmatrix} 1+c+d & 0 & 0 \\ 0 & -1+c+d & 5+c \\ 0 & -10-2c & 14+4c+d \end{bmatrix}$

Since, $6A^{-1} = A^2 + cA + dI$

$\therefore$ $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+c+d & 0 & 0 \\ 0 & -1+c+d & 5+c \\ 0 & -10-2c & 14+4c+d \end{bmatrix}$

$\therefore$ by equality of matrices,

$1 + c + d = 6$ and $5 + c = -1$,

$\therefore$ $c = -6$ and $d = 11$

49. By definition of inverse,

$I_3I_3^{-1} = I_3$

$\Rightarrow I_3^{-1} = I_3$

50. $A^3 = I$

$\Rightarrow A^{-1}A^3 = A^{-1}.I$

$\Rightarrow (A^{-1}A)A^2 = A^{-1}$

$\Rightarrow IA^2 = A^{-1} \Rightarrow A^2 = A^{-1}$

51. $A^2 - A + I = 0$
$\Rightarrow A.A - A + I = 0$
$\Rightarrow A^{-1}.A.A - A^{-1}.A + A^{-1}.\ I = 0$
$\Rightarrow A - I + A^{-1} = 0$
$\Rightarrow A^{-1} = I - A$

52. Given, $B = -A^{-1}BA$
$\therefore$ $AB = -AA^{-1}BA$
$\Rightarrow AB = -I(BA) \Rightarrow AB = -BA$
Now $(A + B)^2 = (A + B)(A + B)$
$= A^2 + AB + BA + B^2$
$= A^2 + B^2 \quad [\because BA = -AB]$

53. $(A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA)$
$= A^{-1}B(AA^{-1})BA$
$= A^{-1}BIBA$
$= A^{-1}B^2A$
$(A^{-1}BA)^3 = (A^{-1}B^2A)(A^{-1}BA)$
$= A^{-1}B^2(AA^{-1})BA$
$= A^{-1}B^2IBA$
$= A^{-1}B^3A$
In general,
$(A^{-1}BA)^n = A^{-1}B^nA$

54. $(M^{-1})^{-1} \neq (M^{-1})^1$
$\therefore$ $(M^{-1})^{-1} = (M^{-1})^1$ is not true

55. $(B^{-1}A^{-1})^{-1} = (A^{-1})^{-1}.(B^{-1})^{-1} = A.B$
$\therefore$ $A.B = \begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 3 \end{bmatrix}$

56. $(A^2 - 5A)A^{-1} = A.A.A^{-1} - 5A.A^{-1}$
$= A - 5I$
$= \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$
$= \begin{bmatrix} -4 & 2 & 3 \\ -1 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

57. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
$\Rightarrow x + y = 2$ and $-x + y = 4$
$\Rightarrow x = -1, y = 3$

58. $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
$\Rightarrow z = 1$
$4y + 5z = 1$
$\Rightarrow y = -1$
$x + 2y - 3z = 1$
$\Rightarrow x = 6$

59. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$
Now $AX = B$
Applying $R_1 \to R_1 + R_2$,
$\begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$
Applying $R_1 \to R_1 + R_3$,
$\begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$
$\therefore$ $2z = 4 \Rightarrow z = 2$
$-y + z = 2 \Rightarrow y = 0$
$-x + y = 1 \Rightarrow x = -1$
$\therefore$ $(x, y, z) = (-1, 0, 2)$

60. Applying $R_2 \to R_2 + 2R_1$,
$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$
Applying $R_1 \to R_1 - R_3$,
$\begin{bmatrix} 0 & -2 & 0 \\ 3 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 4 \end{bmatrix}$
$\therefore$ $-2y = -4 \Rightarrow y = 2$
$3x = 3 \Rightarrow x = 1$
$x + 3y + z = 4 \Rightarrow z = -3$
$\therefore$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

61. Applying $R_1 \to R_1 - R_3$
$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 4 & 4 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 15 \\ 13 \end{bmatrix}$
Applying $R_2 \to R_2 - R_3$
$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 13 \end{bmatrix}$
$\therefore$ $-z = -1 \Rightarrow z = 1$
$y = 2$
$x + 3y + 4z = 13 \Rightarrow x = 3$
$\therefore$ $(x, y, z) = (3, 2, 1)$

62. Let $M = \begin{bmatrix} a & b & c \\ x & y & z \\ l & m & n \end{bmatrix}$, then

$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} b \\ y \\ m \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

$\therefore$ by the equality of matrices,
b = –1, y = 2, m = 3

$M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} a-b \\ x-y \\ l-m \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\therefore$ by the equality of matrices,
$a - b = 1, x - y = 1, l - m = -1$
$\Rightarrow a = 0, x = 3, l = 2$

$M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} a+b+c \\ x+y+z \\ l+m+n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$

$\therefore$ by the equality of matrices,
$a + b + c = 0, x + y + z = 0, l + m + n = 12$
$\Rightarrow c = 1, z = -5, n = 7$

$\therefore$ sum of diagonal elements of M = $a + y + n$
$= 0 + 2 + 7 = 9$

63. Let $U_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$, $U_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ and $U_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$

$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} a_1 \\ 2a_1 + b_1 \\ 3a_1 + 2b_1 + c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ by the equality of matrices,
$a_1 = 1, b_1 = -2$ and $c_1 = 1$
Similarly $a_2 = 2, b_2 = -1$ and $c_2 = -4$
$a_3 = 2, b_3 = -1$ and $c_3 = -3$

$\therefore U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}$

$|U| = \begin{vmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{vmatrix} = 3$

$\therefore$ U^{-1} exists

$\therefore U^{-1} = \begin{bmatrix} \frac{-1}{3} & \frac{-2}{3} & 0 \\ \frac{-7}{3} & \frac{-5}{3} & -1 \\ 3 & 2 & 1 \end{bmatrix}$

$\therefore$ sum of elements of $U^{-1} = 0$

64. $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$
$= \cos\theta(\cos\theta - 0) + \sin\theta[-(-\sin\theta - 0)] + 0(0-0)$
$= \cos^2\theta + \sin^2\theta = 1$

65. $|A| = 1 + \tan^2\frac{\theta}{2} = \sec^2\frac{\theta}{2}$

$\text{adj}A = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{\sec^2\frac{\theta}{2}}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}$

$AB = I \Rightarrow B = IA^{-1} \Rightarrow B = A^{-1}$

$\therefore B = \frac{1}{\sec^2\frac{\theta}{2}}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} = \cos^2\frac{\theta}{2}.A^T$

66. $F(\alpha).F(-\alpha)$
$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore [F(\alpha)]^{-1} = F(-\alpha)$

67. $I + A = \begin{bmatrix} 1 & -\tan\alpha \\ \tan\alpha & 1 \end{bmatrix}$

$I - A = \begin{bmatrix} 1 & \tan\alpha \\ -\tan\alpha & 1 \end{bmatrix}$

$\therefore |I - A| = 1 + \tan^2\alpha = \sec^2\alpha \neq 0$

$$\text{adj}\,(I-A)=\begin{bmatrix}1 & \tan\alpha\\ -\tan\alpha & 1\end{bmatrix}^T=\begin{bmatrix}1 & -\tan\alpha\\ \tan\alpha & 1\end{bmatrix}$$

$$\Rightarrow (I-A)^{-1}=\frac{1}{|I-A|}[\text{adj}\,(I-A)]$$

$$=\begin{bmatrix}\cos^2\alpha & -\sin\alpha.\cos\alpha\\ \sin\alpha.\cos\alpha & \cos^2\alpha\end{bmatrix}$$

$\therefore$ $(I+A)\,(I-A)^{-1}$

$$=\begin{bmatrix}1 & -\tan\alpha\\ \tan\alpha & 1\end{bmatrix}\begin{bmatrix}\cos^2\alpha & -\sin\alpha\cos\alpha\\ \sin\alpha\cos\alpha & \cos^2\alpha\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\alpha-\sin^2\alpha & -2\sin\alpha\cos\alpha\\ 2\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha\end{bmatrix}$$

$\therefore$ $(I+A)\,(I-A)^{-1}=\begin{bmatrix}\cos 2\alpha & -\sin 2\alpha\\ \sin 2\alpha & \cos 2\alpha\end{bmatrix}$ ....(i)

$B(\alpha)=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

$\therefore$ $B(2\alpha)=\begin{bmatrix}\cos 2\alpha & -\sin 2\alpha\\ \sin 2\alpha & \cos 2\alpha\end{bmatrix}$ ....(ii)

$\therefore$ $(I+A)\,(I-A)^{-1}=B(2\alpha)$ ....[From (i) and (ii)]

68. $BB'=(A^{-1}A')\,(A^{-1}A')'$

$=(A^{-1}A')\,(A(A^{-1})')$

$=A^{-1}\,AA'(A^{-1})'$

....[$\because$ $AA'=A'A$ (given)]

$=(A^{-1}A)\,(A'(A^{-1})')$

$=I(A^{-1}\,A)'=I.I=I^2=I$

69. $M^2N^2(M^TN)^{-1}(MN^{-1})^T$

$=M^2N^2N^{-1}(M^T)^{-1}\,(N^{-1})^TM^T$

$=M^2N(M^T)^{-1}(N^T)^{-1}M^T$

$=M^2N(-M)^{-1}(-N)^{-1}(-M)$

....[$\because$ For skew-symmetric matrices M and N, $M^T=-M$, $N^T=-N$]

$=-M^2NM^{-1}N^{-1}M$ ....$\left[\because (kA)^{-1}=\frac{1}{k}A^{-1}\right]$

$=-M(MN)(NM)^{-1}M$

$=-M(NM)(NM)^{-1}M$

....[$\because$ $MN=NM$ (given)]

$=-M.I.M=-M^2$

Evaluation Test

2. $|A|=\begin{vmatrix}1 & 2 & 2\\ -2 & -1 & -1\\ 1 & -4 & -3\end{vmatrix}=1(-1)-2(7)+2(9)$

$=3\neq 0$

$\therefore$ A^{-1} exists.

$\therefore$ $\text{adj}\,A=\begin{bmatrix}-1 & -7 & 9\\ -2 & -5 & 6\\ 0 & -3 & 3\end{bmatrix}^T=\begin{bmatrix}-1 & -2 & 0\\ -7 & -5 & -3\\ 9 & 6 & 3\end{bmatrix}$

$\therefore$ $A^{-1}=\frac{1}{3}\begin{bmatrix}-1 & -2 & 0\\ -7 & -5 & -3\\ 9 & 6 & 3\end{bmatrix}$

$\therefore$ sum of the elements of A^{-1}

$=\frac{1}{3}(-1-2+0-7-5-3+9+6+3)=0$

3. $(\text{adj}\,A)\,A=|A|\,I_n$

$$A=\begin{vmatrix}2 & 0 & 3\\ 1 & -1 & 2\\ 3 & -2 & 0\end{vmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

4. Let $A=\begin{bmatrix}1 & 2 & 4\\ 3 & -19 & 7\\ 2 & 4 & 8\end{bmatrix}$

$\Rightarrow |A|=0$

$\therefore$ A^{-1} does not exist.

5. $\begin{bmatrix}1 & \tan\theta\\ -\tan\theta & 1\end{bmatrix}^{-1}=\frac{1}{\sec^2\theta}\begin{bmatrix}1 & \tan\theta\\ -\tan\theta & 1\end{bmatrix}^T$

$=\frac{1}{\sec^2\theta}\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}$

$\therefore$ $\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}\cdot\frac{1}{\sec^2\theta}\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}$

$=\frac{1}{\sec^2\theta}\begin{bmatrix}1-\tan^2\theta & -2\tan\theta\\ 2\tan\theta & 1-\tan^2\theta\end{bmatrix}$

$=\begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{bmatrix}$

By equality of matrices, we get

$a=\cos 2\theta$, $b=\sin 2\theta$

6. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 14 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -5 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ -4 \end{bmatrix}$

$\therefore$ $-5z = -5 \Rightarrow z = 1$

$-4y = -4 \Rightarrow y = 1$

$x + 2y + 3z = 6 \Rightarrow x = 1$

7. Let $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & \lambda & 3 \\ -1 & 0 & 3 \end{bmatrix}$

Matrix will not be invertible if $|A| = 0$

$\therefore \begin{vmatrix} 1 & -2 & -1 \\ 2 & \lambda & 3 \\ -1 & 0 & 3 \end{vmatrix} = 0$

$\Rightarrow 1(3\lambda) + 2(9) - 1(\lambda) = 0$

$\Rightarrow \lambda = -9$

8. Given, $|A| \neq 0$ and $|B| = 0$

$\therefore$ $|AB| = |A|\,|B| = 0$

and $|A^{-1}B| = |A^{-1}|\,|B|$

$= \dfrac{1}{|A|}|B| \quad \ldots\left[\because |A^{-1}| = \dfrac{1}{|A|}\right]$

$= 0$

$\therefore$ AB and $A^{-1}B$ are singular.

9. $(AB)^{-1} = B^{-1}A^{-1}$

$\therefore B^{-1}A^{-1} = \begin{bmatrix} \dfrac{-1}{2} & \dfrac{1}{2} \\ \dfrac{1}{4} & 0 \end{bmatrix}$

10. $(A^2 - 8A)A^{-1} = A.A.A^{-1} - 8A.A^{-1}$

$= A - 8I$

$= \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

$= \begin{bmatrix} -7 & 4 & 4 \\ 4 & -7 & 4 \\ 4 & 4 & -7 \end{bmatrix}$

11. $\det A = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix}$

$= 13$

$\therefore$ $\det(\text{adj}(\text{adj } A)) = (\det A)^{(3-1)^2}$

$\ldots\left[\because |\text{adj}(\text{adj} A)| = |A|^{(n-1)^2}\right]$

$= (\det A)^4 = (13)^4$

12. $A.(\text{adj } A) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ ….(i)

$= 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= 4.I$

Since, $A(\text{adj } A) = |A|.I$

$\therefore$ $|A| = 4$

From (i), $|A|\,.\,|\text{adj } A| = 64$

$\Rightarrow |\text{adj } A| = \dfrac{64}{4} = 16$

Also, $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

$= |A|^{(3-1)^2}$

$= (4)^4 = 256$

$\therefore \dfrac{|\text{adj}(\text{adj} A)|}{|\text{adj} A|} = \dfrac{256}{16} = 16$

13. Since, $A(\text{adj } A) = |A|.I$

Replacing A by adj A, we get

$\text{adj } A\,(\text{adj}(\text{adj } A)) = |\text{adj } A| I$

$\Rightarrow A^{-1}.|A|\,(\text{adj}(\text{adj } A)) = |\text{adj } A| I$

$\ldots\left[\because A^{-1} = \dfrac{1}{|A|}(\text{adj} A)\right]$

$\Rightarrow \alpha A^{-1}(\text{adj}(\text{adj } A)) = |A|^2.I$

$\ldots[\because |\text{adj } A| = |A|^{n-1}]$

$\Rightarrow \alpha A^{-1}(\text{adj}(\text{adj } A)) = \alpha^2 I$

$\Rightarrow A^{-1}(\text{adj}(\text{adj } A)) = \alpha I$

Given, $A^{-1}(\text{adj}(\text{adj } A)) = kI$

$\therefore$ $k = \alpha$

Textbook Chapter No.

03 Trigonometric Functions

Hints

Classical Thinking

2. $\tan\theta = \cot\alpha \Rightarrow \tan\theta = \tan\left(\frac{\pi}{2} - \alpha\right)$

$\Rightarrow \theta = n\pi + \frac{\pi}{2} - \alpha$

....[$\because$ $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$]

3. $\tan 3x = 1$

$\therefore$ $\tan 3x = \tan\frac{\pi}{4} \Rightarrow 3x = n\pi + \frac{\pi}{4}$

....$\begin{bmatrix}\because \tan\theta = \tan\alpha \\ \Rightarrow \theta = n\pi + \alpha\end{bmatrix}$

$\therefore$ $x = \frac{n\pi}{3} + \frac{\pi}{12}, n \in I$

4. $\tan 3x = \cot x \Rightarrow \tan 3x = \tan\left(\frac{\pi}{2} - x\right)$

$\therefore$ $3x = n\pi + \frac{\pi}{2} - x \Rightarrow 4x = n\pi + \frac{\pi}{2}$

$\therefore$ $x = \frac{n\pi}{4} + \frac{\pi}{8} = (2n+1)\frac{\pi}{8}$

5. $\sin^2\theta + \sin\theta = 2$

$\therefore$ $(\sin\theta - 1)(\sin\theta + 2) = 0$

$\therefore$ $\sin\theta = 1, -2$

Since, $\sin\theta \neq -2$

$\therefore$ $\sin\theta = 1 = \sin\left(\frac{\pi}{2}\right)$

$\therefore$ $\theta = n\pi + (-1)^n\frac{\pi}{2}, n \in I$

....$\begin{bmatrix}\because \sin\theta = \sin\alpha \\ \Rightarrow \theta = n\pi + (-1)^n\alpha\end{bmatrix}$

6. $\cot\theta - \tan\theta = 2 \Rightarrow \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = 2$

$\therefore$ $\cos^2\theta - \sin^2\theta = \sin 2\theta \Rightarrow \cos 2\theta = \sin 2\theta$

$\therefore$ $\tan 2\theta = \tan\frac{\pi}{4} \Rightarrow 2\theta = n\pi + \frac{\pi}{4}$

$\therefore$ $\theta = \frac{n\pi}{2} + \frac{\pi}{8}$

7. $\sin^2\theta = \frac{1}{4} = \sin^2\frac{\pi}{6} \Rightarrow \theta = n\pi \pm \frac{\pi}{6}$

....[$\because$ $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$]

8. $4\cos^2 x + 6\sin^2 x = 5$

$\therefore$ $4 + 2\sin^2 x = 5$

$\therefore$ $\sin^2 x = \frac{1}{2} = \sin^2\frac{\pi}{4} \Rightarrow x = n\pi \pm \frac{\pi}{4}$

9. $\sec^2\theta + \tan^2\theta = \frac{5}{3}$ ....(i)

$\therefore$ $1 + \tan^2\theta + \tan^2\theta = \frac{5}{3}$

$\therefore$ $2\tan^2\theta = \frac{2}{3}$

$\therefore$ $\tan^2\theta = \frac{1}{3} = \tan^2\left(\frac{\pi}{6}\right) \Rightarrow \theta = n\pi \pm \frac{\pi}{6}$

....[$\because$ $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$]

10. $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$

$\therefore$ $\tan\theta + \tan 2\theta = \sqrt{3}(1 - \tan\theta\tan 2\theta)$

$\therefore$ $\frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \sqrt{3} \Rightarrow \tan 3\theta = \tan\frac{\pi}{3}$

$\therefore$ $3\theta = n\pi + \frac{\pi}{3} \Rightarrow \theta = (3n+1)\frac{\pi}{9}$

11. By sine rule,

$\frac{\sin A}{a} = \frac{\sin B}{b}$

$\Rightarrow \frac{2/3}{2} = \frac{\sin B}{3}$

$\Rightarrow \sin B = 1 = \sin 90^\circ \Rightarrow B = 90^\circ$

12. $\frac{\sin B}{\sin(A+B)} = \frac{\sin B}{\sin C} = \frac{b}{c}$

....[$\because$ $A + B + C = \pi, A + B = \pi - C$]

13. $2s = a + b + c = 16 + 24 + 20 = 60 \Rightarrow s = 30$

$\therefore$ $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} = \sqrt{\frac{30\times 6}{320}} = \frac{3}{4}$

14. Let a = 4 cm, b = 5 cm, c = 6 cm

$s = \dfrac{a+b+c}{2} = \dfrac{4+5+6}{2} = \dfrac{15}{2}$

$A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{\dfrac{15}{2}\left(\dfrac{15}{2}-4\right)\left(\dfrac{15}{2}-5\right)\left(\dfrac{15}{2}-6\right)} = \dfrac{15}{4}\sqrt{7}$

15. $2ac \sin\dfrac{A-B+C}{2} = 2ac \sin\dfrac{\pi-2B}{2}$

$= 2ac \cos B$

$= 2ac\,\dfrac{c^2+a^2-b^2}{2ca}$

….[By cosine rule]

$= c^2 + a^2 - b^2$

16. $s - a = 3 \Rightarrow b + c - a = 6$ ....(i)

$s - c = 2 \Rightarrow a + b - c = 4$ ....(ii)

Adding (i) and (ii), we get b = 5

Since, $\angle B = 90^\circ$

$\therefore$ $b^2 = a^2 + c^2 \Rightarrow a^2 + c^2 = 25$ ….(iii)

Solving, we get a = 3, c = 4

17. We know that,

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$

$\Rightarrow \dfrac{b}{\frac{1}{2}} = \dfrac{c}{\frac{1}{\sqrt{2}}} \Rightarrow c - \sqrt{2}\ b = 0$ ….(i)

By projection rule,

a = b cos C + c cos B

$\Rightarrow \sqrt{3} + 1 = \dfrac{b}{\sqrt{2}} + \dfrac{\sqrt{3}}{2}\ c$

$\Rightarrow 2(\sqrt{3}+1) = \sqrt{2}\ b + \sqrt{3}\ c$ ….(ii)

From (i) and (ii), we get

$2(\sqrt{3}+1) = (\sqrt{3}+1)\ c \Rightarrow c = 2$

18. $s = \dfrac{a+b+c}{2} = \dfrac{12}{2} = 6$

$\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}} = \sqrt{\dfrac{2\times 3}{12}} = \sqrt{\dfrac{1}{2}}$

$\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ca}} = \sqrt{\dfrac{6\times 1}{12}} = \sqrt{\dfrac{1}{2}}$

$\therefore$ $\sin\dfrac{B}{2} + \cos\dfrac{B}{2} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

19. $\dfrac{\sin\frac{A}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}} = \sqrt{\dfrac{ac(s-b)(s-c)(s-b)(s-a)}{(s-a)(s-c)bc\times ab}}$

$= \dfrac{s-b}{b}$

But a, b and c are in A. P. $\Rightarrow 2b = a + c$

$\Rightarrow 2b + b = a + b + c$

$\Rightarrow 3b = 2s \Rightarrow s = \dfrac{3b}{2}$

$\therefore$ $\dfrac{s-b}{b} = \dfrac{\frac{3b}{2}-b}{b} = \dfrac{1}{2}$

20. By Napier's analogy, we have

$\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2} \Rightarrow x = \dfrac{b-c}{b+c}$

21. $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

$= \dfrac{a-b}{a+b}\tan\left(\dfrac{A+B}{2}\right)$

....$[\because A+B+C=\pi]$

$\therefore$ $\tan\dfrac{A-B}{2}\cot\dfrac{A+B}{2} = \dfrac{a-b}{a+b}$

26. $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\dfrac{1}{2}\right) = 60^\circ - 30^\circ = 30^\circ$

27. $\sin^{-1}\dfrac{1}{2} = \tan^{-1}x$

$\Rightarrow \dfrac{\pi}{6} = \tan^{-1}x \Rightarrow \tan\dfrac{\pi}{6} = x$

$\Rightarrow x = \dfrac{1}{\sqrt{3}}$

28. Let $\theta = \sin^{-1}\left(\dfrac{3}{5}\right)$

$\therefore$ $\sin\left(2\sin^{-1}\left(\dfrac{3}{5}\right)\right) = \sin 2\theta$

$= 2\sin\theta\cos\theta$

$= 2\sin\left(\sin^{-1}\left(\dfrac{3}{5}\right)\right)\cos\left(\sin^{-1}\left(\dfrac{3}{5}\right)\right)$

$= 2\times\dfrac{3}{5}\sqrt{1-\left(\dfrac{3}{5}\right)^2}$ ….$[\because \cos(\sin^{-1}x) = \sqrt{1-x^2}\,]$

$= 2\times\dfrac{3}{5}\times\dfrac{4}{5} = \dfrac{24}{25}$

29. $\sin\left(3\sin^{-1}\left(\frac{2}{5}\right)\right) = \sin 3\theta$,

Where $\theta = \sin^{-1}\left(\frac{2}{5}\right)$

$\ldots\left[\theta = \sin^{-1}\left(\frac{2}{5}\right), \sin\theta = \frac{2}{5}\right]$

$= 3\sin\theta - 4\sin^3\theta$

$= 3\left(\frac{2}{5}\right) - 4\left(\frac{2}{5}\right)^3$

$\ldots[\theta = \sin^{-1}\left(\frac{2}{5}\right), \sin\theta = \frac{2}{5}]$

$= \frac{6}{5} - \frac{32}{125} = \frac{118}{125}$

30. $\cos^{-1}(\cos 12) - \sin^{-1}(\sin 14) = 12 - 14 = -2$

31. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$

$= \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$

$= -\tan^{-1}\left(\tan\frac{\pi}{4}\right) = -\frac{\pi}{4}$

32. If $x = \sec\theta$, then $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$\therefore$ $\cot^{-1}\frac{1}{\sqrt{x^2-1}} = \cot^{-1}(\cot\theta) = \theta = \sec^{-1}x$

33. $\cot^{-1}\left(-\sqrt{3}\right) = \pi - \cot^{-1}\left(\sqrt{3}\right)$

$= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

34. $\cos^{-1}(-1) = \pi - \cos^{-1}1 = \pi - 0 = \pi$

35. $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right)$

$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\frac{\pi}{2} = 1$

37. $\cos^{-1}\left[\cos\frac{5\pi}{3}\right] + \sin^{-1}\left[\cos\frac{5\pi}{3}\right] = \frac{\pi}{2}$

$\ldots\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

38. $\cos\left\{\cos^{-1}\left(\frac{-1}{7}\right) + \sin^{-1}\left(\frac{-1}{7}\right)\right\} = \cos\frac{\pi}{2} = 0$

39. $\cot^{-1}x + \cot^{-1}y = \left(\frac{\pi}{2} - \tan^{-1}x\right) + \left(\frac{\pi}{2} - \tan^{-1}y\right)$

$\ldots\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$

$= \pi - (\tan^{-1}x + \tan^{-1}y)$

$= \pi - \frac{4\pi}{5} = \frac{\pi}{5}$

40. $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$= \tan^{-1}\sqrt{3} - \left[\pi - \cot^{-1}\sqrt{3}\right]$

$= \tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3} - \pi$

$= \frac{\pi}{2} - \pi = -\frac{\pi}{2}$

41. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}}$

$= \tan^{-1}\frac{15}{20} = \tan^{-1}\frac{3}{4}$

42. $\tan^{-1}x - \tan^{-1}y = \tan^{-1}A$

$\Rightarrow \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}A$

$\Rightarrow A = \frac{x-y}{1+xy}$

43. $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right)$

$= \sin^{-1}\left(\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2}\right)$

$\ldots\left[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$= \sin^{-1}\left(\frac{3}{5}\times\frac{15}{17} + \frac{8}{17}\times\frac{4}{5}\right) = \sin^{-1}\left(\frac{77}{85}\right)$

44. $\cos^{-1}\frac{3}{5} - \sin^{-1}\frac{4}{5} = \cos^{-1}x$

$\therefore$ $\cos^{-1}\frac{3}{5} - \cos^{-1}\sqrt{1-\frac{16}{25}} = \cos^{-1}x$

$\therefore$ $\cos^{-1}\frac{3}{5} - \cos^{-1}\frac{3}{5} = \cos^{-1}x$

$\therefore$ $\cos^{-1}x = 0 \Rightarrow x = 1$

Critical Thinking

1. $\tan\theta + \dfrac{1}{\sqrt{3}} = 0 \Rightarrow \tan\theta = -\dfrac{1}{\sqrt{3}}$

$\Rightarrow \tan\theta = -\tan 30°$

$\Rightarrow \tan\theta = \tan(180° - 30°)$ and

$\tan\theta = \tan(360° - 30°)$

$\Rightarrow \tan\theta = \tan 150°$ and $\tan\theta = \tan 330°$

$\Rightarrow \theta = 150°$ and $330°$

2. $\cos\theta = 1 - 2x^2$

$\therefore$ $\cos\theta = 1 - 2\cos^2 40°$ ….[$\because \cos 40° = x$]

$= -(2\cos^2 40° - 1)$

$= -\cos(2 \times 40°) = -\cos 80°$

$\therefore$ $\cos\theta = \cos(180° + 80°) = \cos 260°$

and $\cos\theta = \cos(180° - 80°) = \cos 100°$

$\therefore$ $\theta = 100°$ and $260°$

3. $\tan\theta = \sqrt{3} = \tan\dfrac{\pi}{3} \Rightarrow \theta = n\pi + \dfrac{\pi}{3}$

For $-\pi < \theta < 0$,

Put $n = -1$, we get $\theta = -\pi + \dfrac{\pi}{3} = \dfrac{-2\pi}{3} = \dfrac{-4\pi}{6}$

4. $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta \Rightarrow \dfrac{1}{\sin\theta\cos\theta} = \dfrac{2}{\sin\theta}$

$\Rightarrow \cos\theta = \dfrac{1}{2} = \cos\left(\dfrac{\pi}{3}\right) \Rightarrow \theta = 2n\pi \pm \dfrac{\pi}{3}$

5. $\tan\theta + \tan\left(\dfrac{\pi}{2} - \theta\right) = 2$

$\Rightarrow \tan\theta + \dfrac{1}{\tan\theta} = 2 \Rightarrow \tan^2\theta - 2\tan\theta + 1 = 0$

$\Rightarrow \tan\theta = 1 = \tan\dfrac{\pi}{4} \Rightarrow \theta = n\pi + \dfrac{\pi}{4}$

6. $\sin\theta = -\dfrac{1}{2} = -\sin\left(\dfrac{\pi}{6}\right) = \sin\left(\pi + \dfrac{\pi}{6}\right)$

$\tan\theta = \dfrac{1}{\sqrt{3}} = \tan\left(\dfrac{\pi}{6}\right) = \tan\left(\pi + \dfrac{\pi}{6}\right)$

$\Rightarrow \theta = \left(\pi + \dfrac{\pi}{6}\right)$

Hence, general value of θ is $2n\pi + \dfrac{7\pi}{6}$.

7. $\cos x - \sin x = \dfrac{1}{\sqrt{2}}$

Dividing both sides by $\sqrt{2}$, we get

$\dfrac{1}{\sqrt{2}}\cos x - \dfrac{1}{\sqrt{2}}\sin x = \dfrac{1}{2}$

$\Rightarrow \cos\left(\dfrac{\pi}{4} + x\right) = \cos\dfrac{\pi}{3} \Rightarrow \dfrac{\pi}{4} + x = 2n\pi \pm \dfrac{\pi}{3}$

$\Rightarrow x = 2n\pi + \dfrac{\pi}{3} - \dfrac{\pi}{4} = 2n\pi + \dfrac{\pi}{12}$

or $x = 2n\pi - \dfrac{\pi}{3} - \dfrac{\pi}{4} = 2n\pi - \dfrac{7\pi}{12}$

8. $1 + \cot\theta = \operatorname{cosec}\theta$

$\Rightarrow \dfrac{1}{\sin\theta} = 1 + \dfrac{\cos\theta}{\sin\theta} \Rightarrow \sin\theta + \cos\theta = 1$

Dividing both sides by $\sqrt{2}$, we get

$\sin\theta\sin\dfrac{\pi}{4} + \cos\theta\cos\dfrac{\pi}{4} = \cos\dfrac{\pi}{4}$

$\Rightarrow \cos\left(\theta - \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{4} \Rightarrow \theta - \dfrac{\pi}{4} = 2n\pi \pm \dfrac{\pi}{4}$

$\Rightarrow \theta = 2n\pi$ or $\theta = 2n\pi + \dfrac{\pi}{2}$

9. $\sin x - \cos x = \sqrt{2}$

$\Rightarrow \sin x.\dfrac{1}{\sqrt{2}} - \cos x.\dfrac{1}{\sqrt{2}} = 1$

$\Rightarrow \cos\left(x + \dfrac{\pi}{4}\right) = -1 = \cos\pi$

$\Rightarrow x + \dfrac{\pi}{4} = 2n\pi \pm \pi$

$\Rightarrow x = 2n\pi + \dfrac{3\pi}{4}$ or $2n\pi - \dfrac{5\pi}{4}$

10. $\cot\theta + \cot\left(\dfrac{\pi}{4} + \theta\right) = 2$

$\therefore$ $\dfrac{\cos\theta}{\sin\theta} + \dfrac{\cos\left(\dfrac{\pi}{4} + \theta\right)}{\sin\left(\dfrac{\pi}{4} + \theta\right)} = 2$

$\therefore$ $\sin\left(\dfrac{\pi}{4} + 2\theta\right) = 2\sin\theta\sin\left(\dfrac{\pi}{4} + \theta\right)$

$= \cos\left(\theta - \dfrac{\pi}{4} - \theta\right) - \cos\left(\theta + \dfrac{\pi}{4} + \theta\right)$

$\therefore \quad \sin\left(\frac{\pi}{4}+2\theta\right) = \cos\left(\frac{-\pi}{4}\right) - \cos\left(2\theta+\frac{\pi}{4}\right)$

$\Rightarrow \sin\left(\frac{\pi}{4}+2\theta\right)+\cos\left(\frac{\pi}{4}+2\theta\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow \left(\frac{1}{\sqrt{2}}\cos 2\theta+\frac{1}{\sqrt{2}}\sin 2\theta\right)$

$+\left(\frac{1}{\sqrt{2}}\cos 2\theta-\frac{1}{\sqrt{2}}\sin 2\theta\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow \frac{2}{\sqrt{2}}\cos 2\theta=\frac{1}{\sqrt{2}} \Rightarrow \cos 2\theta=\frac{1}{2}=\cos\left(\frac{\pi}{3}\right)$

$\Rightarrow 2\theta=2n\pi\pm\frac{\pi}{3}\Rightarrow\theta=n\pi\pm\frac{\pi}{6}$

11. $\sin^2 x - 2\cos x + \frac{1}{4} = 0$

$\Rightarrow 1-\cos^2 x - 2\cos x + \frac{1}{4} = 0$

Putting $\cos x = t$, we get

$1 - t^2 - 2t + \frac{1}{4} = 0 \Rightarrow 4t^2 + 8t - 5 = 0$

$\therefore \quad t = \frac{1}{2}$ or $t = -\frac{5}{2}$

Since, $\cos x \neq \frac{-5}{2}$

$\therefore \quad \cos x = \frac{1}{2} = \cos\frac{\pi}{3} \Rightarrow x = 2n\pi \pm \frac{\pi}{3}$

12. We have, $\sec\theta + \tan\theta = \sqrt{3}$ ….(i)

$\Rightarrow \sec\theta - \tan\theta = \frac{1}{\sqrt{3}}$ ....(ii)

….$[\because \sec^2\theta - \tan^2\theta = 1]$

By solving (i) and (ii), we get

$\tan\theta=\frac{1}{2}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}$

$\therefore \quad \tan\theta = \tan\left(\frac{\pi}{6}\right)$

$\Rightarrow \theta = n\pi+\frac{\pi}{6}$

$\therefore \quad \theta = \frac{\pi}{6}$ and $\frac{7\pi}{6}$ in $[0, 2\pi]$

Hence, there are two solutions.

13. $r\sin\theta = 3$,

$r = 4(1+\sin\theta)$

Eliminating r, we get

$\frac{3}{\sin\theta} = 4 + 4\sin\theta$

$\therefore \quad \sin\theta = \frac{1}{2}, -\frac{3}{2}$

$\therefore \quad \sin\theta = \frac{1}{2}$ ....$\left[\because \sin\theta \neq \frac{-3}{2}\right]$

$\Rightarrow \theta = \frac{\pi}{6}, \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ in $[0, 2\pi]$

14. $2\sin^2\theta - 3\sin\theta - 2 = 0$

$\therefore \quad \sin\theta = \frac{3\pm\sqrt{9+16}}{4} = \frac{3\pm 5}{4} = 2, -\frac{1}{2}$

$\therefore \quad \sin\theta = -\frac{1}{2}$ ….$[\because |\sin\theta| \leq 1]$

$\therefore \quad \sin\theta = \sin\left(\frac{-\pi}{6}\right)$

$\therefore \quad \theta = n\pi + (-1)^n\left(\frac{-\pi}{6}\right) = n\pi + (-1)^{n+1}\left(\frac{\pi}{6}\right)$

15. $2\cos^2 x + 3\sin x - 3 = 0$

$\Rightarrow 2 - 2\sin^2 x + 3\sin x - 3 = 0$

$\Rightarrow (2\sin x - 1)(\sin x - 1) = 0$

$\Rightarrow \sin x = \frac{1}{2}$ or $\sin x = 1$

$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$ i.e., 30°, 150°, 90°.

16. $4\sin^2\theta + 2(\sqrt{3}+1)\cos\theta = 4 + \sqrt{3}$

$\Rightarrow 4 - 4\cos^2\theta + 2(\sqrt{3}+1)\cos\theta = 4 + \sqrt{3}$

$\Rightarrow 4\cos^2\theta - 2(\sqrt{3}+1)\cos\theta + \sqrt{3} = 0$

$\Rightarrow \cos\theta = \frac{2(\sqrt{3}+1)\pm\sqrt{4(\sqrt{3}+1)^2-16\sqrt{3}}}{8}$

$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$ or $\frac{1}{2}$

$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{6}$ or $2n\pi \pm \frac{\pi}{3}$

17. $\sin(A+B) = 1$ and $\cos(A-B) = \frac{\sqrt{3}}{2}$

$\Rightarrow A + B = \frac{\pi}{2}$ and $A - B = \frac{\pi}{6}$

$\Rightarrow A = \frac{\pi}{3}, B = \frac{\pi}{6}$

18. $\cos 7\theta = \cos\theta - \sin 4\theta$
$\Rightarrow \sin 4\theta = \cos\theta - \cos 7\theta$
$\Rightarrow \sin 4\theta = 2\sin(4\theta)\sin(3\theta)$
$\Rightarrow \sin 4\theta = 0 \Rightarrow 4\theta = n\pi$ or
$\sin 3\theta = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$
$\Rightarrow 3\theta = n\pi + (-1)^n \frac{\pi}{6}$
$\therefore \quad \theta = \frac{n\pi}{4}, \frac{n\pi}{3} + (-1)^n \frac{\pi}{18}$

19. $\frac{1-\tan^2\theta}{\sec^2\theta} = \frac{1}{2} \Rightarrow \cos^2\theta - \sin^2\theta = \frac{1}{2}$
$\Rightarrow \cos 2\theta = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$
$\Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{3} \Rightarrow \theta = n\pi \pm \frac{\pi}{6}$

20. $\sqrt{3}\tan 2\theta + \sqrt{3}\tan 3\theta + \tan 2\theta\tan 3\theta = 1$
$\Rightarrow \frac{\tan 2\theta + \tan 3\theta}{1-\tan 2\theta\tan 3\theta} = \frac{1}{\sqrt{3}} \Rightarrow \tan 5\theta = \tan\frac{\pi}{6}$
$\Rightarrow 5\theta = n\pi + \frac{\pi}{6} \Rightarrow \theta = \left(n + \frac{1}{6}\right)\frac{\pi}{5}$

21. $\tan\theta + \tan 2\theta = \tan 3\theta(\tan\theta.\tan 2\theta - 1)$
$\Rightarrow \frac{\tan\theta + \tan 2\theta}{1-\tan\theta\tan 2\theta} = -\tan 3\theta$
$\Rightarrow 2\tan 3\theta = 0 \Rightarrow 3\theta = n\pi$
$\Rightarrow \theta = \frac{n\pi}{3}$

22. $2\tan^2\theta = \sec^2\theta \Rightarrow 2\tan^2\theta = \tan^2\theta + 1$
$\Rightarrow \tan^2\theta = 1 = \tan^2\left(\frac{\pi}{4}\right) \Rightarrow \theta = n\pi \pm \frac{\pi}{4}$

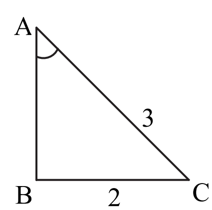
23. $\tan\theta\tan 2\theta = 1$
$\therefore \quad \tan\theta\,\frac{2\tan\theta}{1-\tan^2\theta} = 1$
$\therefore \quad 2\tan^2\theta = 1 - \tan^2\theta$
$\therefore \quad 3\tan^2\theta = 1$
$\therefore \quad \tan^2\theta = \frac{1}{3} = \tan^2\left(\frac{\pi}{6}\right)$
$\therefore \quad \theta = n\pi \pm \frac{\pi}{6}$

24. $\sin 3\alpha = 4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$
$\therefore \quad \sin 3\alpha = 4\sin\alpha(\sin^2 x\cos^2\alpha - \cos^2 x\sin^2\alpha)$
$\therefore \quad 3\sin\alpha - 4\sin^3\alpha = 4\sin\alpha(\sin^2 x - \sin^2\alpha)$
$\therefore \quad \sin^2 x = \left(\frac{3}{4}\right) \Rightarrow \sin^2 x = \sin^2\frac{\pi}{3}$
$\therefore \quad x = n\pi \pm \frac{\pi}{3}$

25. $(\cos\theta + \cos 7\theta) + (\cos 3\theta + \cos 5\theta) = 0$
$\Rightarrow 2\cos 4\theta\cos 3\theta + 2\cos 4\theta\cos\theta = 0$
$\Rightarrow 2\cos 4\theta(\cos 3\theta + \cos\theta) = 0$
$\Rightarrow 4\cos 4\theta\cos 2\theta\cos\theta = 0$
$\Rightarrow 4\frac{\sin 2^3\theta}{2^3\sin\theta} = 0$
$\ldots\left[\because \cos A\cos 2A\cos 2^2A\cos 2^3A\ldots\cos 2^{n-1}A = \frac{\sin 2^n A}{2^n\sin A}\right]$
$\Rightarrow \sin 8\theta = 0$
$\Rightarrow 8\theta = n\pi$
$\Rightarrow \theta = \frac{n\pi}{8}$

26. Given, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ ….(i)
By sine rule,
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ ….(ii)
From (i) and (ii), we get
$\therefore \quad \frac{\cos A}{\cos B} = \frac{\sin A}{\sin B}$
$\therefore \quad \sin(A-B) = 0 \Rightarrow A = B$
Similarly, we get,
$B = C$
$\therefore \quad A = B = C$
Thus, Δ ABC is an equilateral triangle.

27. We know that,
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$
$\Rightarrow \frac{2}{\frac{2}{3}} = \frac{3}{\sin B} = \frac{c}{\sin C} = k$
$\Rightarrow k = 3$



$\therefore \quad \frac{3}{\sin B} = 3$
$\Rightarrow \sin B = 1$
$\Rightarrow B = 90^\circ$
Hence, the triangle is a right angled triangle.
From the figure,
$\cos C = \frac{BC}{AC} = \frac{2}{3}$

28. Since the angles are in A.P., therefore B = 60°
By sine rule,
$$\frac{b}{c} = \frac{\sin B}{\sin C} \Rightarrow \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{2\sin C} \Rightarrow C = 45°$$
$\therefore$ A = 180° – 60° – 45° = 75°

29. B = 60°, C = 75°
$\Rightarrow$A = 180° – 60° – 75° = 45°
By sine rule,
$$\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow \frac{b}{\sin 60^\circ} = \frac{2}{\sin 45^\circ} \Rightarrow b = \sqrt{6}$$

30. Let the angles of the triangle be $2x$, $3x$ and $7x$.
$\therefore$ $2x + 3x + 7x = 180^\circ \Rightarrow 12x = 180^\circ \Rightarrow x = 15^\circ$
$\therefore$ the angles are 30°, 45° and 105°
$\therefore$ a: b: c = sin 30° : sin45° : sin 105°
$$= \frac{1}{2} : \frac{1}{\sqrt{2}} : \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$= \sqrt{2} : 2 : (\sqrt{3} + 1)$$

31. $$\frac{b-c}{a} = \frac{\sin B - \sin C}{\sin A}$$
$$= \frac{2\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$
$$= \frac{\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}$$
$$= \frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}$$
$$\Rightarrow (b - c)\cos\frac{A}{2} = a\sin\frac{B-C}{2}$$

32. $$\frac{1+\cos C\cos(A-B)}{1+\cos(A-C)\cos B} = \frac{1-\cos(A+B)\cos(A-B)}{1-\cos(A-C)\cos(A+C)}$$
$$= \frac{1-\frac{1}{2}(\cos 2A+\cos 2B)}{1-\frac{1}{2}(\cos 2A+\cos 2C)}$$
$$= \frac{1-\frac{1}{2}(1-2\sin^2 A+1-2\sin^2 B)}{1-\frac{1}{2}(1-2\sin^2 A+1-2\sin^2 C)}$$
$$= \frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C} = \frac{a^2+b^2}{a^2+c^2}$$

33. $$\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}} = \frac{\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\sin\frac{B+C}{2}\sin\frac{A}{2}}$$
$$= \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\sin\frac{A}{2}}$$
$$= \frac{\sin B+\sin C}{\sin A} = \frac{b+c}{a}$$

34. $$\cos\theta = \frac{36+100-(14)^2}{2.6.10}$$
$\Rightarrow \theta = 120° \Rightarrow$ Obtuse angled triangle

35. Since A, B and C are in A.P., therefore
$$B = 60° \;\ldots\left[\begin{array}{l}\because A+B+C=180^\circ\\ \Rightarrow A+C=2B\Rightarrow B=60^\circ\end{array}\right]$$
Since sides a, b and c are in G.P., therefore
$b^2 = ac$
$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$
$$\Rightarrow \frac{1}{2} = \frac{a^2+c^2-b^2}{2b^2}, \qquad \ldots[\because b^2 = ac]$$
$\Rightarrow b^2 = a^2 + c^2 - b^2$
$\Rightarrow a^2 + c^2 = 2b^2$
$\Rightarrow a^2, b^2, c^2$ are in A.P.

36. A, B, C are in A. P. then angle B = 60°,
$$\ldots\left[\begin{array}{l}\because A+B+C=180^\circ\\ \Rightarrow A+C=2B\Rightarrow B=60^\circ\end{array}\right]$$
$\therefore$ $$\cos B = \frac{a^2+c^2-b^2}{2ac},$$
$$\Rightarrow \frac{1}{2} = \frac{a^2+c^2-b^2}{2ac} \Rightarrow a^2 + c^2 - b^2 = ac$$
$\Rightarrow b^2 = a^2 + c^2 - ac$

37. $$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$
$$= \frac{b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{2abc}$$
$$= \frac{a^2+b^2+c^2}{2abc}$$

38. We have, a : b : c = 1 : $\sqrt{3}$: 2
i.e. a = λ, b = $\sqrt{3}$ λ, c = 2 λ
$$\cos A = \frac{3\lambda^2+4\lambda^2-\lambda^2}{2(\sqrt{3}\lambda)(2\lambda)} = \frac{6\lambda^2}{4\sqrt{3}\lambda^2} = \frac{\sqrt{3}}{2}$$
$\Rightarrow$ A = 30°

Similarly, $\cos B = \frac{1}{2} \Rightarrow B = 60°$,

$\cos C = 0 \Rightarrow C = 90°$.

Hence, A : B : C = 1 : 2 : 3

39. $(a^2 + b^2 - 2ab)\cos^2\frac{C}{2} + (a^2 + b^2 + 2ab)\sin^2\frac{C}{2}$

$= (a^2 + b^2)\left(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}\right) - 2ab\left(\cos^2\frac{C}{2} - \sin^2\frac{C}{2}\right)$

$= a^2 + b^2 - 2ab\cos C$

$= a^2 + b^2 - (a^2 + b^2 - c^2) = c^2$

40. $\cos A = \frac{\sin B}{2\sin C} \Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{b}{2c}$

$\Rightarrow b^2 + c^2 - a^2 - b^2 = 0 \Rightarrow c^2 = a^2$

$\Rightarrow c = a \Rightarrow$ Triangle is isosceles

41. $a = \sin\theta$, $b = \cos\theta$ and $c = \sqrt{1+\sin\theta\cos\theta}$

Since $\sqrt{1+\sin\theta\cos\theta}$ is greater than $\sin\theta$ and $\cos\theta$.

$\therefore$ C is the greatest angle,

$\therefore$ $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$= \frac{\sin^2\theta + \cos^2\theta - (1+\sin\theta\cos\theta)}{2\sin\theta\cos\theta}$

$= -\frac{1}{2} = \cos 120°$

$\therefore$ $C = 120°$

42. $1 - \tan\frac{A}{2}\tan\frac{B}{2} = \frac{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}$

$= \frac{\cos\left(\frac{A}{2}+\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}}$

$= \frac{\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}$

$= \left[\frac{(s-a)(s-b)bc.ac}{ab.s(s-a)s(s-b)}\right]^{1/2}$

$= \frac{c}{s} = \frac{2c}{a+b+c}$

43. $\frac{s(s-a)}{bc} - \frac{(s-b)(s-c)}{bc}$

$= \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = \cos\frac{2A}{2} = \cos A$

44. $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$

$\therefore$ $a\frac{s(s-c)}{ab} + c\frac{s(s-a)}{bc} = \frac{3b}{2}$

$\therefore$ $2s(s - c + s - a) = 3b^2$

$\therefore$ $2s(b) = 3b^2 \Rightarrow 2s = 3b \Rightarrow a + b + c = 3b$

$\therefore$ $a + c = 2b \Rightarrow$ a, b, c are in A.P.

45. $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = 1$

$\Rightarrow \tan\frac{C}{2} = \tan 45^\circ \Rightarrow \frac{C}{2} = 45^\circ$

$\Rightarrow C = 90°$

46. $\frac{\tan\frac{A}{2} - \tan\frac{B}{2}}{\tan\frac{A}{2} + \tan\frac{B}{2}} = \frac{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} - \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}}{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}}$

$= \frac{(s-b)\sqrt{s(s-c)} - (s-a)\sqrt{s(s-c)}}{(s-b)\sqrt{s(s-c)} + (s-a)\sqrt{s(s-c)}}$

$= \frac{\sqrt{s(s-c)}(s-b-s+a)}{\sqrt{s(s-c)}\ (s-b+s-a)} = \frac{a-b}{c}$

47. $\frac{1}{\sin^2\frac{A}{2}}, \frac{1}{\sin^2\frac{B}{2}}, \frac{1}{\sin^2\frac{C}{2}}$ are in A. P.

$\Rightarrow \frac{1}{\sin^2\frac{C}{2}} - \frac{1}{\sin^2\frac{B}{2}} = \frac{1}{\sin^2\frac{B}{2}} - \frac{1}{\sin^2\frac{A}{2}}$

$\Rightarrow \frac{ab}{(s-a)(s-b)} - \frac{ac}{(s-a)(s-c)} = \frac{ac}{(s-a)(s-c)} - \frac{bc}{(s-b)(s-c)}$

$\Rightarrow \left(\frac{a}{s-a}\right)\left(\frac{b(s-c) - c(s-b)}{(s-b)(s-c)}\right) = \left(\frac{c}{s-c}\right)\left(\frac{a(s-b) - b(s-a)}{(s-a)(s-b)}\right)$

$\Rightarrow abs - abc - acs + abc = acs - abc - bcs + abc$

$\Rightarrow ab - ac = ac - bc \Rightarrow ab + bc = 2ac$

$\Rightarrow \frac{1}{c} + \frac{1}{a} = \frac{2}{b} \Rightarrow$ a,b,c are in H. P.

48. Let $t = \tan\left(\frac{A-B}{2}\right)$

$\cos(A - B) = \frac{1-t^2}{1+t^2} \Rightarrow \frac{4}{5} = \frac{1-t^2}{1+t^2} \Rightarrow t = \frac{1}{3}$

So, $\tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$

Then, $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

$\Rightarrow \frac{1}{3} = \frac{6-3}{6+3}\cot\frac{C}{2} \Rightarrow C = 90^\circ$

$\therefore$ $\Delta = \frac{1}{2}(6)(3)\sin 90^\circ = 9$ square units.

49. Let the common multiple be x.

$\therefore$ the sides are $(2x)$, $\left(\sqrt{6}\,x\right)$, $\left(\sqrt{3}+1\right)x$

$\because$ $\left(\sqrt{3}+1\right)x$ is the largest side.

If θ is the angle opposite to side $\left(\sqrt{3}+1\right)x$, then

$$\cos\theta = \frac{(2x)^2 + \left(\sqrt{6}x\right)^2 - \left[\left(\sqrt{3}+1\right)x\right]^2}{2\times(2x)\times(\sqrt{6}\,x)}$$

$$= \frac{3-\sqrt{3}}{2\sqrt{6}}$$

$\therefore$ $\cos\theta = \frac{\sqrt{3}-1}{2\sqrt{2}} \Rightarrow \theta = 75^\circ$

50. We have,

$$\tan\left(\frac{A-B}{2}\right) = \sqrt{\frac{1-\cos(A-B)}{1+\cos(A-B)}} = \sqrt{\frac{1-\left(\frac{31}{32}\right)}{1+\left(\frac{31}{32}\right)}}$$

$\Rightarrow \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{1}{\sqrt{63}}$

$\Rightarrow \frac{1}{9}\cot\frac{C}{2} = \frac{1}{\sqrt{63}}$

$\Rightarrow \tan\frac{C}{2} = \frac{\sqrt{7}}{3}$

Now, $\cos C = \dfrac{1-\tan^2\left(\frac{C}{2}\right)}{1+\tan^2\left(\frac{C}{2}\right)}$

$\Rightarrow \cos C = \dfrac{1-\left(\frac{7}{9}\right)}{1+\left(\frac{7}{9}\right)} = \frac{1}{8}$

$\therefore$ $c^2 = a^2 + b^2 - 2ab\cos C$

$\Rightarrow c^2 = 25 + 16 - 40 \times \frac{1}{8} = 36 \Rightarrow c = 6$

51. Since $\sin^{-1} x$ cannot be greater than $\frac{\pi}{2}$.

$\therefore$ $\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \frac{\pi}{2}$

Therefore, $x = y = z = 1$

Putting these values in the expression, we get

$1 + 1 + 1 - \frac{9}{1+1+1} = 0$

52. $A = \tan^{-1}\left(\frac{2}{3}\right) \Rightarrow \tan A = \frac{2}{3}$

$B = \operatorname{cosec}^{-1}\left(\frac{5}{3}\right) \Rightarrow \tan B = \frac{3}{4}$

$\cot(A + B) = \frac{1-\tan A\tan B}{\tan A + \tan B}$

$$= \frac{1-\frac{2}{3}\cdot\frac{3}{4}}{\frac{2}{3}+\frac{3}{4}} = \frac{\frac{6}{12}}{\frac{17}{12}} = \frac{6}{17}$$

53. $\sin^2\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)$

$= \sin^2(2\theta)$, where $\theta = \tan^{-1}\sqrt{\frac{1+x}{1-x}}$

$= \left(\frac{2\tan\theta}{1+\tan^2\theta}\right)^2$, where $\tan\theta = \sqrt{\frac{1+x}{1-x}}$

$$= \left\{\frac{\frac{2\sqrt{1+x}}{\sqrt{1-x}}}{1+\left(\frac{1+x}{1-x}\right)}\right\}^2 = \frac{4(1+x)(1-x)}{(1-x+1+x)^2} = 1 - x^2$$

54. The principal value of $\sin^{-1}\left[\sin\left(\frac{2\pi}{3}\right)\right]$

$= \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{3}\right)\right] = \sin^{-1}\left[\sin\left(\frac{\pi}{3}\right)\right] = \frac{\pi}{3}$

55. Let $\sin^{-1}\frac{5}{13} = x \Rightarrow \sin x = \frac{5}{13}$

$\Rightarrow \cos x = \sqrt{1-\frac{25}{169}} = \frac{12}{13}$

$\Rightarrow \cos\left(\sin^{-1}\frac{5}{13}\right) = \cos\left(\cos^{-1}\frac{12}{13}\right) = \frac{12}{13}$

56. $\theta = \sin^{-1}[\sin(-600°)]$

$\Rightarrow \theta = \sin^{-1}[-(\sin 240°)]$

$\Rightarrow \theta = \sin^{-1}[-\sin(180° + 60°)]$

$\Rightarrow \theta = \sin^{-1}(\sin 60°) = \sin^{-1}\left[\sin\left(\frac{\pi}{3}\right)\right] = \frac{\pi}{3}$

57. $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right)$

$= \frac{\pi}{4} - x$

58. $\tan^{-1}\left[\frac{\frac{a\cos x - b\sin x}{b\cos x}}{\frac{b\cos x + a\sin x}{b\cos x}}\right] = \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right]$

$= \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x)$

$= \tan^{-1}\frac{a}{b} - x$

59. $\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) = \tan^{-1}\left[\frac{\sin(\frac{\pi}{2} - x)}{1 + \cos(\frac{\pi}{2} - x)}\right]$

$= \tan^{-1}\left[\frac{2\sin(\frac{\pi}{4} - \frac{x}{2})\cos(\frac{\pi}{4} - \frac{x}{2})}{2\cos^2(\frac{\pi}{4} - \frac{x}{2})}\right]$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right] = \frac{\pi}{4} - \frac{x}{2}$

60. $\tan^{-1}\left\{\frac{3a^2x - x^3}{a(a^2 - 3x^2)}\right\} = \tan^{-1}\left\{\frac{3a^2x - x^3}{a^3 - 3ax^2}\right\}$

$= \tan^{-1}\left(\frac{3\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2}\right)$

Put $\frac{x}{a} = \tan\theta$

$\therefore$ The given expression becomes

$\tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta)$

$= 3\theta = 3\tan^{-1}\frac{x}{a}$

61. $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$

Putting $x = \tan\theta$, we get

$3\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) - 4\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + 2\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \frac{\pi}{3}$

$\Rightarrow 3\sin^{-1}(\sin 2\theta) - 4\cos^{-1}(\cos 2\theta) + 2\tan^{-1}(\tan 2\theta) = \frac{\pi}{3}$

$\Rightarrow 3(2\theta) - 4(2\theta) + 2(2\theta) = \frac{\pi}{3}$

$\Rightarrow 6\theta - 8\theta + 4\theta = \frac{\pi}{3}$

$\Rightarrow \theta = \frac{\pi}{6} \Rightarrow \tan^{-1}x = \frac{\pi}{6}$

$\Rightarrow x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

62. $\tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right)$

$= \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta} - 1}{\tan\theta}\right]$

(Putting $x = \tan\theta$)

$= \tan^{-1}\left[\frac{\sec\theta - 1}{\tan\theta}\right] = \tan^{-1}\left[\frac{1 - \cos\theta}{\sin\theta}\right]$

$= \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$

$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$

63. Let $x = \sin\theta$ and $\sqrt{x} = \sin\phi$

Hence

$\therefore$ $\sin^{-1}(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2})$

$= \sin^{-1}(\sin\theta\sqrt{1-\sin^2\phi} - \sin\phi\sqrt{1-\sin^2\theta})$

$= \sin^{-1}(\sin\theta\cos\phi - \sin\phi\cos\theta)$

$= \sin^{-1}\sin(\theta - \phi)$

$= \theta - \phi = \sin^{-1}(x) - \sin^{-1}(\sqrt{x})$

64. $\cos^{-1}\left(\frac{1}{x}\right) = \theta \Rightarrow \sec^{-1}x = \theta$

$\therefore \quad x = \sec\theta$

$\therefore \quad \tan\theta = \sqrt{\sec^2\theta - 1}$

$= \sqrt{x^2 - 1}$

65. $\sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$

$\therefore \quad \sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1}x = \sin^{-1}x$

$\therefore \quad x = \frac{1}{5}$

66. $\sin^{-1}x + \sin^{-1}\frac{1}{x} + \cos^{-1}x + \cos^{-1}\frac{1}{x}$

$= \{\sin^{-1}(x) + \cos^{-1}(x)\} + \left\{\sin^{-1}\left(\frac{1}{x}\right) + \cos^{-1}\left(\frac{1}{x}\right)\right\}$

$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$

67. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x = \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$

68. $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

Since, $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} > -\tan^{-1}x > -\frac{\pi}{2}$

$\Rightarrow 0 < \frac{\pi}{2} - \tan^{-1}x < \pi$

69. $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow (\tan^{-1}x + \cot^{-1}x)^2$

$- 2\tan^{-1}x\left(\frac{\pi}{2} - \tan^{-1}x\right) = \frac{5\pi^2}{8}$

$\Rightarrow \frac{\pi^2}{4} - 2 \times \frac{\pi}{2}\tan^{-1}x + 2(\tan^{-1}x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow 2(\tan^{-1}x)^2 - \pi\tan^{-1}x - \frac{3\pi^2}{8} = 0$

$\Rightarrow \tan^{-1}x = -\frac{\pi}{4}, \frac{3\pi}{4}$

$\Rightarrow \tan^{-1}x = -\frac{\pi}{4} \Rightarrow x = -1$

70. $\cos\left[\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right] = \cos\left[\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\times\frac{1}{2}}\right)\right]$

$= \cos\{\tan^{-1}(1)\}$

$= \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

71. Let $\alpha = \cos^{-1}\left(\frac{4}{5}\right)$

$\therefore \quad \cos\alpha = \left(\frac{4}{5}\right) \Rightarrow \tan\alpha = \left(\frac{3}{4}\right)$

$\therefore \quad \alpha = \tan^{-1}\left(\frac{3}{4}\right)$

$\therefore \quad \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}$

$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}.\frac{3}{5}}\right) = \tan^{-1}\left(\frac{27}{11}\right)$

72. $\tan^{-1}\frac{x-1}{x+2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

$\therefore \quad \tan^{-1}\left[\frac{\frac{x-1}{x+2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x+2}\right)\left(\frac{x+1}{x+2}\right)}\right] = \frac{\pi}{4}$

$\therefore \quad \left[\frac{2x(x+2)}{x^2+4+4x-x^2+1}\right] = \tan\frac{\pi}{4}$

$\therefore \quad \frac{2x(x+2)}{4x+5} = 1$

$\therefore \quad 2x^2 + 4x = 4x + 5 \Rightarrow x = \pm\sqrt{\frac{5}{2}}$

73. $\tan^{-1}\left[\frac{1}{\sqrt{\cos\alpha}}\right] - \tan^{-1}\left[\sqrt{\cos\alpha}\right] = x$

$\Rightarrow \tan^{-1}\left[\frac{\frac{1}{\sqrt{\cos\alpha}} - \sqrt{\cos\alpha}}{1+\frac{\sqrt{\cos\alpha}}{\sqrt{\cos\alpha}}}\right] = x$

$\Rightarrow \tan x = \frac{1-\cos\alpha}{2\sqrt{\cos\alpha}}$

$\therefore \quad \sin x = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \tan^2\left(\frac{\alpha}{2}\right)$

74. $\tan^{-1}\frac{a+x}{a}+\tan^{-1}\frac{a-x}{a}=\frac{\pi}{6}$

$\therefore\quad \tan^{-1}\left(\frac{\frac{a+x}{a}+\frac{a-x}{a}}{1-\frac{a+x}{a}\cdot\frac{a-x}{a}}\right)=\frac{\pi}{6}$

$\therefore\quad \frac{2a^2}{x^2}=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}\Rightarrow x^2=2\sqrt{3}\,a^2$

75. $\tan^{-1}\left(\frac{a}{b+c}\right)+\tan^{-1}\left(\frac{b}{c+a}\right)$

$=\tan^{-1}\left(\frac{ac+bc+a^2+b^2}{ac+bc+c^2}\right)$

$\ldots\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$=\tan^{-1}(1)$ $\quad\ldots[\because c^2=a^2+b^2]$

$=\frac{\pi}{4}$

76. $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}$

$=\tan^{-1}\left[\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right]-\tan^{-1}\frac{8}{19}$

$=\tan^{-1}\frac{27}{11}-\tan^{-1}\frac{8}{19}=\tan^{-1}\left[\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right]$

$=\tan^{-1}\left(\frac{425}{425}\right)=\tan^{-1}(1)=\frac{\pi}{4}$

77. $\tan^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{xz}{yr}\right)$

$=\tan^{-1}\left[\frac{\frac{xy}{zr}+\frac{yz}{xr}+\frac{xz}{yr}-\frac{xyz}{r^3}}{1-\left(\frac{x^2+y^2+z^2}{r^2}\right)}\right]=\tan^{-1}(\infty)=\frac{\pi}{2}$

78. $\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)$

$+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\frac{1}{c_n}$

$=\tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}\cdot\frac{1}{c_1}}\right)+\tan^{-1}\left(\frac{\frac{1}{c_1}-\frac{1}{c_2}}{1+\frac{1}{c_1c_2}}\right)$

$+\tan^{-1}\left(\frac{\frac{1}{c_2}-\frac{1}{c_3}}{1+\frac{1}{c_2c_3}}\right)+\ldots+\tan^{-1}\frac{1}{c_n}$

$=\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{1}{c_1}\right)+\tan^{-1}\left(\frac{1}{c_1}\right)-\tan^{1}\left(\frac{1}{c_2}\right)$

$+\tan^{-1}\left(\frac{1}{c_2}\right)-\tan^{-1}\left(\frac{1}{c_3}\right)+\cdots+\tan^{-1}\left(\frac{1}{c_{n-1}}\right)$

$-\tan^{-1}\left(\frac{1}{c_n}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$

$=\tan^{-1}\left(\frac{x}{y}\right)$

79. $\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)$

$+\ldots\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)$

$=\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right)$

$+\ldots\ldots+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right)$

$=(\tan^{-1}a_2-\tan^{-1}a_1)+(\tan^{-1}a_3-\tan^{-1}a_2)$
$+\ldots\ldots+(\tan^{-1}a_n-\tan^{-1}a_{n-1})$

$=\tan^{-1}a_n-\tan^{-1}a_1=\tan^{-1}\left(\frac{a_n-a_1}{1+a_na_1}\right)$

$=\tan^{-1}\left(\frac{(n-1)d}{1+a_1a_n}\right)$

80. $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]$

$=\tan\left[\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}-\tan^{-1}(1)\right]$

$=\tan\left[\tan^{-1}\frac{5}{12}-\tan^{-1}(1)\right]$

$=\tan\left[\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)\right]=-\frac{7}{17}$

81. $\sin\left[3\sin^{-1}\left(\frac{1}{5}\right)\right] = \sin\left[\sin^{-1}\left\{3\left(\frac{1}{5}\right)-4\left(\frac{1}{5}\right)^3\right\}\right]$

$= \sin\left[\sin^{-1}\left\{\frac{3}{5}-\frac{4}{125}\right\}\right] = \sin\left[\sin^{-1}\left(\frac{75-4}{125}\right)\right]$

$= \sin\left[\sin^{-1}\frac{71}{125}\right] = \frac{71}{125}$

82. $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right] = \cot\left[\cot^{-1}\left(\frac{7}{24}\right)\right] = \frac{7}{24}$

....$\left[\because \cos^{-1}x = \cot^{-1}\frac{x}{\sqrt{1-x^2}}\right]$

83. Let $\sin^{-1}x = \theta \Rightarrow x = \sin\theta$

$\cos(2\sin^{-1}x) = \frac{1}{9}$ $\quad\Rightarrow \cos 2\theta = \frac{1}{9}$

$\Rightarrow 1 - 2\sin^2\theta = \frac{1}{9}$ $\quad\Rightarrow 1 - 2x^2 = \frac{1}{9}$

$\Rightarrow 2x^2 = 1 - \frac{1}{9} = \frac{8}{9}$ $\quad\Rightarrow x^2 = \frac{4}{9}$

$\Rightarrow x = \pm\frac{2}{3}$

84. $\sin\left[2\tan^{-1}\left(\frac{1}{3}\right)\right] + \cos\left[\tan^{-1}(2\sqrt{2})\right]$

$= \sin\left[\tan^{-1}\frac{2/3}{1-1/9}\right] + \cos\left[\tan^{-1}(2\sqrt{2})\right]$

$= \sin\left[\tan^{-1}\frac{3}{4}\right] + \cos\left[\tan^{-1}2\sqrt{2}\right]$

$= \sin\left[\sin^{-1}\frac{\left(\frac{3}{4}\right)}{\sqrt{1+\left(\frac{3}{4}\right)^2}}\right] + \cos\left[\cos^{-1}\frac{1}{\sqrt{1+\left(2\sqrt{2}\right)^2}}\right]$

$= \frac{3}{5} + \frac{1}{3} = \frac{14}{15}$

85. Given, $\tan^{-1}x = \sin^{-1}\left[\frac{3}{\sqrt{10}}\right]$

$\Rightarrow x = \tan\left\{\sin^{-1}\left[\frac{3}{\sqrt{10}}\right]\right\} = \tan\{\tan^{-1}3\}$

$\Rightarrow x = 3$

86. $\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right)$

$= \tan(\tan^{-1}7 - \tan^{-1}4)$

$= \tan\left[\tan^{-1}\left(\frac{7-4}{1+28}\right)\right] = \frac{3}{29}$

87. $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = \cot^{-1}\left(\frac{\sqrt{1-\frac{1}{5}}}{\frac{1}{\sqrt{5}}}\right) + \cot^{-1}3$

$= \cot^{-1}(2) + \cot^{-1}(3)$

$= \cot^{-1}\left(\frac{2\times3-1}{3+2}\right)$

$= \cot^{-1}(1) = \frac{\pi}{4}$

88. On expanding determinant,
$\cos^2(A+B) + \sin^2(A+B) + \cos 2B = 0$

$\therefore$ $1 + \cos 2B = 0 \Rightarrow \cos 2B = \cos\pi$

$\Rightarrow 2B = 2n\pi + \pi \Rightarrow B = (2n+1)\frac{\pi}{2}$, $n \in Z$.

Competitive Thinking

1. $\tan^2 x = 1$

$\Rightarrow \tan^2 x = \tan^2\frac{\pi}{4} \Rightarrow x = n\pi \pm \frac{\pi}{4}$

2. No solution as $|\sin x| \le 1$, $|\cos x| \le 1$ and both of them do not attain their maximum value for the same angle.

3. $\cot\theta + \tan\theta = 2$

$\therefore$ $\frac{1}{\tan\theta} + \tan\theta = 2 \Rightarrow 1 + \tan^2\theta = 2\tan\theta$

$\therefore$ $\frac{2\tan\theta}{1+\tan^2\theta} = 1 \Rightarrow \sin 2\theta = 1$

$\Rightarrow 2\theta = n\pi + (-1)^n\frac{\pi}{2}$

$\Rightarrow \theta = \frac{n\pi}{2} + (-1)^n\frac{\pi}{4}$

4. $\tan 2\theta = 1$
The value of $\tan\theta$ is positive if θ is in 1st and 3rd quadrant.

$\therefore$ Option (B) is the correct answer.

5. The given equation is defined for $x \ne \frac{\pi}{2}, \frac{3\pi}{2}$.

Now, $\sec x\cos 5x + 1 = 0$
$\Rightarrow \sec x\cos 5x = -1$
$\Rightarrow \cos 5x = -\cos x$
$\Rightarrow \cos 5x + \cos x = 0$
$\Rightarrow 2\cos 3x.\cos 2x = 0$
$\Rightarrow \cos 3x = 0$ or $\cos 2x = 0$
$\Rightarrow 3x = (2n+1)\frac{\pi}{2}$ or $2x = (2n+1)\frac{\pi}{2}$

$\Rightarrow x = \dfrac{(2n+1)\pi}{6}$ or $x = \dfrac{(2n+1)\pi}{4}$

$\Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ in $[0, 2\pi]$

$\therefore$ number of solutions = 8

6. $\cos\theta = \dfrac{-1}{2}$ and $0° < \theta < 360°$

$\therefore$ $\cos\theta = -\cos 60°$

$\therefore$ $\cos\theta = \cos(180° - 60°)$ and
$\cos\theta = \cos(180° + 60°)$
$\Rightarrow \cos\theta = \cos 120°$ and $\cos\theta = \cos 240°$
$\Rightarrow \theta = 120°$ and $240°$

7. $\cos\theta + \sqrt{3}\sin\theta = 2$

$\Rightarrow \dfrac{1}{2}\cos\theta + \dfrac{\sqrt{3}}{2}\sin\theta = 1$

$\Rightarrow \sin\left(\theta + \dfrac{\pi}{6}\right) = 1 = \sin\left(\dfrac{\pi}{2}\right) \Rightarrow \theta = \dfrac{\pi}{2} - \dfrac{\pi}{6} = \dfrac{\pi}{3}$

8. $\operatorname{cosec}\theta + 2 = 0$

$\Rightarrow \sin\theta = -\dfrac{1}{2}$

$\Rightarrow \sin\theta = -\sin 30°$
$\Rightarrow \sin\theta = \sin(180° + 30°)$ and
$\sin\theta = \sin(360° - 30°)$
$\Rightarrow \sin\theta = \sin 210°$ and $\sin\theta = \sin 330°$
$\Rightarrow \theta = 210°$ and $\theta = 330°$

9. $\sin x + \sin y + \sin z = -3$ is satisfied only when $x = y = z = \dfrac{3\pi}{2}$, for $x, y, z \in [0, 2\pi]$.

$\therefore$ option (A) is the correct answer.

10. The given equation is defined for $x \neq \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Now, $\tan x + \sec x = 2\cos x$

$\Rightarrow \dfrac{\sin x}{\cos x} + \dfrac{1}{\cos x} = 2\cos x$

$\Rightarrow (\sin x + 1) = 2\cos^2 x$
$\Rightarrow (\sin x + 1) = 2(1 - \sin^2 x)$
$\Rightarrow (\sin x + 1) = 2(1 - \sin x)(1 + \sin x)$
$\Rightarrow (1 + \sin x)[2(1 - \sin x) - 1] = 0$
$\Rightarrow 2(1 - \sin x) - 1 = 0$

$\ldots\left[\begin{array}{c}\because \sin x \neq -1 \text{ otherwise } \cos x = 0 \text{ and} \\ \tan x, \sec x \text{ will be undefined}\end{array}\right]$

$\Rightarrow \sin x = \dfrac{1}{2}$

$\Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ in $(0, 2\pi)$

$\therefore$ number of solutions = 2

11. $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$

$\Rightarrow \tan(\pi\cos\theta) = \tan\left(\dfrac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow \pi\cos\theta = \dfrac{\pi}{2} - \pi\sin\theta$

$\Rightarrow \sin\theta + \cos\theta = \dfrac{1}{2}$

$\Rightarrow \dfrac{1}{\sqrt{2}}\sin\theta + \dfrac{1}{\sqrt{2}}\cos\theta = \dfrac{1}{2\sqrt{2}}$

$\Rightarrow \cos\theta\cos\dfrac{\pi}{4} + \sin\theta\sin\dfrac{\pi}{4} = \dfrac{1}{2\sqrt{2}}$

$\Rightarrow \cos\left(\theta - \dfrac{\pi}{4}\right) = \dfrac{1}{2\sqrt{2}}$

12. $\cos^2\theta + \sin\theta + 1 = 0$
$\Rightarrow 1 - \sin^2\theta + \sin\theta + 1 = 0$
$\Rightarrow \sin^2\theta - \sin\theta - 2 = 0$
$\Rightarrow (\sin\theta + 1)(\sin\theta - 2) = 0$
$\Rightarrow \sin\theta = 2$, which is not possible and

$\sin\theta = -1 = \sin\dfrac{3\pi}{2}$

Therefore, solution of the given equation lies in the interval $\left(\dfrac{5\pi}{4}, \dfrac{7\pi}{4}\right)$.

13. $2\sin^2\theta = 4 + 3\cos\theta$
$\Rightarrow 2 - 2\cos^2\theta = 4 + 3\cos\theta$
$\Rightarrow 2\cos^2\theta + 3\cos\theta + 2 = 0$

$\Rightarrow \cos\theta = \dfrac{-3 \pm \sqrt{9-16}}{4}$,

which are imaginary, hence no solution.

14. $\cos 2x + k\sin x = 2k - 7$
$\Rightarrow 1 - 2\sin^2 x + k\sin x - 2k + 7 = 0$
$\Rightarrow 2\sin^2 x - k\sin x + 2k - 8 = 0$

$\Rightarrow \sin x = \dfrac{k \pm \sqrt{k^2 - 8(2k-8)}}{4}$

$\Rightarrow \sin x = \dfrac{k \pm (k-8)}{4} = \dfrac{k-4}{2}, 2$

Since, $\sin x \neq 2$
and $-1 < \sin x < 1$

$\therefore$ $-1 < \dfrac{k-4}{2} < 1 \Rightarrow -2 < k - 4 < 2$

$\Rightarrow 2 < k < 6$

15. $(1 + \tan\theta)(1 + \tan\phi) = 2 \Rightarrow \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = 1$

$\Rightarrow \tan(\theta + \phi) = 1 \Rightarrow \theta + \phi = \dfrac{\pi}{4} = 45°$

16. $(1 + \tan\alpha)(1 + \tan 4\alpha) = 2$

$\therefore$ $1 + \tan\alpha + \tan 4\alpha + \tan\alpha.\tan 4\alpha = 2$

$\therefore$ $\tan\alpha + \tan 4\alpha = 1 - \tan\alpha.\tan 4\alpha$

$\therefore$ $\dfrac{\tan\alpha + \tan 4\alpha}{1 - \tan\alpha.\tan 4\alpha} = 1$

$\therefore$ $\tan(\alpha + 4\alpha) = 1$

$\therefore$ $\tan 5\alpha = 1$

$\therefore$ $5\alpha = \dfrac{\pi}{4}$ ....$\left[\because \alpha \in \left(0, \dfrac{\pi}{16}\right)\right]$

$\therefore$ $\alpha = \dfrac{\pi}{20}$

17. $\cos x + \cos y = \dfrac{3}{2}$

$\Rightarrow 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right) = \dfrac{3}{2}$

$\Rightarrow 2\cos\dfrac{\pi}{3}\cos\left(\dfrac{x-y}{2}\right) = \dfrac{3}{2}$

....$\left[\because x + y = \dfrac{2\pi}{3}(\text{given})\right]$

$\Rightarrow \cos\left(\dfrac{x-y}{2}\right) = \dfrac{3}{2}$, which is not possible

....$\left[\because \dfrac{3}{2} > 1\right]$

Hence, the system of equations has no solution.

18. $81^{\sin^2 x} + 81^{\cos^2 x} = 30$ ....(i)

Check by options, put $x = \dfrac{\pi}{6}$ in (i),

$81^{\sin^2\frac{\pi}{6}} + 81^{\cos^2\frac{\pi}{6}} = 30$

$\Rightarrow (81)^{\frac{1}{4}} + (81)^{\frac{3}{4}} = 30 \Rightarrow 30 = 30$

$\therefore$ option (A) is the correct answer.

19. $4\sin^4 x + \cos^4 x = 1$

$\Rightarrow 4\sin^4 x = 1 - \cos^4 x$

$\Rightarrow 4\sin^4 x = (1 - \cos^2 x)(1 + \cos^2 x)$

$\Rightarrow 4\sin^4 x - (\sin^2 x)(1 + 1 - \sin^2 x) = 0$

$\Rightarrow \sin^2 x[4\sin^2 x - 2 + \sin^2 x] = 0$

$\Rightarrow \sin^2 x(5\sin^2 x - 2) = 0$

$\Rightarrow \sin x = 0$ or $\sin x = \pm\sqrt{\dfrac{2}{5}}$

Hence $x = n\pi$ is the required answer.

20. $1 - \cos\theta = \sin\theta.\sin\dfrac{\theta}{2}$

$\Rightarrow 2\sin^2\dfrac{\theta}{2} = 2\sin\dfrac{\theta}{2}.\cos\dfrac{\theta}{2}.\sin\dfrac{\theta}{2}$

$\Rightarrow 2\sin^2\dfrac{\theta}{2} - 2\sin^2\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = 0$

$\Rightarrow 2\sin^2\dfrac{\theta}{2}\left(1 - \cos\dfrac{\theta}{2}\right) = 0$

$\Rightarrow 2\sin^2\dfrac{\theta}{2}\left(2\sin^2\dfrac{\theta}{4}\right) = 0$

$\Rightarrow 2\sin^2\dfrac{\theta}{2} = 0$ or $2\sin^2\dfrac{\theta}{4} = 0$

$\Rightarrow \sin\dfrac{\theta}{2} = 0$ or $\sin\dfrac{\theta}{4} = 0$

$\Rightarrow \dfrac{\theta}{2} = k\pi$ or $\dfrac{\theta}{4} = k\pi$

$\Rightarrow \theta = 2k\pi$ or $\theta = 4k\pi$, $k \in I$

$\therefore$ option (B) is the correct answer.

21. $\sin 5x = \cos 2x$

$\Rightarrow \sin 5x = \sin\left(\dfrac{\pi}{2} - 2x\right)$

$\Rightarrow 5x = n\pi + (-1)^n\left(\dfrac{\pi}{2} - 2x\right)$

$\Rightarrow 5x + (-1)^n 2x = [2n + (-1)^n]\dfrac{\pi}{2}$

$\Rightarrow x[5 + 2(-1)^n] = [2n + (-1)^n]\dfrac{\pi}{2}$

$\Rightarrow x = \left[\dfrac{2n + (-1)^n}{5 + 2(-1)^n}\right]\dfrac{\pi}{2}$

22. $\tan 5\theta = \cot 2\theta$

$\Rightarrow \tan 5\theta = \tan\left(\dfrac{\pi}{2} - 2\theta\right)$

$\Rightarrow 5\theta = n\pi + \dfrac{\pi}{2} - 2\theta$

$\Rightarrow 7\theta = n\pi + \dfrac{\pi}{2}$

$\Rightarrow \theta = \dfrac{n\pi}{7} + \dfrac{\pi}{14}$

23. $\tan\theta = -1 \Rightarrow \tan\theta = \tan\left(2\pi - \dfrac{\pi}{4}\right)$

and $\cos\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \cos\theta = \cos\left(2\pi - \dfrac{\pi}{4}\right)$

$\therefore$ general value is $2n\pi + \left(2\pi - \dfrac{\pi}{4}\right) = 2n\pi + \dfrac{7\pi}{4}$

....$\left[\begin{array}{l}\text{If } \tan\theta = \tan\alpha \text{ and } \cos\theta = \cos\alpha \\ \Rightarrow \theta = 2n\pi + \alpha\end{array}\right]$

24. $\tan\theta = -\dfrac{1}{\sqrt{3}} = \tan\left(\pi - \dfrac{\pi}{6}\right)$,

$\sin\theta = \dfrac{1}{2} = \sin\left(\pi - \dfrac{\pi}{6}\right)$

and $\cos\theta = -\dfrac{\sqrt{3}}{2} = \cos\left(\pi - \dfrac{\pi}{6}\right)$

$\therefore$ principal value of $\theta = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$

25. $\cos p\theta = \cos q\theta \Rightarrow p\theta = 2n\pi \pm q\theta$

$\Rightarrow \theta = \dfrac{2n\pi}{p \pm q}$

26. $(2\cos x - 1)(3 + 2\cos x) = 0$

$\Rightarrow \cos x = \dfrac{1}{2}$ ....$\left[\because \cos x \neq \dfrac{-3}{2}\right]$

$\Rightarrow x = 2n\pi \pm \dfrac{\pi}{3}$

$\Rightarrow x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ in $[0, 2\pi]$

27. $\sin\left(\dfrac{\pi}{4}\cot\theta\right) = \cos\left(\dfrac{\pi}{4}\tan\theta\right)$

$\Rightarrow \sin\left(\dfrac{\pi}{4}\cot\theta\right) = \sin\left(\dfrac{\pi}{2} - \dfrac{\pi}{4}\tan\theta\right)$

$\Rightarrow \dfrac{\pi}{4}\cot\theta = \dfrac{\pi}{2} - \dfrac{\pi}{4}\tan\theta$

$\Rightarrow \tan\theta + \cot\theta = 2$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = 2$

$\Rightarrow \dfrac{1}{\sin\theta\cos\theta} = 2$

$\Rightarrow \sin 2\theta = 1 = \sin\dfrac{\pi}{2}$

$\Rightarrow 2\theta = (4n + 1)\dfrac{\pi}{2}$

$\Rightarrow \theta = n\pi + \dfrac{\pi}{4}$

28. $\tan 2\theta \tan\theta = 1$

$\Rightarrow \dfrac{\tan 2\theta}{\cot\theta} = 1$

$\Rightarrow \tan 2\theta = \cot\theta$

$\Rightarrow \tan 2\theta = \tan\left(\dfrac{\pi}{2} - \theta\right)$

$\Rightarrow 2\theta = n\pi + \dfrac{\pi}{2} - \theta$

$\Rightarrow 3\theta = n\pi + \dfrac{\pi}{2}$

$\Rightarrow \theta = \dfrac{n\pi}{3} + \dfrac{\pi}{6} = \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{3}$

29. $\cos 2\theta = \sin\alpha$

$\Rightarrow \cos 2\theta = \cos\left(\dfrac{\pi}{2} - \alpha\right)$

$\Rightarrow 2\theta = 2n\pi \pm \left(\dfrac{\pi}{2} - \alpha\right)$

$\Rightarrow \theta = n\pi \pm \left(\dfrac{\pi}{4} - \dfrac{\alpha}{2}\right)$

30. $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$

$\Rightarrow \sin 6\theta + \sin 2\theta + \sin 4\theta = 0$

$\Rightarrow 2\sin 4\theta\cos 2\theta + \sin 4\theta = 0$

$\Rightarrow \sin 4\theta(2\cos 2\theta + 1) = 0$

$\Rightarrow \sin 4\theta = 0$ or $2\cos 2\theta + 1 = 0$

$\Rightarrow 4\theta = n\pi$ or $\cos 2\theta = \dfrac{-1}{2}$

$\Rightarrow \theta = \dfrac{n\pi}{4}$ or $\cos 2\theta = -\cos\dfrac{\pi}{3}$

$\cos 2\theta = \cos\left(\pi - \dfrac{\pi}{3}\right)$

$\cos 2\theta = \cos\dfrac{2\pi}{3}$

$2\theta = 2n\pi \pm \dfrac{2\pi}{3}$

$\theta = n\pi \pm \dfrac{\pi}{3}$

31. $\sin 5x + \sin 3x + \sin x = 0$

$\Rightarrow -\sin 3x = \sin 5x + \sin x = 2\sin 3x\cos 2x$

$\Rightarrow \sin 3x = 0 \Rightarrow x = 0$

or $\cos 2x = -\dfrac{1}{2} = -\cos\left(\dfrac{\pi}{3}\right) = \cos\left(\pi - \dfrac{\pi}{3}\right)$

$\Rightarrow 2x = 2n\pi \pm \left(\pi - \dfrac{\pi}{3}\right) \Rightarrow x = n\pi \pm \left(\dfrac{\pi}{3}\right)$

$\Rightarrow x = \dfrac{\pi}{3}$ ....$\left[\because 0 \leq x \leq \dfrac{\pi}{2}\right]$

32. $\sin x + \sin 3x + \sin 5x = 0$

$\Rightarrow \sin 5x + \sin x + \sin 3x = 0$

$\Rightarrow 2\sin 3x\cos 2x + \sin 3x = 0$

$\Rightarrow \sin 3x(2\cos 2x + 1) = 0$

$\Rightarrow \sin 3x = 0$ or $2\cos 2x = -1$

$\Rightarrow 3x = n\pi$ or $\cos 2x = \frac{-1}{2}$

$\Rightarrow x = \frac{n\pi}{3}$ or $\cos 2x = -\cos\frac{\pi}{3}$

$\cos 2x = \cos\left(\pi - \frac{\pi}{3}\right)$

$\cos 2x = \cos\frac{2\pi}{3}$

$2x = 2n\pi \pm \frac{2\pi}{3}$

$x = n\pi \pm \frac{\pi}{3}$

$\Rightarrow x = \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$ $\quad \ldots\left[\because x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right]$

33. $\sin x - \sin 2x + \sin 3x = 2\cos^2 x - \cos x$

$\Rightarrow \sin x + \sin 3x - \sin 2x = \cos x(2\cos x - 1)$

$\Rightarrow 2\sin 2x \cos x - \sin 2x = \cos x(2\cos x - 1)$

$\Rightarrow \sin 2x(2\cos x - 1) = \cos x(2\cos x - 1)$

$\Rightarrow 2\sin x\cos x = \cos x$ or $2\cos x - 1 = 0$

$\Rightarrow \sin x = \frac{1}{2}$ or $\cos x = 0$ or $\cos x = \frac{1}{2}$

$\Rightarrow \sin x = \sin\frac{\pi}{6}$ or $\cos x = 0$ or $\cos x = \cos\frac{\pi}{3}$

$\Rightarrow x = n\pi + (-1)^n\frac{\pi}{6}$ or $x = (2n+1)\frac{\pi}{2}$

or $x = 2n\pi \pm \frac{\pi}{3}$

$\Rightarrow x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{6}$ $\quad \ldots[\because x \in (0, \pi)]$

34. $\sin x \cos x = \frac{1}{4}$

$\Rightarrow \sin 2x = \frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow 2x = n\pi + (-1)^n\frac{\pi}{6}$

$\Rightarrow x = \frac{n\pi}{2} + (-1)^n\frac{\pi}{12}$

$\Rightarrow x = \frac{\pi}{12}$ $\quad \ldots.\left[\because x \in \left(0, \frac{\pi}{2}\right)\right]$

35. $\sin\theta + \cos\theta = 1$

Dividing both sides by $\sqrt{1^2+1^2} = \sqrt{2}$, we get

$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$

$\ldots.[\because \sin(A+B) = \sin A\cos B + \cos A\sin B]$

$\Rightarrow \theta + \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{4}$

$\Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4}$

36. $\sqrt{2}\sec\theta + \tan\theta = 1 \Rightarrow \frac{\sqrt{2}}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = 1$

$\Rightarrow \sin\theta - \cos\theta = -\sqrt{2}$

$\Rightarrow \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = 1$

$\Rightarrow \cos\left(\theta + \frac{\pi}{4}\right) = \cos(0)$

$\Rightarrow \theta + \frac{\pi}{4} = 2n\pi \pm 0 \Rightarrow \theta = 2n\pi - \frac{\pi}{4}$

37. $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$

Dividing both sides by $\sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$,

we get

$\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta = \frac{\sqrt{2}}{2}$

$\Rightarrow \sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3} = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\left(\theta + \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \theta + \frac{\pi}{3} = n\pi + (-1)^n\frac{\pi}{4}$

$\Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{3}$

38. $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$

$\Rightarrow 2\sin 4\theta\cos 2\theta + \sin 4\theta = 0$

$\Rightarrow \sin 4\theta(2\cos 2\theta + 1) = 0$

$\Rightarrow \sin 4\theta = 0$ or $2\cos 2\theta + 1 = 0$

Now, $\sin 4\theta = 0 \Rightarrow 4\theta = n\pi \Rightarrow \theta = \frac{n\pi}{4}$

and $2\cos 2\theta = -1 \Rightarrow \cos 2\theta = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow 2\theta = 2n\pi \pm \frac{2\pi}{3} \Rightarrow \theta = n\pi \pm \frac{\pi}{3}$

$\therefore \quad \theta = \frac{n\pi}{4}$ or $n\pi \pm \frac{\pi}{3}$

39. $\sin 7\theta = \sin 4\theta - \sin\theta$

$\Rightarrow \sin 7\theta + \sin \theta - \sin 4\theta = 0$

$\Rightarrow 2\sin 4\theta \cos 3\theta - \sin 4\theta = 0$

$\Rightarrow \sin 4\theta (2 \cos 3\theta - 1) = 0$

$\Rightarrow \sin 4\theta = 0$ or $\cos 3\theta = \frac{1}{2}$

Now, $\sin 4\theta = 0 \Rightarrow 4\theta = \pi \Rightarrow \theta = \frac{\pi}{4}$

and $\cos 3\theta = \frac{1}{2} \Rightarrow 3\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{9}$

40. $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$

$\Rightarrow \tan (3x - 2x) = 1 \Rightarrow \tan x = 1$

$\Rightarrow \tan x = \tan \frac{\pi}{4}$

$\Rightarrow x = n\pi + \frac{\pi}{4}$

But this value does not satisfy the given equation. Hence, option (A) is the correct answer.

41. $\frac{\tan 3\theta - 1}{\tan 3\theta + 1} = \sqrt{3}$

$\Rightarrow \frac{\tan 3\theta - \tan\left(\frac{\pi}{4}\right)}{1 + \tan 3\theta \tan\left(\frac{\pi}{4}\right)} = \sqrt{3}$

$\Rightarrow \tan\left(3\theta - \frac{\pi}{4}\right) = \tan\frac{\pi}{3}$

$\Rightarrow 3\theta - \frac{\pi}{4} = n\pi + \frac{\pi}{3} \Rightarrow 3\theta = n\pi + \frac{7\pi}{12}$

$\Rightarrow \theta = \frac{n\pi}{3} + \frac{7\pi}{36}$

42. $\cos 3\theta = \sin 2\theta$

$\Rightarrow \cos 3\theta = \cos\left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow 3\theta = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow \theta = \frac{2n\pi}{3} \pm \left(\frac{\pi}{6} - \frac{2\theta}{3}\right)$

$\Rightarrow \theta = \frac{2n\pi}{5} + \frac{\pi}{10}$ or $\theta = 2n\pi - \frac{\pi}{2}$

Since, θ is acute

$\Rightarrow \theta = \frac{\pi}{10} = 18°$

$\Rightarrow \sin\theta = \frac{\sqrt{5} - 1}{4}$ ....$\left[\because \sin 18° = \frac{\sqrt{5} - 1}{4}\right]$

43. $\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$

$\Rightarrow (\sin x + \sin 3x) - 3 \sin 2x - (\cos x + \cos 3x) + 3\cos 2x = 0$

$\Rightarrow 2 \sin 2x \cos x - 3 \sin 2x - 2 \cos 2x \cos x + 3 \cos 2x = 0$

$\Rightarrow \sin 2x(2 \cos x - 3) - \cos 2x(2 \cos x - 3) = 0$

$\Rightarrow (\sin 2x - \cos 2x)(2 \cos x - 3) = 0$

$\Rightarrow \cos 2x = \sin 2x$ ....$\left[\because \cos x \neq \frac{3}{2}\right]$

$\Rightarrow \cos 2x = \cos\left(\frac{\pi}{2} - 2x\right)$

$\Rightarrow 2x = 2n\pi \pm \left(\frac{\pi}{2} - 2x\right)$

Neglecting (–) sign, we get

$x = \frac{n\pi}{2} + \frac{\pi}{8}$

44. $\cot (\alpha + \beta) = 0 \Rightarrow \cos(\alpha + \beta) = 0$

$\Rightarrow \alpha + \beta = (2n + 1)\frac{\pi}{2}$

$\therefore$ $\sin(\alpha + 2\beta) = \sin(2\alpha + 2\beta - \alpha)$

$= \sin[(2n + 1)\pi - \alpha]$

$= \sin(2n\pi + \pi - \alpha)$

$= \sin(\pi - \alpha) = \sin \alpha$

45. $\tan \theta + \tan 2\theta + \tan \theta \,.\, \tan 2\theta = 1$

$\Rightarrow \tan \theta + \tan 2\theta = 1 - \tan \theta \,.\, \tan 2\theta$

$\Rightarrow \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta . \tan 2\theta} = 1$

$\Rightarrow \tan(\theta + 2\theta) = 1$

$\Rightarrow \tan(3\theta) = 1 = \tan \frac{\pi}{4}$

$\Rightarrow 3\theta = n\pi + \frac{\pi}{4} \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{12}$

46. $\sec 4\theta - \sec 2\theta = 2$

$\Rightarrow \frac{1}{\cos 4\theta} - \frac{1}{\cos 2\theta} = 2$

$\Rightarrow \cos 2\theta - \cos 4\theta = 2\cos 4\theta \cos 2\theta$

$\Rightarrow \cos 2\theta - \cos 4\theta = \cos 6\theta + \cos 2\theta$

....[$\because$ 2 cosA cosB = cos (A + B) + cos (A – B)]

$\Rightarrow \cos 6\theta + \cos 4\theta = 0$

$\Rightarrow 2\cos 5\theta \cos \theta = 0$

....$\left[\because \cos C + \cos D = 2\cos\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)\right]$

$\Rightarrow \cos\theta = 0$ or $\cos 5\theta = 0$

$\Rightarrow \theta = (2n+1)\dfrac{\pi}{2}$ or $5\theta = (2n+1)\dfrac{\pi}{2}$

$\Rightarrow \theta = n\pi + \dfrac{\pi}{2}$ or $\theta = \dfrac{n\pi}{5} + \dfrac{\pi}{10}$

47. $\sin 2x + \sin 4x = 2\sin 3x$

$\Rightarrow 2\sin 3x\cos x - 2\sin 3x = 0$

$\Rightarrow \sin 3x = 0$ or $\cos x = 1 \Rightarrow 3x = n\pi$ or $x = 2n\pi$

$\Rightarrow x = \dfrac{n\pi}{3}$ or $x = 2n\pi$

48. $a\sin x + b\cos x = c$

$$\Rightarrow \frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x = \frac{c}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \cos\alpha\sin x + \sin\alpha\cos x = \frac{c}{\sqrt{a^2+b^2}}$$

$\Rightarrow \sin(x+\alpha) = \dfrac{c}{\sqrt{a^2+b^2}} > 1$, which is not possible.

$\therefore$ there is no solution.

49. $\tan\left(\dfrac{p\pi}{4}\right) = \cot\left(\dfrac{q\pi}{4}\right)$

$$\Rightarrow \tan\left(\frac{p\pi}{4}\right) = \tan\left(\frac{\pi}{2} - \frac{q\pi}{4}\right)$$

$$\Rightarrow \frac{p\pi}{4} = n\pi + \frac{\pi}{2} - \frac{q\pi}{4}$$

$$\Rightarrow \frac{p}{4} = n + \frac{1}{2} - \frac{q}{4}$$

$$\Rightarrow \frac{p+q}{4} = \frac{2n+1}{2}$$

$\Rightarrow p + q = 2(2n+1)$

50. $2\sin^2 x + \sin^2 2x = 2$

$\Rightarrow (1 - \cos 2x) + (1 - \cos^2 2x) = 2$

....[$\because \sin^2\theta + \cos^2\theta = 1$ and $2\sin^2\theta = 1 - \cos 2\theta$]

$\Rightarrow \cos 2x(\cos 2x + 1) = 0$

$\Rightarrow \cos 2x = 0$ or $\cos 2x = -1$

$\Rightarrow 2x = (2n+1)\dfrac{\pi}{2}$ or $(2n+1)\pi$

$\Rightarrow x = (2n+1)\dfrac{\pi}{4}$ or $(2n+1)\dfrac{\pi}{2}$

Putting n = – 2, – 1, 0, 1, 2, we get

$$x = \frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$$

and $\dfrac{-3\pi}{2}, \dfrac{-\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}$

Since, $-\pi < x < \pi$

$\therefore \quad x = \pm\dfrac{\pi}{4}, \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{4}$

$\therefore$ option (B) is the correct answer.

51. $\tan(\cot x) = \cot(\tan x)$

$$\Rightarrow \tan(\cot x) = \tan\left(\frac{\pi}{2} - \tan x\right)$$

$$\Rightarrow \cot x = n\pi + \frac{\pi}{2} - \tan x$$

$$\Rightarrow \cot x + \tan x = n\pi + \frac{\pi}{2}$$

$$\Rightarrow \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = n\pi + \frac{\pi}{2}$$

$$\Rightarrow \frac{2\left(\cos^2 x + \sin^2 x\right)}{2\sin x\cos x} = n\pi + \frac{\pi}{2}$$

$$\Rightarrow \frac{2}{\sin 2x} = n\pi + \frac{\pi}{2}$$

$$\Rightarrow \sin 2x = \frac{2}{n\pi + \frac{\pi}{2}} = \frac{4}{(2n+1)\pi}$$

52. Let $\sqrt{3} + 1 = r\cos\alpha$ and $\sqrt{3} - 1 = r\sin\alpha$.

Then $r = \sqrt{\left(\sqrt{3}+1\right)^2 + \left(\sqrt{3}-1\right)^2} = 2\sqrt{2}$

$$\tan\alpha = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{1-\left(\frac{1}{\sqrt{3}}\right)}{1+\left(\frac{1}{\sqrt{3}}\right)} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$\Rightarrow \alpha = \frac{\pi}{12}$$

The given equation reduces to

$2\sqrt{2}\cos(\theta - \alpha) = 2$

$$\Rightarrow \cos\left(\theta - \frac{\pi}{12}\right) = \cos\frac{\pi}{4}$$

$$\Rightarrow \theta - \frac{\pi}{12} = 2n\pi \pm \frac{\pi}{4} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$$

53. $\sec\theta - \operatorname{cosec}\theta = \dfrac{4}{3}$

$\Rightarrow 3(\sin\theta - \cos\theta) = 4\sin\theta\cos\theta$

$\Rightarrow 3(\sin\theta - \cos\theta) = 2\sin 2\theta$

Squaring on both sides, we get $9(1 - s) = 4s^2$,
where $s = \sin 2\theta$

$\Rightarrow 4s^2 + 9s - 9 = 0$

$\Rightarrow (s + 3)(4s - 3) = 0 \Rightarrow s = \dfrac{3}{4}$ ....[$\because \sin 2\theta \neq -3$]

$\Rightarrow \sin 2\theta = \dfrac{3}{4} = \sin \alpha$

$\Rightarrow 2\theta = n\pi + (-1)^n \alpha$

$\Rightarrow \theta = \dfrac{1}{2}\left[n\pi + (-1)^n \sin^{-1}\left(\dfrac{3}{4}\right)\right]$

54. Using $\sec 2\theta = \dfrac{1}{\cos 2\theta} = \dfrac{1+\tan^2\theta}{1-\tan^2\theta}$, we can write the given equation as

$\tan^2\theta + \dfrac{1+\tan^2\theta}{1-\tan^2\theta} = 1$

$\Rightarrow \tan^2\theta(1 - \tan^2\theta) + 1 + \tan^2\theta = 1 - \tan^2\theta$

$\Rightarrow 3\tan^2\theta - \tan^4\theta = 0$

$\Rightarrow \tan^2\theta(3 - \tan^2\theta) = 0$

$\Rightarrow \tan\theta = 0$ or $\tan^2\theta = 3$

$\Rightarrow \tan\theta = 0$ or $\tan^2\theta = \tan^2\dfrac{\pi}{3}$

$\Rightarrow \theta = m\pi$ or $\theta = n\pi \pm \dfrac{\pi}{3}$,

where m and n are integers.

55. $2\sqrt{3}\cos\theta = \tan\theta$

$\Rightarrow 2\sqrt{3}\cos^2\theta = \sin\theta$

$\Rightarrow 2\sqrt{3}\sin^2\theta + \sin\theta - 2\sqrt{3} = 0$

$\Rightarrow \sin\theta = \dfrac{-1\pm 7}{4\sqrt{3}} \Rightarrow \sin\theta = \dfrac{-8}{4\sqrt{3}}$,

which is not possible

and $\sin\theta = \dfrac{6}{4\sqrt{3}} = \dfrac{\sqrt{3}}{2} = \sin\dfrac{\pi}{3}$

$\Rightarrow \theta = n\pi + (-1)^n \dfrac{\pi}{3}$

56. $3\sin^2 x - 7\sin x + 2 = 0$

$\Rightarrow 3\sin^2 x - 6\sin x - \sin x + 2 = 0$

$\Rightarrow 3\sin x(\sin x - 2) - (\sin x - 2) = 0$

$\Rightarrow (3\sin x - 1)(\sin x - 2) = 0$

$\Rightarrow \sin x = \dfrac{1}{3}$ or 2

$\Rightarrow \sin x = \dfrac{1}{3}$ ....[$\because \sin x \neq 2$]

Let $\sin^{-1}\dfrac{1}{3} = \alpha$, $0 < \alpha < \dfrac{\pi}{2}$ are the solutions in $[0, 5\pi]$. Then, α, $\pi - \alpha$, $2\pi + \alpha$, $3\pi - \alpha$, $4\pi + \alpha$, $5\pi - \alpha$ are the solutions in $[0, 5\pi]$.

$\therefore$ number of solutions = 6

57. $\sin 2x + \cos 2x = 0$

$\Rightarrow (\sin 2x + \cos 2x)^2 = 0$

$\Rightarrow \sin^2 2x + \cos^2 2x + 2\sin 2x\cos 2x = 0$

$\Rightarrow 1 + \sin 4x = 0 \Rightarrow \sin 4x = -1$

$\therefore$ $4x = n\pi + (-1)^n\left(\dfrac{-\pi}{2}\right)$

$\therefore$ $4x = n\pi + (-1)^{n+1}\dfrac{\pi}{2}$

$\therefore$ $x = \dfrac{n\pi}{4} + (-1)^{n+1}\dfrac{\pi}{8}$

$\therefore$ For $\pi < x < 2\pi$, the values of x are $\dfrac{11\pi}{8}, \dfrac{15\pi}{8}$.

58. $2\sin^2\theta = 3\cos\theta$

$\Rightarrow 2 - 2\cos^2\theta = 3\cos\theta$

$\Rightarrow 2\cos^2\theta + 3\cos\theta - 2 = 0$

$\Rightarrow \cos\theta = \dfrac{-3\pm\sqrt{9+16}}{4} = \dfrac{-3\pm 5}{4}$

Neglecting (–) sign, we get

$\cos\theta = \dfrac{1}{2} = \cos\left(\dfrac{\pi}{3}\right) \Rightarrow \theta = 2n\pi \pm \dfrac{\pi}{3}$

The values of θ between 0 and 2π are $\dfrac{\pi}{3}, \dfrac{5\pi}{3}$.

59. $5\cos 2\theta + 2\cos^2\dfrac{\theta}{2} + 1 = 0$

$\Rightarrow 5(2\cos^2\theta - 1) + (1 + \cos\theta) + 1 = 0$

$\Rightarrow 10\cos^2\theta + \cos\theta - 3 = 0$

$\Rightarrow (5\cos\theta + 3)(2\cos\theta - 1) = 0$

$\Rightarrow \cos\theta = \dfrac{1}{2}, \cos\theta = -\dfrac{3}{5}$

$\Rightarrow \theta = \dfrac{\pi}{3}, \pi - \cos^{-1}\left(\dfrac{3}{5}\right)$

60. $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$

$\Rightarrow 2 - 2\cos^2\theta + \sqrt{3}\cos\theta + 1 = 0$

$\Rightarrow 2\cos^2\theta - \sqrt{3}\cos\theta - 3 = 0$

$\Rightarrow \cos\theta = \dfrac{\sqrt{3}\pm\sqrt{3+24}}{4} = \dfrac{\sqrt{3}(1\pm 3)}{4} = \sqrt{3}\left(-\dfrac{1}{2}\right)$

$\Rightarrow \theta = \dfrac{5\pi}{6}$

61. $3\sin^2 x + 10\cos x - 6 = 0$
$\Rightarrow 3(1 - \cos^2 x) + 10\cos x - 6 = 0$
$\Rightarrow 3 - 3\cos^2 x + 10\cos x - 6 = 0$
$\Rightarrow 3\cos^2 x - 10\cos x + 3 = 0$
$\Rightarrow 3\cos^2 x - 9\cos x - \cos x + 3 = 0$
$\Rightarrow 3\cos x(\cos x - 3) - 1(\cos x - 3) = 0$
$\Rightarrow (\cos x - 3)(3\cos x - 1) = 0$
$\Rightarrow \cos x = 3$, (which is not possible)
or $\cos x = \frac{1}{3}$
$\Rightarrow \cos x = \frac{1}{3} = \cos\alpha$ (say)
$\Rightarrow x = 2n\pi \pm \alpha$
$\Rightarrow x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$

62. $\cos^2 x - 2\cos x = 4\sin x - \sin 2x$
$\Rightarrow \cos^2 x - 2\cos x = 4\sin x - 2\sin x\cos x$
$\Rightarrow \cos x(\cos x - 2) = 2\sin x(2 - \cos x)$
$\Rightarrow \cos x(\cos x - 2) - 2\sin x(2 - \cos x) = 0$
$\Rightarrow \cos x(\cos x - 2) + 2\sin x(\cos x - 2) = 0$
$\Rightarrow (\cos x - 2)(\cos x + 2\sin x) = 0$
$\Rightarrow \cos x + 2\sin x = 0$ ....[$\because \cos x \neq 2$]
$\Rightarrow \cos x = -2\sin x$
$\Rightarrow \tan x = -\frac{1}{2} = \tan\alpha$ (say)
$\Rightarrow x = n\pi + \alpha$
$\Rightarrow x = n\pi + \tan^{-1}\left(-\frac{1}{2}\right), n \in I$
Since, $0 \le x \le \pi$
$\therefore$ $x = \pi + \tan^{-1}\left(-\frac{1}{2}\right)$

63. $\cos 2\theta = (\sqrt{2}+1)\left(\cos\theta - \frac{1}{\sqrt{2}}\right)$
$\Rightarrow 2\cos^2\theta - 1 = \frac{\sqrt{2}+1}{\sqrt{2}}(\sqrt{2}\cos\theta - 1)$
$\Rightarrow 2\cos^2\theta - 1 - \frac{\sqrt{2}+1}{\sqrt{2}}(\sqrt{2}\cos\theta - 1) = 0$
$\Rightarrow (\sqrt{2}\cos\theta - 1)\left\{(\sqrt{2}\cos\theta + 1) - \left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)\right\} = 0$
$\Rightarrow \sqrt{2}\cos\theta - 1 = 0$ or $\sqrt{2}\cos\theta + 1 = \frac{\sqrt{2}+1}{\sqrt{2}}$
$\Rightarrow \sqrt{2}\cos\theta = 1$ or $\sqrt{2}\cos\theta = \frac{\sqrt{2}+1-\sqrt{2}}{\sqrt{2}}$
$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ or $\sqrt{2}\cos\theta = \frac{1}{\sqrt{2}}$
$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ or $\cos\theta = \frac{1}{2}$
$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ ....$\left[\because \cos\theta \neq \frac{1}{2}\right]$
$\Rightarrow \cos\theta = \cos\frac{\pi}{4}$
$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{4}$

64. $\cos 2\theta = \frac{1}{3}$
$\Rightarrow \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1}{3}$
$\Rightarrow 3 - 3\tan^2\theta = 1 + \tan^2\theta$ $\Rightarrow 2 = 4\tan^2\theta$
$\Rightarrow \tan^2\theta = \frac{1}{2}$
$\Rightarrow \tan^8\theta = \frac{1}{16}$
Now, $32\tan^8\theta = 2\cos^2\alpha - 3\cos\alpha$
$\Rightarrow 32\left(\frac{1}{16}\right) = 2\cos^2\alpha - 3\cos\alpha$
$\Rightarrow 2\cos^2\alpha - 3\cos\alpha - 2 = 0$
$\Rightarrow (2\cos\alpha + 1)(\cos\alpha - 2) = 0$
But $\cos\alpha - 2 \neq 0$
$\therefore$ $2\cos\alpha + 1 = 0$
$\Rightarrow \cos\alpha = -\frac{1}{2}$
$\Rightarrow \cos\alpha = \cos\frac{2\pi}{3}$
$\Rightarrow \alpha = 2n\pi \pm \frac{2\pi}{3}$

65. $\cos 2\theta = \sin\theta \Rightarrow 1 - 2\sin^2\theta = \sin\theta$
$\Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$
$\Rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$
$\Rightarrow \sin\theta = \frac{1}{2}$ or $\sin\theta = -1$
$\therefore$ $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6} \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{6}$
and $\sin\theta = -1 = \sin\frac{3\pi}{2}$
$\Rightarrow \theta = m\pi + (-1)^m\frac{3\pi}{2}$
$\therefore$ $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
$\therefore$ number of solutions = 3

66. $\tan\theta = \cot 5\theta$

$\Rightarrow \tan\theta - \cot 5\theta = 0$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} - \dfrac{\cos 5\theta}{\sin 5\theta} = 0$

$\Rightarrow \cos 5\theta \cos\theta - \sin 5\theta \sin\theta = 0$

$\Rightarrow \cos(5\theta + \theta) = 0$

$\Rightarrow \cos 6\theta = 0 = \cos\dfrac{\pi}{2}$

$\Rightarrow 6\theta = 2n\pi \pm \dfrac{\pi}{2}$

$\Rightarrow 6\theta = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \dfrac{5\pi}{2}$

$\Rightarrow \theta = \pm\dfrac{\pi}{12}, \pm\dfrac{\pi}{4}, \dfrac{5\pi}{12}$

and $\sin 2\theta = \cos 4\theta$

$\Rightarrow \sin 2\theta = 1 - 2\sin^2 2\theta$

$\Rightarrow 2\sin^2 2\theta + \sin 2\theta - 1 = 0$

$\Rightarrow (2\sin 2\theta - 1)(\sin 2\theta + 1) = 0$

$\Rightarrow \sin 2\theta = \dfrac{1}{2}$ or $\sin 2\theta = -1$

$\Rightarrow \sin 2\theta = \sin\left(\dfrac{\pi}{6}\right)$ or $\sin 2\theta = -1$

$\Rightarrow 2\theta = n\pi + (-1)^n\dfrac{\pi}{6}$ or $2\theta = (4n-1)\dfrac{\pi}{2}$

$\Rightarrow 2\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ or $2\theta = -\dfrac{\pi}{2}$

$\Rightarrow \theta = \dfrac{\pi}{12}, \dfrac{5\pi}{12}$ or $\theta = -\dfrac{\pi}{4}$

$\therefore$ the common values of θ are $-\dfrac{\pi}{4}, \dfrac{\pi}{12}$ and $\dfrac{5\pi}{12}$.

Hence, there are 3 values of θ satisfying the given equation.

67. $\cos^2\left(x+\dfrac{\pi}{6}\right) + \cos^2 x - 2\cos\left(x+\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{6}\right) = \sin^2\dfrac{\pi}{6}$

$\Rightarrow \cos^2\left(x+\dfrac{\pi}{6}\right) + \left(\cos^2 x - \sin^2\dfrac{\pi}{6}\right) - 2\cos\left(x+\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{6}\right) = 0$

$\Rightarrow \cos^2\left(x+\dfrac{\pi}{6}\right) + \cos\left(x+\dfrac{\pi}{6}\right)\cos\left(x-\dfrac{\pi}{6}\right) - 2\cos\left(x+\dfrac{\pi}{6}\right)\cos\dfrac{\pi}{6} = 0$

$\Rightarrow \cos\left(x+\dfrac{\pi}{6}\right)\left\{\cos\left(x+\dfrac{\pi}{6}\right) + \cos\left(x-\dfrac{\pi}{6}\right) - 2\cos\dfrac{\pi}{6}\right\} = 0$

$\Rightarrow \cos\left(x+\dfrac{\pi}{6}\right)\left\{2\cos x\cos\dfrac{\pi}{6} - 2\cos\dfrac{\pi}{6}\right\} = 0$

$\Rightarrow 2\cos\left(x+\dfrac{\pi}{6}\right)\cos\dfrac{\pi}{6}(\cos x - 1) = 0$

$\Rightarrow \cos\left(x+\dfrac{\pi}{6}\right)(\cos x - 1) = 0$

$\Rightarrow \cos\left(x+\dfrac{\pi}{6}\right) = 0$ or $\cos x = 1$

$\Rightarrow x + \dfrac{\pi}{6} = (2n+1)\dfrac{\pi}{2}$ or $x = 2n\pi$

$\Rightarrow x + \dfrac{\pi}{6} = \pm\dfrac{\pi}{2}$ or $x = 0$

$\Rightarrow x = \dfrac{\pi}{3}, \dfrac{-2\pi}{3}, 0$

$\Rightarrow x = 0, \dfrac{\pi}{3}$ ....$\left[\because x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)\right]$

$\therefore$ number of solutions = 2.

68. $8\cos x\left[\cos\left(\dfrac{\pi}{6}+x\right)\cdot\cos\left(\dfrac{\pi}{6}-x\right) - \dfrac{1}{2}\right] = 1$

$\Rightarrow 8\cos x\left(\cos^2\dfrac{\pi}{6} - \sin^2 x - \dfrac{1}{2}\right) = 1$

$\Rightarrow 8\cos x\left(\dfrac{3}{4} - \sin^2 x - \dfrac{1}{2}\right) = 1$

$\Rightarrow 8\cos x\left(\dfrac{1}{4} - \left(1 - \cos^2 x\right)\right) = 1$

$\Rightarrow 2(4\cos^3 x - 3\cos x) = 1$

$\Rightarrow 2\cos 3x = 1$

$\Rightarrow \cos 3x = \dfrac{1}{2}$

$\Rightarrow \cos 3x = \cos\dfrac{\pi}{3}$

$\Rightarrow x = \dfrac{2n\pi}{3} \pm \dfrac{\pi}{9}$

$\Rightarrow x = \dfrac{\pi}{9}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}$ ...$[\because x \in [0, \pi]]$

Sum $= \dfrac{\pi}{9} + \dfrac{5\pi}{9} + \dfrac{7\pi}{9} = \dfrac{13\pi}{9}$

$\Rightarrow k = \dfrac{13}{9}$

69. $\sec^2\theta = \frac{4}{3}$

$\Rightarrow \cos^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow \cos^2\theta = \cos^2\left(\frac{\pi}{6}\right)$

$\Rightarrow \theta = n\pi \pm \frac{\pi}{6}$

....[$\because \cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$]

70. $\cot\theta = \sin 2\theta$, $(\theta \neq n\pi)$

$\Rightarrow \frac{\cos\theta}{\sin\theta} = 2\sin\theta\cos\theta$

$\Rightarrow 2\sin^2\theta\cos\theta = \cos\theta$

$\Rightarrow \cos\theta(2\sin^2\theta - 1) = 0$

$\Rightarrow \cos\theta = 0$ or $\sin^2\theta = \frac{1}{2}$

$\Rightarrow \cos\theta = 0$ or $\sin^2\theta = \sin^2\left(\frac{\pi}{4}\right)$

$\Rightarrow \theta = (2n+1)\frac{\pi}{2}$ or $\theta = n\pi \pm \frac{\pi}{4}$

$\Rightarrow \theta = 90°$ and $45°$

...$\left[\begin{array}{l}\because \text{at } \theta = 90° \text{ and } 45°, \\ \text{the given equation is satisfied.}\end{array}\right]$

71. We have, $x - y = \frac{\pi}{4}$ ....(i)

and $\cot x + \cot y = 2$

$\Rightarrow \frac{1}{\tan x} + \frac{1}{\tan y} = 2$ ....(ii)

From (i) and (ii), we get

$\frac{1}{\tan\left(y + \frac{\pi}{4}\right)} + \frac{1}{\tan y} = 2$

$\Rightarrow (1 - \tan y)\tan y + 1 + \tan y = 2\tan y(1 + \tan y)$

$\Rightarrow 3\tan^2 y = 1$

$\Rightarrow \tan^2 y = \frac{1}{3} = \tan^2\frac{\pi}{6}$

$\Rightarrow y = \frac{\pi}{6}$ ....[smallest +ve value]

From (i),

$x = \frac{\pi}{4} + y = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

72. $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$

$\therefore$ $\cos x + \cos 4x + \cos 2x + \cos 3x = 0$

$\therefore$ $2\cos\frac{5x}{2}\cdot\cos\frac{3x}{2} + 2\cos\frac{5x}{2}\cdot\cos\frac{x}{2} = 0$

$\Rightarrow \cos\frac{5x}{2}\cdot\left[\cos\frac{3x}{2} + \cos\frac{x}{2}\right] = 0$

$\Rightarrow \cos\frac{5x}{2}.2\cos x.\cos\frac{x}{2} = 0$

$\Rightarrow x = (2n+1)\frac{\pi}{5}, (2k+1)\frac{\pi}{2}$ or $(2m+1)\pi$

$\Rightarrow x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{2}$ in $0 \leq x < 2\pi$

73. Let the angles of the triangle be x, $2x$ and $3x$.
Then, $x + 2x + 3x = 180° \Rightarrow x = 30°$

$\therefore$ angles of the triangle are 30°, 60° and 90°.

$\therefore$ a : b : c = sin30°: sin60° : sin90°

$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2$

74. Let x be the common multiple.

$\therefore$ $A + B + C = 12x = 180° \Rightarrow x = 15°$

$\therefore$ $A = 45°$, $B = 75°$, $C = 60°$

$\frac{a}{\sin 45°} = \frac{b}{\sin 75°} = \frac{c}{\sin 60°} = k$

$\therefore$ $a = \frac{1}{\sqrt{2}}k$, $b = \frac{\sqrt{3}+1}{2\sqrt{2}}k$, $c = \frac{\sqrt{3}}{2}k$

$\therefore$ $a + b + c\sqrt{2} = \frac{3 + 3\sqrt{3}}{2\sqrt{2}} = 3b$

75. Let the angles of the triangle be $4x$, x and x.

$\therefore$ $4x + x + x = 180° \Rightarrow 6x = 180° \Rightarrow x = 30°$

$\frac{\sin 120°}{a} = \frac{\sin 30°}{b} = \frac{\sin 30°}{c}$

$\therefore$ a : (a + b + c)

= (sin 120°) : (sin 120° + sin 30° + sin 30°)

$= \frac{\sqrt{3}}{2} : \frac{\sqrt{3}+2}{2} = \sqrt{3} : \sqrt{3} + 2$

76. Given, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ ....(i)

By Sine rule,

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ ....(ii)

From (i) and (ii), we get

$\Rightarrow \frac{\cos A}{\sin A} = \frac{\cos B}{\sin B} = \frac{\cos C}{\sin C}$

$\Rightarrow \cot A = \cot B = \cot C$

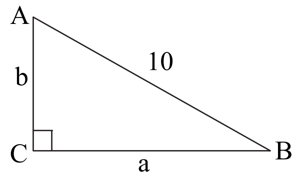
$\Rightarrow A = B = C = 60°$

$\Rightarrow \Delta ABC$ is equilateral.

$\therefore$ $\Delta = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}(2)^2 = \sqrt{3}$

77. $\sin^2 A + \sin^2 B = \sin^2 C$

$\Rightarrow \sin C = 1 \Rightarrow C = \dfrac{\pi}{2}$



$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$\Rightarrow \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{10}{1}$

$\Rightarrow a = 10\sin A,\ b = 10\sin B$

$A(\Delta ABC) = \dfrac{1}{2}ab = \dfrac{1}{2}(10 \sin A)(10 \sin B)$

$= \dfrac{1}{2} \times 100 \times \sin A \times \sin B$

Maximum value of $\sin A \sin B = \dfrac{1}{2}$

$\therefore$ $A(\Delta ABC) = \dfrac{1}{2} \times 100 \times \dfrac{1}{2}$

$= 25$ sq. units

78. $\sin A \sin B = \dfrac{ab}{c^2}$

$\Rightarrow \sin A \sin B = \dfrac{(k\sin A)(k\sin B)}{k^2 \sin^2 C}$

$\Rightarrow \sin^2 C = 1 \Rightarrow \sin C = 1$ ....[$\because \sin C \neq -1$]

$\Rightarrow \angle C = 90°$

$\Rightarrow \Delta ABC$ is right angled.

79. According to the given condition,

$a + b + c = \dfrac{6(\sin A + \sin B + \sin C)}{3}$

$\Rightarrow k(\sin A + \sin B + \sin C) = 2(\sin A + \sin B + \sin C)$

where $k = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$\Rightarrow k = 2$ ....[$\because \sin A + \sin B + \sin C \neq 0$]

$\therefore$ $\dfrac{a}{\sin A} = 2 \Rightarrow \sin A = \dfrac{1}{2}$ ....[$\because a = 1$]

$\Rightarrow A = \dfrac{\pi}{6}$

80. $\dfrac{a+c}{b} = \dfrac{\sin A + \sin C}{\sin B}$

$= \dfrac{2\sin\left(\dfrac{A+C}{2}\right)\cos\left(\dfrac{A-C}{2}\right)}{\sin B}$

$= \dfrac{2\sin B}{\sin B}\cos\left(\dfrac{A-C}{2}\right)$ ....[$\because 2B = A + C$]

$= 2\cos\left(\dfrac{A-C}{2}\right)$

81. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

According to the given condition,

In Δ ABC, $a = 2b$ and

$A - B = 60° \Rightarrow A = 60° + B$

$\Rightarrow \dfrac{\sin(60° + B)}{2b} = \dfrac{\sin B}{b}$

$\Rightarrow \dfrac{\sin B}{\sin(B + 60°)} = \dfrac{1}{2}$

$\Rightarrow 2 \sin B = \sin B \cos 60° + \cos B \sin 60°$

$\Rightarrow \dfrac{3}{2}\sin B = \dfrac{\sqrt{3}}{2}\cos B$

$\therefore$ $\tan B = \dfrac{1}{\sqrt{3}} \Rightarrow B = 30°$

$\therefore$ $A = 30° + 60° = 90°$

$\therefore$ ΔABC is right angled.

82. $\dfrac{\cos 2A}{a^2} - \dfrac{\cos 2B}{b^2}$

$= \dfrac{1 - 2\sin^2 A}{a^2} - \dfrac{1 - 2\sin^2 B}{b^2}$

$= \dfrac{1}{a^2} - \dfrac{1}{b^2} - \dfrac{2\sin^2 A}{a^2} + \dfrac{2\sin^2 B}{b^2}$

$= \dfrac{1}{a^2} - \dfrac{1}{b^2} - 2\left(\dfrac{\sin^2 A}{a^2} - \dfrac{\sin^2 B}{b^2}\right)$

$= \dfrac{1}{a^2} - \dfrac{1}{b^2}$ ....$\left[\text{By sine rule, } \dfrac{a}{\sin A} = \dfrac{b}{\sin B}\right]$.

83. $\cos C = \dfrac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5}$

$\Rightarrow \cos C = -\dfrac{1}{2}$

$\Rightarrow C = 120°$

$\therefore$ option (B) is the correct answer.

84. $2\cos A = \dfrac{\sin B}{\sin C} \Rightarrow \dfrac{2(c^2 + b^2 - a^2)}{2bc} = \dfrac{b}{c}$

$\Rightarrow c^2 = a^2 \Rightarrow c = a$

85. $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac} \Rightarrow \cos B = \dfrac{1}{2} \Rightarrow B = \dfrac{\pi}{3}$

86. $(a + b + c)(a - b + c) = 3ac$

$\Rightarrow a^2 + 2ac + c^2 - b^2 = 3ac$

$\Rightarrow a^2 + c^2 - b^2 = ac$

But $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{1}{2} \Rightarrow B = 60°$

87. $\cos A = \dfrac{8^2+10^2-6^2}{2.8.10} = \dfrac{128}{160} = \dfrac{4}{5}$

$\therefore$ $\sin A = \dfrac{3}{5}$

$\therefore$ $\sin 2A = 2 \sin A . \cos A = 2.\dfrac{3}{5}.\dfrac{4}{5} = \dfrac{24}{25}$

88. $\cos C = \dfrac{81+64-x^2}{2.9.8} \Rightarrow \dfrac{2}{3} = \dfrac{145-x^2}{144}$

$\Rightarrow x^2 = 49 \Rightarrow x = 7$

89. $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{9+16-4}{2\times3\times4} = \dfrac{7}{8}$

$\Rightarrow A = \cos^{-1}\left(\dfrac{7}{8}\right)$

90. Let a = 3, b = 5, c = 7

$\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{9+25-49}{2\times3\times5} = \dfrac{-15}{30} = -\dfrac{1}{2}$

$\therefore$ $\angle C = \dfrac{2\pi}{3}$

91. Since, $c = \sqrt{13}$ is the smallest side.

$\therefore$ C is the smallest angle.

$\therefore$ $\cos C = \dfrac{b^2+a^2-c^2}{2ab} = \dfrac{48+49-13}{2\times7\times4\sqrt{3}}$

$\Rightarrow \cos C = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow C = 30°$

92. $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{\sin A\cos B - \sin B\cos A}{\sin C}$

$= \dfrac{a}{c}\cos B - \dfrac{b}{c}\cos A$

But $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

$\therefore$ $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{1}{2c^2}(a^2+c^2-b^2-b^2-c^2+a^2)$

$= \dfrac{a^2-b^2}{c^2}$

93. $\dfrac{1}{a+c}+\dfrac{1}{b+c} = \dfrac{3}{a+b+c}$

$\Rightarrow \dfrac{a+b+2c}{(a+c)(b+c)} = \dfrac{3}{a+b+c}$

$\Rightarrow (a+b+2c)(a+b+c) = 3(a+c)(b+c)$

$\Rightarrow a^2+b^2-c^2 = ab$

$\therefore$ $\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{ab}{2ab} = \dfrac{1}{2} = \cos 60^\circ$

$\Rightarrow C = 60^\circ$

94. $\angle C = 60°$, $\dfrac{1}{a+c}+\dfrac{1}{b+c} = \dfrac{3}{a+b+c}$

$\therefore$ $\dfrac{3}{a+b+c}-\dfrac{1}{a+c} = \dfrac{1}{b+c}$

95. $A + C = 2B \Rightarrow B = 60°$

$\cos B = \dfrac{a^2+c^2-b^2}{2ac}$

$\Rightarrow \cos 60^\circ = \dfrac{a^2+c^2-b^2}{2ac}$

$\Rightarrow b^2 = a^2 + c^2 - ac$

$\therefore$ $\dfrac{a+c}{\sqrt{a^2-ac+c^2}} = \dfrac{a+c}{b} = \dfrac{\sin A+\sin C}{\sin B}$

$= \dfrac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{2\sin\frac{B}{2}\sin\frac{A+C}{2}} = \dfrac{\cos\frac{A-C}{2}}{\sin\frac{B}{2}}$

$= \dfrac{\cos\frac{A-C}{2}}{\sin 30^\circ} = 2\cos\dfrac{A-C}{2}$

96. $ab^2\cos A + ba^2\cos B + ac^2\cos A + ca^2\cos C + bc^2\cos B + b^2c\cos C$

$= ab(b\cos A + a\cos B) + ac(c\cos A + a\cos C) + bc(c\cos B + b\cos C)$

$= abc + abc + abc = 3abc$

97. Let $\dfrac{b+c}{11} = \dfrac{c+a}{12} = \dfrac{a+b}{13} = k$

$\therefore$ b + c = 11k ….(i)

c + a = 12k ….(ii)

and a + b = 13k ….(iii)

From (i) + (ii) + (iii), 2(a + b + c) = 36k

$\therefore$ a + b + c = 18 k ….(iv)

Now, (iv) – (i) gives, a = 7k

(iv) – (ii) gives, b = 6k

(iv) – (iii) gives, c = 5k

Now,

$\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{(7k)^2+(6k)^2-(5k)^2}{2\times(7k)\times(6k)}$

$= \dfrac{49k^2+36k^2-25k^2}{84k^2} = \dfrac{60k^2}{84k^2} = \dfrac{5}{7}$

98. $a(b\cos C - c\cos B)$

$= a\left(b\dfrac{a^2+b^2-c^2}{2ab} - c\dfrac{c^2+a^2-b^2}{2ca}\right)$ ….[By cosine rule]

$= a\left(\dfrac{a^2+b^2-c^2}{2a} - \dfrac{c^2+a^2-b^2}{2a}\right)$

$= \dfrac{1}{2}\times2(b^2-c^2) = b^2-c^2$

99. $a^2 \cos^2 A - b^2 - c^2 = 0$

$\Rightarrow \cos^2 A = \dfrac{b^2 + c^2}{a^2}$

Since, $\cos^2 A \le 1$ i.e., $\cos^2 A < 1$

$\therefore \quad \dfrac{b^2 + c^2}{a^2} < 1 \Rightarrow b^2 + c^2 - a^2 < 0$

$\therefore \quad \dfrac{b^2 + c^2 - a^2}{2bc} < 0 \quad \ldots[\because 2bc > 0]$

$\therefore \quad \cos A < 0 \Rightarrow A \in \left(\dfrac{\pi}{2}, \pi\right)$

100. Let $a = \alpha - \beta$, $b = \alpha + \beta$, $c = \sqrt{3\alpha^2 + \beta^2}$

Since $\sqrt{3\alpha^2 + \beta^2}$ is the largest side, the largest angle is C.

$\therefore \quad \cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\Rightarrow \cos C = \dfrac{\alpha^2 + \beta^2 - 2\alpha\beta + \alpha^2 + \beta^2 + 2\alpha\beta - 3\alpha^2 - \beta^2}{2(\alpha^2 - \beta^2)}$

$\Rightarrow \cos C = -\dfrac{(\alpha^2 - \beta^2)}{2(\alpha^2 - \beta^2)} = \cos\left(\dfrac{2\pi}{3}\right)$

$\Rightarrow C = \dfrac{2\pi}{3}$

101. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \cos 60° = \dfrac{1}{2} = \dfrac{9 + c^2 - 16}{2 \times 3 \times c}$

$\Rightarrow 3c = c^2 - 7$

$\Rightarrow c^2 - 3c - 7 = 0$

102. We have, $b = \sqrt{3}$, $c = 1$, $\angle A = 30°$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{(\sqrt{3})^2 + 1^2 - a^2}{2.\sqrt{3}.1}$

$\Rightarrow a = 1$, $b = \sqrt{3}$, $c = 1$

$\therefore$ b is the largest side. Therefore, the largest angle B is given by

$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{1 + 1 - 3}{2.1.1} = -\dfrac{1}{2} = \cos 120^{\circ}$

$\Rightarrow B = 120°$

103. $a^4 + b^4 + c^4 = 2c^2(a^2 + b^2)$

$\Rightarrow a^4 + b^4 + c^4 - 2a^2c^2 - 2b^2c^2 + 2a^2b^2 = 2a^2b^2$

$\Rightarrow (a^2 + b^2 - c^2)^2 = 2a^2b^2$

$\Rightarrow a^2 + b^2 - c^2 = \pm\sqrt{2}ab$

$\Rightarrow \dfrac{a^2 + b^2 - c^2}{2ab} = \pm\dfrac{\sqrt{2}ab}{2ab} = \pm\dfrac{1}{\sqrt{2}}$

$\Rightarrow \cos C = \cos 45°$ or $\cos 135°$

$\Rightarrow C = 45°$ or $135°$

104. We have, $b + c = 2a$ ....(i)

$\cos 60° = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{(b + c)^2 - 2bc - a^2}{2bc}$

$\Rightarrow \dfrac{1}{2} = \dfrac{4a^2 - 2bc - a^2}{2bc} \Rightarrow \dfrac{1}{2} = \dfrac{3a^2}{2bc} - 1$

$\Rightarrow \dfrac{3}{2} = \dfrac{3a^2}{2bc}$

$\Rightarrow bc = a^2$ ....(ii)

From (i) and (ii), we get

$b + c = 2\sqrt{b}\sqrt{c}$

$\Rightarrow \left(\sqrt{b} - \sqrt{c}\right)^2 = 0 \Rightarrow b = c$

From (i), $a = b = c$

$\therefore \quad \Delta$ ABC is equilateral.

105. $\dfrac{\sin A}{\sin C} = \dfrac{\sin A \cos B - \cos A \sin B}{\sin B \cos C - \cos B \sin C}$

$\Rightarrow \dfrac{a}{c} = \dfrac{a\cos B - b\cos A}{b\cos C - c\cos B}$

$\Rightarrow ab \cos C - ac \cos B = ac \cos B - bc \cos A$

$\Rightarrow ab \cos C + bc \cos A = 2ac \cos B$

$\Rightarrow \dfrac{a^2 + b^2 - c^2}{2} + \dfrac{b^2 + c^2 - a^2}{2} = \dfrac{c^2 + a^2 - b^2}{1}$

$\Rightarrow b^2 = c^2 + a^2 - b^2 \Rightarrow b^2 = \dfrac{c^2 + a^2}{2}$

$\Rightarrow a^2, b^2, c^2$ are in A.P.

106. $\dfrac{2\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{2\cos C}{c} = \dfrac{a}{bc} + \dfrac{b}{ca}$

$\Rightarrow \dfrac{2(b^2 + c^2 - a^2)}{2abc} + \dfrac{a^2 + c^2 - b^2}{2abc} + \dfrac{2(a^2 + b^2 - c^2)}{2abc} = \dfrac{a}{bc} + \dfrac{b}{ca}$

$\Rightarrow \dfrac{3b^2 + c^2 + a^2}{2abc} = \dfrac{a}{bc} + \dfrac{b}{ca}$

$\Rightarrow \dfrac{3b}{2ac} + \dfrac{c}{2ab} + \dfrac{a}{2bc} = \dfrac{a}{bc} + \dfrac{b}{ca}$

$\Rightarrow b^2 + c^2 = a^2$

Hence, $\angle A = 90°$

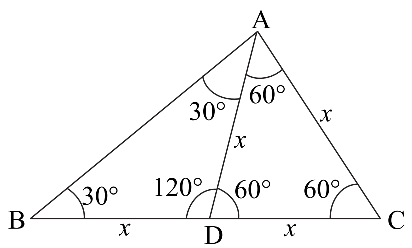
107. 4 sin A = 4 sin B = 3 sin C

$\therefore$ 4a = 4b = 3c or a = b

$\therefore$ $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{a^2+a^2-\left(\frac{4a}{3}\right)^2}{2\times a\times a}$

$= \frac{2a^2-\frac{16a^2}{9}}{2a^2} = \frac{a^2-\frac{8a^2}{9}}{a^2} = 1-\frac{8}{9} = \frac{1}{9}$

108.



From the figure,

$\cos 120° = \frac{x^2+x^2-AB^2}{2x^2}$

$\Rightarrow \frac{2x^2-AB^2}{2x^2} = \frac{-1}{2}$

$\Rightarrow 4x^2-2AB^2 = -2x^2$

$\Rightarrow 3x^2 = AB^2 \Rightarrow AB = x\sqrt{3}$

$\Rightarrow a^2 : b^2 : c^2 = (2x)^2 : x^2 : (x\sqrt{3})^2$

$= 4x^2 : x^2 : 3x^2 = 4 : 1 : 3.$

109. $\frac{\sqrt{3}}{2} < \frac{b}{a} < 1$

$\Rightarrow \frac{b}{a} < 1$

$\Rightarrow b < a$

$\Rightarrow c < b < a$

$\Rightarrow B = 60°$ ….[$\because$ Angles are in A.P.]

Consider $\frac{\sqrt{3}}{2} < \frac{b}{a} < 1$

$\Rightarrow \frac{\sqrt{3}}{2} < \frac{b}{a}$

$\Rightarrow \sqrt{3}\,a < 2b$

$\Rightarrow 3a^2 < 4b^2$

$\Rightarrow 4b^2 - 3a^2 > 0$

Now, $b^2 = a^2 + c^2 - 2ac\cos 60°$

$\Rightarrow c^2 - ac + (a^2 - b^2) = 0$

$\therefore$ $c = \frac{a \pm \sqrt{4b^2-3a^2}}{2}$

110. cot A, cot B and cot C are in A. P.

$\Rightarrow$ cot A + cot C = 2 cot B

$\Rightarrow \frac{\cos A}{\sin A} + \frac{\cos C}{\sin C} = \frac{2\cos B}{\sin B}$

$\Rightarrow \frac{b^2+c^2-a^2}{2bc(ka)} + \frac{a^2+b^2-c^2}{2ab(kc)} = 2\frac{a^2+c^2-b^2}{2ac(kb)}$

$\Rightarrow a^2 + c^2 = 2b^2$

Hence, a^2, b^2, c^2, are in A. P.

111. $\frac{\sin 3B}{\sin B} = \frac{3\sin B - 4\sin^3 B}{\sin B} = 3 - 4\sin^2 B$

$= 3 - 4 + 4\cos^2 B$

$= -1 + \frac{4(a^2+c^2-b^2)^2}{4(ac)^2}$

$= -1 + \frac{\left(\frac{a^2+c^2}{2}\right)^2}{(ac)^2}$ ....[$\because 2b^2 = a^2+c^2$]

$= -1 + \frac{(a^2+c^2)^2}{4(ac)^2}$

$= \frac{(a^2+c^2)^2 - 4a^2c^2}{4(ac)^2} = \left(\frac{c^2-a^2}{2ac}\right)^2$

112. $\cot B + \cot C - \cot A = \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C} - \cot A$

$= \frac{\sin C\cos B + \cos C\sin B}{\sin B\sin C} - \cot A$

$= \frac{\sin(B+C)}{\sin B\sin C} - \frac{\cos A}{\sin A}$

$= \frac{\sin^2 A - \sin B\sin C\cos A}{\sin A\sin B\sin C} = \frac{a^2 - bc\cos A}{(abc)}$

$= \frac{a^2 - bc\frac{(b^2+c^2-a^2)}{2bc}}{(abc)}$

$= \frac{3a^2-b^2-c^2}{2(abc)} = \frac{3a^2-(b^2+c^2)}{2(abc)}$

$\therefore$ $\cot B + \cot C - \cot A = \frac{3a^2-3a^2}{2(abc)} = 0$

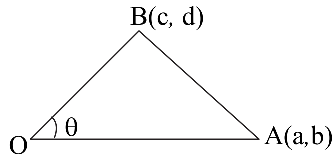
….[$\because b^2 + c^2 = 3a^2$]

113. Largest side is $\sqrt{p^2+pq+q^2}$. If largest angle is θ, then

$\cos\theta = \frac{p^2+q^2-p^2-pq-q^2}{2pq} = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow \theta = \frac{2\pi}{3}$

114.



$(AB)^2 = (a - c)^2 + (b - d)^2$

$(OA)^2 = (a - 0)^2 + (b - 0)^2 = a^2 + b^2$

and $(OB)^2 = c^2 + d^2$

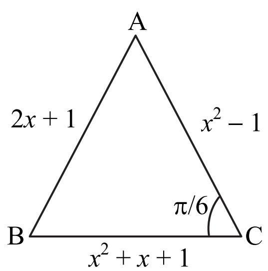
Now from triangle AOB,

$$\cos\theta = \frac{(OA)^2 + (OB)^2 - (AB)^2}{2OA.OB}$$

$$= \frac{a^2 + b^2 + c^2 + d^2 - \{(a-c)^2 + (b-d)^2\}}{2\sqrt{a^2+b^2}.\sqrt{c^2+d^2}}$$

$$= \frac{ac + bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

115.



$$\cos C = \frac{b^2 + a^2 - c^2}{2ba}$$

$$\Rightarrow \cos\frac{\pi}{6} = \frac{(x^2-1)^2 + (x^2+x+1)^2 - (2x+1)^2}{2(x^2-1)(x^2+x+1)}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{(x^2-1)^2 + (x^2+3x+2)(x^2-x)}{2(x^2+x+1)(x^2-1)}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{(x^2-1)^2 + (x+1)(x+2)x(x-1)}{2(x^2+x+1)(x^2-1)}$$

$$\Rightarrow \sqrt{3} = \frac{\left(x^2-1\right)^2 + x\left(x^2-1\right)\left(x+2\right)}{\left(x^2+x+1\right)\left(x^2-1\right)}$$

$$\Rightarrow \sqrt{3} = \frac{x^2 - 1 + x(x+2)}{x^2+x+1}$$

$\Rightarrow \sqrt{3}(x^2+x+1) = 2x^2 + 2x - 1$

$\Rightarrow \left(\sqrt{3}-2\right)x^2 + \left(\sqrt{3}-2\right)x + \left(\sqrt{3}+1\right) = 0$

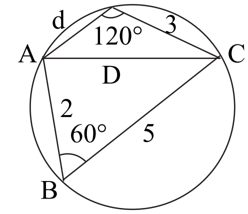
On solving, $x^2 + x - \left(3\sqrt{3}+5\right) = 0$

$\Rightarrow x = \sqrt{3} + 1, -\left(2+\sqrt{3}\right)$

Since, x cannot be negative.

$\therefore \quad x = 1 + \sqrt{3}$

116. Let the fourth side be of length d.



From the figure,

In Δ ADC,

$AC^2 = CD^2 + DA^2 - 2.CD.DA.\cos 120°$ ….[By Cosine rule]

In Δ BAC,

$AC^2 = AB^2 + BC^2 - 2.AB.BC.\cos 60°$ ….[By Cosine rule]

$\therefore \quad 3^2 + d^2 - 2\times3\times d\cos 120° = 2^2 + 5^2 - 2\times2\times5\cos 60°$

$\Rightarrow d^2 + 3d - 10 = 0 \Rightarrow d = -5$ or $d = 2$

$\therefore \quad d = 2$

117. By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\Rightarrow \frac{a}{\sin 2B} = \frac{b}{\sin B}$$

$$\Rightarrow \sin 2B = \frac{a}{b}\sin B$$

$$\Rightarrow 2\sin B\cos B = \frac{a}{b}\sin B$$

$$\Rightarrow \frac{a}{2\cos B} = b$$

$$\Rightarrow \frac{a}{2\left(\frac{a^2 + c^2 - b^2}{2ac}\right)} = b$$

$\Rightarrow a^2c = b(a^2 + c^2 - b^2)$

$\Rightarrow a^2(b - c) - (b + c)(b - c)b = 0$

$\Rightarrow a^2 - b(b + c) = 0$

$\Rightarrow a^2 = b^2 + bc$

Now, $a = \alpha + 1$, $b = \alpha - 1$, $c = \alpha$

$\therefore \quad (\alpha + 1)^2 = (\alpha - 1)^2 + \alpha(\alpha - 1)$

$\Rightarrow \alpha^2 + 2\alpha + 1 = \alpha^2 - 2\alpha + 1 + \alpha^2 - \alpha$

$\Rightarrow \alpha^2 - 5\alpha = 0 \Rightarrow \alpha(\alpha - 5) = 0$

$\Rightarrow \alpha = 0, 5$

118. $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c$

$= (b\cos C + c\cos B) + (c\cos A + a\cos C) + (a\cos B + b\cos A)$

$= a + b + c$ ….[By projection rule]

119. $\dfrac{\sin B\cos C}{\sin A}+\dfrac{\cos B\sin C}{\sin A}$

$=\left(\dfrac{b}{a}\cos C+\dfrac{c}{a}\cos B\right)$

$= 1$ …[By projection rule]

120. $\dfrac{\cos C+\cos A}{c+a}+\dfrac{\cos B}{b}$

$=\dfrac{(b\cos C+c\cos B)+(b\cos A+a\cos B)}{b(c+a)}$

$=\dfrac{a+c}{b(c+a)}$ ….[By projection rule]

$=\dfrac{1}{b}$

121. Since, a, b, c are in A. P.,

∴ $2b = a + c$

$a\cos^2\left(\dfrac{C}{2}\right) + c\cos^2\left(\dfrac{A}{2}\right)$

$=\dfrac{a(1+\cos C)}{2}+\dfrac{c(1+\cos A)}{2}$

$=\dfrac{a+c+a\cos C+c\cos A}{2}$

$=\dfrac{a+c+b}{2}$

$=\dfrac{2b+b}{2}$

$=\dfrac{3b}{2}$

122. $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$

$\Rightarrow bc\sin^2\dfrac{A}{2} = (s-b)(s-c)$

$\Rightarrow x = bc$

123. $\sin\dfrac{A}{2}.\sin\dfrac{C}{2}=\sin\dfrac{B}{2}$

∴ $\sqrt{\dfrac{(s-b)(s-c)}{bc}}\times\sqrt{\dfrac{(s-a)(s-b)}{ab}}=\sqrt{\dfrac{(s-a)(s-c)}{ac}}$

$\Rightarrow \dfrac{(s-b)}{b}\sqrt{\dfrac{(s-a)(s-c)}{ac}}=\sqrt{\dfrac{(s-a)(s-c)}{ac}}$

$\Rightarrow s-b=b\Rightarrow s=2b$

124. $s=\dfrac{a+b+c}{2}=\dfrac{16+24+20}{2}=30$

$\cos\left(\dfrac{B}{2}\right)=\sqrt{\dfrac{s(s-b)}{ac}}$

$=\sqrt{\dfrac{30(30-24)}{16\times 20}}=\sqrt{\dfrac{9}{16}}=\dfrac{3}{4}$

125. $(a-b)^2\cos^2\dfrac{C}{2}+(a+b)^2\sin^2\dfrac{C}{2}$

$=(a^2+b^2-2ab)\cos^2\dfrac{C}{2}+(a^2+b^2+2ab)\sin^2\dfrac{C}{2}$

$=(a^2+b^2)\left[\cos^2\dfrac{C}{2}+\sin^2\dfrac{C}{2}\right]$

$-2ab\left[\cos^2\dfrac{C}{2}-\sin^2\dfrac{C}{2}\right]$

$= a^2 + b^2 - 2ab\cos C$

$= c^2$ ….[By cosine rule]

126. a, b, c are in A.P. $\Rightarrow 2b = a + c$

$\Rightarrow 2s-2b=2s-(a+c)$

$\Rightarrow 2(s-b)=s-a+s-c$

$\Rightarrow 2s(s-b)=s(s-a)+s(s-c)$

$\Rightarrow 4s^2(s-b)^2=s^2(s-a)^2+s^2(s-c)^2 + 2s^2(s-a)(s-c)$

$\Rightarrow \dfrac{4s^2(s-b)^2}{s(s-a)(s-b)(s-c)}=\dfrac{s^2(s-a)^2}{s(s-a)(s-b)(s-c)}+\dfrac{s^2(s-c)^2}{s(s-a)(s-b)(s-c)}+\dfrac{2s^2(s-a)(s-c)}{s(s-a)(s-b)(s-c)}$

Taking square root on both sides, we get

$2\sqrt{\dfrac{s(s-b)}{(s-a)(s-c)}}=\sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\dfrac{s(s-c)}{(s-b)(s-a)}}$

$\Rightarrow 2\cot\dfrac{B}{2}=\cot\dfrac{A}{2}+\cot\dfrac{C}{2}$

127. $\tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, $\tan\dfrac{B}{2}=\sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}$

$(a+b+c)\left(\tan\dfrac{A}{2}+\tan\dfrac{B}{2}\right)$

$=(a+b+c)\left[\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}+\sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}\right]$

$=2s\sqrt{\dfrac{s-c}{s}}\left[\sqrt{\dfrac{s-b}{s-a}}+\sqrt{\dfrac{s-a}{s-b}}\right]$

$$= 2s\sqrt{\frac{s-c}{s}}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right]$$

$$= 2\sqrt{s(s-c)}\left[\frac{2s-a-b}{\sqrt{(s-a)(s-b)}}\right]$$

$$= 2c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \quad \ldots[\because 2s-a-b=c]$$

$$= 2c\cot\frac{C}{2}$$

$\therefore \quad \tan\frac{A}{2}+\tan\frac{B}{2} = \dfrac{2c\cot\frac{C}{2}}{a+b+c}$

128. $\left\{\cot\frac{A}{2}+\cot\frac{B}{2}\right\}\left\{a\sin^2\frac{B}{2}+b\sin^2\frac{A}{2}\right\}$

$$= \left\{\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}\right\}\left\{a\sin^2\frac{B}{2}+b\sin^2\frac{A}{2}\right\}$$

$$= \left\{\cos\frac{C}{2}\right\}\left\{a\frac{\sin\frac{B}{2}}{\sin\frac{A}{2}}+b\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}\right\}$$

$$= \sqrt{\frac{s(s-c)}{ab}}\left\{a\frac{\sqrt{\frac{(s-a)(s-c)}{ac}}}{\sqrt{\frac{(s-b)(s-c)}{bc}}}+b\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{(s-a)(s-c)}{ac}}}\right\}$$

$$= \sqrt{\frac{s(s-c)}{ab}}\left\{\sqrt{\left(\frac{s-a}{s-b}\right)ab}+\sqrt{\left(\frac{s-b}{s-a}\right)ab}\right\}$$

$$= \sqrt{s(s-c)}\left\{\frac{s-a+s-b}{\sqrt{(s-a)(s-b)}}\right\}$$

$$= \sqrt{s(s-c)}\left\{\frac{2s-a-b}{\sqrt{(s-a)(s-b)}}\right\}$$

$$= c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = c\cot\frac{C}{2}$$

Alternate Method :

Let a = 1, b = $\sqrt{3}$, c = 2 and A = 30°, B = 60°, C = 90°.

Hence, the given expression is equal to 2, which is given by option (D).

129. Let $\cot\frac{A}{2}$, $\cot\frac{B}{2}$ and $\cot\frac{C}{2}$ be in A.P.

Then, $2\cot\frac{B}{2} = \cot\frac{C}{2}+\cot\frac{A}{2}$

$\therefore$ we need to prove that

$$2\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}+\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$\text{R.H.S.} = \sqrt{\frac{s}{(s-b)}}\left(\sqrt{\frac{(s-c)}{(s-a)}}+\sqrt{\frac{(s-a)}{(s-c)}}\right)$$

$$= \sqrt{\frac{s}{s-b}}\left(\frac{s-c+s-a}{\sqrt{(s-a)(s-c)}}\right)$$

$$= \sqrt{\frac{s}{s-b}}\left(\frac{2s-a-c}{\sqrt{(s-a)(s-c)}}\right)$$

$$= 2\sqrt{\frac{s}{(s-b)}}\sqrt{\frac{(s-b)^2}{(s-a)(s-c)}}$$

$$\ldots\left[\begin{array}{l}\because 2b = a+c\\ \Rightarrow 2s-2b = 2s-(a+c)\\ \Rightarrow 2(s-b) = 2s-a-c\end{array}\right]$$

$$= 2\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \text{L.H.S.}$$

130. Δ is right angled, $\angle C = 90°$

$\therefore \quad \Delta = \frac{1}{2}ab\sin 90^\circ = \frac{1}{2}ab$

$\therefore \quad 4\Delta^2 = 4\left(\frac{1}{2}ab\right)^2 = a^2b^2$

131. $\Delta = \frac{1}{2}bc\sin A \Rightarrow 9 = \frac{1}{2}.36\sin A$

$\Rightarrow \sin A = \frac{1}{2} \Rightarrow A = 30°$

132. We have, a = 1, b = 2, $\angle C = 60°$

Area of triangle $= \frac{1}{2}ab\sin C$

$$= \frac{1}{2}(1)(2)\sin 60° = \frac{\sqrt{3}}{2}$$

133. $\Delta = \frac{1}{2}ab\sin C$

$$= \frac{1}{2}\times 1\times 2\times\sin 60°$$

$$= \frac{\sqrt{3}}{2}$$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\Rightarrow \cos 60° = \dfrac{1 + 4 - c^2}{2(1)(2)}$

$\Rightarrow \dfrac{1}{2} = \dfrac{5 - c^2}{4}$

$\Rightarrow c^2 = 3$

$\therefore \quad 4\Delta^2 + c^2 = 4\left(\dfrac{\sqrt{3}}{2}\right)^2 + 3$

$= 3 + 3$

$= 6$

134. $a^2 \sin 2C + c^2 \sin 2A$

$= a^2(2 \sin C \cos C) + c^2 (2 \sin A \cos A)$

$= 2a^2 \left(\dfrac{2\Delta}{ab}\cos C\right) + 2c^2\left(\dfrac{2\Delta}{bc}\cos A\right)$

$\ldots.\left[\begin{array}{l} \because \Delta = \dfrac{1}{2}ab \sin C = \dfrac{1}{2}bc \sin A, \\ \therefore \sin C = \dfrac{2\Delta}{ab}, \sin A = \dfrac{2\Delta}{bc} \end{array}\right]$

$= 4\Delta\left\{\dfrac{a\cos C + c\cos A}{b}\right\} = 4\Delta\left(\dfrac{b}{b}\right) = 4\Delta$

135. $\Delta = a^2 - (b - c)^2$

$= 2bc - (b^2 + c^2 - a^2)$

$= 2bc - 2bc\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right) = 2bc (1 - \cos A)$

$\therefore \quad \Delta = 2bc \cdot 2 \sin^2 \dfrac{A}{2}$ ....(i)

Also, $\Delta = \dfrac{1}{2} bc \sin A$

$= \dfrac{1}{2} bc \cdot 2 \sin \dfrac{A}{2} \cos \dfrac{A}{2}$

$\therefore \quad \Delta = bc. \sin \dfrac{A}{2} \cos \dfrac{A}{2}$ ....(ii)

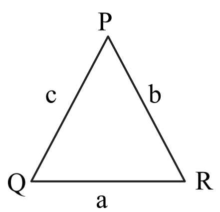
$\therefore \quad \tan \dfrac{A}{2} = \dfrac{1}{4}$ ....[From (i) and (ii)]

136. $a = 2 = QR$,

$b = \dfrac{7}{2} = PR$,

$c = \dfrac{5}{2} = PQ$

$s = \dfrac{a + b + c}{2} = \dfrac{8}{2} = 4$



$\dfrac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \dfrac{2\sin P(1 - \cos P)}{2\sin P(1 + \cos P)}$

$= \dfrac{1 - \cos P}{1 + \cos P} = \dfrac{2\sin^2 \frac{P}{2}}{2\cos^2 \frac{P}{2}} = \tan^2 \dfrac{P}{2}$

$= \dfrac{(s-b)(s-c)}{s(s-a)} = \dfrac{(s-b)^2(s-c)^2}{\Delta^2}$

$= \dfrac{\left(4 - \frac{7}{2}\right)^2\left(4 - \frac{5}{2}\right)^2}{\Delta^2} = \left(\dfrac{3}{4\Delta}\right)^2$

137. $\tan\left(\dfrac{B - C}{2}\right) = \dfrac{b - c}{b + c}\cot\dfrac{A}{2}$

$\Rightarrow \tan\left(\dfrac{90°}{2}\right) = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}\cot\dfrac{A}{2}$

$\Rightarrow \tan\left(\dfrac{A}{2}\right) = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1} = \dfrac{3 + 1 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$

$\Rightarrow \dfrac{A}{2} = 15^{\circ} \Rightarrow A = 30^{\circ}$

139. $\sin^{-1}\left(\dfrac{2x + 1}{3}\right)$ is defined for

$-1 \le \dfrac{2x + 1}{3} \le 1$

$\Rightarrow -3 \le 2x + 1 \le 3 \Rightarrow -4 \le 2x \le 2$

$\Rightarrow -2 \le x \le 1$

140. Given, $\sin^{-1} x = 2 \sin^{-1} 2a$

Since, $-\dfrac{\pi}{2} \le \sin^{-1} x \le \dfrac{\pi}{2}$

$\Rightarrow -\dfrac{\pi}{2} \le 2 \sin^{-1} 2a \le \dfrac{\pi}{2}$

$\Rightarrow -\dfrac{\pi}{4} \le \sin^{-1} 2a \le \dfrac{\pi}{4}$

$\Rightarrow \sin\left(-\dfrac{\pi}{4}\right) \le \sin(\sin^{-1} 2a) \le \sin\dfrac{\pi}{4}$

$\Rightarrow \dfrac{-1}{\sqrt{2}} \le 2a \le \dfrac{1}{\sqrt{2}}$

$\Rightarrow \dfrac{-1}{2\sqrt{2}} \le a \le \dfrac{1}{2\sqrt{2}}$ i.e., $|a| \le \dfrac{1}{2\sqrt{2}}$

141. Let $\cot^{-1}\left(\dfrac{1}{2}\right) = \theta$

$\Rightarrow \cot\theta = \dfrac{1}{2}$ $\quad \therefore \quad \sin\theta = \dfrac{2}{\sqrt{5}}$

Let $\cos^{-1} x = \phi$
$\Rightarrow x = \cos \phi$
Now, $\tan (\cos^{-1} x) = \sin\left(\cot^{-1}\left(\frac{1}{2}\right)\right)$
$\Rightarrow \tan \phi = \sin \theta$
$\Rightarrow \tan \phi = \frac{2}{\sqrt{5}}$
$\therefore$ $x = \cos \phi = \frac{\sqrt{5}}{3}$

142. Let $\tan^{-1} 2 = \alpha \Rightarrow \tan \alpha = 2$
and $\cot^{-1} 3 = \beta \Rightarrow \cot \beta = 3$
$\therefore$ $\sec^2 (\tan^{-1}2) + \text{cosec}^2 (\cot^{-1}3)$
$= \sec^2 \alpha + \text{cosec}^2\beta = 1 + \tan^2 \alpha + 1 + \cot^2 \beta$
$= 2 + (2)^2 + (3)^2 = 15$

143. $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{\pi}{2}$
Put $\sin^{-1} x = \alpha$, $\sin^{-1} y = \beta$, $\sin^{-1} z = \gamma$
$\therefore$ $\alpha + \beta + \gamma = \frac{\pi}{2}$
$\Rightarrow \alpha + \beta = \frac{\pi}{2} - \gamma \Rightarrow \cos (\alpha + \beta) = \cos\left(\frac{\pi}{2} - \gamma\right)$
$\Rightarrow \cos \alpha \cos \beta - \sin \alpha \sin \beta = \sin \gamma$ ….(i)
and, we have
$\sin \alpha = x \Rightarrow \cos \alpha = \sqrt{1 - x^2}$
Similarly, $\cos\beta = \sqrt{1 - y^2}$
$\therefore$ From (i), we get
$\sqrt{1 - x^2} . \sqrt{1 - y^2} = xy + z$
Squaring on both sides, we get
$x^2 + y^2 + z^2 + 2xyz = 1$

144. Let $\sin^{-1} a = A$,
$\sin^{-1} b = B$,
$\sin^{-1} c = C$
$\therefore$ $\sin A = a$, $\sin B = b$, $\sin C = c$
and $A + B + C = \pi$ then
$\sin 2A + \sin 2B + \sin 2C$
$= 4 \sin A \sin B \sin C$
$\Rightarrow \sin A \cos A + \sin B \cos B + \sin C \cos C$
$= 2 \sin A \sin B \sin C$
$\Rightarrow \sin A \sqrt{(1 - \sin^2 A)} + \sin B \sqrt{(1 - \sin^2 B)}$
$+ \sin C \sqrt{1 - \sin^2 C} = 2 \sin A \sin B \sin C$
$\Rightarrow a \sqrt{(1 - a^2)} + b \sqrt{(1 - b^2)} + c \sqrt{(1 - c^2)} = 2abc$

145. Given $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$
$\because$ $0 \le \cos^{-1} x \le \pi$
$\therefore$ $0 \le \cos^{-1} y \le \pi$ and $0 \le \cos^{-1} z \le \pi$
Here, $\cos^{-1} x = \cos^{-1} y = \cos^{-1} z = \pi$
$\Rightarrow x = y = z = \cos \pi = -1$
$\therefore$ $xy + yz + zx = (-1)(-1) + (-1)(-1) + (-1)(-1)$
$= 1 + 1 + 1 = 3$

146. Let $\alpha = \cos^{-1} \sqrt{p}$, $\beta = \cos^{-1} \sqrt{1 - p}$
and $\gamma = \cos^{-1} \sqrt{1 - q}$ $\therefore$ $\cos \alpha = \sqrt{p}$,
$\cos \beta = \sqrt{1 - p}$
and $\cos \gamma = \sqrt{1 - q}$
$\therefore$ $\sin \alpha = \sqrt{1 - p}$, $\sin \beta = \sqrt{p}$ and $\sin \gamma = \sqrt{q}$
The given equation can be written as
$\alpha + \beta + \gamma = \frac{3\pi}{4} \Rightarrow \alpha + \beta = \frac{3\pi}{4} - \gamma$
$\Rightarrow \cos (\alpha + \beta) = \cos\left(\frac{3\pi}{4} - \gamma\right)$
$\Rightarrow \cos \alpha \cos \beta - \sin \alpha \sin \beta$
$= \cos\left\{\pi - \left(\frac{\pi}{4} + \gamma\right)\right\} = -\cos\left(\frac{\pi}{4} + \gamma\right)$
$\Rightarrow \sqrt{p}\ \sqrt{1 - p} - \sqrt{1 - p}\ \sqrt{p}$
$= -\left(\frac{1}{\sqrt{2}}\sqrt{1 - q} - \frac{1}{\sqrt{2}}.\sqrt{q}\right)$
$\Rightarrow 0 = \sqrt{1 - q} - \sqrt{q} \Rightarrow 1 - q = q \Rightarrow q = \frac{1}{2}$

147. $\tan^{-1} \sqrt{x(x + 1)} + \sin^{-1} \sqrt{x^2 + x + 1} = \frac{\pi}{2}$
$\tan^{-1} \sqrt{x(x + 1)}$ is defined when
$x (x + 1) \ge 0$ ….(i)
$\sin^{-1} \sqrt{x^2 + x + 1}$ is defined when
$x(x + 1) + 1 \le 1$ or $x(x + 1) \le 0$ ….(ii)
From (i) and (ii),
$x(x + 1) = 0$ or $x = 0$ and -1.
Hence, number of solutions is 2.

148. Let $\cot^{-1} x = \theta \Rightarrow x = \cot \theta$
Now $\text{cosec}\ \theta = \sqrt{1 + \cot^2 \theta} = \sqrt{1 + x^2}$
$\therefore$ $\sin\theta = \frac{1}{\text{cosec}\,\theta} = \frac{1}{\sqrt{1 + x^2}}$

$\Rightarrow \theta = \sin^{-1}\dfrac{1}{\sqrt{1+x^2}}$

$\therefore \quad \sin(\cot^{-1}x) = \sin\left(\sin^{-1}\dfrac{1}{\sqrt{1+x^2}}\right)$

$= \dfrac{1}{\sqrt{1+x^2}}$

$= (1+x^2)^{\frac{-1}{2}}$

149. $\sin^{-1}\left(\dfrac{2a}{1+a^2}\right) + \sin^{-1}\left(\dfrac{2b}{1+b^2}\right) = 2\tan^{-1}x$

Putting a = tan θ and b = tan ϕ, we get

$\sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) + \sin^{-1}\left(\dfrac{2\tan\phi}{1+\tan^2\phi}\right) = 2\tan^{-1}x$

$\Rightarrow \sin^{-1}[\sin(2\theta)] + \sin^{-1}[\sin(2\phi)] = 2\tan^{-1}x$

$\Rightarrow 2(\theta+\phi) = 2\tan^{-1}x$

$\Rightarrow x = \tan(\theta+\phi)$

$\Rightarrow x = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

Resubstituting the values of a and b, we get

$x = \dfrac{a+b}{1-ab}$

150. $\cos(2\tan^{-1}x) = \dfrac{1}{2}$

$\Rightarrow 2\tan^{-1}x = \dfrac{\pi}{3}, \dfrac{-\pi}{3}$

$\Rightarrow \tan^{-1}x = \dfrac{\pi}{6}, \dfrac{-\pi}{6}$

$\Rightarrow x = \dfrac{1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}}$

151. $\cos\left[\cot^{-1}\left(\dfrac{1}{2}\right)\right] = \cos(\tan^{-1}2)$

$= \cos\left[\cos^{-1}\left(\dfrac{1}{\sqrt{1+(2)^2}}\right)\right] = \dfrac{1}{\sqrt{5}}$

and $\cot(\cos^{-1}x) = \cot\left[\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}\right]$

$= \cot\left(\cot^{-1}\dfrac{x}{\sqrt{1-x^2}}\right) = \dfrac{x}{\sqrt{1-x^2}}$

Given, $\cos\left[\cot^{-1}\left(\dfrac{1}{2}\right)\right] = \cot(\cos^{-1}x)$

$\Rightarrow \dfrac{1}{\sqrt{5}} = \dfrac{x}{\sqrt{1-x^2}} \Rightarrow \dfrac{1}{5} = \dfrac{x^2}{1-x^2}$

$\Rightarrow 6x^2 = 1$

$\Rightarrow x = \pm\dfrac{1}{\sqrt{6}}$

152. Let $\cos^{-1}x = \theta \Rightarrow x = \cos\theta \Rightarrow \sec\theta = \dfrac{1}{x}$

$\Rightarrow \tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\dfrac{1}{x^2}-1} = \dfrac{1}{x}\sqrt{1-x^2}$

Now,

$\sin[\cot^{-1}(\tan\theta)] = \sin\left[\cot^{-1}\left(\dfrac{1}{x}\sqrt{1-x^2}\right)\right]$

Again, putting $x = \sin\theta$

$\therefore \quad \sin\cot^{-1}\left(\dfrac{1}{x}\sqrt{1-x^2}\right) = \sin\cot^{-1}\left(\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right)$

$= \sin\cot^{-1}(\cot\theta)$

$= \sin\theta = x$

153. $\cos^{-1}x + \cos^{-1}(2x) = -\pi$

$\Rightarrow \cos^{-1}2x = -\pi - \cos^{-1}x$

$\Rightarrow 2x = \cos(\pi + \cos^{-1}x)$

$\Rightarrow 2x = (\cos\pi)\cos(\cos^{-1}x) - (\sin\pi)\sin(\cos^{-1}x)$

$\Rightarrow 2x = -x \Rightarrow x = 0$

But $x = 0$ does not satisfy the given equation.

$\therefore$ No solution will exist.

154. $\cos\dfrac{7\pi}{6} = \cos\left(2\pi - \dfrac{7\pi}{6}\right) = \cos\dfrac{5\pi}{6}$

$\Rightarrow \cos\dfrac{7\pi}{6} = \cos\dfrac{5\pi}{6}$

$\therefore \quad \cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) = \cos^{-1}\left(\cos\dfrac{5\pi}{6}\right) = \dfrac{5\pi}{6}$

155. $\cos\dfrac{53\pi}{5} = \cos\left(\dfrac{50\pi}{5} + \dfrac{3\pi}{5}\right)$

$= \cos\left(10\pi + \dfrac{3\pi}{5}\right)$

$= \cos\dfrac{3\pi}{5}$

$= \sin\left(\dfrac{\pi}{2} - \dfrac{3\pi}{5}\right)$

$= \sin\left(\dfrac{-\pi}{10}\right)$

$\therefore \quad \sin^{-1}\left(\cos\dfrac{53\pi}{5}\right) = \sin^{-1}\left(\sin\dfrac{-\pi}{10}\right) = \dfrac{-\pi}{10}$

156. $\tan^{-1}(\cot x) + \cot^{-1}(\tan x)$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - x\right)\right] + \cot^{-1}\left[\cot\left(\frac{\pi}{2} - x\right)\right]$

$= \frac{\pi}{2} - x + \frac{\pi}{2} - x$

$= \pi - 2x$

157. Let $\theta = \cos^{-1}\frac{2}{\sqrt{5}} \Rightarrow \cos\theta = \frac{2}{\sqrt{5}}$

$\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right) = \tan\frac{\theta}{2}$

$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

$= \sqrt{\frac{1-\frac{2}{\sqrt{5}}}{1+\frac{2}{\sqrt{5}}}}$

$= \sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}}$

$= \sqrt{\frac{\left(\sqrt{5}-2\right)^2}{\left(\sqrt{5}+2\right)\left(\sqrt{5}-2\right)}}$

$= \sqrt{\left(\sqrt{5}-2\right)^2}$

$= \sqrt{5} - 2$

158. Putting

a = tanθ, b = tan ϕ and x = tan ψ in the given expression, we get

$\sin^{-1}(\sin 2\theta) - \cos^{-1}(\cos 2\phi) = \tan^{-1}(\tan 2\psi)$

$\Rightarrow 2\theta - 2\phi = 2\psi \Rightarrow \theta - \phi = \psi$

Taking 'tan' on both sides, we get

$\tan(\theta - \phi) = \tan\psi$

$\Rightarrow \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \tan\psi$

$\Rightarrow \frac{a-b}{1+ab} = x$

159. Putting $x = \tan\theta$, we get

$\sin\left[\tan^{-1}\left(\frac{1-x^2}{2x}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$

$= \sin\left[\tan^{-1}\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) + \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\right]$

$= \sin[\tan^{-1}(\cot 2\theta) + \cos^{-1}(\cos 2\theta)]$

$= \sin\left[\tan^{-1}\left\{\tan\left(\frac{\pi}{2} - 2\theta\right)\right\} + \cos^{-1}(\cos 2\theta)\right]$

$= \sin\frac{\pi}{2} = 1$

160. $\cot^{-1}\left[\frac{\sqrt{1-\sin x} + \sqrt{1+\sin x}}{\sqrt{1-\sin x} - \sqrt{1+\sin x}}\right]$

$= \cot^{-1}\left[\frac{(\sqrt{1-\sin x} + \sqrt{1+\sin x})}{(\sqrt{1-\sin x} - \sqrt{1+\sin x})} \times \frac{(\sqrt{1-\sin x} + \sqrt{1+\sin x})}{(\sqrt{1-\sin x} + \sqrt{1+\sin x})}\right]$

$= \cot^{-1}\left[\frac{(1-\sin x) + (1+\sin x) + 2\sqrt{1-\sin^2 x}}{(1-\sin x) - (1+\sin x)}\right]$

$= \cot^{-1}\left[\frac{2(1+\cos x)}{-2\sin x}\right]$

$= \cot^{-1}\left[-\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right]$

$= \cot^{-1}\left(-\cot\frac{x}{2}\right) = \cot^{-1}\left[\cot\left(\pi - \frac{x}{2}\right)\right]$

$= \pi - \frac{x}{2}$

161. Putting $x = \tan\theta$ in the given equation, we get

$\cot^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right) = \cot^{-1}\left(\frac{\sqrt{1+\tan^2\theta} - 1}{\tan\theta}\right)$

$= \cot^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$= \cot^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$= \cot^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$= \cot^{-1}\left(\tan\frac{\theta}{2}\right)$

$= \cot^{-1}\left[\cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]$

$= \frac{\pi}{2} - \frac{\theta}{2} = \frac{\pi}{2} - \frac{\tan^{-1}x}{2}$

162. $\tan^{-1}\dfrac{1-x}{1+x} = \dfrac{1}{2}\tan^{-1}x$

$\Rightarrow \tan^{-1}\left[\dfrac{1-\tan\theta}{1+\tan\theta}\right] = \dfrac{1}{2}.\theta$ ....[Put $x = \tan\theta$]

$\Rightarrow \tan^{-1}\left[\dfrac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right] = \dfrac{\theta}{2}$

$\Rightarrow \tan^{-1}\tan\left(\dfrac{\pi}{4}-\theta\right) = \dfrac{\theta}{2} \Rightarrow \dfrac{\pi}{4}-\theta = \dfrac{\theta}{2}$

$\Rightarrow \theta = \dfrac{\pi}{6} = \Rightarrow \tan^{-1}x = \dfrac{\pi}{6} \Rightarrow x = \tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$

163. $\tan\left[\sin^{-1}\left\{\dfrac{x}{\sqrt{2}}+\dfrac{\sqrt{1-x^2}}{\sqrt{2}}\right\}-\sin^{-1}x\right]$

$= \tan\left[\sin^{-1}\left\{\dfrac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}-\sin^{-1}x\right]$

$= \tan\left[\sin^{-1}\left\{\dfrac{\sin\theta+\cos\theta}{\sqrt{2}}\right\}-\theta\right]$

....$\left[\begin{array}{l}\text{Put } \sin^{-1}x = \theta \\ \Rightarrow x = \sin\theta\end{array}\right]$

$= \tan\left[\sin^{-1}\left[\sin\left(\theta+\dfrac{\pi}{4}\right)\right]-\theta\right]$

$= \tan\left[\theta+\dfrac{\pi}{4}-\theta\right] = \tan\dfrac{\pi}{4} = 1$

164. $\sec^{-1}[\sec(-30°)]$

$= \sec^{-1}(\sec 30°)$ ….[$\because \sec(-\theta) = \sec\theta$]

$= 30°$

165. $\cos^{-1}\left[\cot\left(\dfrac{\pi}{2}\right)\right]+\cos^{-1}\left[\sin\left(\dfrac{2\pi}{3}\right)\right]$

$= \cos^{-1}(0) + \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

$= \dfrac{\pi}{2}+\cos^{-1}\left(\cos\dfrac{\pi}{6}\right)$

$= \dfrac{\pi}{2}+\dfrac{\pi}{6} = \dfrac{2\pi}{3}$

166. $\dfrac{\tan^{-1}\left(\sqrt{3}\right)-\sec^{-1}(-2)}{\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)} = \dfrac{\frac{\pi}{3}-\frac{2\pi}{3}}{\frac{-\pi}{4}+\frac{2\pi}{3}} = \dfrac{-4}{5}$

167. Given,

$\sec^{-1}x = \operatorname{cosec}^{-1}y$

$\Rightarrow \cos^{-1}\left(\dfrac{1}{x}\right) = \sin^{-1}\left(\dfrac{1}{y}\right)$

$\Rightarrow \cos^{-1}\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{1}{y}\right)$

$\Rightarrow \cos^{-1}\left(\dfrac{1}{x}\right) + \cos^{-1}\left(\dfrac{1}{y}\right) = \dfrac{\pi}{2}$

168. $x^2 + 5|x| - 6 = 0$

$\Rightarrow |x|^2 + 5|x| - 6 = 0$

$\Rightarrow |x|^2 + 6|x| - |x| - 6 = 0$

$\Rightarrow (|x|+6)(|x|-1) = 0$

$\Rightarrow |x| = 1$ or $|x| = -6$

But $|x|$ cannot be negative

$\therefore$ $|x| = 1$ $\therefore$ $x = \pm 1$

$\alpha = 1, \beta = -1$

$\left|\tan^{-1}\alpha - \tan^{-1}\beta\right| = \left|\tan^{-1}1 - \tan^{-1}(-1)\right|$

$= \left|\dfrac{\pi}{4}-\left(-\dfrac{\pi}{4}\right)\right|$

$= \dfrac{\pi}{2}$

169. $4\sin^{-1}x + \cos^{-1}x = \pi$

$\Rightarrow 3\sin^{-1}x + \sin^{-1}x + \cos^{-1}x = \pi$

$\Rightarrow 3\sin^{-1}x = \pi - \dfrac{\pi}{2}$

….$\left[\because \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}\right]$

$\Rightarrow 3\sin^{-1}x = \dfrac{\pi}{2} \Rightarrow \sin^{-1}x = \dfrac{\pi}{6}$

$\Rightarrow x = \sin\dfrac{\pi}{6} = \dfrac{1}{2}$

170. $\cos\left(\cos^{-1}\dfrac{1}{5}+\sin^{-1}\dfrac{1}{5}+\cos^{-1}\dfrac{1}{5}\right)$

$= \cos\left(\dfrac{\pi}{2}+\cos^{-1}\dfrac{1}{5}\right) = -\sin\left(\cos^{-1}\dfrac{1}{5}\right)$

$= -\sin\left(\sin^{-1}\sqrt{\dfrac{24}{25}}\right)$

$= -\dfrac{2\sqrt{6}}{5}$

171. $\cos\left[2\left(\tan^{-1}\frac{1}{5}+\tan^{-1}5\right)\right]$

$= \cos\left[2(\cot^{-1}5+\tan^{-1}5)\right]$

$= \cos\left[2\left(\frac{\pi}{2}\right)\right]$

$= \cos\pi$

$= -1$

172. $\tan^{-1}(1+x)+\tan^{-1}(1-x)=\frac{\pi}{2}$

$\Rightarrow \tan^{-1}(1+x)=\frac{\pi}{2}-\tan^{-1}(1-x)$

$\Rightarrow \tan^{-1}(1+x)=\cot^{-1}(1-x)$

$\Rightarrow \tan^{-1}(1+x)=\tan^{-1}\left(\frac{1}{1-x}\right)$

$\Rightarrow 1+x=\frac{1}{1-x} \Rightarrow 1-x^2=1 \Rightarrow x=0$

173. The given equation can be written as

$\tan^{-1}x+\cot^{-1}x+\cot^{-1}x=\frac{2\pi}{3}$

$\Rightarrow \cot^{-1}x=\frac{2\pi}{3}-\frac{\pi}{2}$

$\ldots\left[\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$

$\Rightarrow \cot^{-1}x=\frac{\pi}{6} \Rightarrow x=\cot\frac{\pi}{6} \Rightarrow x=\sqrt{3}$

174. $3\tan^{-1}x+\cot^{-1}x=\pi$

$\Rightarrow 2\tan^{-1}x+\tan^{-1}x+\cot^{-1}x=\pi$

$\Rightarrow 2\tan^{-1}x+\frac{\pi}{2}=\pi$

$\Rightarrow \tan^{-1}x=\frac{\pi}{4}$

$\Rightarrow \tan(\tan^{-1}x)=\tan\frac{\pi}{4}$

$\Rightarrow x=1$

175. $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$

$\Rightarrow \frac{\pi}{2}-\cot^{-1}x+\frac{\pi}{2}-\cot^{-1}y=\frac{4\pi}{5}$

$\Rightarrow \cot^{-1}x+\cot^{-1}y=\frac{\pi}{5}$

176. $\sin^{-1}\left(\frac{x}{13}\right)=\frac{\pi}{2}-\operatorname{cosec}^{-1}\left(\frac{13}{12}\right)$

$=\sec^{-1}\left(\frac{13}{12}\right)=\cos^{-1}\left(\frac{12}{13}\right)$

$\therefore \sin^{-1}\left(\frac{x}{13}\right)=\sin^{-1}\left(\frac{5}{13}\right)$

$\therefore x=5$

177. $\cot^{-1}\alpha+\cot^{-1}\beta=\cot^{-1}x$

$\Rightarrow \cot^{-1}\left(\frac{\alpha\beta-1}{\alpha+\beta}\right)=\cot^{-1}x$

$\ldots\left[\because \cot^{-1}x+\cot^{-1}y=\cot^{-1}\left(\frac{xy-1}{x+y}\right)\right]$

$\Rightarrow x=\frac{\alpha\beta-1}{\alpha+\beta}$

178. $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-(2x)(3x)}\right)=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right)=\tan^{-1}(1)$

$\Rightarrow \frac{5x}{1-6x^2}=1$

$\Rightarrow 1-6x^2=5x \Rightarrow 6x^2+5x-1=0$

$\Rightarrow (x+1)\left(x-\frac{1}{6}\right)=0 \Rightarrow x=-1, \frac{1}{6}$

But $x=-1$ does not hold.

$\therefore x=\frac{1}{6}$

179. $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$

$\Rightarrow \frac{x-1}{x-2}+\frac{x+1}{x+2}=1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)$

$\Rightarrow x^2+x-2+x^2-x-2=x^2-4-x^2+1$

$\Rightarrow 2x^2=1$

$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$

180. $\tan^{-1}\dfrac{x}{y} - \tan^{-1}\left(\dfrac{x-y}{x+y}\right)$

$= \tan^{-1}\dfrac{x}{y} - \tan^{-1}\left(\dfrac{1-\frac{y}{x}}{1+\frac{y}{x}}\right)$

$= \tan^{-1}\dfrac{x}{y} - \left(\tan^{-1}1 - \tan^{-1}\dfrac{y}{x}\right)$

$= \tan^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x} - \dfrac{\pi}{4}$

$= \tan^{-1}\dfrac{x}{y} + \cot^{-1}\dfrac{x}{y} - \dfrac{\pi}{4} = \dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4}$

181. $A + B + C = \pi$

$\Rightarrow \tan^{-1}2 + \tan^{-1}3 + C = \pi$

$\Rightarrow \pi + \tan^{-1}\left(\dfrac{2+3}{1-2\times 3}\right) + C = \pi$

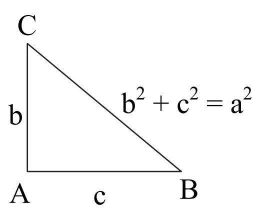
$\Rightarrow \tan^{-1}\left(\dfrac{5}{-5}\right) + C = 0$

$\Rightarrow -\tan^{-1}(1) + C = 0$

$\Rightarrow -\dfrac{\pi}{4} + C = 0$

$\Rightarrow C = \dfrac{\pi}{4}$

182. $\angle A = 90°$



$\tan^{-1}\left(\dfrac{c}{a+b}\right) + \tan^{-1}\left(\dfrac{b}{a+c}\right)$

$= \tan^{-1}\left[\dfrac{\frac{c}{a+b} + \frac{b}{a+c}}{1-\left(\frac{c}{a+b}\right)\left(\frac{b}{a+c}\right)}\right]$

$= \tan^{-1}\left[\dfrac{ca + c^2 + ab + b^2}{a^2 + ab + ca + bc - bc}\right]$

$= \tan^{-1}\left[\dfrac{a^2 + ab + ca}{a^2 + ab + ca}\right]$ ....[$\because b^2 + c^2 = a^2$]

$= \tan^{-1}(1) = \dfrac{\pi}{4}$

183. Since, $\cot^{-1}x - \cot^{-1}y = \cot^{-1}\left(\dfrac{xy+1}{y-x}\right)$

$\therefore$ $\cot^{-1}\dfrac{ab+1}{a-b} + \cot^{-1}\dfrac{bc+1}{b-c} + \cot^{-1}\dfrac{ca+1}{c-a}$

$= \cot^{-1}b - \cot^{-1}a + \cot^{-1}c - \cot^{-1}b + \cot^{-1}a - \cot^{-1}c$

$= 0$

184. $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$

$\Rightarrow \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-xz}\right] = \dfrac{\pi}{2}$

$\Rightarrow \left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right] = \tan\dfrac{\pi}{2}$

$\Rightarrow xy + yz + zx - 1 = 0$

Alternate Method:

Let $x = y = z = \dfrac{1}{\sqrt{3}}$

Then, $\tan^{-1}\dfrac{1}{\sqrt{3}} + \tan^{-1}\dfrac{1}{\sqrt{3}} + \tan^{-1}\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{2}$

Option (D) holds for these values of x, y, z.

185. Since, $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}$

$\therefore$ $4\tan^{-1}\dfrac{1}{5} = 2\left[2\tan^{-1}\dfrac{1}{5}\right] = 2\tan^{-1}\dfrac{\frac{2}{5}}{1-\frac{1}{25}}$

$= 2\tan^{-1}\dfrac{10}{24} = \tan^{-1}\dfrac{\frac{20}{24}}{1-\frac{100}{576}} = \tan^{-1}\dfrac{120}{119}$

$\therefore$ $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239} = \tan^{-1}\dfrac{120}{119} - \tan^{-1}\dfrac{1}{239}$

$= \tan^{-1}\dfrac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}.\frac{1}{239}} = \tan^{-1}\dfrac{(120\times 239) - 119}{(119\times 239) + 120}$

$= \tan^{-1}1 = \dfrac{\pi}{4}$

186. $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$\therefore$ $\tan^{-1}(\cos x) + \tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$\therefore$ $\tan[\tan^{-1}(\cos x) + \tan^{-1}(\cos x)]$

$= \tan\left[\tan^{-1}(2\operatorname{cosec} x)\right]$

$\Rightarrow \dfrac{\tan(\tan^{-1}\cos x) + \tan(\tan^{-1}\cos x)}{1 - \tan(\tan^{-1}\cos x).\tan(\tan^{-1}\cos x)} = 2\operatorname{cosec} x$

$\Rightarrow \dfrac{\cos x + \cos x}{1-\cos^2 x} = 2\operatorname{cosec} x$

$\Rightarrow 2\cos x = 2\operatorname{cosec} x\,.\,(1-\cos^2 x)$

$\Rightarrow \cos x = \operatorname{cosec} x\,.\,\sin^2 x$

$\Rightarrow \cos x = \sin x$

$\Rightarrow x = \dfrac{\pi}{4}$

$\therefore \quad \sin x + \cos x = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

187. $\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$

Let $s^2 = \frac{a+b+c}{abc}$

$\therefore \quad \theta = \tan^{-1}\sqrt{a^2s^2} + \tan^{-1}\sqrt{b^2s^2} + \tan^{-1}\sqrt{c^2s^2}$

$= \tan^{-1}(as) + \tan^{-1}(bs) + \tan^{-1}(cs)$

$= \tan^{-1}\left[\frac{as+bs+cs-abcs^3}{1-abs^2-acs^2-bcs^2}\right]$

$\therefore \quad \tan\theta = \left[\frac{s\left[(a+b+c)-abcs^2\right]}{1-(ab+bc+ca)s^2}\right]$

$= \left[\frac{s[(a+b+c)-(a+b+c)]}{1-s^2(ab+bc+ca)}\right]$ ....$[\because s^2(abc) = (a+b+c)]$

$= 0$

Alternate Method :

Let a = b = c = 1. Then,

$\theta = \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{3} = \pi$

$\Rightarrow \tan\theta = 0$

188. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$

$= \tan^{-1}\frac{120}{119} + \tan^{-1}\frac{1}{99} - \tan^{-1}\frac{1}{70}$

$= \tan^{-1}\left(\frac{120}{119}\right) + \tan^{-1}\left[\frac{\frac{1}{99}-\frac{1}{70}}{1+\frac{1}{99}.\frac{1}{70}}\right]$

$= \tan^{-1}\left(\frac{120}{119}\right) + \tan^{-1}\left(\frac{-29}{6931}\right)$

$= \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{29}{6931}$

$= \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239}$

$= \tan^{-1}1 = \frac{\pi}{4}$

189. $2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8}$

$= 2\left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right) + \sec^{-1}\frac{5\sqrt{2}}{7}$

$= 2\tan^{-1}\left[\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\left(\frac{1}{8}\right)}\right] + \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1}$

...$[\because \sec^{-1}x = \tan^{-1}\sqrt{x^2-1}\,]$

$= 2\tan^{-1}\frac{13}{39} + \tan^{-1}\frac{1}{7}$

$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\left[\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right] + \tan^{-1}\frac{1}{7}$

...$\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2},\ \text{if } -1 < x < 1\right]$

$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left[\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\left(\frac{1}{7}\right)}\right]$

190. Consider option (A),

$\sin(\cos^{-1}x) = \cos(\sin^{-1}x) = \sqrt{1-x^2}$

....$\left[\because \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}, \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\right]$

191. $\sin^{-1}[\cos(\sin^{-1}x)] + \cos^{-1}[\sin(\cos^{-1}x)]$

$= \sin^{-1}\sqrt{1-x^2} + \cos^{-1}\sqrt{1-x^2}$

...$\left[\because \cos\left(\sin^{-1}x\right) = \sin\left(\cos^{-1}x\right) = \sqrt{1-x^2}\right]$

$= \frac{\pi}{2}$

192. $\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3}$

$= \sin^{-1}\left[\frac{1}{3}\sqrt{1-\frac{4}{9}} + \frac{2}{3}\sqrt{1-\frac{1}{9}}\right]$

$= \sin^{-1}\left[\frac{\sqrt{5}+4\sqrt{2}}{9}\right]$

$\therefore \quad x = \frac{\sqrt{5}+4\sqrt{2}}{9}$

193. $\sin^{-1}x + \cos^{-1}y = \dfrac{2\pi}{5}$

$\Rightarrow \dfrac{\pi}{2} - \cos^{-1}x + \dfrac{\pi}{2} - \sin^{-1}y = \dfrac{2\pi}{5}$

$\Rightarrow \pi - \cos^{-1}x - \sin^{-1}y = \dfrac{2\pi}{5}$

$\Rightarrow \cos^{-1}x + \sin^{-1}y = \pi - \dfrac{2\pi}{5}$

$= \dfrac{3\pi}{5}$

194. Given, $\sin^{-1}x + \sin^{-1}y = \dfrac{2\pi}{3}$

$\therefore \quad \cos^{-1}x + \cos^{-1}y = \pi - \dfrac{2\pi}{3} = \dfrac{\pi}{3}$

$\ldots\left[\begin{array}{l}\text{If } \sin^{-1}x + \sin^{-1}y = \theta, \\ \text{then } \cos^{-1}x + \cos^{-1}y = \pi - \theta\end{array}\right]$

195. $\sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right) + \sin^{-1}\left(\dfrac{1}{3}\right)$

$= \sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right) + \cos^{-1}\sqrt{1 - \left(\dfrac{1}{3}\right)^2}$

$\ldots\left[\because \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}\right]$

$= \sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right) + \cos^{-1}\left(\dfrac{2\sqrt{2}}{3}\right)$

$= \dfrac{\pi}{2} \quad \ldots\left[\because \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}\right]$

196. $\left[\sin\left(\tan^{-1}\dfrac{3}{4}\right)\right]^2$

$= \left[\sin\left\{\sin^{-1}\left(\dfrac{\frac{3}{4}}{\sqrt{1+\left(\frac{3}{4}\right)^2}}\right)\right\}\right]^2$

$\ldots\left[\because \tan^{-1}x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)\right]$

$= \left[\sin\left(\sin^{-1}\dfrac{3}{5}\right)\right]^2 = \left(\dfrac{3}{5}\right)^2 = \dfrac{9}{25}$

197. $\sin[\cot^{-1}(x+1)] = \sin\left(\sin^{-1}\dfrac{1}{\sqrt{x^2+2x+2}}\right)$

$= \dfrac{1}{\sqrt{x^2+2x+2}}$

And $\cos(\tan^{-1}x) = \cos\left(\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}\right)$

$= \dfrac{1}{\sqrt{1+x^2}}$

Thus, $\dfrac{1}{\sqrt{x^2+2x+2}} = \dfrac{1}{\sqrt{1+x^2}}$

$\Rightarrow x^2 + 2x + 2 = 1 + x^2 \Rightarrow x = -\dfrac{1}{2}$

198. $\cos(\tan^{-1}x) = \cos\left[\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}\right]$

$\ldots\left[\because \tan^{-1}x = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)\right]$

$= \dfrac{1}{\sqrt{1+x^2}}$

199. $\tan(\cos^{-1}x)$

$= \tan\left[\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}\right]$

$\ldots\left[\because \cos^{-1}x = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}\right]$

$= \dfrac{\sqrt{1-x^2}}{x}$

200. $\cos\left[\tan^{-1}\left(\dfrac{3}{4}\right)\right] = \cos\left[\cos^{-1}\dfrac{1}{\sqrt{1+\left(\frac{3}{4}\right)^2}}\right]$

$= \cos\left[\cos^{-1}\left(\dfrac{4}{5}\right)\right]$

$= \dfrac{4}{5}$

201. Let $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

Now, $\cos^{-1}x + \cos^{-1}\left(\dfrac{x}{2} + \dfrac{1}{2}\sqrt{3-3x^2}\right)$

$= \theta + \cos^{-1}\left(\dfrac{\cos\theta}{2} + \dfrac{\sqrt{3}}{2}\sqrt{1-\cos^2\theta}\right)$

$= \theta + \cos^{-1}\left(\dfrac{1}{2}.\cos\theta + \dfrac{\sqrt{3}}{2}\sin\theta\right)$

$= \theta + \cos^{-1}\left(\cos\frac{\pi}{3}.\cos\theta + \sin\frac{\pi}{3}.\sin\theta\right)$

$= \theta + \cos^{-1}\left[\cos\left(\frac{\pi}{3} - \theta\right)\right] = \theta + \frac{\pi}{3} - \theta$

$= \frac{\pi}{3}$

202. $\tan^{-1} y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$= \tan^{-1}x + 2\tan^{-1}x$

$\therefore \quad \tan^{-1}y = 3\tan^{-1}x$

Since, $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$\therefore \quad y = \frac{3x-x^3}{1-3x^2}$

203. $2\cot^{-1}\left(\frac{1}{2}\right) - \cot^{-1}\left(\frac{4}{3}\right)$

$= 2\tan^{-1}(2) - \cot^{-1}\left(\frac{4}{3}\right)$

....$\left[\because \cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right), \text{ if } x > 0\right]$

$= \pi + \tan^{-1}\left(\frac{4}{-3}\right) - \cot^{-1}\left(\frac{4}{3}\right)$

....$\left[\because 2\tan^{-1}(x) = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{ if } x > 1\right]$

$= \pi - \tan^{-1}\left(\frac{4}{3}\right) - \cot^{-1}\left(\frac{4}{3}\right)$

$= \pi - \left[\tan^{-1}\left(\frac{4}{3}\right) + \cot^{-1}\left(\frac{4}{3}\right)\right]$

$= \pi - \frac{\pi}{2}$

$= \frac{\pi}{2}$

204. $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$

$= \tan^{-1}\left[\frac{1}{2}\left(\frac{2\tan A}{1-\tan^2 A}\right)\right] + \pi + \tan^{-1}\left(\frac{\cot A + \cot^3 A}{1-\cot^4 A}\right)$

$\ldots\left[\begin{array}{l}\because \ 0 \le A \le \frac{\pi}{4}, \\ \tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), \\ \text{for } x, y > 0 \text{ and } xy > 1\end{array}\right]$

$= \pi + \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left[\frac{\cot A(1+\cot^2 A)}{(1+\cot^2 A)(1-\cot^2 A)}\right]$

$= \pi + \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{\cot A}{1-\cot^2 A}\right)$

$= \pi + \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{\tan A}{\tan^2 A - 1}\right)$

$= \pi + \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{-\tan A}{1-\tan^2 A}\right)$

$= \pi + 0$ $\quad\ldots[\tan^{-1}(-x) = -\tan^{-1}x]$

$= \pi$

205. $\cos^{-1}\left(\frac{15}{17}\right) + 2\tan^{-1}\left(\frac{1}{5}\right)$

$= \cos^{-1}\left(\frac{15}{17}\right) + \cos^{-1}\left(\frac{1-\frac{1}{25}}{1+\frac{1}{25}}\right)$

$= \cos^{-1}\left(\frac{15}{17}\right) + \cos^{-1}\left(\frac{12}{13}\right)$

$= \cos^{-1}\left(\frac{15}{17}\times\frac{12}{13} - \sqrt{1-\left(\frac{15}{17}\right)^2}\sqrt{1-\left(\frac{12}{13}\right)^2}\right)$

$= \cos^{-1}\left(\frac{140}{221}\right)$

206. $\tan\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right]$

$= \tan\left[\tan^{-1}\frac{\sqrt{\left(1-\frac{16}{25}\right)}}{\frac{4}{5}} + \tan^{-1}\frac{2}{3}\right]$

$= \tan\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}.\frac{2}{3}}\right)\right]$

$= \tan\left(\tan^{-1}\frac{17}{6}\right)$

$= \frac{17}{6}$

207. $\cot(\cos^{-1}x) = \sec\left[\tan^{-1}\left(\frac{a}{\sqrt{b^2-a^2}}\right)\right]$

$\Rightarrow \cot\left[\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right] = \sec\left[\sec^{-1}\sqrt{1+\frac{a^2}{b^2-a^2}}\right]$

$\Rightarrow \dfrac{x}{\sqrt{1-x^2}} = \dfrac{b}{\sqrt{b^2-a^2}}$

$\Rightarrow \dfrac{x^2}{1-x^2} = \dfrac{b^2}{b^2-a^2}$

$\Rightarrow \dfrac{1-x^2}{x^2} = \dfrac{b^2-a^2}{b^2}$

$\Rightarrow \dfrac{1}{x^2} = \dfrac{2b^2-a^2}{b^2}$

$\Rightarrow x = \dfrac{b}{\sqrt{2b^2-a^2}}$

208. $\tan^{-1}\left\{\sin\left(\cos^{-1}\sqrt{\dfrac{2}{3}}\right)\right\}$

$= \tan^{-1}\left\{\sin\left(\sin^{-1}\sqrt{\dfrac{1}{3}}\right)\right\}$

...$\left[\because \cos^{-1} x = \sin^{-1}\sqrt{1-x^2}\right]$

$= \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

209. We know $\dfrac{5^x+5^{-x}}{2} \geq 1$ ....[$\because$ A.M. $\geq$ G.M.]

Since, $\cos(e^x) \leq 1$

So, there does not exist any solution.

210. Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$ in the given determinant, we get

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

$\Rightarrow 1 + 4\sin 4\theta + \cos^2\theta + \sin^2\theta = 0$

$\Rightarrow 4\sin 4\theta = -2$

$\Rightarrow \sin 4\theta = \dfrac{-1}{2}$

$\Rightarrow 4\theta = \dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$

Since, $0 < \theta < \dfrac{\pi}{2}$

$\Rightarrow 0 < 4\theta < 2\pi$

$\Rightarrow \theta = \dfrac{7\pi}{24}$ or $\dfrac{11\pi}{24}$

211. $\dfrac{(x+1)^2}{x^3+x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$\therefore \dfrac{(x+1)^2}{x(x^2+1)} = \dfrac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}$

$\therefore x^2 + 2x + 1 = (A+B)x^2 + Cx + A$

Equating coefficients on both sides, we get

$A + B = 1$, $C = 2$, $A = 1$

$\Rightarrow B = 0$

$\therefore \operatorname{cosec}^{-1}\left(\dfrac{1}{A}\right) + \cot^{-1}\left(\dfrac{1}{B}\right) + \sec^{-1} C$

$= \dfrac{\pi}{2} + 0 + \dfrac{\pi}{3} = \dfrac{5\pi}{6}$

212. $2y = 1 \Rightarrow y = \dfrac{1}{2} \Rightarrow \sin x = \dfrac{1}{2}$

$\Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, -\dfrac{7\pi}{6}, -\dfrac{11\pi}{6}$ in $-2\pi \leq x \leq 2\pi$

$\therefore$ number of points of intersection = 4

213. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow \dfrac{5}{\sin\left(\dfrac{\pi}{2}+B\right)} = \dfrac{4}{\sin B}$

$\Rightarrow \dfrac{5}{\cos B} = \dfrac{4}{\sin B} \quad \Rightarrow \tan B = \dfrac{4}{5}$

Now, $\tan A = \tan\left(\dfrac{\pi}{2}+B\right) = -\cot B = \dfrac{-5}{4}$

$\tan C = \tan(\pi - (A+B))$

$= -\tan(A+B)$

$= -\dfrac{(\tan A + \tan B)}{1 - \tan A.\tan B} = \dfrac{-\left(-\dfrac{5}{4}+\dfrac{4}{5}\right)}{1-\left(\dfrac{-5}{4}\times\dfrac{4}{5}\right)} = \dfrac{9}{40}$

$\Rightarrow C = \tan^{-1}\left(\dfrac{2.\dfrac{1}{9}}{1-\left(\dfrac{1}{9}\right)^2}\right)$

$\Rightarrow C = 2\tan^{-1}\left(\dfrac{1}{9}\right)$

214. Given, $x = \sin^{-1} K, y = \cos^{-1} K$

$\therefore \sin x = \cos y = K$

$\therefore \sin x = \sin\left(\dfrac{\pi}{2} - y\right)$

$\therefore x = \dfrac{\pi}{2} - y \Rightarrow x + y = \dfrac{\pi}{2}$

215. $A - B = \frac{1}{\pi}\begin{bmatrix} \sin^{-1}(\pi x) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$

$$- \frac{1}{\pi}\begin{bmatrix} -\cos^{-1}(\pi x) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$$

$$= \frac{1}{\pi}\begin{bmatrix} \frac{\pi}{2} & 0 \\ 0 & \frac{\pi}{2} \end{bmatrix}$$

$$= \frac{1}{2}I$$

216. $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P.

$\therefore$ $\cos^2\theta = \frac{1}{6}.\sin\theta.\tan\theta$

$\Rightarrow 6\cos^2\theta = \sin\theta.\frac{\sin\theta}{\cos\theta}$

$\Rightarrow 6\cos^3\theta = \sin^2\theta$

$\Rightarrow 6\cos^3\theta + \cos^2\theta - 1 = 0$

Here, $\cos\theta = \frac{1}{2}$ is the only real root.

$\therefore$ $\theta = 2n\pi \pm \frac{\pi}{3}$

217. $\frac{\sin B}{b} = \frac{\sin A}{a}$

$\Rightarrow \sin B = \frac{b\sin A}{a} = \frac{8\sin 30^\circ}{6} = \frac{2}{3}$

$\Rightarrow \sin(\sin^{-1}x) = \frac{2}{3}$

$\Rightarrow x = \frac{2}{3}$

218. $\cot\left[\sum_{n=1}^{100}\cot^{-1}\left(1 + \sum_{k=1}^{n}2k\right)\right]$

$= \cot\left[\sum_{n=1}^{100}\cot^{-1}(1 + 2 + 4 + 6 + ... + 2n)\right]$

$= \cot\left\{\sum_{n=1}^{100}\cot^{-1}[1 + n(n+1)]\right\}$

$= \cot\left\{\sum_{n=1}^{100}\tan^{-1}\left[\frac{1}{1+n(n+1)}\right]\right\}$

$= \cot\left\{\sum_{n=1}^{100}\tan^{-1}\left[\frac{n+1-n}{1+n(n+1)}\right]\right\}$

$= \cot\left\{\sum_{n=1}^{100}\left[\tan^{-1}(n+1) - \tan^{-1}n\right]\right\}$

$= \cot[(\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + \ldots + (\tan^{-1}101 - \tan^{-1}100)]$

$= \cot(\tan^{-1}101 - \tan^{-1}1)$

$= \cot\left[\tan^{-1}\left(\frac{101-1}{1+101}\right)\right]$

$= \cot\left[\tan^{-1}\left(\frac{100}{102}\right)\right]$

$= \cot\left[\tan^{-1}\left(\frac{50}{51}\right)\right]$

$= \cot\left[\cot^{-1}\left(\frac{51}{50}\right)\right]$

$= \frac{51}{50}$

Evaluation Test

1. $\sin 3x = 3\sin x - 4\sin^3 x$

$\Rightarrow \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$\cos 3x = 4\cos^3 x - 3\cos x$

$\Rightarrow \cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

Given, $\cos 3x\cos^3 x + \sin 3x\sin^3 x = 0$

$\Rightarrow \cos 3x\cdot\frac{1}{4}(\cos 3x + 3\cos x) + \sin 3x\cdot\frac{1}{4}(3\sin x - \sin 3x) = 0$

$\Rightarrow \frac{1}{4}\left[\cos^2 3x + 3\cos x\cos 3x + 3\sin x\sin 3x - \sin^2 3x\right] = 0$

$\Rightarrow \cos^2 3x - \sin^2 3x + 3(\cos 3x\cos x + \sin 3x\sin x) = 0$

$\Rightarrow \cos 6x + 3\cos 2x = 0$

$\Rightarrow 4\cos^3 2x - 3\cos 2x + 3\cos 2x = 0$
$\Rightarrow 4\cos^3 2x = 0$
$\Rightarrow \cos 2x = 0$

$\Rightarrow 2x = (2n+1)\dfrac{\pi}{2}$

$\Rightarrow x = (2n+1)\dfrac{\pi}{4}$

2. $\sin x\sqrt{8\cos^2 x} = 1$

$\therefore$ $\sin x\left|2\sqrt{2}\cos x\right| = 1$ $\quad \ldots\left[\because \sqrt{8} = 2\sqrt{2}\right]$

$\therefore$ $\sin x\,|\cos x| = \dfrac{1}{2\sqrt{2}}$

Case I:

If $\cos x > 0$, $\sin x\cos x = \dfrac{1}{2\sqrt{2}}$

$\therefore$ $\dfrac{1}{2}\sin 2x = \dfrac{1}{2\sqrt{2}}$

$\therefore$ $\sin 2x = \dfrac{1}{\sqrt{2}}$

$\therefore$ $2x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}, \dfrac{11\pi}{4}$

$\ldots[\because x \in (0, 2\pi), \therefore 2x \in (0, 4\pi)]$

$\therefore$ $x = \dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{9\pi}{8}, \dfrac{11\pi}{8}$

But $\cos x > 0$ (x must be in 1st or 4th Quadrant)

$\therefore$ the possible values are $\dfrac{\pi}{8}, \dfrac{3\pi}{8}$.

Case II:

If $\cos x < 0$,

$\sin x(-\cos x) = \dfrac{1}{2\sqrt{2}} \Rightarrow \sin 2x = -\dfrac{1}{\sqrt{2}}$

$\therefore$ $2x = \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

$\therefore$ $x = \dfrac{5\pi}{8}, \dfrac{7\pi}{8}$

$\therefore$ The values of x satisfying the given equation between 0 and 2π are $\dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{7\pi}{8}$.

These are in A.P. with common difference $\dfrac{\pi}{4}$.

3. $16^{\sin^2 x} + 16^{\cos^2 x} = 10$

$\Rightarrow 16^{\sin^2 x} + 16^{1-\sin^2 x} = 10$

$\Rightarrow 16^{\sin^2 x} + \dfrac{16}{16^{\sin^2 x}} = 10$

Let $t = 16^{\sin^2 x}$

$\Rightarrow t + \dfrac{16}{t} = 10$ $\qquad \Rightarrow t^2 + 16 = 10t$

$\Rightarrow t^2 - 10t + 16 = 0$ $\qquad \Rightarrow (t-2)(t-8) = 0$

$\Rightarrow t = 2$ or $t = 8$

$\Rightarrow 16^{\sin^2 x} = 2$ or $16^{\sin^2 x} = 8$

$\Rightarrow 2^{4\sin^2 x} = 2^1$ or $2^{4\sin^2 x} = 2^3$

$\Rightarrow 4\sin^2 x = 1$ or $4\sin^2 x = 3$

$\Rightarrow \sin^2 x = \dfrac{1}{4}$ or $\sin^2 x = \dfrac{3}{4}$

$\Rightarrow \sin^2 x = \sin^2\left(\dfrac{\pi}{6}\right)$ or $\sin^2 x = \sin^2\left(\dfrac{\pi}{3}\right)$

$\Rightarrow x = n\pi \pm \dfrac{\pi}{6}$ or $x = n\pi \pm \dfrac{\pi}{3}$

$\Rightarrow x = \dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{5\pi}{6}, \dfrac{11\pi}{6}$ or $x = \dfrac{\pi}{3}, \dfrac{4\pi}{3}, \dfrac{2\pi}{3}, \dfrac{5\pi}{3}$

$\therefore$ There are 8 solutions in $[0, 2\pi]$.

4. The maximum value of $a\sin x + b\cos x$ is $\sqrt{a^2 + b^2}$.

$\therefore$ Maximum value of $\sin x + \cos x$ is $\sqrt{2}$ and the maximum value of $1 + \sin 2x$ is 2.

$\therefore$ The given equation will be true only when $\sin x + \cos x = \sqrt{2}$ and $1 + \sin 2x = 2$

If $\sin x + \cos x = \sqrt{2}$

$\Rightarrow \cos x \cdot \dfrac{1}{\sqrt{2}} + \sin x \cdot \dfrac{1}{\sqrt{2}} = 1$

$\Rightarrow \cos x\cos\dfrac{\pi}{4} + \sin x\sin\dfrac{\pi}{4} = 1$

$\Rightarrow \cos\left(x - \dfrac{\pi}{4}\right) = 1$

$\Rightarrow x - \dfrac{\pi}{4} = 2n\pi,$

$\Rightarrow x = 2n\pi + \dfrac{\pi}{4}$ $\qquad \ldots$(i)

$1 + \sin 2x = 2 \Rightarrow \sin 2x = 1$

$\Rightarrow \sin 2x = \sin\dfrac{\pi}{2}$

$\Rightarrow 2x = n\pi + (-1)^n . \dfrac{\pi}{2}$

$\Rightarrow x = \dfrac{n\pi}{2} + (-1)^n . \dfrac{\pi}{4}$ $\qquad \ldots$(ii)

The value of $x \in [-\pi, \pi]$ which satisfies both (i) and (ii) is $\dfrac{\pi}{4}$.

5. $\sin^4 x + \cos^4 x = \sin x \cos x$

$\Rightarrow (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = \sin x \cos x$

$\Rightarrow 1 - \frac{1}{2}(2\sin x \cos x)^2 = \frac{1}{2}.2\sin x \cos x$

$\Rightarrow 1 - \frac{1}{2}\sin^2 2x = \frac{1}{2}\sin 2x$

$\Rightarrow \sin^2 2x + \sin 2x - 2 = 0$

$\Rightarrow (\sin 2x + 2)(\sin 2x - 1) = 0$

$\Rightarrow \sin 2x = 1$ ....$[\because \sin 2x \neq -2]$

$\Rightarrow \sin 2x = \sin\frac{\pi}{2}$

$\Rightarrow 2x = n\pi + (-1)^n \frac{\pi}{2}$

$\Rightarrow x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{4}$

$\therefore$ The value of x in $[0, 2\pi]$ are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

$\therefore$ There are 2 solutions.

6. $\tan^4 x - 2\sec^2 x + a^2 = 0$

$\Rightarrow \tan^4 x - 2(1 + \tan^2 x) + a^2 = 0$

$\Rightarrow \tan^4 x - 2\tan^2 x - 2 + a^2 = 0$

$\Rightarrow \tan^4 x - 2\tan^2 x + 1 - 3 + a^2 = 0$

$\Rightarrow (\tan^2 x - 1)^2 = 3 - a^2$

$\Rightarrow 3 - a^2 \geq 0$

$\Rightarrow a^2 \leq 3$

$\Rightarrow |a| \leq \sqrt{3}$

7. $3\cos x + 4\sin x = 5$

$\therefore 3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + 4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = 5$

Let $\tan\frac{x}{2} = t$

$\therefore 3 - 3t^2 + 8t = 5 + 5t^2$ $\quad \Rightarrow 8t^2 - 8t + 2 = 0$

$\Rightarrow 4t^2 - 4t + 1 = 0$ $\quad \Rightarrow (2t - 1)^2 = 0$

$\Rightarrow t = \frac{1}{2}$ $\quad \Rightarrow \tan\frac{x}{2} = \tan\alpha$

$\Rightarrow \frac{x}{2} = n\pi + \alpha$ $\quad \Rightarrow x = 2n\pi + 2\alpha$

8. $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$

$\Rightarrow \tan\theta + \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} + \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} = 3$

$$\Rightarrow \frac{\tan\theta(1 - 3\tan^2\theta) + (\tan\theta + \sqrt{3})(1 + \sqrt{3}\tan\theta) + (\tan\theta - \sqrt{3})(1 - \sqrt{3}\tan\theta)}{1 - 3\tan^2\theta} = 3$$

$\Rightarrow \frac{9\tan\theta - 3\tan^3\theta}{1 - 3\tan^2\theta} = 3$

$\Rightarrow 3\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = 3$

$\Rightarrow 3\tan 3\theta = 3$

$\Rightarrow \tan 3\theta = 1 = \tan\frac{\pi}{4}$

$\Rightarrow 3\theta = n\pi + \frac{\pi}{4}$

$\Rightarrow \theta = (4n + 1)\frac{\pi}{12}$

9. $\frac{5}{6} > \frac{20}{37}$ $\quad \therefore \tan\frac{A}{2} > \tan\frac{B}{2}$

$\therefore \frac{A}{2} > \frac{B}{2}$ $\quad \therefore A > B$

$\therefore \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}}$

$\Rightarrow \tan\left(\frac{A+B}{2}\right) = \frac{\frac{5}{6} + \frac{20}{37}}{1 - \frac{5}{6}\cdot\frac{20}{37}}$

$\Rightarrow \tan\left(\frac{\pi - C}{2}\right) = \frac{185 + 120}{222 - 100}$

$\Rightarrow \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \frac{305}{122}$

$\Rightarrow \cot\frac{C}{2} = \frac{305}{122}$ $\quad \Rightarrow \tan\frac{C}{2} = \frac{122}{305}$

Since, $\frac{20}{37} > \frac{122}{305}$

$\therefore \tan\frac{B}{2} > \tan\frac{C}{2}$

$\Rightarrow \frac{B}{2} > \frac{C}{2}$ $\quad \Rightarrow B > C$

$\Rightarrow A > B > C$ $\quad \Rightarrow a > b > c$

10. $A(\Delta ABC) = \frac{9\sqrt{3}}{2}$

$\Rightarrow \frac{1}{2}bc\sin A = \frac{9\sqrt{3}}{2}$

$\Rightarrow \frac{1}{2}\times bc\times\frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$

....$\left[\because \sin A = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\right]$

$\Rightarrow bc = 18$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \cos\dfrac{2\pi}{3} = \dfrac{(b-c)^2 + 2bc - a^2}{2bc}$

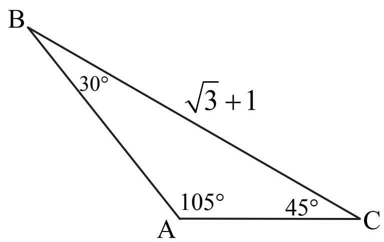
$\Rightarrow -\dfrac{1}{2} = \dfrac{(3\sqrt{3})^2 + 2\times 18 - a^2}{2\times 18}$

$\Rightarrow -18 = 27 + 36 - a^2$

$\Rightarrow a^2 = 27 + 36 + 18 = 81$

$\Rightarrow a = 9$ cm

11.



Let $\angle B = 30°$, $\angle C = 45°$ $\therefore$ $\angle A = 105°$

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$\therefore$ $\dfrac{\sin 105°}{\sqrt{3}+1} = \dfrac{\sin 30°}{b} = \dfrac{\sin 45°}{c}$

$\therefore$ $b = \dfrac{\left(\sqrt{3}+1\right)\sin 30°}{\sin 105°} = \dfrac{\sqrt{3}+1}{2\sin 105°}$

$c = \dfrac{\left(\sqrt{3}+1\right)\sin 45°}{\sin 105°} = \dfrac{\sqrt{3}+1}{\sqrt{2}\sin 105°}$

$A(\Delta ABC) = \dfrac{1}{2} bc \sin A$

$= \dfrac{1}{2} \times \dfrac{\sqrt{3}+1}{2\sin 105°} \times \dfrac{\sqrt{3}+1}{\sqrt{2}\sin 105°} \times \sin 105°$

$= \dfrac{\left(\sqrt{3}+1\right)^2}{4\sqrt{2}\sin(60° + 45°)}$

$= \dfrac{\left(\sqrt{3}+1\right)^2}{4\sqrt{2}\left(\dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{\sqrt{2}} + \dfrac{1}{2}\cdot\dfrac{1}{\sqrt{2}}\right)}$

$= \dfrac{\left(\sqrt{3}+1\right)^2}{4\sqrt{2}\left(\dfrac{\sqrt{3}+1}{2\sqrt{2}}\right)}$

$= \dfrac{\sqrt{3}+1}{2}$

12. Let $\dfrac{b+c}{11} = \dfrac{c+a}{12} = \dfrac{a+b}{13} = k = \dfrac{2(a+b+c)}{36}$

$= \dfrac{a+b+c}{18}$

….(By property of equal ratio)

$\therefore$ $b + c = 11k$, $c + a = 12k$, $a + b = 13k$,
$a + b + c = 18k$

$\therefore$ $a = 7k$, $b = 6k$, $c = 5k$

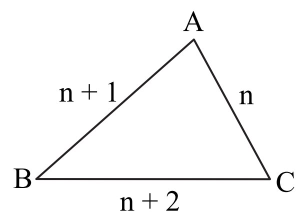
$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$= \dfrac{36k^2 + 25k^2 - 49k^2}{2(6k)(5k)}$

$= \dfrac{12k^2}{60k^2} = \dfrac{1}{5}$

$\therefore$ $\cos A = \dfrac{1}{5}$

13.



Let $AC = n$, $AB = n + 1$, $BC = n + 2$

$\therefore$ Largest angle is A and smallest angle is B.

$\therefore$ $A = 2B$

Since, $A + B + C = 180°$

$\therefore$ $3B + C = 180°$

$\Rightarrow C = 180° - 3B$

$\Rightarrow \sin C = \sin(180° - 3B) = \sin 3B$

$\Rightarrow \dfrac{\sin A}{n+2} = \dfrac{\sin B}{n} = \dfrac{\sin C}{n+1}$

$\Rightarrow \dfrac{\sin 2B}{n+2} = \dfrac{\sin B}{n} = \dfrac{\sin 3B}{n+1}$

$\Rightarrow \dfrac{2\sin B\cos B}{n+2} = \dfrac{\sin B}{n} = \dfrac{3\sin B - 4\sin^3 B}{n+1}$

$\Rightarrow \dfrac{2\cos B}{n+2} = \dfrac{1}{n} = \dfrac{3 - 4\sin^2 B}{n+1}$

$\therefore$ $\cos B = \dfrac{n+2}{2n}$, $3 - 4\sin^2 B = \dfrac{n+1}{n}$

$\therefore$ $3 - 4(1 - \cos^2 B) = \dfrac{n+1}{n}$

$\therefore$ $3 - 4 + 4\left(\dfrac{n+2}{2n}\right)^2 = \dfrac{n+1}{n}$

$\Rightarrow -1 + \dfrac{n^2 + 4n + 4}{n^2} = \dfrac{n+1}{n}$

$\Rightarrow -n^2 + n^2 + 4n + 4 = n^2 + n$

$\Rightarrow n^2 - 3n - 4 = 0$

$\Rightarrow (n + 1)(n - 4) = 0$

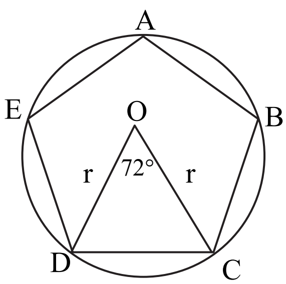
$\Rightarrow n = -1$ or $n = 4$

But n cannot be negative.

$\therefore$ $n = 4$

$\therefore$ The sides of the Δ are 4, 5, 6.

14.



In ΔODC, $OD = OC = r$, $\angle DOC = \dfrac{360°}{5} = 72°$

$\therefore$ $A(\Delta ODC) = \dfrac{1}{2}$ r.r. $\sin 72° = \dfrac{1}{2} r^2 \sin 72°$

$\therefore$ A_2 = Area of pentagon = $\dfrac{5}{2} r^2 \sin 72°$

A_1 = Area of circle = πr^2

$\therefore$ $\dfrac{A_1}{A_2} = \dfrac{\pi r^2}{\frac{5}{2} r^2 \sin 72°}$

$= \dfrac{2\pi}{5\cos 18°} = \dfrac{2\pi}{5}\sec 18° = \dfrac{2\pi}{5}\sec\dfrac{\pi}{10}$

15. Let $a = 4k$, $b = 5k$, $c = 6k$

Now, $s = \dfrac{a+b+c}{2} = \dfrac{4k+5k+6k}{2} = \dfrac{15k}{2}$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{\dfrac{15k}{2}\left(\dfrac{15k}{2} - 4k\right)\left(\dfrac{15k}{2} - 5k\right)\left(\dfrac{15k}{2} - 6k\right)}$

$= \sqrt{\dfrac{15k}{2}\times\dfrac{7k}{2}\times\dfrac{5k}{2}\times\dfrac{3k}{2}} = \dfrac{15\sqrt{7}}{4}k^2$

By sine Rule, $\dfrac{a}{\sin A} = 2R$ $\therefore$ $\sin A = \dfrac{a}{2R}$

$\Delta = \dfrac{1}{2} bc \sin A$

$\Rightarrow \Delta = \dfrac{1}{2} bc \cdot \dfrac{a}{2R} = \dfrac{abc}{4R}$

$\therefore$ $R = \dfrac{abc}{4\Delta} = \dfrac{4k.5k.6k}{15\sqrt{7}k^2} = \dfrac{8}{\sqrt{7}}k$

Also $\Delta = rs$, where r = Radius of incircle of ΔABC

$\therefore$ $r = \dfrac{\Delta}{s} = \dfrac{\frac{15\sqrt{7}}{4}\cdot k^2}{\frac{15k}{2}} = \dfrac{\sqrt{7}}{2}k$

$\therefore$ $\dfrac{R}{r} = \dfrac{8}{\sqrt{7}}k \times \dfrac{2}{\sqrt{7}k} = \dfrac{16}{7}$

$\therefore$ $\dfrac{R}{r} = \dfrac{16}{7}$

16. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \cos 30° = \dfrac{4 + 3 - a^2}{4\sqrt{3}}$

$\Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{7 - a^2}{4\sqrt{3}}$ $\qquad \Rightarrow 7 - a^2 = 6$

$\Rightarrow a^2 = 1$

$\Rightarrow a = 1$ $\qquad$ ....[$\because$ $a \neq -1$]

$\Delta = \dfrac{1}{2} bc \sin A = \dfrac{1}{2} \times 2 \times \sqrt{3} \times \sin 30°$

$= \sqrt{3} \times \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}$

$s = \dfrac{a+b+c}{2} = \dfrac{1+2+\sqrt{3}}{2} = \dfrac{3+\sqrt{3}}{2}$

$\Delta = rs$

$\therefore$ $r = \dfrac{\Delta}{s} = \dfrac{\sqrt{3}}{2} \times \dfrac{2}{3+\sqrt{3}}$

$= \dfrac{\sqrt{3}(3-\sqrt{3})}{9-3} = \dfrac{3\sqrt{3}-3}{6} = \dfrac{\sqrt{3}-1}{2}$

17. $a^4 + b^4 + c^4 = 2a^2(b^2 + c^2)$

$\therefore$ $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 = 0$

$\therefore$ $a^4 + b^4 + c^4 - 2a^2b^2 + 2b^2c^2 - 2a^2c^2 = 2b^2c^2$

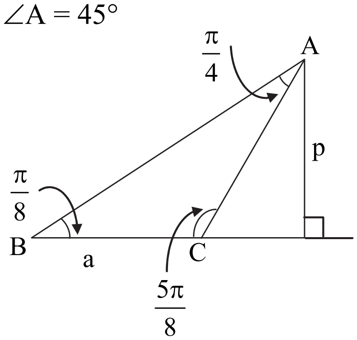
$\therefore$ $\left(b^2 + c^2 - a^2\right)^2 = \left(\sqrt{2}bc\right)^2$

$\therefore$ $b^2 + c^2 - a^2 = \sqrt{2}bc$

$\therefore$ $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{\sqrt{2}bc}{2bc} = \dfrac{1}{\sqrt{2}}$

$\therefore$ $\angle A = 45°$

18.



Let length of altitude = p

Since, A + B + C = π

∴ $A + \frac{\pi}{8} + \frac{5\pi}{8} = \pi$

∴ $A = \pi - \frac{\pi}{8} - \frac{5\pi}{8} = \frac{\pi}{4}$

Area of Δ = $\frac{1}{2}ap = \frac{1}{2}bc \sin A$

∴ $ap = bc \sin\frac{\pi}{4}$

∴ $ap = bc \times \frac{1}{\sqrt{2}}$

∴ $p = \frac{bc}{\sqrt{2}a}$ ….(i)

By sine rule,

$$\frac{a}{\sin\frac{\pi}{4}} = \frac{b}{\sin\frac{\pi}{8}} = \frac{c}{\sin\frac{5\pi}{8}}$$

∴ $b = \frac{a\sin\frac{\pi}{8}}{\frac{1}{\sqrt{2}}} = \sqrt{2}a\sin\frac{\pi}{8}$

$c = \frac{a\sin\frac{5\pi}{8}}{\frac{1}{\sqrt{2}}} = \sqrt{2}\,a\sin\frac{5\pi}{8}$

∴ From (i),

$$p = \frac{\sqrt{2}a\sin\frac{\pi}{8}.\sqrt{2}a\sin\frac{5\pi}{8}}{\sqrt{2}a} = \sqrt{2}a\sin\frac{5\pi}{8}\sin\frac{\pi}{8}$$

$$= \frac{\sqrt{2}a}{2}\left(2\sin\frac{5\pi}{8}\sin\frac{\pi}{8}\right)$$

$$= \frac{a}{\sqrt{2}}\left[\cos\left(\frac{5\pi}{8}-\frac{\pi}{8}\right)-\cos\left(\frac{5\pi}{8}+\frac{\pi}{8}\right)\right]$$

$$= \frac{a}{\sqrt{2}}\left[\cos\frac{\pi}{2}-\cos\frac{3\pi}{4}\right]$$

$$= \frac{a}{\sqrt{2}}\left[0-\left(-\frac{1}{\sqrt{2}}\right)\right]$$

∴ $P = \frac{a}{2}$

19. $\tan\frac{A}{2}$ and $\tan\frac{B}{2}$ are the roots of the quadratic equation $6x^2 - 5x + 1 = 0$

∴ $\tan\frac{A}{2} + \tan\frac{B}{2} = \frac{5}{6}$, $\tan\frac{A}{2}.\tan\frac{B}{2} = \frac{1}{6}$

∴ $\tan\left(\frac{A}{2}+\frac{B}{2}\right) = \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}} = \frac{\frac{5}{6}}{1-\frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$

∴ $\tan\left(\frac{A+B}{2}\right) = 1$

∴ $\frac{A+B}{2} = \frac{\pi}{4}$

∴ $A + B = \frac{\pi}{2}$ ∴ $\angle C = \frac{\pi}{2}$

∴ ΔABC is a right angled triangle.

20. $r = \frac{\Delta}{s} = \frac{\frac{1}{2}ac\sin B}{\frac{1}{2}(a+b+c)} = \frac{ac}{a+b+c}$

….[∵ sin B = sin 90° = 1]

∴ $r = \frac{ac}{a+c+b} \times \frac{a+c-b}{a+c-b}$

$= \frac{ac(a+c-b)}{(a+c)^2-b^2} = \frac{ac(a+c-b)}{a^2+c^2+2ac-b^2}$

$= \frac{a+c-b}{2}$ ….[∵ $a^2 + c^2 = b^2$]

∴ Diameter = a + c – b

21. ∠A = 55°, ∠B = 15°, ∠C = 110°

∴ $\frac{a}{\sin 55°} = \frac{b}{\sin 15°} = \frac{c}{\sin 110°} = k$

∴ a = k sin 55°, b = k sin 15°, c = k sin 110°

∴ $c^2 - a^2 = k^2 \sin^2 110° - k^2 \sin^2 55°$

$= k^2(\sin 110° + \sin 55°)(\sin 110° - \sin 55°)$

$= k^2\left(2\sin\frac{165°}{2}\cos\frac{55°}{2}\right)\left(2\sin\frac{55°}{2}\cos\frac{165°}{2}\right)$

$= k^2 \sin 165° \sin 55°$

$= k^2 \sin 15° \sin 55°$

= (k sin 55°) (k sin 15°)

= ab

22. A, B, C are in A.P.

$\therefore$ A + C = 2B

Also, A + B + C = 180°

$\therefore$ $\angle B = 60°$

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$

$\therefore$ sin A = ak, sin B = bk, sin C = ck

$\therefore$ $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$

$= \frac{a}{c}(2\sin C\cos C) + \frac{c}{a}(2\sin A\cos A)$

$= \frac{a}{c}(2\,ck\cos C) + \frac{c}{a}(2ak\cos A)$

$= 2ka\cos C + 2kc\cos A$

$= 2k(a\cos C + c\cos A)$

$= 2kb$ ....[$\because$ b = a cos C + c cos A]

$= 2\sin B$

$= 2 \times \frac{\sqrt{3}}{2}$ ....[$\because \angle B = 60°$]

$= \sqrt{3}$

23. $2\cot^{-1} 3 = 2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{3}$

$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{3}}{1-\frac{1}{3}\cdot\frac{1}{3}}\right)$

$= \tan^{-1}\left(\frac{3+3}{9-1}\right)$

$= \tan^{-1}\left(\frac{6}{8}\right)$

$= \tan^{-1}\frac{3}{4}$

$\therefore$ $\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right) = \frac{1}{\tan\left(\frac{\pi}{4} - \tan^{-1}\frac{3}{4}\right)}$

$= \frac{1 + \tan\frac{\pi}{4}\tan\left(\tan^{-1}\frac{3}{4}\right)}{\tan\frac{\pi}{4} - \tan\left(\tan^{-1}\frac{3}{4}\right)}$

$= \frac{1 + 1\cdot\frac{3}{4}}{1 - \frac{3}{4}} = \frac{4+3}{4-3} = 7$

24. Let $\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right) = \theta$

$\therefore$ $\cos^{-1}\left(\frac{a}{b}\right) = 2\theta$

$\therefore$ $\cos 2\theta = \frac{a}{b}$

$\therefore$ $\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]$

$= \tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right)$

$= \frac{1+\tan\theta}{1-\tan\theta} + \frac{1-\tan\theta}{1+\tan\theta}$

$= \frac{(1+\tan\theta)^2 + (1-\tan\theta)^2}{1-\tan^2\theta}$

$= \frac{2(1+\tan^2\theta)}{1-\tan^2\theta}$

$= \frac{2}{\frac{1-\tan^2\theta}{1+\tan^2\theta}} = \frac{2}{\cos 2\theta} = \frac{2}{\frac{a}{b}} = \frac{2b}{a}$

25. $\cos^{-1}\alpha - \cos^{-1}\beta = \cos^{-1}\left[\alpha\beta + \sqrt{1-\alpha^2}\sqrt{1-\beta^2}\right]$

Given, $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$

$\therefore$ $\cos^{-1}\left[\frac{xy}{2} + \sqrt{(1-x^2)\left(1-\frac{y^2}{4}\right)}\right] = \alpha$

$\therefore$ $\cos\alpha = \frac{xy}{2} + \sqrt{(1-x^2)\left(1-\frac{y^2}{4}\right)}$

$\therefore$ $\sqrt{(1-x^2)\left(1-\frac{y^2}{4}\right)} = \cos\alpha - \frac{xy}{2}$

$\therefore$ $2\sqrt{(1-x^2)\left(1-\frac{y^2}{4}\right)} = 2\cos\alpha - xy$

Squaring on both sides, we get

$4(1-x^2)\left(1-\frac{y^2}{4}\right) = 4\cos^2\alpha - 4xy\cos\alpha + x^2y^2$

$\therefore$ $4 - y^2 - 4x^2 + x^2y^2 = 4\cos^2\alpha - 4xy\cos\alpha + x^2y^2$

$\therefore$ $4x^2 + y^2 - 4xy\cos\alpha = 4 - 4\cos^2\alpha$

$\therefore$ $4x^2 + y^2 - 4xy\cos\alpha = 4\sin^2\alpha$

26. $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$

$\therefore$ $\sin^{-1} 2x = \frac{\pi}{3} - \sin^{-1} x$

$\therefore$ $2x = \sin\left(\frac{\pi}{3} - \sin^{-1} x\right)$

$= \sin \frac{\pi}{3} \cos(\sin^{-1} x) - \cos\frac{\pi}{3} \sin(\sin^{-1} x)$

$\Rightarrow 2x = \frac{\sqrt{3}}{2} \cos(\sin^{-1} x) - \frac{1}{2} \cdot x$ …. (i)

Let $\sin^{-1} x = \theta$

$\therefore$ $\sin\theta = x$

$\cos\theta = \sqrt{1-x^2}$

$\therefore$ $\cos(\sin^{-1} x) = \sqrt{1-x^2}$ ….(ii)

From (i) and (ii), we get

$2x = \frac{\sqrt{3}}{2} \cdot \sqrt{1-x^2} - \frac{1}{2}x$

$\therefore$ $4x = \sqrt{3}\sqrt{1-x^2} - x$

$\therefore$ $5x = \sqrt{3}\sqrt{1-x^2}$

$\therefore$ $25x^2 = 3 - 3x^2$ (squaring both sides)

$\therefore$ $28x^2 = 3$

$\therefore$ $x^2 = \frac{3}{28}$

$\therefore$ $x = \sqrt{\frac{3}{28}} = \sqrt{\frac{1}{4} \cdot \frac{3}{7}} = \frac{1}{2}\sqrt{\frac{3}{7}}$

(From the given relation it can be seen that x is positive)

27. L.H.S. $= \sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(-\cot\frac{19\pi}{8}\right)$

$= \sin^{-1}\left[\sin\left(5\pi - \frac{2\pi}{7}\right)\right] + \cos^{-1}\left[\cos\left(7\pi - \frac{3\pi}{7}\right)\right] + \tan^{-1}\left[-\tan\left(2\pi - \frac{3\pi}{8}\right)\right] + \cot^{-1}\left[-\cot\left(3\pi - \frac{5\pi}{8}\right)\right]$

$= \sin^{-1}\left(\sin\frac{2\pi}{7}\right) + \cos^{-1}\left(-\cos\frac{3\pi}{7}\right) + \tan^{-1}\left(\tan\frac{3\pi}{8}\right) + \cot^{-1}\left(\cot\frac{5\pi}{8}\right)$

$= \frac{2\pi}{7} + \pi - \frac{3\pi}{7} + \frac{3\pi}{8} + \frac{5\pi}{8}$

….[$\because \cos^{-1}(-x) = \pi - \cos^{-1} x$]

$= \pi - \frac{\pi}{7} + \pi = 2\pi - \frac{\pi}{7} = \frac{13\pi}{7}$

$\therefore$ $\frac{13\pi}{7} = \frac{a\pi}{b}$

$\therefore$ a = 13, b = 7

$\therefore$ a + b = 13 + 7 = 20

28. $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$

$= \sin^{-1}\left[\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right] + \sin^{-1}\frac{16}{65}$

$= \sin^{-1}\left[\frac{4}{5}\times\frac{12}{13} + \frac{5}{13}\times\frac{3}{5}\right] + \sin^{-1}\frac{16}{65}$

$= \sin^{-1}\left(\frac{48+15}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right)$

$= \sin^{-1}\left(\frac{63}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right)$

$= \cos^{-1}\left(\sqrt{1-\left(\frac{63}{65}\right)^2}\right) + \sin^{-1}\left(\frac{16}{65}\right)$

$= \cos^{-1}\left(\frac{16}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right)$

$= \frac{\pi}{2}$

29. $\sqrt{2} = 1.414$

$\therefore$ $2\sqrt{2} - 1 = 2 \times 1.414 - 1 = 2.828 - 1 = 1.828$

$\therefore$ $2\sqrt{2} - 1 > \sqrt{3}$ ….[$\because \sqrt{3} = 1.732$]

$\therefore$ $\tan^{-1}(2\sqrt{2} - 1) > \tan^{-1}(\sqrt{3})$

….[$\because \tan^{-1} x$ is an increasing function]

$\therefore$ $2\tan^{-1}(2\sqrt{2} - 1) > 2 \times \frac{\pi}{3}$

$\therefore$ $A > \frac{2\pi}{3}$ ….(i)

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore$ $3\theta = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

Put $\sin\theta = \frac{1}{3}$

$\therefore$ $\theta = \sin^{-1}\left(\frac{1}{3}\right)$

$\therefore$ $3\sin^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left[3\cdot\frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right]$

$= \sin^{-1}\left(1 - \frac{4}{27}\right)$

$= \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852)$

$\frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.866$, $0.852 < 0.866$

$\therefore$ $\sin^{-1}(0.852) < \sin^{-1}(0.866)$

....[$\because$ $\sin^{-1} x$ is also an increasing function]

$\therefore$ $3\sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\therefore$ $3\sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{3}$ ....(ii)

$\sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}(0.6) < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\therefore$ $\sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{3}$ ....(iii)

From (ii) and (iii), we get

$B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore$ $B < \frac{2\pi}{3}$ ....(iv)

From (i) and (iv), A > B

30. $\cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \frac{\pi}{2}$

$\therefore$ $\frac{\pi}{2} - \tan^{-1} x + \frac{\pi}{2} - \tan^{-1} y + \frac{\pi}{2} - \tan^{-1} z = \frac{\pi}{2}$

$\therefore$ $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$

$\therefore$ $\tan(\tan^{-1} x + \tan^{-1} y + \tan^{-1} z) = \tan\pi = 0$

Let $A = \tan^{-1} x$, $B = \tan^{-1} y$, $C = \tan^{-1} z$

$\therefore$ $\tan(A + B + C) = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$

$= \frac{\frac{\tan A + \tan B}{1 - \tan A\tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A\tan B}\cdot\tan C}$

$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

$\therefore$ $\tan(A + B + C) = 0$

$\Rightarrow \tan A + \tan B + \tan C = \tan A\tan B\tan C$

$\therefore$ $\tan(\tan^{-1} x) + \tan(\tan^{-1} y) + \tan(\tan^{-1} z)$

$= \tan(\tan^{-1} x)\tan(\tan^{-1} y)\tan(\tan^{-1} z)$

$\therefore$ $x + y + z = xyz$

31. $\cos^{-1}\left\{\frac{1}{\sqrt{2}}\left(\cos\frac{9\pi}{10} - \sin\frac{9\pi}{10}\right)\right\}$

$= \cos^{-1}\left[\cos\frac{\pi}{4}\cos\frac{9\pi}{10} - \sin\frac{\pi}{4}\sin\frac{9\pi}{10}\right]$

$= \cos^{-1}\left[\cos\left(\frac{\pi}{4} + \frac{9\pi}{10}\right)\right]$

$= \cos^{-1}\left[\cos\left(\frac{5\pi + 18\pi}{20}\right)\right]$

$= \cos^{-1}\left[\cos\left(\frac{23\pi}{20}\right)\right]$

$= \cos^{-1}\left[\cos\left(2\pi - \frac{23\pi}{20}\right)\right]$

$= \cos^{-1}\left[\cos\left(\frac{17\pi}{20}\right)\right]$ and $0 \le \frac{17\pi}{20} \le \pi$

$= \frac{17\pi}{20}$

$\therefore$ Principal value is $\frac{17\pi}{20}$.

32. $\tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1}\left(\frac{2+3}{1-2\times3}\right)$

....[$\because$ $2 \times 3 > 1$]

$= \pi + \tan^{-1}(-1)$

$= \pi - \tan^{-1} 1$

$\therefore$ $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

33. $\tan^{-1}\frac{1}{1+2x}+\tan^{-1}\frac{1}{4x+1}=\tan^{-1}\frac{2}{x^2}$

$\therefore$ $\tan^{-1}\left(\frac{\frac{1}{1+2x}+\frac{1}{4x+1}}{1-\frac{1}{1+2x}\cdot\frac{1}{4x+1}}\right)=\tan^{-1}\frac{2}{x^2}$

$\therefore$ $\frac{4x+1+2x+1}{(1+2x)(4x+1)-1}=\frac{2}{x^2}$

$\therefore$ $\frac{6x+2}{4x+8x^2+1+2x-1}=\frac{2}{x^2}$

$\therefore$ $x^2(6x+2)=2(8x^2+6x)$

$\therefore$ $6x^3+2x^2-16x^2-12x=0$

$\therefore$ $6x^3-14x^2-12x=0$

$\therefore$ $3x^3-7x^2-6x=0$

$\therefore$ $x(3x^2-7x-6)=0$

$\therefore$ $x(x-3)(3x+2)=0$

$\therefore$ $x=0,3,-\frac{2}{3}$

But $x>0$, $\therefore$ $x=3$

34. $\cot^{-1}x+\sin^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{4}$

$\therefore$ $\tan^{-1}\frac{1}{x}+\tan^{-1}\frac{\frac{1}{\sqrt{5}}}{\sqrt{1-\frac{1}{5}}}=\frac{\pi}{4}$

....$\left[\because \sin^{-1}x=\tan^{-1}\frac{x}{\sqrt{1-x^2}}\right]$

$\therefore$ $\tan^{-1}\frac{1}{x}+\tan^{-1}\frac{1}{2}=\frac{\pi}{4}$

$\therefore$ $\tan^{-1}\left(\frac{\frac{1}{x}+\frac{1}{2}}{1-\frac{1}{x}\cdot\frac{1}{2}}\right)=\frac{\pi}{4}$

$\therefore$ $\frac{2+x}{2x-1}=\tan\frac{\pi}{4}=1$

$\therefore$ $2+x=2x-1$

$\therefore$ $x=3$

Textbook Chapter No.

04 Pair of Straight Lines

Hints

Classical Thinking

1. Joint equation of pair of lines having slopes m_1 and m_2 and passing through the origin is
$y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0$
$\Rightarrow 3x^2 - 4xy + y^2 = 0$
Alternate method:
Equations of the lines are $y = x$ and $y = 3x$ respectively.
i.e. $y - x = 0$ and $y - 3x = 0$
$\therefore$ the combined equation of the pair of lines is
$(y - x)(y - 3x) = 0$
$\therefore$ $y^2 - 3xy - xy + 3x^2 = 0 \Rightarrow 3x^2 - 4xy + y^2 = 0$

2. The required equation is
$y^2 - \left(\frac{8}{3}\right)xy - x^2 = 0$
$\Rightarrow 3x^2 + 8xy - 3y^2 = 0$

3. The required equation is
$y^2 - \frac{3}{2}xy - x^2 = 0$
$\Rightarrow 2x^2 + 3xy - 2y^2 = 0$

4. $x^2 + xy - 12y^2 = 0$
$\Rightarrow x^2 + 4xy - 3xy - 12y^2 = 0$
$\Rightarrow x(x + 4y) - 3y(x + 4y) = 0$
$\Rightarrow (x - 3y)(x + 4y) = 0$
$\Rightarrow x - 3y = 0$ and $x + 4y = 0$

5. It is a homogeneous equation of degree 2 in x and y.
$\therefore$ Correct option is (C).

6. $3x^2 - 10xy - 8y^2 = 0$
$\Rightarrow 3x^2 - 12xy + 2xy - 8y^2 = 0$
$\Rightarrow 3x(x - 4y) + 2y(x - 4y) = 0$
$\Rightarrow (3x + 2y)(x - 4y) = 0$
$\Rightarrow 3x + 2y = 0$ and $x - 4y = 0$

7. $6x^2 - 5xy + y^2 = 0$
$\Rightarrow 6x^2 - 3xy - 2xy + y^2 = 0$
$\Rightarrow 3x(2x - y) - y(2x - y) = 0$
$\Rightarrow (2x - y)(3x - y) = 0$
$\Rightarrow 3x - y = 0$ and $2x - y = 0$

8. Equation of straight lines parallel to $ax^2 + 2hxy + by^2 = 0$ and passing through point (x_1, y_1) is found by shifting the origin to (x_1, y_1)
$\therefore$ The required equation is
$a(x - x_1)^2 + 2h(x - x_1)(y - y_1) + b(y - y_1)^2 = 0$

9. $L_1 = x^2 - y^2 = 0$ represents pair of straight lines passing through the origin
To find equation of pair of straight lines parallel to L_1 and passing through (3, 4), shift the origin to (3, 4)
$\therefore$ $(x - 3)^2 + (y - 4)^2 = 0$
$\Rightarrow x^2 + y^2 - 6x - 8y + 25 = 0$

10. L_1: $ax^2 + 2hxy + by^2 = 0$
Equation of any line passing through origin and perpendicular to L_1 is given by
$bx^2 - 2hxy + ay^2 = 0$
....(interchanging coefficients of x^2 and y^2 and change of sign for xy term)
$\therefore$ The required equation is $ay^2 - 2hxy + bx^2 = 0$

11. The required equation is
$-3x^2 + 7xy + 5y^2 = 0$
i.e. $3x^2 - 7xy - 5y^2 = 0$

12. Comparing given equation with $ax^2 + 2hxy + by^2 = 0$, we get
$h = \frac{-1}{2}$ and $b = -6$
$\therefore$ Sum of slopes $= m_1 + m_2 = \frac{-2h}{b}$
$= \frac{-2\left(\frac{-1}{2}\right)}{-6} = \frac{-1}{6}$

13. Given equation of pair of lines is
$ax^2 + 10xy + y^2 = 0$
$\therefore$ $A = a, H = 5, B = 1$
Let the slopes of the lines given by be m_1 and m_2
$m_1 + m_2 = \frac{-2H}{B}$ and $m_1m_2 = \frac{A}{B}$
Given that $m_2 = 4m_1$
$\therefore$ $m_1 + 4m_1 = \frac{-2H}{B} = -10 \Rightarrow m_1 = -2$
and $m_1 \times 4m_1 = \frac{A}{B} = a \Rightarrow 4m_1^2 = a \Rightarrow a = 16$

14. Given equation of pair of lines is
$ax^2 + 4xy + y^2 = 0$
$\therefore$ A = a, H = 2, B = 1
$m_1 + m_2 = -4$ and $m_1m_2 = a$
Given that $m_1 = 3m_2$
$\therefore$ $3m_2 + m_2 = -4 \Rightarrow m_2 = -1$
Hence, $m_1 = -3$
$\therefore$ $a = (-1)(-3) = 3$

15. Given equation of pair of lines is
$ax^2 + (3a + 1)xy + 3y^2 = 0$
$\therefore$ $A = a, H = \frac{3a+1}{2}, B = 3$
Given that $m_1 = \frac{1}{m_2} \Rightarrow m_1m_2 = 1$
Now, $m_1m_2 = \frac{a}{3} \Rightarrow \frac{a}{3} = 1 \Rightarrow a = 3$
Also, $m_1 + m_2 = -\left(\frac{3a+1}{3}\right) = \frac{-10}{3}$
$\Rightarrow m_1 + \frac{1}{m_1} = \frac{-10}{3} \Rightarrow 3m_1^2 + 10m_1 + 3 = 0$
$\therefore$ $m_1 = \frac{-1}{3}$ or -3.

16. Given equation of pair of lines is
$6x^2 + 41xy - 7y^2 = 0$
$\therefore$ $a = 6, h = \frac{41}{2}, b = -7$
α and β are angles made by the two lines with X-axis
$\therefore$ their slopes m_1 and m_2 respectively are
$m_1 = \tan\alpha$ and $m_2 = \tan\beta$
$\tan\alpha.\tan\beta = m_1m_2 = -\frac{6}{7}$

17. Given equation of pair of lines is
$6x^2 - xy - y^2 = 0$
$\therefore$ $a = 6, h = -\frac{1}{2}, b = -1$
If θ is the acute angle between the pair of lines
$\therefore$ $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$
$\Rightarrow \tan\theta = \left|\frac{2\sqrt{\frac{1}{4}+6}}{5}\right| = 1$
$\Rightarrow \theta = \tan^{-1}(1) = 45°$

18. Given equation of pair of lines is
$\sqrt{3}xy - y^2 = 0$
$\therefore$ $a = 0, h = \frac{\sqrt{3}}{2}, b = -1$
$\therefore$ $\tan\theta = \left|\frac{2\sqrt{\frac{3}{4}-0}}{0-1}\right| = \left|\frac{2\times\frac{\sqrt{3}}{2}}{-1}\right| = \sqrt{3}$
$\Rightarrow \theta = \tan^{-1}(\sqrt{3}) = 60°$

19. Given equation of pair of lines is
$11y^2 - 4xy + 4x^2 = 0$
i.e. $4x^2 - 4xy + 11y^2 = 0$
$\therefore$ a = 4, h = –12, b = 11
$\therefore$ $\tan\theta = \left|\frac{2\sqrt{144-44}}{4+11}\right| = \frac{2(10)}{15} = \frac{4}{3}$
$\Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$

20. Given equation of pair of lines is
$2x^2 - 3xy + y^2 = 0$
$\therefore$ $a = 2, h = \frac{-3}{2}, b = 1$
$\therefore$ $\tan\theta = \left|\frac{2\sqrt{\frac{9}{4}-2}}{3}\right| = \left|\frac{\sqrt{9-8}}{3}\right| = \frac{1}{3}$
$\therefore$ $\cot\theta = 3 \Rightarrow \theta = \cot^{-1}(3)$

21. Given equation of pair of lines is
$x^2(\cos\theta - \sin\theta) + 2xy\cos\theta + y^2(\cos\theta + \sin\theta) = 0$
$\therefore$ $a = \cos\theta - \sin\theta, h = \cos\theta, b = \cos\theta + \sin\theta$
The acute angle α between the pair of lines is given by
$\tan\alpha = \left|\frac{2\sqrt{\cos^2\theta - (\cos^2\theta - \sin^2\theta)}}{2\cos\theta}\right|$
$\Rightarrow \tan\alpha = \tan\theta \Rightarrow \alpha = \theta$

22. Given equation of pair of lines is
$x^2 - 4hxy + 3y^2 = 0$
$\therefore$ A = 1, H = –2h, B = 3
Now, $\theta = 60°$
$\Rightarrow \tan\theta = \sqrt{3}$
$\therefore$ $\tan\theta = \left|\frac{2\sqrt{H^2 - AB}}{A+B}\right|$
$\Rightarrow \sqrt{3} = \left|\frac{2\sqrt{4h^2-3}}{4}\right| \Rightarrow h = \pm\frac{\sqrt{15}}{2}$

23. Given equation of pair of lines is
$3x^2 + 18xy + by^2 = 0$
$\therefore$ a = 3, h = 9, b = b
Now $\theta = \pi \Rightarrow \tan\theta = 0$
$\therefore$ $\tan\theta = \left|\frac{2\sqrt{81-3b}}{3+b}\right|$
$\Rightarrow 0 = \left|\frac{2\sqrt{81-3b}}{3+b}\right|$
$\Rightarrow 81 = 3b \Rightarrow b = 27$

24. Given equation of pair of lines is
$3x^2 + 10xy + 8y^2 = 0$
$\therefore$ a = 3, h = 5, b = 8
Now $\theta = \tan^{-1}(p) \Rightarrow \tan\theta = p$
$\therefore$ $\tan\theta = \left|\frac{2\sqrt{25-24}}{11}\right|$
$\Rightarrow p = \left|\frac{2}{11}\right| = \frac{2}{11}$

25. Given equation of pair of lines is
$3x^2 + 2hxy + y^2 = 0$
$\therefore$ a = 3, h = h, b = 1
The two lines are real and coincident if $h^2 - ab = 0$
$\therefore$ $h^2 - ab = h^2 - 3$
for these lines to be real and coincident,
$h^2 - 3 \geq 0 \Rightarrow h^2 \geq 3$

26. Given equation of pair of lines is
$9x^2 - 12xy + 4y^2 = 0$
a = 9, h = –6, b = 4
Now, $h^2 - ab = (6)^2 - 9 \times 4 = 0$
$\therefore$ The lines are coincident.

27. The condition for a pair of straight lines to be real and coincident is $h^2 - ab = 0$
Consider the equation $4x^2 - 4xy + y^2 = 0$
$\therefore$ a = 4 , h = –2, b = 1
$h^2 - ab = (-2)^2 - (4)(1) = 0$
$\therefore$ Correct option is (A).

28. Given equation of pair of lines is
$6x^2 + hxy + 12y^2 = 0$
$\therefore$ $A = 6, H = \frac{h}{2}, B = 12$
Since lines are parallel,
$\therefore$ $H^2 - AB = 0$
$\Rightarrow \frac{h^2}{4} = 6(12) \Rightarrow h^2 = (24)(12)$
$\Rightarrow h = \pm 12\sqrt{2}$

29. Given equation of pair of lines is
$4x^2 + hxy + y^2 = 0$
The lines are coincident
$\therefore$ $H^2 = AB$
$\Rightarrow \frac{h^2}{4} = 4(1)$
$\Rightarrow h = \pm 4$

30. Given equation of pair of lines is
$x^2 + xy + y^2 = 0$
$\therefore$ $a = 1, h = \frac{1}{2}, b = 1$
Here, $h^2 - ab = \frac{-3}{4} < 0$
Hence, the lines are imaginary.

31. Given equation of pair of lines is
$\lambda y^2 + (1 - \lambda^2)xy - \lambda x^2 = 0$
$\therefore$ $a = -\lambda, b = \lambda$
Now a + b = 0
$\therefore$ the lines are perpendicular
$\therefore$ Angle between the lines is 90°.

32. Given equation of pair of lines is
$xy = 0$
$\therefore$ $a = 0, h = \frac{1}{2}, b = 0$
Now, a + b = 0
$\therefore$ the lines are perpendicular to each other.
$\therefore$ angle between the pair of line is 90°.

33. The condition for a pair of straight lines to be perpendicular is a + b = 0.
Consider the equation $2x^2 = 2y(2x + y)$
i.e. $2x^2 - 4xy - 2y^2 = 0$
$\therefore$ a = 2, b = –2
$\therefore$ a + b = 2 + (–2) = 0
$\therefore$ Correct option is (A).

34. It is a homogeneous equation of degree 2 in x and y
Hence, it represents a pair of lines and
a + b = 0
$\therefore$ lines are perpendicular

35. Given equation of pair of lines is
$3y^2 + 9xy + kx^2 = 0$
i.e. $kx^2 + 9xy + 3y^2 = 0$
$\therefore$ a = k, b = 3
The lines are perpendicular
$\therefore$ a + b = 0
$\Rightarrow k + 3 = 0 \Rightarrow k = -3$

36. Given equation of pair of lines is
$a^2x^2 + bcy^2 = a(b + c)xy$
$\therefore$ $A = a^2, B = bc$
Since the lines are mutually perpendicular,
$\therefore$ $A + B = 0$
$\Rightarrow a^2 + bc = 0$

37. Consider $2x^2 + 3xy - 2y^2 + 5x + 5y + 3 = 0$
Comparing the given equation with
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get
$a = 2, b = -2, c = 3, f = \frac{5}{2}, g = \frac{5}{2}, h = \frac{3}{2}$
Condition for equation to represent pair of lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
$\therefore$ $2(-2)(3) + 2\left(\frac{5}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right) - 2\left(\frac{5}{2}\right)^2 - (-2)\left(\frac{5}{2}\right)^2 - 3\left(\frac{3}{2}\right)^2$
$= -12 + \frac{75}{4} - \frac{50}{4} + \frac{50}{4} - \frac{27}{4} = 0$
$\therefore$ Condition is satisfied
$\therefore$ Correct answer is option (A).

38. Given equation of pair of lines is
$y^2 + xy + px^2 - x - 2y = 0$
$\therefore$ $a = p, b = 1, c = 0, f = -1, g = \frac{-1}{2}, h = \frac{1}{2}$
The given equation represents pair of straight lines if
$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
$\Rightarrow p(1)(0) + 2(-1)\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right) - p(-1)^2 - 1\left(\frac{-1}{2}\right)^2 - 0 = 0$
$\Rightarrow \frac{1}{2} - p - \frac{1}{4} = 0 \Rightarrow p = \frac{1}{4}$

39. Given equation of pair of lines is
$6x^2 + 11xy - 10y^2 + x + 31y + k = 0$
$\therefore$ $a = 6, b = -10, c = k, f = \frac{31}{2}, g = \frac{1}{2}, h = \frac{11}{2}$
Now, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
$\Rightarrow -6(10)k + 2\left(\frac{31}{2}\right)\left(\frac{1}{2}\right)\left(\frac{11}{2}\right) - 6\left(\frac{31}{2}\right)^2 + 10\left(\frac{1}{2}\right)^2 - k\left(\frac{11}{2}\right)^2 = 0$
$\Rightarrow -k\,\frac{361}{4} = \frac{5415}{4} \Rightarrow k = -15$

40. Given equation of pair of lines is
$x^2 - y^2 - x - \lambda y - 2 = 0$
$\therefore$ $a = 1, b = -1, c = -2, f = \frac{-\lambda}{2}, g = \frac{-1}{2}, h = 0$
Now, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
$\therefore$ $2 - \frac{\lambda^2}{4} + \frac{1}{4} = 0 \Rightarrow \frac{\lambda^2}{4} = \frac{9}{4}$
$\Rightarrow \lambda^2 = 9$
$\Rightarrow \lambda = \pm 3$

41. Given equation of pair of lines is
$3x^2 + 2hxy - 3y^2 - 40x + 30y - 75 = 0$
$\therefore$ $A = 3, B = -3, C = -75, F = 15, G = -20, H = h$
Now $ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$
$\Rightarrow (3)(-3)(-75) + 2(15)(-20)(h) - 3(15)^2 - (-3)(-20)^2 - (-75)h^2 = 0$
$\Rightarrow 675 - 600h - 675 + 1200 + 75h^2 = 0$
$\Rightarrow h^2 - 8h + 16 = 0$
$\Rightarrow (h - 4)^2 = 0$
$\Rightarrow h = 4,4$

42. Given equation of pair of lines is
$2x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$
$\therefore$ $a = 2, b = 12, c = -3, f = -8, g = \frac{5}{2}, h = -5$
Equation of perpendicular drawn from origin on $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is
$bx^2 - 2hxy + ay^2 = 0$
$\therefore$ $12x^2 + 10xy + 2y^2 = 0$
i.e., $6x^2 + 5xy + y^2 = 0$

43. Given equation of pair of lines is
$2x^2 - 5xy + 3y^2 + 8x - 9y + 6 = 0$
$\therefore$ $a = 2, b = 3, c = 6, f = -\frac{9}{2}, g = 4, h = \frac{-5}{2}$
The point of intersection is given by
$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$
$$\equiv \left(\frac{\left(\frac{-5}{2}\right)\left(\frac{-9}{2}\right) - 3(4)}{2(3) - \left(\frac{5}{2}\right)^2}, \frac{4\left(\frac{-5}{2}\right) - 2\left(\frac{-9}{2}\right)}{2(3) - \left(\frac{5}{2}\right)^2}\right) \equiv (3, 4)$$

44. Given equation of pair of lines is
$3x^2 + 10xy + 3y^2 - 15x - 21y + 18 = 0$
$a = 3, b = 3, c = 18, f = \frac{-21}{2}, g = \frac{-15}{2}, h = 5$

The point of intersection is

$$\left(\frac{(5)\left(\frac{-21}{2}\right)-(3)\left(\frac{-15}{2}\right)}{(3)(3)-(5)^2}, \frac{\left(\frac{-15}{2}\right)(5)-(3)\left(\frac{-21}{2}\right)}{(3)(3)-(5)^2}\right)$$

$\equiv \left(\frac{15}{8}, \frac{3}{8}\right)$

45. Given equation of pair of lines is
$6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$
$a = 6, b = -12, h = \frac{-1}{2}$
$\therefore$ $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \frac{17}{6} \Rightarrow \theta = \tan^{-1}\left(\frac{17}{6}\right)$

46. Given equation of pair of lines is
$x^2 + 2\sqrt{3}\,xy + 3y^2 - 3x - 3\sqrt{3}\,y - 4 = 0$
$\therefore$ $a = 1, h = \sqrt{3}, b = 3$
Now, $h^2 - ab = \left(\sqrt{3}\right)^2 - (1)(3) = 0$
$\therefore$ the lines are parallel.

47. Given equation of pair of lines is
$4x^2 + 2pxy + 25y^2 + 2x + 5y - 1 = 0$
$\therefore$ $a = 4, b = 25, h = p$
The lines are parallel
$\therefore$ $h^2 - ab = 0 \Rightarrow h^2 = ab$
$\Rightarrow p^2 = 4(25) = 100$
$\Rightarrow p = 10$

48. Given equation of pair of lines is
$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0$
$\therefore$ $a = -3, b = 3$
Now, $a + b = -3 + 3 = 0$,
$\therefore$ The lines are perpendicular to each other.

49. Given equation of pair of lines is
$x^2 - y^2 - 2y - 1 = 0$
$\therefore$ $a = 1, b = -1$
Now, $a + b = 1 + (-1) = 0$
$\therefore$ The lines are perpendicular to each other.

50. Given equation of pair of lines is
$3xy - 4y = 0$
$\therefore$ $a = b = 0$
Now $a + b = 0$
$\therefore$ The lines are perpendicular to each other.

51. Given equation of pair of lines is
$px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$
$a = p, b = 3, c = q, f = 1, g = 7, h = -4$
This lines are perpendicular if $a + b = 0$
$\Rightarrow p + 3 = 0 \Rightarrow p = -3$
Since the equation represents a pair of lines
$\therefore$ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
$\Rightarrow -9q - 56 + 3 - 147 - 16q = 0$
$\Rightarrow -25q - 200 = 0 \Rightarrow q = -8$

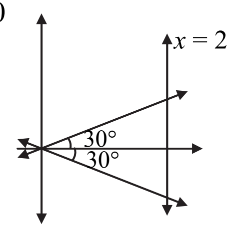
52. Given equation of pair of lines is
$ax^2 + 6xy + by^2 - 10x + 10y - 6 = 0$
$A = a, B = b, C = -6, F = -5, G = 5, H = 3$
The lines are perpendicular
$\therefore$ $a + b = 0 \Rightarrow a = -b$
Also these lines satisfy the condition
$ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$
$\Rightarrow 6a^2 + 2(-75) - 25a + 25a + 54 = 0$
$\Rightarrow 6a^2 - 96 = 0 \Rightarrow a^2 - 16 = 0 \Rightarrow a = \pm 4$

Critical Thinking

1. The lines passing through origin and parallel to the given lines are $y = m_1x$ and $y = m_2x$,
$\therefore$ the combined equation is
$(y - m_1x)(y - m_2x) = 0$
$\Rightarrow m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$

2. Given line $2x - y = 0 \Rightarrow$ Slope = 2
Let the slope of required line be m
$\therefore$ $\tan 30° = \left|\frac{m-2}{1+2m}\right|$
$\Rightarrow \frac{1}{\sqrt{3}} = \left|\frac{m-2}{1+2m}\right|$
$\Rightarrow m^2 + 16m - 11 = 0$ ….(i)
Since, the line passes through origin, its equation is
$y = mx \Rightarrow m = \frac{y}{x}$
Substituting of m in equation (i), we get
$\left(\frac{y}{x}\right)^2 + 16\left(\frac{y}{x}\right) - 11 = 0$
$\Rightarrow 11x^2 - 16xy - y^2 = 0$

3. From the diagram, the required lines are
$y = \frac{x}{\sqrt{3}}$ i.e., $\sqrt{3}\,y - x = 0$
and
$y = \frac{-x}{\sqrt{3}}$ i.e., $\sqrt{3}\,y + x = 0$



$\therefore$ Combined equation is
$(\sqrt{3}\,y - x)(\sqrt{3}\,y + x) = 0$
i.e., $3y^2 - x^2 = 0$

4. Let $y = mx$ be the equation of line.
Slope of the given line $y = -x - \sqrt{3}$ is -1
Since, the pair of straight lines and the given line form an equilateral traingle, angle between them is 60°.
∴ $\tan\frac{\pi}{3} = \left|\frac{m+1}{1-m}\right| \Rightarrow \sqrt{3} = \left|\frac{m+1}{1-m}\right|$
$\Rightarrow 3(1-m)^2 = (1+m)^2$
$\Rightarrow 3(1+m^2-2m) = (1+m^2+2m)$
∴ $m^2 - 4m + 1 = 0$ ….(i)
The equation of line passing through origin is,
$y = mx \Rightarrow m = \frac{y}{x}$
Substituting the value of m in (i), we get
$\left(\frac{y}{x}\right)^2 - 4\left(\frac{y}{x}\right) + 1 = 0 \Rightarrow x^2 - 4xy + y^2 = 0$

5. $ax^2 + (a+b)xy + by^2 + x + y = 0$
$\Rightarrow ax^2 + bxy + x + axy + by^2 + y = 0$
$\Rightarrow x(ax + by + 1) + y(ax + by + 1) = 0$
$\Rightarrow (x + y)(ax + by + 1) = 0$

6. $pq(x^2 - y^2) + (p^2 - q^2)xy = 0$
$\Rightarrow pqx^2 - pqy^2 + p^2xy - q^2xy = 0$
$\Rightarrow px(py + qx) - qy(py + qx) = 0$
$\Rightarrow (px - qy)(py + qx) = 0$
$\Rightarrow px - qy = 0$ and $py + qx = 0$
∴ Required equation of the line is $px - qy = 0$

7. $x^2 + 6xy = 0 \Rightarrow x(x + 6y) = 0$
$\Rightarrow x = 0$ and $x + 6y = 0$ are two straight lines.
$x = 0$ represents Y-axis.

8. $y^2 - x^2 + 2x - 1 = 0$
$\Rightarrow y^2 - (x^2 - 2x + 1) = 0$
$\Rightarrow (y - 0)^2 - (x - 1)^2 = 0$
This is equation of pair of straight lines passing through (1, 0).

9. The given equation represents a pair of straight lines passing through (5, 6).

10. The lines passes through (–2, 2)
Only (–2,2) satisfies the given equation.

11. Given equation of pair of lines is
$ax^2 + xy - by^2 = 0$
Comparing the equations, with
$Ax^2 + 2Hxy + By^2 = 0$
∴ $A = a, H = \frac{1}{2}$ and $B = -a$
∴ the equation represents a pair of straight lines for all real values of 'a'.

12. The combined equation of pair of straight lines passsing through origin and perpendicular to $3x^2 + xy - 2y^2 = 0$ is given by
$-2x^2 - xy + 3y^2 = 0$
i.e. $2x^2 + xy - 3y^2 = 0$
Since the required lines pass through (2, –3)
∴ By shifting the origin to (2, –3), we get
$2(x-2)^2 + (x-2)(y+3) - 3(y+3)^2 = 0$
$\Rightarrow 2x^2 + xy - 3y^2 - 5x - 20y - 25 = 0$

13. Separate equation of lines represented by
$3x^2 - 8xy + 5y^2 = 0$ are
$x - y = 0$ and
$3x - 5y = 0$
Line perpendicular to $x - y = 0$ i.e. $y = x$ and passing through (1, 2) is
$(y - 2) = -1(x - 1)$
i.e. $x + y - 3 = 0$ ….(i)
Line perpendicular to $3x - 5y = 0$
i.e. $y = \frac{3}{5}x$ and passing through (1, 2) is
$(y - 2) = \frac{-5}{3}(x-1)$
i.e. $5x + 3y - 11 = 0$ ….(ii)
∴ combined equation is
$(x + y - 3)(5x + 3y - 11) = 0$

14. Slope of the line $4x + 3y = 0$ is $m = -\frac{4}{3}$
$kx^2 - 5xy - 6y^2 = 0$
$\Rightarrow -6m^2 - 5m + k = 0$
$\Rightarrow -6\left(-\frac{4}{3}\right)^2 - 5\left(-\frac{4}{3}\right) + k = 0$
$\Rightarrow k - \frac{32}{3} + \frac{20}{3} = 0$
$\Rightarrow k = \frac{12}{3} \Rightarrow k = 4$

15. Let $y = mx$ be a line common to the given pair of lines,
It satisfies the given equations
∴ $am^2 + 2m + 1 = 0$ and ….(i)
$m^2 + 2m + a = 0$ ….(ii)
On solving (i) and (ii), we get
$\frac{m^2}{2(1-a)} = \frac{m}{a^2-1} = \frac{1}{2(1-a)}$
∴ $m^2 = 1$ and $m = -\left(\frac{a+1}{2}\right)$
∴ $(a + 1)^2 = 4 \Rightarrow a = 1$ or -3

But for a = 1 the two pair have both the lines common.
So a = –3 and the slope m of the line common to both the pairs is 1.
Now $x^2 + 2xy + ay^2 = x^2 + 2xy - 3y^2$
$= (x - y)(x + 3y)$
and $ax^2 + 2xy + y^2 = -3x^2 + 2xy + y^2$
$= -(x - y)(3x + y)$
Thus, required equation is $(x + 3y)(3x + y) = 0$
i.e., $3x^2 + 10xy + 3y^2 = 0$

16. The equation of given lines are
$ax^2 + 2hxy + by^2 = 0$ ....(i)
$a'x^2 + 2h'xy + b'y^2 = 0$ ....(ii)
Let the line common to both be $y = mx$.
It will satisfy both the above equations.
Hence, $a + 2mh + bm^2 = 0$ ....(iii)
and $a' + 2mh' + b'm^2 = 0$ ....(iv)
Now eliminating 'm' from the equations (iii) and (iv), we get
$$\frac{m^2}{2ha' - 2h'a} = \frac{-m}{ba' - b'a} = \frac{1}{2bh' - 2b'h}$$
$\Rightarrow m^2 = \dfrac{ha' - h'a}{bh' - b'h}$ ....(v)
and $m^2 = \dfrac{(ab' - ba')^2}{4(bh' - b'h)^2}$ ....(vi)
From (v) and (vi), we get the required condition.

17. Given equation of pair of lines
$x^2 - 2xy \tan A - y^2 = 0$
∴ $a = 1, h = -\tan A, b = -1$
$m_1 + m_2 = \dfrac{-2h}{b} \Rightarrow 4 = \dfrac{2\tan A}{-1}$
$\Rightarrow \tan A = -2$
$\Rightarrow \angle A = \tan^{-1}(-2)$

18. Given equation of pair of lines is
$ax^2 + 2hxy + by^2 = 0$
Given that $m_1 = 5m_2$
∴ $m_1 + m_2 = 5m_2 + m_2 = \dfrac{-2h}{b}$
$\Rightarrow m_2 = \dfrac{-h}{3b} \Rightarrow m_2^2 = \dfrac{h^2}{9b^2}$ ....(i)
$m_1m_2 = (5m_2)m_2 = \dfrac{a}{b}$
∴ $m_2^2 = \dfrac{a}{5b}$ ....(ii)
∴ From (i) and (ii), we get
$5h^2 = 9ab$

19. Let the gradient of one line be m.
∴ the gradient of second line is 2m
We know,
$m + 2m = \dfrac{-2h}{b}$
∴ $3m = \dfrac{-2h}{b} \Rightarrow m = \dfrac{-2h}{3b}$ ....(i)
Also, $m \times 2m = \dfrac{a}{b} \Rightarrow 2m^2 = \dfrac{a}{b}$ ....(ii)
∴ from (i) and (ii), we get
$$2\left(\frac{-2h}{3b}\right)^2 = \frac{a}{b} \Rightarrow \frac{8h^2}{9b^2} = \frac{a}{b} \Rightarrow ab = \frac{8h^2}{9}$$

20. Given equation of pair of lines is
$ax^2 + 2hxy + by^2 = 0$
given that $m_2 = \lambda m_1$
Now, $m_1 + m_2 = m_1 + \lambda m_1 = \dfrac{-2h}{b}$
$\Rightarrow m_1 = \dfrac{-2h}{b(1+\lambda)}$ ....(i)
$m_1.m_2 = m_1.\lambda m_1 = \dfrac{a}{b} \Rightarrow m_1 = \sqrt{\dfrac{a}{b\lambda}}$ ....(ii)
∴ from (i) and (ii), we get
$$\sqrt{\frac{a}{b\lambda}} = \frac{-2h}{b(1+\lambda)}$$
Squaring both sides, we get
$4\lambda h^2 = ab(1 + \lambda)^2$

21. Given equation of pair of lines is
$ax^2 + 2hxy + by^2 = 0$
Given that, $m_1 = m_2^2$
$m_1 m_2 = m_2^2 m_2 = \dfrac{a}{b}$
∴ $m_2 = \left(\dfrac{a}{b}\right)^{\frac{1}{3}}$
Also, $m_1 + m_2 = m_2^2 + m_2 = \dfrac{-2h}{b}$
∴ $$\left\{\left(\frac{a}{b}\right)^{\frac{1}{3}}\right\}^2 + \left(\frac{a}{b}\right)^{\frac{1}{3}} = \frac{-2h}{b}$$
Cubing both sides, we get
$$\left(\frac{a}{b}\right)^2 + \frac{a}{b} + 3\left(\frac{a}{b}\right)^{\frac{2}{3}}.\left(\frac{a}{b}\right)^{\frac{1}{3}}.\left\{\left(\frac{a}{b}\right)^{\frac{2}{3}} + \left(\frac{a}{b}\right)^{\frac{1}{3}}\right\}$$
$$= \frac{-8h^3}{b^3}$$

$\therefore \quad \left(\frac{a}{b}\right)^2 + \frac{a}{b} - \frac{6ah}{b^2} = \frac{-8h^3}{b^3}$

$$\ldots.\left\{\because \left(\frac{a}{b}\right)^{\frac{1}{3}} + \left(\frac{a}{b}\right)^{\frac{2}{3}} = \frac{-2h}{b}\right\}$$

$\therefore \quad ab(a + b) - 6abh + 8h^3 = 0$

22. Given equation of pair of lines is
$2x^2 - 5xy + 3y^2 = 0$

$\therefore \quad a = 2, h = \frac{-5}{2}, b = 3$

$\therefore \quad m_1 + m_2 = \frac{5}{6}$ and $m_1.m_2 = \frac{2}{3}$ ….(i)

Slopes of lines $= \frac{1}{m_1}$ and $\frac{1}{m_2}$

$\therefore$ Required equation of pair of lines is

$$y^2 - \left(\frac{1}{m_1} + \frac{1}{m_2}\right)xy + \frac{1}{m_1 m_2}x^2 = 0$$

$$\Rightarrow y^2 - \left(\frac{m_1 + m_2}{m_1 m_2}\right)xy + \frac{1}{m_1 m_2}x^2 = 0$$

$$\Rightarrow y^2 - \left(\frac{\frac{5}{6}}{\frac{2}{3}}\right)xy + \frac{1}{\left(\frac{2}{3}\right)}x^2 = 0$$

$$\Rightarrow 2y^2 - 5xy + 3x^2 = 0$$

23. Let the angle made by one of the lines with X-axis $= \theta$

$\therefore$ The angle made by other line with Y-axis $= \theta$

$\therefore \quad m_1 = \tan\theta$,
$m_2 = \tan(90° - \theta) = \cot\theta$

$\therefore \quad m_1 m_2 = \frac{a}{b} = 1$

$\Rightarrow \frac{a}{b} = 1 \Rightarrow a = b$

24. Given equation of pair of lines is
$x^2(\sec^2\theta - \sin^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$

$\therefore \quad a = \sec^2\theta - \sin^2\theta, h = -\tan\theta, b = \sin^2\theta$

Now, $m_1 + m_2 = \frac{2\tan\theta}{\sin^2\theta}$,

$m_1 m_2 = \frac{\sec^2\theta - \sin^2\theta}{\sin^2\theta}$

$\therefore \quad m_1 - m_2 = \sqrt{(m_1 + m_2)^2 - 4m_1 m_2}$

$$= \sqrt{\left(\frac{2\tan\theta}{\sin^2\theta}\right)^2 - 4\left(\frac{\sec^2\theta - \sin^2\theta}{\sin^2\theta}\right)}$$

$$= \sqrt{\frac{4\tan^2\theta}{\sin^4\theta} - 4(\sec^2\theta\,\mathrm{cosec}^2\theta - 1)}$$

$$= \sqrt{4\sec^2\theta\,\mathrm{cosec}^2\theta - 4\sec^2\theta\,\mathrm{cosec}^2\theta + 4}$$

$$= 2$$

25. Given equation of pair of lines
$(\tan^2\alpha + \cos^2\alpha)x^2 - 2xy\tan\alpha + \sin^2\alpha\, y^2 = 0$
$a = \tan^2\alpha + \cos^2\alpha, h = -\tan\alpha, b = \sin^2\alpha$
If θ_1 and θ_2 are the angles made by lines with X-axis, then $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$

Now, $m_1 + m_2 = \frac{2\tan\alpha}{\sin^2\alpha} = 2\sec\alpha\,\mathrm{cosec}\,\alpha$

$m_1 m_2 = \frac{\tan^2\alpha + \cos^2\alpha}{\sin^2\alpha} = \sec^2\alpha + \cot^2\alpha$

$\therefore \quad m_1 - m_2 = \sqrt{4\sec^2 a\,\mathrm{cosec}^2 a - 4(\sec^2 a + \cot^2 a)}$

$$= \sqrt{4\sec^2 a\,(\mathrm{cosec}^2 a - 1) - 4\cot^2 a}$$

$$= \sqrt{4\cot^2 a\,(\sec^2 a - 1)}$$

$$= \sqrt{4\cot^2\alpha\tan^2\alpha}$$

$$= 2$$

26. The equation of one of the lines passing through origin is $y = mx$.
The line makes an angle α with the line $y = x$

$\therefore \quad \tan\alpha = \pm\left\{\frac{m_1 - m_2}{1 + m_1 m_2}\right\} = \pm\frac{(m-1)}{1+m}$

$\Rightarrow (1 + m)^2\tan^2\alpha = (m - 1)^2$

$\Rightarrow m^2 - 2m\left\{\frac{1 + \tan^2\alpha}{1 - \tan^2\alpha}\right\} + 1 = 0$

$\Rightarrow m^2 - 2m\sec 2\alpha + 1 = 0$

$$\ldots.\left\{\because \frac{1 + \tan^2\alpha}{1 - \tan^2\alpha} = \sec 2\alpha\right\}$$

But $m = \frac{y}{x}$

On eliminating m, we get
$y^2 - 2xy\sec 2\alpha + x^2 = 0$.

27. Let the equation of one of the line which bisects the angle between the co-ordinate axes be $y = x$

$\therefore \quad m_1 = \tan 45^{\circ} = 1$

Let m_2 be the slope of the other line.

Now, $m_1 m_2 = \frac{a}{b}$

Since $m_1 = 1$, we get $m_2 = \frac{a}{b}$;

Also, $m_1 + m_2 = \frac{-2h}{b}$

$\Rightarrow 1 + \frac{a}{b} = \frac{-2h}{b}$

$\Rightarrow a + b = -2h$

28. Let the equation of one of the lines be $y = x$

$\therefore$ $m_1 = \tan 45° = 1$

Now, $m_1m_2 = \frac{a}{c}$

Since $m_1 = 1$, we get $m_2 = \frac{a}{c}$

Also, $m_1 + m_2 = \frac{-b}{c}$

$\therefore$ $1 + \frac{a}{c} = \frac{-b}{c}$

$\Rightarrow \frac{a+b+c}{c} = 0$

$\Rightarrow a + b + c = 0$

29. Let the equation of one of the angle bisector of co-ordinate axes be $x + y = 0 \Rightarrow m_1 = -1$

Now, $m_1m_2 = \frac{a}{b}$

$\Rightarrow m_2 = -\frac{a}{b}$

Also, $m_1 + m_2 = \frac{-2h}{b}$

$\Rightarrow -1-\frac{a}{b} = \frac{-2h}{b} \Rightarrow (a + b)^2 = 4h^2$

30. The line makes angles α and β with X-axis

$\therefore$ $m_1 = \tan \alpha$ and $m_2 = \tan \beta$

$\Rightarrow \cot \alpha = \frac{1}{m_1}$ and $\cot \beta = \frac{1}{m_2}$

Given equation of pair of lines is

$2x^2 - 3xy + y^2 = 0$

$\therefore$ $a = 2, h = \frac{-3}{2}, b = 1$

Now, $m_1 + m_2 = 3$ and $m_1m_2 = 2$

$\therefore$ $\cot^2 \alpha + \cot^2 \beta = \frac{1}{m_1^2} + \frac{1}{m_2^2} = \frac{m_1^2 + m_2^2}{(m_1m_2)^2}$

$= \frac{(m_1 + m_2)^2 - 2m_1m_2}{(m_1m_2)^2}$

$= \frac{(3)^2 - 2(2)}{(2)^2} = \frac{5}{4}$

31. Given equation of pair of lines is

$ax^2 - bxy - y^2 = 0$

$\therefore$ $A = a, H = \frac{-b}{2}, B = -1$

Since lines make angles α and β with X-axis,

$\therefore$ $m_1 = \tan\alpha$ and $m_2 = \tan\beta$

Now, $m_1 + m_2 = \frac{b}{-1} \Rightarrow \tan \alpha + \tan \beta = -b$

and $m_1m_2 = \frac{a}{-1} \Rightarrow \tan \alpha \tan \beta = -a$

We know, $\tan (\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{-b}{1-(-a)} = \frac{-b}{1+a}$

32. Given equation of pair of lines is

$ax^2 + 2hxy + by^2 = 0$

$\therefore$ $A = a, H = h, B = b$

$\tan \theta = \left(\frac{2\sqrt{H^2 - AB}}{A + B}\right)$

$= \left(\frac{\sqrt{4h^2 - 4ab}}{a + b}\right)$

$= \left(\frac{\sqrt{3a^2 + 3b^2 + 10ab - 4ab}}{a + b}\right)$

....[$\because 3a^2 + 3b^2 + 10ab = 4h^2$]

$\therefore$ $\tan \theta = \left(\frac{\sqrt{3(a+b)^2}}{a + b}\right)$

$\Rightarrow \theta = \tan^{-1}\left(\sqrt{3}\right)$

$= 60°$

33. Given equation of pair of lines is

$x^2 - 2pxy + y^2 = 0$

$\therefore$ $a = 1, h = -p, b = 1$

$\therefore$ $\tan \theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$

$\Rightarrow \tan \theta = \frac{\pm 2\sqrt{p^2 - 1}}{1+1} = \pm\sqrt{p^2 - 1}$

$\Rightarrow \tan^2\theta = p^2 - 1$

$\Rightarrow \sec^2\theta - 1 = p^2 - 1$

$\Rightarrow \theta = \sec^{-1} p$

34. Given equation of pair of lines is
$(x^2 + y^2)\sin\theta + 2xy = 0$

$\therefore$ $a = b = \sin\theta$, $h = 1$

$\therefore$ $\tan\theta = \left(\dfrac{2\sqrt{1-\sin^2\theta}}{2\sin\theta}\right)$

$\Rightarrow \theta = \tan^{-1}\left(\dfrac{\cos\theta}{\sin\theta}\right) = \tan^{-1}(\cot\theta)$

$\Rightarrow \theta = \tan^{-1}\left\{\tan\left(\dfrac{\pi}{2} - \theta\right)\right\} = \dfrac{\pi}{2} - \theta$

35. Given equation of pair of lines is
$ax^2 + xy + by^2 = 0$

$\therefore$ $A = a$, $H = \dfrac{1}{2}$, $B = b$

Now, $\theta = 45° \Rightarrow \tan\theta = 1$

$\therefore$ $\tan 45° = \left|\dfrac{2\sqrt{\dfrac{1}{4} - ab}}{a+b}\right|$

$\Rightarrow (a + b)^2 = (1 - 4ab)$

$\Rightarrow a^2 + b^2 + 6ab - 1 = 0$

The above equation is satisfied by
$a = 1$ and $b = -6$

36. Given equation of pair of lines is

$\therefore$ $a = -\tan^2 A$, $h = \dfrac{k}{2}$, $b = 1$

$\therefore$ $\tan 2A = \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right|$

$\tan 2A = \dfrac{2\sqrt{\dfrac{k^2}{4} + \tan^2 A}}{1 - \tan^2 A}$

$\Rightarrow \dfrac{2\tan A}{1 - \tan^2 A} = \dfrac{2\sqrt{\dfrac{k^2}{4} + \tan^2 A}}{1 - \tan^2 A}$

$\Rightarrow \dfrac{k^2}{4} + \tan^2 A = \tan^2 A \Rightarrow k = 0$

37. Here, $a_1 = a$, $h_1 = h$, $b_1 = b$,

$a_2 = 2$, $h_2 = \dfrac{-5}{2}$, $b_2 = 3$

Given that $\theta_1 = \theta_2$

$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\dfrac{2\sqrt{\dfrac{25}{4} - 6}}{5}\right|$

$\Rightarrow \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\dfrac{1}{5}\right|$

Squaring both sides, we get

$4 \times 25(h^2 - ab) = (a + b)^2$

$100(h^2 - ab) = (a + b)^2$

Comparing with given condition,
$k(h^2 - ab) = (a + b)^2$, we get
$k = 100$

38. Comparing the given equations with
$ax^2 + 2hxy + by^2 = 0$, we get,

$a_1 = 3$, $h_1 = \dfrac{-7}{2}$, $b_1 = 4$

$a_2 = 6$, $h_2 = \dfrac{-5}{2}$, $b_2 = 1$

If θ_1 and θ_2 are acute angles between the two pairs of lines, then

$\tan\theta_1 = \left(\dfrac{2\sqrt{\dfrac{49}{4} - 12}}{3+4}\right) = \dfrac{1}{7}$

$\Rightarrow \theta_1 = \tan^{-1}\left(\dfrac{1}{7}\right)$

$\tan\theta_2 = \left(\dfrac{2\sqrt{\dfrac{25}{4} - 6}}{6+1}\right) = \left(\dfrac{1}{7}\right)$

$\Rightarrow \theta_2 = \tan^{-1}\left(\dfrac{1}{7}\right)$

Hence, $\theta_1 = \theta_2$.

39. Given equation of pair of lines is
$a^2x^2 + bcy^2 = a(b + c)xy$

$\therefore$ $A = a^2$, $H = \dfrac{-a(b+c)}{2}$, $B = bc$

Since the lines are coincident

$\therefore$ $H^2 - AB = 0$

$\Rightarrow \left\{\dfrac{-a(b+c)}{2}\right\}^2 - a^2(bc) = 0$

$\Rightarrow a^2(b - c)^2 = 0$

$\Rightarrow a = 0$ or $b = c$

40. Given equation of pair of lines is
$(p - q)x^2 + 2(p + q)xy + (q - p)y^2 = 0$

$\therefore$ $a = p - q$, $h = p + q$, $b = q - p$

Since, the lines are mutually perpendicular

$\therefore$ $a + b = 0$

$\Rightarrow (p - q) + (q - p) = 0$

The above equation is true for all values of p and q.

41. Given equation of pair of lines is
$3ax^2 + 5xy + (a^2 - 2)y^2 = 0$

$\therefore$ $A = 3a,\ H = \dfrac{5}{2},\ B = a^2 - 2$

Since the lines are perpendicular

$\therefore$ $A + B = 0$

$\Rightarrow 3a + (a^2 - 2) = 0$

$\Rightarrow a^2 + 3a - 2 = 0$

Since, the equation is a quadratic equation in 'a' and $B^2 - 4AC > 0$,

The roots of 'a' are real and distinct.

$\therefore$ Lines are perpendicular to each other for two values of 'a'.

42. Given equation of pair of lines is
$ay^2 + (-1 - \lambda^2)xy - ax^2 = 0$

$\therefore$ $A = -a,\ H = \dfrac{-1-\lambda^2}{2},\ B = a$

$A + B = (-a) + a = 0$

$\Rightarrow$ Angle between the given lines is 90°.

Now, consider $xy = 0$. Here, $A = B = 0$

$\Rightarrow A + B = 0$

$\therefore$ the angle between the lines is 90°

$\therefore$ Correct option is (C).

43. Given equation of pair of lines is
$x^2 + y^2 + 2gx + 2fy + 1 = 0$

$A = 1, B = 1, C = 1, F = f, G = g, H = 0$

The given equation represents a pair of lines

$\therefore$ $ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$

$\Rightarrow (1)(1)(1) + 2fg(0) - (1)f^2 - 1(g)^2 - (1)(0)^2 = 0$

$\Rightarrow f^2 + g^2 = 1$

44. Given equation of pair of lines is
$ax^2 + by^2 + cx + cy = 0$

$\therefore$ $A = a,\ B = b,\ C = 0,\ F = \dfrac{c}{2},\ G = \dfrac{c}{2},\ H = 0$

Now $ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$

$$\Rightarrow ab(0) + 2\left(\frac{c}{2}\right)\left(\frac{c}{2}\right)(0) - a\left(\frac{c}{2}\right)^2 - b\left(\frac{c}{2}\right)^2 - 0(0)^2 = 0$$

$\Rightarrow ac^2 + bc^2 = 0$

$\Rightarrow c^2(a + b) = 0$

$\Rightarrow c(a + b) = 0$

45. Given equation of pair of lines is
$hxy + gx + fy + c = 0$

$A = B = 0,\ C = c,\ F = \dfrac{f}{2},\ G = \dfrac{g}{2},\ H = \dfrac{h}{2}$

Now, $ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$

$$\Rightarrow 0 + 2\left(\frac{f}{2}\right)\left(\frac{g}{2}\right)\left(\frac{h}{2}\right) - 0 - 0 - c\left(\frac{h}{2}\right)^2 = 0$$

$$\Rightarrow \frac{fgh}{4} - \frac{ch^2}{4} = 0$$

$\Rightarrow fg = ch$

46. Given equation of pair of lines is
$2x^2 + 5xy + 2y^2 + 3x + 3y + 1 = 0$

$a = 2,\ b = 2,\ c = 1,\ f = \dfrac{3}{2},\ g = \dfrac{3}{2},\ h = \dfrac{5}{2}$

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{\left(\frac{25}{4}\right) - 4}}{2+2}\right| = \frac{3}{4}$$

$$\therefore \cos\theta = \frac{4}{5} \Rightarrow \theta = \cos^{-1}\left(\frac{4}{5}\right)$$

47. Given equation of pair of lines is
$x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$

$a = 1,\ b = \lambda,\ c = 2,\ f = \dfrac{-5}{2},\ g = \dfrac{3}{2},\ h = \dfrac{-3}{2}$

$$\theta = \tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \tan\theta = \frac{1}{3}$$

$$\text{Since, } \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$\Rightarrow \frac{1}{3} = \left|\frac{2\sqrt{\left(\frac{-3}{2}\right)^2 - \lambda}}{\lambda + 1}\right|$$

$\Rightarrow (\lambda + 1)^2 = 9(9 - 4\lambda) \Rightarrow \lambda^2 + 38\lambda - 80 = 0$

$\Rightarrow (\lambda + 40)(\lambda - 2) = 0 \Rightarrow \lambda = -40, 2$

48. Given equation of pair of lines is
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$\theta = \frac{\pi}{4} \Rightarrow \tan\theta = 1$$

$$\therefore 1 = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$\Rightarrow 4(h^2 - ab) = (a + b)^2$

$\Rightarrow 4h^2 - 4ab = a^2 + 2ab + b^2$

$\Rightarrow a^2 + 6ab + b^2 = 4h^2$

49. Given equation of pair of lines is
$x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$
$a = 1, b = \lambda, c = 2, f = \frac{-5}{2}, g = \frac{3}{2}, \ h = \frac{-3}{2}$
Now, $abc + 2\,fgh - af^2 - bg^2 - ch^2 = 0$
$\Rightarrow 2\lambda + 2\left(-\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right) - \frac{25}{4} - \frac{9\lambda}{4} - \frac{18}{4} = 0$
$\Rightarrow \lambda = 2$
$\tan\theta = \dfrac{2\sqrt{\frac{9}{4} - 2}}{1+2} = \dfrac{1}{3}$
$\Rightarrow \cot\theta = 3$
$\therefore$ $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + 9 = 10$

50. Given equation of pair of lines is
$9x^2 + y^2 + 6xy - 4 = 0$
$\therefore$ $a = 9, b = 1, h = 3$
$h^2 - ab = 3^2 - 9(1) = 0$
$\therefore$ The lines are parallel
Now, $9x^2 + 6xy + y^2 = 4$
$\Rightarrow (3x + y)^2 = 4 \Rightarrow 3x + y = \pm 2$
Hence, the lines are parallel and not coincident.

51. Given equation of pair of lines is
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
$\therefore$ $A = a, B = b, H = h$
The lines are parallel
$\therefore$ $H^2 = AB$
$\Rightarrow h = \sqrt{ab}$
Now $ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$
$\Rightarrow abc + 2fg\sqrt{ab} - af^2 - bg^2 - abc = 0$
$\Rightarrow (\sqrt{a}\,f - \sqrt{b}\,g)^2 = 0 \Rightarrow af^2 = bg^2$

52. Given equation of pair of lines is
$x^2 + k_1y^2 + 2k_2y = a^2$
$a = 1, b = k_1, c = -a^2, f = k_2, g = 0, h = 0$
The lines are perpendicular
$\therefore$ $a + b = 0 \Rightarrow k_1 = -1$
Substituting value of k_1 in the given equation of lines, we get
$x^2 - y^2 + 2k_2y - a^2 = 0$
$\Rightarrow a^2 - k_2^2 = 0 \Rightarrow k_2 = \pm a$

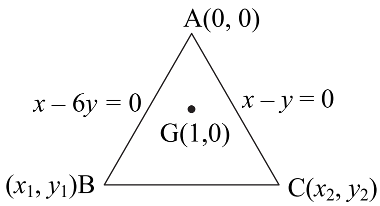
53. $(x^2 + y^2)(h^2 + k^2 - a^2) = (hx + ky)^2$
$\Rightarrow x^2(h^2 + k^2 - a^2) + y^2(h^2 + k^2 - a^2)$
$= h^2x^2 + k^2y^2 + 2hkxy$
$\Rightarrow x^2(k^2 - a^2) + y^2(h^2 - a^2) - 2hkxy = 0$
$\therefore$ $A = k^2 - a^2, B = h^2 - a^2$
The lines are perpendicular
$\therefore$ $A + B = 0$
$\Rightarrow k^2 - a^2 + h^2 - a^2 = 0 \Rightarrow h^2 + k^2 = 2a^2$

54. Given equation of pair of lines is
$2x^2 - 4xy - py^2 + 4x + qy + 1 = 0$
$a = 2, b = -p, c = 1, f = \frac{q}{2}, g = 2, h = -2$
The lines are perpendicular,
$\therefore$ $a + b = 0$
$\Rightarrow 2 - p = 0 \Rightarrow p = 2$
The equations represents pair of lines
$\therefore$ $2(-2)(1) + 2\left(\frac{q}{2}\right)(2)(2) - 2\left(\frac{q}{2}\right)^2$
$+ 2(2)^2 - 1(2)^2 = 0$
$\Rightarrow q^2 - 8q = 0 \Rightarrow q = 0$ or 8

55. Given equation of pair of lines is
$12x^2 + 7xy + by^2 + gx + 7y - 1 = 0$
$\therefore$ $A = 12, B = b, C = -1, F = \frac{7}{2}, G = \frac{g}{2}, H = \frac{7}{2}$
The lines are perpendicular
$\therefore$ $A + B = 0 \Rightarrow 12 + b = 0 \Rightarrow b = -12$
Also, $ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$
$\Rightarrow (12)(-12)(-1) + 2\left(\frac{7}{2}\right)\left(\frac{g}{2}\right)\left(\frac{7}{2}\right)$
$- (12)\left(\frac{7}{2}\right)^2 - (-12)\left(\frac{g}{2}\right)^2 - (-1)\left(\frac{7}{2}\right)^2 = 0$
$\Rightarrow 12g^2 + 49g + 37 = 0$
$\Rightarrow (g + 1)(12g + 37) = 0$
$\Rightarrow g = -1$ or $-\frac{37}{12}$

56. Given equation of pair of lines is
$12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$
$a = 12, b = -p, c = 6, f = \frac{q}{2}, g = -9, h = \frac{7}{2}$
The lines are be perpendicular
$\therefore$ $a + b = 0.$
$\Rightarrow 12 - p = 0 \Rightarrow p = 12$
Also, $abc + 2\,fgh - af^2 - bg^2 - ch^2 = 0$
$\Rightarrow 12(-12)6 + 2\left(\frac{q}{2}\right)(-9)\left(\frac{7}{2}\right) - 12\left(\frac{q}{2}\right)^2$
$- (-12)(-9)^2 - 6\left(\frac{7}{2}\right)^2 = 0$
$\Rightarrow -864 - \frac{63q}{2} - 3q^2 + 972 - \frac{147}{2} = 0$
$\Rightarrow 23 - 21q - 2q^2 = 0$
$\Rightarrow (q - 1)(2q + 23) = 0 \Rightarrow q = 1$ or $-\frac{23}{2}$

57. The separate equations of lines represented by $x^2 - 7xy + 6y^2 = 0$ are
$x - 6y = 0$ and $x - y = 0$
Let the 3 points be as shown in figure.



We know $\dfrac{0 + x_1 + x_2}{3} = 1$

$\Rightarrow x_1 + x_2 = 3$ ....(i)
and $y_1 + y_2 = 0$ ....(ii)
Also, $x_1 - 6y_1 = 0$ ....(iii)
$x_2 - y_2 = 0$ ....(iv)
[Since the points (x_1, y_1) and (x_2, y_2) lie on the lines AB and AC respectively]
On solving, we get the co-ordinates of B and C.

∴ $B \equiv \left(\dfrac{18}{5}, \dfrac{3}{5}\right)$ and $C \equiv \left(\dfrac{-3}{5}, \dfrac{-3}{5}\right)$

Hence, the equation of third side i.e., BC is

$$\frac{y - \frac{3}{5}}{x - \frac{18}{5}} = \frac{\frac{-3}{5} - \frac{3}{5}}{\frac{-3}{5} - \frac{18}{5}}$$

$\Rightarrow 2x - 7y - 3 = 0.$

58. The given pair of lines can be separated as:
$L_1 = (l + \sqrt{3}\,m)x + (m - \sqrt{3}\,l)y = 0$
$L_2 = (l - \sqrt{3}\,m)x + (m + \sqrt{3}\,l)y = 0$
and $L_3 = lx + my + n = 0$

∴ The slopes S_1, S_2 and S_3 of the three lines respectively are,

$$S_1 = \frac{-(l + \sqrt{3}m)}{(m - \sqrt{3}l)},\ S_2 = \frac{-(l - \sqrt{3}m)}{(m + \sqrt{3}l)},\ S_3 = \frac{-l}{m}$$

Angle between L_1 and L_3 is

$$\theta_{13} = \tan^{-1}\left|\frac{S_1 - S_3}{1 + S_1S_3}\right|$$

$$= \tan^{-1}\left|\frac{-\left(\dfrac{l + \sqrt{3}m}{m - \sqrt{3}l}\right) + \dfrac{l}{m}}{1 + \left(\dfrac{l + \sqrt{3}m}{m - \sqrt{3}l}\right)\left(\dfrac{l}{m}\right)}\right|$$

$$= \tan^{-1}\left|\frac{-\sqrt{3}m^2 - \sqrt{3}l^2}{l^2 + m^2}\right| = \tan^{-1}\left(\sqrt{3}\right) = 60^\circ$$

Angle between L_2 and L_3 is

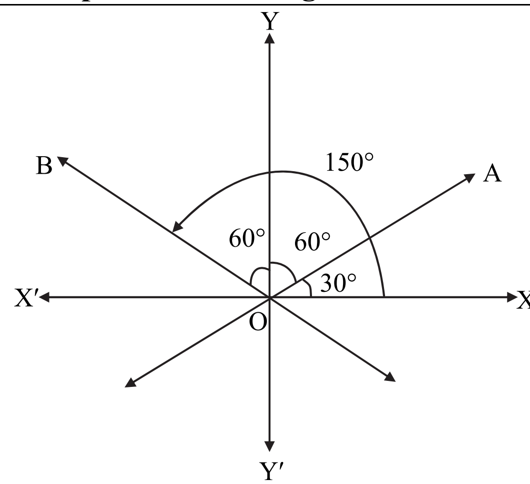
$$\theta_{23} = \tan^{-1}\left|\frac{S_2 - S_3}{1 + S_2S_3}\right| = \tan^{-1}\left|\frac{-\left(\dfrac{l - \sqrt{3}m}{m + \sqrt{3}l}\right) + \dfrac{l}{m}}{1 + \left(\dfrac{l - \sqrt{3}m}{m + \sqrt{3}l}\right)\left(\dfrac{l}{m}\right)}\right|$$

$$= \tan^{-1}\left|\frac{\sqrt{3}m^2 + \sqrt{3}l^2}{m^2 + l^2}\right| = \tan^{-1}(\sqrt{3}) = 60^\circ$$

∴ Angle between the lines L_1 and $L_2 = 60^\circ$
Hence, the triangle is equilateral.

Competitive Thinking

2.



Let OA and OB be the required lines.

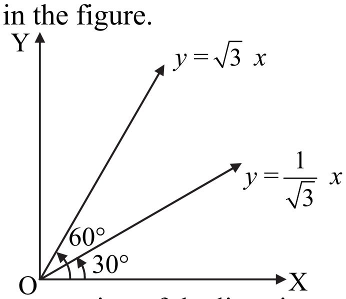
∴ angles made by OA and OB with X-axis are 30° and 150° respectively.

∴ their equations are $y = \dfrac{1}{\sqrt{3}}x$ and $y = -\dfrac{1}{\sqrt{3}}x$

i.e., $x - \sqrt{3}y = 0$ and $x + \sqrt{3}y = 0$

∴ The joint equations of the lines is
$\left(x - \sqrt{3}y\right)\left(x + \sqrt{3}y\right) = 0 \Rightarrow x^2 - 3y^2 = 0$

3. The lines trisecting the first quadrant are as shown in the figure.



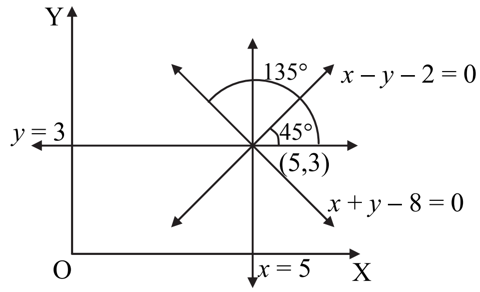
∴ The joint equation of the lines is

$$\left(y - \frac{1}{\sqrt{3}}x\right)\left(y - \sqrt{3}x\right) = 0$$

$$\Rightarrow \left(\sqrt{3}y - x\right)\left(y - \sqrt{3}x\right) = 0$$

$$\Rightarrow \sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0$$

4.



The equations of bisectors are,

$y - 3 = (1)(x - 5)$ and $y - 3 = (-1)(x - 5)$

$\Rightarrow x - y - 2 = 0$ and $x + y - 8 = 0$

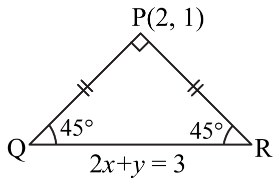
$\therefore$ The joint equation of the bisectors is

$(x - y - 2)(x + y - 8) = 0$

$\Rightarrow x^2 - y^2 - 10x + 6y + 16 = 0$

5. Slope of QR $= -2$.

Slope of PQ $= m_1$



$\therefore \tan 45° = \left|\dfrac{m_1 + 2}{1 + m_1(-2)}\right|$

$\Rightarrow 1 = \left|\dfrac{m_1 + 2}{1 - 2m_1}\right|$

$\Rightarrow m_1 = -\dfrac{1}{3}$

$\therefore$ Equation of PQ passing through point P (2, 1) and having slope m_1 is

$y - 1 = -\dfrac{1}{3}(x - 2)$

$\Rightarrow 3(y - 1) + (x - 2) = 0$ ….(i)

Slope of PR $= m_2 = 3$ …. [$\because$ PQ $\perp$ PR]

$\therefore$ equation of PR is

$y - 1 = 3(x - 2)$

$\Rightarrow (y - 1) - 3(x - 2) = 0$ ….(ii)

$\therefore$ The joint equation of the lines is

$[3(y - 1) + (x - 2)][(y - 1) - 3(x - 2)] = 0$

$\Rightarrow 3(y - 1)^2 - 8(y - 1)(x - 2) - 3(x - 2)^2 = 0$

$\Rightarrow 3(x^2 - 4x + 4) + 8(xy - x - 2y + 2) - 3(y^2 - 2y + 1) = 0$

$\Rightarrow 3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$

6. $x^2 - 7xy + 12y^2 = 0$

$\Rightarrow (x - 3y)(x - 4y) = 0$

Hence, the lines are intersecting and non-perpendicular.

7. $\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 4$

i.e. $\sqrt{(x-2)^2 + y^2} = 4 - \sqrt{(x+2)^2 + y^2}$

Squaring both sides, we get

$(x - 2)^2 + y^2 = 16 - 8\sqrt{(x+2)^2 + y^2} + (x + 2)^2 + y^2$

$\Rightarrow x^2 - 4x + 4 + y^2 = 16 + x^2 + 4x + 4 + y^2 - 8\sqrt{(x+2)^2 + y^2}$

$\Rightarrow x + 2 = \sqrt{(x+2)^2 + y^2}$

Again squaring both sides, we get

$(x + 2)^2 = (x + 2)^2 + y^2$

$\Rightarrow y^2 = 0$

This is an equation of pair of two coincident straight lines.

8. The required lines are parallel to $x^2 - 4xy + 3y^2 = 0$, which pass through $(3, -2)$.

$\therefore$ the combined equation of lines is

$(x - 3)^2 - 4(x - 3)(y + 2) + 3(y + 2)^2 = 0$

$\Rightarrow x^2 - 6x + 9 - 4(xy + 2x - 3y - 6) + 3(y^2 + 4y + 4) = 0$

$\Rightarrow x^2 - 6x + 9 - 4xy - 8x + 12y + 24 + 3y^2 + 12y + 12 = 0$

$\Rightarrow x^2 - 4xy + 3y^2 - 14x + 24y + 45 = 0$

9. The required equation is $-2x^2 - 3xy + 5y^2 = 0$

i.e., $2x^2 + 3xy - 5y^2 = 0$

10. Given equation of pair of lines is

$4xy + 2x + 6y + 3 = 0$

$\Rightarrow 2x(2y + 1) + 3(2y + 1) = 0$

$\Rightarrow (2y + 1)(2x + 3) = 0$

$\therefore$ Separate equations of lines are $2x + 3 = 0$ and $2y + 1 = 0$

i.e. $x = \dfrac{-3}{2}$ and $y = \dfrac{-1}{2}$

The equation of line passing through (2, 1) and perpendicular to $x = \dfrac{-3}{2}$ is $y = 1$ i.e. $y - 1 = 0$

The equation of line passing through (2, 1) and perpendicular to $y = \dfrac{-1}{2}$ is $x = 2$ i.e. $x - 2 = 0$

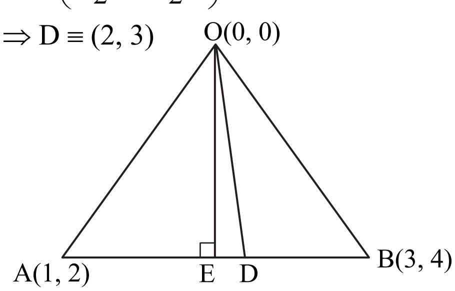
$\therefore$ Combined equation of pair of lines is $(x - 2)(y - 1) = 0$

$\Rightarrow xy - x - 2y + 2 = 0$

11. OD is the median

$\therefore$ D $\equiv \left(\dfrac{1+3}{2}, \dfrac{2+4}{2}\right)$

$\Rightarrow$ D $\equiv (2, 3)$



Equation of OD is $y = mx$

$\Rightarrow y = \frac{3}{2}x \Rightarrow 3x - 2y = 0$

Slope of line AB $= \frac{2}{2} = 1$

Given, OE $\perp$ AB

$\therefore$ Slope of OE $= -1$

Equation of OE is $y = mx$

$\Rightarrow y = -x \Rightarrow x + y = 0$

$\therefore$ Joint equation of median and altitude is

$(3x - 2y)(x + y) = 0$

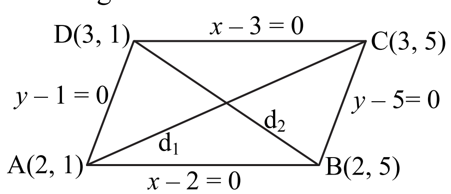
$\Rightarrow 3x^2 + xy - 2y^2 = 0$

12. We have, $x^2 - 5x + 6 = 0$ and $y^2 - 6y + 5 = 0$

$\Rightarrow (x - 3)(x - 2) = 0$ and $(y - 1)(y - 5) = 0$

$\therefore$ One pair of opposite sides of parallelogram is $x - 3 = 0$ and $x - 2 = 0$ and the other pair is $y - 1 = 0$ and $y - 5 = 0$

$\therefore$ The vertices of the parallelogram are as shown in the figure below.



$\therefore$ equation of diagonal d_1 is

$y - 1 = \frac{5-1}{3-2}(x - 2)$

$\Rightarrow y - 1 = 4(x - 2) \Rightarrow y = 4x - 7$

and equation of diagonal d_2 is

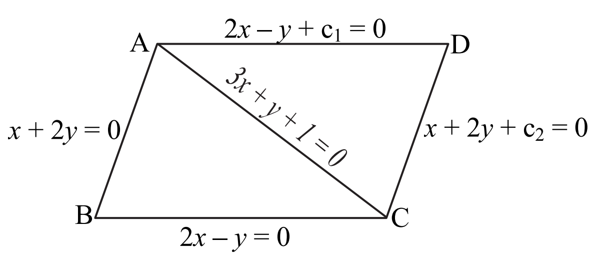
$y - 1 = \frac{5-1}{2-3}(x - 3)$

$\Rightarrow y - 1 = -4(x - 3) \Rightarrow 4x + y = 13$

$\therefore$ the equations are $4x + y = 13$ and $y = 4x - 7$.

13. $2x^2 + 3xy - 2y^2 = 0$

$\Rightarrow x + 2y = 0$ and $2x - y = 0$



From the figure,

$A\left(\frac{-2}{5}, \frac{1}{5}\right)$, B(0, 0), $C\left(\frac{-1}{5}, \frac{-2}{5}\right)$

Now, equation of side AD is

$2x - y + c_1 = 0$

Substituting $x = \frac{-2}{5}$, $y = \frac{1}{5}$ in above equation, we get

$c_1 = 1$

$\therefore$ equation of AD becomes $2x - y + 1 = 0$

Similarly equation of side DC is $x + 2y + c_2 = 0$

i.e., $x + 2y + 1 = 0$

$\therefore$ $D\left(\frac{-3}{5}, \frac{-1}{5}\right)$

Now, equation of diagonal BD is

$y - 0 = \frac{0 + \frac{1}{5}}{0 + \frac{3}{5}}(x - 0)$

$\Rightarrow \frac{3}{5}y = \frac{1}{5}x$

$\Rightarrow x - 3y = 0$

14. Substituting the value of y in the equation $ax^2 + 2hxy + by^2 = 0$.

$\Rightarrow ax^2 + 2hx(mx) + b(mx)^2 = 0$

$\Rightarrow a + 2hm + bm^2 = 0$

15. One of the lines is $3x + 4y = 0$

i.e., $\frac{y}{x} = -\frac{3}{4}$

The given joint equation is $6x^2 - xy + 4cy^2 = 0$

$\Rightarrow 4c\left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right) + 6 = 0$ ….(i)

Substituting value of $\frac{y}{x}$ in equation (i), we get

$4c\left(\frac{-3}{4}\right)^2 - \left(-\frac{3}{4}\right) + 6 = 0$

$\Rightarrow 4c \times \frac{9}{16} + \frac{3}{4} + 6 = 0$

$\Rightarrow \frac{9c}{4} + \frac{3 + 24}{4} = 0 \Rightarrow 9c + 27 = 0$

$\Rightarrow c = -3$

16. Given equation of pair of lines is

$kx^2 - 5xy - 3y^2 = 0$

$\Rightarrow k - 5\frac{y}{x} - 3\left(\frac{y}{x}\right)^2 = 0$

$\Rightarrow k - 5m - 3m^2 = 0$ ….(i)

Now, slope of line $x - 2y + 3 = 0$ is $m_1 = \frac{1}{2}$.

$\therefore$ slope of the line perpendicular to $x - 2y + 3 = 0$ is $m = -2$.

Substituting value of m in equation (i), we get

$k - 5(-2) - 3(-2)^2 = 0$

$\Rightarrow k = -10 + 12 \Rightarrow k = 2$

17. $6x^2 + xy - y^2 = 0$
$\Rightarrow 6x^2 + 3xy - 2xy - y^2 = 0$
$\Rightarrow 2x + y = 0$ and $3x - y = 0$
let $a = \frac{1}{2}$
$\therefore$ equation $3x^2 - axy - y^2 = 0$ becomes
$3x^2 - \frac{1}{2}xy - y^2 = 0$
$\Rightarrow 6x^2 - xy - 2y^2 = 0$
$\Rightarrow 3x - 2y = 0$ and $2x + y = 0$
$\therefore$ given pair of lines have common line $2x + y = 0$
$\therefore$ Option (A) is correct answer.

19. Given equation of pair of lines is
$3x^2 + 5xy - 2y^2 = 0$
$\therefore$ $a = 3, h = \frac{5}{2}, b = -2$
Now, $m_1 + m_2 = \frac{-2h}{b} = \frac{5}{2}$

20. Given equation of pair of lines is
$4x^2 + 2hxy - 7y^2 = 0$
$\therefore$ $A = 4, H = h, B = -7$
Now, $m_1 + m_2 = -\frac{2H}{B} = \frac{2h}{7}$ and
$m_1m_2 = \frac{A}{B} = \frac{4}{-7}$
Given that, $m_1 + m_2 = m_1m_2$
$\Rightarrow \frac{2h}{7} = \frac{4}{-7} \Rightarrow h = -2$

21. Given equation of pair of lines is
$x^2 - 2cxy - 7y^2 = 0$
$\therefore$ $a = 1, h = -c, b = -7$
$\therefore$ $m_1 + m_2 = \frac{-2c}{7}$ and $m_1m_2 = \frac{-1}{7}$
Given that, $m_1 + m_2 = 4m_1m_2$
$\Rightarrow \frac{-2c}{7} = \frac{-4}{7} \Rightarrow c = 2$

22. Given equation of pair of lines is
$ax^2 - 6xy + y^2 = 0$
$\therefore$ $A = a, H = -3, B = 1$
Given that, $m_1 = 2m_2$
$m_1 + m_2 = -\frac{2(-3)}{1} = 6$
$\Rightarrow 2m_2 + m_2 = 6 \Rightarrow m_2 = 2 \Rightarrow m_1 = 4$
Now, $m_1m_2 = \frac{a}{1} = a$
$\Rightarrow a = (4)(2) = 8$

23. Given equation of pair of lines is
$x^2 + hxy + 2y^2 = 0$
$\therefore$ $A = 1, H = \frac{h}{2}, B = 2$
Given that $m_1 = 2m_2$
Now, $m_1 + m_2 = \frac{-h}{2}$ and $m_1m_2 = \frac{1}{2}$
$\therefore$ $(2m_2)m_2 = \frac{1}{2} \Rightarrow 2(m_2)^2 = \frac{1}{2} \Rightarrow m_2 = \pm\frac{1}{2}$
Also, $2m_2 + m_2 = \frac{-h}{2} \Rightarrow m_2 = \frac{-h}{6}$
$\Rightarrow \pm\frac{1}{2} = \frac{-h}{6} \Rightarrow h = \pm 3$

24. Given equation of pairs of lines is
$ax^2 + 2hxy + by^2$
$\therefore$ $m_1 + m_2 = \frac{-2h}{b}$ and $m_1m_2 = \frac{a}{b}$
Given that, $m_1 = 2m_2$
$\therefore$ $2m_2 + m_2 = \frac{-2h}{b}$ and $2m_2m_2 = \frac{a}{b}$
$\therefore$ $m_2 = \frac{-2h}{3b}$ and $m_2^2 = \frac{a}{2b}$
$\therefore$ $\left(-\frac{2h}{3b}\right)^2 = \frac{a}{2b}$
$\therefore$ $\frac{4h^2}{9b^2} = \frac{a}{2b}$
$\therefore$ $8h^2 = 9ab$

25. Given equation of pairs of lines is
$kx^2 + 5xy + y^2 = 0$
$\therefore$ $a = k, b = 1, h = \frac{5}{2}$
$\therefore$ $m_1 + m_2 = \frac{-2h}{b} = -5$
$m_1m_2 = \frac{a}{b} = k$
Given that, $m_1 - m_2 = 1$
Now, $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$
$\Rightarrow 1^2 = (-5)^2 - 4k \Rightarrow 4k = 24$
$\Rightarrow k = 6$

26. If the gradients of two lines are in ratio 1 : n.
then $\frac{h^2}{ab} = \frac{(n+1)^2}{4n} = \frac{(3+1)^2}{4.3} = \frac{4}{3}$
Alternate Method:
Gradients $\frac{m_1}{m_2} = 1 : 3$
$\Rightarrow m_1 = m, m_2 = 3m$

$m_1 + m_2 = -\dfrac{2h}{b} \Rightarrow m + 3m = -\dfrac{2h}{b}$

$\Rightarrow m = \dfrac{-h}{2b}$

$m_1.m_2 = \dfrac{a}{b} \Rightarrow m.3m = \dfrac{a}{b}$

$\Rightarrow 3m^2 = \dfrac{a}{b} \Rightarrow 3.\dfrac{h^2}{4b^2} = \dfrac{a}{b} \Rightarrow \dfrac{h^2}{ab} = \dfrac{4}{3}$

27. $m_1 : m_2 = 1 : 2$

$\therefore$ $\dfrac{h^2}{ab} = \dfrac{(2+1)^2}{4(2)} = \dfrac{9}{8}$

$\Rightarrow \dfrac{ab}{h^2} = \dfrac{8}{9}$

28. Given equation of pair of lines is
$x^2 + 4xy + y^2 = 0$

$\therefore$ a = 1, h = 2, b = 1

$\therefore$ $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\dfrac{2\sqrt{(2)^2 - (1)(1)}}{1+1}\right| = \sqrt{3}$

$\Rightarrow \theta = \tan^{-1}\left(\sqrt{3}\right) = 60°$

29. Given equation of pair of lines is
$(x^2 + y^2)\sqrt{3} = 4xy$

$\therefore$ a = $\sqrt{3}$, h = – 2, b = $\sqrt{3}$

$\therefore$ $\tan\theta = \left|\dfrac{2\sqrt{4-3}}{2\sqrt{3}}\right| = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \theta = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

30. Given equation of pair of lines is
$x^2 + 4y^2 - 7xy = 0$

$\therefore$ a = 1, h = $-\dfrac{7}{2}$, b = 4

$\therefore$ $\tan\theta = \left|\dfrac{2\sqrt{\left(\frac{-7}{2}\right)^2 - (1)(4)}}{1+(4)}\right|$

$= \left|\dfrac{2\sqrt{\frac{49}{4} - 4}}{5}\right| = \dfrac{\sqrt{33}}{5}$

$\Rightarrow \theta = \tan^{-1}\left(\dfrac{\sqrt{33}}{5}\right)$

31. Given equation of pair of lines is
$4x^2 - 24xy + 11y^2 = 0$

$\therefore$ a = 4, h = – 12, b = 11

$\therefore$ $\tan\theta = \pm 2\dfrac{\sqrt{h^2 - ab}}{a+b} = \pm 2\dfrac{\sqrt{144-44}}{15} = \pm\dfrac{4}{3}$

$\Rightarrow \theta = \tan^{-1}\left(\pm\dfrac{4}{3}\right)$

32. Given equation of pair of lines is
$x^2 + 2xy\sec\theta + y^2 = 0$

$\therefore$ a = 1, h = sec θ, b = 1

Let φ be the angle between the lines.

$\therefore$ $\tan\phi = \left|\dfrac{2\sqrt{\sec^2\theta - 1}}{2}\right|$

$\Rightarrow \tan\phi = \tan\theta \Rightarrow \phi = \theta$

33. Let m_1 and m_2 be the slopes of the lines given by $x^2 + 4xy + y^2 = 0$

$\therefore$ $m_1 + m_2 = -4 \Rightarrow m_2 = -4 - m_1$

and $m_1.m_2 = 1 \Rightarrow m_1(-4 - m_1) = 1$

$\Rightarrow m_1^2 + 4m_1 + 1 = 0$

$\therefore$ $m_1, m_2 = -2 \pm \sqrt{3}$

Slope of line $x - y = 4$ is

$m_3 = 1$

$\therefore$ Angle between first two lines,

$\tan^{-1}\theta_{12} = \left|\dfrac{m_1 - m_2}{1 + m_1.m_2}\right| = \left|\dfrac{(-2+\sqrt{3}) - (-2-\sqrt{3})}{1 + (-2+\sqrt{3})(-2-\sqrt{3})}\right|$

$\Rightarrow \theta_{12} = \tan^{-1}\left(\sqrt{3}\right) = 60°$

Angle between second and third line

$\theta_{23} = \tan^{-1}\left(\dfrac{-2-\sqrt{3}-1}{1+(-2-\sqrt{3})1}\right) = \tan^{-1}\left(\sqrt{3}\right) = 60°$

Similarly, we have, $\theta_{31} = 60°$

$\therefore$ The triangle formed by the lines is equilateral triangle.

34. Let m_1 and m_2 be the slopes of the lines given by $23x^2 - 48xy + 3y^2 = 0$

$\therefore$ $m_1 + m_2 = \dfrac{48}{3} = 16 \Rightarrow m_2 = 16 - m_1$

and $m_1m_2 = \dfrac{23}{3} \Rightarrow m_1(16 - m_1) = \dfrac{23}{3}$

$\Rightarrow -m_1^2 + 16m_1 - \dfrac{23}{3} = 0$

$\Rightarrow 3m_1^2 - 48m_1 + 23 = 0$

$\Rightarrow m_1, m_2 = \dfrac{24 \pm 13\sqrt{3}}{3}$

slope of line is $2x + 3y + 4 = 0$ is

$m_3 = \frac{-2}{3}$

$\therefore$ Angle between first two lines,

$$\tan^{-1}\theta_{12} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\left(\frac{24+13\sqrt{3}}{3}\right) - \left(\frac{24-13\sqrt{3}}{3}\right)}{1+\left(\frac{24+13\sqrt{3}}{3}\right)\left(\frac{24-13\sqrt{3}}{3}\right)}\right|$$

$$= \left|\frac{\frac{26\sqrt{3}}{3}}{\frac{9+576-507}{9}}\right| = \left|\frac{\frac{26\sqrt{3}}{3}}{\frac{78}{9}}\right|$$

$\therefore$ $\tan^{-1}\theta_{12} = \sqrt{3}$

$\Rightarrow \theta_{12} = \tan^{-1}\left(\sqrt{3}\right) = 60°$

Angle between second and third line

$$\theta_{23} = \tan^{-1}\left(\frac{\frac{24-13\sqrt{3}}{3} - \left(-\frac{2}{3}\right)}{1+\left(\frac{24-13\sqrt{3}}{3}\right)\left(-\frac{2}{3}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{26-13\sqrt{3}}{3}}{\frac{9-48+26\sqrt{3}}{9}}\right) = \tan^{-1}\left(\frac{\frac{26-13\sqrt{3}}{3}}{\frac{-39+26\sqrt{3}}{9}}\right)$$

$$= \tan^{-1}\left(\frac{26-13\sqrt{3}}{3} \times \frac{9}{-39+26\sqrt{3}}\right)$$

$$= \tan^{-1}\left(\frac{13\left(2-\sqrt{3}\right)\times 3}{13\sqrt{3}\left(2-\sqrt{3}\right)}\right)$$

$$= \tan^{-1}\left(\sqrt{3}\right) = 60°$$

Similarly, we have, $\theta_{31} = 60°$

$\therefore$ The triangle formed by the lines is equilateral triangle.

35. Given equation of pair of lines is

$4x^2 + 12xy + 9y^2 = 0$

$a = 4, h = 6, b = 9$

Here,

$h^2 - ab = (6)^2 - (4)(9) = 36 - 36 = 0$

Hence, the lines are real and coincident.

36. Given equation of pair of lines is

$x^2 + ky^2 + 4xy = 0$

$\therefore$ $a = 1, h = \frac{k}{2}, b = 4$

The pair of lines are coincident if $h^2 - ab = 0$

$\Rightarrow h^2 = ab \Rightarrow \frac{k^2}{4} = 4(1)$

$\Rightarrow k = \pm 4$

37. Given equation of pair of lines is

$px^2 - qy^2 = 0$

$\therefore$ $a = p, b = -q, c = 0$

Since, the lines are real and distinct

$\therefore$ $h^2 - ab > 0$

$\Rightarrow 0 - p(-q) > 0$

$\Rightarrow pq > 0$

38. Given equation of pair of lines is

$y^2 \sin^2\theta - xy \sin^2\theta + x^2(\cos^2\theta - 1) = 0$

$\therefore$ $a = \sin^2\theta, b = \cos^2\theta - 1 = -(1 - \cos^2\theta)$

$= -\sin^2\theta$

Now, $a + b = \sin^2\theta - \sin^2\theta = 0$

$\therefore$ The lines are perpendicular.

$\therefore$ $\theta = \frac{\pi}{2}$

39. Consider option (C)

Given equation is $y^2 + x + 1 = 0$

$\therefore$ $a = 0, b = 1, c = 0, f = 0, g = \frac{-1}{2}, h = 0$

Now, $abc + 2fgh - af^2 - bg^2 - ch^2$

$= 0 + 0 - 0 - \left(\frac{1}{4}\right) + 0 = \frac{-1}{4} \neq 0$

$\therefore$ The equation does not represent a pair of straight lines.

40. Given equation of pair of lines is

$3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$

$a = 3, b = 2, c = 2, f = \frac{5}{2}, g = \frac{5}{2}, h = \frac{7}{2}$

Consider $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= (3)(2)(2) + 2\left(\frac{5}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right) - 3\left(\frac{5}{2}\right)^2 - 2\left(\frac{5}{2}\right)^2 - 2\left(\frac{7}{2}\right)^2 = 0$$

$\therefore$ the given equation represents a pair of straight lines.

41. Given equation of pair of lines is
$xy + a^2 = ax + ay$
i.e. $ax + ay - xy - a^2 = 0$

$\therefore$ $A = 0, B = 0, C = -a^2, F = \frac{a}{2}, G = \frac{a}{2}, H = -\frac{1}{2}$

Now, $ABC + 2FGH - AF^2 - BG^2 - CH^2$

$= 0 - 2\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{-1}{2}\right) - (a^2)\left(\frac{-1}{2}\right)^2 = 0$

$\therefore$ the given equation represents a pair of straight lines.

42. Given equation of pair of lines is
$ax^2 - y^2 + 4x - y = 0$

$\therefore$ $A = a, B = -1, C = 0, F = \frac{-1}{2}, G = 2, H = 0$

The given equation represents a pair of straight lines,

$\therefore$ $ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$

$\Rightarrow 0 - 0 - a\left(\frac{1}{4}\right) - (-1)(4) = 0$

$\Rightarrow -\frac{a}{4} + 4 = 0 \Rightarrow a = 16$

43. Given equation of pair of lines is
$kxy + 10x + 6y + 4 = 0$

$\therefore$ $a = b = 0, c = 4, f = 3, g = 5, h = \frac{k}{2}$

Now, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow 0 + 2(3)(5)\left(\frac{k}{2}\right) - 0 - 0 - 4\left(\frac{k}{2}\right)^2 = 0$

$\Rightarrow 15k - k^2 = 0 \Rightarrow k(15 - k) = 0$

$\Rightarrow k = 0$ or $k = 15$

44. Given equation of pair of lines is
$x^2 + kxy + y^2 - 5x - 7y + 6 = 0$

$\therefore$ $a = 1, b = 1, c = 6, f = \frac{-7}{2}, g = \frac{-5}{2}, h = \frac{k}{2}$

Now, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow (1)(1)(6) + 2\left(\frac{-7}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{k}{2}\right) - 1\left(-\frac{7}{2}\right)^2 - 1\left(\frac{-5}{2}\right)^2 - 6\left(\frac{k}{2}\right)^2 = 0$

$\Rightarrow 6 + \frac{35k}{4} - \frac{49}{4} - \frac{25}{4} - \frac{6k^2}{4} = 0$

$\Rightarrow -6k^2 + 35k - 50 = 0$

$\Rightarrow (2k - 5)(3k - 10) = 0$

$\Rightarrow k = \frac{5}{2}$ or $k = \frac{10}{3}$

45. Given equation of pair of lines is
$x^2 - y^2 + x + 3y - 2 = 0$

$\therefore$ $a = 1, b = -1, g = \frac{1}{2}, f = \frac{3}{2}, c = -2$

$\therefore$ point of intersection of the lines is
$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right) = \left(\frac{-1}{2}, \frac{3}{2}\right)$

46. Given equation of pair of lines is
$2x^2 - 10xy + 2\lambda y^2 + 5x - 16y - 3 = 0$

$\therefore$ $a = 2, b = 2\lambda, c = -3, f = -8, g = \frac{5}{2}, h = -5$

Since the equation represents pair of lines,

$\therefore$ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow 2(2\lambda)(-3) + 2(-8)\left(\frac{5}{2}\right)(-5) - 2(64) - 2\lambda\left(\frac{25}{4}\right) + 3(25) = 0$

$\Rightarrow \frac{49\lambda}{2} = 147 \Rightarrow \lambda = 6$

$\therefore$ Point of intersection of the lines is
$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

$\equiv \left(\frac{(-5)(-8) - 2(6)\left(\frac{5}{2}\right)}{2(12) - (-5)^2}, \frac{\frac{5}{2}(-5) - 2(-8)}{2(12) - (-5)^2}\right)$

$\equiv \left(-10, \frac{-7}{2}\right)$

47. Given equation of pair of lines is
$2x^2 - 3xy - 2y^2 + 10x + 5y = 0$

$\therefore$ $a = 2.\ b = -2, c = 0, f = \frac{5}{2}, g = 5, h = \frac{-3}{2}$

$\therefore$ Point of intersection of the lines is
$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right) \equiv (-1, 2)$

Slope of line joining origin and $(-1, 2)$ $m = -2$

Slope of $kx + y + 3 = 0$ is $-k$

Now, $(-k)(-2) = -1 \Rightarrow k = \frac{-1}{2}$

48. The line $5x + y - 1 = 0$ is coincides
$5x^2 + xy - kx - 2y + 2 = 0$

$\therefore$ $a = 5, b = 0, c = 2, f = -1, g = -\frac{k}{2}, h = \frac{1}{2}$

$m_1 + m_2 = \frac{-2h}{b}$

As b = 0, this case is not defined
Slope of line $5x + y - 1 = 0$ is $m = -5$
$\therefore$ Slope of another line must be infinite
$\therefore$ equation of another line is $x = k_1$
$\therefore$ Combine equation is $(5x + y - 1)(x - k_1) = 0$
$\Rightarrow 5x^2 - 5xk_1 + xy - yk_1 - x + k_1 = 0$
$\Rightarrow 5x^2 + xy - (5k_1 + 1)x - yk_1 + k_1 = 0$
Comparing this equation with the given equation, we get k = 11

49. Given equation of pair of lines is
$3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$
$\therefore$ $a = 3, b = 2, h = \dfrac{7}{2}$

$\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\dfrac{2\sqrt{\frac{49}{4} - 6}}{3+2}\right| = \left|\dfrac{2\sqrt{\frac{25}{4}}}{5}\right|$
$\Rightarrow \tan\theta = 1$
$\Rightarrow \theta = \tan^{-1}(1) = \dfrac{\pi}{4}$

50. Given equation of pair of lines is
$x^2 - xy - 6y^2 - 7x + 31y - 18 = 0$
$\therefore$ $a = 1, b = -6, h = -\dfrac{1}{2}$
$\therefore$ $\tan\theta = \left|\dfrac{2\sqrt{\left(\frac{-1}{2}\right)^2 - 1(-6)}}{1-6}\right| = \left|\dfrac{2\sqrt{\frac{1}{4}+6}}{-5}\right| = |-1| = 1$
$\Rightarrow \theta = \tan^{-1}(1) = 45°$

51. Given equation of pair of lines is
$x^2 - 3xy + \lambda y^2 + 3x + 5y + 2 = 0$
$\therefore$ $a = 1, b = \lambda, h = -\dfrac{3}{2}$
$\theta = \tan^{-1} 3 \Rightarrow \tan\theta = 3$
$\therefore$ $\tan\theta = \left|\dfrac{2\sqrt{\left(-\frac{3}{2}\right)^2 - (1)\lambda}}{1+\lambda}\right|$
$\Rightarrow 3 = \left|\dfrac{2\sqrt{\frac{9-4\lambda}{4}}}{1+\lambda}\right| = \left|\dfrac{\sqrt{9-4\lambda}}{1+\lambda}\right|$
$\Rightarrow \dfrac{9-4\lambda}{(1+\lambda)^2} = 9$
$\Rightarrow 9 - 4\lambda = 9(1 + \lambda)^2$
$\Rightarrow 9\lambda^2 + 22\lambda = 0$
$\Rightarrow \lambda(9\lambda + 22) = 0$
$\Rightarrow \lambda = 0$ or $\lambda = -\dfrac{22}{9}$
But λ is non-negative
$\therefore$ $\lambda = 0$

52. Given equation of pair of lines is
$2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$
$a = 2, b = 3, h = \dfrac{5}{2}$
$\theta = \tan^{-1} m \Rightarrow \tan\theta = m$
$\tan\theta = \left|\dfrac{2\sqrt{\frac{25}{4} - 6}}{2+3}\right| \Rightarrow m = \dfrac{1}{5}$

53. Given equation of pair of lines is
$x^2 + y^2 - 2x - 1 = 0$ ….(i)
$x + y = 1$ intersects the above pair of lines
$\therefore$ It satisfies equation (i)
$\therefore$ $x^2 + y^2 - 2x(x + y) - (x + y)^2 = 0$
$\Rightarrow 2x^2 + 4xy = 0 \Rightarrow x^2 + 2xy = 0$
$\therefore$ $a = 1, b = 0, h = 1$
$\therefore$ $\tan\theta = \dfrac{2\sqrt{1^2 - 0}}{1}$
$\Rightarrow \theta = \tan^{-1}(2)$

54. The joint equation of the pair of straight lines joining the origin to the points of intersection of the line $lx + my + n = 0$ and
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is
$ax^2 + 2hxy + by^2 + 2g\left(\dfrac{lx + my}{-n}\right)x$
$+ 2f\left(\dfrac{lx + my}{-n}\right)y + c\left(\dfrac{lx + my}{-n}\right)^2 = 0$
Here, $l = 2, m = 1, n = -1$ and
$a = 3, b = 0, c = 1, f = 0, g = -2, h = 2$
$\therefore$ $3x^2 + 4xy - 4x(2x + y) + (2x + y)^2 = 0$
$\Rightarrow 3x^2 + 4xy - 8x^2 - 4xy + 4x^2 + y^2 + 4xy = 0$
$\Rightarrow x^2 - 4xy - y^2 = 0$
$\therefore$ $A = 1, B = -1, H = -2$
$\therefore$ $\tan\theta = \dfrac{2\sqrt{4+1}}{0} = \infty$
$\Rightarrow \theta = \dfrac{\pi}{2}$

55. Given, $ax^2 + 2hxy + by^2 = -2gx$
$a_1x^2 + 2h_1xy + b_1y^2 = -2g_1x$
$\therefore$ $\dfrac{ax^2 + 2hxy + by^2}{a_1x^2 + 2h_1xy + b_1y^2} = \dfrac{g}{g_1}$
We have,
$(ag_1 - a_1g)x^2 + 2(hg_1 - h_1g)xy + (bg_1 - b_1g)y^2 = 0$
$\therefore$ $A = (ag_1 - a_1g), B = (bg_1 - b_1g)$
The lines are perpendicular
$\therefore$ $A + B = 0$
$\Rightarrow (ag_1 - a_1g) + (bg_1 - b_1g) = 0$
$\Rightarrow (a + b)g_1 = (a_1 + b_1)g$

56. The equation of line is $y = 2\sqrt{2}\,x + c$

$\Rightarrow \left(\dfrac{y - 2\sqrt{2}x}{c}\right) = 1$ ....(i)

Given equation of circle is

$x^2 + y^2 = 2\,(1)^2$ ....(ii)

$\therefore$ from (i) and (ii), we get

$x^2 + y^2 = 2\left(\dfrac{y - 2\sqrt{2}x}{c}\right)^2$

$\Rightarrow c^2(x^2 + y^2) = 2(y^2 - 4\sqrt{2}\,xy + 8x^2)$

$\Rightarrow (c^2 - 16)x^2 + (c^2 - 2)y^2 + 8\sqrt{2}xy = 0$

The lines are perpendicular if A + B = 0.

$\therefore$ $c^2 - 16 + c^2 - 2 = 0$

$\Rightarrow 2c^2 - 18 = 0$ $\Rightarrow c^2 - 9 = 0$

57. Lines represented by the equation $2y^2 - xy - 6x^2 = 0$ are

$y = 2x$ and $y = -\dfrac{3}{2}x$

The co-ordinates of the vertices of the triangle formed by above lines with $x + y = 1$ are

$(0, 0)$, $\left(\dfrac{1}{3}, \dfrac{2}{3}\right)$ and $(-2, 3)$

The altitude from vertex (0, 0) on $x + y = 1$ is $y = x$. ....(i)

The altitude from vertex $\left(\dfrac{1}{3}, \dfrac{2}{3}\right)$ on $y = \dfrac{-3}{2}x$

is $y - \dfrac{2}{3} = \dfrac{2}{3}\left(x - \dfrac{1}{3}\right)$

$\Rightarrow 6x - 9y + 4 = 0$ ....(ii)

Solving (i) and (ii), we get

$x = \dfrac{4}{3}$ and $y = \dfrac{4}{3}$,

$\therefore$ Orthocentre is $\left(\dfrac{4}{3}, \dfrac{4}{3}\right)$

58. Given equations of pair of lines are

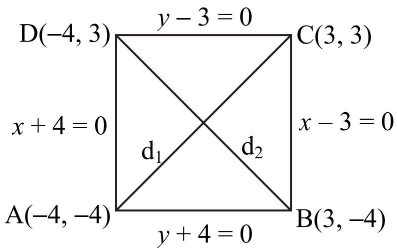
$xy + 4x - 3y - 12 = 0$ and

$xy - 3x + 4y - 12 = 0$

$\therefore$ $x(y + 4) - 3(y + 4) = 0$ and $x(y - 3) + 4(y - 3) = 0$

$\therefore$ $(y + 4)(x - 3) = 0$ and $(x + 4)(y - 3) = 0$

$\therefore$ The vertices of the square are as shown in the figure



Equation of diagonal d_1 is

$y + 4 = \dfrac{-4 - 3}{-4 - 3}(x + 4)$

$\Rightarrow y + 4 = x + 4$

$\Rightarrow x - y = 0$

and equation of diagonal d_2 is

$y + 4 = \dfrac{3 + 4}{-4 - 3}(x - 3)$

$\Rightarrow y + 4 = -1\,(x - 3)$

$\Rightarrow y + 4 = -x + 3$

$\Rightarrow x + y + 1 = 0$

$\therefore$ Combined equation of diagonals d_1 and d_2 is

$(x - y)(x + y + 1) = 0$

$\Rightarrow x^2 - y^2 + x - y = 0$

Evaluation Test

1. L_1: $ax^2 + 2hxy + by^2 = 0$

Equation of any line passing through origin and perpendicular to L_1 is given by

$bx^2 - 2hxy + ay^2 = 0$

....(interchanging coefficients of x^2 and y^2 and change of sign for xy term)

$\therefore$ The required equation of pair of lines is

$-15x^2 + 7xy + 2y^2 = 0$

i.e. $15x^2 - 7xy - 2y^2 = 0$

2. Here, $m_1 + m_2 = \dfrac{-2h}{b}$ .....(i)

and $m_1 m_2 = \dfrac{a}{b}$

$\therefore$ $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$

$= \dfrac{4h^2 - 4ab}{b^2}$

$= \dfrac{4h^2 - 3h^2}{b^2}$ ....[$\because$ $4ab = 3h^2$ (given)]

$= \dfrac{h^2}{b^2}$

$\therefore$ $m_1 - m_2 = \dfrac{h}{b}$ .....(ii)

On solving (i) and (ii), we get

$m_1 = \dfrac{-h}{2b}$ and $m_2 = \dfrac{-3h}{2b}$

$\therefore$ $m_1 : m_2 = 1 : 3$

3. The lines are parallel, if $af^2 = bg^2$

$\therefore$ $4f^2 = 9g^2$

$\Rightarrow f = \frac{3}{2}g$

Let g = 2 and f = 3

$\therefore$ $abc + 2fgh - af^2 - bg^2 - ch^2$

$= 4(9)(c) + 2(3)(2)(6) - 4(3)^2 - 9(2)^2 - c(6)^2 = 0$

$\Rightarrow$ c is any number.

4. Given equation is $x^2 - y^2 - x - \lambda y - 2 = 0$.

$\therefore$ $a = 1, b = -1, c = -2, f = \frac{-\lambda}{2}, g = \frac{-1}{2}, h = 0$

This equation represents a pair of straight lines, if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow 2 - \frac{\lambda^2}{4} + \frac{1}{4} = 0 \Rightarrow \frac{\lambda^2}{4} = \frac{9}{4} \Rightarrow \lambda^2 = 9 \Rightarrow \lambda = \pm 3$

5. The given equation of pair of lines is

$x^2 + 2\sqrt{2}\,xy - y^2 = 0$

$\therefore$ $a = 1, b = -1, h = \sqrt{2}$

Now, $a + b = 1 + (-1) = 0$

$\therefore$ The lines are perpendicular

6. The joint equation of the lines through the point (x_1, y_1) and at right angles to the lines $ax^2 + 2hxy + by^2 = 0$ is

$b(x - x_1)^2 - 2h(x - x_1)(y - y_1) + a(y - y_1)^2 = 0$

$\therefore$ joint equation of pair of lines drawn through (1, 1) and perpendicular to the pair of lines $3x^2 - 7xy + 2y^2 = 0$ is

$2(x - 1)^2 + 7(x - 1)(y - 1) + 3(y - 1)^2 = 0$

7. The given equations are $x - y - 1 = 0$ and $2x + y - 6 = 0$

$\therefore$ The joint equation is given by

$(x - y - 1)(2x + y - 6) = 0$

$\Rightarrow 2x^2 + xy - 6x - 2xy - y^2 + 6y - 2x - y + 6 = 0$

$\Rightarrow 2x^2 - y^2 - xy - 8x + 5y + 6 = 0$

8. Let the equation of one of the angle bisector of the co-ordinate axes be $x + y = 0 \Rightarrow m_1 = -1$

Given equation of pair of lines is

$2x^2 + 2hxy + 3y^2 = 0$

$\therefore$ A = 2, H = h, B = 3

Now, $m_1m_2 = \frac{a}{b} \Rightarrow m_2 = \frac{-2}{3}$

Also $m_1 + m_2 = \frac{-2h}{b} \Rightarrow -1 - \frac{2}{3} = \frac{-2h}{3}$

$\Rightarrow h = \frac{5}{2}$

9. The given equation of pair of lines is

$3x^2 - 2y^2 + \lambda xy - x + 5y - 2 = 0$

$\therefore$ $a = 3, b = -2, c = -2, f = \frac{5}{2}, g = \frac{-1}{2}, h = \frac{\lambda}{2}$

Now $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$12 - \frac{5\lambda}{4} - \frac{75}{4} + \frac{1}{2} + \frac{\lambda^2}{2} = 0$

$\Rightarrow 2\lambda^2 - 5\lambda - 25 = 0 \Rightarrow (\lambda - 5)(2\lambda + 5) = 0$

$\Rightarrow \lambda = 5$ or $\frac{-5}{2}$

10. Let $y = mx$ be the common line and let $y = m_1x$ and $y = m_2x$ be the other lines given by $2x^2 + axy + 3y^2 = 0$ and $2x^2 + bxy - 3y^2 = 0$ respectively. Then,

$m + m_1 = -\frac{a}{3}, mm_1 = \frac{2}{3}$, and

$m + m_2 = \frac{b}{3}, mm_2 = -\frac{2}{3}$

$\therefore$ $(mm_1)(mm_2) = \frac{2}{3}\left(-\frac{2}{3}\right)$

$\Rightarrow m^2(m_1m_2) = -\frac{4}{9}$

$\Rightarrow m^2 = \frac{4}{9}$ ....[$\because$ $m_1m_2 = -1$ (given)]

$\Rightarrow m = \pm\frac{2}{3}$

When $m = \frac{2}{3}$,

$mm_1 = \frac{2}{3}$ and $mm_2 = -\frac{2}{3} \Rightarrow m_1 = 1$ and $m_2 = -1$

$\therefore$ $m + m_1 = -\frac{a}{3}$ and $m + m_2 = \frac{b}{3}$

$\Rightarrow a = -5$ and $b = -1$

When $m = -\frac{2}{3}$,

$mm_1 = \frac{2}{3}$ and $mm_2 = -\frac{2}{3} \Rightarrow m_1 = -1$ and $m_2 = 1$

$\therefore$ $m + m_1 = -\frac{a}{3}$ and $m + m_2 = \frac{b}{3}$

$\Rightarrow a = 5$ and $b = 1$

11. Given equation of pair of lines is

$3x^2 - 48xy + 23y^2 = 0$

$\therefore$ a = 3, h = –24, b = 23

$\therefore$ $\tan\theta = \left|\frac{2\sqrt{576 - 69}}{3 + 23}\right|$

$\Rightarrow \tan\theta = \left|\frac{2\sqrt{507}}{26}\right| = \left|\frac{2 \times 13\sqrt{3}}{26}\right| = \sqrt{3}$

$\Rightarrow \theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

Textbook Chapter No.

05 Vectors

Hints

Classical Thinking

1. Since the vectors are collinear,

$\therefore$ $\overline{b} = \lambda\overline{a}$

$\Rightarrow (-2\hat{i} + m\hat{j}) = \lambda(\hat{i} - \hat{j})$

On comparing, we get

$\lambda = -2$ and $-\lambda = m$

$\Rightarrow m = 2$

2. $\overline{c} = \lambda\overline{d}$

$\Rightarrow (x-2)\overline{a} + \overline{b} = \lambda(2x+1)\overline{a} - \lambda\overline{b}$

On comparing, we get

$\lambda = -1$ and

$(x-2) = \lambda(2x+1)$

$\Rightarrow x - 2 = -2x - 1$

$\Rightarrow x = \dfrac{1}{3}$

3. Let $\overline{a} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ and $\overline{b} = -2\hat{i} + p\hat{j} - q\hat{k}$

Two vector are collinear if

$\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

$\Rightarrow \dfrac{3}{-2} = \dfrac{-2}{p} = \dfrac{5}{-q}$

$\Rightarrow p = \dfrac{4}{3}, q = \dfrac{10}{3}$

4. For the points to be collinear,

$\overline{AB} \times \overline{BC} = 0$

$\Rightarrow (\overline{b} - \overline{a}) \times (\overline{c} - \overline{b}) = \overline{0}$

$\Rightarrow \overline{b} \times \overline{c} - \overline{a} \times \overline{c} + \overline{a} \times \overline{b} = \overline{0}$

$\Rightarrow \overline{a} \times \overline{b} + \overline{b} \times \overline{c} + \overline{c} \times \overline{a} = \overline{0}$

5. Here $\overline{a} = \hat{i} + \hat{j}$, $\overline{b} = 2\hat{i} - \hat{j}$ and $\overline{r} = 2\hat{i} - 4\hat{j}$

Let $\overline{r} = t_1\overline{a} + t_2\overline{b}$

$\Rightarrow 2\hat{i} - 4\hat{j} = t_1(\hat{i} + \hat{j}) + t_2(2\hat{i} - \hat{j})$

$= (t_1 + 2t_2)\hat{i} + (t_1 - t_2)\hat{j}$

Comparing the coefficients, we get

$t_1 + 2t_2 = 2$ ….(i)

$t_1 - t_2 = -4$ ….(ii)

On solving (i) and (ii), we get

$t_1 = -2, t_2 = 2$

6. Given, $3\overline{A} = 2\overline{B}$

$\therefore$ $3(x + 4y) = 2(y - 2x + 2)$

$\Rightarrow 7x + 10y = 4$ ….(i)

and $3(2x + y + 1) = 2(2x - 3y - 1)$

$\Rightarrow 2x + 9y = -5$ ….(ii)

On solving (i) and (ii), we get

$x = 2, y = -1$

8. $1(\overline{a}) + 1(\overline{b}) = \overline{a} + \overline{b}$.

$\therefore$ $1(\overline{a}) + 1(\overline{b}) - 1(\overline{a} + \overline{b}) = 0$

$\therefore$ The vectors are coplanar.

9. Let $R(\overline{r})$ be the point dividing PQ internally in the ratio 2 : 5

$\therefore$ $\overline{r} = \dfrac{5\overline{p} + 2\overline{q}}{7}$

10. Let $R(\overline{r})$ divide line AB internally in the ratio 2 : 3

$\therefore$ $\overline{r} = \dfrac{2\overline{b} + 3\overline{a}}{2+3}$

$= \dfrac{2(3\hat{i} + \hat{j} + 4\hat{k}) + 3(2\hat{i} + 3\hat{j} - \hat{k})}{5}$

$= \dfrac{12\hat{i} + 11\hat{j} + 5\hat{k}}{5}$

$\therefore$ Co-ordinates of R are $\left(\dfrac{12}{5}, \dfrac{11}{5}, 1\right)$

11. $C \equiv \left(\dfrac{2-4}{2}, \dfrac{-1+3}{2}\right) \equiv (-1, 1)$

$\therefore$ $\overline{OC} = -\hat{i} + \hat{j}$

12. If $M(\overline{m})$ is the mid-point of AB, then

$\overline{m} = \dfrac{\overline{a} + \overline{b}}{2}$

$\Rightarrow \dfrac{\hat{i} + 3\hat{j} - \hat{k} + 3\hat{i} - \hat{j} - 3\hat{k}}{2} = 2\hat{i} + \hat{j} - 2\hat{k}$

13. Let $R(\overline{r})$ divide AB externally in the ratio 5:2

$\therefore$ $\overline{r} = \dfrac{5(\hat{i} - \hat{j} + 2\hat{k}) - 2(2\hat{i} + \hat{j} - \hat{k})}{5-2} = \dfrac{\hat{i} - 7\hat{j} + 12\hat{k}}{3}$

14. Let $R(\bar{r})$ divide PQ externally in the ratio 2 : 1

$\therefore$ $\bar{r} = \dfrac{2\bar{q} - \bar{p}}{2-1}$

$= \dfrac{2(3\hat{i} + 2\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 4\hat{k})}{1}$

$= 4\hat{i} + 5\hat{j} - 2\hat{k}$

$\therefore$ Co-ordinates of R are (4, 5, –2)

15. Let P divide AB in the ratio $\lambda : 1$

$\therefore$ $\left(\dfrac{17}{4}, \dfrac{11}{4}, 0\right) \equiv \left(\dfrac{2\lambda+5}{\lambda+1}, \dfrac{-7\lambda+a}{\lambda+1}, \dfrac{k\lambda-1}{\lambda+1}\right)$

$\Rightarrow \dfrac{17}{4} = \dfrac{2\lambda+5}{\lambda+1} \Rightarrow \lambda = \dfrac{1}{3}$

16. If $A(\bar{a}), B(\bar{b}), C(\bar{c})$ are the vertices and $G(\bar{g})$ is the centroid of ΔABC, then

$\bar{g} = \dfrac{\bar{a} + \bar{b} + \bar{c}}{3}$

$\therefore$ $3\hat{i} + 2\hat{j} + n\hat{k} = \dfrac{(2\hat{i} + 3\hat{j} - 4\hat{k}) + (m\hat{i} + \hat{j} - \hat{k}) + (3\hat{i} + 2\hat{j} + 2\hat{k})}{3}$

$\Rightarrow 3(3\hat{i} + 2\hat{j} + n\hat{k}) = (5 + m)\hat{i} + 6\hat{j} + (-3)\hat{k}$

On comparing, we get

$9 = 5 + m \Rightarrow m = 4$, and

$3n = -3 \Rightarrow n = -1$

17. $G \equiv \left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}, \dfrac{z_1 + z_2 + z_3}{3}\right)$

$\Rightarrow (2, 1, c) \equiv \left(\dfrac{2a+1}{3}, \dfrac{4+b}{3}, \dfrac{1}{3}\right)$

$\Rightarrow 2 = \dfrac{2a+1}{3}, 1 = \dfrac{4+b}{3}, c = \dfrac{1}{3}$

$\Rightarrow a = \dfrac{5}{2}, b = -1, c = \dfrac{1}{3}$

18. $[\hat{i}\ \hat{k}\ \hat{j}] = \hat{i}.(\hat{k} \times \hat{j}) = \hat{i}.(-\hat{i}) = -1.$

19. $2\hat{i} \cdot \left[3\hat{j} \times (-5\hat{k})\right] = -30\ [\hat{i} \cdot (\hat{j} \times \hat{k})]$

$= -30(\hat{i} \cdot \hat{i}) = -30(1)$

$= -30$

20. $(\hat{i} + \hat{j}) \cdot [(\hat{j} + \hat{k}) \times (\hat{k} + \hat{i})]$

$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$

$= 1(1) - 1(-1) = 2$

21. $[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -4 \\ -1 & 2 & -1 \end{vmatrix}$

$= 1(-1 + 8) + 1(-1 - 4) + 1(2 + 1)$

$= 5$

22. $\bar{a}.(\bar{b} \times \bar{c}) = \begin{vmatrix} 3 & -2 & 2 \\ 6 & 4 & -2 \\ 3 & -2 & -4 \end{vmatrix}$

$= 3(-16 - 4) + 2(-24 + 6) + 2(-12 - 12)$

$= -144$

24. Since $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}] = -[\bar{b}\ \bar{a}\ \bar{c}]$

25. $[\hat{i}\ \hat{k}\ \hat{j}] + [\hat{k}\ \hat{j}\ \hat{i}] + [\hat{j}\ \hat{k}\ \hat{i}] = [\hat{i}\ \hat{k}\ \hat{j}] + [\hat{i}\ \hat{k}\ \hat{j}] - [\hat{i}\ \hat{k}\ \hat{j}]$

$= [\hat{i}\ \hat{k}\ \hat{j}] = -1$

27. $[\bar{a} + 2\bar{b}\quad \bar{a} + \bar{c}\quad \bar{b}]$

$= [\bar{a}\quad \bar{a} + \bar{c}\quad \bar{b}] + [2\bar{b}\quad \bar{a} + \bar{c}\quad \bar{b}]$

$= [\bar{a}\ \bar{a}\ \bar{b}] + [\bar{a}\ \bar{c}\ \bar{b}] + [2\bar{b}\ \bar{a}\ \bar{b}] + [2\bar{b}\ \bar{c}\ \bar{b}]$

$= 0 - [\bar{a}\ \bar{b}\ \bar{c}] + 2(0) + 2(0)$

$= -[\bar{a}\ \bar{b}\ \bar{c}]$

28. Let $\bar{p} = \bar{a} - 2\bar{b} + 3\bar{c}$, $\bar{q} = 2\bar{a} + m\bar{b} - 4\bar{c}$ and $\bar{r} = -7\bar{b} + \bar{c}$

Since the points are collinear.

$\therefore$ $[\bar{p}\quad \bar{q}\quad \bar{r}] = 0$

$\Rightarrow \begin{vmatrix} 1 & -2 & 3 \\ 2 & m & -4 \\ 0 & -7 & 10 \end{vmatrix} = 0$

$\Rightarrow 1(10m - 28) + 2(20 - 0) + 3(-14 - 0) = 0$

$\Rightarrow 10m - 30 = 0 \Rightarrow m = 3$

29. Let $\bar{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{b} = \lambda\hat{i} + 4\hat{j} + 7\hat{k}$, and $\bar{c} = -3\hat{i} - 2\hat{j} - 5\hat{k}$

Since the vectors are collinear,

$\therefore$ $\begin{vmatrix} 1 & 2 & 3 \\ \lambda & 4 & 7 \\ -3 & -2 & -5 \end{vmatrix} = 0$

$\Rightarrow -6 + 10\lambda - 42 - 6\lambda + 36 = 0$

$\Rightarrow \lambda = 3$

30. We know that,

$[\bar{a} - \bar{b}\quad \bar{b} - \bar{c}\quad \bar{c} - \bar{a}] = 0$

$\therefore$ Vectors $\bar{a} - \bar{b}$, $\bar{b} - \bar{c}$ and $\bar{c} - \bar{a}$ are coplanar

31. Since, the vectors are coplanar,

$\therefore$ $\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right] = 0$

$$\Rightarrow \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & p & 5 \end{vmatrix} = 0$$

$\Rightarrow 10 + p + 5 + 3 + p - 6 = 0$

$\Rightarrow p = -6$

32. Let $\bar{a} = \hat{i} + 3\hat{j} + 2\hat{k}$, $\bar{b} = 2\hat{i} - \hat{j} + 4\hat{k}$, and $\bar{c} = 3\hat{i} + 2\hat{j} + x\hat{k}$

Since the vectors are coplanar,

$$\therefore \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 3 & 2 & x \end{vmatrix} = 0$$

$\Rightarrow -x - 8 - 6x + 36 - 14 = 0$

$\Rightarrow x = 2$

33. Let $\bar{a} = \hat{i} + \hat{j} + \hat{k}, \bar{b} = -\hat{i} + \hat{j}$ and $c = \hat{i} + 2\hat{j} + a\hat{k}$

Since $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar.

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & a \end{vmatrix} = 0$$

$\Rightarrow 1(a - 0) - 1(-a - 0) + 1(-2 - 1) = 0$

$\Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$

35. We have $[\bar{a}\ \ \bar{b}\ \ \bar{a} \times \bar{b}] = \bar{a} \cdot \left[\bar{b} \times (\bar{a} \times \bar{b})\right]$

$= (\bar{a} \times \bar{b}).(\bar{a} \times \bar{b})$

$= |\bar{a} \times \bar{b}|^2$

36. $[\bar{a}\ \ \bar{c}\ \ \bar{b}] = \bar{a}.(\bar{c} \times \bar{b})$

$= \bar{c}.(\bar{b} \times \bar{a})$

$= 0$ ….[$\because$ $\bar{a}$ and $\bar{b}$ are parallel]

37. $\bar{a}.(\bar{b} \times \bar{c}) = 0$ or $(\bar{a} \times \bar{b}).\ \bar{c} = 0$

38. Volume of parallelopiped $= \left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$= \left(\frac{11}{2}\right)(12)\left(\frac{13}{3}\right)\left[\hat{i}\ \hat{j}\ \hat{k}\right]$

$= 286$ cu. unit.

39. Let $\bar{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\bar{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\bar{c} = -3\hat{i} - \hat{j} + \hat{k}$

$$\therefore \text{volume of parallelopiped} = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -3 & -1 & 1 \end{vmatrix}$$

$= 2(2 + 3) - 1(1 + 9) - 1(1 - 6)$

$= 5$ cu. unit.

40. $$\text{Volume of parallelopiped} = \begin{vmatrix} -3 & 7 & 5 \\ -3 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$= (-3)(-21-15) - 7(9 + 21) + 5(15 - 49)$

$= 108 - 210 - 170 = -272$

But volume cannot be negative.

$\therefore$ Volume of parallelopiped = 272 cu. unit.

41. A, B, C, D are vertices of tetrahedron.

$\therefore$ $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ are its edges.

Now, $\overline{AB} = -2\hat{i} - 2\hat{j} - 3\hat{k}$

$\overline{AC} = 4\hat{i} - 9\hat{k}$

$\overline{AD} = 6\hat{i} - 3\hat{j} - 3\hat{k}$

$\therefore$ Volume of tetrahedron $= \frac{1}{6}\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right]$

$$= \frac{1}{6}\begin{vmatrix} -2 & -2 & -3 \\ 4 & 0 & -9 \\ 6 & -3 & -3 \end{vmatrix}$$

$= \frac{1}{6}[-2(0 - 27) + 2(-12 + 54) - 3(-12 - 0)]$

$= \frac{1}{6}(174) = 29$ cu. unit.

42. Let A, B, C, D be the given points

$\therefore$ $\overline{AB} = 2\hat{i} - 3\hat{j} + 6\hat{k}$, $\overline{AC} = -4\hat{i} - 5\hat{j} + 9\hat{k}$ and $\overline{AD} = -6\hat{i} - 2\hat{j} + 6\hat{k}$

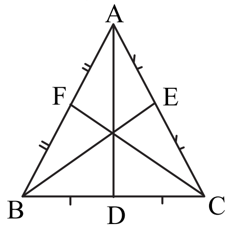
$$\therefore \text{Volume of tetrahedron} = \frac{1}{6}\begin{vmatrix} 2 & -3 & 6 \\ -4 & -5 & 9 \\ -6 & -2 & 6 \end{vmatrix}$$

$= \frac{-66}{6} = -11$

But volume cannot be negative

$\therefore$ Volume of tetrahedron = 11 cu.unit.

43. Consider ΔABC,
AD, BE and CF are its medians.



$\therefore$ $\overline{AD} + \overline{BE} + \overline{CF} = \bar{d} - \bar{a} + \bar{e} - \bar{b} + \bar{f} - \bar{c}$

$= \dfrac{\bar{b}+\bar{c}}{2} - \bar{a} + \dfrac{\bar{a}+\bar{c}}{2} - \bar{b} + \dfrac{\bar{a}+\bar{b}}{2} - \bar{c} = 0$

Critical Thinking

1. Let $A(\bar{a}), B(\bar{b}), C(\bar{c})$ be the given points
$\bar{a} = 60\hat{i} + 3\hat{j}, \ \bar{b} = 40\hat{i} - 8\hat{j}, \ \bar{c} = a\hat{i} - 52\hat{j}$
$\therefore$ $\overline{AB} = k(\overline{BC})$
$\Rightarrow -20\hat{i} - 11\hat{j} = k\left[(a-40)\hat{i} - 44\hat{j}\right]$
On comparing, we get
$-11 = -44k \Rightarrow k = \dfrac{1}{4}$
and $-20 = \dfrac{1}{4}(a-40) \Rightarrow a = -40$

2. Let $\bar{a} = \hat{i} + 2\hat{k}$, $\bar{b} = \hat{j} + \hat{k}$ and $\bar{c} = \lambda\hat{i} + \mu\hat{j}$
$\therefore$ $\overline{AB} = m.\ \overline{BC}$
$\Rightarrow -\hat{i} + \hat{j} - \hat{k} = m[(\lambda\hat{i} + (\mu-1)\hat{j} - \hat{k})]$
On comparing, we get
$-1 = -m \Rightarrow m = 1,$
$-1 = \lambda \cdot m \Rightarrow \lambda = -1,$
and $1 = m(\mu - 1) \Rightarrow \mu = 2$

3. Let $\bar{a} = -\hat{i} + 3\hat{j} + 2\hat{k}$, $\bar{b} = -4\hat{i} + 2\hat{j} - 2\hat{k}$ and
$\bar{c} = 5\hat{i} + \lambda\hat{j} + \mu\hat{k}$
$\therefore$ $\overline{AB} = m.\overline{BC}$
$\Rightarrow -3\hat{i} - \hat{j} - 4\hat{k} = m\left[9\hat{i} + (\lambda-2)\hat{j} + (\mu+2)\hat{k}\right]$
On comparing, we get
$9m = -3 \Rightarrow m = \dfrac{-1}{3}$,
$-1 = m(\lambda - 2) \Rightarrow \lambda = 5$
and $-4 = m(\mu + 2) \Rightarrow \mu = 10$

4. Here $\bar{a} = \hat{i} + x\hat{j} + 3\hat{k}$, $\bar{b} = 3\hat{i} + 4\hat{j} + 7\hat{k}$, and
$\bar{c} = y\hat{i} - 2\hat{j} - 5\hat{k}$
$\therefore$ $\overline{AB} = \lambda\overline{BC}$
$\Rightarrow 2\hat{i} + (4-x)\hat{j} + 4\hat{k} = \lambda\left[(y-3)\hat{i} - 6\hat{j} - 12\hat{k}\right]$
On comparing, we get
$4 = -12\lambda \Rightarrow \lambda = \dfrac{-1}{3}$,
$4 - x = -6\lambda \Rightarrow x = 2$, and
$2 = \lambda(y-3) \Rightarrow -6 = y - 3 \Rightarrow y = -3$

5. Here $\bar{a} = \hat{i} + \hat{j}, \ \bar{b} = \hat{i} - \hat{j}, \ \bar{c} = a\hat{i} + b\hat{j} + c\hat{k}$
The points are collinear
$\therefore$ $\overline{AB} = \lambda\overline{BC}$
$\Rightarrow -2\hat{j} = \lambda[(a-1)\hat{i} + (b+1)\hat{j} + c\hat{k}]$
On comparing, we get
$\lambda(a-1) = 0, \lambda(b+1) = -2, \lambda c = 0$
Hence a = 1, c = 0 and b is arbitrary scalar.

6. Let A, B, C be the three collinear point.
$\therefore$ $\overline{AB} = \lambda\overline{BC}$
Here, $\overline{AB} = -2b$, $\overline{BC} = (k+1)\bar{b}$
$\therefore$ $\forall\ k \in R \Rightarrow \overline{AB} = \lambda\overline{BC}$

7. Since, $\bar{a} + 2\bar{b}$ is collinear with $\bar{c}$, and $\bar{b} + 3\bar{c}$ is collinear with $\bar{a}$.
$\therefore$ $\bar{a} + 2\bar{b} = x\bar{c}$ and $\bar{b} + 3\bar{c} = y\bar{a}$ $\forall\ x, y \in R$
$\therefore$ $\bar{a} + 2\bar{b} + 6\bar{c} = (x+6)\bar{c}$
Also, $\bar{a} + 2\bar{b} + 6\bar{c} = \bar{a} + 2(\bar{b} + 3\bar{c}) = (1 + 2y)\bar{a}$
$\therefore$ $(x+6)\bar{c} = (1+2y)\bar{a}$
Since, $\bar{a}$ and $\bar{c}$ are non-collinear.
$\therefore$ $x + 6 = 0$ and $1 + 2y = 0$
$\Rightarrow x = -6$ and $y = -\dfrac{1}{2}$
Now, $\bar{a} + 2\bar{b} = x\bar{c}$
$\Rightarrow \bar{a} + 2\bar{b} + 6\bar{c} = \bar{0}$

8. $\overline{AB} = \bar{a} + \bar{b}$
$\overline{BD} = 3\bar{a} + 3\bar{b} = 3\overline{AB}$
$\therefore$ Points A, B, D are collinear.

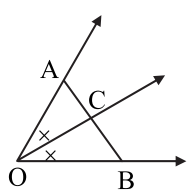
9. Let $\bar{R} = x\bar{a} + y\bar{b} + z\bar{c}$
$\Rightarrow \bar{R} = x(2\bar{p} + 3\bar{q} - \bar{r}) + y(\bar{p} - 2\bar{q} + 2\bar{r})$
$+ z(-2\bar{p} + \bar{q} - 2\bar{r})$
$\Rightarrow 3\bar{p} - \bar{q} + 2\bar{r} = (2x + y - 2z)\bar{p}$
$+ (3x - 2y + z)\bar{q} + (-x + 2y - 2z)\bar{r}$
On comparing, we get
$2x + y - 2z = 3,$ ….(i)
$3x - 2y + z = -1,$ ….(ii)
$-x + 2y - 2z = 2$ ….(iii)
Solving above equations, we get
$x = 2, y = 5, z = 3$
$\therefore$ $\bar{R} = 2\bar{a} + 5\bar{b} + 3\bar{c}$

10. $\bar{a}+\bar{b}+\bar{c}+\bar{d} = (1+\lambda)\bar{d}$
Also, $\bar{a}+\bar{b}+\bar{c}+\bar{d} = (1+\mu)\bar{a}$
$\Rightarrow (1+\lambda)\bar{d} = (1+\mu)\bar{a}$
if $\lambda \neq -1$, then
$\bar{d} = \left(\dfrac{1+\mu}{1+\lambda}\right)\bar{a}$
Now, $\bar{a} + \bar{b} + \bar{c} + \bar{d} = (1+\mu)\bar{a}$
$\therefore \quad \bar{a} + \bar{b} + \bar{c} + \left(\dfrac{1+\mu}{1+\lambda}\right)\bar{a} = (1+\mu)\bar{a}$
$\Rightarrow \left[1+\left(\dfrac{1+\mu}{1+\lambda}\right)-(1+\mu)\right]\bar{a} + \bar{b} + \bar{c} = 0$
This contradicts the fact that $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar
$\Rightarrow \lambda = -1$
$\therefore \quad \bar{a} + \bar{b} + \bar{c} + \bar{d} = \bar{0}$

11. The position vector of A is $6\bar{b}-2\bar{a}$ and the position vector of P is $\bar{a}-\bar{b}$
Let the position vector of B be $\bar{r}$
Since, P divides AB in the ratio 1 : 2
$\therefore \quad \bar{a}-\bar{b} = \dfrac{1(\bar{r})+2(6\bar{b}-2\bar{a})}{3}$
$\Rightarrow 3\bar{a}-3\bar{b}-12\bar{b}+4\bar{a} = \bar{r}$
$\Rightarrow \bar{r} = 7\bar{a}-15\bar{b}$

12. $2\bar{a}+3\bar{b}-5\bar{c} = 0$
$\Rightarrow 5\bar{c} = 2\bar{a}+3\bar{b} \Rightarrow \bar{c} = \dfrac{3\bar{b}+2\bar{a}}{3+2}$
$\therefore$ point C divides segment AB internally in the ratio 3:2.

13. $|\overline{OA}| = \sqrt{1+9+4} = \sqrt{14}$
$|\overline{OB}| = \sqrt{9+1+4} = \sqrt{14}$
$\therefore$ OA = OB
Let C be any point on angle bisector and on line AB
$\therefore$ C is midpoint of AB



$\therefore \quad \bar{c} = \dfrac{\bar{a}+\bar{b}}{2} = 2\hat{i}+2\hat{j}-2\hat{k}$

14. P($\bar{p}$) divide AB internally in the ratio 3 : 1.
$\therefore \quad \bar{p} = \dfrac{3\bar{b}+\bar{a}}{4}$
Q($\bar{q}$) is midpoint of AP
$\therefore \quad \bar{q} = \dfrac{\bar{a}+\bar{p}}{2} = \dfrac{\bar{a}+\dfrac{3\bar{b}+\bar{a}}{4}}{2} = \dfrac{5\bar{a}+3\bar{b}}{8}$

15. $2\bar{a}+\bar{b} = 3\bar{c}$
$\Rightarrow 2\bar{a} = 3\bar{c}-\bar{b}$
$\Rightarrow \bar{a} = \dfrac{3\bar{c}-\bar{b}}{2} = \dfrac{3\bar{c}-\bar{b}}{3-1}$
$\therefore$ A divides BC in the ratio 3 :1 externally.

16. P($\bar{p}$) is midpoint of BC
$\therefore \quad \bar{p} = \dfrac{\bar{b}+\bar{c}}{2}$
$\Rightarrow 2\bar{p} = \bar{b}+\bar{c}$ ….(i)
Q($\bar{q}$) divides CA internally in the ratio 2:1
$\therefore \quad \bar{q} = \dfrac{2\bar{a}+\bar{c}}{3}$
$\Rightarrow 3\bar{q} = 2\bar{a}+\bar{c}$ ….(ii)
R($\bar{r}$) divides AB externally in the ratio 1:2
$\bar{r} = \dfrac{\bar{b}-2\bar{a}}{1-2}$
$= \dfrac{2\bar{p}-3\bar{q}}{-1}$ ….[From (i) and (ii)]
$\therefore \quad \bar{r} = -2\bar{p}+3\bar{q}$
$\therefore$ points P, Q and R are collinear.

17. $[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$
Applying, $C_3 \Rightarrow C_3 + C_1$
$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix} = 1(1+x-x) = 1$

18. Let A ≡ (1,1, 2), B ≡ (2, 1, p), C ≡ (1, 0, 3) and D ≡ (2, 2, 0).
$\therefore \quad \overline{AB} = \hat{i}+(p-2)\hat{k}$
$\overline{AC} = -\hat{j}+\hat{k}$, and
$\overline{AD} = \hat{i}+\hat{j}-2\hat{k}$
The points are coplanar.
$\therefore$ $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ are coplanar
$\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right] = 0$
$\Rightarrow \begin{vmatrix} 1 & 0 & p-2 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 0$
$\Rightarrow 1(2-1)+(p-2)(1) = 0$
$\Rightarrow 1+p-2 = 0 \Rightarrow p = 1$

19. Since the points are coplanar,

$\therefore$ $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ \lambda-1 & 2 & 3 \end{vmatrix} = 0$

$\Rightarrow 1(3-8) - 2[(0-4(\lambda-1)] = 0$

$\Rightarrow -5 + 8\lambda - 8 = 0 \Rightarrow \lambda = \dfrac{13}{8}$

20. Since, the given vectors are coplanar,

$\therefore$ $\begin{vmatrix} a & 1 & 1 \\ 1 & -b & 1 \\ 1 & 1 & -c \end{vmatrix} = 0$

$\Rightarrow a(bc-1) - 1(-c-1) + 1(1+b) = 0$

$\Rightarrow abc - a + c + 1 + 1 + b = 0$

$\Rightarrow abc + 2 = a - b - c$

21. Since the given vectors are coplanar,

$\therefore$ $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = 0$

$\Rightarrow (ab+bc+ca)^3 = 0 \Rightarrow ab + bc + ca = 0.$

22. Let $P(\bar{p})$, $Q(\bar{q})$, $R(\bar{r})$ be the three points.

$\therefore$ $\bar{p} = \bar{a} - \bar{b} + \bar{c}$, $\bar{q} = 4\bar{a} - 7\bar{b} - \bar{c}$ and $\bar{r} = 3\bar{a} + 6\bar{b} + 6\bar{c}$

$\overline{PQ}$ is not scalar multiple of $\overline{PR}$

$\therefore$ they are not collinear

$\left[\bar{p}\ \ \bar{q}\ \ \bar{r}\right] = \begin{vmatrix} 1 & -1 & 1 \\ 4 & -7 & -1 \\ 3 & 6 & 6 \end{vmatrix}$

$= 36 \neq 0$

$\therefore$ they are not coplanar.

23. $\dfrac{(\bar{b}\times\bar{c}).(\bar{a}+\bar{b}+\bar{c})}{\lambda}$

$= \dfrac{(\bar{b}\times\bar{c}).\bar{a} + (\bar{b}\times\bar{c}).\bar{b} + (\bar{b}\times\bar{c}).\bar{c}}{\lambda}$

$= \dfrac{(\bar{b}\times\bar{c}).\bar{a} + \bar{0} + \bar{0}}{\lambda} = \dfrac{\bar{a}.(\bar{b}\times\bar{c})}{\lambda} = \dfrac{\lambda}{\lambda} = 1$

24. $(\bar{a} - \bar{b})\cdot[(\bar{b} + \bar{c}) \times (\bar{c} + \bar{a})]$

$= (\bar{a} - \bar{b})\cdot[\bar{b} \times \bar{c} + \bar{b} \times \bar{a} + \bar{c} \times \bar{c} + \bar{c} \times \bar{a}]$

$= \bar{a}\cdot(\bar{b} \times \bar{c}) + \bar{a}.(\bar{b} \times \bar{a}) + \bar{a}\cdot(\bar{c} \times \bar{a})$

$\quad - \bar{b}\cdot(\bar{b} \times \bar{c}) - \bar{b}\cdot(\bar{b} \times \bar{a}) - \bar{b}\cdot(\bar{c} \times \bar{a})$

$= [\bar{a}\ \bar{b}\ \bar{c}] - [\bar{a}\ \bar{b}\ \bar{c}] = 0$

25. $\left[\bar{a}+\bar{b}+\bar{c}\right] \cdot \left[\left(\bar{a}+\bar{b}\right) \times \left(\bar{a}+\bar{c}\right)\right]$

$= \left(\bar{a}+\bar{b}\right) \cdot \left[\left(\bar{a}+\bar{b}\right) \times \left(\bar{a}+\bar{c}\right)\right]$

$\quad + \bar{c} \cdot \left[\left(\bar{a}+\bar{b}\right) \times \left(\bar{a}+\bar{c}\right)\right]$

$= 0 + \left[\bar{c}\ \ \bar{a}+\bar{b}\ \ \bar{a}+\bar{c}\right]$

$= \left[\bar{c}\ \bar{a}\ \ \bar{a}+\bar{c}\right] + \left[\bar{c}\ \ \bar{b}\ \ \bar{a}+\bar{c}\right]$

$= \left[\bar{c}\ \ \bar{a}\ \ \bar{a}\right] + \left[\bar{c}\ \ \bar{a}\ \ \bar{c}\right] + \left[\bar{c}\ \ \bar{b}\ \ \bar{a}\right] + \left[\bar{c}\ \ \bar{b}\ \ \bar{c}\right]$

$= 0 + 0 + \left[\bar{c}\ \ \bar{b}\ \ \bar{a}\right] + 0 = -\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

26. $\bar{r} = l(\bar{b}\times\bar{c}) + m(\bar{c}\times\bar{a}) + n(\bar{a}\times\bar{b})$

$\therefore$ $\bar{a}\cdot\bar{r} = l\,\bar{a}.\left(\bar{b}\times\bar{c}\right) + m\,\bar{a}.\left(\bar{c}\times\bar{a}\right) + n\,\bar{a}\cdot\left(\bar{a}\times\bar{b}\right)$

$= l\left[\bar{a}\ \bar{b}\ \bar{c}\right] + 0 + 0$

$\bar{a}\cdot\bar{r} = 2l$ .... $\left[\because \left[\bar{a}\ \bar{b}\ \bar{c}\right] = 2\right]$ ….(i)

Similarly,

$\bar{b}\cdot\bar{r} = 2m,$ ….(ii)

$\bar{c}.\bar{r} = 2n$ ….(iii)

$\therefore$ On adding equations (i), (ii) and (iii) we get

$\left(\bar{a}+\bar{b}+\bar{c}\right).\bar{r} = 2(l + m + n)$

$\therefore$ $l + m + n = \dfrac{1}{2}\left(\bar{a}+\bar{b}+\bar{c}\right).\bar{r}$

27. Volume of parallelopiped

$= \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{vmatrix} \left[\bar{a}\ \bar{b}\ \bar{c}\right] = k\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$\Rightarrow 1(1-0) - 2(-1-0) - 1(-1+1) = k$

$\Rightarrow 1 + 2 - 0 = k \Rightarrow k = 3$

28. Volume of parallelopiped $= \begin{vmatrix} -p & 0 & 5 \\ 1 & -1 & q \\ 3 & -5 & 0 \end{vmatrix} = 8$

$\Rightarrow -p(0 + 5q) + 5(-5 + 3) = 8$

$\Rightarrow -5pq - 18 = 0$

$\Rightarrow 5pq + 18 = 0$

29. Let $A \equiv (1, 2, 0)$, $B \equiv (2, 0, 4)$, $C \equiv (-1, 2, 0)$ and $D \equiv (-1, 1, \lambda)$ be the vertices of the tetrahedron

$\therefore$ $\overline{AB} = \hat{i} - 2\hat{j} + 4\hat{k}$

$\overline{AC} = -2\hat{i}$

$\overline{AD} = -2\hat{i} - \hat{j} + \lambda\hat{k}$

Volume of tetrahedron = $\frac{1}{6}\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right]$

$\Rightarrow \frac{2}{3} = \frac{1}{6}\begin{vmatrix} 1 & -2 & 4 \\ -2 & 0 & 0 \\ -2 & -1 & \lambda \end{vmatrix}$

$\Rightarrow 2(-2\lambda) + 4(2) = 4$

$\Rightarrow \lambda = 1$

30. $\overline{AB} + \overline{BC} + \overline{AC} = \overline{b} - \overline{a} + \overline{c} - \overline{b} + \overline{c} - \overline{a}$

$= 2(\overline{c} - \overline{a})$

$= 2(\overline{c} - \overline{d} + \overline{d} - \overline{a})$

$= 2(\overline{DC} + \overline{AD})$

$= 2(\overline{DC} - \overline{BD})$

....[$\because$ D is mid-point of AB]

31. If AD is the median,

$\therefore$ $\overline{AD} = \overline{d} - \overline{a} = \frac{\overline{b}+\overline{c}}{2} - \overline{a}$

$= \frac{(\overline{b}-\overline{a})+(\overline{c}-\overline{a})}{2} = \frac{1}{2}(\overline{AB} + \overline{AC})$

$= 4\hat{i} - \hat{j} + 4\hat{k}$

$\therefore$ $l\,(AD) = \sqrt{16+1+16} = \sqrt{33}$

32. $\overline{AA'} + \overline{BB'} + \overline{CC'} = \overline{a'} - \overline{a} + \overline{b'} - \overline{b} + \overline{c'} - \overline{c}$

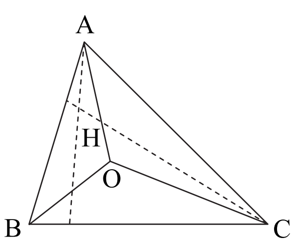
$= (\overline{a'} + \overline{b'} + \overline{c'}) - (\overline{a} + \overline{b} + \overline{c})$

$= 3\overline{g'} - 3\overline{g}$

....[$\because$ G′ and G are centroids]

$= 3\ \overline{GG'}$

33.



$\overline{HA} = \overline{HO} + \overline{OA}$ ...(i)

$\overline{HB} = \overline{HO} + \overline{OB}$ ....(ii)

$\overline{HC} = \overline{HO} + \overline{OC}$ ....(iii)

$\therefore$ Adding (i), (ii) and (iii), we get

$\overline{HA} + \overline{HB} + \overline{HC}$

$= 3\overline{HO} + \overline{OA} + \overline{OB} + \overline{OC}$

Since, $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH} = -\overline{HO}$

$\therefore$ $\overline{HA} + \overline{HB} + \overline{HC} = 2\overline{HO}$

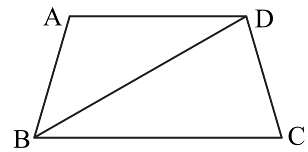
34. Let $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$ be the position vectors of A, B, C, D respectively

For parallelogram:

$\overline{a} + \overline{c} = \overline{b} + \overline{d}$

$\Rightarrow \overline{d} = \overline{a} + \overline{c} - \overline{b} \Rightarrow \overline{d} = -\hat{i} + \hat{j} + \hat{k}$

35.



We have, $\overline{p} = \overline{AC} + \overline{BD}$

$= \overline{AC} + \overline{BC} + \overline{CD}$

$= \overline{AC} + \lambda\,\overline{AD} + \overline{CD}$

$= \lambda\,\overline{AD} + (\overline{AC} + \overline{CD})$

$= \lambda\,\overline{AD} + \overline{AD}$

$= (\lambda + 1)\,\overline{AD}$

Also, $\overline{p} = \mu\overline{AD}$

$\therefore$ $\mu = \lambda + 1$

Competitive Thinking

1. Here, $\overline{a} = \hat{i}$, $\overline{b} = \hat{j}$, $\overline{c} = x\hat{i} + 8\hat{j}$

$\overline{AB} = -\hat{i} + \hat{j}$, $\overline{BC} = x\hat{i} + 7\hat{j}$

Since the points are collinear,

$\therefore$ $\overline{AB} = \lambda\ \overline{BC}$

$\Rightarrow -\hat{i} + \hat{j} = \lambda\left(x\hat{i} + 7\hat{j}\right)$

On comparing, we get

$7\lambda = 1 \Rightarrow \lambda = \frac{1}{7}$

$\lambda x = -1 \Rightarrow x = -7$

2. Let $A(\overline{a})$, $B(\overline{b})$, $C(\overline{c})$ be the given points

$\therefore$ $\overline{a} = 20\hat{i} + p\hat{j}$, $\overline{b} = 5\hat{i} - \hat{j}$, $\overline{c} = 10\hat{i} - 13\hat{j}$

$\therefore$ $\overline{AB} = k\,\overline{BC}$

$\Rightarrow -15\hat{i} - (p+1)\hat{j} = k(5\hat{i} - 12\hat{j})$

On comparing, we get

$-15 = 5k \Rightarrow k = -3$ and

$-(p+1) = -12k$

$\Rightarrow -(p+1) = 36$

$\Rightarrow p = -37$

3. $\overline{PQ} = k\,\overline{QR}$

$\overline{a} + \overline{b} - \overline{c} = k(-2\overline{a} - 2\overline{b} + t\overline{c})$

On comparing, we get

$1 = -2k \Rightarrow k = \frac{-1}{2}$ and $-1 = kt \Rightarrow t = 2$

4. Here $\overline{AB} = \overline{b} - \overline{a}$ and
$\overline{AC} = 2\overline{a} - 2\overline{b} = -2(\overline{b} - \overline{a})$
$\therefore$ $\overline{AC} = m\overline{AB}$
Hence A, B, C are collinear.

5. Since, $\overline{a} + 3\overline{b}$ is collinear with $\overline{c}$, and $\overline{b} + 2\overline{c}$ is collinear with $\overline{a}$,
$\therefore$ $\overline{a} + 3\overline{b} = x\overline{c}$ and $\overline{b} + 2\overline{c} = y\overline{a}$ $\forall\, x, y \in R$.
$\therefore$ $\overline{a} + 3\overline{b} + 6\overline{c} = (x+6)\overline{c}$
Also, $\overline{a} + 3\overline{b} + 6\overline{c} = \overline{a} + 3(\overline{b} + 2\overline{c}) = (1+3y)\overline{a}$
$\therefore$ $(x+6)\overline{c} = (1+3y)\overline{a}$
$\Rightarrow (x+6)\overline{c} - (1+3y)\overline{a} = 0$
$\therefore$ $x + 6 = 0$ and $1 + 3y = 0$
$\Rightarrow x = -6$ and $y = -\dfrac{1}{3}$
Now, $\overline{a} + 3\overline{b} = x\overline{c} \Rightarrow \overline{a} + 3\overline{b} + 6\overline{c} = \overline{0}$

6. Let $\overline{a} = 3\hat{i} + 2\hat{j} - \hat{k}$ and $\overline{b} = 6\hat{i} - 4x\hat{j} + y\hat{k}$
Since, $\overline{a}$ and $\overline{b}$ are parallel,
$\dfrac{3}{6} = \dfrac{2}{-4x} = \dfrac{-1}{y}$
$\Rightarrow x = -1$ and $y = -2$

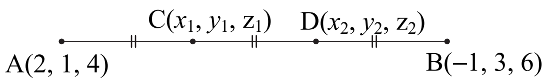
7. The given vectors are collinear.
$\therefore$ $\dfrac{3}{a} = \dfrac{1}{b} = \dfrac{-5}{-15}$
$\Rightarrow a = 9, b = 3$

8. $x = 0$, $y = 0$, otherwise one vector will be a scalar multiple of the other and hence collinear which is a contradiction.

11. $\overline{c} = m\overline{a} + n\overline{b}$
$\therefore$ $3\hat{i} - \hat{k} = m(\hat{i} + \hat{j} - 2\hat{k}) + n(2\hat{i} - \hat{j} + \hat{k})$
Comparing the co-efficients of $\hat{i}$ and $\hat{j}$, we get
$3 = m + 2n$, and ….(i)
$m = n$ ….(ii)
$\therefore$ Solving the above two equations, we get
$m = n = 1$
$\therefore$ $m + n = 1 + 1 = 2$

12. Let $P(\overline{p})$ divide the line internally in the ratio 2 : 3
$\therefore$ $\overline{p} = \dfrac{3(2\overline{a} - 3\overline{b}) + 2(3\overline{a} - 2\overline{b})}{2+3} = \dfrac{12\overline{a} - 13\overline{b}}{5}$

13. $A \equiv (1, -1, 2)$, $B \equiv (2, 3, -1)$
Point P divides AB internally in the ratio 2 : 3.
$\therefore$ $P \equiv \left[\dfrac{2(2)+3(1)}{2+3}, \dfrac{2(3)+3(-1)}{2+3}, \dfrac{2(-1)+3(2)}{2+3}\right]$
$\equiv \left(\dfrac{7}{5}, \dfrac{3}{5}, \dfrac{4}{5}\right)$
$\therefore$ the position vector of P is $\dfrac{1}{5}(7\hat{i} + 3\hat{j} + 4\hat{k})$

14.


C divides AB internally in the ratio 1 : 2 and D divides AB internally in the ratio 2 : 1.
$\therefore$ $z_1 + z_2 = \dfrac{1(6)+2(4)}{1+2} + \dfrac{2(6)+1(4)}{2+1}$
$= \dfrac{14}{3} + \dfrac{16}{3} = \dfrac{30}{3}$
$= 10$

15. Let position vector of B be $\overline{r}$
Since, $\overline{a}$ divides AB in the ratio 2 : 3
$\therefore$ $\dfrac{2\overline{r} + 3(\overline{a} + 2\overline{b})}{2+3} = \overline{a}$
$\Rightarrow 2\overline{r} = 5\overline{a} - 3\overline{a} - 6\overline{b} = 2\overline{a} - 6\overline{b}$
$\Rightarrow \overline{r} = \overline{a} - 3\overline{b}$

16. We know that, centroid of a triangle divides the line segment joining the orthocentre and circumcentre in the ratio 2 : 1.
The co-ordinates of orthocentre and circumcentre are (–1, 3, 2), (5, 3, 2) respectively.
$\therefore$ Co-ordinates of centroid
$\equiv \left(\dfrac{2(5)+1(-1)}{2+1}, \dfrac{2(3)+1(3)}{2+1}, \dfrac{2(2)+1(2)}{2+1}\right)$
$\equiv (3, 3, 2)$

17. Let the co-ordinates of circumcentre be (x, y, z).
Co-ordinates of orthocentre and centroid are (–3, 5, 2) and (3, 3, 4) respectively.
We know that, centroid of triangle divides the line segment joining its orthocentre and circumcentre in the ratio 2 : 1.
$\therefore$ $\left(\dfrac{2x-3}{3}, \dfrac{2y+5}{3}, \dfrac{2z+2}{3}\right) \equiv (3, 3, 4)$
$\Rightarrow \dfrac{2x-3}{3} = 3, \dfrac{2y+5}{3} = 3, \dfrac{2z+2}{3} = 4$
$\Rightarrow x = 6, y = 2, z = 5$

18. Let N $(\bar{n})$ divide line segment LM externally in the ratio 2 : 1.

$\therefore$ $\bar{n} = \dfrac{2(\bar{a} + 2\bar{b}) - (2\bar{a} - \bar{b})}{2 - 1}$

$= \dfrac{2\bar{a} + 4\bar{b} - 2\bar{a} + \bar{b}}{1} = 5\,\bar{b}$

19. $R(\bar{r})$ divides PQ externally in the ratio 2 : 1

$\therefore$ $\bar{r} = \dfrac{2(-\hat{i} + \hat{j} - \hat{k}) - 1(\hat{i} + 2\hat{j} - \hat{k})}{2 - 1}$

$= -2\hat{i} + 2\hat{j} - 2\hat{k} - \hat{i} - 2\hat{j} + \hat{k}$

$\therefore$ $\bar{r} = -3\hat{i} - \hat{k}$

20. $3\bar{P} + 2\bar{R} - 5\bar{Q} = \bar{0}$

$\Rightarrow 3\bar{P} + 2\bar{R} = 5\bar{Q}$

$\Rightarrow \bar{Q} = \dfrac{3\bar{P} + 2\bar{R}}{5}$

$\therefore$ $\bar{Q}$ is the position vector of the point dividing P and R in the ratio 3 : 2 internally.
Thus, P, Q and R are collinear.

21. Let the point B divide AC in the ratio $\lambda : 1$

$\therefore$ $5\hat{i} - 2\hat{k} = \dfrac{\lambda(11\hat{i} + 3\hat{j} + 7\hat{k}) + \hat{i} - 2\hat{j} - 8\hat{k}}{\lambda + 1}$

$\Rightarrow \lambda(5\hat{i} - 2\hat{k}) + (5\hat{i} - 2\hat{k})$

$= \lambda(11\hat{i} + 3\hat{j} + 7\hat{k}) + (\hat{i} - 2\hat{j} - 8\hat{k})$

$\Rightarrow -6\lambda\hat{i} - 3\lambda\hat{j} - 9\lambda\hat{k} = -4\hat{i} - 2\hat{j} - 6\hat{k}$

$\Rightarrow -6\lambda = -4$

$\Rightarrow \lambda = \dfrac{2}{3}$ i.e. ratio = 2 : 3

22. M and N are the midpoints of sides PQ and RS

$\therefore$ $\bar{m} = \dfrac{\bar{p} + \bar{q}}{2}$ and $\bar{n} = \dfrac{\bar{r} + \bar{s}}{2}$

$\Rightarrow 2\bar{m} = \bar{p} + \bar{q}$ and $2\bar{n} = \bar{r} + \bar{s}$

$\overline{PS} + \overline{QR} = \bar{s} - \bar{p} + \bar{r} - \bar{q}$

$= \bar{r} + \bar{s} - (\bar{p} + \bar{q})$

$= 2\bar{n} - 2\bar{m}$

$= 2\,\overline{MN}$

23. Let the position vectors of A, B, C, L, M, N and K be $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{l}$, $\bar{m}$, $\bar{n}$ and $\bar{k}$ respectively.

$\bar{l} = \dfrac{2\bar{b} + \bar{c}}{3}$, $\bar{m} = \dfrac{2\bar{a} + 3\bar{c}}{5}$, $\bar{n} = \dfrac{3\bar{b} + 5\bar{a}}{8}$,

$\bar{k} = \dfrac{5\bar{b} + 3\bar{a}}{8}$

$\dfrac{\left|\overline{AL} + \overline{BM} + \overline{CN}\right|}{\left|\overline{CK}\right|}$

$= \dfrac{\left|\dfrac{2\bar{b} + \bar{c}}{3} - \bar{a} + \dfrac{2\bar{a} + 3\bar{c}}{5} - \bar{b} + \dfrac{3\bar{b} + 5\bar{a}}{8} - \bar{c}\right|}{\left|\dfrac{5\bar{b} + 3\bar{a}}{8} - \bar{c}\right|}$

$= \dfrac{\left|\dfrac{1}{120}\left[80\bar{b} + 40\bar{c} - 120\bar{a} + 48\bar{a} + 72\bar{c} - 120\bar{b} + 45\bar{b} + 75\bar{a} - 120\bar{c}\right]\right|}{\left|\dfrac{1}{8}\left[5\bar{b} + 3\bar{a} - 8\bar{c}\right]\right|}$

$= \dfrac{\left|\dfrac{1}{120}\left[3\bar{a} + 5\bar{b} - 8\bar{c}\right]\right|}{\left|\dfrac{1}{8}\left[3\bar{a} + 5\bar{b} - 8\bar{c}\right]\right|}$

$= \dfrac{1}{15}$

24. G is the centroid.

$\therefore$ $\overline{OG} = \dfrac{\overline{OA} + \overline{OB} + \overline{OC}}{3}$

$\Rightarrow \overline{OA} + \overline{OB} + \overline{OC} = 3\,\overline{OG}$

25. $\overline{GA} + \overline{GB} + \overline{GC} = (\bar{a} - \bar{g}) + (\bar{b} - \bar{g}) + (\bar{c} - \bar{g})$

$= \bar{a} + \bar{b} + \bar{c} - 3\bar{g}$

$= \bar{a} + \bar{b} + \bar{c} - 3\left(\dfrac{\bar{a} + \bar{b} + \bar{c}}{3}\right) = \bar{0}$

26. $\bar{a}.(\bar{b} \times \bar{c}) = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix}$

$= 2(4 + 1) - 1(2 - 1) - 1(-1 - 2)$
$= 12$

27. $\bar{a}.(\bar{b} \times \bar{c}) = 10$

$\therefore$ $\begin{vmatrix} 1 & 1 & 1 \\ 2 & \lambda & 1 \\ 1 & -1 & 4 \end{vmatrix} = 10$

$\Rightarrow (4\lambda + 1) - (8 - 1) + (-2 - \lambda) = 10$
$\Rightarrow \lambda = 6$

28. Let $\hat{n}$ be the unit vector perpendicular to $\bar{a}$ and $\bar{b}$

$\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \bar{a}.(\bar{b} \times \bar{c}) = \bar{a}.(|\bar{b}||\bar{c}|\sin\theta\ \hat{n})$

$= \bar{a}.(3 \times 4 \sin \frac{2\pi}{3}.\hat{n}) = \bar{a}.\left(12 \times \frac{\sqrt{3}}{2}\hat{n}\right)$

$= 6\sqrt{3}|\bar{a}||\hat{n}|\cos 0 = 6\sqrt{3} \times 2 \times 1 \Rightarrow 12\sqrt{3}$.

29. $|\bar{a}| = 1$, $|\bar{b}| = 2$, $|\bar{c}| = 3$

$[\bar{a} + \bar{b} + \bar{c}\quad \bar{b} - \bar{a}\quad \bar{c}]$

$= \left(\bar{a} + \bar{b} + \bar{c}\right) \cdot [\left(\bar{b} - \bar{a}\right) \times \bar{c}]$

$= \left(\bar{a} + \bar{b} + \bar{c}\right).\ \left(\bar{b} \times \bar{c} - \bar{a} \times \bar{c}\right)$

$= [\bar{a}\ \bar{b}\ \bar{c}] - [\bar{b}\ \bar{a}\ \bar{c}]$

$= 2\ [\bar{a}\ \bar{b}\ \bar{c}]$

$= 2\ \bar{a}.\left(\bar{b} \times \bar{c}\right)$

$= 2|\bar{a}|.\ |\bar{b} \times \bar{c}|\ \cos 0°$

$= 2|\bar{a}|.\ |\bar{b} \times \bar{c}|$

$= 2\ |\bar{a}|.\ |\bar{b}|.\ |\bar{c}|\ \sin 90°$

$= 2 \times 1 \times 2 \times 3 = 12$

30. $\left[\bar{a} - \bar{b}\ \ \bar{b} - \bar{c}\ \ \bar{c} - \bar{a}\right] = \left[\bar{a}\ \bar{b}\ \bar{c}\right] - \left[\bar{b}\ \bar{c}\ \bar{a}\right]$

$= \left[\bar{a}\ \bar{b}\ \bar{c}\right] - \left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$

31. $\left(\bar{a} - \bar{b}\right).\left[\left(\bar{b} - \bar{c}\right) \times \left(\bar{c} - \bar{a}\right)\right]$

$= \left[\bar{a} - \bar{b}\ \ \bar{b} - \bar{c}\ \ \bar{c} - \bar{a}\right] = 0$

32. Vector $\bar{\alpha}$ lies in the plane of $\bar{\beta}$ and $\bar{\gamma}$

$\therefore$ $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are coplanar

$\Rightarrow [\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}] = 0$

33. $\left[\bar{a} + \bar{b}\ \ \bar{b} + \bar{c}\ \ \bar{c} + \bar{a}\right] = 2\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$= 0$ ....[$\because$ $\bar{a}, \bar{b}, \bar{c}$ are coplanar]

34. $\left[\bar{a} + \bar{b}\ \ \bar{b} + \bar{c}\ \ \bar{c} + \bar{a}\right] = 2\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

35. Here $\bar{C} = C_1\hat{i} - \hat{j} + \hat{k}$

To make three vectors coplanar $[\bar{A}\,\bar{B}\,\bar{C}] = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ C_1 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow 1(0 - 0) - 1(1 - 0) + 1(-1 - 0) = 0$

$\therefore$ The value of $[\bar{A}\,\bar{B}\,\bar{C}]$ is independent of C_1

Hence no value of C_1 can be found.

36. Let $\bar{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\bar{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\bar{c} = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$

Since, $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar,

$\therefore$ $\left[\bar{a}\,\bar{b}\,\bar{c}\right] = 0$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$$

$\Rightarrow 2(10 + 3\lambda) + 1(5 + 9) + 1\ (\lambda - 6) = 0$

$\Rightarrow \lambda = -4$

37. Let $\bar{a}, \bar{b}$ and $\bar{c}$ be the given vectors

The given vectors are coplanar

$$\therefore \begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda(\lambda^2 - 1) - (\lambda + 2) + 2(-1 - 2\lambda) = 0$

$\Rightarrow \lambda^3 - 6\lambda - 4 = 0$

$\Rightarrow (\lambda + 2)(\lambda^2 - 2\lambda - 2) = 0$

$\Rightarrow \lambda = -2$ or $\lambda = \dfrac{2 \pm \sqrt{4 + 8}}{2} = 1 \pm \sqrt{3}$

38. Let $\bar{a} = 4\hat{i} + 11\hat{j} + m\hat{k}$, $\bar{b} = 7\hat{i} + 2\hat{j} + 6\hat{k}$ and $\bar{c} = \hat{i} + 5\hat{j} + 4\hat{k}$.

Since $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar,

$\therefore$ $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

$$\Rightarrow \begin{vmatrix} 4 & 11 & m \\ 7 & 2 & 6 \\ 1 & 5 & 4 \end{vmatrix} = 0$$

$\Rightarrow 4\ (8 - 30) - 11\ (28 - 6) + m\ (35 - 2) = 0$

$\Rightarrow -330 + 33m = 0$

$\Rightarrow m = 10$

39. Here $\bar{a} = 2\hat{i} + 2\hat{j} + 6\hat{k}$, $\bar{b} = 2\hat{i} + \lambda\hat{j} + 6\hat{k}$, $\bar{c} = 2\hat{i} - 3\hat{j} + \hat{k}$

Since $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar,

$$\therefore \begin{vmatrix} 2 & 2 & 6 \\ 2 & \lambda & 6 \\ 2 & -3 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2\ (\lambda + 18) - 2\ (2 - 12) + 6\ (-6 - 2\lambda) = 0$

$\Rightarrow -10\lambda = -20$

$\Rightarrow \lambda = 2$

40. Let $\bar{a} = -2\hat{i} + \hat{j} + \hat{k}$, $\bar{b} = \hat{i} + \hat{j} + \hat{k}$, $\bar{c} = \hat{j} - \hat{k}$ and $\bar{d} = \lambda\hat{j} + \hat{k}$

Since the given points are coplanar.

$\therefore$ $\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right] = 0$

$\Rightarrow \begin{vmatrix} 3 & 0 & 0 \\ 2 & 0 & -2 \\ 2 & \lambda-1 & 0 \end{vmatrix} = 0$

$\Rightarrow 3(2\lambda - 2) + 0 + 0 = 0$

$\Rightarrow 6\lambda - 6 = 0$

$\Rightarrow \lambda = 1$

41. Since $\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\bar{c} = x\hat{i} + (x-2)\hat{j} - \hat{k}$ are coplanar vectors

$\therefore$ $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$

$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x-2 & -1 \end{vmatrix} = 0$

$\Rightarrow 1\,[1 - 2(x-2)] - 1\,(-1 - 2x) + 1(x - 2 + x) = 0$

$\Rightarrow 1 - 2x + 4 + 1 + 2x + 2x - 2 = 0$

$\Rightarrow 2x = -4$

$\Rightarrow x = -2$

42. Let $\bar{a} = 3\hat{i} - 2\hat{j} - \hat{k}$, $\bar{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$, $\bar{c} = -\hat{i} + \hat{j} + 2\hat{k}$ and $\bar{d} = 4\hat{i} + 5\hat{j} + \lambda\hat{k}$

Since, the given points are coplanar,

$\therefore$ $\left[\overline{AB}\quad \overline{AC}\quad \overline{AD}\right] = 0$

$\Rightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix} = 0$

$\Rightarrow -1(3\lambda + 3 - 21) - 5(-4\lambda - 4 - 3)$
$-3(-28 - 3) = 0$

$\Rightarrow -3\lambda + 18 + 20\lambda + 35 + 93 = 0$

$\Rightarrow 17\,\lambda = -146$

$\Rightarrow \lambda = \dfrac{-146}{17}$

43. Let $\bar{s} = 2\bar{a} + 3\bar{b} - \bar{c}$, $\bar{t} = \bar{a} - 2\bar{b} + 3\bar{c}$, $\bar{u} = 3\bar{a} + 4\bar{b} - 2\bar{c}$, $\bar{v} = k\bar{a} - 6\bar{b} + 6\bar{c}$

$\therefore$ $\overline{ST} = -\bar{a} - 5\bar{b} + 4\bar{c}$, $\overline{SU} = \bar{a} + \bar{b} - \bar{c}$

$\overline{SV} = (k - 2)\bar{a} - 9\bar{b} + 7\bar{c}$

Since, the given points are coplanar,

$\therefore$ $\left[\overline{ST}\ \overline{SU}\ \overline{SV}\right] = 0$

$\Rightarrow \begin{vmatrix} -1 & -5 & 4 \\ 1 & 1 & -1 \\ k-2 & -9 & 7 \end{vmatrix} = 0$

$\Rightarrow 2 + 5\,(7 + k - 2) + 4\,(-9 - k + 2) = 0$

$\Rightarrow 2 + 25 + 5k - 28 - 4k = 0$

$\Rightarrow -1 + k = 0$

$\Rightarrow k = 1$

44. Since, $a\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ are coplanar,

$\therefore$ $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$\Rightarrow a\,(bc - 1) - 1\,(c - 1) + 1\,(1 - b) = 0$

$\Rightarrow abc - a - b - c + 2 = 0$

$\Rightarrow abc - (a + b + c) = -2$

45. Let $\bar{a}, \bar{b}$ and $\bar{c}$ be the given vectors.

The vectors are coplanar

$\therefore$ $\begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$

$\Rightarrow -\lambda^2(\lambda^4 - 1) - 1(-\lambda^2 - 1) + 1(1 + \lambda^2) = 0$

$\Rightarrow \lambda^6 - 3\lambda^2 - 2 = 0$

$\Rightarrow (1 + \lambda^2)^2\,(\lambda^2 - 2) = 0$

$\Rightarrow \lambda = \pm\sqrt{2}$

46. The given vectors are coplanar

$\therefore$ $\begin{vmatrix} \lambda^3 & 0 & 1 \\ 1 & -\lambda^3 & 0 \\ 1 & 2\lambda - \sin\lambda & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3(\lambda^4 - 0) + 1(2\lambda - \sin\lambda + \lambda^3) = 0$

$\Rightarrow \lambda^7 + \lambda^3 + 2\lambda = \sin\lambda$ ….(i)

This is true for $\lambda = 0$.

For non-zero values of λ, equation (i) is

$\lambda^6 + \lambda^2 + 2 = \dfrac{\sin\lambda}{\lambda}$ ....(ii)

We know that $\dfrac{\sin x}{x} < 1$ for all $x \neq 0$.

$\therefore$ L.H.S. of (ii) is greater than 2 and R.H.S. is less than 1.

So, (ii) is not true for any non-zero λ.

Hence, there is only one value of λ.

47. Let $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ be the given vectors
$\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ are coplanar

$\therefore \begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & (2\lambda-1) \end{vmatrix} = 0$

$\Rightarrow \lambda(2\lambda - 1) = 0 \Rightarrow \lambda = 0, \dfrac{1}{2}$

Hence, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are non-coplanar for all values of λ except 0 and $\dfrac{1}{2}$.

48. Since, O(0, 0, 0), P(2, 3, 4), Q(1, 2, 3), R(x, y, z) are co-planar

$\therefore \left[\overline{OR}\ \ \overline{OP}\ \ \overline{OQ}\right] = 0$

$\Rightarrow \begin{vmatrix} x & y & z \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} = 0$

$\Rightarrow x(9-8) - y(6-4) + z(4-3) = 0$
$\Rightarrow x - 2y + z = 0$

49. Let the vector be $a\hat{i} + b\hat{j} + c\hat{k}$.
It is perpendicular to $2\hat{i} + \hat{j} + \hat{k}$.
$\therefore$ $2a + b + c = 0$ ….(i)
The vector is coplanar with $\hat{i} + 2\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + 2\hat{k}$

$\therefore \begin{vmatrix} a & b & c \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0$

$\therefore$ $3a - b - c = 0$ ….(ii)
On solving (i) and (ii), we get
$a = 0$, $b = 5$, $c = -5$
$\therefore$ The required vector is $5(\hat{j} - \hat{k})$

50. $\left[\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\right] = \begin{vmatrix} 5 & 6 & 7 \\ 7 & -8 & 9 \\ 3 & 20 & 5 \end{vmatrix}$

$= 5(-40 - 180) - 6(35 - 27) + 7(140 + 24) = 0$
$\therefore$ the given vectors are coplanar.

51. Since $\bar{x}$ is a non-zero vector, the given conditions will be satisfied, if either
i. at least one of the vectors $\bar{a}, \bar{b}, \bar{c}$ is zero or
ii. $\bar{x}$ is perpendicular to all the vectors $\bar{a}, \bar{b}, \bar{c}$

In case (ii), $\bar{a}, \bar{b}, \bar{c}$ are coplanar
$\Rightarrow \left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$

54. options (A), (B) and (D) = $\left[\bar{u}\ \bar{v}\ \bar{w}\right]$,
while option (C) $= -\left[\bar{u}\ \bar{v}\ \bar{w}\right]$

55. $\bar{a}.(\bar{a} \times \bar{b}) = (\bar{a} \times \bar{a}).b = 0$

56. $\dfrac{\bar{a}.\bar{b}\times\bar{c}}{\bar{c}\times\bar{a}.\bar{b}} + \dfrac{\bar{b}.\bar{a}\times\bar{c}}{\bar{c}.\bar{a}\times\bar{b}} = \dfrac{\bar{a}.\bar{b}\times\bar{c}}{\bar{c}.\bar{a}\times\bar{b}} + \dfrac{\bar{b}.\bar{a}\times\bar{c}}{\bar{c}.\bar{a}\times\bar{b}}$

$= \dfrac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{c}\ \bar{a}\ \bar{b}]} + \dfrac{[\bar{b}\ \bar{a}\ \bar{c}]}{[\bar{c}\ \bar{a}\ \bar{b}]}$

$= \dfrac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{c}\ \bar{a}\ \bar{b}]} - \dfrac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{c}\ \bar{a}\ \bar{b}]} = 0$

57. $\bar{a}, \bar{b}$ and $\bar{c}$ are non-coplanar.
So, $\left[\bar{a}\,\bar{b}\,\bar{c}\right] \neq 0$

$\bar{a}.\left\{\dfrac{\bar{b}\times\bar{c}}{3\bar{b}.(\bar{c}\times\bar{a})}\right\} - \bar{b}.\left\{\dfrac{\bar{c}\times\bar{a}}{2\bar{c}.(\bar{a}\times\bar{b})}\right\}$

$= \dfrac{[\bar{a}\,\bar{b}\,\bar{c}]}{3[\bar{b}\,\bar{c}\,\bar{a}]} - \dfrac{[\bar{b}\,\bar{c}\,\bar{a}]}{2[\bar{c}\,\bar{a}\,\bar{b}]} = \dfrac{1}{3} - \dfrac{1}{2} = -\dfrac{1}{6}$

58. $\bar{a}.[(\bar{b}+\bar{c})\times(\bar{a}+\bar{b}+\bar{c})]$

$= \bar{a}.\left[\bar{b}\times\bar{a} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a} + \bar{c}\times\bar{b}\right]$

....$\left[\because \bar{b}\times\bar{b} = 0, \bar{c}\times\bar{c} = 0\right]$

$= [\bar{a}\ \bar{b}\ \bar{a}] + [\bar{a}\ \bar{b}\ \bar{c}] + [\bar{a}\ \bar{c}\ \bar{a}] + [\bar{a}\ \bar{c}\ \bar{b}]$
$= 0 + [\bar{a}\ \bar{b}\ \bar{c}] + 0 - [\bar{a}\ \bar{b}\ \bar{c}]$
$= 0$

59. $(\bar{a}+\bar{b}).(\bar{b}+\bar{c})\times(\bar{a}+\bar{b}+\bar{c})$

$= (\bar{a}+\bar{b}).\left[\bar{b}\times\bar{a} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a} + \bar{c}\times\bar{b}\right]$

$= \left[\bar{a}\,\bar{b}\,\bar{a}\right] + \left[\bar{a}\,\bar{b}\,\bar{c}\right] + \left[\bar{a}\,\bar{c}\,\bar{a}\right] + \left[\bar{a}\,\bar{c}\,\bar{b}\right]$
$\quad + \left[\bar{b}\,\bar{b}\,\bar{a}\right] + \left[\bar{b}\,\bar{b}\,\bar{c}\right] + \left[\bar{b}\,\bar{c}\,\bar{a}\right] + \left[\bar{b}\,\bar{c}\,\bar{b}\right]$

$= 0 + \left[\bar{a}\,\bar{b}\,\bar{c}\right] + 0 + \left[\bar{a}\,\bar{c}\,\bar{b}\right] + 0 + 0 + \left[\bar{b}\,\bar{c}\,\bar{a}\right] + 0$

$= \left[\bar{a}\,\bar{b}\,\bar{c}\right] - \left[\bar{a}\,\bar{b}\,\bar{c}\right] + \left[\bar{a}\,\bar{b}\,\bar{c}\right] = \left[\bar{a}\,\bar{b}\,\bar{c}\right]$

60. Since, $\bar{a}.\bar{b} = 0$
$\therefore$ $\bar{a}$ and $\bar{b}$ are perpendicular unit vectors.

Now, $\left(2\bar{a}-\bar{b}\right).\left\{\left(\bar{a}\times\bar{b}\right)\times\left(\bar{a}+2\bar{b}\right)\right\}$

$= \left[2\bar{a}-\bar{b}\ \ \bar{a}\times\bar{b}\ \ \bar{a}+2\bar{b}\right]$

$= -\left[\bar{a}\times\bar{b}\ \ 2\bar{a}-\bar{b}\ \ \bar{a}+2\bar{b}\right]$

$= -\left(\bar{a}\times\bar{b}\right).\left\{\left(2\bar{a}-\bar{b}\right)\times\left(\bar{a}+2\bar{b}\right)\right\}$

$= -(\bar{a}\times\bar{b}).5(\bar{a}\times\bar{b})$

$= -5|\bar{a}\times\bar{b}| = -5|\bar{a}|^2|\bar{b}|^2$ ....$[\because \bar{a}\perp\bar{b}]$

$= -5$ ....$[\because |\bar{a}| = |\bar{b}| = 1]$

61. $\bar{p}+\bar{q}+\bar{r} = \dfrac{\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$

$(\bar{a}+\bar{b}+\bar{c}).(\bar{p}+\bar{q}+\bar{r}) = \dfrac{[\bar{a}\ \bar{b}\ \bar{c}]+[\bar{b}\ \bar{c}\ \bar{a}]+[\bar{c}\ \bar{a}\ \bar{b}]}{[\bar{a}\ \bar{b}\ \bar{c}]}$

$= 3$

62. $(\bar{u}+\bar{v}-\bar{w})\cdot[(\bar{u}-\bar{v})\times(\bar{v}-\bar{w})]$

$= \bar{u}\cdot(\bar{u}\times\bar{v}) - \bar{u}\cdot(\bar{u}\times\bar{w}) + \bar{u}\cdot(\bar{v}\times\bar{w}) + \bar{v}\cdot(\bar{u}\times\bar{v})$

$- \bar{v}\cdot(\bar{u}\times\bar{w}) + \bar{v}\cdot(\bar{v}\times\bar{w}) - \bar{w}\cdot(\bar{u}\times\bar{v})$

$+ \bar{w}\cdot(\bar{u}\times\bar{w}) - \bar{w}\cdot(\bar{v}\times\bar{w})$

$= [\bar{u}\ \bar{v}\ \bar{w}] - [\bar{v}\,\bar{u}\,\bar{w}] - [\bar{w}\,\bar{u}\,\bar{v}]$

$= [\bar{u}\ \bar{v}\ \bar{w}] + [\bar{u}\,\bar{v}\,\bar{w}] - [\bar{u}\ \bar{v}\,\bar{w}] = \bar{u}\cdot(\bar{v}\times\bar{w})$

63. $\bar{p}.(\bar{a}+\bar{b}) = \bar{p}.\bar{a}+\bar{p}.\bar{b}$

$= \dfrac{(\bar{b}\times\bar{c}).\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]} + \dfrac{(\bar{b}\times\bar{c}).\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$

$= \dfrac{[\bar{b}\ \bar{c}\ \bar{a}]}{[\bar{a}\ \bar{b}\ \bar{c}]} + \dfrac{[\bar{b}\ \bar{c}\ \bar{b}]}{[\bar{a}\ \bar{b}\ \bar{c}]}$

$= 1 + 0$

$= 1$

Similarly, $\bar{q}.(\bar{b}+\bar{c}) = 1$ and $\bar{r}.(\bar{a}+\bar{c}) = 1$

$(\bar{a}+\bar{b}).\bar{p} + (\bar{b}+\bar{c}).\bar{q} + (\bar{c}+\bar{a}).\bar{r}$

$= 1+1+1 = 3$

64. Since $d = \lambda a + \mu b + \nu c$

$\therefore$ $d.(b\times c) = \lambda a.(b\times c) + \mu b.(b\times c) + \nu c.(b\times c)$

$\Rightarrow d.(b\times c) = \lambda[a\ b\ c]$

$\Rightarrow \lambda = \dfrac{[d\,b\,c]}{[a\,b\,c]} = \dfrac{[b\,c\,d]}{[b\,c\,a]}$

65. $[3\bar{u}\ p\bar{v}\ p\bar{w}] - [p\bar{v}\ \bar{w}\ q\bar{u}] - [2\bar{w}\ q\bar{v}\ q\bar{u}] = 0$

$\Rightarrow 3p^2[\bar{u}\ \bar{v}\,\bar{w}] - pq[\bar{v}\,\bar{w}\,\bar{u}] - 2q^2[\bar{w}\,\bar{v}\,\bar{u}] = 0$

$\Rightarrow 3p^2[\bar{u}\,\bar{v}\,\bar{w}] - pq[\bar{u}\,\bar{v}\,\bar{w}] + 2q^2[\bar{u}\,\bar{v}\,\bar{w}] = 0$

$\Rightarrow (3p^2 - pq + 2q^2)[\bar{u}\ \bar{v}\ \bar{w}] = 0$

But, $[\bar{u}\,\bar{v}\,\bar{w}] \neq 0$

$\therefore$ $3p^2 - pq + 2q^2 = 0$

$\Rightarrow p^2 - \dfrac{1}{3}pq + \dfrac{2}{3}q^2 = 0$

$\Rightarrow \left(p^2 - \dfrac{1}{3}pq + \dfrac{1}{36}q^2\right) - \dfrac{1}{36}q^2 + \dfrac{2}{3}q^2 = 0$

$\Rightarrow \left(p - \dfrac{q}{6}\right)^2 + \dfrac{23}{36}q^2 = 0$

$\Rightarrow p - \dfrac{q}{6} = 0,\ q = 0$

$\Rightarrow p = 0,\ q = 0$

Hence, there is exactly one value of (p, q).

66. $[\lambda(\bar{a}+\bar{b})\ \ \lambda^2\bar{b}\ \ \lambda\bar{c}] = [\bar{a}\ \ \bar{b}+\bar{c}\ \ \bar{b}]$

$\Rightarrow \lambda^4[\bar{a}+\bar{b}\ \ \bar{b}\ \ \bar{c}] = [\bar{a}\ \bar{b}+\bar{c}\ \bar{b}]$

$\Rightarrow \lambda^4\{[\bar{a}\,\bar{b}\,\bar{c}] + [\bar{b}\,\bar{b}\,\bar{c}]\} = \{[\bar{a}\,\bar{b}\,\bar{b}] + [\bar{a}\,\bar{c}\,\bar{b}]\}$

$\Rightarrow \lambda^4[\bar{a}\,\bar{b}\,\bar{c}] = -[\bar{a}\,\bar{b}\,\bar{c}]$

$\Rightarrow (\lambda^4 + 1)[\bar{a}\,\bar{b}\,\bar{c}] = 0$

But, $[\bar{a}\,\bar{b}\,\bar{c}] \neq 0.$

$\therefore$ $\lambda^4 + 1 = 0$

This is not true for any real value of λ.

67. Let $\bar{a} = 2\hat{i} - 3\hat{j}$, $\bar{b} = \hat{i} + \hat{j} - \hat{k}$ and $\bar{c} = 3\hat{i} - \hat{k}$

Volume of parallelopiped $= [\bar{a}\ \bar{b}\ \bar{c}]$

$= \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$

$= 2(-1) + 3(-1+3) = 4$ cu.unit.

68. Volume of parallelopiped $= \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$

$= 2(1-2) + 3(-1-4) + 1(1+2) = -14$

But, volume cannot be negative.

$\therefore$ Volume of parallelopiped = 14 cu. units.

69. Volume of tetrahedron $= \dfrac{1}{6}[\bar{a}\ \bar{b}\ \bar{c}]$

$\Rightarrow 4 = \dfrac{1}{6}[\bar{a}\ \bar{b}\ \bar{c}] \Rightarrow [\bar{a}\ \bar{b}\ \bar{c}] = 24$

Edges of parallelopiped are $\bar{a}\times\bar{b},\ \bar{b}\times\bar{c},\ \bar{c}\times\bar{a}$

$\therefore$ Volume of parallelopiped $= [\bar{a}\times\bar{b}\ \ \bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}]$

$= [\bar{a}\ \bar{b}\ \bar{c}]^2$

$= 24^2$

$= 576$ sq. units

70. Volume of parallelopiped = $\begin{vmatrix} -12 & 0 & \alpha \\ 0 & 3 & -1 \\ 2 & 1 & -15 \end{vmatrix}$

$\Rightarrow 546 = -12(-45+1) + \alpha(0-6)$

$\Rightarrow \alpha = -3$

71. Volume of parallelopiped = [a – b b – c c – a]

$= \left[\bar{a}\ \bar{b}\ \bar{c}\right] - \left[\bar{b}\ \bar{c}\ \bar{a}\right]$

$= \left[\bar{a}\ \bar{b}\ \bar{c}\right] - \left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$= 0$

72. Volume of parallelopiped

$= \left[\bar{a}+\bar{b}\ \ \bar{b}+\bar{c}\ \ \bar{c}+\bar{a}\right] = 2\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$= 2\begin{vmatrix} 2 & -3 & 5 \\ 3 & -4 & 5 \\ 5 & -3 & -2 \end{vmatrix}$

$= 2\left[2(8+15)+3(-6-25)+5(-9+20)\right]$

$= 2\left[46-93+55\right]$

= 16 cu. Unit

73. Let A, B, C and D be the given points.

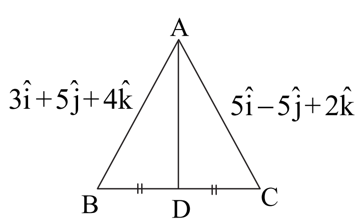
$\therefore$ $\overline{AB} = -4\hat{i} - 6\hat{j}$, $\overline{AC} = -\hat{i} + 4\hat{j} + 3\hat{k}$, and $\overline{AD} = -6\hat{i} - \hat{j} + 3\hat{k}$

Volume of tetrahedron $= \dfrac{1}{6}\begin{vmatrix} -4 & -6 & 0 \\ -1 & 4 & 3 \\ -6 & -1 & 3 \end{vmatrix}$

$= \dfrac{30}{6}$

$= 5$

74. AD is the median



$\therefore$ $\overline{AD} = \dfrac{\overline{AB}+\overline{AC}}{2}$

$\Rightarrow \overline{AD} = \dfrac{(3+5)\hat{i}+(5-5)\hat{j}+(4+2)\hat{k}}{2}$

$= \dfrac{8\hat{i}+6\hat{k}}{2}$

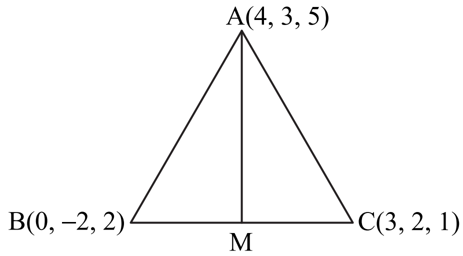
$= 4\hat{i} + 3\hat{k}$

$\therefore$ $l(AD) = |\overline{AD}| = \sqrt{16+9} = 5$ units.

75. Let AM be the angle bisector of $\angle$BAC

$|\overline{AB}| = \sqrt{16+25+9} = \sqrt{50} = 5\sqrt{2}$

$|\overline{AC}| = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$



$\therefore$ M divides BC internally in the ratio 5 : 3

$\overline{M} = \dfrac{5\bar{c}+3\bar{b}}{8} = \dfrac{5(3\hat{i}+2\hat{j}+\hat{k})+3(-2\hat{j}+2\hat{k})}{8}$

$= \dfrac{15\hat{i}+4\hat{j}+11\hat{k}}{8}$

$\therefore$ $\overline{M} = \left(\dfrac{15}{8}, \dfrac{4}{8}, \dfrac{11}{8}\right)$

76. Let AM be the angle bisector of angle A

$|\overline{AB}| = 6$ and $(\overline{AC}) = 3$

$\therefore$ M divides BC internally is the ratio 2 : 1

$\therefore$ $\overline{M} = \dfrac{2(2\hat{i}+5\hat{j}+7\hat{k})+1(2\hat{i}+3\hat{j}+4\hat{k})}{2+1}$

$= \dfrac{6\hat{i}+13\hat{j}+18\hat{k}}{3}$

77. $\overline{AB} = -6\hat{i}-2\hat{j}+3\hat{k}$, $\overline{BC} = -2\hat{i}+3\hat{j}-6\hat{k}$

$\overline{CD} = 6\hat{i}+2\hat{j}-3\hat{k}$, $\overline{DA} = 2\hat{i}-3\hat{j}+6\hat{k}$

$\overline{AC} = -8\hat{i}+\hat{j}-3\hat{k}$ and $\overline{BD} = 4\hat{i}+5\hat{j}-9\hat{k}$

Here, $\left|\overline{AB}\right| = \left|\overline{BC}\right| = \left|\overline{CD}\right| = \left|\overline{DA}\right| = 7$

and $\overline{AC}.\overline{BD} = 0 \Rightarrow \overline{AC} \perp \overline{BD}$

Hence, ABCD is a rhombus.

78. In ΔABC, hypotenuse AB = p

$\therefore$ $\overline{AC} \perp \overline{CB}$

$\therefore$ $\overline{AC}.\overline{CB} = 0$ ....(i)

Now, $\overline{AB}.\overline{AC} + \overline{BC}.\overline{BA} + \overline{CA}.\overline{CB}$

$= \overline{AB}.\overline{AC} + \overline{BC}.\left(-\overline{AB}\right) + \left(-\overline{AC}\right).\overline{CB}$

$= \overline{AB}.\overline{AC} - \overline{BC}.\overline{AB} - \overline{AC}.\overline{CB}$

$= \overline{AB}.\left(\overline{AC} - \overline{BC}\right) - 0$ ....[From (i)]

$= \overline{AB}.\left(\overline{AC}+\overline{CB}\right)$

$= \overline{AB}.\overline{AB}$ ....$\left[\because \overline{AC}+\overline{CB}=\overline{AB}\right]$

$= |\overline{AB}|^2 = p^2$

79. Since, $\overline{c}$ is coplanar with $\overline{a}$ and $\overline{b}$

$\therefore$ $\overline{c} = x\overline{a} + y\overline{b}$

$\Rightarrow \overline{c} = x(2\hat{i}+\hat{j}+\hat{k}) + y(\hat{i}+2\hat{j}-\hat{k})$

$\Rightarrow \overline{c} = (2x+y)\hat{i} + (x+2y)\hat{j} + (x-y)\hat{k}$

Also, $\overline{a}\,.\,\overline{c} = 0$ ....$\left[\because \overline{c} \perp \overline{a}\right]$

$\therefore$ $2(2x+y) + x + 2y + x - y = 0$

$\Rightarrow y = -2x$

$\overline{c} = -3x\hat{j} + 3x\hat{k} = 3x(-\hat{j}+\hat{k})$

Now, $|\overline{c}| = 1$

$\therefore$ $9x^2 + 9x^2 = 1$

$\Rightarrow x = \pm\dfrac{1}{3\sqrt{2}}$

$\therefore$ $\overline{c} = \dfrac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$

Evaluation Test

1. Since, $\overline{a}+\overline{b}$ and $\overline{b}+\overline{c}$ are collinear with $\overline{c}$ and $\overline{a}$ respectively

$\therefore$ $\overline{a}+\overline{b} = t\overline{c}$ …(i)

$\overline{b}+\overline{c} = s\overline{a}$ …(ii)

From (i) and (ii), we get

$\overline{a}-\overline{c} = t\overline{c} - s\overline{a} \Rightarrow \overline{a}(1+s) = \overline{c}(1+t)$

But $\overline{a}$ and $\overline{c}$ are non-collinear

$\therefore$ $1 + s = 0,\ 1 + t = 0 \Rightarrow s = -1,\ t = -1$

Substituting value of t in (i) and value of s in (ii), we get

$\overline{a}+\overline{b} = -\overline{c}$ and $\overline{b}+\overline{c} = -\overline{a}$

Hence, $\overline{a}+\overline{b}+\overline{c} = 0$.

2. Given, $\overline{r} = \lambda_1\overline{r}_1 + \lambda_2\overline{r}_2 + \lambda_3\overline{r}_3$

$\Rightarrow 2\overline{a} - 3\overline{b} + 4\overline{c} = (\lambda_1 - \lambda_2 + \lambda_3)\overline{a}$
$+(-\lambda_1 + \lambda_2 + \lambda_3)\overline{b} + (\lambda_1 + \lambda_2 + \lambda_3)\overline{c}$

$\Rightarrow \lambda_1 - \lambda_2 + \lambda_3 = 2, -\lambda_1 + \lambda_2 + \lambda_3 = -3,$

$\lambda_1 + \lambda_2 + \lambda_3 = 4$

$\Rightarrow \lambda_1 = \dfrac{7}{2},\ \lambda_2 = 1,\ \lambda_3 = -\dfrac{1}{2}$

$\therefore$ $\lambda_1 + \lambda_3 = 3$

3. Since, the given vectors are coplanar

$\therefore$ $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$

Applying $C_2 \to C_2 - C_1$,

$\begin{vmatrix} a & 0 & c \\ 1 & -1 & 1 \\ c & 0 & b \end{vmatrix} = 0$

$\Rightarrow a(-b) + c(c) = 0 \Rightarrow c^2 = ab$

Hence, c is the geometric mean of a and b.

4. $\hat{a}\cdot\hat{a} = 1,\ \hat{b}\cdot\hat{b} = 1,\ \hat{c}\cdot\hat{c} = 1,$

$\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{a} = \dfrac{1}{2}$

$\therefore$ $\left[\overline{a}\,\overline{b}\,\overline{c}\right]^2 = \begin{vmatrix} \hat{a}\cdot\hat{a} & \hat{a}\cdot\hat{b} & \hat{a}\cdot\hat{c} \\ \hat{b}\cdot\hat{a} & \hat{b}\cdot\hat{b} & \hat{b}\cdot\hat{c} \\ \hat{c}\cdot\hat{a} & \hat{c}\cdot\hat{b} & \hat{c}\cdot\hat{c} \end{vmatrix}$

$= \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = \dfrac{1}{2}$

$\therefore$ $\left[\overline{a}\,\overline{b}\,\overline{c}\right] = \dfrac{1}{\sqrt{2}}$ cubic units

5. Since, $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$\begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 1-a & 0 & c-1 \end{vmatrix} = 0$

$\Rightarrow a(b-1)(c-1) - (1-a)(c-1) - (1-a)(b-1) = 0$

Dividing by $(1-a)(1-b)(1-c)$, we get

$\dfrac{a}{1-a} + \dfrac{1}{1-b} + \dfrac{1}{1-c} = 0$ ….(i)

Consider, $\dfrac{1}{1-a} + \dfrac{1}{1-b} + \dfrac{1}{1-c}$

$= \dfrac{1}{1-a} - \dfrac{a}{1-a}$ ….[From (i)]

$= 1$

6. Volume of the parallelopiped formed by vectors is

i.e., $V = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 1 - a + a^3$

$\therefore \quad \dfrac{dV}{da} = -1 + 3a^2, \ \dfrac{d^2V}{da^2} = 6a$

For max. or min. of V, $\dfrac{dV}{da} = 0$

$\therefore \quad a^2 = \dfrac{1}{3} \qquad \therefore \quad a = \dfrac{1}{\sqrt{3}}$

$\dfrac{d^2V}{da^2} = 6a > 0$ for $a = \dfrac{1}{\sqrt{3}}$

$\therefore$ V is minimum for $a = \dfrac{1}{\sqrt{3}}$

7. Given, $\overline{a}.\overline{b} = \overline{b}.\overline{c} = \overline{c}.\overline{a} = 0$

The scalar triple product of three vectors is

$[\overline{a}\ \overline{b}\ \overline{c}] = (\overline{a} \times \overline{b}).\overline{c}$

$\because \quad \overline{a}.\overline{b} = 0 \qquad \therefore \quad \overline{a} \perp \overline{b}$

$\therefore$ angle between $\overline{a}$ and $\overline{b}$ is $\theta = 90°$

Similarly, $[\overline{a}\ \overline{b}\ \overline{c}] = |\overline{a}||\overline{b}|\hat{n}.\overline{c}$ where $\hat{n}$ is a normal vector.

$\therefore$ $\hat{n}$ and $\overline{c}$ are parallel to each other

$\therefore \quad [\overline{a}\ \overline{b}\ \overline{c}] = |\overline{a}||\overline{b}||\hat{n}|.|\overline{c}| = |a||b||c|$.

8. Given, $\overline{r} \times \overline{b} = \overline{c} \times \overline{b}$

$\Rightarrow (\overline{r} - \overline{c}) \times \overline{b} = \overline{0}$

$\therefore$ $\overline{r} - \overline{c}$ is parallel to $\overline{b}$

$\Rightarrow \overline{r} - \overline{c} = \lambda\overline{b}$ for some scalar λ

$\Rightarrow \overline{r} = \overline{c} + \lambda\overline{b}$ ….(i)

$\Rightarrow \overline{r}.\overline{a} = \overline{c}.\overline{a} + \lambda(\overline{b} \cdot \overline{a})$

$\Rightarrow 0 = \overline{c}.\overline{a} + \lambda(\overline{b} \cdot \overline{a})$ ….$[\because \overline{r} \cdot \overline{a} = 0 \text{ (given)}]$

$\Rightarrow \lambda = -\dfrac{\overline{a}.\overline{c}}{\overline{a}.\overline{b}}$

Substituting the value of λ in (i), we get

$\overline{r} = \overline{c} - \dfrac{\overline{a}.\overline{c}}{\overline{a}.\overline{b}}\overline{b}$

$\Rightarrow \overline{r}.\overline{b} = \overline{c}.\overline{b} - \dfrac{\overline{a}.\overline{c}}{\overline{a}.\overline{b}}(\overline{b}.\overline{b})$

$\Rightarrow \overline{r}.\overline{b} = 1 - \dfrac{(-4)}{1} \times 2 = 9$

9. Let $\overline{c} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore \quad \overline{a} \times \overline{c} = \overline{c} \times \overline{b}$

$\Rightarrow \overline{a} \times \overline{c} = -\overline{b} \times \overline{c} \Rightarrow (\overline{a} + \overline{b}) \times \overline{c} = 0 \Rightarrow (\overline{a} + \overline{b}) \parallel \overline{c}$

Let $(\overline{a} + \overline{b}) = \lambda\overline{c}$

$\Rightarrow |\overline{a} + \overline{b}| = |\lambda|\,|\overline{c}| \Rightarrow \sqrt{29} = |\lambda|.\sqrt{29} \Rightarrow \lambda = \pm 1$

$\therefore \quad \overline{a} + \overline{b} = \pm(2\hat{i} + 3\hat{j} + 4\hat{k})$

Now, $(\overline{a} + \overline{b}).(-7\hat{i} + 2\hat{j} + 3\hat{k}) = \pm(-14 + 6 + 12)$

$= \pm 4$

10. Given,

$[l\overline{a} + m\overline{b} + n\overline{c}\quad l\overline{b} + m\overline{c} + n\overline{a}\quad l\overline{c} + m\overline{a} + n\overline{b}] = 0$

$\Rightarrow [l\overline{a} + m\overline{b} + n\overline{c}\quad n\overline{a} + l\overline{b} + m\overline{c}\quad m\overline{a} + n\overline{b} + l\overline{c}] = 0$

$\Rightarrow \begin{vmatrix} l & m & n \\ n & l & m \\ m & n & l \end{vmatrix} [\overline{a}\ \overline{b}\ \overline{c}] = 0$

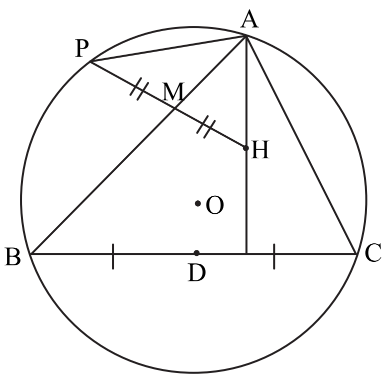
$\Rightarrow \begin{vmatrix} l & m & n \\ n & l & m \\ m & n & l \end{vmatrix} = 0$ ….$[\because [\overline{a}\ \overline{b}\ \overline{c}] \neq 0]$

$\Rightarrow l^3 + m^3 + n^3 - 3lmn = 0$

$\Rightarrow (l + m + n)(l^2 + m^2 + n^2 - lm - mn - nl) = 0$

$\Rightarrow l + m + n = 0$

11.



Let point O be the circumcentre of ΔABC.

Let $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{p}$, $\overline{d}$, $\overline{h}$, $\overline{m}$ be the position vectors of the respective points.

Since, $\overline{h} = \overline{a} + \overline{b} + \overline{c}$ ….(Standard formula)

$\therefore \quad \overline{m} = \dfrac{\overline{p} + \overline{h}}{2} = \dfrac{\overline{p} + \overline{a} + \overline{b} + \overline{c}}{2}$

$\therefore \quad \overline{DM} = \overline{m} - \overline{d} = \dfrac{\overline{p} + \overline{a} + \overline{b} + \overline{c}}{2} - \dfrac{\overline{b} + \overline{c}}{2} = \dfrac{\overline{p} + \overline{a}}{2}$

$\therefore \quad \overline{DM} \cdot \overline{PA} = \left(\dfrac{\overline{p} + \overline{a}}{2}\right) \cdot (\overline{a} - \overline{p}) = \dfrac{1}{2}(a^2 - p^2) = 0$

….[$\because$ O is circumcentre, $\therefore$ OA = OP i.e., a = p]

$\therefore$ DM is perpendicular to PA.

15. Let position vector of Q be $\bar{r}$

Since, $\bar{p}$ divides PQ in the ratio 3 : 4

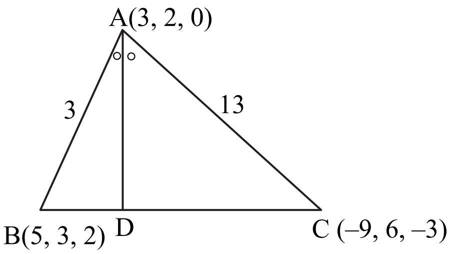
$\therefore \quad \dfrac{3\bar{r} + 4(3\bar{p} + \bar{q})}{3+4} = \bar{p}$

$\Rightarrow 7\bar{p} = 3\bar{r} + 12\bar{p} + 4\bar{q}$

$\Rightarrow -5\bar{p} - 4\bar{q} = 3\bar{r}$

$\Rightarrow \bar{r} = \dfrac{-1}{3}\left(5\bar{p} + 4\bar{q}\right)$

16.



By distance formula,

$AB = \sqrt{(5-3)^2 + (3-2)^2 + (2-0)^2}$

$= \sqrt{4+1+4} = \sqrt{9} = 3$

$AC = \sqrt{(3+9)^2 + (2-6)^2 + (0+3)^2}$

$= \sqrt{144+16+9} = \sqrt{169} = 13$

$\therefore$ Point D divides seg BC in the ratio of 3 : 13

$\therefore$ By section formula,

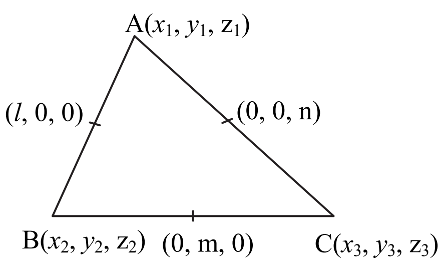
$D \equiv \left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}, \dfrac{mz_2 + nz_1}{m+n}\right)$

$\equiv \left(\dfrac{3(-9)+13(5)}{3+13}, \dfrac{3(6)+13(3)}{3+13}, \dfrac{3(-3)+13(2)}{3+13}\right)$

$\equiv \left(\dfrac{-27+65}{16}, \dfrac{18+39}{16}, \dfrac{-9+26}{16}\right)$

$\equiv \left(\dfrac{38}{16}, \dfrac{57}{16}, \dfrac{17}{16}\right) \equiv \left(\dfrac{19}{8}, \dfrac{57}{16}, \dfrac{17}{16}\right)$

17.



$x_1 + x_2 = 2l$, $x_2 + x_3 = 0$, $x_3 + x_1 = 0$

On solving we get $x_1 = l$, $x_2 = l$, $x_3 = -l$

$y_1 + y_2 = 0$, $y_2 + y_3 = 2m$, $y_3 + y_1 = 0$

On solving we get $y_1 = -m$, $y_2 = m$, $y_3 = m$

$z_1 + z_2 = 0$, $z_2 + z_3 = 0$, $z_3 + z_1 = 2n$

On solving we get $z_1 = n$, $z_3 = n$, $z_2 = -n$

$\therefore$ A(l, –m, n), B(l, m, –n), C(–l, m, n)

By distance formula,

$AB^2 = (l - l)^2 + (-m - m)^2 + (n + n)^2 = 4m^2 + 4n^2$

$BC^2 = (l + l)^2 + (m - m)^2 + (-n - n)^2 = 4l^2 + 4n^2$

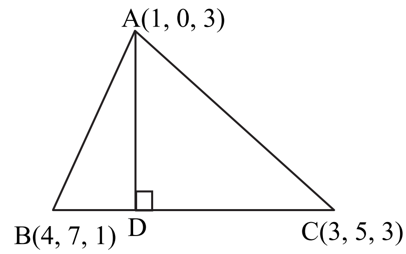
$CA^2 = (l + l)^2 + (-m - m)^2 + (n - n)^2 = 4l^2 + 4m^2$

$\therefore \quad \dfrac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2}$

$= \dfrac{4m^2 + 4n^2 + 4l^2 + 4n^2 + 4l^2 + 4m^2}{l^2 + m^2 + n^2}$

$= 8\dfrac{\left(l^2 + m^2 + n^2\right)}{l^2 + m^2 + n^2} = 8$

18.



Let D be the foot of perpendicular and let it divide BC in the ratio $\lambda : 1$ internally

$\therefore \quad D \equiv \left(\dfrac{3\lambda+4}{\lambda+1}, \dfrac{5\lambda+7}{\lambda+1}, \dfrac{3\lambda+1}{\lambda+1}\right)$

$\overline{AD} = \bar{d} - \bar{a}$

$= \left(\dfrac{3\lambda+4}{\lambda+1}\right)\hat{i} + \left(\dfrac{5\lambda+7}{\lambda+1}\right)\hat{j} + \left(\dfrac{3\lambda+1}{\lambda+1}\right)\hat{k} - \hat{i} - 3\hat{k}$

$= \left(\dfrac{2\lambda+3}{\lambda+1}\right)\hat{i} + \left(\dfrac{5\lambda+7}{\lambda+1}\right)\hat{j} - \left(\dfrac{2}{\lambda+1}\right)\hat{k}$

$\overline{BC} = 3\hat{i} + 5\hat{j} + 3\hat{k} - \left(4\hat{i} + 7\hat{j} + \hat{k}\right)$

$= -\hat{i} - 2\hat{j} + 2\hat{k}$

Since, $\overline{AD} \perp \overline{BC}$.

$\overline{AD} \cdot \overline{BC} = 0$

$\Rightarrow \left(\dfrac{2\lambda+3}{\lambda+1}\right)(-1) + \left(\dfrac{5\lambda+7}{\lambda+1}\right)(-2) + \left(\dfrac{-2}{\lambda+1}\right)(2) = 0$

$\Rightarrow -2\lambda - 3 - 10\lambda - 14 - 4 = 0$

$\Rightarrow -12\lambda - 21 = 0 \Rightarrow \lambda = -\dfrac{7}{4}$

$\therefore \quad D \equiv \left(\dfrac{3\left(-\frac{7}{4}\right)+4}{-\frac{7}{4}+1}, \dfrac{5\left(-\frac{7}{4}\right)+7}{-\frac{7}{4}+1}, \dfrac{3\left(-\frac{7}{4}\right)+1}{-\frac{7}{4}+1}\right)$

$\equiv \left(\dfrac{-21+16}{-7+4}, \dfrac{-35+28}{-7+4}, \dfrac{-21+4}{-7+4}\right) \equiv \left(\dfrac{5}{3}, \dfrac{7}{3}, \dfrac{17}{3}\right)$

Textbook Chapter No.

06 Three Dimensional Geometry

Hints

Classical Thinking

1. For every point (x, y, z) on X-axis $y = 0$, z = 0

2. Let the direction cosines of the line be l, m, n

$\therefore$ $l = \cos 45^\circ$, m = $\cos 60^\circ$, n = $\cos 60^\circ$

$\Rightarrow l = \dfrac{1}{\sqrt{2}}$, m = $\dfrac{1}{2}$ and n = $\dfrac{1}{2}$

$\therefore$ d.c.s are $\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}, \dfrac{1}{2}$.

3. Let the d.c.s of the line be l, m, n

$\therefore$ $l = \cos 90^\circ$, m = $\cos 60^\circ$, n = $\cos 30^\circ$

$\Rightarrow l = 0$, m = $\dfrac{1}{2}$, n = $\dfrac{\sqrt{3}}{2}$

$\therefore$ d.c.s are 0, $\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}$

4. The d.c.s of Y-axis are

cos90°, cos0°, cos90°

i.e. 0, 1, 0

5. The d.c.s of X-axis are 1, 0, 0.

7. For option (B),

$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma \neq 1$

$\therefore$ option (B) is correct answer.

8. Since, $l^2 + m^2 + n^2 = 1$

$\therefore$ $k^2 + \left(\dfrac{1}{2}\right)^2 + 0^2 = 1$

$\Rightarrow k^2 = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

$\Rightarrow k = \pm \dfrac{\sqrt{3}}{2}$

9. Since, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

$\therefore$ $\cos^2 45^\circ + \cos^2 60^\circ + \cos^2 \gamma = 1$

$\Rightarrow \cos^2 \gamma = 1 - \dfrac{1}{2} - \dfrac{1}{4} = \dfrac{1}{4}$

$\Rightarrow \cos \gamma = \pm\dfrac{1}{2}$

$\Rightarrow \gamma = 60^\circ$ or 120°

10. $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

$\Rightarrow \cos^2 \alpha + \cos^2(90^\circ - \alpha) + \cos^2 \gamma = 1$

….[$\because$ $\alpha + \beta = 90^\circ$]

$\Rightarrow \cos^2 \alpha + \sin^2 \alpha + \cos^2 \gamma = 1$

$\Rightarrow \cos^2 \gamma + 1 = 1$

$\Rightarrow \cos^2 \gamma = 0$

$\Rightarrow \gamma = 90^\circ$

11. Let l, m, n be the d.c.s of the line.

$\therefore$ $l = \cos \alpha$; m = $\cos 60^\circ$; n = $\cos 45^\circ$

Since, $\cos^2 \alpha + \cos^2 60^\circ + \cos^2 45^\circ = 1$

$\Rightarrow \cos^2\alpha = 1 - \dfrac{1}{2} - \dfrac{1}{4} = \dfrac{1}{4}$

$\Rightarrow \cos \alpha = \pm \dfrac{1}{2}$

$\therefore$ the d.c.s are $\pm \dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{\sqrt{2}}$

13. Let $\bar{r} = 2\hat{i} + 2\hat{j} - \hat{k}$

$|\bar{r}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$

$\therefore$ The d.c.s are $\dfrac{x}{|\bar{r}|}, \dfrac{y}{|\bar{r}|}, \dfrac{z}{|\bar{r}|}$

i.e., $\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{-1}{3}$

14. Let $\bar{r} = 3\hat{i} + 4\hat{k}$.

$|\bar{r}| = \sqrt{3^2 + 0^2 + 4^2} = 5$

$\therefore$ The d.c.s are $\dfrac{3}{5}$, 0, $\dfrac{4}{5}$

15. D.c.s are $\dfrac{a}{|\bar{r}|}, \dfrac{b}{|\bar{r}|}, \dfrac{c}{|\bar{r}|}$

i.e., $\dfrac{2}{7}, \dfrac{-3}{7}, \dfrac{6}{7}$

16. A ≡ (1, 2, 6) and B ≡ (–4, 5, 0)

$\therefore$ D.r.s of AB are –4 – 1, 5 – 2, 0 – 6

i.e., –5, 3, –6

17. On Y-axis, x and z co-ordinates are zero.

Hence, (B) is the correct option.

18. Since $(-l)^2 + (-m)^2 + (-n)^2 = 1$, we can say that $-l, -m, -n$ are the direction cosines of the line.

Also that $\frac{-l}{l} = \frac{-m}{m} = \frac{-n}{n} = -1$

Hence, we can say that $-l, -m, -n$ are the d.r.s. of the line.

19. Let a, b, c be the d.r.s of the line.

$\therefore$ The d.c.s are given by

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$$

i.e., $\frac{2}{\sqrt{2^2+(-1)^2+(-2)^2}}, \frac{-1}{\sqrt{2^2+(-1)^2+(-2)^2}}, \frac{-2}{\sqrt{2^2+(-1)^2+(-2)^2}}$

i.e., $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$

20. The direction cosines are

$$\frac{\sqrt{2}}{\sqrt{2+5+2}}, \frac{-\sqrt{5}}{\sqrt{2+5+2}}, \frac{\sqrt{2}}{\sqrt{2+5+2}}$$

i.e., $\frac{\sqrt{2}}{3}, \frac{-\sqrt{5}}{3}, \frac{\sqrt{2}}{3}$

21. The d.r.s of line through $(1, 2, -3)$ and $(-2, 3, 1)$ are $-2-1, 3-2, 1-(-3)$

i.e. $-3, 1, 4$

$\therefore$ d.c.s are

$$\frac{-3}{\sqrt{9+1+16}}, \frac{1}{\sqrt{9+1+16}}, \frac{4}{\sqrt{9+1+16}}$$

i.e. $\frac{-3}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}$

22. The d.r.s of AB are $2-14, -3-5, 1+3$

i.e. $-12, -8, 4$ i.e., $3, 2, -1$

$\therefore$ The d.c.s are $\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$

23. Let O(0, 0, 0) and P(1, 2, 3) be two points.

$\therefore$ Then the d.r.s of OP are 1, 2, 3

$\therefore$ The d.c.s of OP are

$$\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$$

24. D.r.s. of line through A(3, 1, 2), B(4, λ, 0) are $4-3, \lambda-1, 0-2 \Rightarrow 1, \lambda-1, -2 \equiv a_1, b_1, c_1$

D.r.s. of line through C(1, 2, 1), D(2, 3, -1) are $2-1, 3-2, -1-1 \Rightarrow 1, 1, -2 \equiv a_2, b_2, c_2$

Since, AB || CD,

$\therefore$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\Rightarrow \frac{1}{1} = \frac{\lambda-1}{1} = \frac{-2}{-2}$

i.e., $\lambda - 1 = 1 \Rightarrow \lambda = 2$

25. Let $A \equiv (5, 2, 4)$, $B \equiv (6, -1, 2)$ and $C \equiv (8, -7, k)$

$\therefore$ The d.r.s of AB are $6-5, -1-2, 2-4$

i.e., $1, -3, -2$, and

The d.r.s of BC are $8-6, -7+1, k-2$

i.e. $2, -6, k-2$

Since, the points A, B, C are collinear, AB || BC

$\therefore$ $\frac{2}{1} = \frac{-6}{-3} = \frac{k-2}{-2}$

$\Rightarrow k - 2 = -4$

$\Rightarrow k = 2 - 4 = -2$

26. Let $A \equiv (-2, 4, \lambda)$, $B \equiv (3, -6, -8)$, $C \equiv (1, -2, -2)$

The d.r.s of AB are

$-5, 10, \lambda+8$, and

The d.r.s of AC are

$-3, 6, \lambda+2$

Since, the points A,B,C are collinear, AB || AC

$\therefore$ $\frac{-5}{-3} = \frac{10}{6} = \frac{\lambda+8}{\lambda+2}$

$\therefore$ $5(\lambda + 2) = 3(\lambda + 8)$

$\Rightarrow 5\lambda + 10 = 3\lambda + 24$

$\Rightarrow 2\lambda = 14$

$\Rightarrow \lambda = 7$

27. Let, $l_1 = \frac{1}{\sqrt{6}}, m_1 = \frac{-1}{\sqrt{6}}, n_1 = \frac{2}{\sqrt{6}}$

and $l_2 = \frac{2}{\sqrt{6}}, m_2 = \frac{1}{\sqrt{6}}, n_2 = \frac{-1}{\sqrt{6}}$

$\therefore$ angle between the lines is

$\cos\theta = | l_1l_2 + m_1m_2 + n_1n_2 |$

$\Rightarrow \cos\theta = \left|\frac{1}{\sqrt{6}}\left(\frac{2}{\sqrt{6}}\right) + \left(\frac{-1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{2}{\sqrt{6}}\right)\left(\frac{-1}{\sqrt{6}}\right)\right|$

$\Rightarrow \cos\theta = \left|\frac{-1}{6}\right|$

$\therefore$ $\theta = \cos^{-1}\left(\frac{1}{6}\right)$

28. Let, $a_1, b_1, c_1 = 5, -12, 13$
and $a_2, b_2, c_2 = -3, 4, 5$

$\therefore$ $$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$
$$= \left|\frac{5(-3)+(-12)4+13(5)}{\sqrt{5^2+(-12)^2+13^2}\cdot\sqrt{(-3)^2+4^2+5^2}}\right|$$
$$= \left|\frac{-15-48+65}{13\sqrt{2}\cdot 5\sqrt{2}}\right|$$
$$= \frac{1}{65}$$
$\therefore$ $\theta = \cos^{-1}\left(\frac{1}{65}\right)$

29. Here, $A \equiv (1, 2, 3)$, $B \equiv (4, 5, 7)$,
$C \equiv (-4, 3, -6)$ and $D \equiv (2, 9, 2)$

$\therefore$ d.r.s of lines AB and CD are 3, 3, 4 and 6, 6, 8 respectively.

$\therefore$ $$\theta = \cos^{-1}\left[\frac{(3)(6)+(3)(6)+(4)(8)}{\sqrt{34}.\sqrt{136}}\right]$$
$$= \cos^{-1}\left[\frac{68}{2\times 34}\right] = 0°$$

30. $$\cos 45° = \left|\frac{2a-3+10}{\sqrt{2^2+(-1)^2+2^2}\sqrt{a^2+3^2+5^2}}\right|$$
$$\Rightarrow \frac{1}{\sqrt{2}} = \left|\frac{2a+7}{3\sqrt{a^2+34}}\right|$$
$\Rightarrow 9(a^2 + 34) = 2(2a + 7)^2$
$\Rightarrow 9a^2 + 306 = 8a^2 + 56a + 98$
$\Rightarrow a^2 - 56a + 208 = 0$
$\Rightarrow a = 4$

31. Let $a_1, b_1, c_1 = 1, -2, 1$ and
$a_2, b_2, c_2 = 2, 3, 4$
Consider,
$a_1a_2 + b_1b_2 + c_1c_2 = 1(2) + (-2)(3) + 1(4)$
$= 0$

$\therefore$ $OP \perp OQ$.

Critical Thinking

1. If α, β, λ are direction angles of any vector $\overline{OL}$, then those of $\overline{OL'}$ are $\pi - \alpha, \pi - \beta, \pi - \gamma$ respectively

$\therefore$ correct answer is option (B).

2. We know that, $l^2 + m^2 + n^2 = 1$
Consider option (D)
$$\left(\frac{2}{\sqrt{25}}\right)^2 + \left(\frac{3}{\sqrt{25}}\right)^2 + \left(\frac{4}{\sqrt{25}}\right)^2 = \frac{4+9+16}{25}$$
$$= \frac{29}{25} \neq 1$$
$\therefore$ correct answer is option (D).

3. Consider option (B)

$\therefore$ $\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{3} + \cos^2\frac{\pi}{3}$
$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$

$\therefore$ correct answer is option (B).

4. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore$ $$\cos\gamma = \pm\sqrt{1-\left(\frac{14}{15}\right)^2-\left(\frac{1}{3}\right)^2} = \pm\sqrt{\frac{8}{9}-\left(\frac{196}{225}\right)} = \pm\frac{2}{15}$$

5. Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore$ $\cos^2\alpha + \cos^2 60° + \cos^2 60° = 1$

$\therefore$ $\cos^2\alpha = 1 - \frac{1}{4} - \frac{1}{4} = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{2}}$
$\Rightarrow \alpha = 45°$ or $\alpha = 135°$

6. Since, the line lies in ZOX plane, it makes an angle 90° with Y-axis
Also, line makes angle 30° and $\pi - 30°$ with positive Z-axis and 60° and $\pi - 60°$ with positive X-axis

$\therefore$ d.c.s of the required line are
$\pm\cos\alpha, \pm\cos\beta, \pm\cos\gamma$
i.e., $\pm\cos 60°, \pm\cos 0°, \pm\cos 30°$
i.e. $\pm\frac{1}{2}, 0, \pm\frac{\sqrt{3}}{2}$

7. $\cos\gamma = \sqrt{1-\frac{3}{4}-\frac{1}{2}} = \sqrt{\frac{-1}{4}}$ which is not possible.

8. Let l, m, n be the d.c.s of $\bar{r}$.
$l = m = n$
....[$\because \alpha = \beta = \gamma \Rightarrow \cos\alpha = \cos\beta = \cos\gamma$]
Now, $l^2 + m^2 + n^2 = 1$
$\Rightarrow l = \pm\frac{1}{\sqrt{3}}$

9. Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$ $(\because \alpha = \beta = \gamma)$
$\Rightarrow \cos^2\alpha = \frac{1}{3} \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$
Now, sum of d.c.s. $= l + m + n$
$= \cos\alpha + \cos\alpha + \cos\alpha$
$= 3\cos\alpha = \sqrt{3}$

10. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$
$= 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1$
$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3$
$= 2(1) - 3 = -1$

11. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$
$= (1 - \cos^2\alpha) + (1 - \cos^2\beta) + (1 - \cos^2\gamma)$
$= 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3 - (1) = 2$

12. Let $\alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{4}$
$\therefore$ $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\cos\beta = \frac{1}{\sqrt{2}}$
Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
$\therefore$ $\frac{3}{4} + \frac{1}{2} + \cos^2\gamma = 1$
$\Rightarrow \cos^2\gamma = -\frac{1}{4}$
Square of a real number cannot be negative.
$\therefore$ option (A) is the correct answer.

13. The line makes angle θ with X–axis and Z–axis and β with Y–axis.
$\therefore$ $l = \cos\theta$, $m = \cos\beta$, $n = \cos\theta$
$\cos^2\theta + \cos^2\beta + \cos^2\theta = 1$
$\Rightarrow 2\cos^2\theta = 1 - \cos^2\beta$
$\Rightarrow 2\cos^2\theta = \sin^2\beta$ …(i)
But $\sin^2\beta = 3\sin^2\theta$ …(ii)
From (i) and (ii), we get
$3\sin^2\theta = 2\cos^2\theta$
$\Rightarrow 3(1 - \cos^2\theta) = 2\cos^2\theta$
$\Rightarrow 3 = 5\cos^2\theta \Rightarrow \cos^2\theta = \frac{3}{5}$

14. Let the length of the line segment be r and its d.c.s be l, m, n.
$\therefore$ The projections on the co-ordinate axes are lr, mr, nr.
$\therefore$ $l r = 4$, $mr = 6$ and $nr = 12$
$\therefore$ $l^2r^2 + m^2r^2 + n^2r^2 = (4)^2 + (6)^2 + (12)^2$
$\Rightarrow r^2(l^2 + m^2 + n^2) = 16 + 36 + 144$
$\Rightarrow r^2 = 196$ ….$[\because l^2 + m^2 + n^2 = 1]$
$\Rightarrow r = 14$
$\therefore$ The d.c.s. of line are $\frac{4}{r}, \frac{6}{r}, \frac{12}{r}$
i.e., $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

15. Let α, β, γ be the angles which OP makes with the co-ordinates axes,
$\therefore$ $x = r\cos\alpha$, $y = r\cos\beta$, $z = r\cos\gamma$
$\therefore$ $\cos\alpha = \frac{x}{r}$; $\cos\beta = \frac{y}{r}$; $\cos\gamma = \frac{z}{r}$
So, the direction cosines are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$.

16. We have $l = \cos 45° = \frac{1}{\sqrt{2}}$,
$m = \cos 60° = \frac{1}{2}$ and $n = \cos\gamma$
We know that $l^2 + m^2 + n^2 = 1$
$\therefore$ $\frac{1}{2} + \frac{1}{4} + n^2 = 1$
$\Rightarrow n^2 = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow n = \pm\frac{1}{2}$
$\Rightarrow \cos\gamma = \pm\frac{1}{2}$
$\bar{r} = r\left(l\hat{i} + m\hat{j} + n\hat{k}\right)$
$\Rightarrow \bar{r} = 12\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} \pm \frac{1}{2}\hat{k}\right)$

17. Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$ $(\because \alpha = \beta = \gamma)$
$\Rightarrow \cos^2\alpha = \frac{1}{3} \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$
$\therefore$ The d.c.s are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$.
The magnitude of the given vector is 6.
$\therefore$ $\bar{r} = 6(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})$
$= \frac{\pm 6}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) = \pm 2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

18. For a line passing through origin, d.r.s are the co-ordinates of the point.

19. D.c.s. of the line are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
$\therefore$ $\cos\alpha = \frac{1}{\sqrt{3}}$, $\cos\beta = \frac{1}{\sqrt{3}}$, $\cos\gamma = \frac{1}{\sqrt{3}}$
Hence, line is equally inclined to axes.

20. $\cos\gamma = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$

$= \dfrac{5}{\sqrt{9+16+25}} = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \gamma = \dfrac{\pi}{4}$

21. The d.r.s. of the given line are

$2 - 6, -3 + 7, 1 + 1$

i.e., $-2, 2, 1$. i.e., $2, -2, -1$

$\because$ angle α is acute, $\cos\alpha > 0$

$\Rightarrow \cos\alpha = \dfrac{2}{3}$

Thus, required d.c.s are $\dfrac{2}{3}, \dfrac{-2}{3}, \dfrac{-1}{3}$

22. $l^2 + m^2 + n^2 = 1$

$\therefore$ $\left(\dfrac{2}{7}\right)^2 + \left(\dfrac{-3}{7}\right)^2 + n^2 = 1$

$\therefore$ $n^2 = 1 - \dfrac{13}{49} = \dfrac{36}{49}$

Let a, b, c be the d.r.s. of the line.

$\therefore$ $a = 2, b = -3, c = z$

Since, $n = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$

$\therefore$ $\dfrac{z}{\sqrt{4+9+z^2}} = \pm\dfrac{6}{7}$

$\Rightarrow \dfrac{z^2}{13+z^2} = \dfrac{36}{49}$

$\Rightarrow 49z^2 - 36z^2 = 13\times 36$

$\Rightarrow z^2 = 36$

$\Rightarrow z = \pm 6$

23. Let $A \equiv (2, a, -1)$, $B \equiv (3, 4, b)$ and $C \equiv (1, -2, 3)$

d.r.s of AB are $3 - 2, 4 - a, b - (-1)$

i.e., $1, 4 - a, b + 1$

d.r.s of BC are $1 - 3, -2 - 4, 3 - b$

i.e., $-2, -6, 3 - b$

Since, the points A, B and C are collinear, AB || BC

$\therefore$ $\dfrac{1}{-2} = \dfrac{4-a}{-6} = \dfrac{b+1}{3-b}$

$\Rightarrow \dfrac{4-a}{-6} = \dfrac{1}{-2}, \dfrac{b+1}{3-b} = \dfrac{1}{-2}$

$\Rightarrow 4 - a = 3, -2b - 2 = 3 - b$

$\Rightarrow a = 1, b = -5$

24. Let $A \equiv (1, a, 1)$, $B \equiv (3, -1, 2)$ and $C \equiv (1, a^2, 1)$

d.r.s of AB are $3 - 1, -1 - a, 2 - 1$

i.e. $2, -1 - a, 1$

d.r.s of BC are $1 - 3, a^2 + 1, 1 - 2$

i.e. $-2, a^2 + 1, -1$

Since, the points are collinear, AB || BC

$\therefore$ $\dfrac{2}{-2} = \dfrac{-1-a}{a^2+1} = \dfrac{1}{-1}$

$\Rightarrow \dfrac{-1-a}{a^2+1} = -1$

$\Rightarrow a^2 + 1 = 1 + a$

$\Rightarrow a^2 - a = 0$

$\Rightarrow a(a - 1) = 0$

$\Rightarrow a = 0$ or $a = 1$

25. Here, $\dfrac{3-(-2)}{1-3} = \dfrac{-6-4}{-2-(-6)} = \dfrac{-8-7}{-2-(-8)}$

$\Rightarrow -\dfrac{5}{2} = -\dfrac{5}{2} = -\dfrac{5}{2}$

$\therefore$ the given points are collinear.

26. Let, $l_1, m_1, n_1 = a, \dfrac{-2}{3}, \dfrac{1}{3}$ and

$l_2, m_2, n_2 = \dfrac{2}{3}, \dfrac{1}{3}, \dfrac{-2}{3}$

$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

$\therefore$ $\cos\dfrac{\pi}{2} = \left|a\left(\dfrac{2}{3}\right) + \left(\dfrac{-2}{3}\right)\left(\dfrac{1}{3}\right) + \dfrac{1}{3}\left(\dfrac{-2}{3}\right)\right|$

$\Rightarrow 0 = \dfrac{2a}{3} - \dfrac{2}{9} - \dfrac{2}{9}$

$\Rightarrow \dfrac{2a}{3} = \dfrac{4}{9}$

$\Rightarrow a = \dfrac{2}{3}$

27. Let, $a_1, b_1, c_1 = 5, 4, 1$

$a_2, b_2, c_2 = -3, 2, 1$

$\cos\theta = \left|\dfrac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}}\right|$

$\therefore$ $\cos\theta = \left|\dfrac{5(-3)+4(2)+1(1)}{\sqrt{5^2+4^2+1^2}.\sqrt{(-3)^2+(2)^2+(1)^2}}\right|$

$= \left|\dfrac{-15+8+1}{\sqrt{42}\sqrt{14}}\right| = \dfrac{6}{14\sqrt{3}} = \dfrac{\sqrt{3}}{7}$

$\therefore$ $\theta = \cos^{-1}\left(\dfrac{\sqrt{3}}{7}\right)$

28. $\theta = \cos^{-1}\left|\dfrac{1(2)+2(-3)+1(4)}{\sqrt{1^2+2^2+1^2}\sqrt{2^2+(-3)^2+4^2}}\right|$

$\theta = \cos^{-1}(0) = \dfrac{\pi}{2}$

29. Given, A ≡ (1, 2, –1), B ≡ (2, 0, 3), C ≡ (3, –1, 2)
The d.r.s of AB = 1, –2, 4 and d.r.s of AC = 2, –3, 3

∴ $\cos\theta = \left|\dfrac{1(2)+(-2)(-3)+4(3)}{\sqrt{1+4+16}\sqrt{4+9+9}}\right|$

$\Rightarrow \cos\theta = \dfrac{2+6+12}{\sqrt{21}\sqrt{22}} = \dfrac{20}{\sqrt{462}}$

$\Rightarrow \sqrt{462}\ \cos\theta = 20$

30. $l + m + n = 0$
$\Rightarrow l = -(m+n)$ and $lm = 0 \Rightarrow -(m+n)m = 0$
$\Rightarrow m = 0$ or $m + n = 0 \Rightarrow m = 0$ or $m = -n$
If m = 0, then $l = -n$

∴ $\dfrac{l}{-1} = \dfrac{m}{0} = \dfrac{n}{1}$

If m = –n, then $l = 0$

∴ $\dfrac{l}{0} = \dfrac{m}{-1} = \dfrac{n}{1}$

∴ the d.r.s of the lines are proportional to –1, 0, 1 and 0, –1, 1

∴ angle between them is

$\cos\theta = \left|\dfrac{0+0+1}{\sqrt{1+0+1}\sqrt{0+1+1}}\right| = \dfrac{1}{2}$

∴ $\theta = \dfrac{\pi}{3}$

31. $l + m - n = 0$ and $l^2 + m^2 - n^2 = 0$
$\Rightarrow l + m = n$ and $l^2 + m^2 = n^2$
Putting $l + m = n$ in $l^2 + m^2 = n^2$, we get
$l^2 + m^2 = (l + m)^2$
$\Rightarrow 2lm = 0 \Rightarrow l = 0$ or $m = 0$
If $l = 0$, then m = n

∴ $\dfrac{l}{0} = \dfrac{m}{1} = \dfrac{n}{1}$

If m = 0, then $l = n$

∴ $\dfrac{l}{1} = \dfrac{m}{0} = \dfrac{n}{1}$

∴ the d.r.s of the lines are proportional to 0, 1, 1 and 1, 0, 1.

∴ $\cos\theta = \left|\dfrac{0(1)+1(0)+1(1)}{\sqrt{0+1+1}\sqrt{1+0+1}}\right| = \dfrac{1}{2}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{2}\right) \Rightarrow \theta = \dfrac{\pi}{3}$

32. Since, the three lines are mutually perpendicular

∴ $l_1l_2 + m_1m_2 + n_1n_2 = 0$
$l_2l_3 + m_2m_3 + n_2n_3 = 0$
$l_3l_1 + m_3m_1 + n_3n_1 = 0$

Also, $l_1^2 + m_1^2 + n_1^2 = 1$,
$l_2^2 + m_2^2 + n_2^2 = 1$,
$l_3^2 + m_3^2 + n_3^2 = 1$

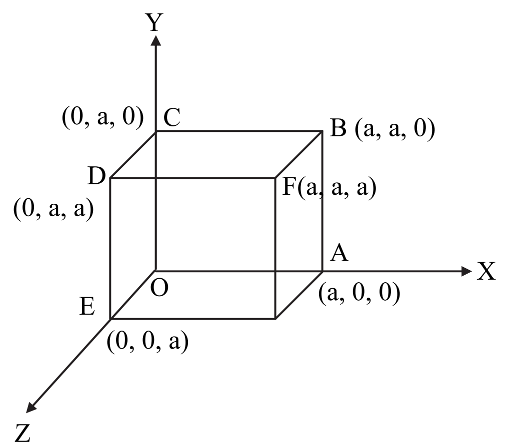
Now, $(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2$
$= (l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) + (l_3^2 + m_3^2 + n_3^2) + 2(l_1l_2 + m_1m_2 + n_1n_2) + 2(l_2l_3 + m_2m_3 + n_2n_3) + 2(l_3l_1 + m_3m_1 + n_3n_1)$
$= 3$

$\Rightarrow (l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2 = 3$

Hence, direction cosines of required line are :

$\dfrac{l_1+l_2+l_3}{\sqrt{3}}, \dfrac{m_1+m_2+m_3}{\sqrt{3}}, \dfrac{n_1+n_2+n_3}{\sqrt{3}}$

33.



The d.r.s of diagonal EB = a, a, –a
The d.r.s of diagonal AD = –a, a, a

∴ Angle between EB and AD is

$\cos\theta = \left|\dfrac{-a^2 + a^2 - a^2}{3a^2}\right|$

∴ $\theta = \cos^{-1}\left(\dfrac{1}{3}\right)$

34. As d.r.s. are proportional, the required lines are parallel to the given lines.

∴ $(\text{d.r.s.})_1 \equiv 2, 3, -6$ and $(\text{d.r.s.})_2 \equiv 3, -4, 5$

∴ $\cos\theta = \left|\dfrac{6-12-30}{\sqrt{49}\sqrt{50}}\right| = \dfrac{36}{7\left(5\sqrt{2}\right)} = \dfrac{18\sqrt{2}}{7(5)}$

∴ $\theta = \cos^{-1}\left(\dfrac{18\sqrt{2}}{35}\right)$

35. Let A ≡ (–2, 1, –8) and B ≡ (a, b, c)
∴ the d.r.s of the line AB are a + 2, b – 1, c + 8
Since, AB is parallel to the line whose d.r.s are 6, 2, 3.
∴ $\frac{a+2}{6}=\frac{b-1}{2}=\frac{c+8}{3}$
Only option (A) satisfies this condition.

36. The d.r.s of AB and CD are 1, 2, –2 and 2, 3, 4 respectively
Now, $a_1a_2 + b_1b_2 + c_1c_2 = 1(2) + 2(3) + (-2)(4) = 0$
∴ AB ⊥ CD,
∴ projection of AB on CD is 0.

37. As $\frac{a}{\left(\frac{1}{bc}\right)}=\frac{b}{\left(\frac{1}{ca}\right)}=\frac{c}{\left(\frac{1}{ab}\right)}$
the lines are parallel.

38. $\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix}$
$= \hat{i}(m_1n_2 - m_2n_1) + \hat{j}(n_1l_2 - l_1n_2) + \hat{k}(l_1m_2 - m_1l_2)$
∴ The d.c.s are
$m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1,\ l_1m_2 - l_2m_1$

39. $\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -3\\ 1 & -2 & 1\end{vmatrix} = -5\hat{i} - 5\hat{j} - 5\hat{k}$
∴ d.r.s of line are –5, –5, –5
i.e. –1, –1,–1
∴ the d.c.s are $\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}$

40. The vectors $4\hat{i}+\hat{j}+3\hat{k}$ and $2\hat{i}-3\hat{j}+\hat{k}$ will lie along the given lines.
The vector perpendicular to these vectors is given by
$(4\hat{i}+\hat{j}+3\hat{k})\times(2\hat{i}-3\hat{j}+\hat{k})$
$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 1 & 3\\ 2 & -3 & 1\end{vmatrix} = 10\hat{i}+2\hat{j}-14\hat{k}$
∴ The d.r.s of required line are 10, 2, –14.
∴ The d.c.s. are $\frac{1}{\sqrt{3}},\frac{1}{5\sqrt{3}},\frac{-7}{5\sqrt{3}}$

Competitive Thinking

2. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$
$= 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1$
$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3 = 2(1) - 3 = -1$

3. $l^2 + m^2 + n^2 = 1$
∴ $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + n^2 = 1$
$\Rightarrow n^2 = \frac{23}{36} \Rightarrow n = \pm\frac{\sqrt{23}}{6}$

4. $l^2 + m^2 + n^2 = 1$
∴ $\frac{1}{c^2}+\frac{1}{c^2}+\frac{1}{c^2} = 1$
$\Rightarrow c^2 = 3 \Rightarrow c = \pm\sqrt{3}$

5. Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
∴ $\cos^2 45° + \cos^2 120° + \cos^2\theta = 1$
$\Rightarrow \cos^2\theta = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2}$
Since, θ is an acute angle.
∴ $\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$

6. Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
∴ $\cos^2 120° + \cos^2\beta + \cos^2 60° = 1$
$\Rightarrow \left(\frac{-1}{2}\right)^2 + \cos^2\beta + \left(\frac{1}{2}\right)^2 = 1$
$\Rightarrow \cos^2\beta = 1 - \frac{1}{4} - \frac{1}{4}$
$\Rightarrow \cos^2\beta = \frac{1}{2}$
$\Rightarrow \cos\beta = \pm\frac{1}{\sqrt{2}} \Rightarrow \beta = 45°$ or $135°$

7. Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
∴ $\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} + \cos^2\gamma = 1$
$\Rightarrow \cos^2\gamma = 1 - \frac{1}{2} - \frac{1}{2} = 0$
$\Rightarrow \cos\gamma = 0 \Rightarrow \gamma = \frac{\pi}{2}$

8. Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
∴ $\cos^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{5\pi}{4}\right) + \cos^2\gamma = 1$
$\Rightarrow \cos^2\gamma = 1 - \frac{1}{2} - \frac{1}{2}$
$\Rightarrow \cos^2\gamma = 0 \Rightarrow \gamma = \frac{\pi}{2}$

9. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \cos^2 45° + \cos^2\beta + \cos^2\beta = 1$ ....($\because \beta = \gamma$)

$\Rightarrow 2\cos^2\beta = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \cos^2\beta = \frac{1}{4}$

$\therefore$ $\beta = 60° = \gamma$

$\Rightarrow \alpha + \beta + \gamma = 165°$

10. Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore$ $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$ ....($\alpha = \beta = \gamma$)

$\Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = \frac{1}{3}$

$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

Now, $l = m = n = \cos\alpha$

$\therefore$ $l = m = n = \pm\frac{1}{\sqrt{3}}$

11. Since,

$\alpha = \beta = \gamma \Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow \cos\alpha = \left(\pm\frac{1}{\sqrt{3}}\right)$

So, there are four lines whose direction cosines are

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$,

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$.

12. Since, the vector is equally inclined to the co-ordinate axes,

$l = m = n = \pm\frac{1}{\sqrt{3}}$

13. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \frac{1+\cos 2\alpha}{2} + \frac{1+\cos 2\beta}{2} + \frac{1+\cos 2\gamma}{2} = 1$

$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma + 3 = 2$

$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 0$

14. $\cos^2\frac{\alpha}{2} + \cos^2\frac{\beta}{2} + \cos^2\frac{\gamma}{2} = 1$

Now, $\cos\alpha + \cos\beta + \cos\gamma$

$= 2\cos^2\frac{\alpha}{2} - 1 + 2\cos^2\frac{\beta}{2} - 1 + 2\cos^2\frac{\gamma}{2} - 1$

$= 2(1) - 3 = -1$

15. Let the length of the line segment be r and its direction cosines be l, m, n.

$\therefore$ The projections on the co-ordinate axes are lr, mr, nr.

$\therefore$ lr = 3, mr = 4 and nr = 5

$\therefore$ $l^2r^2 + m^2r^2 + n^2r^2 = 3^2 + 4^2 + 5^2$

$\Rightarrow r^2(l^2 + m^2 + n^2) = 9 + 16 + 25$

$\Rightarrow r^2 = 50$ ....[$\because l^2 + m^2 + n^2 = 1$]

$\Rightarrow r = \sqrt{50} = 5\sqrt{2}$

16. The projections on the co-ordinate axes are lr, mr, nr.

$\therefore$ lr = 2, mr = 3 and nr = 6

$\therefore$ $l^2r^2 + m^2r^2 + n^2r^2 = 4 + 9 + 36$

$\Rightarrow r^2(l^2 + m^2 + n^2) = 49$

$\Rightarrow r = 7$

17. d.r.s. of line are $-2 - 4$, $1 - 3$, $-8 - (-5)$

i.e., $-6, -2, -3$

i.e. 6, 2, 3

18. AD is the median

$\therefore$ $D \equiv \left(\frac{\lambda-1}{2}, \frac{5+3}{2}, \frac{\mu+2}{2}\right) \equiv \left(\frac{\lambda-1}{2}, 4, \frac{\mu+2}{2}\right)$

$\therefore$ d.r.s. of AD are $\frac{\lambda-1}{2} - 2,\ 4 - 3,\ \frac{\mu+2}{2} - 5$

i.e. $\frac{\lambda-5}{2}, 1, \frac{\mu-8}{2}$ ...(i)

Since AD is equally inclined to co-ordinate axes, its d.r.s. are $\pm1, \pm1, \pm1$

Option (D) satisfies (i).

19. The d.c.s. are

$\frac{1}{\sqrt{1+9+4}}, \frac{-3}{\sqrt{1+9+4}}, \frac{2}{\sqrt{1+9+4}}$

$\Rightarrow \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}$.

20. d.r.s. of line are $-2 - 4$, $1 - 3$, $-8 + 5$

i.e., $-6, -2, -3$ i.e., 6, 2, 3

$\therefore$ The d.c.s. are $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$

21. The d.r.s of OP are 3, 12, 4

$\therefore$ The required d.c.s. are

i.e., $\frac{3}{13}, \frac{12}{13}, \frac{4}{13}$

22. Let the length of the line segment be r and its direction cosines be l, m, n.

$\therefore$ The projections on the co-ordinate axes are lr, mr, nr.

$\therefore$ lr = 6, mr = –3 and nr = 2

$\therefore$ $l^2r^2 + m^2r^2 + n^2r^2 = (6)^2 + (-3)^2 + (2)^2$

$\Rightarrow r^2(l^2 + m^2 + n^2) = 36 + 9 + 4$

$\Rightarrow r^2 = 49$ ....$[\because l^2 + m^2 + n^2 = 1]$

$\Rightarrow r = 7$

Now, d.c.s. of line are $\frac{6}{r}, \frac{-3}{r}, \frac{2}{r}$

i.e., $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$.

23. Here, $\bar{a} = 3\hat{i} + 5\hat{j} - 2\hat{k}$, $\bar{b} = 6\hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore$ Projection = $\frac{\bar{a}.\bar{b}}{|\bar{b}|} = \frac{18+10-6}{7} = \frac{22}{7}$

24. For option (C), $\frac{4-(-2)}{-3-4} \neq \frac{-3-4}{-2-(-3)}$

$\therefore$ option (C) is the correct answer.

25. Let A(5, –2, 7), B(2, 2, β), C(–1, 6, –1) be the given points

d.r.s. of AB are 2 – 5, 2 + 2, β – 7

i.e., –3, 4, β – 7

d.r.s. of BC are –1 – 2, 6 – 2, –1 – β

i.e., –3, 4, –1 – β

Since, the points are collinear

$\therefore$ AB || BC

$\therefore$ $\frac{4}{4} = \frac{\beta - 7}{-1-\beta} \Rightarrow \beta - 7 = -1 - \beta \Rightarrow \beta = 3$

26. Let A (–1, 2, –3), B (4, a, 1) and C (b, 8, 5)

Since, the given points are collinear.

$\therefore$ AB || BC

$\therefore$ $\frac{4-(-1)}{b-4} = \frac{a-2}{8-a} = \frac{1-(-3)}{5-1}$

$\Rightarrow \frac{5}{b-4} = 1, \frac{a-2}{8-a} = 1$

$\Rightarrow$ b = 9, a = 5

27. P(4, 5, x), Q(3, y, 4) and R(5, 8, 0)

Since, the points are collinear

$\therefore$ PQ || QR

$\therefore$ $\frac{-1}{2} = \frac{y-5}{8-y} = \frac{4-x}{-4}$

$\Rightarrow \frac{-1}{2} = \frac{y-5}{8-y}$ and $\frac{4-x}{-4} = \frac{-1}{2}$

$\Rightarrow y - 8 = 2y - 10$ and $8 - 2x = 4$

$\Rightarrow y = 2$ and $x = 2$

$\therefore$ $x + y = 4$

28. d.r.s. of AB and BC are (–2, 2, 2) and (1, –1, –1) respectively.

$\therefore$ $\frac{-2}{1} = \frac{2}{-1} = \frac{2}{-1}$

$\therefore$ the given points are collinear.

29. The d.r.s. of the diagonal of the line joining the origin to the opposite corner of cube are a – 0, a – 0, a – 0 i.e. 1, 1, 1.

30. Here, $a_1, b_1, c_1 = 1, 1, 2$ and

$a_2, b_2, c_2 = \sqrt{3} - 1, -\sqrt{3} - 1, 4$

$\therefore$ $\cos\theta = \frac{1(\sqrt{3}-1) + 1(-\sqrt{3}-1) + 2(4)}{\sqrt{1+1+4}\sqrt{(\sqrt{3}-1)^2 + (-\sqrt{3}-1)^2 + 16}}$

$= \frac{6}{\sqrt{6}\sqrt{4+4+16}} = \frac{6}{\sqrt{6}\sqrt{24}} = \frac{1}{2}$

$\therefore$ θ = 60°

31. D.r.s. are 2, 2, 1 and

7 – 3, 2 – 1, 12 – 4 ≡ 4, 1, 8

$\therefore$ $\cos\theta = \frac{2\times4 + 2\times1 + 1\times8}{\sqrt{2^2+2^2+1^2}\sqrt{4^2+1^2+8^2}} = \frac{18}{3\times9} = \frac{2}{3}$

$\therefore$ $\theta = \cos^{-1}\left(\frac{2}{3}\right)$

32. The direction ratios of AB = 1, 2, – 2 and the direction ratios of CD = 2, 3, 4

$a_1a_2 + b_1b_2 + c_1c_2 = (1)(2) + (2)(3) + (-2)(4) = 0$

$\therefore$ AB ⊥ CD $\therefore$ $\theta = \frac{\pi}{2}$

33. Putting $l = -m - n$ in $l^2 = m^2 + n^2$, we get

$(-m-n)^2 = m^2 + n^2$

$\Rightarrow mn = 0 \Rightarrow m = 0$ or $n = 0$

If m = 0, then $l = -n$

$\therefore$ $\frac{l}{-1} = \frac{m}{0} = \frac{n}{1}$

If n = 0, then $l = -m$

$\therefore$ $\frac{l}{-1} = \frac{m}{1} = \frac{n}{0}$

$\therefore$ $a_1, b_1, c_1 = -1, 0, 1$ and

$a_2, b_2, c_2 = -1, 1, 0$

$\therefore$ The angle between the lines is given by

$\cos\theta = \frac{1+0+0}{\sqrt{1+0+1}\sqrt{1+1+0}} = \frac{1}{2}$

$\therefore$ $\theta = \frac{\pi}{3}$

34. Let the direction ratios of the line perpendicular to both the lines be a, b, c.
The line is perpendicular to the lines with Direction ratios −1, 2, 2 and 0, 2, 1

$\therefore$ $-a + 2b + 2c = 0$ ….(i)

$2b + c = 0$ ….(ii)

Solving (i) and (ii), we get

$\frac{a}{-2} = \frac{b}{1} = \frac{c}{-2}$

$\therefore$ The d.r.s. of the line are 2, −1, 2.

$\therefore$ The required d.c.s. of the line are $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$.

35. The d.r.s. of the two lines are 1, −1, 2 and 2, 1, −1
Let d.r.s. of the line be a, b, c.

$\therefore$ $a - b + 2c = 0$ ….(i)

and $2a + b - c = 0$ ….(ii)

Solving (i) and (ii), we get

$\frac{a}{-1} = \frac{b}{5} = \frac{c}{3}$

$\therefore$ d.r.s. of the line are −1, 5, 3.

$\therefore$ the required d.c.s. are $\frac{-1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}$.

36. If the straight line makes angles α, β, γ, δ with diagonals of a cube, then

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

$\therefore$ $\frac{1+\cos 2\alpha}{2} + \frac{1+\cos 2\beta}{2} + \frac{1+\cos 2\gamma}{2} + \frac{1+\cos 2\delta}{2} = \frac{4}{3}$

$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta = \frac{8}{3} - 4$

$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta = \frac{-4}{3}$

37. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

$\therefore$ $1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma + 1 - \sin^2\delta = \frac{4}{3}$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = 4 - \frac{4}{3} = \frac{8}{3}$

Evaluation Test

1. $\alpha = \beta = 2\gamma$

$\Rightarrow \beta = \alpha, \gamma = \frac{\alpha}{2}$

Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore$ $\cos^2\alpha + \cos^2\alpha + \cos^2\frac{\alpha}{2} = 1$

$\Rightarrow 2\cos^2\alpha + \frac{1+\cos\alpha}{2} = 1$

$\Rightarrow 4\cos^2\alpha + \cos\alpha - 1 = 0$

$\Rightarrow \cos\alpha = \frac{-1 \pm \sqrt{1+16}}{2(4)} = \frac{-1 \pm \sqrt{17}}{8}$

If α is acute, then cos α is positive.

$\therefore$ $\cos\alpha = \frac{\sqrt{17}-1}{8}$

2. $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$\sin^2\theta = 1 - \cos^2\theta$

$= 1.1 - \cos^2\theta$

$= \left(l_1^2 + m_1^2 + n_1^2\right)\left(l_2^2 + m_2^2 + n_2^2\right) - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$= l_1^2l_2^2 + l_1^2m_2^2 + l_1^2n_2^2 + l_2^2m_1^2 + m_1^2m_2^2 + m_1^2n_2^2 + l_2^2n_1^2 + m_2^2n_1^2 + n_1^2n_2^2 - l_1^2l_2^2 - m_1^2m_2^2 - n_1^2n_2^2 - 2l_1l_2m_1m_2 - 2m_1m_2n_1n_2 - 2n_1n_2l_1l_2$

$= l_1^2m_2^2 - 2l_1l_2m_1m_2 + l_2^2m_1^2 + m_1^2n_2^2 - 2m_1m_2n_1n_2 + m_2^2n_1^2 + l_2^2n_1^2 - 2l_1l_2n_1n_2 + l_1^2n_2^2$

$= (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2$

3. Given, A(2, 3, 7), B(−1, 3, 2), C(p, 5, r)
Let D be the midpoint of BC.

$\therefore$ $D \equiv \left(\frac{-1+p}{2}, \frac{3+5}{2}, \frac{2+r}{2}\right) \equiv \left(\frac{p-1}{2}, 4, \frac{r+2}{2}\right)$

$\therefore$ d.r.s. of AD are $\frac{p-1}{2} - 2, 4 - 3, \frac{r+2}{2} - 7$

i.e., $\frac{p-5}{2}, 1, \frac{r-12}{2}$

Since, AD is equally inclined to the axes

$\therefore$ $\frac{p-5}{2} = 1 = \frac{r-12}{2}$

$\Rightarrow p = 7, r = 14$

4. The d.r.s of AB are $3 - 1, 2 - 4, 6 - 5$
i.e. $2, -2, 1$
Let $a_1, b_1, c_1 = 2, -2, 1$
d.r.s. of BC are $1 - 3, 4 - 5, 5 - 3$
i.e., $-2, -1, 2$
Let $a_2, b_2, c_2 = -2, -1, 2$
$\therefore$ $a_1a_2 + b_1b_2 + c_1c_2 = 2(-2) + (-2)(-1) + 1(2)$
$= -4 + 2 + 2 = 0$
$\Rightarrow$ AB and BC are perpendicular.
$\therefore$ $m\angle ABC = 90°$

5. The given equations are
$6mn - 2nl + 5lm = 0$, and ....(i)
$3l + m + 5n = 0$
$\Rightarrow m = -3l - 5n$ ....(ii)
Substituting value of m in equation (i), we get
$6(-3l - 5n)n - 2nl + 5l(-3l - 5n) = 0$
$\Rightarrow -18ln - 30n^2 - 2nl - 15l^2 - 25nl = 0$
$\Rightarrow 15l^2 + 45ln + 30n^2 = 0$
$\Rightarrow l^2 + 3ln + 2n^2 = 0$
$\Rightarrow (l + n)(l + 2n) = 0$
$\Rightarrow l = -n$ or $l = -2n$
If $l = -n$, then $m = -2n$
$\Rightarrow \dfrac{l}{1} = \dfrac{n}{-1}$ and $\dfrac{m}{2} = \dfrac{n}{-1}$
$\Rightarrow \dfrac{l}{1} = \dfrac{m}{2} = \dfrac{n}{-1}$
$\therefore$ d.r.s. of the 1st line are $1, 2, -1$.
If $l = -2n$, then $m = n$
$\Rightarrow \dfrac{l}{-2} = \dfrac{n}{1}$ and $\dfrac{m}{1} = \dfrac{n}{1}$
$\Rightarrow \dfrac{l}{-2} = \dfrac{m}{1} = \dfrac{n}{1}$
$\therefore$ d.r.s. of the 2nd line are $-2, 1, 1$.
$\therefore$ $\cos\theta = \dfrac{1\times(-2) + 2\times 1 + (-1)\times 1}{\sqrt{1^2 + 2^2 + (-1)^2}\sqrt{(-2)^2 + 1^2 + 1^2}}$
$= \dfrac{-2 + 2 - 1}{\sqrt{6}\sqrt{6}} = \dfrac{-1}{6}$
$\therefore$ $\theta = \cos^{-1}\left(\dfrac{-1}{6}\right)$

6. Since, $(l - m)^2 \geq 0$
$\therefore$ $l^2 - 2lm + m^2 \geq 0$
$\therefore$ $l^2 + m^2 \geq 2lm$ ....(i)
Similarly, $m^2 + n^2 \geq 2mn$ ....(ii)
and $n^2 + l^2 \geq 2nl$ ....(iii)
Adding (i), (ii) and (iii), we get
$2(l^2 + m^2 + n^2) \geq 2(lm + mn + nl)$
$\therefore$ $lm + mn + nl \leq 1$
$\therefore$ The maximum value of $lm + mn + nl$ is 1.

7. Let $A = (a, 2, 3)$, $B \equiv (3, b, 7)$ and $C \equiv (-3, -2, -5)$
d.r.s of AB are $3 - a, b - 2, 4$
d.r.s of BC are $-6, -2-b, -12$
Since the points are collinear
$\therefore$ $\dfrac{3-a}{-6} = \dfrac{b-2}{-2-b} = \dfrac{4}{-12}$
$\Rightarrow a = 2, b = 4$

8. Let the d.r.s of the line perpendicular to both the lines be a, b, c.
d.r.s of lines is $1, -1, 0$ and $2, -1, 1$
$\therefore$ $a - b = 0$ ....(i)
$2a - b + c = 0$ ....(ii)
On solving (i) and (ii), we get
$\dfrac{a}{-1} = \dfrac{b}{-1} = \dfrac{c}{1}$
$\therefore$ d.r.s of the line are $-1, -1, 1$
$\therefore$ the required d.c.s are $\dfrac{-1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

9. Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
$\therefore$ $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$ ....[$\because \alpha = \beta = \gamma$]
$\Rightarrow 3\cos^2\alpha = 1$
$\Rightarrow \cos\alpha = \pm\dfrac{1}{\sqrt{3}}$
$\therefore$ $l = m = n = \cos\alpha = \pm\dfrac{1}{\sqrt{3}}$

Textbook Chapter No.

07 Line

Hints

Classical Thinking

1. On X-axis, $y = 0$ and $z = 0$

2. On Y-axis, the co-ordinates of x and $z = 0$

3. Equation of X-axis is $y = 0$, $z = 0$.
Hence y and z remain fixed.

4. Vector equation of line passing through $\bar{a}$ and parallel to $\bar{b}$ is
$\bar{r} = \bar{a} + \lambda\bar{b}$
$\therefore$ $\bar{r} = (\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 5\hat{k})$

5. Let $A \equiv (2, 1, -1)$
$\therefore$ $\bar{a} = 2\hat{i} + \hat{j} - \hat{k}$
$\bar{b} = \hat{i} + 2\hat{j} + \hat{k}$
Now, $\bar{r} = (2\hat{i} + \hat{j} - \hat{k}) + \lambda(\hat{i} + 2\hat{j} + \hat{k})$

6. The given line passes through $(3, -4, 6)$
The d.r.s. of line are 2, 5, 3
$\therefore$ The given line is parallel to $2\hat{i} + 5\hat{j} + 3\hat{k}$
$\therefore$ The vector equation of the line is
$\bar{r} = (3\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(2\hat{i} + 5\hat{j} + 3\hat{k})$

7. The given line passes through $(5, -4, 6)$
The d.r.s. of line are 3, 7, 2
$\therefore$ The given line is parallel to $3\hat{i} + 7\hat{j} + 2\hat{k}$
$\therefore$ The vector equation of the line is
$\bar{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

8. Given, cartesian equation of the line is
$3x - 2 = 2y + 1 = 3z - 3$
$$\Rightarrow 3\left(x - \frac{2}{3}\right) = 2\left(y + \frac{1}{2}\right) = 3(z - 1)$$
$$\Rightarrow \frac{x - \frac{2}{3}}{2} = \frac{y + \frac{1}{2}}{3} = \frac{z - 1}{2}$$
$\therefore$ The given line passes through $\left(\frac{2}{3}, \frac{-1}{2}, 1\right)$, and has direction ratios proportional to 2, 3, 2.
$\therefore$ The vector equation is
$$\bar{r} = \left(\frac{2}{3}\hat{i} - \frac{1}{2}\hat{j} + \hat{k}\right) + \lambda(2\hat{i} + 3\hat{j} + 2\hat{k})$$

9. Given cartesian equation of the line is
$6x - 2 = 3y + 1 = 1 - 2z$
$$\Rightarrow 6\left(x - \frac{1}{3}\right) = 3\left(y + \frac{1}{3}\right) = -2\left(z - \frac{1}{2}\right)$$
$$\Rightarrow \frac{x - \frac{1}{3}}{1} = \frac{y + \frac{1}{3}}{2} = \frac{z - \frac{1}{2}}{-3}$$
The given line passes through $\left(\frac{1}{3}, \frac{-1}{3}, \frac{1}{2}\right)$ and the direction ratios are proportional to 1, 2, – 3
$\therefore$ The vector equation is
$$\bar{r} = \left(\frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{1}{2}\hat{k}\right) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$$

11. The given vector equation is
$\bar{r} = 3\hat{i} - 5\hat{j} + 7\hat{k} + \lambda(2\hat{i} + \hat{j} - 3\hat{k})$
$\therefore$ The line passes through $(3, -5, 7)$ and has direction ratios 2, 1, –3
$\therefore$ The equation of line is $\frac{x - 3}{2} = \frac{y + 5}{1} = \frac{z - 7}{-3}$

12. The required lines passes through $(2, -1, 1)$ and has d.r.s. proportional to 2, 7, –3
$\therefore$ The equation of line is
$\bar{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} + 7\hat{j} - 3\hat{k})$

13. The line is parallel to $\frac{x - 2}{3} = \frac{y - 3}{-1} = \frac{z + 1}{2}$
$\therefore$ d.r.s of line are 3, –1, 2
also, the line passes through origin
$\therefore$ The equation of line is
$\bar{r} = \lambda(3\hat{i} - \hat{j} + 2\hat{k})$

14. $\frac{2x-1}{2}=\frac{1-y}{1}=\frac{z}{3}$ $\Rightarrow \frac{x-\frac{1}{2}}{1}=\frac{y-1}{-1}=\frac{z}{3}$

$\therefore$ The direction ratios of the required line are 1, –1, 3.

Also line passes through (2, –1, 3)

$\therefore$ Equation of the line is $\frac{x-2}{1}=\frac{y+1}{-1}=\frac{z-3}{3}$

15. Let $\bar{a}$ and $\bar{b}$ be the position vectors of the points

$\therefore$ $\bar{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$ and $\bar{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\therefore$ $\bar{b}-\bar{a} = 3\hat{i} - 2\hat{j} + 6\hat{k} - 3\hat{i} + 2\hat{j} + 5\hat{k}$

$= 11\hat{k}$

The vector equation of line is given by

$\bar{r} = \bar{a} + \lambda(\bar{b}-\bar{a})$

$\Rightarrow \bar{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda(11\hat{k})$

16. Let $\bar{a}=-2\hat{i}+\hat{j}+3\hat{k}$ and $\bar{b}=\hat{i}-2\hat{j}+5\hat{k}$

$\therefore$ $\bar{b}-\bar{a}=3\hat{i}-3\hat{j}+2\hat{k}$

The vector equation of the line is $\bar{r}=\bar{a}+\lambda\left(\bar{b}-\bar{a}\right)$

$\bar{r}=-2\hat{i}+\hat{j}+3\hat{k}+\lambda(3\hat{i}-3\hat{j}+2\hat{k})$

17. The equation of line passing through (x_1, y_1, z_1) and (x_2, y_2, z_2)

$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$\therefore$ The equation of line passing through (4, –5, –2) and (–1, 5, 3) is

$\frac{x-4}{-1-4}=\frac{y+5}{5+5}=\frac{z+2}{3+2}$

$\Rightarrow \frac{x-4}{1}=\frac{y+5}{-2}=\frac{z+2}{-1}$

18. The required equation of line which passes through the points (1, 2, 3) and (0, 0, 0) is

$\frac{x-1}{0-1}=\frac{y-2}{0-2}=\frac{z-3}{0-3}$

$\Rightarrow \frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$

19. The equation of the line joining the points (– 2, 4, 2) and (7, – 2, 5) is

$\frac{x+2}{7-(-2)}=\frac{y-4}{-2-4}=\frac{z-2}{5-2}$

$\Rightarrow \frac{x+2}{3}=\frac{y-4}{-2}=\frac{z-2}{1}$

20. $2x + z - 4 = 0$

$\Rightarrow 2x + z = 4$

$\Rightarrow z = 4 - 2x$ ….(i)

$2y + z = 0$

$\Rightarrow z = -2y$ ….(ii)

$\therefore$ $4 - 2x = -2y = z$ ….[From (i) and (ii)]

$\Rightarrow -2(x-2) = -2y = z$

$\Rightarrow x-2 = y = \frac{z}{-2}$

$\Rightarrow x-2+2 = y+2 = \frac{z}{-2}+2$

$\Rightarrow \frac{x}{1}=\frac{y+2}{1}=\frac{z-4}{-2}$

21. $a_1, b_1, c_1 = 1, 2, 2$ and $a_2, b_2, c_2 = 3, 2, 6$

$\therefore$ $\cos\theta = \left|\frac{1\times3+2\times2+2\times6}{\sqrt{1^2+2^2+2^2}\sqrt{3^2+2^2+6^2}}\right|$

$= \frac{19}{3\times7} = \frac{19}{21}$

22. $a_1, b_1, c_1 = 2, 2, -1$ and $a_2, b_2, c_2 = 1, 2, 2$

$\cos\theta = \left|\frac{2\times1+2\times2+(-1)\times2}{\sqrt{2^2+2^2+(-1)^2}\sqrt{1^2+2^2+2^2}}\right|$

$= \left|\frac{2+4-2}{3\times3}\right| = \frac{4}{9}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{4}{9}\right)$

23. $a_1, b_1, c_1 = 3, -2, 0$ and $a_2, b_2, c_2 = 2, 3, 4$

$\Rightarrow \cos\theta = \left|\frac{3\times2+(-2)\times3+0\times4}{\sqrt{3^2+(-2)^2+0}.\sqrt{2^2+3^2+4^2}}\right|$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \frac{\pi}{2}$

24. $a_1, b_1, c_1 = 1, 2, 3$ and $a_2, b_2, c_2 = 2, 2, -2$

$a_1a_2 + b_1b_2 + c_1c_2 = 1(2) + 2(2) + 3(-2) = 0$

$\therefore$ The lines are at right angles.

25. $a_1, b_1, c_1 = 1, 2, 3$ and $a_2, b_2, c_2 = -5, 1, 1$

$a_1a_2 + b_1b_2 + c_1c_2 = (1)(-5) + (2)(1) + (3)(1)$

$= 0$

$\therefore$ Lines are at right angle.

26. The given equation of line is,

$\frac{x-2}{3}=\frac{y-3}{4}$; $z = 4$

$\therefore$ The line is perpendicular to Z-axis.

Hence parallel to XY-plane.

27. Line L_1: $\bar{r} = (2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$
Line L_2: $\bar{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$
L_1 and L_2 can be written in cartesian form as
L_1: $\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ and
L_2: $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$
The point (2, 6, 3) satisfies both the equations.
∴ it is the point of intersection.
Alternate method:
L_1: $\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3} = \lambda$
$\Rightarrow x = \lambda, y = 2\lambda + 2, z = 3\lambda - 3.$
L_2: $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4} = \mu$
$\Rightarrow x = 2\mu + 2, y = 3\mu + 6, z = 4\mu + 3$
Co-ordinates of a point on the line L_1 are
$(\lambda, 2\lambda + 2, 3\lambda - 3)$
Co-ordinates of a point on the line L_2 are
$(2\mu + 2, 3\mu + 6, 4\mu + 3)$
They intersect. Therefore, their co-ordinates must be same.
∴ $\lambda = 2\mu + 2, 2\lambda + 2 = 3\mu + 6, 3\lambda - 3 = 4\mu + 3$
$\Rightarrow \lambda - 2\mu = 2$ ….(i)
$2\lambda - 3\mu = 4$ .....(ii)
$3\lambda - 4\mu = 6$ ….(iii)
Solving equations (i) and (ii), we get
$\lambda = 2, \mu = 0.$
Equation (i) holds true for these values.
∴ Intersection is (2, 6, 3).

28. The point (–1, –1, –1) satisfies both the equations so it is the point of intersection
Alternate method:
Let $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$
$\Rightarrow x = 1 + 2\lambda, y = 2 + 3\lambda, z = 3 + 4\lambda.$
Let $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{1} = \mu$
$\Rightarrow x = 4 + 5\mu, y = 1 + 2\mu, z = \mu$
Co-ordinates of a point on the first line are
$(1 + 2\lambda, 2 + 3\lambda, 3 + 4\lambda)$
Co-ordinates of a point on the second line are
$(4 + 5\mu, 1 + 2\mu, \mu)$
They intersect. Therefore, their co-ordinates must be same.
∴ $1 + 2\lambda = 4 + 5\mu, 2 + 3\lambda = 1 + 2\mu, 3 + 4\lambda = \mu$
$\Rightarrow 2\lambda - 5\mu = 3$ ….(i)
$3\lambda - 2\mu = -1$ .....(ii)
$4\lambda - \mu = -3$ ….(iii)
Solving equations (ii) and (iii), we get
$\lambda = -1, \mu = -1.$
Equation (i) holds true for these values.
∴ Intersection is (– 1, – 1, – 1).

29. The point (4, 0, –1) satisfies both equations.
∴ The two lines intersect at (4, 0, –1)
Alternate method:
Let $\frac{x-1}{3} = \frac{y-1}{-1} = \lambda; z = -1$
$\Rightarrow$ general point on this line is
$(3\lambda + 1, -\lambda + 1, -1)$
Also, $\frac{x-4}{2} = \frac{z+1}{3} = \mu; y = 0$
$\Rightarrow$ general point on this line is
$(2\mu + 4, 0, 3\mu - 1)$
For $\lambda = 1$ and $\mu = 0$, they have a common point on them. i.e., (4, 0, –1)

30. Co-ordinate of any point on Y-axis is
$x = 0, z = 0$ i.e. $(0, y, 0)$
∴ The foot of perpendicular from the point (α, β, γ) on Y-axis is $(0, \beta, 0)$

31. Any point on Z-axis is (0, 0, z)
∴ The foot of perpendicular from the point (a, b, c) on Z-axis is (0, 0, c)

32. Distance from X-axis = $\sqrt{y^2 + z^2} = \sqrt{b^2 + c^2}$

33. Distance = $\sqrt{y^2 + z^2} = \sqrt{9 + 16} = 5$

34. Distance from Z-axis = $\sqrt{x^2 + y^2} = 5$

35. Distance from Y-axis = $\sqrt{1 + 9} = \sqrt{10}$

36. Let $p(\bar{\alpha}) = 2\hat{i} + \hat{j} + \hat{k}$
Comparing the equation of line with
$\bar{r} = \bar{a} + \lambda\bar{b}$, we get
$\bar{a} = -\hat{i} + 2\hat{j} + 2\hat{k}$, $\bar{b} = 3\hat{i} + \hat{k}$
Now,
$\bar{\alpha} - \bar{a} = 3\hat{i} - \hat{j} - \hat{k}$
$|\bar{\alpha} - \bar{a}| = \sqrt{3^2 + (-1)^2 + (-1)^2}$
$= \sqrt{11}$
$(\bar{\alpha} - \bar{a}).\bar{b} = (3\hat{i} - \hat{j} - \hat{k}).(3\hat{i} + \hat{k})$
$= 9 - 1$
$= 8$

$\therefore$ The distance of point from the line is

$$d = \sqrt{\left|\bar{\alpha} - \bar{a}\right|^2 - \left[\frac{(\bar{\alpha} - \bar{a}).\bar{b}}{\left|\bar{b}\right|}\right]^2}$$

$$= \sqrt{11 - \frac{8\times 8}{10}} = \sqrt{\frac{46}{10}} = \sqrt{\frac{23}{5}}$$

37. Let $A \equiv (2, 4, -1)$

Let $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = \lambda$

Any point on the line is

$P \equiv (\lambda - 5, 4\lambda - 3, -9\lambda + 6)$

The d.r.s. of the line AP are

$2 - \lambda + 5, 4 - 4\lambda + 3, -1 + 9\lambda - 6$

Since, AP is perpendicular to the given line,

$1(2 - \lambda + 5) + 4(4 - 4\lambda + 3) - 9(-1 + 9\lambda - 6) = 0$

$\therefore$ $2 - \lambda + 5 + 16 - 16\lambda + 12 + 9 - 81\lambda + 54 = 0$

$\therefore$ $98 - 98\lambda = 0 \Rightarrow \lambda = 1$

The point P is $(1 - 5, 4 - 3, -9 + 6) \equiv (-4, 1, -3)$

$$AP = \sqrt{\left(2-(-4)\right)^2 + (4-1)^2 + (-1+3)^2}$$

$$= \sqrt{36+9+4} = 7$$

Alternate method:

Since the point is $(2, 4, -1)$

$\therefore$ $a = 2, b = 4, c = -1$

Given equation of line is

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$$

Comparing with

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c},$$

$x_1 = -5, y_1 = -3, z_1 = 6$

d.r.s. are $1, 4, -9$

$\therefore$ d.c.s. are $\frac{1}{\sqrt{98}}, \frac{4}{\sqrt{98}}, \frac{-9}{\sqrt{98}}$

$\therefore$ Perpendicular distance of point from the line is

$$\sqrt{\left[(a-x_1)^2 + (b-y_1)^2 + (c-z_1)^2\right] - \left[(a-x_1)l + (b-y_1)m + (c-z_1)n\right]^2}$$

$$= \sqrt{\left[(2+5)^2 + (4+3)^2 + (-1-6)^2\right] - \left[(2+5)\frac{1}{\sqrt{98}} + (4+3)\frac{4}{\sqrt{98}} + (-1-6)\frac{-9}{\sqrt{98}}\right]^2}$$

$$= \sqrt{49 + 49 + 49 - \frac{98\times 98}{98}}$$

$$= \sqrt{49}$$

$$= 7$$

38. $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Any point on the line is $P(\lambda, 2\lambda + 1, 3\lambda + 2)$

Given point is $A(1, 6, 3)$

$\therefore$ the d.r.s of the line AP are

$\lambda - 1, 2\lambda + 1 - 6, 3\lambda + 2 - 3$

$\Rightarrow \lambda - 1, 2\lambda - 5, 3\lambda - 1$

Since, AP is perpendicular to the given line,

$(1)(\lambda - 1) + (2)(2\lambda - 5) + (3)(3\lambda - 1) = 0$

$\Rightarrow \lambda - 1 + 4\lambda - 10 + 9\lambda - 3 = 0$

$\Rightarrow 14\lambda - 14 = 0 \Rightarrow \lambda = 1$

$\therefore$ $P \equiv (1, 3, 5)$

$\therefore$ $AP = \sqrt{(1-1)^2 + (6-3)^2 + (3-5)^2} = \sqrt{13}$

40. First line passes through

$(x_1, y_1, z_1) = (4, -1, 0)$ and has d.r.s

$a_1, b_1, c_1 = 1, 2, -3$

Second line passes through

$(x_2, y_2, z_2) = (1, -1, 2)$ and has d.r.s

$a_2, b_2, c_2 = 2, 4, -5$

$\therefore$ Shortest distance between them is

$$d = \left|\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}\right|$$

$$\Rightarrow d = \left|\frac{\begin{vmatrix} 1-4 & -1+1 & 2-0 \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}}{\sqrt{(-10+12)^2 + (-6+5)^2 + (4-4)^2}}\right|$$

$$= \left|\frac{-3(2) + 0 + 2(0)}{\sqrt{5}}\right| = \frac{6}{\sqrt{5}}$$

Alternate method:

Shortest distance between the lines

$\bar{r}_1 = \bar{a}_1 + \lambda\bar{b}_1$ and $\bar{r}_2 = \bar{a}_2 + \mu\bar{b}_2$ is given by

$$d = \left|\frac{(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2)}{\left|\bar{b}_1 \times \bar{b}_2\right|}\right|$$

Here $\bar{a}_1 = 4\hat{i} - \hat{j}$, $\bar{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$

$\bar{b}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$, $\bar{b}_2 = 2\hat{i} + 4\hat{j} - 5\hat{k}$

Now $\bar{a}_2 - \bar{a}_1 = -3\hat{i} + 2\hat{k}$

$$\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\hat{i} - \hat{j}$$

$\therefore$ Shortest distance (d) $= \left|\dfrac{\left(-3\hat{i}+2\hat{k}\right).\left(2\hat{i}-\hat{j}\right)}{\sqrt{4+1+0}}\right|$

$= \left|-\dfrac{6}{\sqrt{5}}\right|$

$= \dfrac{6}{\sqrt{5}}$

41. Here, $(x_1, y_1, z_1) = (1, -1, 0)$

$(x_2, y_2, z_2) = (2, -1, 0)$

$(a_1, b_1, c_1) = (2, 0, 1)$

$(a_2, b_2, c_2) = (1, -1, -1)$

$$d = \left|\dfrac{\begin{vmatrix} 2-1 & -1+1 & 0-0 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix}}{\sqrt{(0+1)^2+(1+2)^2+(-2-0)^2}}\right|$$

$= \left|\dfrac{1(0+1)}{\sqrt{14}}\right|$

$= \dfrac{1}{\sqrt{14}}$

42. Here, $(x_1, y_1, z_1) = (3, 5, 7)$

$(x_2, y_2, z_2) = (-1, -1, -1)$

$(a_1, b_1, c_1) = (1, -2, 1)$

$(a_2, b_2, c_2) = (7, -6, 1)$

$$d = \left|\dfrac{\begin{vmatrix} -4 & -6 & -8 \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix}}{\sqrt{(-2+6)^2+(7-1)^2+(-6+14)^2}}\right|$$

$= \left|\dfrac{-16-36-64}{2\sqrt{29}}\right|$

$= \dfrac{116}{2\sqrt{29}}$

$= 2\sqrt{29}$

43. Here, $(x_1, y_1, z_1) = (1, 2, 3)$

$(x_2, y_2, z_2) = (2, 4, 5)$

$(a_1, b_1, c_1) = (2, 3, 4)$

$(a_2, b_2, c_2) = (3, 4, 5)$

$$d = \left|\dfrac{\begin{vmatrix} 2-1 & 4-2 & 5-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}}{\sqrt{(15-16)^2+(12-10)^2+(8-9)^2}}\right|$$

$= \left|\dfrac{1(-1)-2(-2)+2(-1)}{\sqrt{(-1)^2+(2)^2+(-1)^2}}\right|$

$= \dfrac{1}{\sqrt{6}}$

44. The given equation of lines are

$x+a = 2y = -12z$ and $x = y+2a = 6z-6a$

i.e., $\dfrac{x+a}{-12} = \dfrac{y}{-6} = \dfrac{z}{1}$ and $\dfrac{x}{6} = \dfrac{y+2a}{6} = \dfrac{z-a}{1}$

$$d = \left|\dfrac{\begin{vmatrix} -a & 2a & -a \\ -12 & -6 & 1 \\ 6 & 6 & 1 \end{vmatrix}}{\sqrt{(-6-6)^2+(6+12)^2+(-72+36)^2}}\right|$$

$= \left|\dfrac{-a(-12)-2a(-12-6)-a(-72+36)}{\sqrt{12^2+18^2+36^2}}\right|$

$= \dfrac{12a+36a+36a}{\sqrt{1764}} = \dfrac{84a}{42} = 2a$

45. Since, the line intersect each other,

$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

$\Rightarrow \begin{vmatrix} 2-1 & 2-k & -1+1 \\ 3 & 6 & -2 \\ -1 & 4 & -1 \end{vmatrix} = 0$

$\Rightarrow 1(-6+8) - (2-k)(-3-2) + 0 = 0$

$\Rightarrow 2 + (2-k)5 = 0$

$\Rightarrow 12 - 5k = 0$

$\Rightarrow k = \dfrac{12}{5}$

46. Comparing the given equations with

$\bar{r} = \bar{a}_1 + \lambda\bar{b}_1$ and

$\bar{r} = \bar{a}_2 + \lambda\bar{b}_2$ we get

$\bar{a}_1 = -\hat{i}+3\hat{j}+\hat{k}$, and $\bar{a}_2 = 3\hat{i}+\hat{j}$

$\bar{b}_1 = \bar{b}_2 = \bar{b} = 5\hat{i}+\hat{j}+4\hat{k}$

$\therefore$ The lines are parallel

$\bar{a}_2 - \bar{a}_1 = 4\hat{i} - 2\hat{j} - \hat{k}$

$$\left(\bar{a}_2 - \bar{a}_1\right) \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -1 \\ 5 & 1 & 4 \end{vmatrix}$$

$$= \hat{i}(-8+1) - \hat{j}(16+5) + \hat{k}(4+10)$$

$$= -7\hat{i} - 21\hat{j} + 14\hat{k}$$

$\therefore$ The distance between the parallel lines is

$$d = \left|\frac{\left(\bar{a}_2 - \bar{a}_1\right) \times \bar{b}}{|\bar{b}|}\right|$$

$\therefore$ $$d = \left|\frac{-7\hat{i} - 21\hat{j} + 14\hat{k}}{\sqrt{25+1+16}}\right|$$

$$= \sqrt{\frac{49+441+196}{42}} = \frac{7}{\sqrt{3}}$$

Critical Thinking

1. The d.r.s. of line are 1, –2, 3 and it passes through point (1, 2, 3)

$\therefore$ the vector equation of the line is

$\bar{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$

The cartesian equation of the line is

$$\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{3}$$

2. The d.r.s. of line are 3, 2, –8 and its passes through (5, 2, –4)

$\therefore$ the vector equation of line is

$\bar{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$

The cartesian equation of the line is

$$\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8}$$

3. The line passes through (2, – 3, 4) and has direction ratios proportional to 3, 4, –5.

$\therefore$ the cartesian equation of the line is

$$\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-4}{-5}$$

$\therefore$ $4x - 8 = 3y + 9$ and $-5y - 15 = 4z - 16$

i.e., $4x - 3y = 17$ and $5y + 4z = 1$

4. Line $||$ Z-axis

$\therefore$ d.r.s. are 0, 0, 1

$\therefore$ Required equation is

$\bar{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + \lambda(0.\hat{i} + 0.\hat{j} + 1\hat{k})$

$\Rightarrow \bar{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda\hat{k}$

5. Let a, b, c, be the direction ratios of the required line.

Since, the line is perpendicular to the lines with d.r.s 3, –16, 7 and 3, 8, –5

$\therefore$ $3a - 16b + 7c = 0$ ....(i)

and $3a + 8b - 5c = 0$ ....(ii)

$\Rightarrow \frac{a}{2} = \frac{b}{3} = \frac{c}{6}$ ....[From (i) and (ii)]

$\therefore$ Equation of the required line is

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

6. Let A ≡ (–1, 3, –2) and B ≡ (–5, 3, –6)

Midpoint of AB = (–3, 3,–4)

Since the line is equally inclined to the axis

$\therefore$ d.r.s. are 1, 1, 1.

$\therefore$ equation of the line is

$$\frac{x+3}{1} = \frac{y-3}{1} = \frac{z+4}{1}$$

$\Rightarrow x + 3 = y - 3 = z + 4$

7. Co-ordinates of G ≡ (1, 1, 1)

D.r.s of OG are 1, 1, 1 and it passes through (0, 0, 0)

$\therefore$ equation of line OG is

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1}$$

$\Rightarrow x = y = z$

8. $\bar{r} = (3\hat{i} + 4\hat{j} + \hat{k}) + t(2\hat{i} - 3\hat{j} + 5\hat{k})$

$= (3 + 2t)\hat{i} + (4 - 3t)\hat{j} + (1 + 5t)\hat{k}$

When the line crosses XY plane $\Rightarrow$ Z = 0

$\therefore$ $1 + 5t = 0 \Rightarrow t = \frac{-1}{5}$

9. The equation of the line joining the points (– 2, 1, – 8) and (a, b, c) is

$$\frac{x-(-2)}{a+2} = \frac{y-1}{b-1} = \frac{z-(-8)}{c+8}$$

The above line is in the direction of vector $6\hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore$ $a + 2 = 6$, $b - 1 = 2$, $c + 8 = 3$

$\Rightarrow a = 4$, $b = 3$ and $c = -5$

10. The equation of the line joining the points (2, 2, 1) and (5, 1, – 2) is

$$\frac{x-2}{5-2} = \frac{y-2}{1-2} = \frac{z-1}{-2-1}$$

$\Rightarrow \frac{x-2}{3} = \frac{y-2}{-1} = \frac{z-1}{-3}$ ....(i)

Since, x co-ordinate is 4

$\therefore$ It satisfies (i)

$\therefore$ $\frac{4-2}{3} = \frac{y-2}{-1} = \frac{z-1}{-3}$

$\therefore$ $\frac{z-1}{-3} = \frac{2}{3}$

$\therefore$ $3z - 3 = -6$

$\therefore$ $z = -1$

11. The equation of the line joining the points (3, 4, 1) and (5, 1, 6) is

$\frac{x-3}{5-3} = \frac{y-4}{1-4} = \frac{z-1}{6-1}$

$\Rightarrow \frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5}$ ....(i)

Co-ordinate of any point on the XY-plane is $z = 0$

$\therefore$ $\frac{x-3}{2} = \frac{y-4}{-3} = \frac{0-1}{5}$

$\therefore$ $\frac{x-3}{2} = \frac{-1}{5}$

$\Rightarrow x - 3 = -\frac{2}{5}$

$\Rightarrow x = \frac{13}{5}$

Also we have $\frac{y-4}{-3} = -\frac{1}{5}$

$\Rightarrow y - 4 = \frac{3}{5} \Rightarrow y = \frac{23}{5}$

$\therefore$ The line meets the XY-plane at $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

12. Here, $(x_1, y_1, z_1) = (3, -6, 10)$ and $|r| = \sqrt{17}$

$x_2 = x_1 + lr = 3 - \frac{2}{\sqrt{17}}\left(\sqrt{17}\right) = 1$

$y_2 = y_1 + mr = -6 + \frac{3}{\sqrt{17}}\left(\sqrt{17}\right) = -3$

$z_2 = z_1 + nr = 10 - \frac{2}{\sqrt{17}}\left(\sqrt{17}\right) = 8$

13. The d.r.s. of the two lines are 2, –1, 1 and 4, –1, λ

Since, the lines are perpendicular

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow 2(4) + (-1)(-1) + (1)(\lambda) = 0$

$\Rightarrow \lambda + 9 = 0$

$\Rightarrow \lambda = -9$

14. $a_1, b_1, c_1 = 2, p, 5$ and

$a_2, b_2, c_2 = 3, -p, p$

Since, the given lines are perpendicular.

$\therefore$ $(2)(3) + p(-p) + (5)(p) = 0$

$\Rightarrow 6 - p^2 + 5p = 0$

$\Rightarrow p^2 - 5p - 6 = 0$

$\Rightarrow (p - 6)(p + 1) = 0$

$\Rightarrow p = 6$ or $p = -1$

15. $a_1, b_1, c_1 = 2, \lambda, 0$ and $a_2, b_2, c_2 = 1, 3, 1$

Since, the lines are perpendicular.

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore$ $2(1) + \lambda(3) + 0(1) = 0$

$\therefore$ $2 + 3\lambda = 0$

$\therefore$ $\lambda = \frac{-2}{3}$

16. Given lines pass through common point (1, 2, 3)

Also, $a_1a_2 + b_1b_2 + c_1c_2 = 2(3) + 3(4) + 4(5) \neq 0$

$\therefore$ lines are intersecting

17. Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$\bar{r} \times \bar{a} = \bar{b} \times \bar{a} \Rightarrow (\bar{r} - \bar{b}) \times \bar{a} = 0$

$\therefore$ $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-2 & y & z+1 \\ 1 & 1 & 0 \end{vmatrix} = 0$

$\Rightarrow (-z-1)\hat{i} - (-z-1)\hat{j} + (x - y - 2)\hat{k} = 0$

$\Rightarrow z = -1, x - y = 2$ ....(i)

Now, $\bar{r} \times \bar{b} = \bar{a} \times \bar{b} \Rightarrow (\bar{r} - \bar{a}) \times \bar{b} = 0$

$\therefore$ $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-1 & y-1 & z \\ 2 & 0 & -1 \end{vmatrix} = 0$

$\Rightarrow (1 - y)\hat{i} - (1 - x - 2z)\hat{j} + (2 - 2y)\hat{k} = 0$

$\Rightarrow y = 1, x + 2z = 1$ ....(ii)

Solving (i) and (ii), we get

$x = 3, y = 1, z = -1$

18. Let P (x, y, z) be any point

Now by the given condition, we get

$\left[\sqrt{(x^2+y^2)}\right]^2 + \left[\sqrt{(y^2+z^2)}\right]^2 + \left[\sqrt{(z^2+x^2)}\right]^2 = 36$

i.e., $x^2 + y^2 + z^2 = 18$

$\therefore$ The distance from origin

$= \sqrt{x^2 + y^2 + z^2} = \sqrt{18} = 3\sqrt{2}$

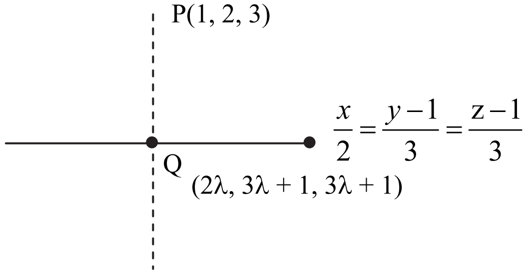
19. Let $\frac{x}{2} = \frac{y-1}{3} = \frac{z-1}{3} = \lambda$

$\therefore$ Any general point on this line is

Q $(2\lambda, 3\lambda+1, 3\lambda+1)$

Let P $\equiv (1, 2, 3)$.

$\therefore$ D.r.s. of PQ are $2\lambda - 1, 3\lambda - 1, 3\lambda - 2$



Since, PQ is perpendicular to given line

$\therefore$ $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

$\therefore$ $(2\lambda - 1)2 + (3\lambda - 1)3 + (3\lambda - 2)3 = 0$

$\therefore$ $\lambda = \frac{1}{2}$

$\therefore$ $Q \equiv \left(1, \frac{5}{2}, \frac{5}{2}\right)$

20. Let $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$

Any point on the line is

$P \equiv (2\lambda, 3\lambda + 2, 4\lambda + 3)$

Given point is A $(3, -1, 11)$

$\therefore$ The d.r.s. of AP are

$2\lambda - 3, 3\lambda + 3, 4\lambda - 8$

Since, the line AP is perpendicular to the given line

$\therefore$ $2(2\lambda - 3) + 3(3\lambda + 3) + 4(4\lambda - 8) = 0$

$\Rightarrow 29\lambda - 29 = 0$

$\Rightarrow \lambda = 1$

$\therefore$ $P \equiv (2, 5, 7)$

21. Let $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3} = \lambda$

$\therefore$ Any general point on this line is

Q $(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$

Let P $\equiv (0, 2, 3)$.

$\therefore$ The d.r.s. of PQ are $5\lambda - 3, 2\lambda - 1, 3\lambda - 7$

Since, PQ is perpendicular to given line

$\therefore$ $5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0$

$\Rightarrow \lambda = 1$

$\therefore$ $Q \equiv (2, 3, -1)$

22. Distance of point $P(\bar{\alpha})$ from the line $\bar{r} = \bar{a} + \lambda\bar{b}$ is

$$\sqrt{|\bar{\alpha} - \bar{a}|^2 - \left[\frac{(\bar{\alpha} - \bar{a}).\bar{b}}{|\bar{b}|}\right]^2}$$

Given, $P(\bar{\alpha}) \equiv (0,0,0)$ and

$\bar{t} = 4\hat{i} + 2\hat{j} + 4\hat{k} + \lambda(3\hat{i} + 4\hat{j} - 5\hat{k})$

$\therefore$ $\bar{a} = 4\hat{i} + 2\hat{j} + 4\hat{k}$ and

$\bar{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

$\therefore$ Distance of point

$$= \sqrt{\left[(-4)^2 + (-2)^2 + (-4)^2\right] - \left[\frac{-4(3) - 2(4) - 4(-5)}{\sqrt{3^2 + 4^2 + (-5)^2}}\right]^2}$$

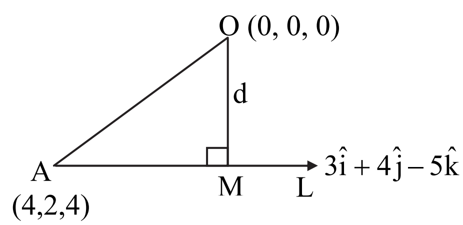
$= \sqrt{16 + 4 + 16}$

$= 6$

Alternate method:

$\overline{AO} = 4\hat{i} + 2\hat{j} + 4\hat{k}$

$\therefore$ $OA = \sqrt{16 + 4 + 16} = 6$



AM = Projection of OA on AL

$= \frac{12 + 8 - 20}{\sqrt{9 + 16 + 25}} = 0$

In right angled ΔOAM, $d^2 = OA^2 - AM^2$

$\Rightarrow d^2 = 6^2 - 0 \Rightarrow d = 6$

23. Any point on the line $\frac{x-1}{2} = \frac{y}{9} = \frac{z}{5} = \lambda$ is

P $(2\lambda + 1, 9\lambda, 5\lambda)$

Let A $\equiv (5, 4, -1)$

The d.r.s. of the line AP are

$2\lambda + 1 - 5, 9\lambda - 4, 5\lambda - (-1)$

$\Rightarrow 2\lambda - 4, 9\lambda - 4, 5\lambda + 1$

Since, AP is perpendicular to the given line

$\therefore$ $2(2\lambda - 4) + 9(9\lambda - 4) + 5(5\lambda + 1) = 0$

$\Rightarrow 4\lambda - 8 + 81\lambda - 36 + 25\lambda + 5 = 0$

$\Rightarrow \lambda = \frac{39}{110}$

$\therefore$ $P \equiv \left(\frac{188}{110}, \frac{351}{110}, \frac{195}{110}\right)$

$\therefore$ $AP = \sqrt{\left(5-\frac{188}{110}\right)^2+\left(4-\frac{351}{110}\right)^2+\left(-1-\frac{195}{110}\right)^2}$

$= \frac{1}{\sqrt{110^2}}\sqrt{131044+7921+93025}$

$= \sqrt{\frac{2109}{110}}$

24. Let $\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11} = \lambda$

Any point on the line is

$P(10\lambda + 11, -4\lambda - 2, -11\lambda - 8)$

Let $A \equiv (2, -1, 5)$

The d.r.s. of the line AP are

$10\lambda + 11 - 2, -4\lambda - 2 - (-1), -11\lambda - 8 - 5$

i.e., $10\lambda + 9, -4\lambda - 1, -11\lambda - 13$

Since, AP is perpendicular to the given line

$\therefore$ $10(10\lambda + 9) - 4(-4\lambda - 1) - 11(-11\lambda - 13) = 0$

$\Rightarrow 100\lambda + 90 + 16\lambda + 4 + 121\lambda + 143 = 0$

$\Rightarrow 237\lambda + 237 = 0 \Rightarrow \lambda = -1$

$\therefore$ $P \equiv (1, 2, 3)$

$\therefore$ $AP = \sqrt{(2-1)^2+(-1-2)^2+(5-3)^2}$

$= \sqrt{1+9+4} = \sqrt{14}$

25. The given equation of line is

$\frac{x-1}{2} = \frac{y+1}{-3} = z$

The co-ordinates of any point on the given line are $(2\lambda + 1, -3\lambda - 1, \lambda)$

The distance of this point from the point $(1, -1, 0)$ is $4\sqrt{14}$.

$\therefore$ $(2\lambda)^2 + (-3\lambda)^2 + (\lambda)^2 = (4\sqrt{14})^2 \Rightarrow \lambda = \pm 4$

$\therefore$ The co-ordinates of the required point are $(9, -13, 4)$ or $(-7, 11, -4)$

The point nearer to the origin is $(-7, 11, -4)$.

26. The equation of the line joining the points $A(2, -3, -1)$ and $B(8, -1, 2)$ is

$\frac{x-2}{8-2} = \frac{y+3}{-1+3} = \frac{z+1}{2+1}$

$\Rightarrow \frac{x-2}{6} = \frac{y+3}{2} = \frac{z+1}{3} = \lambda$

Any point on the line is

$(6\lambda + 2, 2\lambda - 3, 3\lambda - 1)$

The distance of this point from the point $A(2, -3, -1)$ is 14 units.

$\therefore$ $(6\lambda)^2 + (2\lambda)^2 + (3\lambda)^2 = (14)^2$

$\therefore$ $49\lambda^2 = 196$

$\therefore$ $\lambda^2 = 4 \Rightarrow \lambda = \pm 2$

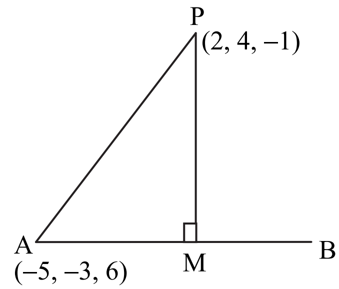
$\therefore$ The points are $(14, 1, 5)$ and $(-10, -7, -7)$

$\therefore$ The point nearer to the origin is $(-10, -7, -7)$.

27. Any point on the line

$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = \lambda$ is given by

$M \equiv (\lambda - 5, 4\lambda - 3, -9\lambda + 6)$.



The d.r.s. of PM are

$\lambda - 7, 4\lambda - 7, -9\lambda + 7$

Since, PM is perpendicular to AM,

$\therefore$ $1(\lambda - 7) + 4(4\lambda - 7) - 9(-9\lambda + 7) = 0$

$\Rightarrow 98\lambda - 98 = 0 \Rightarrow \lambda = 1$

$\therefore$ $M = (-4, 1, -3)$

Now, Equation of perpendicular passing through $P(2, 4, -1)$ and $M(-4, 1, -3)$ is

$\frac{x-2}{-4-2} = \frac{y-4}{1-4} = \frac{z+1}{-3+1}$

$\Rightarrow \frac{x-2}{6} = \frac{y-4}{3} = \frac{z+1}{2}$

28. The direction ratios are same. Also both lines pass through origin.

$\therefore$ Given lines are coinciding lines.

29. The lines can be rewritten as

$\bar{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - \hat{k})$ and

$\bar{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k})$

Here, $(x_1, y_1, z_1) = (1, -2, 3)$

$(x_2, y_2, z_2) = (1, -1, -1)$

$(a_1, b_1, c_1) = (-1, 1, -1)$

$(a_2, b_2, c_2) = (1, 2, -2)$

$\therefore$ Shortest distance (d)

$$d = \left|\frac{\begin{vmatrix} 1-1 & -1+2 & -1-3 \\ -1 & 1 & -1 \\ 1 & 2 & -2 \end{vmatrix}}{\sqrt{(-2+2)^2+(-1-2)^2+(-2-1)^2}}\right|$$

$$= \left|\frac{0-1(3)-4(-3)}{3\sqrt{2}}\right| = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}}$$

30. The equations of the given lines are

$\bar{r} = (-\hat{i}+\hat{j}-\hat{k})+\lambda(\hat{i}+\hat{j}-\hat{k})$ and

$\bar{r} = \left(\hat{i}-\hat{j}+2\hat{k}\right)+\mu\left(-\hat{i}+2\hat{j}+\hat{k}\right)$

Here, $(x_1, y_1, z_1) = (-1, 1, -1)$

$(x_2, y_2, z_2) = (1, -1, 2)$

$(a_1, b_1, c_1) = (1, 1, -1)$

$(a_2, b_2, c_2) = (-1, 2, 1)$

$\therefore$ Shortest Distance (d)

$$= \left|\frac{\begin{vmatrix} 1+1 & -1-1 & 2+1 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix}}{\sqrt{(1+2)^2+(1-1)^2+(2+1)^2}}\right|$$

$$= \left|\frac{2(3)+2(0)+3(3)}{3\sqrt{2}}\right| = \frac{5}{\sqrt{2}}$$

31. The given equation of lines are

$\frac{x-1}{k}=\frac{y+1}{3}=\frac{z-1}{4}$ and

$\frac{x-3}{1}=\frac{2y-9}{2k}=\frac{z}{1}$

i.e. $\frac{x-3}{1}=\frac{y-\frac{9}{2}}{k}=\frac{z}{1}$

Since the line intersect,

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

$\therefore$ $\begin{vmatrix} 2 & \frac{11}{2} & -1 \\ k & 3 & 4 \\ 1 & k & 1 \end{vmatrix}=0$

$\therefore$ $2(3-4k)-\frac{11}{2}(k-4)-1(k^2-3)=0$

$\therefore$ $6-8k-\frac{11}{2}k+22-k^2+3=0$

$\therefore$ $2k^2+27k-62=0$

$\therefore$ $2k^2-4k+31k-62=0$

$\therefore$ $2k(k-2)+31(k-2)=0$

$\therefore$ $k=2$ or $k=\frac{-31}{2}$

32. $\bar{a}_2-\bar{a}_1=-3\hat{i}+2\hat{k}$

$$\bar{b}_1\times\bar{b}_2=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 1 & 4 & -5 \end{vmatrix}$$

$= \hat{i}(-10+12)-\hat{j}(-5+3)+\hat{k}(4-2)$

$= 2\hat{i}+2\hat{j}+2\hat{k}$

$\therefore$ $\left(\bar{a}_2-\bar{a}_1\right).\left(\bar{b}_1\times\bar{b}_2\right) = \left(-3\hat{i}+2\hat{k}\right).\left(2\hat{i}+2\hat{j}+2\hat{k}\right)$

$= -6+4 = -2$

33. Let the components of the line vector be a, b, c.

$a^2+b^2+c^2=(63)^2$ ....(i)

$\frac{a}{3}=\frac{b}{-2}=\frac{c}{6}=\lambda$(say)

$\Rightarrow a=3\lambda,\ b=-2\lambda,\ c=6\lambda$

$\therefore$ $9\lambda^2+4\lambda^2+36\lambda^2=(63)^2$ ....[From (i)]

$\therefore$ $49\lambda^2=(63)^2 \Rightarrow \lambda=\pm\frac{63}{7}=\pm 9$

Since, as the line makes an obtuse angle with X-axis, $a=3\lambda<0,\ \lambda=-9$

$\therefore$ The required components are $-27, 18, -54$.

Competitive Thinking

1. The line passes through $(1, -2, -1)$

Let other point be (x_2, y_2, z_2)

Direction ratio are $0, 6, -1$

$\therefore$ $x_2-1=0\Rightarrow x_2=1$

$y_2-(-2)=6\Rightarrow y_2=4$

$z_2-(-1)=-1\Rightarrow z_2=-2$

2. The equation of line passing through (a, b, c) and having d.r.s. 0, 0, 1 is $\frac{x-a}{0}=\frac{y-b}{0}=\frac{z-c}{1}$

3. Let a, b, c be the d.r.s. of the required line

d.r.s. of the given lines are $2, -2, 1$ and $1, -2, 2$.

$\therefore$ $2a-2b+c=0$ ...(i)

$a-2b+2c=0$ ...(ii)

$\therefore$ $\frac{a}{-4+2}=\frac{-b}{4-1}=\frac{c}{-4+2}$

$\Rightarrow \frac{a}{-2}=\frac{b}{-3}=\frac{c}{-2}$

$\therefore$ Equation of the required line is

$\frac{x-3}{-2}=\frac{y+1}{-3}=\frac{z-2}{-2}$

$\Rightarrow \frac{x-3}{2}=\frac{y+1}{3}=\frac{z-2}{2}$

4. $\dfrac{x+2}{2}=\dfrac{2y-5}{3}$, $z=-1$

$\therefore$ $\dfrac{x+2}{2}=\dfrac{y-\frac{5}{2}}{\frac{3}{2}}$, $z=-1$

$\therefore$ $\dfrac{x+2}{4}=\dfrac{y-\frac{5}{2}}{3}$, $z=-1$

$\therefore$ d.r.s of given line are 4, 3, 0

$\therefore$ d.c.s of the line are

$\dfrac{4}{\sqrt{4^2+3^2}},\dfrac{3}{\sqrt{4^2+3^2}},0 \Rightarrow \dfrac{4}{5},\dfrac{3}{5},0$

5. d.r.s. of given line are 1, 1, 1

$\therefore$ d.c.s. are $\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}}$

6. Given equation of line $x = 4z + 3$, $y = 2 - 3z$

$\Rightarrow z = \dfrac{x-3}{4}$, $z = \dfrac{y-2}{-3}$

$\therefore$ Equation of line is $\dfrac{x-3}{4}=\dfrac{y-2}{-3}=\dfrac{z-0}{1}$

d.r.s of line are 4, –3, 1

$\therefore$ $\cos\alpha = \dfrac{4}{\sqrt{4^2+(-3)^2+1^2}}=\dfrac{4}{\sqrt{26}}$,

$\cos\beta = \dfrac{-3}{\sqrt{26}}$, $\cos\gamma = \dfrac{1}{\sqrt{26}}$

$\therefore$ $\cos\alpha+\cos\beta+\cos\gamma = \dfrac{4}{\sqrt{26}}-\dfrac{3}{\sqrt{26}}+\dfrac{1}{\sqrt{26}}$

$= \dfrac{2}{\sqrt{26}}$

7. The given equation is $\dfrac{x-3}{3}=\dfrac{y-2}{1}=\dfrac{z-1}{0}$

The direction ratios of the above line are 3, 1, 0

$\Rightarrow n = \cos\gamma = 0 \Rightarrow \gamma = 90°$

$\therefore$ The given straight line is perpendicular to Z-axis.

8. Let a, b, c be the direction ratios of the line.

$\therefore$ $a - b + c = 0$ and ....(i)

$a - 3b = 0$ ....(ii)

$\therefore$ $\dfrac{a}{3}=\dfrac{b}{1}=\dfrac{c}{-2}$

$\therefore$ the direction ratios of the line are 3, 1, –2.

9. If a line is equally inclined to axes, then $l = m = n = \pm\dfrac{1}{\sqrt{3}}$

$\therefore$ d.r.s. of the line are 1, 1, 1

Given that the line passes through the point (– 3, 2, – 5)

$\therefore$ The equation of line is $\dfrac{x+3}{1}=\dfrac{y-2}{1}=\dfrac{z+5}{1}$

10. Here, $(x_1, y_1, z_1) \equiv (a, b, c)$

and $(x_2, y_2, z_2) \equiv (a - b, b - c, c - a)$

Required equation of line is

$\dfrac{x-a}{a-b-a}=\dfrac{y-b}{b-c-b}=\dfrac{z-c}{c-a-c}$

i.e., $\dfrac{x-a}{b}=\dfrac{y-b}{c}=\dfrac{z-c}{a}$

11. Given equation is $\dfrac{x-1}{l}=\dfrac{y-2}{m}=\dfrac{z+1}{n}$

The equation of line passing through (1, 2, –1) and (–1, 0, 1) is

$\dfrac{x-1}{-1-1}=\dfrac{y-2}{0-2}=\dfrac{z+1}{1+1}$

$\Rightarrow \dfrac{x-1}{-2}=\dfrac{y-2}{-2}=\dfrac{z+1}{2}$

$\Rightarrow \dfrac{x-1}{1}=\dfrac{y-2}{1}=\dfrac{z+1}{-1}$ ....(i)

Comparing (i) with given equation, we get

$l = 1$, $m = 1$, $n = -1$

12. Equation of line AB in vector form is

$\bar{r} = 6\bar{a}-4\bar{b}+4\bar{c}+\lambda\left(-4\bar{c}-\{6\bar{a}-4\bar{b}+4\bar{c}\}\right)$

$\Rightarrow \bar{r} = 6\bar{a}-4\bar{b}+4\bar{c}+\lambda\left(-6\bar{a}+4\bar{b}-8\bar{c}\right)$ ....(i)

Equation of line CD in vector form is

$\bar{r}' = \bar{a}+2\bar{b}-5\bar{c}+\lambda'\left(-\bar{a}-2\bar{b}-3\bar{c}-\{\bar{a}+2\bar{b}-5\bar{c}\}\right)$

$\Rightarrow \bar{r}' = \bar{a}+2\bar{b}-5\bar{c}+\lambda'\left(-2\bar{a}-4\bar{b}+2\bar{c}\right)$ ....(ii)

The point of intersection of AB and CD will satisfy

$\bar{r} = \bar{r}'$

$\Rightarrow 6\bar{a}-4\bar{b}+4\bar{c}+\lambda(-6\bar{a}+4\bar{b}-8\bar{c})$

$= \bar{a}+2\bar{b}-5\bar{c}+\lambda'(-2\bar{a}-4\bar{b}+2\bar{c})$

Comparing the coefficients of $\bar{a}$ and $\bar{b}$, we get

$6\lambda - 2\lambda' = 5$ ...(iii)

$2\lambda + 2\lambda' = 3$ ...(iv)

$\Rightarrow \lambda = 1$ and $\lambda' = \dfrac{1}{2}$

Substituting value of λ in equation (i), we get the point of intersection

$\therefore$ Point of intersection $\bar{r} = -4\bar{c}$ i.e. point B.

13. The equation of the line joining the points (3, 5, – 7) and (– 2, 1, 8) is

$$\frac{x-3}{-2-3}=\frac{y-5}{1-5}=\frac{z-(-7)}{8-(-7)}$$

Let $\frac{x-3}{-5}=\frac{y-5}{-4}=\frac{z+7}{15}=\lambda$

$\Rightarrow x = 3 - 5\lambda,\ y = 5 - 4\lambda,\ z = -7 + 15\lambda$

For YZ plane, $x = 0$

$\therefore$ $3 - 5\lambda = 0 \Rightarrow \lambda = \frac{3}{5}$

Now, $y = 5 - 4\lambda = 5 - 4\left(\frac{3}{5}\right) = 5-\frac{12}{5}=\frac{13}{5}$

$z = -7 + 15\lambda = -7 + 15\left(\frac{3}{5}\right) = 2$

$\therefore$ The required point is $\left(0,\frac{13}{5},2\right)$

14. Given equations of line are

$\bar{r} = (\hat{i}+2\hat{j}-\hat{k})+\lambda(3\hat{i}-4\hat{k})$ ....(i)

and $\bar{r} = (1-t)\ (4\hat{i}-\hat{j})+t(2\hat{i}+\hat{j}-3\hat{k})$

i.e., $\bar{r} = (4\hat{i}-\hat{j})+t(-2\hat{i}+2\hat{j}-3\hat{k})$ ....(ii)

Now, d.r.s. of line (i) and (ii) are

$a_1, b_1, c_1 = 3, 0, -4$

and $a_2, b_2, c_2 = -2, 2, -3$

$$\cos\theta = \left|\frac{3(-2)+0(2)+(-4)(-3)}{\sqrt{9+0+16}\ \sqrt{4+4+9}}\right|$$

$\Rightarrow \cos\theta = \frac{6}{5\sqrt{17}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{6}{5\sqrt{17}}\right)$

15. The d.r.s. of the lines are 2, 5, –3 and –1, 8, 4

$$\therefore \quad \cos\theta = \left|\frac{2(-1)+5(8)+(-3)(4)}{\sqrt{2^2+5^2+(-3)^2}\ \sqrt{(-1)^2+8^2+4^2}}\right|$$

$\Rightarrow \cos\theta = \frac{26}{9\sqrt{38}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$

16. The d.r.s. of the lines are 1, 0, –1 and 3, 4, 5

$$\therefore \quad \cos\theta = \left|\frac{1(3)+0(4)+(-1)(5)}{\sqrt{1^2+0^2+(-1)^2}\ \sqrt{3^2+4^2+5^2}}\right| = \left|\frac{-2}{10}\right|$$

$\therefore$ $\theta = \cos^{-1}\left(\frac{1}{5}\right)$

17. $$\cos\theta = \left|\frac{2(1)+2(2)+(-1)(2)}{\sqrt{4+4+1}\ \sqrt{1+4+4}}\right| = \left|\frac{4}{\sqrt{9}.\sqrt{9}}\right| = \frac{4}{9}$$

$\Rightarrow \theta = \cos^{-1}\left(\frac{4}{9}\right)$

18. The d.r.s. of the line joining the points (2, 1, –3) and (–3, 1, 7) are –5, 0, 10

The d.r.s. of the line parallel to line $\frac{x-1}{3}=\frac{y}{4}=\frac{z+3}{5}$ are 3, 4, 5

$\therefore$ The angle between the lines having d.r.s. –5, 0, 10 and 3, 4, 5 is

$$\cos\theta = \left|\frac{-5(3)+0(4)+10(5)}{\sqrt{25+0+100}\sqrt{9+16+25}}\right|$$

$\Rightarrow \cos\theta = \frac{35}{25\sqrt{10}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{7}{5\sqrt{10}}\right)$

19. $a_1a_2 + b_1b_2 + c_1c_2 = (2)\ (1) + (5)\ (2) + (4)\ (-3)$
$= 0$

$\therefore$ Lines are perpendicular

$\therefore$ $\theta = 90°$

20. The equation of given lines are

$\frac{x}{3}=\frac{y}{2}=\frac{z}{-6}$ and $\frac{x}{2}=\frac{y}{-12}=\frac{z}{-3}$

$a_1a_2 + b_1b_2 + c_1c_2 = 3(2) + 2(-12) + (-6)\ (-3)$
$= 0$

$\therefore$ Lines are perpendicular

$\therefore$ $\theta = 90°$

21. The first line is parallel to Z-axis and the second line is parallel to X-axis.

$\therefore$ The angle between them is 90°.

22. Let the d.r.s of the given line be a, b, c

Then, according to given condition of perpendicularity,

$-1.a + 2.b + 2.c = 0$ ....(i)

$0.a + 2.b + 1.c = 0$ ....(ii)

On solving (i) and (ii), we get

$a = 2,\ b = -1$ and $c = 2$

23. $a_1, b_1, c_1, = -3, 2k, 2$ and $a_2, b_2, c_2 = 3k, 1, -5$

Since, the lines are perpendicular to each other,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore$ $(-3)(3k) + (2k)(1) + (2)(-5) = 0$

$\Rightarrow -9k + 2k - 10 = 0$

$\Rightarrow k = \frac{-10}{7}$

24. Given equations of lines are
$x = ay + b, z = cy + d$
$\Rightarrow \frac{x-b}{a} = \frac{y}{1}, \frac{z-d}{c} = \frac{y}{1}$
$\Rightarrow \frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c}$
and $x = a'y + b', z = c'y + d'$
$\Rightarrow \frac{x-b'}{a'} = \frac{y}{1}, \frac{z-d'}{c'} = \frac{y}{1}$
$\Rightarrow \frac{x-b'}{a'} = \frac{y}{1} = \frac{z-d'}{c'}$
Since, the lines are perpendicular to each other.
$a_1a_2 + b_1b_2 + c_1c_2 = 0$
$\Rightarrow aa' + 1(1) + cc' = 0$
$\Rightarrow aa' + cc' = -1$

25. Equation of line BC is
$\frac{x-0}{2-0} = \frac{y+11}{-3+11} = \frac{z-4}{1-4}$
$\Rightarrow \frac{x}{2} = \frac{y+11}{8} = \frac{z-4}{-3}$
Let $\frac{x}{2} = \frac{y+11}{8} = \frac{z-4}{-3} = \lambda$
Any point D on the line is
$\equiv (2\lambda, 8\lambda - 11, -3\lambda + 4)$
Given point $A \equiv (1, 8, 4)$
$\therefore$ d.r.s of AD are $2\lambda - 1, 8\lambda - 11 - 8, -3\lambda + 4 - 4$
$= 2\lambda - 1, 8\lambda - 19, -3\lambda$
Since, $AD \perp BC$,
$\therefore$ $aa_1 + bb_1 + cc_1 = 0$
$\Rightarrow 2(2\lambda - 1) + 8(8\lambda - 19) - 3(-3\lambda) = 0$
$\Rightarrow 4\lambda - 2 + 64\lambda - 152 + 9\lambda = 0$
$\Rightarrow 77\lambda = 154$
$\Rightarrow \lambda = 2$
$\therefore$ $D \equiv (4, 5, -2)$

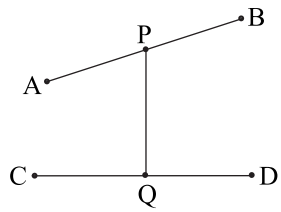
26. Given equation of line is
$\bar{r} = (3+t)\hat{i} + (1-t)\hat{j} + (-2-2t)\hat{k}$
$\Rightarrow \bar{r} = 3\hat{i} + \hat{j} - 2\hat{k} + (\hat{i} - \hat{j} - 2\hat{k})t$, where $t \in R$
$\therefore$ The line passes through $(3, 1, -2)$ and is parallel to the vector $\hat{i} - \hat{j} - 2\hat{k}$.
Equation of second line is
$x = 4 + k, y = -k, z = -4 - 2k,$
$\Rightarrow \frac{x-4}{1} = \frac{y}{-1} = \frac{z+4}{-2} = k$, where $k \in R$
$\therefore$ d.r.s. of the line are $1, -1, -2$. Also, it passes through $(3, 1, -2)$.
$\therefore$ Both lines are coincident.

27. Consider option (A)
point $\left(21, \frac{5}{3}, \frac{10}{3}\right)$ satisfies both the equations of line
$\therefore$ option (A) is correct answer
Alternate method:
Let $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1} = \lambda$
and $\frac{x+3}{-36} = \frac{y-3}{2} = \frac{z-6}{4} = \mu$
$\therefore$ $x = 3\lambda + 5,\ y = -\lambda + 7, z = \lambda - 2$ and
$x = -36\mu - 3, y = 2\mu + 3, z = 4\mu + 6$
On solving, we get $x = 21, y = \frac{5}{3}, z = \frac{10}{3}$

28. Consider option (B)
Point $(-2, -4, -5)$ satisfies both the equations of the line.
$\therefore$ Option (B) is the correct answer.

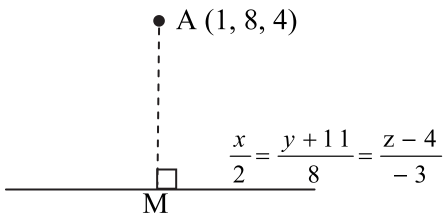
29. Consider option (B)
point $(-11, -4, 5)$ satisfies both the equations of line
$\therefore$ option (B) is correct answer

30. Consider option (B)
point $(2, 3, 4)$ satisfies both the equations of line
$\therefore$ option (B) is correct answer

31.


Let the two lines be AB and CD having equations $\frac{x}{1} = \frac{y+a}{1} = \frac{z}{1} = \lambda$ and $\frac{x+a}{2} = \frac{y}{1} = \frac{z}{1} = \mu$.
Then, $P \equiv (\lambda, \lambda - a, \lambda)$ and $Q \equiv (2\mu - a, \mu, \mu)$
According to the given condition,
$\frac{\lambda - 2\mu + a}{2} = \frac{\lambda - a - \mu}{1} = \frac{\lambda - \mu}{2}$
$\Rightarrow \mu = a$ and $\lambda = 3a$
$\therefore$ $P \equiv (3a, 2a, 3a)$ and $Q \equiv (a, a, a)$

32. d.r.s. of the line joining $(0, -11, 4)$ and $(2, -3, 1)$ are $2, 8, -3$.
$\therefore$ Equation of line is $\frac{x}{2} = \frac{y+11}{8} = \frac{z-4}{-3}$



Let $\frac{x}{2} = \frac{y+11}{8} = \frac{z-4}{-3} = \lambda$

Any general point on this line is

$M \equiv (2\lambda, 8\lambda - 11, -3\lambda + 4)$

Let $A \equiv (1, 8, 4)$

d.r.s. of AM are $2\lambda - 1, 8\lambda - 19, -3\lambda$

Since, AM is perpendicular to the given line,

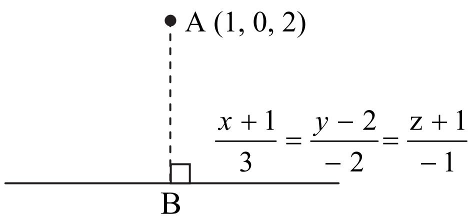
$\therefore$ $2(2\lambda - 1) + 8(8\lambda - 19) - 3(-3\lambda) = 0$

$\Rightarrow 77\lambda = 154$

$\Rightarrow \lambda = 2$

$\therefore$ $M \equiv (4, 5, -2)$

33. Let $\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+1}{-1} = \lambda$



Any general point on this line is

$B\ (3\lambda - 1, -2\lambda + 2, -\lambda - 1)$

Let $A \equiv (1, 0, 2)$

$\therefore$ d.r.s. of AB are $3\lambda - 2, -2\lambda + 2, -\lambda - 3$

Since, AB is perpendicular to the given line,

$\therefore$ $3(3\lambda - 2) - 2(-2\lambda + 2) - 1(-\lambda - 3) = 0$

$\Rightarrow 14\lambda = 7$

$\Rightarrow \lambda = \frac{1}{2}$

$\therefore$ $B \equiv \left(\frac{1}{2}, 1, \frac{-3}{2}\right)$

34. Let M be the foot of perpendicular drawn from the point P(1, 2, 3) to the line and $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2} = \lambda$

$\therefore$ The co-ordinates of any point on the line are

$M \equiv (3\lambda + 6, 2\lambda + 7, -2\lambda + 7)$

$\therefore$ The d.r.s of PM are

$3\lambda + 6 - 1,\ 2\lambda + 7 - 2, -2\lambda + 7 - 3$

i.e., $3\lambda + 5, 2\lambda + 5, -2\lambda + 4$.

Since, PM is perpendicular to the given line whose d.r.s. are 3, 2, –2,

$\therefore$ $3(3\lambda + 5) + 2(2\lambda + 5) - 2(-2\lambda + 4) = 0$

$\Rightarrow 9\lambda + 15 + 4\lambda + 10 + 4\lambda - 8 = 0$

$\Rightarrow 17\lambda + 17 = 0$

$\Rightarrow \lambda = -1$

$\therefore$ $M \equiv (3, 5, 9)$

$\therefore$ $PM = \sqrt{(3-1)^2 + (5-2)^2 + (9-3)^2}$

$= \sqrt{4 + 9 + 36} = 7$

35. Since the point is (–2, 4, –5),

$\therefore$ $a = -2, b = 4, c = -5$

Given equation of line is

$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

$\therefore$ $x_1 = -3, y_1 = 4, z_1 = -8$

d.r.s of the line are 3, 5, 6

$\therefore$ d.c.s are $\frac{3}{\sqrt{70}}, \frac{5}{\sqrt{70}}, \frac{6}{\sqrt{70}}$

Perpendicular distance of point from the line is

$$\sqrt{\begin{array}{l}\left[(a - x_1)^2 + (b - y_1)^2 + (c - z_1)^2\right] \\ \quad - \left[(a - x_1)l + (b - y_1)m + (c - z_1)n\right]^2\end{array}}$$

$$= \sqrt{1^2 + 0 + 3^2 - \left[\frac{3(1)}{\sqrt{70}} + \frac{0(5)}{\sqrt{70}} + \frac{3(6)}{\sqrt{70}}\right]^2}$$

$$= \sqrt{1 + 9 - \left(\frac{3}{\sqrt{70}} + \frac{18}{\sqrt{70}}\right)^2}$$

$$= \sqrt{\frac{37}{10}} \text{ units}$$

36. Let M be the foot of perpendicular drawn from the point P(2, 3, 4) to the line and $\frac{x-1}{-1} = \frac{y-0}{2} = \frac{z+1}{3} = \lambda$

$\therefore$ $M \equiv (-\lambda + 1, 2\lambda, 3\lambda - 1)$.

The d.r.s of PM are $-\lambda - 1, 2\lambda - 3, 3\lambda - 5$

Since, PM is perpendicular to the given line,

$-1(-\lambda - 1) + 2(2\lambda - 3) + 3(3\lambda - 5) = 0$

$\Rightarrow \lambda + 1 + 4\lambda - 6 + 9\lambda - 15 = 0$

$\Rightarrow 14\lambda = 20$

$\Rightarrow \lambda = \frac{10}{7}$

$\therefore$ $M \equiv \left(\frac{-3}{7}, \frac{20}{7}, \frac{23}{7}\right)$

$\therefore$ PM $= \sqrt{\left(2+\frac{3}{7}\right)^2+\left(3-\frac{20}{7}\right)^2+\left(4-\frac{23}{7}\right)^2}$

$= \sqrt{\frac{289}{49}+\frac{1}{49}+\frac{25}{49}}$

$= \frac{3}{7}\sqrt{35}$

37. The equation of the line joining the points $(-9, 4, 5)$ and $(11, 0, -1)$ is

$\frac{x+9}{11+9} = \frac{y-4}{0-4} = \frac{z-5}{-1-5}$

$\Rightarrow \frac{x+9}{20} = \frac{y-4}{-4} = \frac{z-5}{-6}$

$\Rightarrow \frac{x+9}{10} = \frac{y-4}{-2} = \frac{z-5}{-3}$

$\therefore$ The d.r.s. of the given line are $10, -2, 3$

Let $\frac{x+9}{10} = \frac{y-4}{-2} = \frac{z-5}{-3} = \lambda$

$\therefore$ Any point on the line is

$P \equiv (10\lambda - 9, -2\lambda + 4, -3\lambda + 5)$

$\therefore$ The d.r.s.of OP are

$10\lambda - 9, -2\lambda + 4, -3\lambda + 5$

Since, the given line is perpendicular to OP,

$10(10\lambda - 9) - 2(-2\lambda + 4) - 3(-3\lambda + 5) = 0$

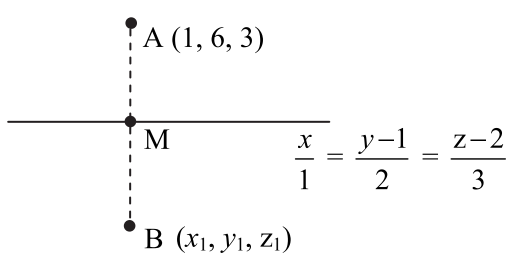
$\Rightarrow 100\lambda - 90 + 4\lambda - 8 + 9\lambda - 15 = 0$

$\Rightarrow 113\lambda = 113$

$\Rightarrow \lambda = 1$

$\therefore$ $P \equiv (1, 2, 2)$

38. Let $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda$



Any general point on this line is

$M \equiv (\lambda, 2\lambda + 1, 3\lambda + 2)$

Let $A \equiv (1, 6, 3)$

d.r.s. of AM are $\lambda - 1,\ 2\lambda - 5,\ 3\lambda - 1$

Since, AM is perpendicular to the given line,

$\therefore$ $1(\lambda - 1) + 2(2\lambda - 5) + 3(3\lambda - 1) = 0$

$\Rightarrow 14\lambda = 14$

$\Rightarrow \lambda = 1$

$\therefore$ $M = (1, 3, 5)$

Now, M is the midpoint of AB.

$\therefore$ $\left(\frac{1+x_1}{2}, \frac{6+y_1}{2}, \frac{3+z_1}{2}\right) = (1, 3, 5)$

$\Rightarrow x_1 = 1,\ y_1 = 0,\ z_1 = 7$

39. Let $\frac{x-1}{2} = \frac{y+1}{-2} = \frac{z+1}{1} = \lambda$

any point on the line is

$P \equiv (2\lambda + 1, -2\lambda - 1, \lambda - 1)$

Let $A \equiv (1, -1, -1)$

Now, PA = 3

$\therefore$ $\sqrt{(2\lambda+1-1)^2+(-2\lambda-1+1)^2+(\lambda-1+1)^2} = 3$

$\Rightarrow \sqrt{4\lambda^2+4\lambda^2+\lambda^2} = 3$

$\Rightarrow 9\lambda^2 = 9$

$\Rightarrow \lambda = \pm 1$

$\therefore$ $P \equiv (3, -3, 0)$ or $P \equiv (-1, 1, -2)$

40. First line passes through $(x_1, y_1, z_1) = (3, 8, 3)$ and has d.r.s. $(a_1, b_1, c_1) = (3, -1, 1)$

Second line passes through $(x_2, y_2, z_2) = (-3, -7, 6)$ and has d.r.s. $(a_2, b_2, c_2) = (-3, 2, 4)$

$\therefore$ Shortest distance (d) between them is

$$d = \left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right|$$

$$= \left|\frac{\begin{vmatrix} -6 & -15 & 3 \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}}{\sqrt{(-4-2)^2+(-3-12)^2+(6-3)^2}}\right|$$

$$= \left|\frac{-6(-4-2)+15(12+3)+3(6-3)}{\sqrt{36+225+9}}\right|$$

$$= \frac{270}{\sqrt{270}} = 3\sqrt{30}$$

41. Here, $(x_1, y_1, z_1) = (-1, -2, -1)$

$(x_2, y_2, z_2) = (2, -2, 3)$

$(a_1, b_1, c_1) = (3, 1, 2)$

$(a_2, b_2, c_2) = (1, 2, 3)$

$$d = \left|\frac{\begin{vmatrix} 3 & 0 & 4 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix}}{\sqrt{(3-4)^2+(2-9)^2+(6-1)^2}}\right|$$

$$= \frac{17}{\sqrt{75}} = \frac{17}{5\sqrt{3}}$$

42. Since, the given lines intersect each other,

$\therefore \quad \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 3-1 & k+1 & 0-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow 2(3-8) - (k+1)(2-4) - 1(4-3) = 0$

$\Rightarrow -10 + 2k + 2 - 1 = 0$

$\Rightarrow k = \dfrac{9}{2}$

43. Since, the given lines intersect each other,

$\therefore \quad \begin{vmatrix} 2-1 & 3-2 & 1-3 \\ k & 2 & 3 \\ 3 & k & 2 \end{vmatrix} = 0$

$\Rightarrow 1(4-3k) - 1(2k-9) - 2(k^2-6) = 0$

$\Rightarrow 2k^2 + 5k - 25 = 0$

$\Rightarrow k = \dfrac{5}{2}, -5$

44. Let the equation of a line passing through the origin be $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

This meets the lines

$\dfrac{x-2}{1} = \dfrac{y-1}{-2} = \dfrac{z+1}{1}$ and $\dfrac{x-\frac{8}{3}}{2} = \dfrac{y+3}{-1} = \dfrac{z-1}{1}$

$\therefore \quad \begin{vmatrix} 2 & 1 & -1 \\ a & b & c \\ 1 & -2 & 1 \end{vmatrix} = 0$ and $\begin{vmatrix} \frac{8}{3} & -3 & 1 \\ a & b & c \\ 2 & -1 & 1 \end{vmatrix} = 0$

$\Rightarrow a + 3b + 5c = 0$ and $3a + b - 5c = 0$

$\Rightarrow \dfrac{a}{5} = \dfrac{b}{-5} = \dfrac{c}{2}$

Thus, the equation of the line through the origin intersecting the given lines is

$\dfrac{x}{5} = \dfrac{y}{-5} = \dfrac{z}{2} = \lambda\text{(say)}$

The co-ordinates of any point on this line are $(5\lambda, -5\lambda, 2\lambda)$.

The co-ordinates of any point on

$\dfrac{x-2}{1} = \dfrac{y-1}{-2} = \dfrac{z+1}{1} = \lambda_1\text{(say)}$ are

$(\lambda_1 + 2, -2\lambda_1 + 1, \lambda_1 - 1)$.

If these two lines intersect, then

$5\lambda = \lambda_1 + 2$, $-5\lambda = -2\lambda_1 + 1$ and $2\lambda = \lambda_1 - 1$

$\Rightarrow \lambda_1 = 3$ and $\lambda = 1$

So, the co-ordinates of P are $(5, -5, 2)$.

Similarly, co-ordinates of Q are $\left(\dfrac{10}{3}, \dfrac{-10}{3}, \dfrac{8}{3}\right)$

$\therefore \quad PQ^2 = \left(\dfrac{10}{3} - 5\right)^2 + \left(\dfrac{-10}{3} + 5\right)^2 + \left(\dfrac{8}{3} - 2\right)^2 = 6$

45. Lines L_1 and L_2 are parallel to the vectors $\bar{b}_1 = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\bar{b}_2 = \hat{i} + 2\hat{j} + 3\hat{k}$ respectively.

$\therefore$ The unit vector perpendicular to both L_1 and L_2 is

$\hat{n} = \dfrac{\bar{b}_1 \times \bar{b}_2}{|\bar{b}_1 \times \bar{b}_2|}$

Now, $\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\hat{i} - 7\hat{j} + 5\hat{k}$

$\therefore \quad \hat{n} = \dfrac{1}{5\sqrt{3}}\left(-\hat{i} - 7\hat{j} + 5\hat{k}\right)$

Evaluation Test

1. Let $\dfrac{x+1}{-1} = \dfrac{y-12}{5} = \dfrac{z-7}{2} = r$

$\therefore \quad x = -r - 1$, $y = 5r + 12$, $z = 2r + 7$

$\therefore$ Co-ordinates of any point on the line are $(-r-1, 5r+12, 2r+7)$.

This point lies on the curve $11x^2 - 5y^2 + z^2 = 0$

$\therefore \quad 11(-r-1)^2 - 5(5r+12)^2 + (2r+7)^2 = 0$

$\Rightarrow 11r^2 + 22r + 11 - 125r^2 - 600r - 720 + 4r^2 + 28r + 49 = 0$

$\Rightarrow -110r^2 - 550r - 660 = 0$

$\Rightarrow r^2 + 5r + 6 = 0$

$\Rightarrow (r+2)(r+3) = 0$

$\Rightarrow r = -2$ or $r = -3$

If $r = -2$, then the point is $(1, 2, 3)$

and if $r = -3$, then the point is $(2, -3, 1)$

$\therefore$ option (A) is correct.

2. The given equation of line is
$x = 4y + 5$, $z = 3y - 6$.
It can be written as
$\frac{x-5}{4} = y = \frac{z+6}{3} = r$, say

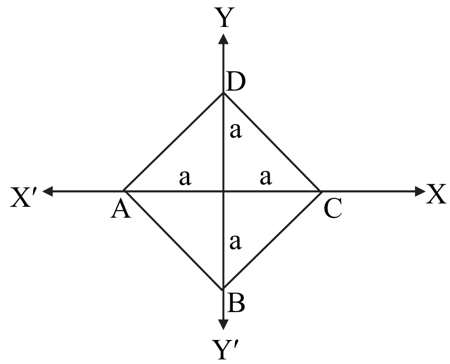
$\therefore$ co-ordinates of the any point on the line are $(4r + 5, r, 3r - 6)$.
This point is at a distance of $3\sqrt{26}$ from the point $(5, 0, -6)$

$\therefore$ $(4r + 5 - 5)^2 + (r - 0)^2 + (3r - 6 + 6)^2 = \left(3\sqrt{26}\right)^2$
$\Rightarrow 16r^2 + r^2 + 9r^2 = 234$
$\Rightarrow 26r^2 = 234$
$\Rightarrow r^2 = 9$
$\Rightarrow r = \pm 3$
If $r = 3$, then the point is
$(4 \times 3 + 5, 3, 3 \times 3 - 6) \equiv (17, 3, 3)$

3. Let the components of the line vector be a, b, c.
$\therefore$ $a^2 + b^2 + c^2 = (63)^2$ ....(i)
Also, $\frac{a}{3} = \frac{b}{-2} = \frac{c}{6} = k$, say
$\therefore$ $a = 3k$, $b = -2k$, $c = 6k$
Substituting value of a, b and c in equation (i), we get
$9k^2 + 4k^2 + 36k^2 = 63^2$
$\therefore$ $49k^2 = 63 \times 63$
$\therefore$ $k^2 = \frac{63 \times 63}{49} = 81$
$\therefore$ $k = \pm 9$
Since, the line makes obtuse angle with X-axis component along X-axis is negative.
$\therefore$ $k = -9$
$\therefore$ The components of the line vector are $3k, -2k, 6k$
i.e., $-27, 18, -54$

4. Let M be the foot of the perpendicular drawn from the point $P(3, -1, 11)$ to the given line.
Let $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$
$\Rightarrow x = 2\lambda$, $y = 3\lambda + 2$, $z = 4\lambda + 3$
$\therefore$ $M \equiv (2\lambda, 3\lambda + 2, 4\lambda + 3)$
d.r.s. of PM are $2\lambda - 3, 3\lambda + 3, 4\lambda - 8$
Since, PM is perpendicular to the given line
$\therefore$ $(2\lambda - 3)(2) + (3\lambda + 3)(3) + (4\lambda - 8)(4) = 0$
$\Rightarrow 4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 = 0$
$\Rightarrow \lambda = 1$
$\therefore$ $M \equiv (2, 5, 7)$
$\therefore$ length of perpendicular (PM)
$= \sqrt{(3-2)^2 + (-1-5)^2 + (11-7)^2}$
$= \sqrt{1 + 36 + 16}$
$= \sqrt{53}$

5. When square is folded co-ordinates will be $D(0, 0, a)$, $C(a, 0, 0)$, $A(-a, 0, 0)$, $B(0, -a, 0)$.



Equation AB is, $\frac{x+a}{a} = \frac{y}{-a} = \frac{z}{0}$

and equation of DC is $\frac{x}{a} = \frac{y}{0} = \frac{z-a}{-a}$

$\therefore$ shortest distance

$$= \left| \frac{\begin{vmatrix} -a & 0 & -a \\ a & -a & 0 \\ a & 0 & -a \end{vmatrix}}{\sqrt{(a^2 - 0)^2 + (0 + a^2)^2 + (0 + a^2)^2}} \right|$$

$$= \left| \frac{-a(a^2) - a(a^2)}{\sqrt{a^4 + a^4 + a^4}} \right| = \left| \frac{-2a^3}{\sqrt{3a^4}} \right| = \frac{2a}{\sqrt{3}}$$

6. Given equation of motion of a rocket is
$x = 2t$, $y = -4t$, $z = 4t$
i.e., the equation of the path is $\frac{x}{2} = \frac{y}{-4} = \frac{z}{4}$
i.e., $\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$
Thus, the path of the rocket represents a straight line passing through the origin.
For t = 10 sec.
we have, $x = 20$, $y = -40$, $z = 40$
Let $M(20, -40, 40)$
$\therefore$ $\left|\overline{OM}\right| = \sqrt{x^2 + y^2 + z^2}$
$= \sqrt{400 + 1600 + 1600} = 60$ km
$\therefore$ Rocket will be at 60 km from the starting point $O(0, 0, 0)$ in 10 seconds.

7. d.r.s. of L_1 are 3, 1, 2 and d.r.s. of L_2 are 1, 2, 3

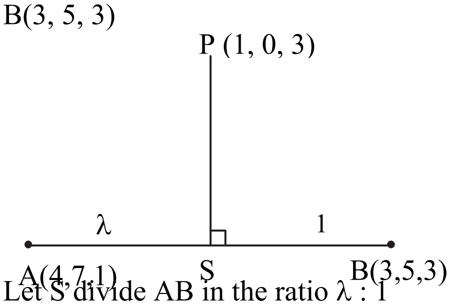
$\therefore$ vector perpendicular to L_1 and L_2 = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix}$

$= \hat{i}(3-4) - \hat{j}(9-2) + \hat{k}(6-1)$

$= -\hat{i} - 7\hat{j} + 5\hat{k}$

$\therefore$ unit vector = $\dfrac{-\hat{i}-7\hat{j}+5\hat{k}}{\sqrt{1+49+25}} = \dfrac{-\hat{i}-7\hat{j}+5\hat{k}}{5\sqrt{3}}$

8. Let S be the foot of perpendicular drawn from P(1, 0, 3) to the join of points A(4, 7, 1) and B(3, 5, 3)



Let S divide AB in the ratio $\lambda : 1$

$\therefore$ $S \equiv \left(\dfrac{3\lambda+4}{\lambda+1}, \dfrac{5\lambda+7}{\lambda+1}, \dfrac{3\lambda+1}{\lambda+1}\right)$ ….(i)

Now, d.r.s. of PS are

$\dfrac{3\lambda+4}{\lambda+1} - 1,\ \dfrac{5\lambda+7}{\lambda+1} - 0,\ \dfrac{3\lambda+1}{\lambda+1} - 3$

i.e., $\dfrac{2\lambda+3}{\lambda+1}, \dfrac{5\lambda+7}{\lambda+1}, \dfrac{-2}{\lambda+1}$

i.e., $2\lambda + 3, 5\lambda + 7, -2$

Also, d.r.s. of AB are $-1, -2, 2$

Since, PS $\perp$ AB

$\therefore$ $(2\lambda + 3)(-1) + (5\lambda + 7)(-2) + (-2)(2) = 0$

$\Rightarrow -2\lambda - 3 - 10\lambda - 14 - 4 = 0$

$\Rightarrow \lambda = -\dfrac{7}{4}$

Substituting the value of λ in (i), we get

$S = \left(\dfrac{5}{3}, \dfrac{7}{3}, \dfrac{17}{3}\right)$

9. Equation of the line passing through the points (5, 1, a) and (3, b, 1) is

$\dfrac{x-3}{5-3} = \dfrac{y-b}{1-b} = \dfrac{z-1}{a-1}$ ….(i)

The line passes through the point $\left(0, \dfrac{17}{2}, \dfrac{-13}{2}\right)$

$\therefore$ $\dfrac{-3}{2} = \dfrac{\dfrac{17}{2} - b}{1-b} = \dfrac{\dfrac{-13}{2} - 1}{a-1}$ ….[From (i)]

$\therefore$ $a - 1 = \dfrac{\dfrac{-15}{2}}{\dfrac{-3}{2}} = 5$

$\therefore$ $a = 5 + 1 = 6$

and $-3 + 3b = 17 - 2b$

$\therefore$ $5b = 20 \Rightarrow b = 4$

$\therefore$ $a = 6, b = 4$

Textbook Chapter No.

08 Plane

Hints

Classical Thinking

1. Here, $\overline{n} = \hat{i} - 2\hat{j} + 3\hat{k}$ and p = 1

$$\hat{n} = \frac{\overline{n}}{|\overline{n}|} = \frac{\hat{i} - 2\hat{j} + 3\hat{k}}{\sqrt{1+4+9}} = \frac{\hat{i} - 2\hat{j} + 3\hat{k}}{\sqrt{14}}$$

$\therefore$ The vector equation of the plane is

$$\overline{r}.\hat{n} = p$$

$$\Rightarrow \overline{r}.\left(\frac{\hat{i} - 2\hat{j} + 3\hat{k}}{\sqrt{14}}\right) = 1$$

$$\Rightarrow \overline{r}.\left(\hat{i} - 2\hat{j} + 3\hat{k}\right) = \sqrt{14}$$

2. The given vector equation is

$$\overline{r}.\left(3\hat{i} - 2\hat{j} + 2\hat{k}\right) = 12 \quad \text{....(i)}$$

$\overline{r}.\overline{n} = 12$, where $\overline{n} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

$$\hat{n} = \frac{\overline{n}}{|\overline{n}|} = \frac{3\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{9+4+4}} = \frac{3\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{17}}$$

$$\Rightarrow \hat{n} = \frac{3}{\sqrt{17}}\hat{i} - \frac{2}{\sqrt{17}}\hat{j} + \frac{2}{\sqrt{17}}\hat{k}$$

$\therefore$ Normal form is

$$\overline{r}.\left(\frac{3}{\sqrt{17}}\hat{i} - \frac{2}{\sqrt{17}}\hat{j} + \frac{2}{\sqrt{17}}\hat{k}\right) = \frac{12}{\sqrt{17}}$$

3. Given equation of plane is

$$\overline{r}.\left(2\hat{i} - 3\hat{j} + \hat{k}\right) + 9 = 0$$

$$\Rightarrow \overline{r}.\left(2\hat{i} - 3\hat{j} + \hat{k}\right) = -9 \quad \text{....(i)}$$

$$\overline{n} = 2\hat{i} - 3\hat{j} + \hat{k}$$

$$\therefore \quad \hat{n} = \frac{2\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{4+9+1}} = \frac{2\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{14}}$$

$\therefore$ The d.c.s. of normal to the plane are

$$\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$$

4. Given that $lx + my + nz = p$ is the equation of the plane in normal form.

$\therefore$ l, m, n are the direction cosines.

Also $l^2 + m^2 + n^2 = 1$,

Since, p is the distance from the origin, p should be greater than zero.

$\therefore$ All the statements are true,

$\therefore$ correct answer is option (D)

6. Equation of XY plane is z = 0,

$\therefore$ d.c.s. of its normal are 0, 0, 1

7. $$\frac{x}{\frac{7}{a}} + \frac{y}{7} + \frac{z}{7} = 1$$

For equal intercepts, $\frac{7}{a} = 7 \quad \Rightarrow a = 1$

8. Equation of plane in intercept form is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Here, a = b = c and point (1, –1, 2) lies in the plane,

$$\therefore \quad \frac{1}{a} + \frac{-1}{a} + \frac{2}{a} = 1 \Rightarrow a = 2$$

$\therefore$ the required equation of a plane is $x + y + z = 2$.

9. Here, $\overline{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\overline{n} = 3\hat{i} - 2\hat{j} + 3\hat{k}$

The vector equation of the plane is

$$\overline{r}.\overline{n} = \overline{a}.\overline{n}$$

$$\Rightarrow \overline{r}.\left(3\hat{i} - 2\hat{j} + 3\hat{k}\right) = \left(\hat{i} + \hat{j} + 2\hat{k}\right).\left(3\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

$$\Rightarrow \overline{r}.\left(3\hat{i} - 2\hat{j} + 3\hat{k}\right) = 7$$

10. Let $\overline{a} = \hat{j} - 3\hat{k}$ and $\overline{n} = \hat{i} + 2\hat{j} + 4\hat{k}$

The vector equation of plane is

$$\overline{r}.\left(\hat{i} + 2\hat{j} + 4\hat{k}\right) = \left(\hat{j} - 3\hat{k}\right).\left(\hat{i} + 2\hat{j} + 4\hat{k}\right)$$

$$\Rightarrow \overline{r}.\left(\hat{i} + 2\hat{j} + 4\hat{k}\right) = -10$$

11. The plane passes through (2, –1, 1)

This point satisfies the equation of plane in option (D)

Also, it has d.r.s. 1, 1, –2.

$\therefore$ option (D) is correct answer.

Alternate method:

Let A ≡ (2, –1, 1)

The d.r.s. of line joining the points (2, 3, –1) and (1, 2, 1) are 1, 1, –2

∴ the equation of the required plane is

$1(x-2)+1(y+1)-2(z-1)=0$

$\Rightarrow x+y-2z+1=0$

12. The plane passes through (3, 2, –1)
This point satisfies the equation of plane in option (C).
Also, it has d.r.s. 2, 2, –3
∴ option (C) is correct answer.

13. The plane passes through (–10, 5, 4)
This point satisfies the equation of plane in option (B)
Also, it has d.r.s. 7, –3, –1
∴ option (B) is correct answer.

14. The plane passes through (1, 2, –3)
This point satisfies the equation of plane in option (A)
Also, it has d.r.s. 1, 2, –3.
∴ option (A) is correct answer.

Alternate method:

Let M (1, 2, –3) be the foot of perpendicular from the origin O (0, 0, 0) to the plane

D. r. s of normal are 1, 2, –3

∴ the equation of the required plane is

$1(x-1)+2(y-2)-3(z+3)=0$

$\Rightarrow x+2y-3z=14$

15. The plane passes through (2, 4, –3)
This point satisfies the equation of plane in option (C)
Also, it has d.r.s. 2, 4, –3.
∴ option (C) is correct answer.

16. The plane passes through (1, –1, 1)
This point satisfies the equation of plane in option (D)

Also, it has d.r.s $=\bar{b}\times\bar{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 1\\ 0 & 1 & 2\end{vmatrix}$

$=\hat{i}(2-1)-\hat{j}(4-0)+\hat{k}(2-0)$

$=\hat{i}-4\hat{j}+2\hat{k}$

i.e., 1, –4, 2

∴ option (D) is correct answer.

Alternate Method

Let $\bar{a}=\hat{i}-\hat{j}+\hat{k}$, $\bar{b}=2\hat{i}+\hat{j}+\hat{k}$ and $\bar{c}=\hat{j}+2\hat{k}$

Now, $\bar{b}\times\bar{c}=\hat{i}-4\hat{j}+2\hat{k}$

∴ the vector equation of required plane is

$\bar{r}.(\bar{b}\times\bar{c})=\bar{a}.(\bar{b}\times\bar{c})$

$\Rightarrow \bar{r}.(\hat{i}-4\hat{j}+2\hat{k})=(\hat{i}-\hat{j}+\hat{k}).(\hat{i}-4\hat{j}+2\hat{k})$

$\Rightarrow \bar{r}.(\hat{i}-4\hat{j}+2\hat{k})=7$

17. Let $(x_1, y_1, z_1)=(0, 1, 2)$,
$a_1, b_1, c_1 = 3, 1, 1$ and
$a_2, b_2, c_2 = -1, 2, -5$
∴ the equation of required plane is

$$\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix}x-0 & y-1 & z-2\\ 3 & 1 & 1\\ -1 & 2 & -5\end{vmatrix}=0$$

$\Rightarrow -7x+14y-14+7z-14=0$

$\Rightarrow x-2y-z+4=0$

18. Let $(x_1, y_1, z_1)=(1, 2, -1)$,
$a_1, b_1, c_1 = 2, 1, 3$ and $a_2, b_2, c_2 = 4, 1, 2$
∴ the equation of required plane is

$$\begin{vmatrix}x-1 & y-2 & z+1\\ 2 & 1 & 3\\ 4 & 1 & 1\end{vmatrix}=0$$

$\Rightarrow (x-1)(-2)+(y-2)(10)+(z+1)(-2)=0$

$\Rightarrow -2x+2+10y-20-2z-2=0$

$\Rightarrow x-5y+z+10=0$

19. Required plane passes through point $(x_1, y_1, z_1)\equiv(1, -3, -2)$ and is perpendicular to planes $x+2y+2z=5$ and $3x+3y+2z=8$
∴ their normals are parallel to the required plane
∴ $a_1, b_1, c_1 = 1, 2, 2$ and $a_2, b_2, c_2 = 3, 3, 2$
∴ the equation of required plane is

$$\begin{vmatrix}x-1 & y+3 & z+2\\ 1 & 2 & 2\\ 3 & 3 & 2\end{vmatrix}=0$$

$\Rightarrow 2x-4y+3z-8=0$

20. The equation $\bar{r}=\bar{a}+\lambda\bar{b}+\mu\bar{c}$ represents a plane passing through vector $\bar{a}$ and parallel to $\bar{b}$ and $\bar{c}$
∴ $\bar{a}=3\hat{i}+\hat{j}$, $\bar{b}=-\hat{j}+\hat{k}$, $\bar{c}=\hat{i}+2\hat{j}+3\hat{k}$

Now, $\bar{b}\times\bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$= -5\hat{i}+\hat{j}+\hat{k}$

∴ the equation of required plane is

$\bar{r}.(-5\hat{i}+\hat{j}+\hat{k}) = (3\hat{i}+\hat{j}).(-5\hat{i}+\hat{j}+\hat{k})$

$\Rightarrow \bar{r}.(-5\hat{i}+\hat{j}+\hat{k}) = -14$

21. Consider option (B)

$\bar{r}.\left(\hat{i}+11\hat{j}+3\hat{k}\right) = 14$

Its Cartesian form is

$x + 11y + 3z = 14$

Since, the given points (1, 2, –3), (3, 1, 0) and (0, 1, 1) satisfiy the above plane,

∴ correct answer is option (B)

Alternate method:

Equation of a plane passing through three points is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z+3 \\ 2 & -1 & 3 \\ -1 & -1 & 4 \end{vmatrix} = 0$$

$\Rightarrow (x-1)(-1) - (y-2)(11) + (z+3)(-3) = 0$

$\Rightarrow -x - 11y - 3z + 14 = 0$

$\Rightarrow x + 11y + 3z = 14$

Its vector form is

$\bar{r}.\left(\hat{i}+11\hat{j}+3\hat{k}\right) = 14$

22. Consider option (B)

$\bar{r}.(3\hat{i} + \hat{j} - \hat{k}) + 4 = 0$

Its Cartesian form is

$3x + y - z = -4$

Since the given points A(1, –2, 5), B(0, –5, –1) and C(–3, 5, 0) satisfy the above plane,

∴ correct answer is option (B).

23. Consider option (B)

$\bar{r}.(9\hat{i} + 3\hat{j} - \hat{k}) = 14$

Its Cartesian form is

$9x + 3y - z = 14$

Since, given points (1, 1, –2), (2, –1, 1) and (1, 2, 1) satisfy the above plane,

∴ correct answer is option (B)

24. Consider option (D)

$2x + 2y - 5z = 0$

Since, the given points (4, 1, 2), (1, –1, 0) and (0, 0, 0) satisfy the above plane,

∴ correct answer is option (D)

25. Consider option (C)

$3x - 4z + 1 = 0$

Since, the given points (1, 1, 1), (1, –1, 1) and (–7, –3, –5) satisfy the above plane,

∴ correct answer is option (C)

26. Here $\bar{n}_1 = \hat{i}-\hat{j}+2\hat{k}$ and $\bar{n}_2 = 3\hat{i}-\hat{j}-\hat{k}$

The vector equation of plane passing through intersection of $\bar{r}.\bar{n}_1 = p_1$ and $\bar{r}.\bar{n}_2 = p_2$ is

$\bar{r}.\left(\bar{n}_1 + \lambda\bar{n}_2\right) = p_1 + \lambda p_2$

$\Rightarrow \bar{r}.\left[\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}-\hat{j}-\hat{k}\right)\right] = 3+\lambda(4)$

$\Rightarrow \bar{r}.\left[(1+3\lambda)\hat{i}-(1+\lambda)\hat{j}+(2-\lambda)\hat{k}\right] = 3+4\lambda$

27. Consider option (B)

$\bar{r}\cdot(10\hat{i} + 11\hat{j} + 12\hat{k}) = 33$

Its Cartesian form is

$10x + 11y + 12z = 33$

Since, the given point (1, 1, 1) is satisfies the above plane

∴ correct answer is option (B)

Alternate method:

The equation of plane through the intersection of given planes is

$(x + y + z - 4) + \lambda(x + 2y + 3z + 3) = 0$

Since, it passes through (1, 1, 1)

∴ $(1 + 1 + 1 - 4) + \lambda(1 + 2 + 3 + 3) = 0 \Rightarrow \lambda = \frac{1}{9}$

$\Rightarrow (x + y + z - 4) + \frac{1}{9}(x + 2y + 3z + 3) = 0$

$\Rightarrow 10x + 11y + 12z - 33 = 0$

∴ the equation of plane in vector form is

$\bar{r}\cdot(10\hat{i} + 11\hat{j} + 12\hat{k}) = 33$

28. Consider option (D)

$\bar{r}.\left(11\hat{i}+3\hat{j}-5\hat{k}\right) = 22$

Its Cartesian form is

$11x + 3y - 5z = 22$

Since, the given point (1, 2, –1) is satisfies the above plane,

∴ correct answer is option (D)

29. Equation of plane passing through intersection of given planes is,
$(x + y + z - 1) + \lambda(2x + 3y - z + 4) = 0$
$(1 + 2\lambda)x + (1 + 3\lambda)y + (1 - \lambda)z + 4\lambda - 1 = 0$
Since, the plane is parallel to X-axis,
$\therefore$ $(1 + 2\lambda) = 0 \Rightarrow \lambda = -\frac{1}{2}$
Hence, the equation of required plane is
$y - 3z + 6 = 0$

30. Plane passes through (1, 2, 3)
The point (1, 2, 3) satisfies the equation of plane represented by option (B)
$\therefore$ option (B) is correct
Alternate method:
Any plane parallel to $2x + 4y + 2z = 5$ is
$2x + 4y + 2z = k$
It passes through $(1, 2, 3) \Rightarrow k = 16$
$\therefore$ Equation of plane is $x + 2y + z = 8$

31. Plane passes through (0, 0, 0)
The point (0, 0, 0) satisfies the equation of plane represented by option (A)
$\therefore$ option (A) is correct.

32. Equation of plane parallel to ZX-plane is $y = b$.
It is passes through (0, 2, 0)
$\therefore$ its equation is $y = 2$.

33. Equation of plane parallel to YZ-plane is $x = a$
Since, it is passes through (–1, 3, 4)
$\therefore$ equation of required plane is $x = -1$
i.e., $x + 1 = 0$

34. Since, the plane is parallel to X-axis,
$\therefore$ the d.r.s. of the normal to the plane are 0, b, c
$\therefore$ The equation of required plane is $by + cz + d = 0$

35. Since, the plane is parallel to $ax + by + cz = 0$, their d.r.s will be same and
It passes through (α, β, γ)
$\therefore$ The equation of the plane is
$a(x - \alpha) + b(y - \beta) + c(z - \gamma) = 0$
$\Rightarrow ax + by + cz = a\alpha + b\beta + c\gamma$

36. Equation of the plane through the origin is $ax + by + cz = 0$
The required plane passes through the line $\frac{x-1}{5} = \frac{y-2}{4} = \frac{z-3}{5}$
$\therefore$ $5a + 4b + 5c = 0$ ….(i)
The plane passes through the point (1, 2, 3)
$\therefore$ $a + 2b + 3c = 0$ ….(ii)
Solving (i) and (ii), we get
$\therefore$ $\frac{a}{12-10} = \frac{b}{5-15} = \frac{c}{10-4}$
$\Rightarrow \frac{a}{1} = \frac{b}{-5} = \frac{c}{3}$
$\therefore$ The equation of the required plane is $x - 5y + 3z = 0$

37. Since, line is perpendicular to the plane
$\therefore$ d.r.s. of the line are a, b, c
It passes through (α, β, γ)
$\therefore$ equation of perpendicular is
$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$

38. Since, line is perpendicular to the plane
$\therefore$ d.r.s. of the line are 2, –3, 1
It passes through (1, 1, 1)
$\therefore$ the equation of required line is
$\frac{x-1}{2} = \frac{y-1}{-3} = \frac{z-1}{1}$

39. Since, line is perpendicular to the plane
$\therefore$ d.r.s. of the line are 1, –2, –3
It passes through (1, 1, –1)
$\therefore$ the equation of required line is
$\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z+1}{-3}$

40. D.r.s of line perpendicular to YZ-plane are 1, 0, 0
It passes through (1, 2, 3)
$\therefore$ equation of required line is
$\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-3}{0}$

41. D.r.s of the normal to the XZ plane are a, 0, c
The required line passes through (1, 2, 3)
$\therefore$ The equation of required line is
$\frac{x-1}{a} = \frac{y-2}{0} = \frac{z-3}{c}$

42. Equation of line passing through point (1, 1, 1) is
$\frac{x-1}{a} = \frac{y-1}{b} = \frac{z-1}{c}$
Also, the line is parallel to the plane
$2x + 3y + z + 5 = 0$
$\therefore$ $2a + 3b + c = 0$
The above equation is satisfied by –1, 1, –1
$\therefore$ correct answer is option (A)

43. The line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$.

$\therefore$ the point (4, 2, k) lies on the line and hence lies in the plane

$\therefore$ $2(4) - 4(2) + k = 7$

$\Rightarrow k = 7$

44. $\bar{n}_1 = 2\hat{i} - \hat{j} + \hat{k}$ and $\bar{n}_2 = \hat{i} + \hat{j} + 2\hat{k}$

$\therefore$ $\cos\theta = \left|\frac{\bar{n}_1 . \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}\right|$

$= \left|\frac{2(1) - 1(1) + 1(2)}{\sqrt{4+1+1}\sqrt{1+1+4}}\right| = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

45. Let $a_1, b_1, c_1 = 1, 2, -3$ and $a_2, b_2, c_2 = 4, 1, 2$

$\therefore$ The angle between the planes is

$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} . \sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$

$= \left|\frac{1(4) + 2(1) + (-3)(2)}{\sqrt{1+4+9} . \sqrt{16+1+4}}\right| = 0$

$\Rightarrow \theta = \frac{\pi}{2}$

48. The d.r.s. of normal to first plane are a, b, c and the d.r.s. of normal to second plane are a′, b′, c′

Since the two planes are perpendicular,

$\therefore$ $aa' + bb' + cc' = 0$

49. The d.r.s of the normal to the plane are 0, 2, 3.

The d.r.s of X axis are 1, 0, 0

Now, $a_1a_2 + b_1b_2 + c_1c_2 = 0(1) + 2(0) + 3(0)$
$= 0$

$\therefore$ The plane $2y + 3z = 0$ passes through X-axis.

50. Comparing the equations of line and plane with $\bar{r} = \bar{a} + \lambda\bar{b}$ and $\bar{r}.\bar{n} = p$, we get

$\bar{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\bar{n} = 2\hat{i} - \hat{j} + \hat{k}$

$\therefore$ The angle between the line and plane is

$\sin\theta = \left|\frac{\bar{b}.\bar{n}}{|\bar{b}|.|\bar{n}|}\right|$

$= \left|\frac{1(2) + 2(-1) - 1(1)}{\sqrt{1+4+1}\ \sqrt{4+1+1}}\right| = \frac{1}{6}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{6}\right)$

51. Here, $\bar{b} = \hat{i} - \hat{j} + \hat{k}$ and $\bar{n} = 3\hat{i} - 4\hat{k}$

$\therefore$ Angle between the line and plane is

$\sin\theta = \left|\frac{(\hat{i} - \hat{j} + \hat{k}).(3\hat{i} - 4\hat{k})}{\sqrt{1+1+1} . \sqrt{9+16}}\right| = \left|\frac{-1}{5\sqrt{3}}\right|$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{5\sqrt{3}}\right)$

52. Let a, b, c = 2, 3, 4 and $a_1, b_1, c_1 = 3, 2, -3$

$\therefore$ $\sin\theta = \frac{aa_1 + bb_1 + cc_1}{\sqrt{a^2 + b^2 + c^2}\sqrt{a_1^2 + b_1^2 + c_1^2}}$

$= \frac{2(3) + 3(2) + 4(-3)}{\sqrt{2^2 + 3^2 + 4^2}\sqrt{(3)^2 + (2)^2 + (-3)^2}}$

$\Rightarrow \sin\theta = 0$

$\Rightarrow \theta = 0°$

53. Let a, b, c = 3, 2, 4 and $a_1, b_1, c_1 = 2, 1, -3$

$\therefore$ $\sin\theta = \frac{6 + 2 - 12}{\sqrt{9+4+16}\sqrt{4+1+9}}$

$\Rightarrow \sin\theta = \frac{-4}{\sqrt{29 \times 14}} = \frac{-4}{\sqrt{406}}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{-4}{\sqrt{406}}\right)$

54. The d.r.s. of line and plane are a, b, c

$\therefore$ $\sin\theta = \frac{aa + bb + cc}{\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{a^2 + b^2 + c^2}}$

$= \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2} = 1$

$\Rightarrow \theta = 90°$

55. Given equation of line is $6x = 4y = 3z$

i.e. $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$

$\therefore$ the d.r.s. of line are 2, 3, 4

the d.r.s. of plane are 3, 2, –3

$\therefore$ $\sin\theta = \frac{2(3) + 3(2) + 4(-3)}{\sqrt{4+9+16} . \sqrt{9+4+9}} = 0$

$\Rightarrow \theta = 90°$

57. Since the line $\bar{r} = \hat{i} + \lambda(2\hat{i} - m\hat{j} - 3\hat{k})$ is parallel to the plane $\bar{r}.(m\hat{i} + 3\hat{j} + \hat{k}) = 0$

$\therefore$ $\bar{b}.\bar{n} = 0$

$\Rightarrow (2\hat{i} - m\hat{j} - 3\hat{k}) . (m\hat{i} + 3\hat{j} + \hat{k}) = 0$

$\Rightarrow 2(m) - m(3) - 3(1) = 0$

$\Rightarrow -m = 3$

$\Rightarrow m = -3$

58. The line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{4}$ lie on the plane $4x + 4y - kz = 0$

Since the given line lies on the plane, it is parallel to the plane

$\therefore$ $aa_1 + bb_1 + cc_1 = 0$

$\Rightarrow 4(2) + 4(3) - k(4) = 0$

$\Rightarrow 4k = 20 \Rightarrow k = 5$

59. The equation of plane is

$3x - 2y + 6z - 5 = 0$ and the point is (2, 3, 4)

$\therefore$ The distance of point from the plane is

$$d = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left|\frac{2(3) + 3(-2) + 4(6) - 5}{\sqrt{3^2 + 2^2 + 6^2}}\right| = \frac{19}{7}$$

Alternate method:

Let $A(\bar{a}) = (2,3,4)$

Given equation of plane is

$\bar{r}.(3\hat{i} - 2\hat{j} + 6\hat{k}) = 5$

$\therefore$ $\bar{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, and $\bar{n} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\therefore$ The distance of point from plane is

$$d = \frac{|(\bar{a}.\bar{n}) - p|}{|\bar{n}|} = \left|\frac{2(3) + 3(-2) + 4(6) - 5}{\sqrt{3^2 + 2^2 + 6^2}}\right| = \frac{19}{7}$$

60. Here, a = 1, b = 1, c = 1, d = –3 and $x = y = z = 0$

$\therefore$ $d = \left|\frac{-3}{\sqrt{1^2 + 1^2 + 1^2}}\right| = \sqrt{3}$

61. Here, a = 3, b = –6, c = 2, z = 11 and $x = 2$, $y = 3$, z = 4

$\therefore$ $d = \left|\frac{3(2) + (-6)(3) + 2(4) + 11}{\sqrt{3^2 + (-6)^2 + 2^2}}\right| = 1$

62. Let the intercepts made by the plane a, b, c = 2, 1, –2

$\therefore$ The distance of plane from origin is

$$d = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{4} + 1 + \frac{1}{4}}} = \frac{2}{\sqrt{6}}$$

Alternate method:

The equation of plane is

$\frac{x}{2} + \frac{y}{1} + \frac{z}{-2} = 1$

i.e. $x + 2y - z - 2 = 0$

$\therefore$ distance of plane from the origin is

$$d = \left|\frac{-2}{\sqrt{1 + 4 + 1}}\right|$$

$$d = \frac{2}{\sqrt{6}}$$

63. Let a, b, c = –6, 3, 4

$\therefore$ The length of perpendicular from origin is

$$d = \frac{1}{\sqrt{\frac{1}{(-6)^2} + \frac{1}{3^2} + \frac{1}{4^2}}} = \frac{1}{\sqrt{\frac{29}{144}}} = \frac{12}{\sqrt{29}}$$

64. The distance of (1,1,1) from the origin is

$d = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$

Distance of (1,1,1) from $x + y + z + k = 0$ is

$$d_1 = \left|\frac{(1) + (1) + (1) + k}{\sqrt{(1)^2 + (1)^2 + (1)^2}}\right| = \pm\frac{k + 3}{\sqrt{3}}$$

Now, $\sqrt{3} = \pm\frac{1}{2}\left(\frac{k + 3}{\sqrt{3}}\right)$ ….(given)

$\Rightarrow 6 = \pm (k + 3)$

$\Rightarrow k = 3, -9$

65. Since, the points (1, 1, k) and (–3, 0 , 1) are equidistance from the given plane

$\therefore$ $\left|\frac{3 + 4 - 12k + 13}{\sqrt{9 + 16 + 144}}\right| = \left|\frac{-9 - 12 + 13}{\sqrt{9 + 16 + 144}}\right|$

$\Rightarrow |3 + 4 - 12k + 13| = |-9 - 12 + 13|$

$\Rightarrow 20 - 12k = \pm 8 \Rightarrow k = 1, \frac{5}{7}$

66. Given line passes through (1, –2, 1) and the d.r.s. of normal to the plane are 2, 2, –1

$\therefore$ $d = \left|\frac{2(1) + 2(-2) - 1(1) - 6}{\sqrt{2^2 + 2^2 + (-1)^2}}\right| = \frac{9}{\sqrt{9}} = 3$

67. Given planes are parallel and can be written as

$2x - 2y + z + 3 = 0$ and $2x - 2y + z + \frac{5}{2} = 0$

$\therefore$ the distance between these planes is

$$d = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left|\frac{3 - \frac{5}{2}}{\sqrt{4 + 4 + 1}}\right| = \frac{\left(\frac{1}{2}\right)}{3} = \frac{1}{6}$$

68. Given planes are parallel, and can be written as

$x + 2y + 3z + 7 = 0$ and $x + 2y + 3z + \frac{7}{2} = 0$

$\therefore$ the distance between these planes is

$$d = \left|\frac{7-\frac{7}{2}}{\sqrt{1+4+9}}\right| = \frac{\sqrt{7}}{2\sqrt{2}}$$

69. The plane passes through points $(1, -2, 3)$ and $(4, 0, -1)$

This points satisfies the equation of plane in option (A)

$\therefore$ option (A) is correct answer.

70. The plane passes through $(1, 2, -1)$

This point satisfies the equation of plane in option (A)

Also, it has d.r.s = $\bar{b}\times\bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 1 & 1 & 3 \end{vmatrix}$

$= 7\hat{i} - 4\hat{j} - \hat{k}$

i.e., 7, –4, –1

$\therefore$ option (A) is correct answer.

Alternate Method

Let $\bar{a} = \hat{i} - \hat{j} + \hat{k}$, $\bar{b} = 2\hat{i} + \hat{j} + \hat{k}$ and $\bar{c} = \hat{j} + 2\hat{k}$

Now, $\bar{b}\times\bar{c} = \hat{i} - 4\hat{j} + 2\hat{k}$

$\therefore$ the vector equation of required plane is

$\bar{r}.(\bar{b}\times\bar{c}) = \bar{a}.(\bar{b}\times\bar{c})$

$\Rightarrow \bar{r}(7\hat{i} - 4\hat{j} - \hat{k}) = (\hat{i} - 2\hat{j} - \hat{k}).(7\hat{i} - 4\hat{j} - \hat{k})$

$\Rightarrow \bar{r}.(7\hat{i} - 4\hat{j} - \hat{k}) = 0$

Critical Thinking

1. Let x_1, y_1, z_1 be the intercepts made by the plane

$\therefore$ Equation of plane is

$$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$$

Since it passes through (a, b, c),

$$\Rightarrow \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 1$$

$\therefore$ Locus of (x_1, y_1, z_1) is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$

2. Since, the plane contains the X-axis, it passes through the origin

$\therefore$ d = 0

$\therefore$ The equation of the plane is

$ax + by + cz = 0$ ....(i)

Also, plane passes through (1, 1, 1)

$\therefore$ $a + b + c = 0$ ....(ii)

The equation of the X-axis is $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

As the plane contains the X-axis, the d.r.s of the normal to the plane are perpendicular to X-axis

$\therefore$ $a(1) + b(0) + c(0) = 0$

$\Rightarrow a = 0$

Substituting value of a in (ii) we get

$b + c = 0 \Rightarrow b = -c$

$\therefore$ The equation of the required plane is

$by - bz = 0$

$\Rightarrow y - z = 0$

3. The plane passes through $(1, -1, 3)$ and $(2, 3 -4)$

The points satisfies the equation of plane in option (B)

$\therefore$ option (B) is correct answer.

Alternate method:

Let $ax + by + cz + d = 0$ be the equation of the required plane.

Since, the plane is parallel to X-axis,

$\therefore$ a = 0

The points $(1, -1, 3)$ and $(2, 3, -4)$ lie in the plane,

$\therefore$ $-b + 3c + d = 0$, and ....(i)

$3b - 4c + d = 0$ ....(ii)

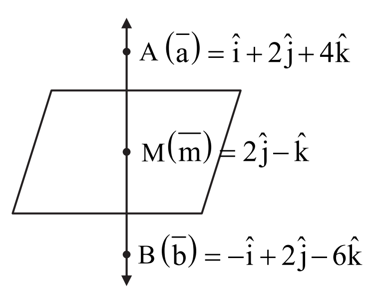
Solving the equations (i) and (ii), we get

$$\frac{b}{3-(-4)} = \frac{c}{3+1} = \frac{d}{4-9}$$

$$\Rightarrow \frac{b}{7} = \frac{c}{4} = \frac{d}{-5}$$

$\therefore$ Equation of the required plane is $7y + 4z - 5 = 0$

4.



$\therefore$ $M(\overline{m}) = \frac{(1-1)}{2}\hat{i} + \frac{(2+2)}{2}\hat{j} + \frac{(4-6)}{2}\hat{k}$

$= 2\hat{j} - \hat{k}$

$\therefore$ equation of plane passing through the vector $2\hat{j} - \hat{k}$ and perpendicular to $\overline{AB} = -2\hat{i} - 10\hat{k}$ is

$\overline{r}.(-2\hat{i} - 10\hat{k}) = (2\hat{j} - \hat{k}).(-2\hat{i} - 10\hat{k})$

$\Rightarrow \overline{r}.(\hat{i} + 5\hat{k}) = -10$

5. P be the point (a, b, c).

$\therefore$ The d.r.s of OP are a, b, c.

$\therefore$ Equation of the plane passing through the point (a, b, c) is

$a(x - a) + b(y - b) + c(z - c) = 0$

$\Rightarrow ax + by + cz = a^2 + b^2 + c^2$

6. Mid-point of the line segment joining the points (–1, 2, 3) and (3, – 5, 6) is

$M \equiv \left(\frac{-1+3}{2}, \frac{2-5}{2}, \frac{3+6}{2}\right)$

$M \equiv \left(1, \frac{-3}{2}, \frac{9}{2}\right)$

The plane passes through point M

It satisfies option (C)

Alternate method:

The required plane bisects the line segment perpendicularly.

$\therefore$ the d.r.s. of the normal to the plane are

$3 - (-1), -5 - 2, 6 - 3$

i.e. 4, –7, 3

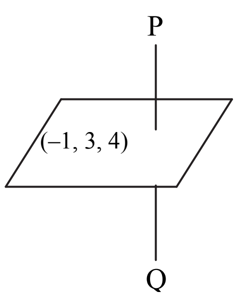
Since, the mid-point $\left(1, -\frac{3}{2}, \frac{9}{2}\right)$ lies in the plane,

$\therefore$ The equation of the plane is

$4(x - 1) - 7\left(y + \frac{3}{2}\right) + 3\left(z - \frac{9}{2}\right) = 0$

$\Rightarrow 4x - 7y + 3z = 28$

7.



Mid-point of line joining P(1, 2, 3) and Q(–3, 4, 5) is (–1, 3, 4)

It lies on the plane

The d.r.s. of normal to the plane are –4, 2, 2 i.e. –2, 1, 1

$\therefore$ The equation of the plane is

$-2(x + 1) + 1(y - 3) + 1(z - 4) = 0$

$\Rightarrow 2x - y - z = -9$

$\Rightarrow \frac{x}{\frac{-9}{2}} + \frac{y}{9} + \frac{z}{9} = 1$

$\therefore$ Intercepts are $\frac{-9}{2}$, 9, 9

8. (2, –1, 0) lies on the plane $9x - 2y - 3z = k$

$\therefore$ $9(2) - 2(-1) - 3(0) = k$

$\Rightarrow k = 20$

9. Since, the point $(1, 0, z_1)$ lies on the plane $\overline{r}.(-\hat{i} + 3\hat{k}) = 2$

i.e. $-x + 3z = 2$

$\Rightarrow z_1 = 1$

10. (3, 2, –1) lies on the plane $5x + 3y - 2z = \lambda$

$\therefore$ $5(3) + 3(2) - 2(-1) = \lambda$

$\Rightarrow \lambda = 23$

11. The equation of the plane passing through the intersection of the planes $\overline{r} \cdot \overline{a} = p$ and $\overline{r} \cdot \overline{b} = q$ is

$\overline{r} \cdot (\overline{a} + \lambda \overline{b}) = p + \lambda q$ ….(i)

Since, the plane passes through the origin,

$p + \lambda q = 0$

$\Rightarrow \lambda = \frac{-p}{q}$

Substituting the value of λ in (i), we get

$\overline{r} \cdot \left(\overline{a} - \frac{p}{q}\overline{b}\right) = p + \left(\frac{-p}{q}\right)(q)$

$\Rightarrow \overline{r} \cdot (\overline{a}q - \overline{b}p) = pq - pq$

$\Rightarrow \overline{r} \cdot (q\overline{a} - p\overline{b}) = 0$

12. The line of intersection of the planes $\overline{r}.(3\hat{i} - \hat{j} + \hat{k}) = 1$ and $\overline{r}.(\hat{i} + 4\hat{j} - 2\hat{k}) = 2$ is perpendicular to each of the normal vectors $\overline{n_1} = 3\hat{i} - \hat{j} + \hat{k}$ and $\overline{n_2} = \hat{i} + 4\hat{j} - 2\hat{k}$.

$\therefore$ The line is parallel to the vector $\overline{n}_1 \times \overline{n}_2$

$\therefore$ $\overline{n}_1 \times \overline{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 1 & 4 & -2 \end{vmatrix}$

$= -2\hat{i} + 7\hat{j} + 13\hat{k}$

13. The equation of the required plane is

$x + 2y + 3z - 4 + \lambda(2x + y - z + 5) = 0$

$\Rightarrow (1 + 2\lambda)x + (2 + \lambda)y + (3 - \lambda)z - 4 + 5\lambda = 0$ ….(i)

Let a, b, c be the d.r.s of the required plane

$\therefore$ From equation (i), $a = 1 + 2\lambda$; $b = 2 + \lambda$; $c = 3 - \lambda$

The required plane is perpendicular to $5x + 3y - 6z + 8 = 0$

$\therefore$ $5a + 3b - 6c = 0$

$\Rightarrow 5(1 + 2\lambda) + 3(2 + \lambda) - 6(3 - \lambda) = 0$

$\Rightarrow 5 + 10\lambda + 6 + 3\lambda - 18 + 6\lambda = 0$

$\Rightarrow -7 + 19\lambda = 0$

$\Rightarrow \lambda = \dfrac{7}{19}$

Substituting the value of λ in equation (i), we get

$$\left(1+2\times\frac{7}{19}\right)x + \left(2+\frac{7}{19}\right)y + \left(3-\frac{7}{19}\right)z - 4 + 5\left(\frac{7}{19}\right) = 0$$

$\Rightarrow \dfrac{33}{19}x + \dfrac{45}{19}y + \dfrac{50}{19}z - \dfrac{41}{19} = 0$

$\Rightarrow 33x + 45y + 50z - 41 = 0$

14. The equation of the plane passing through the origin is $ax + by + cz = 0$.

The required plane is perpendicular to the line $x = 2y = 3z$

i.e., $\dfrac{x}{6} = \dfrac{y}{3} = \dfrac{z}{2}$

$\therefore$ the d.r.s. of the line are 6, 3, 2

$\therefore$ the d.r.s. of the normal to the plane are 6, 3 and 2.

$\therefore$ the equation of the required plane is

$6x + 3y + 2z = 0$

15. Let a, b, c be the d.r.s. of the required plane.

Since, the plane passes through Z-axis,

$\therefore$ $a(0) + b(0) + c(1) = 0$

$\Rightarrow c = 0$

Given that the required plane is perpendicular to $\dfrac{x-1}{\cos\theta} = \dfrac{y+2}{\sin\theta} = \dfrac{z-3}{0}$

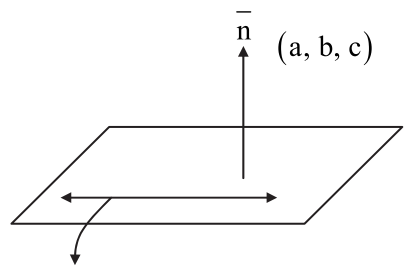
$\therefore$ d.r.s of normal to plane are $\cos\theta$, $\sin\theta$, 0

$\therefore$ the equation of required plane is

$x\cos\theta + y\sin\theta = 0$

$\Rightarrow x + y\tan\theta = 0$

16.



$$\frac{x-1}{1} = \frac{y-2}{3} = \frac{z-3}{2}$$

$$\bar{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ 2 & 7 & 5 \end{vmatrix} = \hat{i} - \hat{j} + \hat{k}$$

$\therefore$ the d.r.s of the normal to the plane are 1, –1, 1

$\therefore$ the equation of plane passing through the point (1, 2, 3)

$1(x - 1) - 1(y - 2) + 1(z - 3) = 0$

$\Rightarrow x - y + z = 2$

17. Equation of any plane through (x_1, y_1, z_1) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ….(i)

it contains the line

$\dfrac{x-x_2}{d_1} = \dfrac{y-y_2}{d_2} = \dfrac{z-z_2}{d_3} = 0$

i.e. it passes through (x_2, y_2, z_2)

$\therefore$ $a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$ ….(ii)

Also, $ad_1 + bd_2 + cd_3 = 0$ ….(iii)

Eliminating a, b, c from (i), (ii), (iii), we get the equation of the required plane as

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

18. Vector perpendicular to plane is

$\bar{n} = 6\hat{i} - 3\hat{j} + 5\hat{k}$

Thus, the line perpendicular to the given line will be parallel to $\bar{n}$

$\therefore$ The equation of line which passes through $\bar{a} = 2\hat{i} - 3\hat{j} - 5\hat{k}$ and parallel to $\bar{n}$ is

$\bar{r} = \bar{a} + \lambda\bar{n}$

$\Rightarrow \bar{r} = (2\hat{i} - 3\hat{j} - 5\hat{k}) + \lambda(6\hat{i} - 3\hat{j} + 5\hat{k})$

19. The d.r.s. of the line are 3, – 4, 5 and it passes through is 3, – 5, 7

$\therefore$ The equation of line is

$\bar{r} = 3\hat{i} - 5\hat{j} + 7\hat{k} + \lambda\left(3\hat{i} - 4\hat{j} + 5\hat{k}\right)$

20. The line is perpendicular to the plane
$x + 2y - 5z + 9 = 0$
$\therefore$ the d.r.s are 1, 2, –5
Also it passes through (1, 2, 3)
$\therefore$ The equation of line is $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{-5}$

21. $\bar{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{vmatrix} = -3\hat{i} + 5\hat{j} + 4\hat{k}$

$\therefore$ the d.r.s. of line are – 3, 5, 4
$\therefore$ The equation of the line passing through (1, 2, 3) and having d.r.s. –3, 5, 4 is
$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$

22. Here, $\bar{n}_1 = \left(x\hat{i} + \hat{j} - \hat{k}\right)$, and
$\bar{n}_2 = \left(\hat{i} + x\hat{j} - \hat{k}\right)$

$\therefore$ $\cos\theta = \left|\frac{\bar{n}_1 . \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}\right|$

$\Rightarrow \cos\frac{\pi}{3} = \left|\frac{\left(x\hat{i} + \hat{j} - \hat{k}\right).\left(\hat{i} + x\hat{j} + \hat{k}\right)}{\sqrt{x^2+1+1}.\sqrt{1+x^2+1}}\right|$

$\Rightarrow \frac{1}{2} = \pm\left(\frac{x + x - 1}{x^2+2}\right)$

$\Rightarrow \frac{2x-1}{x^2+2} = \frac{1}{2}$ ....(considering positive value)

$\Rightarrow x^2 + 2 - 4x + 2 = 0$
$\Rightarrow (x - 2)^2 = 0$
$\Rightarrow x = 2$

23. Consider plane OPQ
the equation of plane is
$ax + by + cz = 0$
The plane passes through P(1, 2, 1) and Q(2, 3, 0)
$\therefore$ $a + 2b + c = 0$ and ....(i)
$2a + 3b = 0$ ....(ii)
On solving (i) and (ii), we get
$\frac{a}{-3} = \frac{b}{2} = \frac{c}{-1}$
$\therefore$ The equation of plane OPQ is
$-3x + 2y - z = 0$ ....(iii)
The equation of plane PQR is
$a_1 (x - 1) + b_1 (y - 2) + c_1 (z - 1) = 0$
On solving for a_1, b_1, c_1, we get
$a_1 = -3, b_1 = 3, c_1 = 0$
$\therefore$ The equation of PQR is
$x - y + 1 = 0$ ....(iv)
$\therefore$ The angle between the planes represented by equations (iii) and (iv) is

$\cos\theta = \left|\frac{(-3)(1) + 2(-1)}{\sqrt{9+4+1}.\sqrt{1+1}}\right|$

$= \left|\frac{-5}{\sqrt{14}.\sqrt{2}}\right|$

$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{\sqrt{28}}\right)$

24. The d.r.s. of normal to the plane are 2, 3, –1
The d.r.s. of X-axis are 1, 0, 0
$\therefore$ the angle between the plane and X-axis is

$\sin\theta = \frac{aa_1 + bb_1 + cc_1}{\sqrt{a^2+b^2+c^2}.\sqrt{a_1^2+b_1^2+c_1^2}}$

$\Rightarrow \sin\theta = \frac{2(1)+0+0}{\sqrt{4+9+1}.\sqrt{1}}$

$\Rightarrow \sin\theta = \frac{2}{\sqrt{14}}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{2}{\sqrt{14}}\right)$

25. Here $a = 1, b = k, c = 4$ and
$a_1 = 1, b_1 = -3, c_1 = 2$
The angle between the line and plane is

$\sin\theta = \frac{aa_1 + bb_1 + cc_1}{\sqrt{a^2+b^2+c^2}.\sqrt{a_1^2+b_1^2+c_1^2}}$

Now, $\theta = \sin^{-1}\left(\frac{3}{7\sqrt{6}}\right) \Rightarrow \sin\theta = \frac{3}{7\sqrt{6}}$

$\therefore$ $\frac{3}{7\sqrt{6}} = \frac{1-3k+8}{\sqrt{1+k^2+16}.\sqrt{1+9+4}}$

$\Rightarrow k^2 + 21k - 46 = 0$
$\Rightarrow k = 2$ or -23

26. Equation of the line L: $\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{2}$ and equation of the plane P: $4x - 2y - z = 1$.
The d.r.s of the line are 2, 3, 2, and
The d.r.s of the normal to the plane are 4, –2, –1.
Now consider
$a_1a_2 + b_1b_2 + c_1c_2 = 8 - 6 - 2 = 0$
$\therefore$ Line L and plane P are parallel.
Since the point (1, 0, 3), which lies on the line L also satisfies the equation of the plane,
$\therefore$ The line L lies in the plane P.

27. Equation of the line
L: $\frac{x+3}{2} = \frac{y-4}{3} = \frac{z+5}{1}$
and equation of the plane
P : $2x - 3y + 5z = 1$.
The d.r.s of the line are 2, 3, 1
The d.r.s of the normal to the plane are 2, –3, 5.
Now consider
$a_1a_2 + b_1b_2 + c_1c_2 = 4 - 9 + 5 = 0$
∴ Line L is parallel to the plane P.

28. Since, the line $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ lies in the plane $4x + 4y - cz - d = 0$,
∴ $aa_1 + bb_1 + cc_1 = 0$
$\Rightarrow 2(4) + 3(4) + 4(-c) = 0$
$\Rightarrow 20 - 4c = 0$
$\Rightarrow c = 5$
Also, the plane passes through (3, 4, 5)
∴ $4(3) + 4(4) - 5(5) - d = 0$
$\Rightarrow d = 3$

29. Given equation of plane
$\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+2}{2}$
∴ The line passes through (1, 1, –2)
The above point lies on the plane
$x + By - 3z + D = 0$
∴ $1 + B + 6 + D = 0$
$\Rightarrow B + D = -7$ ....(i)
Also the given line is perpendicular to the normal to the plane
$a_1a_2 + b_1b_2 + c_1c_2 = 0$
$\Rightarrow 2(1) + 3(B) + 2(-3) = 0$
$\Rightarrow B = \frac{4}{3}$
Substituting value of B is equation (i), we get
$D = \frac{-25}{3}$

30. Since both the given lines pass through the point with position vector $\hat{i}+\hat{j}$, the required plane also passes through $\hat{i}+\hat{j}$, and normal to the plane is perpendicular to the vectors
$\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} - 2\hat{k}$.
Let a, b, c be the d.r.s. of the normal to the plane.
∴ $\bar{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{vmatrix}$
$\Rightarrow \bar{n} = -3\hat{i} + 3\hat{j} + 3\hat{k}$
i.e. $\bar{n} = -\hat{i} + \hat{j} + \hat{k}$
∴ Vector equation of the plane passing through $\hat{i} + \hat{j}$ and containing the given lines is
$\bar{r}.(-\hat{i} + \hat{j} + \hat{k}) = (\hat{i} + \hat{j}).(-\hat{i} + \hat{j} + \hat{k})$
$\Rightarrow \bar{r}.(-\hat{i} + \hat{j} + \hat{k}) = 0$

31. The plane passes through
(0, 2, –3) and (2, 6, 3)
The two points satisfy the equation of plane is option (A)
∴ option (A) is correct.
Alternate Method:
The equation of the plane is
$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$
$\Rightarrow \begin{vmatrix} x & y-2 & z+3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0$
$\Rightarrow -x - (y-2)(-2) + (z+3)(-1) = 0$
$\Rightarrow -x + 2y - 4 - z - 3 = 0$
$\Rightarrow x - 2y + z + 7 = 0$

32. The plane passes through
(5, 7, –3) and (8, 4, 5)
The two points satisfy the equation of plane is option (A)
∴ option (A) is correct.

33. Let a, b, c be the d.r.s of the normal to the plane
∴ $\bar{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = \hat{i} - 2\hat{j} + \hat{k}$
Since, the plane passes through (– 1, – 3, – 5)
∴ $1(x + 1) - 2(y + 3) + 1(z + 5) = 0$
$\Rightarrow x - 2y + z = 0$
From the given options only (0, 0, 0) satisfies the equation of the plane.
∴ The plane passes through (0, 0, 0).

34. Here $x_1, y_1, z_1 = -l, -3, -5$ and $x_2, y_2, z_2 = 2, 4, 6$
$a_1, b_1, c_1 = 3, 5, 7$ and $a_2, b_2, c_2 = 1, 3, 5$
Since the given lines are coplanar
∴ $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} -l-2 & -3-4 & -5-6 \\ 3 & 5 & 7 \\ 1 & 3 & 5 \end{vmatrix} = 0$$

$\Rightarrow (-l-2)(25-21) - (-3-4)(15-7) + (-5-6)(9-5) = 0$

$\Rightarrow 12 = 4(l+2)$

$\Rightarrow l = 1.$

35. The lines are coplanar

$$\therefore \begin{vmatrix} -1-2 & -3-4 & -5-6 \\ 1 & 4 & k \\ 3 & 5 & k \end{vmatrix} = 0$$

$\Rightarrow -3(4k-5k) + 7(k-3k) - 11(-7) = 0$

$\Rightarrow k = 7$

36. Since the given lines are coplanar, then

$$\therefore \begin{vmatrix} 3-1 & 1-2 & 3-1 \\ 1 & 2 & -\lambda \\ \lambda & 3 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -\lambda \\ \lambda & 3 & 4 \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 + 2\lambda + 26 = 0$

$\Delta = 4 - 4(1)(26) < 0$

$\therefore$ Roots are imaginary

So no real value of λ exists.

37. $\dfrac{x}{1} = \dfrac{y-2}{2} = \dfrac{z+3}{3}$ and

$\dfrac{x-2}{2} = \dfrac{y-6}{3} = \dfrac{z-3}{4}$

The d.r.s. of the first line are 1, 2, 3 and

The d.r.s. of the second line are 2, 3, 4

Ratio of the d.r.s. are not same

i.e. $\dfrac{2}{1} \neq \dfrac{3}{2} \neq \dfrac{4}{3}$

$\therefore$ The lines are not parallel.

Sum of the products of the d.r.s. is not equal to 0 i.e., $2(1) + 2(3) + 3(4) \neq 0$

$\therefore$ The lines are not perpendicular.

$$\text{Consider} \begin{vmatrix} 0+2 & -2+6 & 3+3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0 \quad (\because \text{ the two rows are same})$$

$\therefore$ The two lines are coplanar.

38. Let d_1 be the distance of the point (1, 2, −1) from the plane $2x - 3y + z + k = 0$

$$\therefore \quad d_1 = \left|\frac{2(1)-3(2)+(-1)+k}{\sqrt{2^2+(-3)^2+1^2}}\right| = \left|\frac{-5+k}{\sqrt{4+9+1}}\right|$$

$$= \left|\frac{k-5}{\sqrt{14}}\right|$$

Let d_2 be the distance of the point (1, 2, −1) from the plane $x + 2y + 3z = 0$

$$\therefore \quad d_2 = \left|\frac{(1)+2(2)+3(-1)}{\sqrt{1^2+2^2+3^2}}\right| = \left|\frac{2}{\sqrt{14}}\right|$$

Given that $d_1.d_2 = 1$.

$$\therefore \quad \left|\frac{k-5}{\sqrt{14}}\right|\left|\frac{2}{\sqrt{14}}\right| = 1$$

$\Rightarrow (k-5)\,2 = 14$

$\Rightarrow k - 5 = 7$

$\Rightarrow k = 12$

39. $$P_1 = \left|\frac{3(2)-6(3)+2(4)+11}{\sqrt{3^2+(-6)^2+(2)^2}}\right| = 1$$

$$P_2 = \left|\frac{3(1)-6(1)+2(4)+11}{\sqrt{3^2+(-6)^2+(2)^2}}\right| = \frac{16}{7}$$

the equation P_1 and P_2 satisfies

$7P^2 - 23P + 16 = 0.$

$\therefore$ P_1 and P_2 are the roots of the equation (B).

40. Equation of plane parallel to $x - 2y + 2z = 5$ is

$x - 2y + 2z + k = 0$ ....(i)

distance of the above plane from (1, 2, 3) is 1.

$$\therefore \quad \left|\frac{1-4+6+k}{\sqrt{9}}\right| = 1$$

i.e. $k + 3 = \pm 3$

$\Rightarrow k = 0$ or -6

41. Let x, y, z be any point

$d_1^2 + d_2^2 + d_3^2 = 36$

$$\therefore \quad \left|\frac{x-z}{\sqrt{2}}\right|^2 + \left|\frac{x-2y+z}{\sqrt{6}}\right|^2 + \left|\frac{x+y+z}{\sqrt{3}}\right|^2 = 36$$

$$\Rightarrow \frac{1}{6}[3x^2 - 6xz + 3z^2 + x^2 + 4y^2 + z^2 - 4xy - 4yz + 2xz + 2x^2 + 2y^2 + 2z^2 + 4xy + 4yz + 4xz] = 36$$

$\therefore \quad \Rightarrow x^2 + y^2 + z^2 = 36$

42. Since all the planes are parallel,

$\therefore$ $p_1 = \dfrac{|2-6|}{\sqrt{2^2+(-3)^2+4^2}} = \dfrac{4}{\sqrt{29}}$

Equation of the plane $4x - 6y + 8z + 3 = 0$ can be written as $2x - 3y + 4z + \dfrac{3}{2} = 0$

$\therefore$ $p_2 = \dfrac{\left|2-\dfrac{3}{2}\right|}{\sqrt{2^2+(-3)^2+4^2}} = \dfrac{1}{2\sqrt{29}}$

and $p_3 = \dfrac{|2+6|}{\sqrt{2^2+(-3)^2+4^2}} = \dfrac{8}{\sqrt{29}}$

Now consider $p_1 + 8p_2 - p_3$

$= \dfrac{4}{\sqrt{29}} + \dfrac{4}{\sqrt{29}} - \dfrac{8}{\sqrt{29}}$

$= 0$

43. Let $A \equiv (a, 0, 0)$, $B \equiv (0, b, 0)$ and $c \equiv (0, 0, c)$
The equation of the plane in intercept form is

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

Since, centroid is (3, 3, 3)

$\therefore$ $3 = \dfrac{x_1+x_2+x_3}{3} = \dfrac{a+0+0}{3} = 3$

$\Rightarrow a = 9$

Similarly $\dfrac{0+b+0}{3} = 3 \Rightarrow b = 9$, and

$\dfrac{0+0+c}{3} = 3 \Rightarrow c = 9$

$\therefore$ The equation of plane is $\dfrac{x}{9} + \dfrac{y}{9} + \dfrac{z}{9} = 1$

$\Rightarrow x + y + z = 9$

44. Let $A \equiv (a, 0, 0)$, $B \equiv (0, b, 0)$ and $C \equiv (0, 0, c)$.
Since, centroid is (α, β, γ)

$\therefore$ $a = 3\alpha$, $b = 3\beta$, $c = 3\gamma$

$\therefore$ the equation of the plane is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

$\dfrac{x}{3\alpha} + \dfrac{y}{3\beta} + \dfrac{z}{3\gamma} = 1$

$\Rightarrow \dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 3$

45. The given equation of plane is $6x - 3y + 2z = 18$

i.e. $\dfrac{x}{3} + \dfrac{y}{-6} + \dfrac{z}{9} = 1$

If a, b, c are intercepts made by the plane, then

Centroid $\equiv \left(\dfrac{a+0+0}{3}, \dfrac{0+b+0}{3}, \dfrac{0+0+c}{3}\right)$

$\therefore$ $G \equiv \left(\dfrac{3+0+0}{3}, \dfrac{0-6+0}{3}, \dfrac{9+0+0}{3}\right)$

$\Rightarrow G \equiv (1, -2, 3)$

46. The given equations of plane is $ax + by + cz = 1$

i.e. $\dfrac{x}{\frac{1}{a}} + \dfrac{y}{\frac{1}{b}} + \dfrac{z}{\frac{1}{c}} = 1$

$\therefore$ The intercepts made by the plane are $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$

$\therefore$ $A \equiv \left(\dfrac{1}{a}, 0, 0\right)$; $B \equiv \left(0, \dfrac{1}{b}, 0\right)$; $C \equiv \left(0, 0, \dfrac{1}{c}\right)$

$\therefore$ centroid $\equiv \left(\dfrac{\frac{1}{a}+0+0}{3}, \dfrac{0+\frac{1}{b}+0}{3}, \dfrac{0+0+\frac{1}{c}}{3}\right)$

$\Rightarrow G \equiv \left(\dfrac{1}{3a}, \dfrac{1}{3b}, \dfrac{1}{3c}\right)$

47. Let equation of plane be $lx + my + nz = p$

i.e., $\dfrac{x}{\frac{p}{l}} + \dfrac{y}{\frac{p}{m}} + \dfrac{z}{\frac{p}{n}} = 1$

$\therefore$ $A \equiv \left(\dfrac{p}{l}, 0, 0\right)$, $B \equiv \left(0, \dfrac{p}{m}, 0\right)$, $C \equiv \left(0, 0, \dfrac{p}{n}\right)$

If centroid of ΔABC is (x_1, y_1, z_1), then

$x_1 = \dfrac{p}{3l}$, $y_1 = \dfrac{p}{3m}$, $z_1 = \dfrac{p}{3n}$

Now, $l^2 + m^2 + n^2 = 1$

$\therefore$ $\dfrac{p^2}{9x_1^2} + \dfrac{p^2}{9y_1^2} + \dfrac{p^2}{9z_1^2} = 1$

$\Rightarrow \dfrac{1}{x_1^2} + \dfrac{1}{y_1^2} + \dfrac{1}{z_1^2} = \dfrac{9}{p^2}$

48. The equation of line perpendicular to given plane passing through (2, 2, 2) is

$\dfrac{x-2}{1} = \dfrac{y-2}{1} = \dfrac{z-2}{1} = \lambda$ (say)

Any general point on it is $P \equiv (\lambda + 2, \lambda + 2, \lambda + 2)$
Since, P lies the plane $x + y + z = 0$

$\therefore$ $\lambda + 2 + \lambda + 2 + \lambda + 2 = 9 \Rightarrow \lambda = 1$

$\therefore$ The foot of perpendicular is (3, 3, 3).

49. The required plane is perpendicular to the line

$\dfrac{x-2}{1} = \dfrac{y-4}{2} = \dfrac{z-5}{2} = \lambda$ (say)

the d.r.s of normal to the plane are proportional to 1, 2, 2

$\therefore$ Equation of the plane is

$x + 2y + 2z + d = 0$ ….(i)

Since it passes through the point (5, 1, 2), we have

$(5) + 2(1) + 2(2) + d = 0$

$\Rightarrow d = -11$

$\therefore$ The equation (i) becomes $x + 2y + 2z - 11 = 0$

Any general point on the given line is given by

$\lambda + 2, 2\lambda + 4, 2\lambda + 5$.

This point lies in the required plane

$\therefore$ $\lambda + 2 + 2(2\lambda + 4) + 2(2\lambda + 5) - 11 = 0$

$\Rightarrow \lambda + 2 + 4\lambda + 8 + 4\lambda + 10 - 11 = 0$

$\Rightarrow 9\lambda + 9 = 0 \Rightarrow \lambda = -1$

$\therefore$ The point of intersection is

$[(-1) + 2, 2(-1) + 4, 2(-1)+5]$

$\Rightarrow (1, 2, 3)$

50. The equation of plane passing through the intersection of the given planes is

$(2x - 5y + z - 3) + \lambda(x + y + 4z - 5) = 0$

$\Rightarrow (2 + \lambda)x + (-5 + \lambda)y + (1 + 4\lambda) z - 3 - 5\lambda = 0$ ....(i)

This plane is parallel to the plane $x + 3y + 6z = 1$

$\therefore$ $\dfrac{2+\lambda}{1} = \dfrac{-5+\lambda}{3} = \dfrac{1+4\lambda}{6}$

$\Rightarrow \lambda = \dfrac{-11}{2}$

$\therefore$ Substituting value of λ in equation (i), we get

$-\dfrac{7}{2}x - \dfrac{21}{2}y - \dfrac{42}{2}z + \dfrac{49}{2} = 0$

$\therefore$ $x + 3y + 6z = 7$

Comparing with $x + 3y + 6z = k$, we get

$k = 7$

51. The equation of the plane through the line of intersection of the planes,

$4x + 7y + 4z + 81 = 0$ and $5x + 3y + 10z = 25$

is $(4x + 7y + 4z + 81) + \lambda(5x + 3y + 10z - 25) = 0$

$\Rightarrow (4+ 5\lambda) x +(7+ 3\lambda) y + (4 +10\lambda)z + 81- 25\lambda =0$ ....(i)

It is parallel to $x - 4y + 6z = k$,

$\therefore$ $\dfrac{4+5\lambda}{1} = \dfrac{7+3\lambda}{-4} = \dfrac{4+10\lambda}{6}$

$\Rightarrow \lambda = -1$

Substituting value of λ in equation (i), we get

$-x + 4y - 6z + 106 = 0$

$\Rightarrow x - 4y + 6z = 106$

Hence $k = 106$

52. The equations of the planes bisecting the angle between the given planes are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\Rightarrow \frac{2x-y+2z+3}{\sqrt{2^2+(-1)^2+2^2}} = \pm \frac{3x-2y+6z+8}{\sqrt{3^2+(-2)^2+6^2}}$$

$\Rightarrow 7(2x - y + 2z + 3) = \pm 3(3x - 2y + 6z + 8)$

$\Rightarrow 7(2x - y + 2z + 3) = 3(3x - 2y + 6z + 8)$

or $7(2x - y + 2z + 3) = -3(3x - 2y + 6z + 8)$

$\Rightarrow 5x - y - 4z - 3 = 0$ or $23x - 13y + 32z + 45 = 0$

53. The point (3, –2, 1) satisfies both the equations so it is the point of intersection

Alternate method:

Line is $\dfrac{x+3}{3} = \dfrac{y-2}{-2} = \dfrac{z+1}{1} = \lambda$ (say)

$x = 3\lambda - 3;\ y = -2\lambda + 2;\ z = \lambda - 1$

Line intersects plane,

$4x + 5y + 3z - 5 = 0$

$\therefore$ $4(3\lambda - 3) + 5(-2\lambda + 2) + 3(\lambda - 1) - 5 = 0$

$\Rightarrow \lambda = 2$.

$\therefore$ The point of intersection is (3, –2, 1)

54. The point (1, –2, 7) satisfies the given equation of plane. So it is the point of intersection.

Alternate method:

The d.r.s ratios of the line joining the points (2, –3, 1) and (3, –4, –5) are 1, –1, –6

$\therefore$ The equation of line is

$\dfrac{x-2}{1} = \dfrac{y+3}{-1} = \dfrac{z-1}{-6} = \lambda$(say)

Any general point on the line is

$(\lambda + 2, -\lambda - 3, -6\lambda + 1)$

The above point lies on the plane

$2x + y + z = 7$

$\therefore$ $2(\lambda + 2) + (-\lambda - 3) + (-6\lambda + 1) = 7$

$\Rightarrow -5\lambda + 2 = 7$

$\Rightarrow \lambda = -1$

$\therefore$ The point is (1, –2, 7)

55. The equations of line is

$\dfrac{x-3}{1} = \dfrac{y-4}{2} = \dfrac{z-5}{2} = \lambda$(say)

Any point on the line is $(\lambda + 3, 2\lambda + 4, 2\lambda + 5)$

Since the point lies on the plane $x + y + z = 17$

$\therefore$ $\lambda + 3 + 2\lambda + 4 + 2\lambda + 5 = 17 \Rightarrow \lambda = 1$

$\therefore$ The point is (4, 6, 7).

Hence, the required distance is

$\sqrt{(3-4)^2+(4-6)^2+(5-7)^2}$

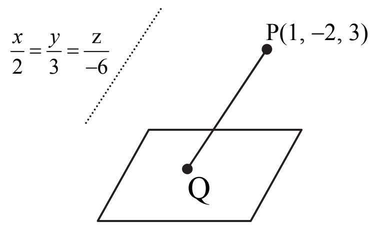
$= \sqrt{1^2+2^2+2^2} = 3$

56. The d.r.s ratios of the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ are 2, 3, − 6.

∴ The d.r.s of any line parallel to it are also 2, 3, −6.

∴ The equation of the line passing through P(1, −2, 3) and parallel to the given line is

$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = \lambda(\text{say})$ ….(i)



Any point on the line is

Q ≡ (2λ + 1, 3λ − 2, −6λ + 3)

The point Q lies on the plane $x - y + z = 5$.

∴ (2λ + 1) − (3λ − 2) + (−6λ + 3) = 5

$\Rightarrow -7\lambda = -1 \Rightarrow \lambda = \frac{1}{7}$

∴ $Q \equiv \left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$

∴ Required distance = l(PQ) = d

∴ $d = \sqrt{\left(\frac{9}{7}-1\right)^2 + \left(\frac{-11}{7}+2\right)^2 + \left(\frac{15}{7}-3\right)^2}$

$= \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{-6}{7}\right)^2}$

$= \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = \sqrt{\frac{49}{49}} = 1$

57. Let $\left.\begin{array}{l}\pi_1 : x + 2y + 3z = 5 \\ \pi_2 : x + 2y + 3z = 7\end{array}\right\}$ be two given planes

Any plane parallel to the given planes and equidistant from these is given by

$x + 2y + 3z = \frac{d_1 + d_2}{2} = \frac{5+7}{2}$

i.e. $x + 2y + 3z = 6$

58. Given planes are parallel,

∴ the required plane is also parallel to them

Let $3x + 4y + 5z + \lambda = 0$ be the required plane

$\lambda = \frac{d_1 + d_2}{2} = \frac{-6+6}{2} = 0$

∴ the equation of required plane is

$3x + 4y + 5z = 0$

Competitive Thinking

1. $\bar{n} = 2\hat{i} - 3\hat{j} + \hat{k}$

$\Rightarrow \hat{n} = \frac{1}{\sqrt{14}}\left(2\hat{i} - 3\hat{j} + \hat{k}\right)$

The equation of required plane is

$\bar{r} \cdot \hat{n} = d$

$\Rightarrow \bar{r} \cdot \frac{1}{\sqrt{14}}\left(2\hat{i} - 3\hat{j} + \hat{k}\right) = \frac{3}{\sqrt{14}}$

$\Rightarrow \bar{r} \cdot \left(2\hat{i} - 3\hat{j} + \hat{k}\right) = 3$

2. Let A ≡ (−1, 1, 2)

∴ $\bar{a} = -\hat{i} + \hat{j} + 2\hat{k}$

$\bar{n} = \hat{i} + \hat{j} + \hat{k}$

∴ equation of plane is $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$

$\Rightarrow \bar{r} \cdot \left(\hat{i} + \hat{j} + \hat{k}\right) = \left(-\hat{i} + \hat{j} + 2\hat{k}\right) \cdot \left(\hat{i} + \hat{j} + \hat{k}\right)$

$\Rightarrow \bar{r} \cdot \left(\hat{i} + \hat{j} + \hat{k}\right) = 2$

3. The d.r.s. of the normal to the plane are 1, 2, −3

∴ the d.c.s. of the normal to the plane are

$\frac{1}{\sqrt{1^2 + 2^2 + (-3)^2}}, \frac{2}{\sqrt{1^2 + 2^2 + (-3)^2}}, \frac{-3}{\sqrt{1^2 + 2^2 + (-3)^2}}$

i.e., $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}$.

4. d.c.s of normal to the plane are

$\cos\frac{\pi}{4}, \cos\frac{\pi}{4}, \cos\frac{\pi}{2} = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$

Equation of the plane is $lx + my + nz = p$

$\Rightarrow \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \sqrt{2}$

$\Rightarrow x + y = 2$

5. The equation of plane passing through (1, 2, −3) and (2, −2, 1) and parallel to X-axis is

$\begin{vmatrix} x-1 & y-2 & z+3 \\ 2-1 & -2-2 & 1+3 \\ 1 & 0 & 0 \end{vmatrix} = 0$

$\Rightarrow (y - 2)(4) + (z + 3)(4) = 0$

$\Rightarrow y + z + 1 = 0$

6. The plane passes through (2, 3, 4)

This point satisfies the equation of plane in option (D)

Also, it has d.r.s. 1, 2, 4.

∴ option (D) is correct answer.

Alternate method:
The equation of the required plane parallel to the plane $x + 2y + 4z = 5$ is
$x + 2y + 4z + k = 0$
The plane passes through (2, 3, 4)
$\therefore$ $2 + 2(3) + 4(4) + k = 0$
$\Rightarrow k = -24$
$\therefore$ the equation of the required plane is
$x + 2y + 4z = 24$

7. The plane passes through (2, 3, 4)
This point satisfies the equation of plane in option (B)
Also, it has d.r.s. 5, –6, 7.
$\therefore$ option (B) is correct answer.

8. The plane passes through (1, 2, 3)
This point satisfies the equation of plane in option (D)
Also, it has d.r.s. 2, 3, –4.
$\therefore$ option (D) is correct answer.

9. $5x - 3y + 6z = 60$
$\Rightarrow \frac{5x}{60} - \frac{3y}{60} + \frac{6z}{60} = 1 \Rightarrow \frac{x}{12} + \frac{y}{-20} + \frac{z}{10} = 1$
$\therefore$ the intercepts are (12, –20, 10).

10. The plane $x - 3y + 5z = d$ passes through (1, 2, 4).
$\therefore$ $d = 15$
$\therefore$ the equation of plane becomes $x - 3y + 5z = 15$
$\Rightarrow \frac{x}{15} + \frac{y}{-5} + \frac{z}{3} = 1$
$\therefore$ length of intercept cut by plane on the X, Y, Z axes are 15, –5, 3 respectively.

11. The plane π is parallel to Y-axis.
$\therefore$ Y intercept is zero
$\therefore$ the equation of plane is $\frac{x}{4} + \frac{z}{3} = 1$
$\Rightarrow 3x + 4z = 12$

12. Here, $a = b = c = 1$
$\therefore$ the equation of the required plane is $\frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1$
$\Rightarrow x + y + z = 1$

13. The intercepts made by the plane are
$a, b, c = l, m, n$
$\therefore$ The distances of plane from origin is
$$d = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$
$$\Rightarrow k = \frac{1}{\sqrt{\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}}} \Rightarrow \frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{k^2}$$

14. Let $P \equiv (2, 3, 4)$ and $Q \equiv (6, 7, 8)$
If R is the mid-point of PQ,
$\therefore$ $R \equiv (4, 5, 6)$
This point satisfies the equation of plane in option (D)
$\therefore$ option (D) is correct answer
Alternate method:
$\overline{n} = 4\hat{i} + 4\hat{j} + 4\hat{k}$, $\overline{a} = 4\hat{i} + 5\hat{j} + 6\hat{k}$
$\therefore$ equation of plane is
$\overline{r} \cdot \overline{n} = \overline{a} \cdot \overline{n}$
$\Rightarrow \overline{r} \cdot (4\hat{i} + 4\hat{j} + 4\hat{k}) = (4\hat{i} + 5\hat{j} + 6\hat{k}).(4\hat{i} + 4\hat{j} + 4\hat{k})$
$\Rightarrow 4x + 4y + 4z = 16 + 20 + 24$
$\Rightarrow x + y + z - 15 = 0$

15. The plane passes through (1, 2, 2)
This point satisfies the equation of plane in option (B)
Also, it has d.r.s. 1, 2, 2.
$\therefore$ option (B) is correct answer.

16. Let M (1, 2, 3) be the foot of perpendicular from the origin O(0, 0, 0) to the plane
d.r.s. of normal are 1, 2, 3
$\therefore$ the equation of the required plane is
$1(x - 1) + 2(y - 2) + 3(z - 3) = 0$
$\Rightarrow x - 1 + 2y - 4 + 3z - 9 = 0$
$\Rightarrow x + 2y + 3z - 14 = 0$
Consider the option (B)
point (7, 2, 1) satisfies the above equation of plane.
$\therefore$ option (B) is correct answer.

17. The plane is $y = \frac{-8}{5}$ which is parallel to XZ-plane
$\therefore$ Foot of the perpendicular drawn from the origin $\equiv \left(0, \frac{-8}{5}, 0\right)$

18. The plane passes through (2, 6, 3)
It satisfies option (D)
Alternate Method:
The d.r.s of OP are 2 – 0, 6 – 0, 3 – 0 i.e., 2, 6, 3
The plane passes through P(2, 6, 3).
$\therefore$ the equation of the required plane is
$2(x - 2) + 6(y - 6) + 3(z - 3) = 0$
$\Rightarrow 2x + 6y + 3z = 49$

19. The plane passes through (1, 1, 1) and (1, –1, –1)
The above points satisfies the equation of plane in option (B)
$\therefore$ option (B) is correct answer.

20. The plane passes through A(–2, 2, 2) and B(2, –2, –2)
The above points satisfies the equation of plane in option (A)
$\therefore$ option (A) is correct answer.

21. The plane passes through (0, 1, 2) and (–1, 0, 3)
The above points satisfies the equation of plane in option (D)
$\therefore$ option (D) is correct answer.

22. The plane passes through (2, –3, 1)
This point satisfies the equation of plane in option (A)
Also, it has d.r.s. 3 – 2 , 4 + 1, –1 –5
i.e. 1, 5, –6.
$\therefore$ option (A) is correct answer.
Alternate method:
The d.r.s. of the line joining the points (3, 4, –1) and (2, –1, 5) are 1, 5, –6.
The plane passes through (2, –3, 1)
$\therefore$ the equation of required plane is
$1(x-2)+5(y+3)-6(z-1)=0$
$\Rightarrow x+5y-6z+19=0$

23. The d.r.s. of the line joining the points (4, –1, 2) and (–3, 2, 3) are 7, –3, –1
The plane passes through (–10, 5, 4)
$\therefore$ The equation of required plane is
$7(x+10)-3(y-5)-1(z-4)=0$
$\Rightarrow 7x+70-3y+15-z+4=0$
$\Rightarrow 7x-3y-z+89=0$

24. The equation of the plane is
$b(x-1)+c(y-1)+a(z-1)=0$ ….(i)
Now, 2001 = 3 × 23 × 29
Since, a < b < c
$\therefore$ a = 3, b = 23 and c = 29
Substituting the values of a, b, c in equation (i), we get
$23x+29y+3z=55$

25. $\bar{r}=(1-p-q)\bar{a}+p\bar{b}+q\bar{c}$
$\Rightarrow \bar{r}=\bar{a}+p\left(\bar{b}-\bar{a}\right)+q\left(\bar{c}-\bar{a}\right)$ ....(i)
Comparing with $\bar{r}=\bar{A}+\lambda\bar{B}+\mu\bar{C}$,
the equation (i) represents a plane passing through a point having position vector $\bar{a}$ and parallel to the vectors $\bar{b}-\bar{a}$ and $\bar{c}-\bar{a}$.

26. Equation of plane passing through (1, 0, 2), (–1, 1, 2) and (5, 0, 3) is
$$\begin{vmatrix} x-1 & y-0 & z-2 \\ -1-1 & 1-0 & 2-2 \\ 5-1 & 0-0 & 3-2 \end{vmatrix}=0$$
$$\Rightarrow \begin{vmatrix} x-1 & y & z-2 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{vmatrix}=0$$
$\Rightarrow (x-1)-y(-2)+(z-2)(-4)=0$
$\Rightarrow x-1+2y-4z+8=0$
$\Rightarrow x+2y-4z+7=0$

27. Equation of plane passing through (1, 2, 3), (–1, 4, 2) and (3, 1, 1) is
$$\begin{vmatrix} x-1 & y-2 & z-3 \\ -1-1 & 4-2 & 2-3 \\ 3-1 & 1-2 & 1-3 \end{vmatrix}=0$$
$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ -2 & 2 & -1 \\ 2 & -1 & -2 \end{vmatrix}=0$$
$\Rightarrow (x-1)(-4-1)-(y-2)(4+2)+(z-3)(2-4)=0$
$\Rightarrow -5x+5-6y+12-2z+6=0$
$\Rightarrow -5x-6y-2z+23=0$
$\Rightarrow 5x+6y+2z=23$

28. Equation of plane passing through (1, 2, 3), (2, 3, 1) and (3, 1, 2) is
$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2-1 & 3-2 & 1-3 \\ 3-1 & 1-2 & 2-3 \end{vmatrix}=0$$
$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix}=0$$
$\Rightarrow (x-1)(-3)-(y-2)(3)+(z-3)(-3)=0$
$\Rightarrow -3x+3-3y+6-3z+9=0$
$\Rightarrow x+y+z=6$
Comparing the above equation with
$ax+by+cz=d$, we get
a = 1, b = 1, c = 1
Now, a + 2b + 3c = (1) + 2(1) + 3(1) = 6

29. The equation of the required plane is
$(x+2y+3z+4)+\lambda(4x+3y+2z+1)=0$ ....(i)
The plane passes through origin i.e., (0, 0, 0)
$\therefore$ $4+\lambda=0 \Rightarrow \lambda=-4$

Substituting value of λ in equation (i), we get
$-15x - 10y - 5z = 0$
$\Rightarrow 3x + 2y + z = 0$

30. The plane passes through (2, 1, 0)
It satisfies option (C)
The equation of the required plane is
$(x - 2y + 3z - 4) + \lambda(x - y + z - 3) = 0$ ....(i)
The plane passes through (2, 1, 0).
$\therefore$ $(2 - 2 + 0 - 4) + \lambda(2 - 1 + 0 - 3) = 0$
$\Rightarrow \lambda = -2$
Substituting value of λ in (i), we get
$-x + z + 2 = 0$
$\Rightarrow x - z = 2$

31. The d.r.s. of the line are 1, 2, 3.
The line is perpendicular to the plane
$\therefore$ The d.r.s. of plane are 1, 2, 3
$\therefore$ The equation of plane passing through (2, 3, 4) is
$a(x - 2) + b(y - 3) + c(z - 4) = 0$ ….(i)
$\Rightarrow 1(x - 2) + 2(y - 3) + 3(z - 4) = 0$
$\Rightarrow x + 2y + 3z = 20$

32. The plane passes through the line $\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}$ i.e. through (3, 6, 4)
The points (3, 2, 0) and (3, 6, 4) satisfies option (A)
$\therefore$ option (A) is correct answer.
Alternate method:
The equation of plane passing through (3, 2, 0) is
$a(x - 3) + b(y - 2) + c(z - 0) = 0$ ….(i)
$\therefore$ $a(3 - 3) + b(6 - 2) + c(4 - 0) = 0$
$\Rightarrow 0.a + 4b + 4c = 0$ ….(ii)
and $1.a + 5b + 4c = 0$ ….(iii)
On solving (ii) and (iii), we get
$a = -4, b = 4, c = -4$
$\therefore$ equation of required plane is $x - y + z = 1$

33. The equation of plane passing through (2, –1, –3) is
$a(x - 2) + b(y + 1) + c(z + 3) = 0$
Also, as the plane is parallel is the given two lines,
$\therefore$ $3a + 2b - 4c = 0$ and $2a - 3b + 2c = 0$
$\Rightarrow a = -8, b = -14, c = -13$
$\therefore$ The equation of the required plane is
$-8(x - 2) - 14(y + 1) - 13(z + 3) = 0$
$\Rightarrow 8x + 14y + 13z + 37 = 0$

34. Point (2, 1, –2) lies in the plane
$x + 3y - \alpha z + \beta = 0$
$\therefore$ $2 + 3(1) - \alpha(-2) + \beta = 0$
$\Rightarrow 2\alpha + \beta = -5$ ….(i)
Also, the d.r.s of the normal are perpendicular to the given plane.
$\therefore$ $3(1) + (-5)(3) + (2)(-\alpha) = 0$
$\Rightarrow 3 - 15 - 2\alpha = 0$
$\Rightarrow \alpha = -6$
Substituting value of α in equation (i), we get
$\beta = 7$

35. The d.r.s of normal to the given planes are 1, 2, 2 and –5, 3, 4
$\therefore$ $\cos\theta = \frac{(1)(-5)+(2)(3)+(2)(4)}{\sqrt{1^2+2^2+2^2}\sqrt{(-5)^2+3^2+4^2}} = \frac{3\sqrt{2}}{10}$
$\Rightarrow \theta = \cos^{-1}\left(\frac{3\sqrt{2}}{10}\right)$

36. $\cos\theta = \frac{3(2)-4(-1)+5(-2)}{\sqrt{9+16+25}\sqrt{4+1+4}}$
$\therefore$ $\cos\theta = 0$
$\Rightarrow \theta = \frac{\pi}{2}$

37. Given equation of locus $xy + yz = 0$
$\Rightarrow y(x + z) = 0$
$\Rightarrow y = 0$ or $x + z = 0$
The planes $y = 0$ and $x + z = 0$ perpendicular to each other.

38. $\bar{r}\cdot\left(m\hat{i} - \hat{j} + 2\hat{k}\right) + 3 = 0 \Rightarrow \bar{r}\cdot\left(m\hat{i} - \hat{j} + 2\hat{k}\right) = -3$
$\bar{r}\cdot\left(2\hat{i} - m\hat{j} - \hat{k}\right) - 5 = 0 \Rightarrow \bar{r}\cdot\left(2\hat{i} - m\hat{j} - \hat{k}\right) = 5$
Here, $\overline{n_1} = m\hat{i} - \hat{j} + 2\hat{k}$ and $\overline{n_2} = 2\hat{i} - m\hat{j} - \hat{k}$
$\therefore$ $\cos\theta = \left|\frac{\overline{n_1}\cdot\overline{n_2}}{\left|\overline{n_1}\right|\left|\overline{n_2}\right|}\right|$
$\Rightarrow \cos\frac{\pi}{3} = \left|\frac{\left(m\hat{i} - \hat{j} + 2\hat{k}\right)\cdot\left(2\hat{i} - m\hat{j} - \hat{k}\right)}{\sqrt{m^2+1+4}\sqrt{4+m^2+1}}\right|$
$\Rightarrow \frac{1}{2} = \frac{2m+m-2}{m^2+5}$
…(Cosidering positive value)
$\Rightarrow m^2 + 5 = 6m - 4$
$\Rightarrow m^2 - 6m + 9 = 0$
$\Rightarrow (m - 3)^2 = 0$
$\Rightarrow m = 3$

39. Here, $\overline{n}_1 = p\hat{i} - \hat{j} + 2\hat{k}$ and $\overline{n}_2 = 2\hat{i} - p\hat{j} - \hat{k}$

$\therefore$ $\cos\theta = \left|\dfrac{\overline{n}_1 . \overline{n}_2}{\left|\overline{n}_1\right|\left|\overline{n}_2\right|}\right|$

$\Rightarrow \cos\dfrac{\pi}{3} = \left|\dfrac{\left(p\hat{i} - \hat{j} + 2\hat{k}\right)\left(2\hat{i} - p\hat{j} - \hat{k}\right)}{\sqrt{p^2+1+4}\sqrt{4+p^2+1}}\right|$

$\Rightarrow \dfrac{1}{2} = \pm\left(\dfrac{2p+p-2}{p^2+5}\right)$

$\Rightarrow \dfrac{1}{2} = \dfrac{3p-2}{p^2+5}$

…. (considering positive value)

$\Rightarrow p^2 + 5 = 6p - 4$

$\Rightarrow p^2 - 6P + 9 = 0$

$\Rightarrow (p-3)^2 = 0$

$\Rightarrow p = 3$

40. Let the d.r.s of the normal to the plane be proportional to a, b, c.

It passes through (1, 0, 0)

$\therefore$ the equation of the plane is

$a(x-1) + b(y-0) + c(z-0) = 0$ ....(i)

Also, the plane passses through (0, 1, 0).

$\therefore$ $a(-1) + b(1) + c(0) = 0$

$\Rightarrow a = b$ ....(ii)

Now, the angle between the required plane and the plane $x + y = 3$ is $\dfrac{\pi}{4}$.

$\therefore$ $\cos\dfrac{\pi}{4} = \dfrac{a(1)+b(1)+c(0)}{\sqrt{a^2+b^2+c^2}\sqrt{1+1}}$

$\Rightarrow \dfrac{1}{\sqrt{2}} = \dfrac{a+b}{\sqrt{a^2+b^2+c^2}\sqrt{2}}$

Squaring both sides, we get

$\Rightarrow a^2 + b^2 + c^2 = a^2 + b^2 + 2ab$

$\Rightarrow c^2 = 2ab$ ....(iii)

From (ii) and (iii), we get

$a : b : c = a : a : \sqrt{2}a = 1 : 1 : \sqrt{2}$

41. For perpendicular planes, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow 2(1) + 1(2) - 2(k) = 0$

$\Rightarrow k = 2$

42. Since the planes are perpendicular,

$\therefore$ $(3)(2) + (-6)(1) + (-2)(-k) = 0$

$\Rightarrow k = 0$

43. Since, the planes are perpendicular to each other.

$\therefore$ $3(4) + (-2)(3) + 2 \times (-k) = 0$

$\Rightarrow k = 3$

44. The equation of plane passing through (4, 4, 0) is $a(x-4) + b(y-4) + c(z-0) = 0$

$\Rightarrow a(x-4) + b(y-4) + cz = 0$ …(i)

Since, plane (i) is perpendicular to the planes $2x + y + 2z + 3 = 0$ and $3x + 3y + 2z - 8 = 0$

$\therefore$ $2a + b + 2c = 0$, and …(ii)

$3a + 3b + 2c = 0$ …(iii)

On solving (i) and (ii), we get

$a = -4, b = 2, c = 3$

Substituting the values of a, b, c in (i), we get

$-4(x-4) + 2(y-4) + 3z = 0$

$\Rightarrow -4x + 16 + 2y - 8 + 3z = 0$

$\Rightarrow 4x - 2y - 3z = 8$

45. Comparing with $\overline{r} = \overline{a} + \lambda\overline{b}$ and $\overline{r}.\overline{n} = p$, we get

$\overline{b} = -\hat{i} + \hat{j} + \hat{k}$ and $\overline{n} = 3\hat{i} + 2\hat{j} - \hat{k}$

$\therefore$ Angle between the line and plane is given by

$\sin\theta = \dfrac{\overline{b}.\overline{n}}{\left|\overline{b}\right|\left|\overline{n}\right|}$

$= \dfrac{(-1)(3)+(1)(2)+(1)(-1)}{\sqrt{3}\sqrt{14}} = \dfrac{-2}{\sqrt{42}}$

$\Rightarrow \theta = \sin^{-1}\left(\dfrac{-2}{\sqrt{42}}\right)$

46. The d.r.s. of line are 3, 4, 5 and

the d.r.s. of normal to the plane are 2, –2, 1

$\therefore$ The angle between line and plane is

$\sin\theta = \dfrac{aa_1 + bb_1 + cc_1}{\sqrt{a^2+b^2+c^2}\cdot\sqrt{a_1^2+b_1^2+c_1^2}}$

$= \dfrac{(2)(3)+(-2)(4)+(1)(5)}{\sqrt{2^2+(-2)^2+(1)^2}\cdot\sqrt{3^2+4^2+5^2}}$

$= \dfrac{3}{\sqrt{9}\sqrt{50}} = \dfrac{1}{5\sqrt{2}} = \dfrac{\sqrt{2}}{10}$

47. The d.r.s. of line are 1, 2, 2 and

the d.r.s. of normal to the plane are 2, –1, $\sqrt{\lambda}$

$\therefore$ $\sin\theta = \dfrac{1(2)+2(-1)+2(\sqrt{\lambda})}{\sqrt{1+4+4}\cdot\sqrt{4+1+\lambda}}$

$\Rightarrow \dfrac{1}{3} = \dfrac{2\sqrt{\lambda}}{3\sqrt{5+\lambda}}$

$\Rightarrow 2\sqrt{\lambda} = \sqrt{5+\lambda}$

$\Rightarrow 4\lambda = 5 + \lambda$

$\Rightarrow \lambda = \dfrac{5}{3}$

48. d.r.s. of normal to the plane are 2, –3, 6
d.r.s. of X-axis are 1, 0, 0.

$\therefore$ The angle between the plane and X-axis is

$$\sin\theta = \frac{aa_1 + bb_1 + cc_1}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}$$

$$= \frac{2(1)-3(0)+6(0)}{\sqrt{4+9+36}\cdot\sqrt{1}}$$

$$= \frac{2}{7}$$

$\therefore$ $\theta = \sin^{-1}\left(\frac{2}{7}\right)$

But $\theta = \sin^{-1}\alpha$

$\therefore$ $\alpha = \frac{2}{7}$

49. The d.r.s. of line are 1, 2, λ and
The d.r.s. of normal to the plane are 1, 2, 3.

$\therefore$ $\sin\theta = \frac{1(1)+2(2)+\lambda(3)}{\sqrt{1+4+9}\sqrt{1+4+\lambda^2}}$

$\Rightarrow \sin\theta = \frac{5+3\lambda}{\sqrt{14}\sqrt{5+\lambda^2}}$

$\Rightarrow \sin^2\theta = \frac{(5+3\lambda)^2}{14(5+\lambda^2)}$

$\Rightarrow 1 - \frac{5}{14} = \frac{(5+3\lambda)^2}{14(5+\lambda^2)}$

....$\left[\because \cos\theta = \sqrt{\frac{5}{14}} \text{ (given)}\right]$

$\Rightarrow \frac{9}{14} = \frac{25+30\lambda+9\lambda^2}{14(5+\lambda^2)}$

On solving, we get

$\lambda = \frac{2}{3}$

50. Let a, b, c = 3, 2 + λ, –1 and a_1, b_1, c_1 = 1, –2, 0
Since, the line lies on the plane

$\therefore$ $aa_1 + bb_1 + cc_1 = 0$

$\Rightarrow 3(1) + (2 + \lambda)(-2) + (-1)(0) = 0$

$\Rightarrow \lambda = \frac{-1}{2}$

51. The line is parallel to the plane if
$aa_1 + bb_1 + cc_1 = 0$
Consider option (B), 2(3) + 1(4) – 2(5) = 0

$\therefore$ $2x + y - 2z = 0$ is the required plane.

52. The equation of the plane is $ax + by + cz + d = 0$

$\therefore$ the d.r.s. of the normal to the plane are a, b, c
Since the given line is parallel to the plane,

$\therefore$ $al + bm + cn = 0$

53. line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies on the plane $2x - 4y + z = 7$.

$\therefore$ Point (4, 2, k) lies on the plane $2x - 4y + z = 7$

$\therefore$ $2(4) - 4(2) + k = 7$

$\Rightarrow k = 7$

54. Line is perpendicular to normal of plane

$\Rightarrow \left(2\hat{i} - \hat{j} + 3\hat{k}\right).\left(l\hat{i} + m\hat{j} - \hat{k}\right) = 0$

$2l - m - 3 = 0$ ....(i)

(3, – 2, –4) lies on the plane $lx + my - z = 9$

$\therefore$ $3l - 2m + 4 = 9$

$\Rightarrow 3l - 2m = 5$ ....(ii)

Solving (i) and (ii)

$l = 1, m = -1$

$l^2 + m^2 = 2$

55. The d.r.s. of the XY-plane are 0, 0, 1
the d.r.s. of the given line are l, m, n
Since, the line is parallel to the plane

$\therefore$ $aa_1 + bb_1 + cc_1 = 0$

$\Rightarrow l(0) + m(0) + n(1) = 0$

$\Rightarrow n = 0$

56. Let the position vector of Q be

$\left(\hat{i} - \hat{j} + 2\hat{k}\right) + \mu\left(-3\hat{i} + \hat{j} + 5\hat{k}\right)$

$= (-3\mu + 1)\hat{i} + (\mu - 1)\hat{j} + (5\mu + 2)\hat{k}$

$\therefore$ $\overline{PQ} = (-3\mu - 2)\hat{i} + (\mu - 3)\hat{j} + (5\mu - 4)\hat{k}$

Since, $\overline{PQ}$ is parallel to the plane

$\therefore$ $(-3\mu-2)(1) + (\mu-3)(-4) + (5\mu-4)(3) = 0$

$\Rightarrow \mu = \frac{1}{4}$

57. The plane passes through points (– 3, 0, 2) and (3, 2, 6)
This points satisfies the equation of plane in option (D)

$\therefore$ option (D) is correct answer.

58. Lines are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

$\therefore$ $\begin{vmatrix} 1-2 & 4-3 & 5-4 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow -1(1 + 2k) -1(1 + k^2) + 1(2 - k) = 0$

$\Rightarrow k^2 + 3k = 0$

$\Rightarrow k = 0, -3$

59. The planes are concurrent,

$$\therefore \quad \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow a^2 + b^2 + c^2 + 2abc - 1 = 0$

$\Rightarrow a^2 + b^2 + c^2 = 1 - 2abc$

60. The equation of the plane is $\frac{x}{8} + \frac{y}{4} + \frac{z}{4} = 1$

i.e., $x + 2y + 2z = 8$

$\therefore$ The length of the perpendicular from origin to the plane is

$$d = \left|\frac{-8}{\sqrt{1+4+4}}\right| = \frac{8}{3}$$

61. The equations of the plane with reference to the two systems of rectangular axes are

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{....(i)}$$

and $$\frac{X}{a'} + \frac{Y}{b'} + \frac{Z}{c'} = 1 \qquad \text{....(ii)}$$

Since the origin of axes is same.

$\therefore$ Length of the perpendicular from (0, 0, 0) on plane (i)

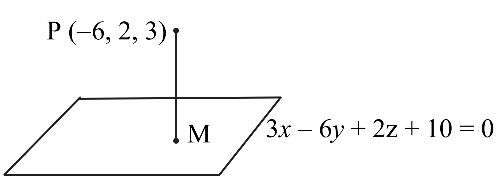
= Length of the perpendicular from (0, 0, 0) on plane (ii)

$$\Rightarrow \left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| = \left|\frac{-1}{\sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}}\right|$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2} = 0$$

62. Since the line is parallel to XY-plane, the distance of the point P (6, 7, 8) from this plane is equal to its Z co-ordinate i.e. 8 units.

63.



Distance of point P from the given plane is given by

$$d = \left|\frac{3(-6) - 6(2) + 2(3) + 10}{\sqrt{(3)^2 + (-6)^2 + (2)^2}}\right|$$

$$= \left|\frac{-18 - 12 + 6 + 10}{\sqrt{9 + 36 + 4}}\right|$$

$$= \left|\frac{-14}{\sqrt{49}}\right| = \frac{14}{7}$$

$\therefore$ d = 2

64. Given equation of plane is $\bar{r} \cdot \left(3\hat{i} + 2\hat{j} + 6\hat{k}\right) = 13$

The vector form of the equation is

$3x + 2y + 6z = 13$

$\Rightarrow 3x + 2y + 6z - 13 = 0$

Given point $\equiv (2, 3, \lambda)$

$\therefore$ Distance of the point from the plane

$$= \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$\Rightarrow 5 = \left|\frac{3(2) + 2(3) + 6(\lambda) - 13}{\sqrt{9 + 4 + 36}}\right|$$

$$\Rightarrow 5 = \left|\frac{6\lambda - 1}{7}\right|$$

$\Rightarrow 6\lambda - 1 = \pm 35$

$\Rightarrow \lambda = 6, \frac{-17}{3}$

65. Here, a = 2, b = 1, c = 2, d = 5, $x = 2$, $y = 1$, z = 0

$$\therefore \quad d = \left|\frac{2(2) + 1(1) + 2(0) + 5}{\sqrt{2^2 + 1^2 + 2^2}}\right|$$

$$= \left|\frac{10}{\sqrt{9}}\right| = \frac{10}{3}$$

66. Normal vector $\hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix}$

$$= \hat{i}(2+3) - \hat{j}(-1-6) + \hat{k}(-1+4)$$

$$= 5\hat{i} + 7\hat{j} + 3\hat{k}$$

Let A ≡ (1, −1, −1)

$\therefore \quad \bar{a} = \hat{i} - \hat{j} - \hat{k}$

$\therefore$ Equation of the plane is

$5(x - 1) + 7(y + 1) + 3(z + 1) = 0$

$\Rightarrow 5x + 7y + 3z + 5 = 0$

Distance of (1, 3, –7) from the above plane is

$$d = \left|\frac{5(1)+7(3)+3(-7)+5}{\sqrt{25+49+9}}\right|$$

$$= \frac{10}{\sqrt{83}} \text{ units}$$

67. $\bar{n} = 2\hat{i}+\hat{j}+2\hat{k}$ and p = 5

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$$

∴ The vector equation of the plane is $\bar{r}.\hat{n} = p$

$$\Rightarrow \bar{r}.\left(\frac{2\hat{i}+\hat{j}+2\hat{k}}{3}\right) = 5$$

$$\Rightarrow \bar{r}.\left(2\hat{i}+\hat{j}+2\hat{k}\right) = 15$$

68. Let a, b, c = –3, 2, 6

∴ the equation of plane is

∴ $-3x + 2y + 6z + d = 0$ ….(i)

Now, the perpendicular distance (D) from origin is

$$D = \left|\frac{d}{\sqrt{(-3)^2+2^2+6^2}}\right|$$

$$\Rightarrow 7 = \frac{|d|}{7} \Rightarrow d = \pm 49$$

∴ The equation of plane is

$-3x + 2y + 6z + 49 = 0$

or $-3x + 2y + 6z - 49 = 0$

69. The equation of a plane passing through (1, – 2, 1) is

$a(x-1) + b(y+2) + c(z-1) = 0$ ….(i)

Plane (i) is perpendicular to planes

$2x - 2y + z = 0$ and $x - y + 2z = 4$.

∴ $2a - 2b + c = 0$, and ….(ii)

$a - b + 2c = 0$ ….(iii)

Solving (ii) and (iii), we get

a = –3, b = –3, c = 0

Substituting the values of a, b, c in equation (i), we get

$x + y + 1 = 0$

∴ The distance of this plane from (1, 2, 2) is

$$d = \left|\frac{1+2+1}{\sqrt{1+1}}\right| = 2\sqrt{2}$$

70. Equation of L_1 i.e., the line of intersection of the first two given planes is

$(2x - 2y + 3z - 2) + \lambda(x - y + z + 1) = 0$

$\Rightarrow (\lambda + 2)x - (2 + \lambda)y$

$+ (\lambda + 3)z + (\lambda - 2) = 0$ …(i)

Equation of L_2 i.e., the line of intersection of the next two given planes is

$(1 + 3\mu)x + (2 - \mu)y$

$+ (2\mu - 1)z - (\mu + 3) = 0$ …(ii)

Since, equations (i) and (ii) represent the same plane.

∴ by comparing, we get

$$\frac{2+\lambda}{1+3\mu} = \frac{-(2+\lambda)}{2-\mu}$$

$\Rightarrow 1 + 3\mu = \mu - 2$ $\Rightarrow \mu = -\frac{3}{2}$

Substituting $\mu = -\frac{3}{2}$ in (ii), we get

$7x - 7y + 8z + 3 = 0$

Perpendicular distance from the origin (0, 0, 0)

$$= \left|\frac{7(0)-7(0)+8(0)+3}{\sqrt{7^2+7^2+8^2}}\right| = \frac{3}{\sqrt{162}}$$

$$= \frac{3}{9\sqrt{2}} = \frac{1}{3\sqrt{2}}$$

71. The equation of a plane passing through the line of intersection of the planes

$x + 2y + 3z = 2$ and $x - y + z = 3$ is

$(x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$

$\Rightarrow x(1 + \lambda) + y(2 - \lambda) + z(3 + \lambda) - 2 - 3\lambda = 0$ …. (i)

This plane is at a distance of $\frac{2}{\sqrt{3}}$ units from (3, 1, – 1).

$$\therefore \frac{|3(1+\lambda)+1(2-\lambda)-1(3+\lambda)-2-3\lambda|}{\sqrt{(1+\lambda)^2+(2-\lambda)^2+(3+\lambda)^2}} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \frac{|-2\lambda|}{\sqrt{3\lambda^2+4\lambda+14}} = \frac{2}{\sqrt{3}}$$

Squaring both sides, we get

$$3\lambda^2 + 4\lambda + 14 = 3\lambda^2 \Rightarrow 4\lambda = -14 \Rightarrow \lambda = \frac{-7}{2}$$

Substituting value of λ in equation (i), we get

$$-\frac{5}{2}x + \frac{11}{2}y - \frac{z}{2} + \frac{17}{2} = 0$$

$\Rightarrow 5x - 11y + z - 17 = 0$

72. The equation of the plane passing through (–1, 3, 0) is

$a(x + 1) + b(y - 3) + c(z - 0) = 0$ ....(i)

Also, the plane passes through the points (2, 2, 1) and (1, 1, 3).

∴ $3a - b + c = 0$ ....(ii)

$2a - 2b + 3c = 0$ ....(iii)

Solving (ii) and (iii), we get
a = – 1, b = – 7, c = – 4
Substituting the values of a, b, c in equation (i), we get
$-1(x + 1) - 7(y - 3) - 4(z) = 0$
$\Rightarrow x + 7y + 4z - 20 = 0$

∴ The distance of this plane from the point (5, 7, 8) is

$$d = \left|\frac{1(5)+7(7)+4(8)-20}{\sqrt{1^2+7^2+4^2}}\right| = \frac{66}{\sqrt{66}} = \sqrt{66}$$

73. Given planes are
$2x + y + 2z - 8 = 0$
$4x + 2y + 4z - 16 = 0$ ….(i)
and $4x + 2y + 4z + 5 = 0$ ….(ii)
The distance between two parallel planes is

$$d = \left|\frac{d_1 - d_2}{\sqrt{a^2+b^2+c^2}}\right| = \left|\frac{-16-5}{\sqrt{4^2+2^2+4^2}}\right| = \frac{21}{6} = \frac{7}{2}$$

74. $x^2 - 5x + 6 = 0$
$\Rightarrow (x - 2) = 0$ or $(x - 3) = 0$, which represents a plane.

75. Here, the co-ordinates of A, B, C are (3a, 0, 0) (0, 3b, 0) and (0, 0, 3c) respectively.
∴ The centroid is (a, b, c).

76. Let A ≡ (a, 0, 0), B ≡ (0, b, 0) and C ≡ (0, 0, c)
The equation of the plane in intercept form is
$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
Since, centriod is (6, 6, 3)
∴ $6 = \frac{x_1 + x_2 + x_3}{3}$
$\Rightarrow 6 = \frac{a+0+0}{3} \Rightarrow a = 18$
Similarly $\frac{0+b+0}{3} = 6 \Rightarrow b = 18$
$\frac{0+0+c}{3} = 3 \Rightarrow c = 9$
∴ The equation of plane is $\frac{x}{18} + \frac{y}{18} + \frac{z}{9} = 1$
$\Rightarrow x + y + 2z - 18 = 0$

77. Given equation of plane is $ax + by + cz = 1$
∴ $A \equiv \left(\frac{1}{a}, 0, 0\right)$, $B \equiv \left(0, \frac{1}{b}, 0\right)$ and
$C \equiv \left(0, 0, \frac{1}{c}\right)$
∴ Centroid $\equiv \left(\frac{1}{3a}, \frac{1}{3b}, \frac{1}{3c}\right) = \left(\frac{1}{6}, \frac{-1}{3}, 1\right)$
∴ $3a = 6 \Rightarrow a = 2$
$3b = -3 \Rightarrow b = -1$
$3c = 1 \Rightarrow c = \frac{1}{3}$
∴ $a + b + 3c = 2 - 1 + 3\left(\frac{1}{3}\right) = 2$

78. $[a\ b\ c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0]$
$\Rightarrow a + 8b + 7c = 0,\ 9a + 2b + 3c = 0,$
$7a + 7b + 7c = 0$
$\Rightarrow a = 1,\ b = 6,\ c = -7$
P(a, b, c) lies on the plane $2x + y + z = 1$.
∴ $7a + b + c = 7 + 6 - 7 = 6$

79. The equation of the required plane is
$(2x - 5y + z - 3) + \lambda(x + y + 4z - 5) = 0$ ….(i)
∴ $(2 + \lambda)x + (-5 + \lambda)y + (1 + 4\lambda)z + (-3 - 5\lambda) = 0$
Since, this plane is parallel to $x + 3y + 6z = 1$
∴ $\frac{2+\lambda}{1} = \frac{-5+\lambda}{3} = \frac{1+4\lambda}{6}$
On solving, we get
$\lambda = -\frac{11}{2}$
Substituting the value of λ in equation (i), we get
$(2x - 5y + z - 3) - \frac{11}{2}(x + y + 4z - 5) = 0$
$\Rightarrow -7x - 21y - 42z + 49 = 0$
$\Rightarrow x + 3y + 6z - 7 = 0$

80. The point $\left(\frac{-1}{11}, \frac{9}{11}, \frac{-25}{11}\right)$ satisfies both the equations
∴ it is the point of intersection.
Alternate method:
Let $\frac{x}{1} = \frac{y-1}{2} = \frac{z+2}{3} = \lambda$(say)
Any general point on the line is $(\lambda, 2\lambda+1, 3\lambda-2)$
This point lies on the plane $2x + 3y + z = 0$
∴ $2\lambda + 3(2\lambda + 1) + (3\lambda - 2) = 0$
$\Rightarrow \lambda = \frac{-1}{11}$
∴ $x = \frac{-1}{11},\ y = \frac{9}{11},\ z = \frac{-25}{11}$
∴ The point is $\left(\frac{-1}{11}, \frac{9}{11}, \frac{-25}{11}\right)$

81. The point (5, –1, 1) satisfies both the equations

$\therefore$ it is the point of intersection

$\therefore$ option (D) is correct

82. The point (10, 10, 3) satisties both the equations.

$\therefore$ it is the point of intersection.

$\therefore$ option (B) is correct

83. The point (–4, –3, 0) satisfies the given equations

$\therefore$ correct answer is option (D).

84. Let $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = \lambda$

$\therefore$ the co-ordinates of any point on the line are

$P \equiv (3\lambda + 2, 4\lambda - 1, 12\lambda + 2)$

This point lies on the plane $x - y + z = 16$

$\therefore$ $3\lambda + 2 - 4\lambda + 1 + 12\lambda + 2 = 16$

$\Rightarrow 11\lambda = 11 \Rightarrow \lambda = 1$

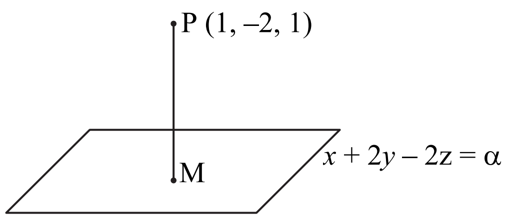
$\therefore$ $P \equiv (5, 3, 14)$

$\therefore$ Let $Q \equiv (1, 0, 2)$

$\therefore$ distance PQ is given by

$d = \sqrt{(5-1)^2 + (3-0)^2 + (14-2)^2} = 13$

85.



Distance of point P (1, –2, 1) from the $x + 2y - 2z = \alpha$ plane is 5

$\therefore$ $\frac{|1-4-2-\alpha|}{\sqrt{1+4+4}} = 5$

$\Rightarrow |\alpha + 5| = 15$

$\Rightarrow \alpha + 5 = \pm 15$

$\Rightarrow \alpha = 10, -20$

$\Rightarrow \alpha = 10$ ...($\because$ $\alpha > 0$)

The equation of line PM whose d.r.s. are 1, 2, –2 is

$\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-1}{-2} = \lambda$(say)

The co-ordinates of M are $(\lambda + 1, 2\lambda - 2, -2\lambda + 1)$

Since, M lies on the plane

$x + 2y - 2z = 10$

$\therefore$ $\lambda + 1 + 4\lambda - 4 + 4\lambda - 2 = 10$

$\Rightarrow 9\lambda = 15 \Rightarrow \lambda = \frac{5}{3}$

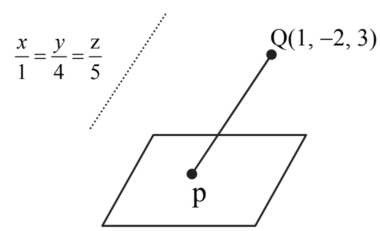
Hence, the co-ordinates of M are $\left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$.

86. The d.r.s. of the line $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ are 1, 4, 5

$\therefore$ The d.r.s. of any line parallel to it are also 1, 4, 5

The equation of the line passing through Q (1, –2, 3)

$\frac{x-1}{1} = \frac{y+2}{4} = \frac{z-3}{5} = \lambda$(say) ...(i)



Any point on the line is

$P \equiv (\lambda + 1, 4\lambda - 2, 5\lambda + 3)$

The point P lies on the plane

$2x + 3y - 4z + 22 = 0$

$\therefore$ $2(\lambda + 1) + 3(4\lambda - 2) - 4(5\lambda + 3) + 22 = 0$

$\Rightarrow 6\lambda = 6 \Rightarrow \lambda = 1$

$\therefore$ $P = (2, 2, 8)$

$\therefore$ Required distance = l(PQ) = d

$d = \sqrt{(2-1)^2 + (2+2)^2 + (8-3)^2}$

$= \sqrt{1+16+25}$

$\therefore$ $d = \sqrt{42}$ units

87. Since line PQ is parallel to line $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$

$\therefore$ d.r.s. of PQ are 1, 4, 5

$\therefore$ Equation of line PQ passing through P(1, –2, 3) is

$\frac{x-1}{1} = \frac{y+2}{4} = \frac{z-3}{5}$

Let $\frac{x-1}{1} = \frac{y+2}{4} = \frac{z-3}{5} = \lambda$

Any point R on PQ $\equiv (\lambda + 1, 4\lambda - 2, 5\lambda + 3)$

Since point R lies in the plane

$2x + 3y - 4z + 22 = 0$

$\therefore$ $2(\lambda + 1) + 3(4\lambda - 2) - 4(5\lambda + 3) + 22 = 0$

$\Rightarrow -6y + 6 = 0$

$\Rightarrow \lambda = 1$

$\therefore$ $R \equiv (2, 2, 8)$

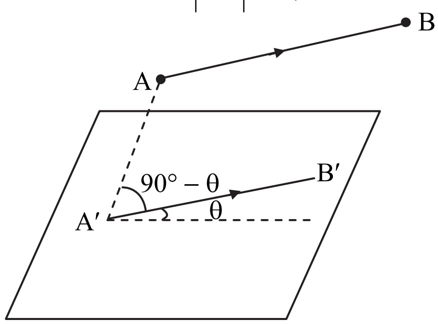
$PQ = 2PR$

$= 2\sqrt{(2-1)^2 + (2+2)^2 + (8-3)^2}$

$= 2\sqrt{42}$ units

88. Let $A = (5, -1, 4)$, $B = (4, -1, 3)$

$\overline{AB} = -\hat{i} - \hat{k} \Rightarrow \left|\overline{AB}\right| = \sqrt{2}$



$x + y + z = 7$

Projection of $\overline{AB}$ in the plane $x + y + z = 7$

is $\left|\overline{AB}\right| \cos\theta = \left|\overline{A'B'}\right| \cos\theta$

Direction ratios of normal to the given plane is 1, 1, 1.

$$\cos(90^\circ - \theta) = \left|\frac{1(-1) + 1(0) + 1(-1)}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{1^2 + 0^2 + 1^2}}\right|$$

$$\Rightarrow \sin\theta = \frac{2}{\sqrt{6}} \Rightarrow \cos\theta = \sqrt{1 - \frac{4}{6}} = \sqrt{\frac{1}{3}}$$

Required projection $= \left|\overline{AB}\right| \cos\theta$

$$= \sqrt{2} \times \frac{1}{\sqrt{3}} = \sqrt{\frac{2}{3}}$$

89. The line of intersection of first two planes is

$$\frac{x-5}{0} = \frac{y}{-3} = \frac{z + \frac{8}{3}}{-5a}$$

It must lie on third plane.

$\therefore$ $3b(0) + (-3)(1) + (-3)(-5a) = 0$

and $3b(5) + 0(1) + (-3)\left(\frac{-8}{3}\right) = 0$

$$\Rightarrow a = \frac{1}{5} \text{ and } 15b + 8 = 0$$

$$\Rightarrow a = \frac{1}{5} \text{ and } b = -\frac{8}{15}$$

Evaluation Test

1. Given planes are

$x - cy - bz = 0$ ….(i)

$cx - y + az = 0$ ….(ii)

$bx + ay - z = 0$ ….(iii)

Equation of a plane passing through the line of intersection of planes (i) and (ii) is

$x - cy - bz + k(cx - y + az) = 0$

$\Rightarrow (1 + ck)x - (c + k)y - (b - ak)z = 0$ ….(iv)

Now, planes (iii) and (iv) are same for some value of k,

$$\therefore \frac{1+ck}{b} = -\frac{c+k}{a} = \frac{-(b-ak)}{-1}$$

$$\Rightarrow \frac{1+ck}{b} = -\frac{c+k}{a}$$

$\Rightarrow a + ack = -bc - bk$

$\Rightarrow k(b + ac) = -(a + bc)$

$$\Rightarrow k = -\left(\frac{a+bc}{b+ac}\right)$$

Also, $-\dfrac{c+k}{a} = b - ak$

$$\Rightarrow -\left(\frac{c - \frac{a+bc}{b+ac}}{a}\right) = b + a\left(\frac{a+bc}{b+ac}\right)$$

$$\Rightarrow \frac{-bc - ac^2 + a + bc}{a} = b^2 + abc + a^2 + abc$$

$\Rightarrow 1 - c^2 = a^2 + b^2 + 2abc$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

2. Let a, b, c be the intercepts form by the plane on co-ordinate axes.

Since, $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{2}$

$$\therefore \frac{2}{a} + \frac{2}{b} + \frac{2}{c} = 1$$

$\therefore$ The point (2, 2, 2) satisfies the equation of the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

$\therefore$ the required point is (2, 2, 2).

3. Given euation of line and plane are

$\bar{r} = \hat{i}+\hat{j}+\lambda\left(2\hat{i}+\hat{j}+4\hat{k}\right)$, and

$\bar{r}.\left(\hat{i}+2\hat{j}-\hat{k}\right)=3$

$\therefore$ $\bar{b}=2\hat{i}+\hat{j}+4\hat{k}$ and

$\bar{n}=\hat{i}+2\hat{j}-\hat{k}$

Consider $\bar{b}\cdot\bar{n} = \left(2\hat{i}+\hat{j}+4\hat{k}\right)\cdot\left(\hat{i}+2\hat{j}-\hat{k}\right)$

$= 2 + 2 - 4$

$= 0$

$\therefore$ the line lies in the plane.

4. The equation of the given line is

$x = 2 + t, y = 1 + t, z = -\frac{1}{2}-\frac{1}{2}t$

$\Rightarrow \frac{x-2}{1}=\frac{y-1}{1}=\frac{z+\frac{1}{2}}{-\frac{1}{2}}$

$\therefore$ The given line passes through the point $\left(2,1,-\frac{1}{2}\right)$ and it's d. r.s are 1, 1, $-\frac{1}{2}$

The equation of the given plane is $x + 2y + 6z = 10$

$\therefore$ d.r.s of the normal to the plane are 1, 2, 6

$\therefore$ $p = \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$

$= \left|\frac{1(2)+2(1)+6\left(-\frac{1}{2}\right)-10}{\sqrt{1^2+2^2+6^2}}\right|$

$= \left|\frac{2+2-3-10}{\sqrt{1+4+36}}\right| = \frac{9}{\sqrt{41}}$

$\therefore$ $\frac{\lambda}{\sqrt{\mu}}=\frac{9}{\sqrt{41}}$

$\Rightarrow \lambda = 9, \mu = 41$

$\therefore$ $5\lambda - \mu = 5(9) - 41 = 45 - 41 = 4$

5. Let $\bar{a}$ be the vector along the line of intersection of the planes $3x - 7y - 5z = 1$ and $8x - 11y + 2z = 0$. the d.r.s of the normals to the planes are 3, –7, –5 and 8, –11, 2.

$\therefore$ $\bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -7 & -5 \\ 8 & -11 & 2 \end{vmatrix}$

$= \hat{i}(-14-55)-\hat{j}(6+40)+\hat{k}(-33+56)$

$= -69\hat{i}-46\hat{j}+23\hat{k}$

Similarly, let $\bar{b}$ the vector along the line of intersection of the planes $5x - 13y + 3z + 2 = 0$ and $8x - 11y + 2z = 0$

the d.r.s of the normals to the planes are 5, –13, 3 and 8, –11, 2

$\therefore$ $\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -13 & 3 \\ 8 & -11 & 2 \end{vmatrix}$

$= \hat{i}(-26+33)-\hat{j}(10-24)+\hat{k}(-55+104)$

$= 7\hat{i}+14\hat{j}+49\hat{k}$

Consider,

$\bar{a}\,.\,\bar{b} = \left(-69\hat{i}-46\hat{j}+23\hat{k}\right).\left(7\hat{i}+14\hat{j}+49\hat{k}\right)$

$= -69 \times 7 + (-46) \times 14 + 23 \times 49$

$= -483 - 644 + 1127$

$= -1127 + 1127$

$= 0$

$\therefore$ $\bar{a}$ and $\bar{b}$ are perpendicular

$\Rightarrow \theta = 90°$

$\therefore$ $\sin\theta = \sin 90° = 1$

6. The equation of the given plane is

$2\lambda x - (1 + \lambda)y + 3z = 0$

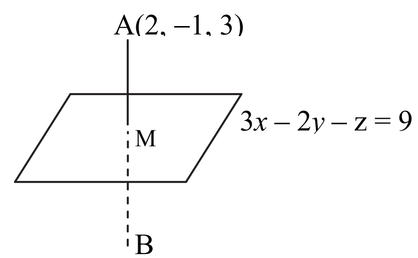
$\Rightarrow 2\lambda x - y - \lambda y + 3z = 0$

$\Rightarrow (2x - y)\lambda - (y - 3z) = 0$

$\Rightarrow (2x - y) - \frac{1}{\lambda}(y - 3z) = 0$

$\therefore$ The plane passes through the point of intersection of the planes $2x - y = 0$ and $y - 3z = 0$

7.



Let A ≡ (2, –1, 3), AM be ⊥ to the given plane and let B ≡ (x, y, z) be the image of A in the Plane.

the d.r.s. of the normal to the plane are 3, –2, –1

$\therefore$ The equation of the line AM is

$$\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-3}{-1}=k, \text{ say}$$

$\therefore$ $x = 3k + 2, y = -2k - 1, z = -k + 3$

Let $M \equiv (3k + 2, -2k - 1, -k + 3)$

$\therefore$ equation of plane becomes

$3(3k + 2) - 2(-2k - 1) - (-k + 3) = 9$

$\therefore$ $k = \frac{2}{7}$

$\therefore$ $M \equiv \left(\frac{6}{7}+2,-\frac{4}{7}-1,-\frac{2}{7}+3\right) \equiv \left(\frac{20}{7},-\frac{11}{7},\frac{19}{7}\right)$

Since, M is the mid point of AB.

$\therefore$ $\frac{x_1+2}{2}=\frac{20}{7}, \frac{y_1-1}{2}=-\frac{11}{7}, \frac{z_1+3}{2}=\frac{19}{7}$

$\therefore$ $x_1 = \frac{26}{7}, y_1 = -\frac{15}{7}, z_1 = \frac{17}{7}$

Image of A is $B\left(\frac{26}{7},-\frac{15}{7},\frac{17}{7}\right)$

8. Since, $\bar{a}$ and $\bar{b}$ are coplanar, $\bar{a}\times\bar{b}$ is a vector perpendicular to the plane containing $\bar{a}$ and $\bar{b}$. Similarly, $\bar{c}\times\bar{d}$ is a vector perpendicular to the plane containing $\bar{c}$ and $\bar{d}$.
The two planes will be parallel, if their normals $\bar{a}\times\bar{b}$ and $\bar{c}\times\bar{d}$ are parallel.

$\therefore$ $\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)=\bar{0}$

9. Equation of the plane containing the given lines is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$\Rightarrow (x - 1)(15 - 16) - (y - 2)(10 - 12) + (z - 3)(8 - 9) = 0$

$\Rightarrow (x - 1)(-1) - (y - 2)(-2) + (z - 3)(-1) = 0$

$\Rightarrow -x + 1 + 2y - 4 - z + 3 = 0$

$\Rightarrow -x + 2y - z = 0$

$\Rightarrow x - 2y + z = 0$ ....(i)

Given equation of plane is

$Ax - 2y + z = d$ ....(ii)

The planes given by equation (i) and (ii) are parallel.

$\therefore$ $A = 1$

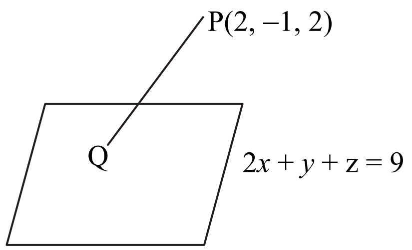
distance between the planes (D) is

$$D = \left|\frac{d}{\sqrt{1^2+(-2)^2+1^2}}\right| = \left|\frac{d}{\sqrt{6}}\right|$$

$\therefore$ $\left|\frac{d}{\sqrt{6}}\right| = \sqrt{6}$

$\Rightarrow |d| = 6$

10.



Since, direction cosines of PQ are equal and positive

$\therefore$ the d.r.s. of PQ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

$\therefore$ The equation of the line PQ is

$$\frac{x-2}{\frac{1}{\sqrt{3}}}=\frac{y+1}{\frac{1}{\sqrt{3}}}=\frac{z-2}{\frac{1}{\sqrt{3}}}$$

$\Rightarrow x - 2 = y + 1 = z - 2 = k$, say

$\therefore$ Co-ordinate of the point Q are $(k + 2, k - 1, k + 2)$

The point Q lies on the plane $2x + y + z = 9$

$\therefore$ $2(k + 2) + k - 1 + k + 2 = 9$

$\Rightarrow 4k + 5 = 9$ $\Rightarrow k = 1$

$\therefore$ $Q \equiv (3, 0, 3)$

$\therefore$ $PQ = \sqrt{(3-2)^2+(0+1)^2+(3-2)^2}$

$= \sqrt{1+1+1} = \sqrt{3}$

11. Let $A \equiv (a, 0, 0), B \equiv (0, b, 0), C \equiv (0, 0, c)$

$\therefore$ $G \equiv (x, y, z) \equiv \left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$

$\Rightarrow \frac{a}{3} = x, \frac{b}{3} = y, \frac{c}{3} = z$

$\Rightarrow a = 3x, b = 3y, c = 3z$ ....(i)

The equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Since, this plane is at a distance of 1 unit from the origin,

$$\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=1$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=1$$

$$\Rightarrow \frac{1}{9x^2}+\frac{1}{9y^2}+\frac{1}{9z^2}=1 \quad \text{....[From (i)]}$$

$$\Rightarrow \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=9$$

$\Rightarrow k = 9$

12. Let the equation of the plane OAB be $ax + by + cz = d$

This plane passes through the points A(1, 2, 1) and B(2, 1, 3)

$\therefore$ $a + 2b + c = 0$, …(i)

and $2a + b + 3c = 0$ …(ii)

$\therefore$ on solving (i) and (ii), we get

$$\frac{a}{5}=\frac{b}{-1}=\frac{c}{-3}$$

Similarly, let the equation of the plane ABC be $a'(x + 1) + b'(y - 1) + c'(z - 2) = 0$

Substituting the co-ordinates of A and B, we get

$2a' + b' - c' = 0$,

and $3a' + c' = 0$

$$\therefore \frac{a'}{1}=\frac{b'}{-5}=\frac{c'}{-3}$$

If θ is the angle between two planes, then it is the angle between their normals.

$$\therefore \cos\theta=\frac{5\times1+(-1)\times(-5)+(-3)\times(-3)}{\sqrt{25+1+9}\ \sqrt{1+25+9}}$$

$$=\frac{5+5+9}{\sqrt{35}\sqrt{35}}$$

$$=\frac{19}{35}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{19}{35}\right)$$

13. The equation of the given plane can be written as $\frac{x}{20}+\frac{y}{15}+\frac{z}{-12}=1$

Let the plane intersects the x, y and z axes in the points A(20, 0, 0), B(0, 15, 0), C(0, 0, –12)

$\therefore$ $\bar{a}=20\hat{i}, \bar{b}=15\hat{j}$, and $\bar{c}=-12\hat{k}$

$\therefore$ Volume of tetrahedron $=\frac{1}{6}\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$

$$=\frac{1}{6}\begin{vmatrix}20 & 0 & 0\\ 0 & 15 & 0\\ 0 & 0 & -12\end{vmatrix}=|-600|=600$$

14. Given lines are coplanar.

$$\therefore \begin{vmatrix}1-2 & 4-3 & 5-4\\ 1 & 1 & -k\\ k & 2 & 1\end{vmatrix}=0$$

$\Rightarrow -1(1 + 2k) - 1(1 + k^2) + 1(2 - k) = 0$

$\Rightarrow -1 - 2k - 1 - k^2 + 2 - k = 0$

$\Rightarrow -k^2 - 3k = 0 \Rightarrow k(k + 3) = 0$

$\Rightarrow k = 0$ or $k = -3$

Textbook Chapter No.

09 Linear Programming

Hints

Classical Thinking

3. Option D is the only option which is non-linear.

4. 'p' is a linear inequality and 'q' is a non-linear inequality

5. Since the profit should be maximum, the objective function is
Maximum profit, $z = 40x + 25y$.

9. Let x = number of table clothes produced in a day, and
y = number of curtains produced in a day

$\therefore$ $x \geq 0, y \geq 0$ ....[$\because$ both items cannot be negative]

Representing the given information in tabular form, we get

	Table cloth (x)	Curtain (y)	Total availability
Money earned (₹)	50	250	500
Hours of work	1	3	z

$\therefore$ $50x + 250y \geq 500$

$\therefore$ total hours = $z = x + 3y$

$\therefore$ Required LPP is formulated as
Minimize, $z = x + 3y$, subject to $50x + 250y \geq 500$, $x \geq 0$, $y \geq 0$

15. At (800, 400), P = 12 (800) + 6 (400) = 12000
At (1050, 150), P = 12 (1050) + 6 (150) = 13500
At (600, 0), P = 12 (600) + 6 (0) = 7200

$\therefore$ Maximum value of P is 13500.

16. The corner points of feasible region are O(0, 0), A(7, 0), B(3, 4) and D(0, 2)
At A(7, 0), z = 5(7) + 7(0) = 35
At B(3, 4), z = 5(3) + 7(4) = 43
At C(0, 2), z = 5(0) + 7(2) = 14

$\therefore$ Maximum value of z is 43.

17. The corners of feasible region are O(0, 0), A(25, 0), B(16, 16) and C(0, 24)
At O(0, 0), z = 0
At A(25, 0), z = 4(25) + 3(0) = 100
At B (16, 16), z = 4(16) + 3(16) = 112
At C(0, 24), z = 4(0) + 3(24) = 72

$\therefore$ Maximum value of z is 112.

18. The corners of feasible region are O (0,0), A (52, 0), E (44, 16) and D (0, 38).

$\therefore$ At A(52, 0), z = 3(52) + 4(0) = 156
At E(44, 16), z = 3(44) + 4(16) = 196
At D(0, 38), z = 3(0) + 4(38) = 152

$\therefore$ Maximum value of z is 196

19. At A (50, 50), P = $\frac{5}{2}(50) + \frac{3}{2}(50) + 410 = 610$

At B (10, 50), P = $\frac{5}{2}(10) + \frac{3}{2}(50) + 410 = 510$

At C (60, 0), P = $\frac{5}{2}(60) + \frac{3}{2}(0) + 410 = 560$

At D (60, 40), P = $\frac{5}{2}(60) + \frac{3}{2}(40) + 410 = 620$

$\therefore$ Minimum value of P is 510 at B (10, 50)

20. The corners of given feasible region are A(12, 0), B(4, 2), C(1, 5) and D(0, 10)
At A(12, 0), z = 3(12) + 2(0) = 36
At B(4, 2), z = 3(4) + 2(2) = 16
At C(1, 5), z = 3(1) + 2(5) = 13
At D(0, 10), z = 3(0) + 2(10) = 20
Minimum value of z is 13

21. The corner points of feasible region are (0, 3), (0, 5) and (3, 2)

$\therefore$ At (0, 3), z = 11(0) + 7(3) = 21
At (0, 5), z = 11(0) + 7(5) = 35
At (3, 2), z = 11(3) + 7(2) = 47

$\therefore$ Minimum value of z is 21

22. At P$\left(\frac{3}{13}, \frac{24}{13}\right)$, z = $\frac{3}{13} + 2\left(\frac{24}{13}\right) = \frac{51}{13} = 3.923$

At Q$\left(\frac{3}{2}, \frac{15}{4}\right)$, z = $\frac{3}{2} + 2\left(\frac{15}{4}\right) = 9$

At R$\left(\frac{7}{2}, \frac{3}{4}\right)$, z = $\frac{7}{2} + 2\left(\frac{3}{4}\right) = 5$

At S$\left(\frac{18}{7}, \frac{2}{7}\right)$, z = $\frac{18}{7} + 2\left(\frac{2}{7}\right) = \frac{22}{7} = 3.143$

$\therefore$ Maximum value of z is 9, and

Minimum value of z is $\frac{22}{7}$.

23. Assume that x and y take arbitrary large values. So the objective function can be made as large as we want. Hence the problem has unbounded solution.

24. The feasible region is unbounded. x and y can take arbitrary large values.
Hence the problem has unbounded solution.

25. Since there are two disjoint feasible regions, the LPP has no solution.

26. The feasible region is disjoint.

$\therefore$ There is no solution.

Critical Thinking

1. From the given table the constraints are $2x + 3y \le 36$; $5x + 2y \le 50$; $2x + 6y \le 60$
   Also $x \ge 0, y \ge 0$ ....[$\because$ number of magazines cannot be negative]

$\therefore$ The number of constraints are 5.

2. Repersenting the given information in table form, we get

	Shirt (x)	Pants (y)	Total availability
Work time on machine (hours)	2	3	70
Man labour (hours)	3	2	75

   Linear constraints are $2x + 3y \le 70$, $3x + 2y \le 75$.
   Also, $x \ge 0, y \ge 0$ ....[$\because$ number of shirts and pants cannot be negative]

3. Let the factory owner purchase x units of machine A and y units of machine B for his factory.

$\therefore$ $x \ge 0$, $y \ge 0$ ....[$\because$ number of machines cannot be negative]

   Representing the given information in tabular form, we get

	Machine A(x)	Machine B(y)	Total Availability
Machine Area (m^2)	1000	1200	7600
Skilled men	12	8	72
Daily output (no. of units)	50	40	z

$\therefore$ $1000x + 1200y \le 7600$
   $12x + 8y \le 72$

4. Let, x = number of necklaces, and y = number of bracelets
   $\therefore$ $x \ge 0, y \ge 0$ ....[$\because$ number of necklaces and bracelets cannot be negative]
   Representing the given information in tabular form, we get

	Necklace (x)	Bracelet (y)	Total availability
Time required (hrs)	$\frac{1}{2}$	1	16
Profit (₹)	100	200	z

$\therefore$ $\frac{1}{2}x + y \le 16 \Rightarrow x + 2y \le 32$
   $x + y \le 24$
   total profit z = $100x + 300y$

$\therefore$ Required LPP is formulated as
   Maximize z = $100x + 300y$, subject to
   $x + y \le 24$, $x + 2y \le 32$, $x \ge 0$, $y \ge 0$

5. Let the consumption per day be, x grams of food X and Y grams of food Y.

$\therefore$ $x \ge 0$ and $y \ge 0$ ....[$\because$ the quantities cannot be negative]

   Representing the given information in table form, we get

Type of food	Food X (x)	Food Y (y)	Minimum requirement
Vitamin A per gram (units)	4	6	90
Vitamin B per gram (units)	7	11	130
Cost per gram (paise)	15	22	z

$\therefore$ $4x + 6y \ge 90$,
   $7x + 11y \ge 130$, and z = $15x + 22y$

$\therefore$ Required LLP is formulated as,
   Minimize z = $15x + 22y$, subject to constraints
   $4x + 6y \ge 90$, $7x + 11y \ge 130$, $x \ge 0$, $y \ge 0$

6. Suppose x kg of food A and y kg of food B are consumed to form a weekly diet.

$\therefore$ $x \geq 0, y \geq 0$. ….[Since quantity of food cannot be negative]

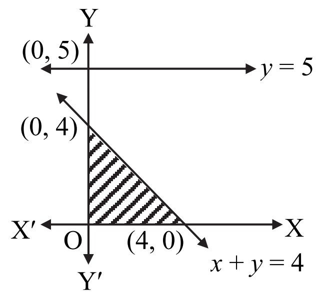
Representing the given information in table form, we get

	Food A (x)	**Food B (y)**	**Minimum requirement**
Fats (units)	4	12	18
Carbohydrates (units)	16	4	24
Protein (units)	8	6	16
Cost (₹)	6	5	z

$\therefore$ Required LPP is formulated as

Minimize, $z = 6x + 5y$ subject to constraints,

$4x + 12y \geq 18$, $16x + 4y \geq 24$, $8x + 6y \geq 24$, $x \geq 0$, $y \geq 0$



7. Converting the given inequalities into equations, we get $x + y = 4$

The equation intersects the axes at (4 , 0) and (0 , 4)

The feasible region lies on origin side of lines $y = 5$ and $x + y = 4$ and in first quadrant.

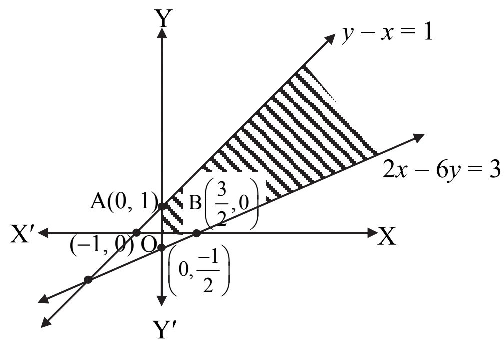
It is bounded in first quadrant.

8. Converting given inequalities into equations, we get

$y - x = 1$ i.e. $\dfrac{x}{(-1)} + \dfrac{y}{1} = 1$ ….(i)

$2x - 6y = 3$ i.e. $\dfrac{x}{\frac{3}{2}} + \dfrac{y}{\left(\frac{-1}{2}\right)} = 1$ ….(ii)

$x = 0, y = 0$



$\therefore$ Equation (i) intersects the axes at (–1, 0) and (0, 1)

Equation (ii) intersects the axes at $\left(\dfrac{3}{2}, 0\right)$ and $\left(0, \dfrac{-1}{2}\right)$

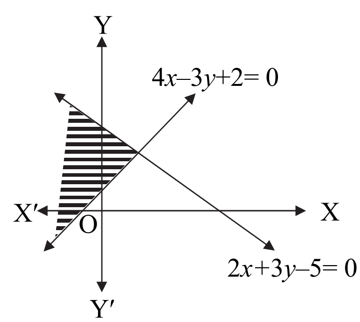
Substituting $x = 0$, $y = 0$ in given inequalities, we get

$(0) - (0) = 0 \leq 1$, and $2(0) - 6(0) = 0 \leq 3$

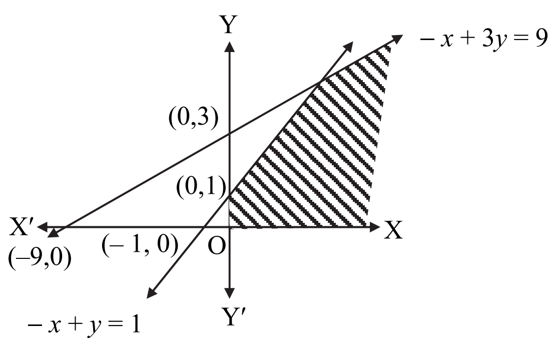
$\therefore$ Feasible region lies on the origin side of both the lines, in first quadrant.

It is unbounded and convex.

9. The feasible region lies on origin side of line $2x + 3y - 5 = 0$ and non-origin side of line $4x - 3y + 2 = 0$. However, it is not bounded by any axes.



10.



The feasible region lies on origin side of the lines $-x + 3y = 9$ and $-x + y = 1$, and in first quadrant.

It is unbounded.

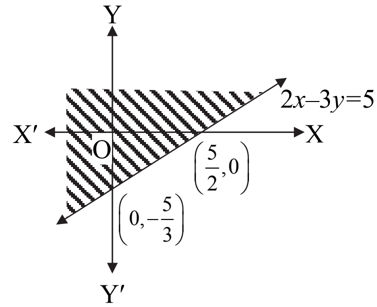
11. Feasible region lies on origin side of line $2x - 3y = 5$.

$\therefore$ O lies inside the region

Substituting P (2, –2) in given inequality, we get

$2(2) - 3(-2) = 10 \nleq 5$

$\therefore$ P lies outside the region.



12. It is clear from the graph that origin is not there in the feasible region. Out of the 4 options, only option (B) satisfies this condition i.e., $4(0) - 2(0) = 0 \leq -3$ is correct.

13. The shaded region lies;

On origin side of line $x + 2y = 8 \Rightarrow x + 2y \leq 8$,

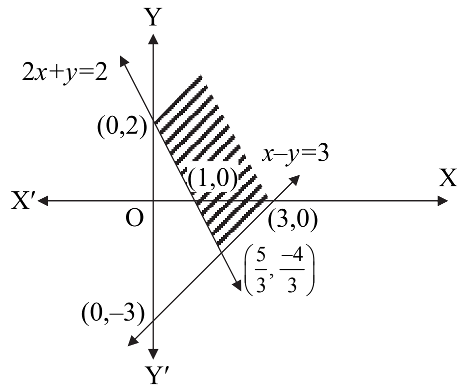
On non-origin side of line $2x + y = 2 \Rightarrow 2x + y \geq 2$,

On origin side of line $x - y = 1 \Rightarrow x - y \leq 1$

and in first quadrant $\Rightarrow x \geq 0,\ y \geq 0$.

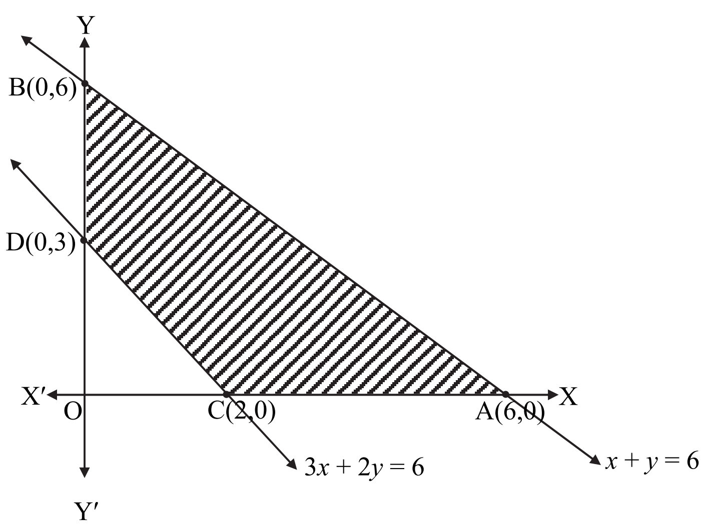
14. The feasible region lies on non-origin side of line $2x + y = 2$ and origin side of line $x - y = 3$ as shown in the figure.

By solving the two equations, we get the point of intersection $\left(\frac{5}{3}, \frac{-4}{3}\right)$, which is the vertex of the common graph.

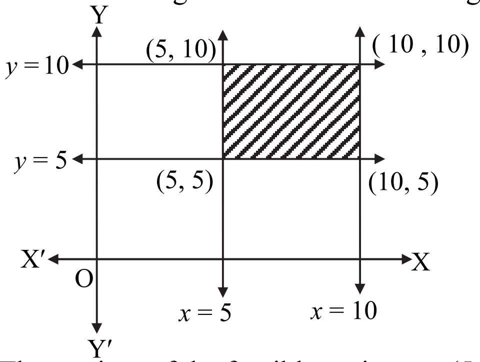


15. Feasible region lies on origin side of line $x + y = 6$, non-origin side of line $3x + 2y = 6$ and in the first quadrant.

$\therefore$ Vertices of the feasible region are (0, 6), (0,3), (2, 0) and (6, 0)



16. Converting the given inequalities into equations, we get $x = 5$, $x = 10$, $y = 5$ and $y = 10$
The feasible region is as shown in the figure

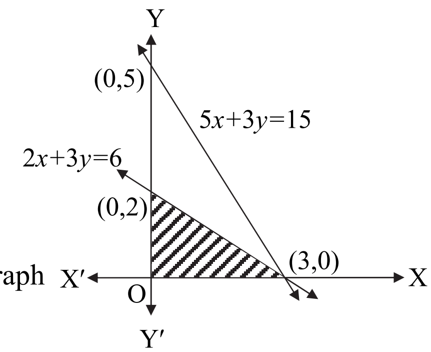


∴ The vertices of the feasible region are (5, 5), (10, 5), (10,10) and (5, 10)

17. Converting the given inequations into equations, we get

$2x + 3y = 6$ i.e. $\frac{x}{3} + \frac{y}{2} = 1$ ....(i)

$5x + 3y = 15$ i.e. $\frac{x}{3} + \frac{y}{5} = 1$ ....(ii)



∴ Equation (i) intersects the axes at points (3, 0) and (0, 2)
Equation (ii), intersects at points (3, 0) and (0, 5).
Also substituting origin (0, 0) in both in equalities we get,
$2(0) + 3(0) = 0 \le 6$ and $5(0) + 3(0) = 0 \le 15$

∴ Feasible region lies on origin side of both the lines as shown in the graph

∴ the vertices of feasible region are (0, 2), (0, 0) and (3, 0)

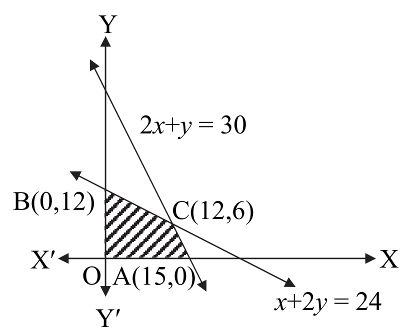
∴ (0, 5) is not a vertex of feasible region.

18. Using two point form we have, equation of line AB : $x + 2y = 8$ and equation of line CD : $3x + 2y = 12$
Since, the shaded region lies on, origin side of line AB, non-origin side of line CD and above X− axis.

∴ $x + 2y \le 8$, $3x + 2y \ge 12$ and $y \ge 0$

19. Take a test point (1, 1) that lies within the feasible region. Since $(1) + (1) = 2 \le 5$, is true we have $x + y \le 5$.
Since $1 \le 4$ and $1 \le 3$ are true, we have $x \le 4$ and $y \le 3$. Since $4(1) + 1 = 5 \ge 4$, we have $4x + y \ge 4$

20. The feasible region lies on the origin side of $2x + y = 30$ and $x + 2y = 24$, in the first quadrant.
The corners of the feasible region are O (0, 0), A (15, 0), B (0, 12) and C (12, 6)
At A(15, 0), z = 90
At B(0, 12), z = 96
At C(12, 6), z = 120



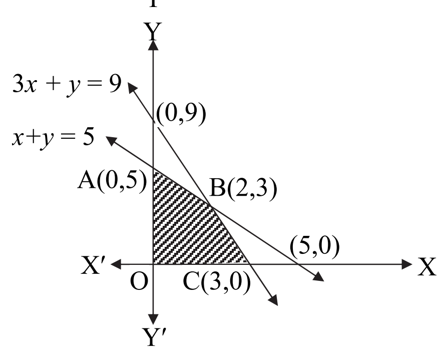
∴ Maximum value of z is 120.

21. The feasible region lies on origin side of lines $x + y = 5$ and $3x + y = 9$, in first quadrant.

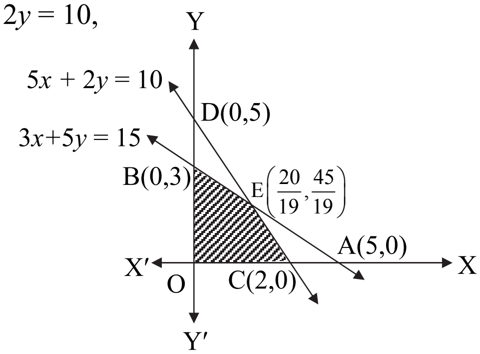
∴ The corners of feasible region are
O (0, 0), A (0, 5), B (2, 3) and C (3, 0)

∴ Maximum value of objective function
$z = 12x + 3y$ is at C (3, 0)

∴ z =12 (3) + 3 (0) = 36



22. The feasible region lies on the origin side of $3x + 5y = 15$ and $5x + 2y = 10$, in first quadrant.



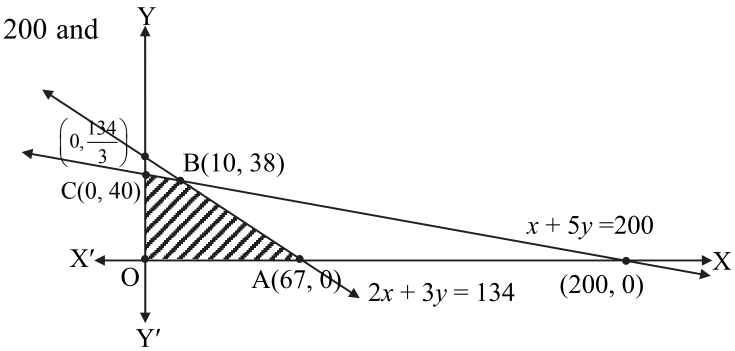
The corners of the feasible region are

O (0, 0), B (0, 3), E $\left(\frac{20}{19}, \frac{45}{19}\right)$ and C (2, 0)

The maximum value of $z = 5x + 3y$ is at E $\left(\frac{20}{19}, \frac{45}{19}\right)$

∴ Maximum $z = 5\left(\frac{20}{19}\right) + 3\left(\frac{45}{19}\right) = \frac{235}{19}$

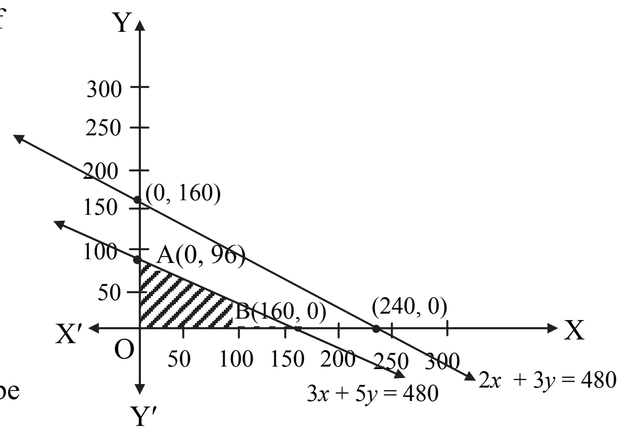
23. Feasible region lies on the origin side of $x + 5y = 200$ and $2x + 3y = 134$, in first quadrant.



The corner points of the feasible region are
O (0, 0), A (67, 0), B (10, 38) and C (0, 40)
At A (67, 0), z = 268
At A (10, 38), z = 382
At A (0, 40), z = 360
∴ Maximum value of z is at B (10, 38)

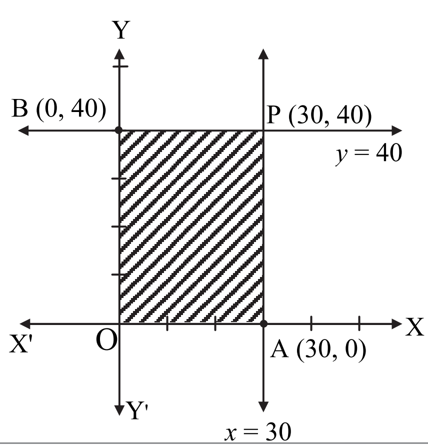
24. $z = px + qy$
At (15, 15), z = 15p + 15q
At (0, 20), z = 0 + 20q = 20q
∵ Maximum z occurs at both the points,
⇒ 15p + 15q = 20q ⇒ 15p = 5q ⇒ 3p = q

25. Suppose that the manufacturer produces x soaps of type I and y soaps of type II.



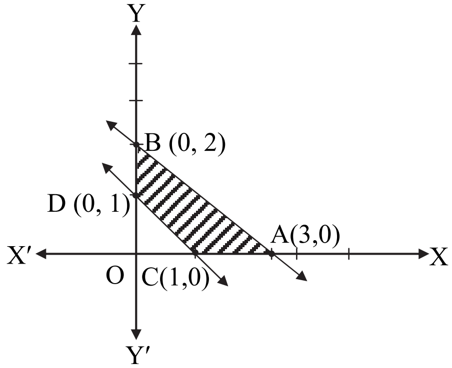
∴ $x \geq 0$; $y \geq 0$; $2x + 3y \leq 480$ and $3x + 5y \leq 480$
Feasible region lies on origin side on both inequalities, in first quadrant.
The corners of the feasible region are
O (0, 0), A (0, 96) and B (160, 0)
Maximum profit, $P = 0.25x + 0.5y$
∴ At A (0, 96), P = 0.25(0) + 0.5(96) = 48
At B (160, 0), P = 0.25(160) + 0.5(0) = 40
For maximum profit of ₹ 48, 96 soaps of type II must be manufactured.

26. The feasible region lies on the origin side of both the lines.

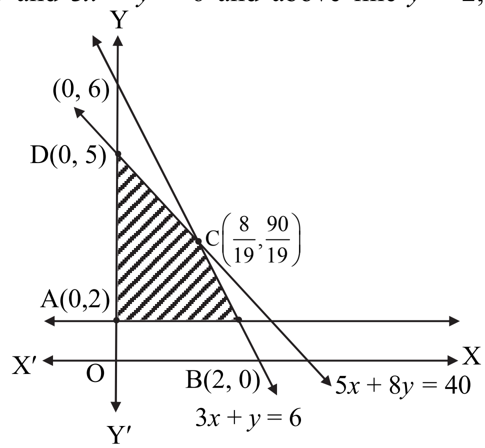


The corner points of feasible region are
O (0, 0), A (30, 0), B (0, 40) and P (30, 40)
∴ At O (0, 0), z = 4(0) + 5(0) = 0
At A (30, 0), z = 4(30) + 5(0) = 120
At B (0, 40), z = 4(0) + 5(40) = 200
At P (30, 40), z = 4(30) + 5(40) = 320
∴ The minimum value of z is 0

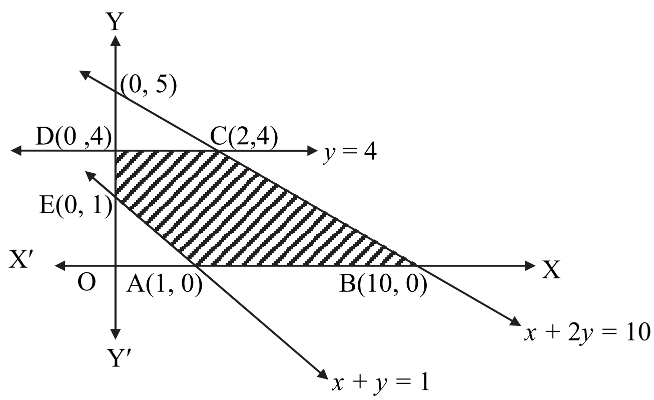
27. The feasible region lies on origin side of line $2x + 3y = 6$ and non-origin side of line $x + y = 1$
The corners of feasible region are
A (3, 0), B (0, 2), C (1, 0) and D (0, 1)
$\therefore$ z = $3x + y$ will be minimum at C or D.
$\therefore$ At C (1, 0), z = 3 (1) + (0) = 3
At D (0, 1), z = 3 (0) + 1 = 1
$\therefore$ Minimum value of z is 1



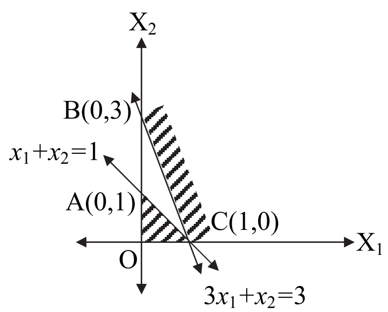
28. Feasible region lies on origin side of lines $5x + 8y = 40$ and $3x + y = 6$ and above line $y = 2$, in first quadrant.
The corner points of the feasible region
A(0, 2), B$\left(\frac{4}{3}, 2\right)$, C $\left(\frac{8}{19}, \frac{90}{19}\right)$ and D(0, 5)
At A (0, 2), z = 14
At B ($\frac{4}{3}$, 2), z = 22
At C $\left(\frac{8}{19}, \frac{90}{19}\right)$, z = $\frac{678}{19}$
At D (0, 5), z = 35
$\therefore$ Minimum value of z is 14



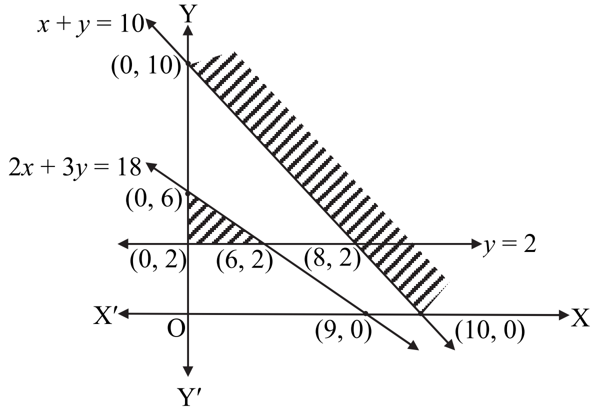
30. The corner points of feasible region are
A(1, 0), B(10, 0), C (2, 4), D(0, 4) and E (0, 1)
At A (1, 0), z = 1 + 0 = 1 = 1
At B (10, 0), z = 10 + 0 = 10
At C (2, 4), z = 2 + 4 = 6
At D (0, 4), z = 0 + 4 = 4
At E (0, 1), z = 0 + 1 = 1
z has minimum value at both A (1, 0) and E (0, 1).
$\therefore$ z has infinite solutions on seg AE.



31. Feasible region lies on origin side of line $x_1 + x_2 = 1$ and non-origin side of line $3x_1 + x_2 = 3$ in first quadrant.
$\therefore$ there is no feasible region.



32.



The feasible regions are is disjoint. Hence there is no point in common.

$\therefore$ There is no optimum value of the objective function.

Competitive Thinking

5. Condition (i),

i = 1, $x_{11} + x_{12} + x_{13} + \ldots + x_{1n}$

i = 2, $x_{21} + x_{22} + x_{23} + \ldots + x_{2n}$

i = 3, $x_{31} + x_{32} + x_{33} + \ldots + x_{3n}$

....

i = m, $x_{m1} + x_{m2} + x_{m3} + \ldots + x_{mn} \rightarrow$ m constraints

Condition (ii),

j = 1, $x_{11} + x_{21} + x_{31} + \ldots + x_{m1}$

j = 2, $x_{12} + x_{22} + x_{32} + \ldots + x_{m1}$

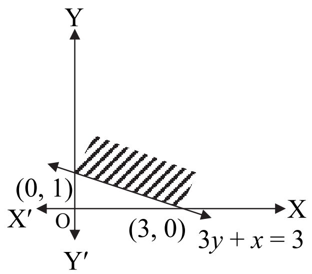
....

j = n, $x_{1n} + x_{2n} + x_{3n} + \ldots + x_{mn} \rightarrow$ n constraints

$\therefore$ Total constraints = m + n

7. In linear programming problem, concave region is not used. Convex region is used in linear programming.

8.

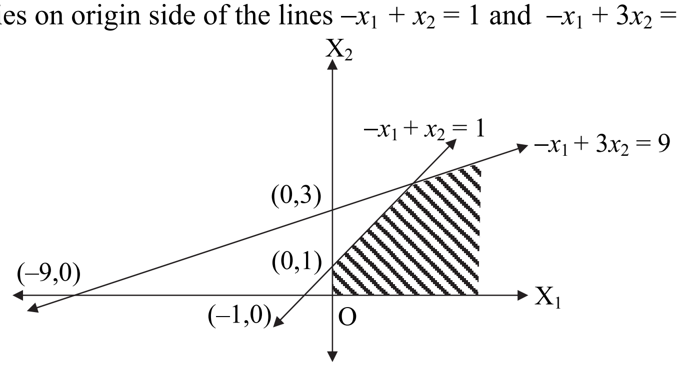


Feasible region is on non-origin side of $3y + x = 3$ and in first quadrant.

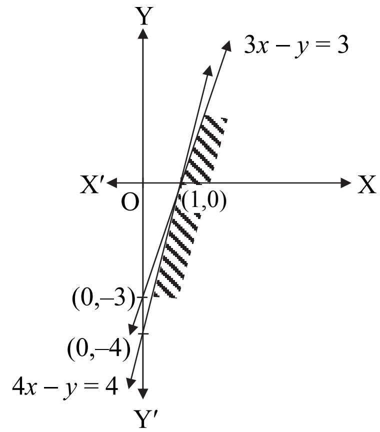
Hence, it is unbounded.

9. The feasible region lies on origin side of the lines $-x_1 + x_2 = 1$ and $-x_1 + 3x_2 = 9$, in first quadrant.

It is unbounded.



10. Feasible region lies on non-origin side of both lines and is true for positive values of x and both positive and negative values of y.



11. Since shaded region lies on origin side of lines $x + y = 20$ and $2x + 5y = 80$ and is in first quadrant

$\therefore$ $x + y \le 20$, $2x + 5y \le 8$, $x \ge 0$, $y \ge 0$

12. Shaded region lies on origin side of $x + 2y = 8$ and $x - y = 1$, and on non-origin side of $2x + y = 2$.

$\therefore$ $x + 2y \le 8$, $x - y \le 1$, $2x + y \ge 2$

13. Take a test point (2, 1) which lies within the feasible region.

Since, $2 - 1 = 1 \ge 0$, $2 \le 5$, $1 \le 3$ and $2,1 \ge 0$

$\therefore$ $x, y \ge 0$, $x - y \ge 0$, $x \le 5$, $y \le 3$.

14. Since shaded region lies on non-origin side of $5x + 4y = 20$, and on origin side of the lines $x = 6$ and $y = 3$

$\therefore$ $5x + 4y \ge 20$, $x \le 6$, $y \le 3$, $x \ge 0$, $y \ge 0$

17. The feasible region lies on the origin side of $x + y = 40$ and $x + 2y = 6$, in fist quadrant.

The corners of feasible region are

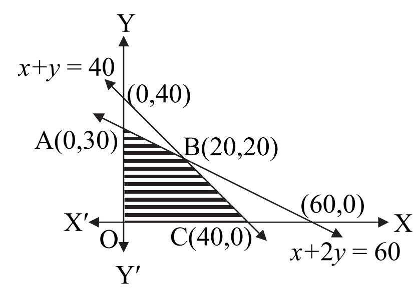
O(0, 0), A(0, 30), B(20, 20) and C(40, 0)

$\therefore$ At A(0, 30), P = 0 + 4 (30) = 120

At B(20, 20), P = 3(20) + 4 (20) = 140

At C(40, 0), P = 3(40) + 0 = 120

$\therefore$ Maximum value of P is 140.



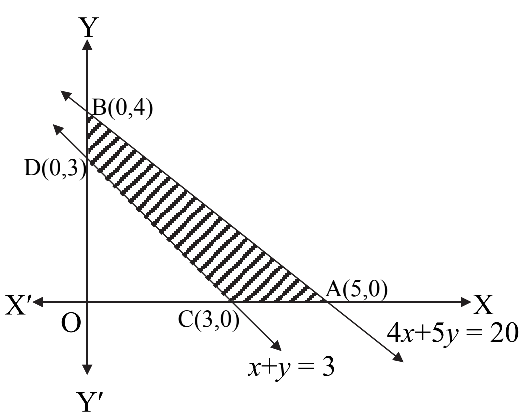
18. The feasible region lies on origin side of $4x + 5y = 20$, non-origin side of $x + y = 3$ and in first quadrant.

$\therefore$ The corners of feasible region are

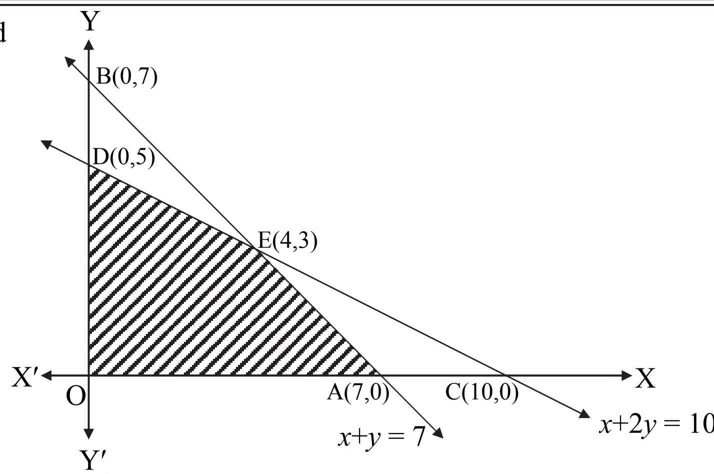
A(5, 0), B(0, 4), C(3, 0) and D(0, 3)

$\therefore$ Maximum $2x + 3y$ is at B (0, 4)

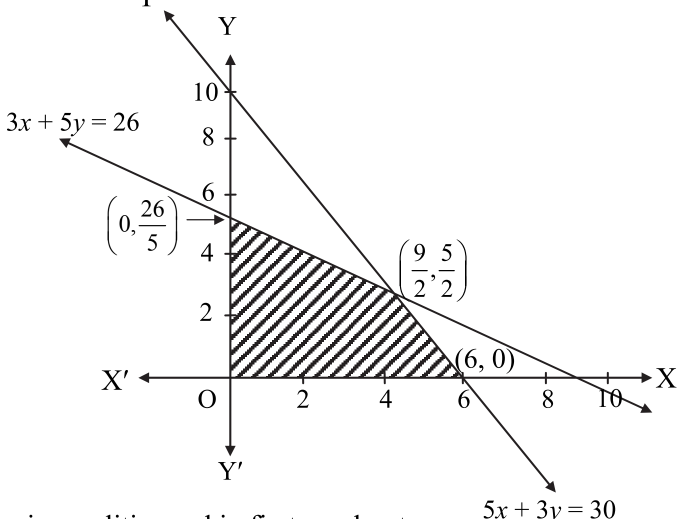
$\therefore$ Maximum $2x + 3y = 2 (0) + 4 (3) = 12$



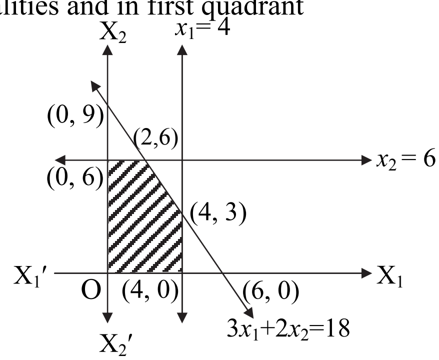
19. Feasible region lies on origin side of $x + y = 7$ and $x + 2y = 10$, and in first quadrant.
The corners of feasible region are
O(0, 0), A(7, 0), E(4, 3) and D(0, 5)
$\therefore$ Maximum $z = 5x + 2y$ is at A (7, 0)
$\therefore$ Maximum, $z = 5(7) + 2(0) = 35$



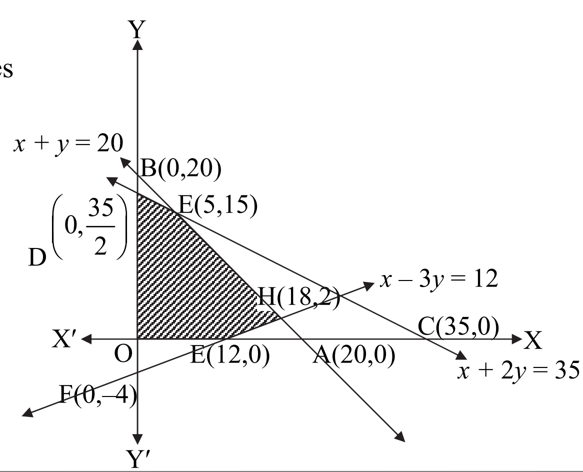
20. Corner points of the feasible region are
$(0, 0)$ $(6, 0)$, $\left(\frac{9}{2}, \frac{5}{2}\right)$ and $\left(0, \frac{26}{5}\right)$
At $(0, 0)$, $z = 2(0) + 0 = 0$
At $(6, 0)$, $z = 2(6) + 0 = 12$
At $\left(\frac{9}{2}, \frac{5}{2}\right)$, $z = 2\left(\frac{9}{2}\right) + \frac{5}{2} = 11.5$
At $\left(0, \frac{26}{5}\right)$, $z = 2(0) + \frac{26}{5} = 5.2$
$\therefore$ Maximum value of z is 12 at (6, 0).



21. The feasible region lies on origin side of all the inequalities and in first quadrant
The corners of feasible region are
(0, 0), (4, 0), (4, 3), (2, 6) and (0, 6).
At (4, 0), $z = 3(4) + 0 = 2$
At (4, 3), $z = 3(4) + 5(3) = 27$
At (2, 6), $z = 3(2) + 5(6) = 36$
At (0, 6), $z = 0 + 5(6) = 30$
$\therefore$ Maximum value of 2 is 36 at (2, 6)



22. The feasible region lies on the origin side of the lines $x + y = 20$, $x + 2y = 35$ and $x - 3y = 12$
The corners of the feasible region are
O (0, 0), E (12, 0), H (18, 2), G (5, 15), D $\left(0, \frac{35}{2}\right)$
$\therefore$ The maximum value of $4x + 5y$ is at G (5, 15)
$\therefore$ Maximum $4x + 5y = 4(5) + 5(15) = 95$



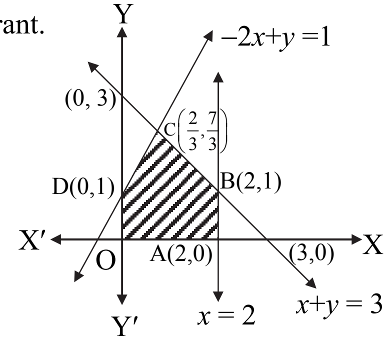
23. The feasible region lies on origin side of all the lines and in first quadrant.

$\therefore$ The corners of feasible region are

O (0, 0), A (2, 0), B (2, 1), C $\left(\frac{2}{3}, \frac{7}{3}\right)$ and D (0, 1)

Maximum value of z = $3x + 2y$ is at B (2, 1)

$\therefore$ Maximum z = 3 (2) + 2 (1) = 8

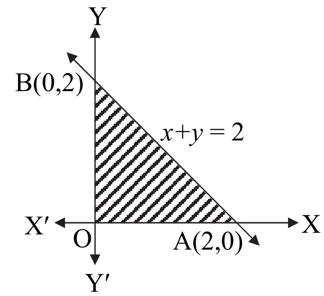


24. The feasible region lies on origin side of $x + y = 2$

The corners of feasible region are

A (2, 0), B (0, 2) and O (0, 0).

At A (2, 0), the value of z is maximum = 6



25. The feasible region lies on the origin side of the lines

$6x + 4y = 120$ and $3x + 10y = 180$

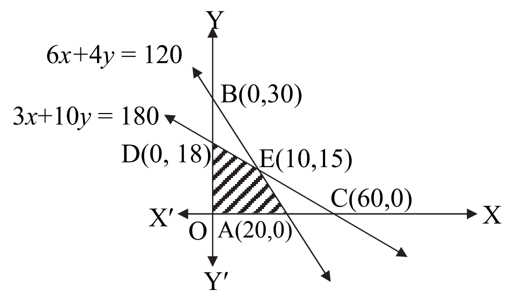
The corners of feasible region are

O (0, 0), A (20, 0), E (10, 15) and D (0, 18)

$\therefore$ The maximum value of $45x + 55y$ is at E (10, 15)

Max $(45x + 55y) = 45(10) + 55(15)$

$= 1275$



26. Feasible region lies on origin side of all lines and in first quadrant.

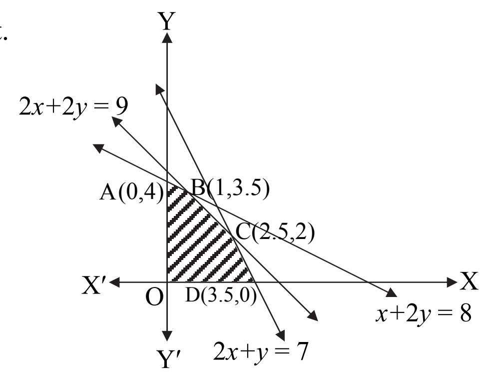
The corners of feasible region are

O (0, 0), A (0, 4), B $\left(1, \frac{7}{2}\right)$, C $\left(\frac{5}{2}, 2\right)$, D $\left(\frac{7}{2}, 0\right)$

Substituting the above points in P = $2x + 3y$, we get

Max P = 12.5 at B $\left(1, \frac{7}{2}\right)$

$\therefore$ B ≡ (1, 3.5)



27. At (15, 15), z = 15p + 15q

At (0, 20), z = 20q

Since, maximum occurs at (15, 15) and (0, 20),

$\therefore$ z_{max} = 15p + 15q = 20q

$\Rightarrow$ 15p + 15q = 20q

$\Rightarrow$ 15p = 5q $\Rightarrow$ 3p = q

28. z = px + qy

At (25, 20), z = 25p + 20q

At (0, 30), z = 0 + 30q = 30q

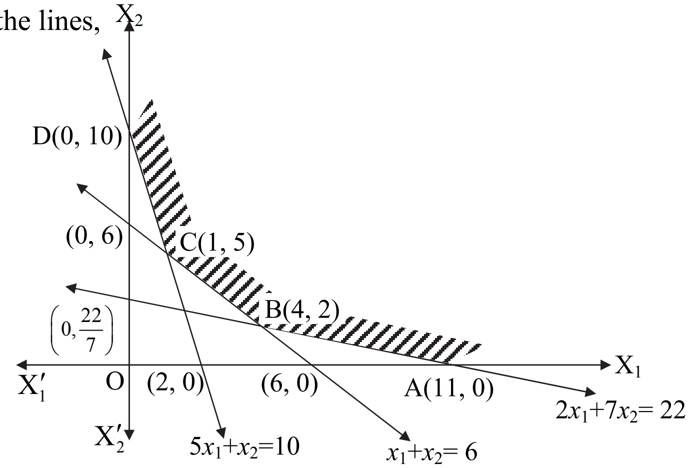
Since maximum z occurs at both the points,

25p + 20q = 30q

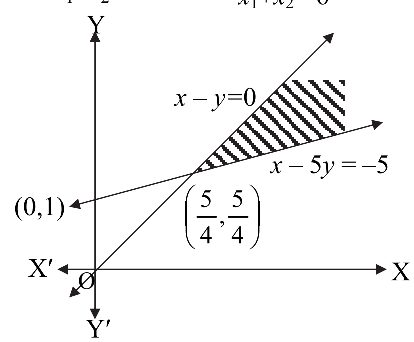
$\Rightarrow$ 25p = 10q $\Rightarrow$ 5p = 2q

29. At (5, 5), z = 3(5) + 9(5) = 60
At (0, 10), z = 3(0) + 9(10) = 90
At (0, 20), z = 3(0) + 9(20) = 180
At (15, 15), z = 3(15) + 9(15) = 180
∴ Minimum value of z is 60 at (5, 5).

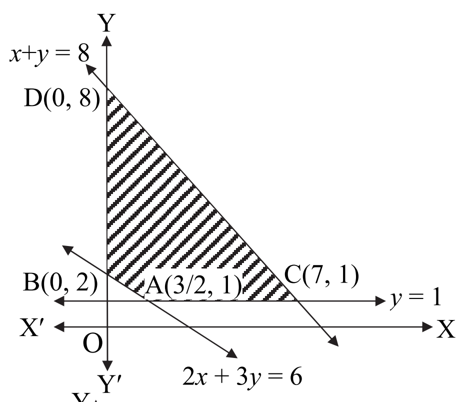
30. The feasible region lies on non-origin side of all the lines, in first quadrant
The corners of feasible region are
A(11, 0), B(4, 2), C(1, 5) and D(0, 10).
∴ At A(11,0), z = 2(11) + 0 = 22
At B(4, 2), z = 2(4) + 3(2) = 14
At C(1, 5), z = 2(1) + 3(5) = 17
At D(0, 1), z = 0 + 3(10) = 30
∴ Maximum value of z is 14



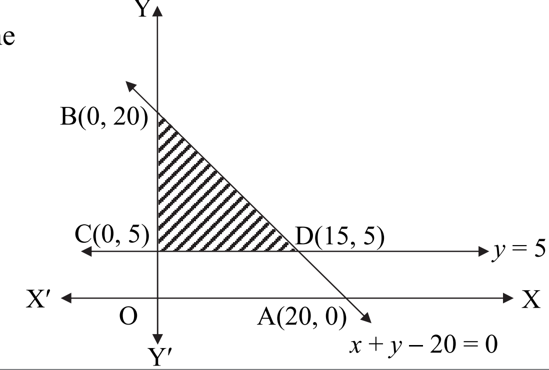
31. The feasible region is unbounded whose vertex is $\left(\frac{5}{4}, \frac{5}{4}\right)$.
∴ Minimum z = $2x + 10y$ is at $\left(\frac{5}{4}, \frac{5}{4}\right)$
∴ $z = 2\left(\frac{5}{4}\right) + 10\left(\frac{5}{4}\right) = 15$



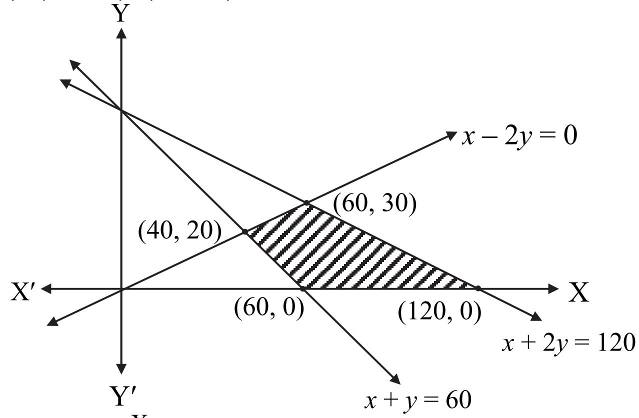
32. The feasible region region lies on the non-origin side of $2x + 3y = 6$ and $y = 1$ and on origin side of $x + y = 8$
The corners of feasible region are
$A\left(\frac{3}{2}, 1\right)$, B(0, 2), C(7, 1) and D(0, 8).
Substituting above points in z = $4x + 6y$, we get
Min. z = 12 at A $\left(\frac{3}{2}, 1\right)$ and B (0, 2).



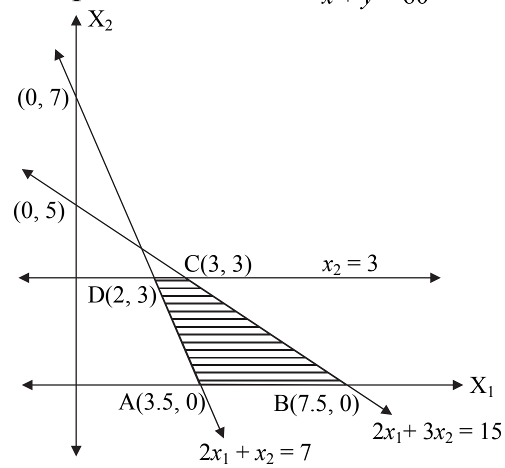
33. The feasible region lies on origin side of line $x + y - 20 = 0$ and above the line $y = 5$.
The corners of feasible region are
B (0, 20), C (0, 5) and D (15, 5)
∴ The minimum value of z = $7x - 8y$ is at B (0, 20)
z = 7 (0) – 8 (20) = – 160



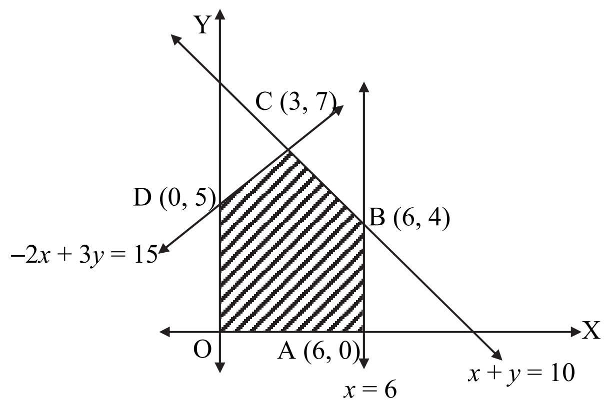
34. Corner points of the feasible region are (60, 0), (120, 0), (60, 30), (40, 20).
At (60, 0), z = 5(60) + 10(0) = 300
At (120, 0), z = 5(120) + 10(0) = 600
At (60, 30), z = 5(60) + 10(30) = 600
At (40, 20), z = 5(40) + 10(20) = 400
∴ Minimum value of z is 300 at (60, 0).



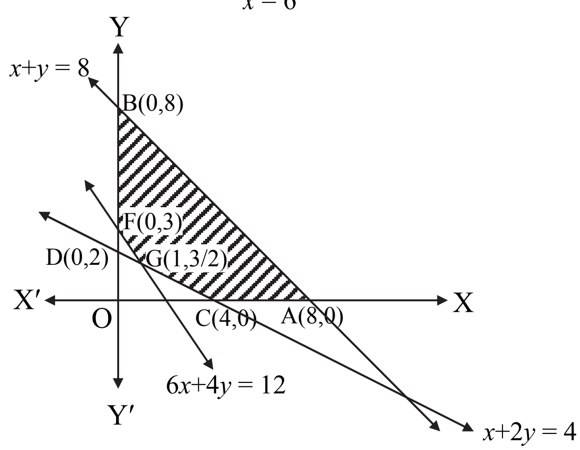
35. The corner points of the feasible region are
A(3.5, 0), B(7.5, 0), C(3, 3) and D(2, 3)
At A(3.5, 0), z = 4(3.5) + 5(0) = 14
At B(7.5, 0), z = 4(7.5) + 5(0) = 30
At C(3, 3), z = 4(3) + 5(3) = 27
At D(2, 3), z = 4(2) + 3(3) = 17
∴ z is minimum at A(3.5, 0).



36. The corner points of feasible region are
A (6, 0), B (6, 4), C (3, 7) and D (0, 5)
∴ At A (6, 0), z = 6 + 0 = 6
At B (6, 4), z = 6 + 4 = 10
At C (3, 7), z = 3 + 7 = 10
At D (0, 5), z = 0 + 5 = 5
∴ z is maximum at B (6, 4) and C (3, 7)
∴ Infinite number of solutions exists along BC.

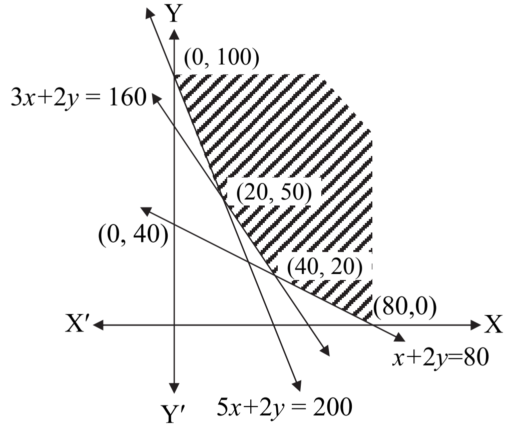


37. The corners of feasible region are
A(8,0), B(0, 8), F(0, 3), G$\left(1, \frac{3}{2}\right)$ and C(4, 0)
At F(0,3), z = 30(0) + 20(3) = 60
At G(1,3/2), z = 30(1) + 20(3/2) = 30 + 30 = 60
At A (8, 0), z = 30 (8) + 0 = 240
At A (0, 8), z = 0 + 20 (8) = 160
At C (4, 0), z = 30 (4) + 0 = 120
∴ z has minimum value at F (0, 3) and G $\left(1, \frac{3}{2}\right)$
∴ z has infinite solution on seg FG.



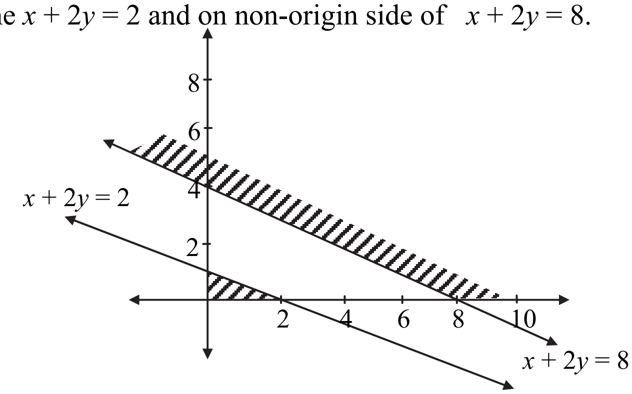
38. The feasible region is unbounded.

$\therefore$ its maximum value does not exist.



39. The feasible region lies on the origin side of the line $x + 2y = 2$ and on non-origin side of $x + 2y = 8$.

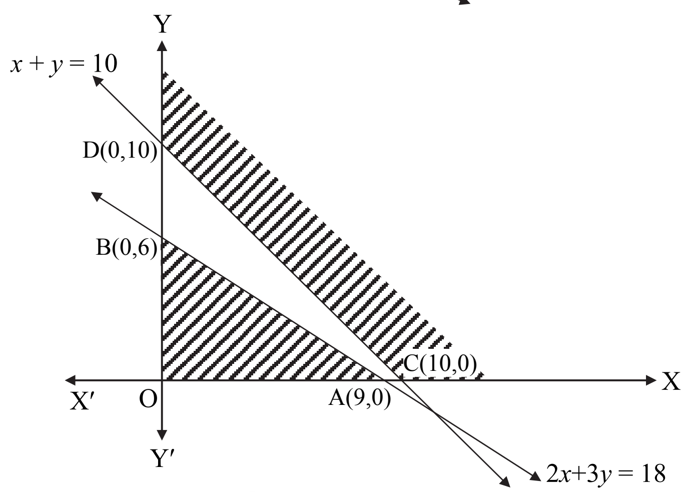
$\therefore$ There is no feasible solution.



40. The feasible region is disjoint.

$\therefore$ there is no point common to all inequations.

$\therefore$ There is no maximum value of z.



Evaluation Test

1. Let no. of model $M_1 = x$ and no. of model $M_2 = y$

$\therefore$ $x \geq 0, y \geq 0$

Constraints are $4x + 2y \leq 80 \Rightarrow 2x + y \leq 40$, $2x + 5y \leq 180$

Maximize $z = 3x + 4y$

The corners of feasible region are

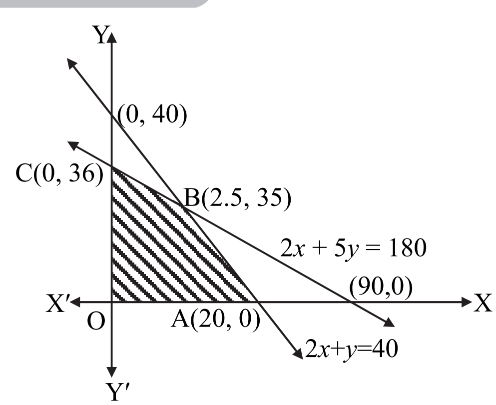
O(0, 0), A(20, 0), B(2.5, 35), C(0, 36)

$\therefore$ At A (20, 0), $z = 3(20) + 0 = 60$

At B (2.5,35), $z = 3(2.5) + 4(35) = 147.5$

At C (0, 36), $z = 0 + 3(36) = 108$

$\therefore$ z is maximum at B(2.5, 35).



3. Objective function $P = 2x + 3y$

The corner points of feasible region are

B(12, 12), C(3,3), D(20, 3), E(20, 10), F(18, 12)

At B = $P_B = 2(12) + 3(12) = 60$

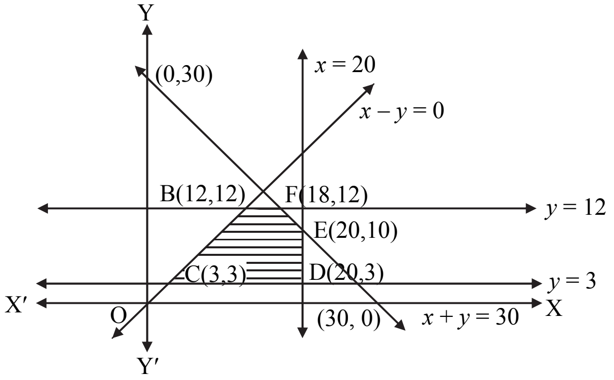
At C = $P_C = 2(3) + 3(3) = 15$

At D = $P_D = 2(20) + 3(3) = 49$

At E = $P_E = 2(20) + 3(10) = 70$

At F = $P_F = 2(18) + 3(12) = 72$

∴ P is maximum at F(18, 12).



4. For (1, 3), $3x + 2y = 3 + 6 > 0$,

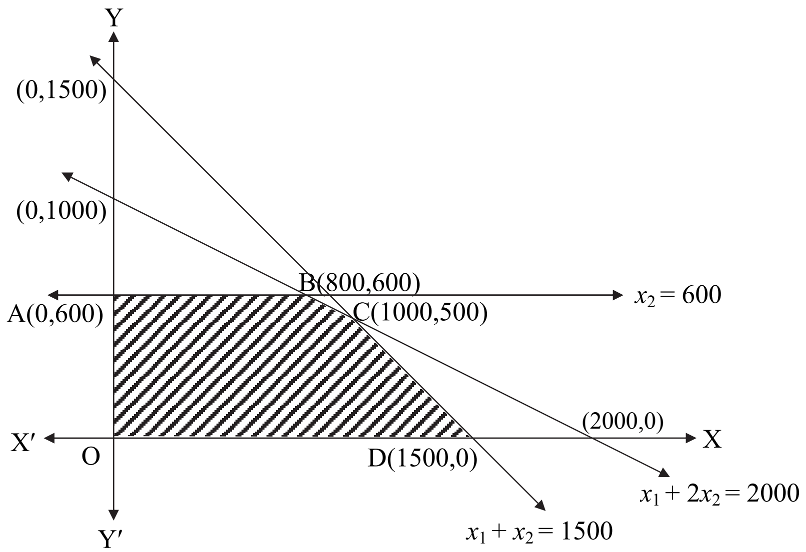
for (5, 0), $3 \times 5 + 0 > 0$,

and for (–1, 2), $-3 + 4 > 0$

Similarly, other inequalities satisfies the given points.

∴ Option (D) is the correct answer.

5.



OABCD is the feasible region

∴ O(0, 0), A(0, 600), B(800, 600), C(1000, 500), D(1500, 0)

$z = x_1 + x_2$

At point C and D, z is maximum. Max z = 1500

∴ Infinite optimal solutions exist along CD.

6. Consider option (C)

$3 + 2(4) \geq 11$

$3(3) + 4(4) \leq 30$

$2(3) + 5(4) \leq 30$

∴ All the above three in-equalities hold for point (3, 4).

∴ Option (C) is the correct answer.

7. Let the manufacturer produce x and y bottles of medicines A and B.

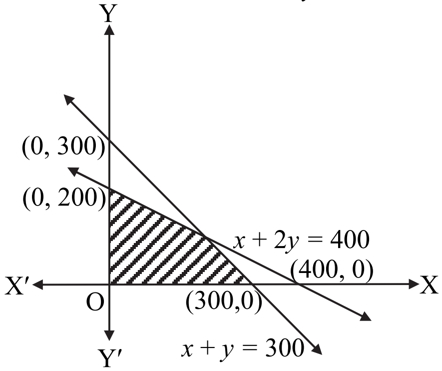
He must have $\frac{3x}{1000} + \frac{y}{1000} \leq 66$, $x + y \leq 45000$, $x \leq 20000$, $y \leq 40,000$, $x \geq 0$, $y \geq 0$.

∴ the number of constraints is 6.

8. Let the company produce x telephones of A type and y telephones of B type.

$\therefore$ Constraints are $2x + 4y \le 800 \Rightarrow x + 2y \le 400$, $x + y \le 300$

Maximize $z = 300x + 400y$



$\therefore$ the feasible region of the LPP is bounded.

9. Given that $4x + 2y \le 8$, $2x + 5y \le 10$

$\therefore$ the feasible region lies on origin side of $4x + 2y = 8$ and $2x + 5y = 10$.

Also, $x, y \ge 0$

$\therefore$ the feasible region lies in first quadrant.

$\therefore$ option (C) is correct.

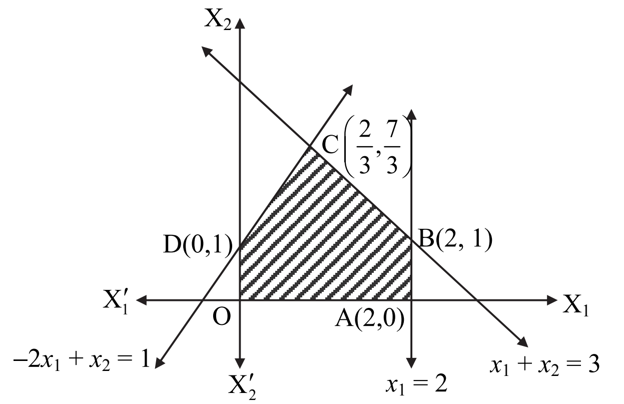
10. Objective function $z = x_1 + x_2$

The corner points of feasible region are

O(0, 0), A(2, 0), B(2, 1), $C\left(\frac{2}{3}, \frac{7}{3}\right)$ and D(0, 1)

At B(2, 1) and $C\left(\frac{2}{3}, \frac{7}{3}\right)$, z is maximum. Max z = 3

$\therefore$ Infinite number of solutions exists along BC.



11. Objective function $z = 3x + 2y$

The corner points of feasible region are

$A\left(\frac{1}{4}, \frac{5}{4}\right)$, $B\left(\frac{1}{6}, \frac{5}{6}\right)$, C(1, 0), D(3, 0), E(3, 3), $F\left(\frac{5}{2}, \frac{7}{2}\right)$

At A = $z_A = 3\left(\frac{1}{4}\right) + 2\left(\frac{5}{4}\right) = 3.25$

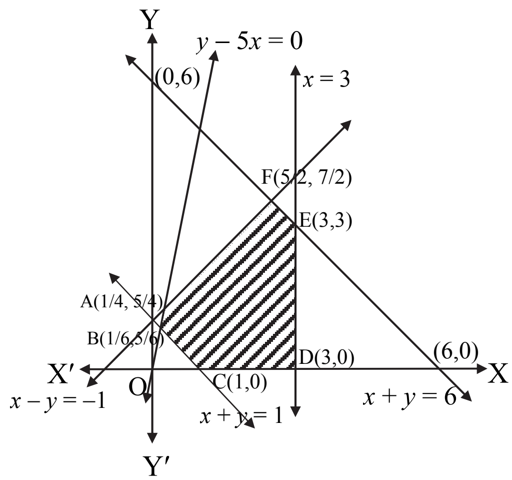
At B = $z_B = 3\left(\frac{1}{6}\right) + 2\left(\frac{5}{6}\right) = 2.167$

At C = $z_C = 3(1) + 2(0) = 3$

At D = $z_D = 3(3) + 2(0) = 9$

At E = $z_E = 3(3) + 2(3) = 15$

At F = $z_F = 3\left(\frac{5}{2}\right) + 2\left(\frac{7}{2}\right) = 14.5$



$\therefore$ Maximum value of z at (3,3) is 15.

Textbook Chapter No.

01 Continuity

Hints

Classical Thinking

1. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left(\dfrac{\sin x}{x} + \cos x\right)$

$= 1 + 1 = 2 = f(0)$

$\therefore$ f(x) is continuous at $x = 0$.

2. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{1}{2}\sin x^2 = 0 = f(0)$

$\therefore$ f(x) is continuous at $x = 0$.

3. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left(1+\dfrac{4x}{5}\right)^{\frac{1}{x}}$

$= \left[\lim\limits_{x\to 0}\left(1+\dfrac{4x}{5}\right)^{\frac{5}{4x}}\right]^{\frac{4}{5}} = e^{\frac{4}{5}} = f(0)$

$\therefore$ f(x) is continuous at $x = 0$.

4. Since, f(x) is continuous $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x)$

$= \lim\limits_{x\to 0}(\sin x - \cos x)$

$= \sin 0 - \cos 0 = -1$

5. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x)$

$= \lim\limits_{x\to 0} \dfrac{2x + \tan x}{x}$

$= \lim\limits_{x\to 0}\left(2 + \dfrac{\tan x}{x}\right) = 2 + 1 = 3$

6. Since, f(x) is continuous at $x = 1$.

$\therefore$ $f(1) = \lim\limits_{x\to 1} f(x)$ $\quad \Rightarrow k = \lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$

$\Rightarrow k = \lim\limits_{x\to 1}(x+1)$ $\quad \Rightarrow k = 2$

7. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x)$

$\Rightarrow \dfrac{k}{2} = \lim\limits_{x\to 0}\dfrac{\sin 3x}{x} = \lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}\cdot 3$

$\Rightarrow \dfrac{k}{2} = 3$ $\quad \Rightarrow k = 6$

8. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x)$

$\Rightarrow k = \lim\limits_{x\to 0}\dfrac{\sin \pi x}{5x}$

$\Rightarrow k = \lim\limits_{x\to 0}\left(\dfrac{\sin \pi x}{\pi x}\right).\dfrac{\pi}{5}$

$\Rightarrow k = (1).\dfrac{\pi}{5}$

$\Rightarrow k = \dfrac{\pi}{5}$

9. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\dfrac{(e^{3x}-1)\sin x}{x^2}$

$= \lim\limits_{x\to 0}\dfrac{e^{3x}-1}{3x}\times 3\times\dfrac{\sin x}{x} = 1\times 3\times 1$

$\therefore$ $f(0) = 3$

10. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(2x+1) = 3 \neq f(1)$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(x^2+1) = 2 = f(1)$

11. $f\left(\dfrac{1}{2}\right) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

$\lim\limits_{x\to \frac{1}{2}^-} f(x) = \lim\limits_{x\to \frac{1}{2}^-}(x) = \dfrac{1}{2}$

$\lim\limits_{x\to \frac{1}{2}^+} f(x) = \lim\limits_{x\to \frac{1}{2}^+}(1-x) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

$\therefore$ $\lim\limits_{x\to \frac{1}{2}^-} f(x) = \lim\limits_{x\to \frac{1}{2}^+} f(x) = f\left(\dfrac{1}{2}\right)$

$\therefore$ f(x) is continuous at $x = \dfrac{1}{2}$.

12. Since, f(x) is continuous at $x = 1$.

$\therefore$ $\lim\limits_{x\to 1^-} f(x) = f(1)$

$\Rightarrow \lim\limits_{x\to 1^-}(8x-1) = k$

$\Rightarrow k = 7$

13. Since, f(x) is continuous at $x = 2$.

$\therefore$ $f(2) = \lim_{x\to 2^-} f(x)$

$\Rightarrow 3 = \lim_{x\to 2}(kx - 1)$

$\Rightarrow 3 = 2k - 1$

$\Rightarrow k = 2$

14. Since, f(x) is continuous at $x = 1$.

$\therefore$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\Rightarrow 2 = \lim_{x\to 1} (c - 2x)$

$\Rightarrow 2 = c - 2$

$\Rightarrow c = 4$

15. Since, f(x) is continuous at $x = 0$.

$\therefore$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\Rightarrow \lim_{x\to 0^-} (-x^2 - k) = \lim_{x\to 0^+} (x^2 + k)$

$\Rightarrow -k = k$

$\Rightarrow k = 0$

16. Since, f(x) is continuous at $x = 1$.

$\therefore$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\Rightarrow \lim_{x\to 1^-} (2x + 1) = \lim_{x\to 1^+} (3 - kx^2)$

$\Rightarrow 2 + 1 = 3 - k(1)^2$

$\Rightarrow k = 0$

17. Since, f(x) is continuous at $x = 3$.

$\therefore$ $f(3) = \lim_{x\to 3^-} f(x)$

$\Rightarrow 4 = \lim_{h\to 0} f(3 - h) \Rightarrow 4 = \lim_{h\to 0} (3 - h + \lambda)$

$\Rightarrow 3 + \lambda = 4 \Rightarrow \lambda = 1$

18. Since, f(x) is continuous at $x = 2$.

$\therefore$ $f(2) = \lim_{x\to 2^-} f(x)$

$\Rightarrow f(2) = \lim_{x\to 2} \left(\frac{x^2 - 4}{x - 2} + a\right) \Rightarrow 8 = 4 + a$

$\Rightarrow a = 4$

Also, $f(2) = \lim_{x\to 2^+} f(x)$

$\Rightarrow f(2) = \lim_{x\to 2} (x + b + 4) \Rightarrow 8 = 6 + b$

$\Rightarrow b = 2$

19. Since, f(x) is continuous at $x = \frac{\pi}{2}$.

$\therefore$ $\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}^+} f(x)$

$\Rightarrow \lim_{x\to \frac{\pi}{2}} (ax + 1) = \lim_{x\to \frac{\pi}{2}} (\sin x + b)$

$\Rightarrow a.\frac{\pi}{2} + 1 = 1 + b \qquad \Rightarrow b = \frac{a\pi}{2}$

20. $\lim_{x\to 1} f(x) = \lim_{x\to 1} x^2 = 1$ and $f(1) = 2$

$\therefore$ f(x) is discontinuous at $x = 1$.

21. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1} x^2 = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(x + 5) = 6$

$\therefore$ f(x) is discontinuous at $x = 1$.

22. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x + 1) = 1 + 1 = 2$

$\therefore$ $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

$\therefore$ f(x) is discontinuous at $x = 1$.

23. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} (x - 1) = -1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} x^2 = 0$

$\therefore$ $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\therefore$ f(x) is discontinuous at $x = 0$.

24. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2}\left(\frac{5}{2} - x\right) = \frac{1}{2}$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}\left(x - \frac{3}{2}\right) = \frac{1}{2}$ and $f(2) = 1$

$\therefore$ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \neq f(2)$

$\therefore$ f(x) is discontinuous at $x = 2$.

25. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(1 - x) = 0$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(1 + x^2) = 1 + 1^2 = 2$

$\therefore$ $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$

$\therefore$ f(x) is discontinuous at $x = 1$.

26. $\lim_{y\to 2^-} f(y) = \lim_{y\to 2^-} (y^2 - y - 1) = 4 - 2 - 1 = 1$

$\lim_{y\to 2^+} f(y) = \lim_{y\to 2^+} (4y + 1) = 8 + 1 = 9$

$\therefore$ $\lim_{y\to 2^-} f(y) \neq \lim_{y\to 2^+} f(y)$

$\therefore$ f(y) is discontinuous at $y = 2$.

28. $\lim_{x\to 3} f(x) = \lim_{x\to 3} \sqrt{x - 2} = 1$

$f(3) = \sqrt{3 - 2} = 1$

$\therefore$ $\lim_{x\to 3} f(x) = f(3)$

$\therefore$ f(x) is continuous at $x = 3$.

Since, $3 \in (2, 4)$

$\therefore$ f(x) is continuous in (2, 4).

29. For $x > 0$, $f(x) = x$
Since f is a polynomial function, it is continuous for all $x > 0$.
For $x < 0$, $f(x) = x^2$
Since f is a polynomial function, it is continuous for all $x < 0$.
$\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0} x^2 = 0$
$\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} x = 0$
$f(0) = 0$
$\therefore$ $f(x)$ is continuous at $x = 0$.
$\therefore$ $f(x)$ is continuous on R.

30. $f(x)$ being a rational function, is continuous in [0, 1] except at those points where the denominator $(x - 2)(x - 5) = 0$
i.e., when $x = 2$ or $x = 5$
Since $2, 5 \notin [0, 1]$
$\therefore$ $f(x)$ is continuous in [0, 1].

31. For $x < 2$, $f(x) = x - 1$
Since f is a polynomial function,
it is continuous for all $x < 2$.
For $x > 2$, $f(x) = 2x - 3$
Since f is a polynomial function,
it is continuous for all $x > 2$.
$\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2} (x - 1) = 1$
$\lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2} (2x - 3) = 1$
$f(2) = 1$
$\therefore$ $f(x)$ is continuous for all real values of x.

32. Since, $f(x)$ is continuous in [0, 3].
$\therefore$ it is continuous at $x = 2$.
$\therefore$ $\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^+} f(x)$
$\Rightarrow \lim\limits_{x\to2^-} (3x - 4) = \lim\limits_{x\to2^+} (2x + k)$
$\Rightarrow 3(2) - 4 = 2(2) + k$
$\Rightarrow 2 = 4 + k \Rightarrow k = -2$

33. Since, $f(x)$ is continuous in $[-2, 2]$.
$\therefore$ it is continuous at $x = 0$ and $x = 1$.
$\therefore$ $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^+} f(x)$
$\Rightarrow \lim\limits_{x\to0^-} (x + a) = \lim\limits_{x\to0^+} x$
$\Rightarrow a = 0$
Also, $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x)$
$\Rightarrow \lim\limits_{x\to1^-} x = \lim\limits_{x\to1^+} (b - x)$
$\Rightarrow 1 = b - 1$
$\Rightarrow b = 2$

34. Since, $f(x)$ is continuous on $[-4, 2]$.
$\therefore$ it is continuous at $x = -2$.
$\therefore$ $\lim\limits_{x\to-2^-} f(x) = \lim\limits_{x\to-2^+} f(x)$
$\Rightarrow \lim\limits_{x\to-2^-} (6b - 3ax) = \lim\limits_{x\to-2^+} (4x + 1)$
$\Rightarrow 6b - 3a(-2) = 4(-2) + 1$
$\Rightarrow 6b + 6a = -7$
$\Rightarrow a + b = -\dfrac{7}{6}$

Critical Thinking

1. Since, $f(x)$ is continuous at $x = 5$.
$\therefore$ $f(5) = \lim\limits_{x\to5} f(x)$
$$= \lim_{x\to5} \frac{x^2 - 10x + 25}{x^2 - 7x + 10}$$
$$= \lim_{x\to5} \frac{(x-5)^2}{(x-2)(x-5)}$$
$$= \frac{5-5}{5-2} = 0$$

2. Since, $f(x)$ is continuous at $x = \dfrac{1}{2}$.
$\therefore$ $f\left(\dfrac{1}{2}\right) = \lim\limits_{x\to\frac{1}{2}} f(x)$
$$\Rightarrow k = \lim_{x\to\frac{1}{2}} \frac{x^6 - \frac{1}{64}}{x^3 - \frac{1}{8}}$$
Applying L'Hospital rule on R.H.S., we get
$$k = \lim_{x\to\frac{1}{2}} \frac{6x^5}{3x^2} = \lim_{x\to\frac{1}{2}} 2x^3 = 2\left(\frac{1}{2}\right)^3 = \frac{1}{4}$$

3. $\lim\limits_{x\to0} f(x) = \sin^{-1}(0) = 0 = f(0)$
$\therefore$ $f(x)$ is continuous at $x = 0$.

4. $\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0} x^2 \sin\dfrac{1}{x}$, but $-1 \le \sin\dfrac{1}{x} \le 1$ and $x \to 0$
$\therefore$ $\lim\limits_{x\to0^+} f(x) = 0 = \lim\limits_{x\to0^-} f(x) = f(0)$
$\therefore$ $f(x)$ is continuous at $x = 0$.

5. Since, $f(x)$ is continuous at $x = 0$.
$\therefore$ $\lim\limits_{x\to0} f(x) = f(0)$
$\therefore$ $\lim\limits_{x\to0} x^a \sin\dfrac{1}{x} = 0$, if $a > 0$

6. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} \frac{-h}{e^{\frac{-1}{h}}+1} = \lim_{h\to 0}\frac{-h}{1+\frac{1}{e^{\frac{1}{h}}}} = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{h}{e^{\frac{1}{h}}+1} = 0$$

$\therefore$ $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$.

7. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x} = 2 \neq f(0)$

$\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e \neq f(0)$

$\lim\limits_{x\to 0} e^{\frac{-1}{x}} = \lim\limits_{x\to 0}\dfrac{1}{e^{\frac{1}{x}}} = \dfrac{1}{e^{\infty}} = \dfrac{1}{\infty} = 0 \neq f(0)$

$\lim\limits_{x\to 0}\left(\dfrac{3x}{x}+\dfrac{4\tan x}{x}\right) = 3 + 4 = 7 = f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$.

8. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 5^{\frac{1}{x}} = \lim\limits_{h\to 0} 5^{-\frac{1}{h}} = 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \lambda[x] = 0$, for all $\lambda \in R$

$f(0) = \lambda(0) = 0$

$\therefore$ f is continuous at $x = 0$, whatever λ may be.

9. Since, $f(x)$ is continuous at $x = a$.

$\therefore$ $f(a) = \lim\limits_{x\to a} f(x)$

$$= \lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}\times\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$$

$$= \lim_{x\to a}\left(\sqrt{x}+\sqrt{a}\right) = \sqrt{a} + \sqrt{a} = 2\sqrt{a}$$

10. Since, $f(x)$ is continuous at $x = 1$.

$\therefore$ $f(1) = \lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}\dfrac{\sqrt{x+3}-2}{x^3-1}$

$$= \lim_{x\to 1}\frac{\sqrt{x+3}-2}{x^3-1^3}\times\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$= \lim_{x\to 1}\frac{x-1}{(x-1)\left(x^2+x+1\right)\left(\sqrt{x+3}+2\right)} = \frac{1}{3(4)} = \frac{1}{12}$$

11. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}$

By rationalising, we get

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\frac{2kx}{x\left(\sqrt{1+kx}+\sqrt{1-kx}\right)}$$

$$= 2\lim_{x\to 0}\frac{k}{\sqrt{1+kx}+\sqrt{1-kx}} = k$$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0}(2x^2+3x-2) = -2$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$

$\Rightarrow k = -2$

12. Since, $f(x)$ is continuous at $x = 4$.

$\therefore$ $f(4) = \lim\limits_{x\to 4} f(x)$

$$= \lim_{x\to 4}\frac{x^4-64x}{\sqrt{x^2+9}-5}$$

$$= \lim_{x\to 4}\frac{x\left(x^3-64\right)\left(\sqrt{x^2+9}+5\right)}{\left(x^2+9\right)-25}$$

$$= \left(\lim_{x\to 4}\frac{x^3-4^3}{x^2-4^2}\right)\left[\lim_{x\to 4} x\left(\sqrt{x^2+9}+5\right)\right]$$

$$= \frac{3}{2}(4)\left[4\left(\sqrt{16+9}+5\right)\right]$$

$= 240$

13. Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\dfrac{\tan(x^2-x)}{x}$

$$= \lim_{x\to 0}\frac{\tan[x(x-1)]}{x(x-1)}\times(x-1) = 1\times(-1) = -1$$

14. Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x)$

$\Rightarrow k = \lim\limits_{x\to 0}\dfrac{1-\cos 4x}{8x^2} = \lim\limits_{x\to 0}\dfrac{2\sin^2 2x}{8x^2}$

$\Rightarrow k = \lim\limits_{x\to 0}\dfrac{\sin^2 2x}{4x^2} = 1$

15. Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to 0^-} f(x)$

$\Rightarrow a = \lim\limits_{x\to 0}\dfrac{1-\cos 4x}{x^2}$

$$= \lim_{x\to 0}\frac{2\sin^2 2x}{x^2}$$

$$= 2\lim_{x\to 0}\frac{\sin^2 2x}{(2x)^2}\times 4 = 2\times 4 = 8$$

16. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to0} f(x)$

$$\Rightarrow k = \lim_{x\to0} \frac{1-\cos 3x}{x\tan x}$$

$$\Rightarrow k = \lim_{x\to0} \frac{1-\cos 3x}{x^2} \times \frac{1}{\frac{\tan x}{x}}$$

$$\Rightarrow k = \frac{3^2}{2} \times 1 \qquad \ldots\left[\because \lim_{x\to0}\left(\frac{1-\cos kx}{x^2}\right) = \frac{k^2}{2}\right]$$

$$\Rightarrow k = \frac{9}{2}$$

17. Since, f(x) is continuous at $x = \frac{\pi}{2}$.

$\therefore$ $f\left(\frac{\pi}{2}\right) = \lim\limits_{x\to\frac{\pi}{2}} f(x)$

$$\Rightarrow 3 = \lim_{x\to\frac{\pi}{2}} \left(\frac{k\cos x}{\pi - 2x}\right)$$

Applying L'Hospital rule on R.H.S., we get

$$3 = \lim_{x\to\frac{\pi}{2}} \frac{k(-\sin x)}{-2}$$

$$\Rightarrow 3 = \frac{k}{2} \quad \Rightarrow k = 6$$

18. Since, f(x) is continuous at $x = \frac{\pi}{4}$.

$\therefore$ $f\left(\frac{\pi}{4}\right) = \lim\limits_{x\to\frac{\pi}{4}} f(x)$

$$\Rightarrow k = \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos 2x}$$

$$\Rightarrow k = \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos^2 x - \sin^2 x}$$

$$\Rightarrow k = \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{(\cos x - \sin x)(\cos x + \sin x)}$$

$$\Rightarrow k = \lim_{x\to\frac{\pi}{4}} \frac{1}{\cos x + \sin x} = \frac{1}{\sqrt{2}}$$

19. Since, f(x) is continuous $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to0} f(x)$

$$\Rightarrow \lambda = \lim_{x\to0} \frac{\cos 3x - \cos x}{x^2}$$

Applying L'Hospital rule on R.H.S., we get

$$\lambda = \lim_{x\to0} \frac{-3\sin 3x + \sin x}{2x}$$

Applying L'Hospital rule on R.H.S., we get

$$\lambda = \lim_{x\to0} \frac{-9\cos 3x + \cos x}{2} \quad \Rightarrow \lambda = \frac{-9+1}{2} = -4$$

20. Since, f(x) is continuous at $x = \frac{\pi}{4}$.

$\therefore$ $f\left(\frac{\pi}{4}\right) = \lim\limits_{x\to\frac{\pi}{4}} f(x)$

$$\Rightarrow k = \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x}$$

Applying L'Hospital rule on R.H.S., we get

$$k = \lim_{x\to\frac{\pi}{4}} \frac{-\sec^2 x}{-\sqrt{2}\cos x} = \frac{-2}{-1} = 2$$

21. Since, f(x) is continuous at $x = \frac{\pi}{6}$,

$\therefore$ $\lim\limits_{x\to\frac{\pi}{6}} f(x) = f\left(\frac{\pi}{6}\right)$

$$\Rightarrow \lim_{x\to\frac{\pi}{6}} \frac{3\sin x - \sqrt{3}\cos x}{6x - \pi} = a$$

Applying L'Hospital rule to L.H.S, we get

$$\lim_{x\to\frac{\pi}{6}} \frac{3\cos x + \sqrt{3}\sin x}{6} = a$$

$$\Rightarrow \frac{3\left(\frac{\sqrt{3}}{2}\right) + \sqrt{3}\left(\frac{1}{2}\right)}{6} = a$$

$$\Rightarrow \frac{4\sqrt{3}}{12} = a \Rightarrow a = \frac{1}{\sqrt{3}}$$

22. Since, f(x) is continuous at $x = \frac{\pi}{2}$.

$\therefore$ $f\left(\frac{\pi}{2}\right) = \lim\limits_{x\to\frac{\pi}{2}} f(x)$

$$\Rightarrow \lambda = \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{(\pi - 2x)^2}$$

Applying L'Hospital rule on R.H.S., we get

$$\lambda = \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{-4(\pi - 2x)}$$

Applying L'Hospital rule on R.H.S., we get

$$\lambda = \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{-4(-2)} \qquad \Rightarrow \quad \lambda = \frac{1}{8}$$

23. For f(x) to be continuous at $x = 0$,

$f(0) = \lim_{x\to 0} f(x)$

$\Rightarrow f(0) = \lim_{x\to 0} \frac{(a+x)^2 \sin(a+x) - a^2 \sin a}{x}$

Applying L'Hospital rule on R.H.S., we get

$f(0) = \lim_{x\to 0} \frac{2(a+x)\sin(a+x) + (a+x)^2 \cos(a+x)}{1}$

$\Rightarrow f(0) = 2a \sin a + a^2 \cos a$

24. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{2-\sqrt{x+4}}{\sin 2x}$

Applying L'Hospital rule on R.H.S., we get

$f(0) = \lim_{x\to 0} \frac{\left(-\frac{1}{2\sqrt{x+4}}\right)}{2\cos 2x} = -\frac{1}{8}$

25. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(27-2x)^{\frac{1}{3}} - 3}{9 - 3(243+5x)^{\frac{1}{5}}}$

Applying L'Hospital rule on R.H.S., we get

$f(0) = \lim_{x\to 0} \frac{\frac{1}{3}(27-2x)^{\frac{-2}{3}}(-2)}{-\frac{3}{5}(243+5x)^{\frac{-4}{5}}(5)} = 2$

26. For f(x) to be continuous at $x = \frac{\pi}{2}$,

$f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}} f(x)$

$= \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$

$= \lim_{x\to\frac{\pi}{2}} \frac{2-(1+\sin x)}{(1-\sin^2 x)\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$

$= \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{(1-\sin x)(1+\sin x)\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$

$= \lim_{x\to\frac{\pi}{2}} \frac{1}{(1+\sin x)\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$

$= \frac{1}{(1+1)\left(\sqrt{2}+\sqrt{1+1}\right)} = \frac{1}{4\sqrt{2}}$

27. For f(x) to be continuous at $x = 0$,

$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$

Applying L'Hospital rule on R.H.S., we get

$f(0) = \lim_{x\to 0} \frac{\frac{\cos x}{2\sqrt{1+\sin x}} + \frac{\cos x}{2\sqrt{1-\sin x}}}{1}$

$= \frac{1}{2}(1+1) = 1$

28. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$\Rightarrow f(0) = \lim_{x\to 0} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1} - 1}$

$= \lim_{x\to 0} \frac{(\cos^2 x - 1) - \sin^2 x}{\sqrt{x^2+1}-1} \times \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}$

$= \lim_{x\to 0} \frac{\left(-\sin^2 x - \sin^2 x\right)\left(\sqrt{x^2+1}+1\right)}{x^2+1-1}$

$= \lim_{x\to 0} \frac{-2\sin^2 x}{x^2} \times \left(\sqrt{x^2+1}+1\right)$

$= -2(1)^2\left(\sqrt{0^2+1}+1\right) = -4$

29. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{2^x - 2^{-x}}{x}\right)$

Applying L'Hospital rule on R.H.S., we get

$f(0) = \lim_{x\to 0}\left[\frac{(2^x + 2^{-x})\log_e 2}{1}\right]$

$= (2^0 + 2^0)\log_e 2$

$\therefore$ $f(0) = 2\log_e 2 = \log_e 4$

30. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{3^x + 3^{-x} - 2}{x^2}$

$= \lim_{x\to 0} \frac{\frac{(3^x-1)^2}{x^2}}{3^x}$

$= \frac{(\log 3)^2}{1} = (\log 3)^2$

31. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$\Rightarrow 2 = \lim_{x\to 0} \frac{8^x - 2^x}{k^x - 1}$

$\Rightarrow 2 = \lim_{x\to 0} \frac{2^x\left(\frac{4^x-1}{x}\right)}{\frac{k^x-1}{x}}$ $\Rightarrow 2 = \frac{2^0 \log 4}{\log k}$

$\Rightarrow 2\log k = \log 4$

$\Rightarrow 2\log k = 2\log 2$

$\Rightarrow k = 2$

32. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to0} f(x)$

$$= \lim_{x\to0} \frac{(e^{3x}-1)\sin x^\circ}{x^2}$$

$$= \lim_{x\to0} \frac{e^{3x}-1}{3x} \times 3 \times \frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}} \times \frac{\pi}{180}$$

$$= 1 \times 3 \times 1 \times \frac{\pi}{180} = \frac{\pi}{60}$$

33. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to0} f(x)$

$$\Rightarrow \frac{k}{2} = \lim_{x\to0} \frac{e^{5x}-e^{2x}}{\sin 3x}$$

Applying L'Hospital rule on R.H.S., we get

$$\frac{k}{2} = \lim_{x\to0} \frac{5e^{5x}-2e^{2x}}{3\cos 3x}$$

$$\Rightarrow \frac{k}{2} = \frac{5e^0-2e^0}{3\cos 0} = \frac{5-2}{3} = 1$$

$\Rightarrow k = 2$

34. For f(x) to be continuous at $x = 0$,

$f(0) = \lim_{x\to0} f(x)$

$$= \lim_{x\to0} \frac{e^x - e^{\sin x}}{2(x-\sin x)}$$

$$= \frac{1}{2}\lim_{x\to0} e^{\sin x}\left(\frac{e^{x-\sin x}-1}{x-\sin x}\right)$$

$$= \frac{1}{2} \times e^0 \times 1 \qquad \ldots\left[\because \lim_{x\to0}\frac{e^x-1}{x} = 1\right]$$

$$= \frac{1}{2}$$

35. For f(x) to be continuous at $x = 0$,

$f(0) = \lim_{x\to0} f(x)$

$$= \lim_{x\to0} (x+1)^{\cot x}$$

$$= \lim_{x\to0} \left\{(1+x)^{\frac{1}{x}}\right\}^{x\cot x}$$

$$= \lim_{x\to0} \left\{(1+x)^{\frac{1}{x}}\right\}^{\lim_{x\to0}\left(\frac{x}{\tan x}\right)}$$

$= e^1 = e$

36. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to0} f(x)$

$$= \lim_{x\to0}\left(\frac{4x+1}{1-4x}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to0}\frac{\left[(1+4x)^{\frac{1}{4x}}\right]^4}{\left[(1-4x)^{-\frac{1}{4x}}\right]^{-4}}$$

$$= \frac{e^4}{e^{-4}} = e^8$$

37. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to0} f(x)$

$$\Rightarrow k = \lim_{x\to0} (\sec^2 x)^{\cot^2 x}$$

$$\Rightarrow k = \lim_{x\to0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}}$$

$\Rightarrow k = e$

38. Since, f(x) is continuous at $x = \frac{\pi}{2}$,

$$\therefore \quad f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}} f(x)$$

$$= \lim_{x\to\frac{\pi}{2}} (\sin x)^{\frac{1}{\pi-2x}}$$

$$= \lim_{x\to\frac{\pi}{2}} \left[1+(\sin x-1)\right]^{\frac{1}{\pi-2x}}$$

$$= e^{\lim_{x\to\frac{\pi}{2}}\left(\frac{\sin x-1}{\pi-2x}\right)}$$

$$= e^{-\frac{1}{2}\lim_{x\to\frac{\pi}{2}}\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)}}$$

$= e^0$

$= 1$

39. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to0} f(x)$

$$\Rightarrow 5 = \lim_{x\to0} \frac{\log(1+kx)}{\sin x}$$

$$\Rightarrow 5 = \lim_{x\to0} \frac{\frac{\log(1+kx)}{kx}\times k}{\frac{\sin x}{x}}$$

$$\Rightarrow 5 = \frac{1\times k}{1} \Rightarrow k = 5$$

40. Since, f(x) is continuous at $x = 7$.

$\therefore$ $f(7) = \lim_{x\to 7} f(x)$

$\Rightarrow k = \lim_{x\to 7} \dfrac{\log x - \log 7}{x-7}$

Applying L'Hospital rule on R.H.S., we get

$k = \lim_{x\to 7} \dfrac{\frac{1}{x}}{1} = \dfrac{1}{7}$

41. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$\Rightarrow k = \lim_{x\to 0} \dfrac{\log(1+2ax) - \log(1-bx)}{x}$

$\Rightarrow k = \lim_{x\to 0}\left[\dfrac{\log(1+2ax)}{2ax}\times 2a + \dfrac{\log(1-bx)}{-bx}\times b\right]$

$\Rightarrow k = 2a + b$

42. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$\Rightarrow k = \lim_{x\to 0} \dfrac{(3^{\sin x}-1)^2}{x\log(1+x)}$

$\Rightarrow k = \lim_{x\to 0} \dfrac{\left(\dfrac{3^{\sin x}-1}{\sin x}\right)^2 \cdot \left(\dfrac{\sin x}{x}\right)^2}{\dfrac{\log(1+x)}{x}}$

$\Rightarrow k = \dfrac{(\log 3)^2 \times (1)^2}{1} = (\log 3)^2$

43. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$= \lim_{x\to 0} \dfrac{\log(\sec^2 x)}{x\sin x}$

$= \lim_{x\to 0} \dfrac{\log(1+\tan^2 x)}{\tan^2 x} \times \dfrac{\tan^2 x}{x\sin x}$

$= \lim_{x\to 0} \dfrac{\log(1+\tan^2 x)}{\tan^2 x} \times \dfrac{\dfrac{\tan^2 x}{x^2}}{\dfrac{\sin x}{x}}$

$= 1 \times \dfrac{1^2}{1} = 1$

44. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$\Rightarrow 12(\log 4)^3 = \lim_{x\to 0} \dfrac{\left(4^x-1\right)^3}{\sin\frac{x}{p}\log\left(1+\frac{x^2}{3}\right)}$

$\Rightarrow 12(\log 4)^3$

$= \lim_{x\to 0} \left(\dfrac{4^x-1}{x}\right)^3 \times \dfrac{\left(\dfrac{x}{p}\right)}{\left(\sin\dfrac{x}{p}\right)} \times \dfrac{p}{\dfrac{\log\left(1+\frac{1}{3}x^2\right)}{\frac{x^2}{3}\times 3}}$

$\Rightarrow 12(\log 4)^3 = (\log 4)^3(1)\left(\dfrac{3p}{1}\right)$

$\Rightarrow p = 4$

45. Since, f(x) is continuous at $x = 3$.

$\therefore$ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

$\lim_{x\to 3^-} f(x) = f(3)$

$\Rightarrow \lim_{x\to 3^-} (2x^2 + 3x + b) = 5$

$\Rightarrow 2(3)^2 + 3(3) + b = 5$

$\Rightarrow b = -22$

Also, $\lim_{x\to 3^+} f(x) = f(3)$

$\Rightarrow \lim_{x\to 3^+}\left(\dfrac{x^2-9}{x-3} + a\right) = 5$

$\Rightarrow (3 + 3 + a) = 5$

$\Rightarrow a = -1$

46. $f\left(\dfrac{1}{2}\right) = \left(\dfrac{1}{2}\right)^2 + a$

$\Rightarrow 2 = \dfrac{1}{4} + a \Rightarrow a = \dfrac{7}{4}$ ....(i)

Since, f(x) is continuous at $x = 0$.

$\therefore$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\Rightarrow \lim_{x\to 0^-}\left(2\sqrt{x^2+1} + b\right) = \lim_{x\to 0^+}(x^2 + a)$

$\Rightarrow 2\sqrt{0+1} + b = 0 + a$

$\Rightarrow 2 + b = \dfrac{7}{4}$ ....[From (i)]

$\Rightarrow b = -\dfrac{1}{4}$

47. $\lim_{x\to 4^-} f(x) = \lim_{h\to 0} f(4-h)$

$= \lim_{h\to 0} \dfrac{4-h-4}{|4-h-4|} + a$

$= \lim_{h\to 0}\left(-\dfrac{h}{h} + a\right) = a - 1$

$\lim\limits_{x\to4^+} f(x) = \lim\limits_{h\to0} f(4+h)$

$= \lim\limits_{h\to0} \dfrac{4+h-4}{|4+h-4|} + b = b+1$

and $f(4) = a + b$

Since, $f(x)$ is continuous at $x = 4$.

$\therefore$ $\lim\limits_{x\to4^-} f(x) = f(4) = \lim\limits_{x\to4^+} f(x)$

$\Rightarrow a - 1 = a + b = b + 1$

$\Rightarrow b = -1$ and $a = 1$

48. $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} \dfrac{\sin(a+1)x + \sin x}{x}$

$= \lim\limits_{x\to0^-}\left[\dfrac{\sin(a+1)x}{(a+1)x}\times(a+1) + \dfrac{\sin x}{x}\right]$

$= a + 1 + 1$

$= a + 2$

$\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{b\sqrt{x}}$

$= \lim\limits_{x\to0^+} \dfrac{\sqrt{x}\left(\sqrt{1+bx}-1\right)}{b\sqrt{x}}$

$= \lim\limits_{x\to0^+} \dfrac{\sqrt{1+bx}-1}{b} = \dfrac{0}{b} = 0$, if $b \neq 0$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^+} f(x) = f(0)$

$\Rightarrow a + 2 = 0 = c$

$\Rightarrow a = -2, c = 0$

$\therefore$ $a = -2$, $b \neq 0$ and $c = 0$

49. $\lim\limits_{x\to0^-} f(x) = \lim\limits_{h\to0} e^{-1/h} = 0$

$\lim\limits_{x\to0^+} f(x) = \lim\limits_{h\to0} e^{1/h} = \infty$

$\therefore$ $\lim\limits_{x\to0^-} f(x) \neq \lim\limits_{x\to0^+} f(x)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

50. $\lim\limits_{x\to2} f(x) = \lim\limits_{x\to2} \dfrac{x^4-16}{x-2}$

$= \lim\limits_{x\to2} \dfrac{(x-2)(x+2)(x^2+4)}{x-2}$

$= \lim\limits_{x\to2} (x+2)(x^2+4) = 32$ and $f(2) = 16$

$\therefore$ $\lim\limits_{x\to2} f(x) \neq f(2)$

$\therefore$ $f(x)$ is discontinuous at $x = 2$.

51. As $\dfrac{|x|}{x}$ is discontinuous at $x = 0$.

$\therefore$ $\dfrac{|3x-4|}{3x-4}$ is discontinuous at $3x - 4 = 0$.

$\therefore$ $x = \dfrac{4}{3}$

52. $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0} |x| = \lim\limits_{x\to0} (-x) = 0$

$\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0} x = 0$

$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1} x = 1$

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1} 1 = 1$

53. When $x < 0$, $|x| = -x$

$\therefore$ $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0} \dfrac{-x}{x} = \lim\limits_{x\to0} (-1) = -1$

When $x > 0$, $|x| = x$

$\therefore$ $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0} \dfrac{x}{x} = \lim\limits_{x\to0} (1) = 1$

$\therefore$ $\lim\limits_{x\to0^-} f(x) \neq \lim\limits_{x\to0^+} f(x)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

54. $\lim\limits_{x\to0^+} \dfrac{\sin x}{|x|} = \lim\limits_{x\to0} \dfrac{\sin x}{x} = 1$

and $\lim\limits_{x\to0^-} \dfrac{\sin x}{|x|} = \lim\limits_{x\to0} \dfrac{\sin x}{-x} = -1$

$\therefore$ the given function is discontinuous at $x = 0$.

55. $\lim\limits_{x\to0^-} f(x) = 1 + 1 = 2$

$\lim\limits_{x\to0^+} f(x) = 0$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

56. $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} \dfrac{|x|}{x^2+2x}$

$= \lim\limits_{x\to0} \dfrac{-x}{x^2+2x}$

$= -\dfrac{1}{2}$

$\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} \dfrac{|x|}{x^2+2x}$

$= \lim\limits_{x\to0} \dfrac{x}{x^2+2x} = \dfrac{1}{2}$

$\therefore$ $\lim\limits_{x\to0} f(x)$ does not exist.

57. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} \dfrac{e^{\frac{-1}{h}}-1}{e^{\frac{-1}{h}}+1} = \lim\limits_{h\to 0} \dfrac{\frac{1}{e^{\frac{1}{h}}}-1}{\frac{1}{e^{\frac{1}{h}}}+1} = \dfrac{0-1}{0+1} = -1$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} \dfrac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1} = \lim\limits_{h\to 0} \dfrac{1-\frac{1}{e^{\frac{1}{h}}}}{1+\frac{1}{e^{\frac{1}{h}}}} = \dfrac{1-0}{1+0} = 1$

∴ $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$

∴ $f(x)$ is not continuous at $x = 0$.

58. $f(x)$ is discontinuous, when $x^2 - 3x + 2 = 0$
i.e., $(x-1)(x-2) = 0 \Rightarrow x = 1, x = 2$

59. $f(x) = \dfrac{x+1}{(x-3)(x+4)}$

∴ $f(x)$ is discontinuous at $x = 3, -4$.

60. $f(x) = \dfrac{4-x^2}{x(4-x^2)} = \dfrac{4-x^2}{x(2+x)(2-x)}$

Since, $f(x)$ does not exist at $x = 0, 2, -2$.

∴ there are three points of discontinuity.

61. $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (4-3x) = 4-6 = -2$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (2x-6) = 4-6 = -2$

$f(2) = 4 - 3(2) = -2$

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (2x-6) = 6-6 = 0$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (x+5) = 3+5 = 8$

∴ $\lim\limits_{x\to 3^-} f(x) \neq \lim\limits_{x\to 3^+} f(x)$

∴ $f(x)$ is continuous at $x = 2$ and discontinuous at $x = 3$.

62. $\lim\limits_{x\to \frac{\pi}{2}^-} f(x) = \lim\limits_{x\to \frac{\pi}{2}} x\sin x = \dfrac{\pi}{2}$

$\lim\limits_{x\to \frac{\pi}{2}^+} f(x) = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\pi}{2}\sin(\pi + x) = \dfrac{-\pi}{2}$

∴ $f(x)$ is discontinuous at $x = \dfrac{\pi}{2}$.

63. $\lim\limits_{y\to 0} f(y) = \lim\limits_{y\to 0} \dfrac{(e^{2y}-1)}{2y}\times 2\times\dfrac{\sin y}{y}$

$= \log e \times 2 \times 1 = 2$

and $f(0) = 4$

∴ $f(y)$ is discontinuous at $y = 0$.

64. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sin^2 ax}{(ax)^2} a^2 = a^2$ and $f(0) = 1$.

∴ $f(x)$ is discontinuous at $x = 0$, when $a \neq \pm 1$

65. Let $f(x) = \tan x$
The point of discontinuity of $f(x)$ are those points where $\tan x$ is infinite.
i.e., $\tan x = \infty$

$\Rightarrow x = (2n+1)\dfrac{\pi}{2}, n \in I$

66. $\lim\limits_{x\to \frac{\pi}{2}^-} f(x) = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}} = \dfrac{\sin\pi}{\sqrt{1-\cos\pi}} = 0$

$\lim\limits_{x\to \frac{\pi}{2}^+} f(x) = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\cos x}{\pi - 2x}$

$= \lim\limits_{h\to 0} \dfrac{\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)}$

$= \lim\limits_{h\to 0} \dfrac{-\sin h}{-2h}$

$= \dfrac{1}{2}(1) = \dfrac{1}{2}$

∴ $\lim\limits_{x\to \frac{\pi}{2}^-} f(x) \neq \lim\limits_{x\to \frac{\pi}{2}^+} f(x)$

∴ $f(x)$ is discontinuous at $x = \dfrac{\pi}{2}$.

67. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{x\cos x - 3\tan x}{x^2 + 2\sin x}$

$= \lim\limits_{x\to 0} \dfrac{\frac{x\cos x - 3\tan x}{x}}{\frac{x^2+2\sin x}{x}}$

$= \lim\limits_{x\to 0} \dfrac{\cos x - \frac{3\tan x}{x}}{x + \frac{2\sin x}{x}}$

$= \dfrac{1-3}{0+2} = -1$

∴ $\lim\limits_{x\to 0} f(x) \neq f(0)$

∴ $f(x)$ is discontinuous at $x = 0$.

68. $\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0}\dfrac{5^x - e^x}{\sin 2x} = \lim\limits_{x\to0}\dfrac{5^x - 1 + 1 - e^x}{\sin 2x}$

$$= \lim_{x\to0}\frac{\frac{5^x-1}{x} - \frac{e^x-1}{x}}{\frac{\sin 2x}{2x}\times 2}$$

$$= \frac{\log 5 - \log e}{2}$$

$$= \frac{1}{2}\left(\log 5 - 1\right)$$

$\therefore$ $\lim\limits_{x\to0} f(x) \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

69. Applying L'Hospital rule, we get

$$\lim_{x\to\frac{\pi}{2}}\frac{5^{\cos x}-1}{\frac{\pi}{2}-x} = \lim_{x\to\frac{\pi}{2}}\frac{5^{\cos x}.\log 5\left(-\sin x\right)}{-1}$$

$$= 5^{\cos\frac{\pi}{2}}.\log 5\sin\frac{\pi}{2}$$

$$= \log 5$$

and $f\left(\dfrac{\pi}{2}\right) = 2\log 5$

$\therefore$ $f(x)$ is discontinuous at $x = \dfrac{\pi}{2}$.

Here, $\lim\limits_{x\to\frac{\pi}{2}} f(x)$ exists but not equal to $f\left(\dfrac{\pi}{2}\right)$.

$\therefore$ the discontinuity at $x = \dfrac{\pi}{2}$ is removable.

70. $\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0}\dfrac{\left(2^x-1\right)^2}{\tan x\ .\ \log(1+x)}$

$$= \lim_{x\to0}\frac{\left(2^x-1\right)^2}{x^2}\times\frac{1}{\frac{\tan x}{x}}\times\frac{1}{\frac{\log\left(1+x\right)}{x}}$$

$$= (\log 2)^2 \times 1 = (\log 2)^2$$

and $f(0) = \log 4$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

Here, $\lim\limits_{x\to0} f(x)$ exists but not equal to $f(0)$.

$\therefore$ the discontinuity at $x = 0$ is removable.

71. Since, $f(x)$ is continuous in $\left(0, \dfrac{\pi}{2}\right)$.

$\therefore$ $f(x)$ is continuous at $x = \dfrac{\pi}{4}$.

$\therefore$ $f\left(\dfrac{\pi}{4}\right) = \lim\limits_{x\to\frac{\pi}{4}} f(x)$

$$\Rightarrow f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}}\frac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}$$

Applying L'Hospital rule on R.H.S., we get

$$f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}}\frac{-\sec^2\left(\frac{\pi}{4}-x\right)}{-2\operatorname{cosec}^2 2x} = \frac{-1}{-2} = \frac{1}{2}$$

72. Since, $f(x)$ is continuous in $[-1, 1]$.

$\therefore$ it is continuous at $x = 0$.

$\therefore$ $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^+} f(x)$

$$\Rightarrow \lim_{x\to0}\frac{\sqrt{1+px}-\sqrt{1-px}}{x} = \lim_{x\to0}\frac{2x+1}{x-2}$$

$$\Rightarrow \lim_{x\to0}\frac{\left(1+px\right)-\left(1-px\right)}{x\left(\sqrt{1+px}+\sqrt{1-px}\right)} = \frac{-1}{2}$$

$$\Rightarrow p = \frac{-1}{2}$$

73. For all $x \in R$, $-1 \leq \sin x \leq 1$

$\therefore$ $f(x)$ is continuous for all real values of x.

74. Since, $f(x)$ is continuous in $[0, 8]$.

$\therefore$ it is continuous at $x = 2$ and $x = 4$.

$\therefore$ $\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^+} f(x)$

$\Rightarrow \lim\limits_{x\to2^-}(x^2 + ax + 6) = \lim\limits_{x\to2^+}(3x + 2)$

$\Rightarrow (2)^2 + 2a + 6 = 3(2) + 2$

$\Rightarrow 10 + 2a = 8$

$\Rightarrow a = -1$ ….(i)

Also, $\lim\limits_{x\to4^-} f(x) = \lim\limits_{x\to4^+} f(x)$

$\Rightarrow \lim\limits_{x\to4^-}(3x + 2) = \lim\limits_{x\to4^+}(2ax + 5b)$

$\Rightarrow 3(4) + 2 = 2a(4) + 5b$

$\Rightarrow 14 = 8a + 5b$

$\Rightarrow b = \dfrac{22}{5}$ ....[From (i)]

75. Since, $f(x)$ is continuous in $[-2, 2]$.

$\therefore$ it is continuous at $x = 0$ and $x = 1$.

$\therefore$ $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^+} f(x)$

$$\Rightarrow \lim_{x\to0^-}\left(\frac{\sin ax}{x}-2\right) = \lim_{x\to0^+}(2x+1)$$

$\Rightarrow a - 2 = 0 + 1 \Rightarrow a = 3$

Also, $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x)$

$\Rightarrow \lim\limits_{x\to1^-} (2x+1) = \lim\limits_{x\to1^+} \left(2b\sqrt{x^2+3}-1\right)$

$\Rightarrow 2(1)+1 = 2b\sqrt{1+3}-1$

$\Rightarrow 3 = 4b - 1$

$\Rightarrow b = 1$

$\therefore$ $a + b = 3 + 1 = 4$

76. Since, $f(x)$ is continuous on its domain.

$\therefore$ it is continuous at $x = 2$ and $x = 9$.

$\therefore$ $\lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2^-} f(x)$

$\Rightarrow \lim\limits_{x\to2^+} (ax+b) = 7$

$\Rightarrow 2a + b = 7$ ….(i)

Also, $\lim\limits_{x\to9^-} f(x) = \lim\limits_{x\to9^+} f(x)$

$\Rightarrow \lim\limits_{x\to9^-} (ax+b) = 21$

$\Rightarrow 9a + b = 21$ ….(ii)

Solving (i) and (ii), we get a = 2, b = 3

77. $f(x)$ is continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ except at $x = 0$.

For $f(x)$ to be continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$f(0) = \lim\limits_{x\to0} f(x)$

$\Rightarrow f(0) = \lim\limits_{x\to0} \dfrac{e^x + e^{-x} - 2}{x\sin x}$

Applying L'Hospital rule on R.H.S., we get

$f(0) = \lim\limits_{x\to0} \dfrac{e^x - e^{-x}}{x\cos x + \sin x}$

Applying L'Hospital rule on R.H.S., we get

$f(0) = \lim\limits_{x\to0} \dfrac{e^x + e^{-x}}{-x\sin x + \cos x + \cos x}$

$= \dfrac{e^0 + e^0}{0 + 2\cos 0} = \dfrac{1+1}{2(1)} = 1$

78. $f(x) = \dfrac{(x-1)(x+1)(x-2)(x+2)}{|x-1||x-2|}$

Since, $\lim\limits_{x\to1} \dfrac{x-1}{|x-1|}$ does not exist.

Also, $\lim\limits_{x\to2} \dfrac{x-2}{|x-2|}$ does not exist

$\therefore$ $f(x)$ is discontinuous at $x = 1, 2$.

For any $x \neq 1, 2$, $f(x)$ is the quotient of two polynomials and a polynomial is everywhere continuous. Therefore, $f(x)$ is continuous for all $x \neq 1, 2$.

$\therefore$ $f(x)$ is continuous on $R - \{1, 2\}$.

79. Since, $f(x)$ is continuous in $[0, \pi]$.

$\therefore$ it is continuous at $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

$\therefore$ $\lim\limits_{x\to\left(\frac{\pi}{4}\right)^-} f(x) = \lim\limits_{x\to\left(\frac{\pi}{4}\right)^+} f(x)$

$\Rightarrow \lim\limits_{x\to\left(\frac{\pi}{4}\right)^-} (x + a\sqrt{2}\sin x) = \lim\limits_{x\to\left(\frac{\pi}{4}\right)^+} (2x\cot x + b)$

$\Rightarrow \frac{\pi}{4} + a\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2\left(\frac{\pi}{4}\right)(1) + b$

$\Rightarrow \frac{\pi}{4} + a = \frac{\pi}{2} + b$

$\Rightarrow a - b = \frac{\pi}{4}$ ….(i)

Also, $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} f(x) = \lim\limits_{x\to\left(\frac{\pi}{2}\right)^+} f(x)$

$\Rightarrow \lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} (2x\cot x + b) = \lim\limits_{x\to\left(\frac{\pi}{2}\right)^+} (a\cos 2x - b\sin x)$

$\Rightarrow 2\left(\frac{\pi}{2}\right)(0) + b = a(-1) - b(1)$

$\Rightarrow b = -a - b$

$\Rightarrow a + 2b = 0$ ....(ii)

From (i) and (ii), we get

$a = \frac{\pi}{6}$ and $b = \frac{-\pi}{12}$

80. Since, $f(x)$ is continuous in $(-\infty, 6)$.

$\therefore$ it is continuous at $x = 1$ and $x = 3$.

$\therefore$ $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x)$

$\Rightarrow \lim\limits_{x\to1^-} \left(1 + \sin\frac{\pi x}{2}\right) = \lim\limits_{x\to1^+} (ax+b)$

$\Rightarrow 1 + \sin\frac{\pi}{2} = a + b$

$\Rightarrow a + b = 2$ .....(i)

Also, $\lim\limits_{x\to3^-} f(x) = \lim\limits_{x\to3^+} f(x)$

$\Rightarrow \lim\limits_{x\to3^-} (ax+b) = \lim\limits_{x\to3^+} \left(6\tan\frac{\pi x}{12}\right)$

$\Rightarrow 3a + b = 6\tan\frac{3\pi}{12}$

$\Rightarrow 3a + b = 6$ .....(ii)

From (i) and (ii), we get a = 2, b = 0

Competitive Thinking

1. $f(2) = k(2)^2$
$= 4k$
$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} 3 = 3$
Since the function is continous at $x = 2$,
$\lim\limits_{x\to 2^+} f(x) = f(2)$
$\Rightarrow 4k = 3$
$\Rightarrow k = \dfrac{3}{4}$

2. Since, f(x) is continuous at x = a.
$\therefore$ $f(a) = \lim\limits_{x\to a} f(x)$
$\Rightarrow b = \lim\limits_{x\to a} \dfrac{x^3 - a^3}{x - a}$
$\Rightarrow b = 3a^{3-1} = 3a^2$

3. Since, f(x) is continuous at x = 2.
$\therefore$ $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(2)$
$\Rightarrow \lim\limits_{x\to 2^-} (x^2 - 1) = \lim\limits_{x\to 2^+} (2x - 1) = k$
$\Rightarrow 3 = 3 = k \quad \Rightarrow k = 3$

4. $\lim\limits_{x\to 5^-} f(x) = \lim\limits_{x\to 5} (3x - 8) = 7$
$\lim\limits_{x\to 5^+} f(x) = \lim\limits_{x\to 5} 2k = 2k$
Since, f(x) is continuous at x = 5.
$\therefore$ $\lim\limits_{x\to 5^-} f(x) = \lim\limits_{x\to 5^+} f(x)$
$\Rightarrow 7 = 2k \Rightarrow k = \dfrac{7}{2}$

5. Since, f(x) is continuous at x = 0,
$\therefore$ $f(0) = \lim\limits_{x\to 0^-} f(x)$
$\Rightarrow 0^2 + \alpha = \lim\limits_{x\to 0} 2\sqrt{x^2+1} + \beta$
$\Rightarrow \alpha = 2 + \beta$
$\Rightarrow \beta = \alpha - 2$
Also, $f\left(\dfrac{1}{2}\right) = 2$
$\Rightarrow \left(\dfrac{1}{2}\right)^2 + \alpha = 2$
$\Rightarrow \dfrac{1}{4} + \alpha = 2 \qquad \Rightarrow \alpha = \dfrac{7}{4}$
$\beta = \alpha - 2 = \dfrac{7}{4} - 2 = -\dfrac{1}{4}$
$\therefore$ $\alpha^2 + \beta^2 = \left(\dfrac{7}{4}\right)^2 + \left(-\dfrac{1}{4}\right)^2 = \dfrac{50}{16} = \dfrac{25}{8}$

6. For f(x) to be continuous at x = 0,
$f(0) = \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} (1 + x)^{\frac{1}{x}} = e$

7. Since, f(x) is continuous at x = – 5.
$\therefore$ $f(-5) = \lim\limits_{x\to -5} f(x)$
$\Rightarrow a = \lim\limits_{x\to -5} \dfrac{x^2 + 3x - 10}{x^2 + 2x - 15}$
$\Rightarrow a = \lim\limits_{x\to -5} \dfrac{(x-2)(x+5)}{(x+5)(x-3)}$
$\Rightarrow a = \lim\limits_{x\to -5} \dfrac{x-2}{x-3} = \dfrac{7}{8}$

8. Since, f(x) is continuous at x = 3.
$\therefore$ $f(3) = \lim\limits_{x\to 3} f(x)$
$\Rightarrow 2(3) + k = \lim\limits_{x\to 3} \dfrac{x^2 - 9}{x - 3}$
$\Rightarrow 6 + k = \lim\limits_{x\to 3} \dfrac{(x+3)(x-3)}{x-3}$
$\Rightarrow 6 + k = \lim\limits_{x\to 3} (x+3)$
$\Rightarrow 6 + k = 6 \Rightarrow k = 0$

9. Since, f(x) is continuous at x = 0,
$\therefore$ $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$
$\Rightarrow \lim\limits_{x\to 0} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x} = \lim\limits_{x\to 0} \dfrac{2x+1}{x-1}$
Applying L'Hospital rule on L.H.S, we get
$\lim\limits_{x\to 0} \dfrac{\dfrac{k}{2\sqrt{1+kx}} - \dfrac{(-k)}{2\sqrt{1-kx}}}{1} = -1$
$\Rightarrow \dfrac{k}{2} + \dfrac{k}{2} = -1 \qquad \Rightarrow k = -1$

10. Since, f(x) is continuous at x = 0.
$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x)$
$\Rightarrow k = \lim\limits_{x\to 0} \dfrac{\sin 2x}{5x}$
$\Rightarrow k = \lim\limits_{x\to 0} \dfrac{\sin 2x}{2x} \times \dfrac{2}{5}$
$\Rightarrow k = \dfrac{2}{5}$

11. Since, f(x) is continuous at x = 0.
$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x)$
$\Rightarrow 2k = \lim\limits_{x\to 0} \dfrac{3\sin \pi x}{5x} = \lim\limits_{x\to 0} \dfrac{3\sin \pi x}{5(\pi x)} \times \pi = \dfrac{3\pi}{5}$
$\therefore$ $k = \dfrac{3\pi}{10}$

12. Here, f(2) = 0

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{h \to 0} f(2 - h) = \lim\limits_{h \to 0} |2 - h - 2| = 0$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{h \to 0} f(2 + h) = \lim\limits_{h \to 0} |2 + h - 2| = 0$

$\therefore$ f(x) is continuous at $x = 2$.

13. Here, f(b) = 0

$\lim\limits_{x \to b^-} f(x) = \lim\limits_{h \to 0} f(b - h) = \lim\limits_{h \to 0} |b - h - b| = 0$

$\lim\limits_{x \to b^+} f(x) = \lim\limits_{h \to 0} f(b + h) = \lim\limits_{h \to 0} |b + h - b| = 0$

$\therefore$ f(x) is continuous at x = b.

14. Here, $f\left(\frac{3\pi}{4}\right) = 1$ and $\lim\limits_{x \to \frac{3\pi}{4}^-} f(x) = 1$

$$\lim\limits_{x \to \frac{3\pi}{4}^+} f(x) = \lim\limits_{h \to 0} 2\sin\frac{2}{9}\left(\frac{3\pi}{4} + h\right)$$

$$= 2\sin\frac{\pi}{6} = 1$$

$\therefore$ f(x) is continuous at $x = \frac{3\pi}{4}$.

15. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x \to 0} f(x)$

$$= \lim\limits_{x \to 0} \frac{1 - \cos x}{x} = \lim\limits_{x \to 0} \frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}} \times \frac{x}{4}$$

$\therefore$ f(0) = 2(1)(0) = 0

Alternate method:

Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x \to 0} f(x)$

$\Rightarrow f(0) = \lim\limits_{x \to 0} \frac{1 - \cos x}{x}$

Applying L'Hospital rule on R.H.S., we get

$f(0) = \lim\limits_{x \to 0} \sin x = 0$

16. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x \to 0} f(x)$

$\Rightarrow k = \lim\limits_{x \to 0} (\cos x)^{\frac{1}{x}}$

$\Rightarrow \log k = \lim\limits_{x \to 0} \frac{1}{x} \log(\cos x)$

Applying L'Hospital rule on R.H.S., we get

$$\log k = \lim\limits_{x \to 0} \frac{-\frac{\sin x}{\cos x}}{1}$$

$\Rightarrow \log k = 0$

$\Rightarrow k = e^0 = 1$

17. Since, f(x) is continuous at $x = 1$.

$\therefore$ $f(1) = \lim\limits_{x \to 1} f(x)$

$\Rightarrow k = \lim\limits_{x \to 1} \frac{\log x}{x - 1}$

Applying L'Hospital rule on R.H.S., we get

$$k = \lim\limits_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

18. For f(x) to be continuous at $x = \pi$,

$$f(\pi) = \lim\limits_{x \to \pi} f(x) = \lim\limits_{x \to \pi} \frac{1 - \sin x + \cos x}{1 + \sin x + \cos x}$$

Applying L'Hospital rule on R.H.S., we get

$$f(\pi) = \lim\limits_{x \to \pi} \frac{-\cos x - \sin x}{\cos x - \sin x}$$

$\Rightarrow f(\pi) = -1$

19. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x \to 0} f(x)$

$$\Rightarrow 4 = \lim\limits_{x \to 0} \frac{\left(e^{kx} - 1\right)^2 \sin x}{x^3}$$

$$\Rightarrow 4 = \lim\limits_{x \to 0} \left[k^2 \times \frac{\left(e^{kx} - 1\right)^2}{k^2 x^2} \cdot \frac{\sin x}{x}\right]$$

$\Rightarrow 4 = k^2$

$\Rightarrow k = \pm 2$

20. For f(x) to be continuous at $x = 0$,

$f(0) = \lim\limits_{x \to 0} f(x)$

$$= \lim\limits_{x \to 0} \frac{e^{x^2} - \cos x}{x^2}$$

$$= \lim\limits_{x \to 0} \frac{e^{x^2} - 1 + 1 - \cos x}{x^2}$$

$$= \lim\limits_{x \to 0} \frac{e^{x^2} - 1}{x^2} + \lim\limits_{x \to 0} \frac{1 - \cos x}{x^2}$$

$$= 1 + \frac{1}{2} = \frac{3}{2}$$

21. For f(x) to be continuous at $x = 0$,

$f(0) = \lim\limits_{x \to 0} f(x)$

$$\Rightarrow k = \lim\limits_{x \to 0} \frac{\log(1 + 2x) \sin x^\circ}{x^2}$$

$$\Rightarrow k = \lim\limits_{x \to 0} \frac{\log(1 + 2x)}{2x} \times 2 \times \frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}} \times \frac{\pi}{180}$$

$$\Rightarrow k = 1 \times 2 \times 1 \times \frac{\pi}{180} = \frac{\pi}{90}$$

22. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to0} f(x)$

$\Rightarrow k = \lim\limits_{x\to0} \log_{(1-3x)}(1+3x)$

$\Rightarrow k = \lim\limits_{x\to0} \dfrac{\log(1+3x)}{\log(1-3x)}$

$\Rightarrow k = \dfrac{\lim\limits_{x\to0} \dfrac{\log(1+3x)}{3x} \times 3}{\lim\limits_{x\to0} \dfrac{\log(1-3x)}{-3x} \times -3}$

$\Rightarrow k = -1$

23. Since the function is continuous at $x = 0$,

$\lim\limits_{x\to0} f(0) = f(x)$

$\therefore$ $k = \lim\limits_{x\to0} \log(\sec^2 x)^{\cot^2 x}$

$= \lim\limits_{x\to0} \cot^2 x \log\sec^2 x$

$= \lim\limits_{x\to0} \dfrac{\log(1+\tan^2 x)}{\tan^2 x}$

$= 1$

24. Since, f(x) is continuous at $x = 0$

$\therefore$ $f(0) = \lim\limits_{x\to0} f(x)$

$\Rightarrow k = \lim\limits_{x\to0}\left[\tan\left(\dfrac{\pi}{4}+x\right)\right]^{\frac{1}{x}}$

$= \lim\limits_{x\to0}\left(\dfrac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}}$

$= \lim\limits_{x\to0} \dfrac{\left[(1+\tan x)^{\frac{1}{\tan x}}\right]^{\frac{\tan x}{x}}}{\left[(1-\tan x)^{\frac{1}{-\tan x}}\right]^{\frac{-\tan x}{x}}}$

$= \dfrac{e^1}{e^{-1}} = e^2$

25. $\lim\limits_{x\to1} (\log_2 2x)^{\log_x 8}$

$= \lim\limits_{x\to1} [\log_2 2 + \log_2 x]^{\log_x 2^3}$

$= \lim\limits_{x\to1} [1+\log_2 x]^{3\log_x 2}$

$= \lim\limits_{x\to1} [1+\log_2 x]^{\frac{3}{\log_2 x}}$

$= e^{\lim\limits_{x\to1} \log_2 x \times \frac{3}{\log_2 x}}$

$= e^3$

Since the function is continuous at $x = 1$,

$\therefore$ $\lim\limits_{x\to1} f(x) = f(1)$

$\Rightarrow e^3 = (k-1)^3$

$\Rightarrow e = k - 1$

$\Rightarrow k = e + 1$

26. For f(x) to be continuous at $x = 0$,

$f(0) = \lim\limits_{x\to0} f(x)$

$= \lim\limits_{x\to0} \dfrac{\log_e(1+x) - \log_e(1-x)}{x}$

Applying L'Hospital rule on R.H.S., we get

$f(0) = \lim\limits_{x\to0} \dfrac{\dfrac{1}{1+x} + \dfrac{1}{1-x}}{1}$

$\Rightarrow f(0) = 2$

27. For f(x) to be continuous at $x = 0$,

$f(0) = \lim\limits_{x\to0} f(x)$

$\Rightarrow f(0) = \lim\limits_{x\to0} \dfrac{\log(1+ax) - \log(1-bx)}{x}$

Applying L'Hospital rule on R.H.S., we get

$f(0) = \lim\limits_{x\to0} \dfrac{\dfrac{a}{1+ax} + \dfrac{b}{1-bx}}{1}$

$\Rightarrow f(0) = a + b$

28. Since, f(x) is continuous at $x = 2$.

$\therefore$ $f(2) = \lim\limits_{x\to2} f(x)$

$\Rightarrow 2 = \lim\limits_{x\to2} \dfrac{x^2 - (A+2)x + A}{x-2}$

$\Rightarrow 2 = \lim\limits_{x\to2} \dfrac{x(x-2) - A(x-1)}{x-2}$,

which is true if A = 0

29. If $x \to 0$, then the value of $\sin\dfrac{1}{x}$ passes through [–1,1] infinitely many ways, therefore limit of the function does not exist at $x = 0$. Hence, there is no value of k for which the function is continuous at $x = 0$.

30. $\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0} x\sin\dfrac{1}{x}$, but $-1 \le \sin\dfrac{1}{x} \le 1$ and $x \to 0$

$\therefore$ $\lim\limits_{x\to0} f(x) = 0$

Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to0} f(x)$

$\Rightarrow k = 0$

31. Since, f(x) is continuous at $x = 1$.

$\therefore$ $f(1) = \lim\limits_{x\to1^-} f(x)$

$\Rightarrow 2 = \lim\limits_{x\to1} (ax^2 - b)$

$\Rightarrow 2 = a - b$

The values of a and b in options (A), (B) and (C) satisfies this relation.

$\therefore$ option (D) is the correct answer.

32. $f(x) = \sin x$

$\therefore$ $f(0) = \sin 0 = 0$

$\lim\limits_{x\to0^+} x^2 + a^2 = 0^2 + a^2 = a^2$

Since the function is continuous at $x = 0$,

$\lim\limits_{x\to0^+} f(x) = f(0)$

$\Rightarrow 0 = a^2$

$\Rightarrow a = 0$

$\lim\limits_{x\to1^-} x^2 + a^2 = 1^2 + a^2$

$= 1$

$f(x) = bx + 2$

$\therefore$ $f(1) = b + 2$

Now, $\lim\limits_{x\to1^-} f(x) = f(1)$

$\Rightarrow 1 = b + 2$

$\Rightarrow b = -1$

$a + b + ab = 0 - 1 + 0(-1) = -1$

33. Since, f(x) is continuous at $x = \frac{\pi}{2}$.

$\therefore$ $f\left(\frac{\pi}{2}\right) = \lim\limits_{x\to\frac{\pi}{2}} f(x)$

$\Rightarrow \lambda = \lim\limits_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\pi - 2x}$

Applying L'Hospital rule on R.H.S., we get

$\lambda = \lim\limits_{x\to\frac{\pi}{2}} \frac{-\cos x}{-2}$

$\Rightarrow \lambda = \lim\limits_{x\to\frac{\pi}{2}} \frac{\cos x}{2} = \frac{\cos\frac{\pi}{2}}{2} = 0$

34. Since, f(x) is continuous at $x = \frac{\pi}{4}$,

$\therefore$ $f\left(\frac{\pi}{4}\right) = \lim\limits_{x\to\frac{\pi}{4}} f(x)$

$\Rightarrow k = \lim\limits_{x\to\frac{\pi}{4}} \frac{1-\sqrt{2}\sin x}{\pi - 4x}$

Applying L'Hospital rule on R.H.S., we get

$k = \lim\limits_{x\to\frac{\pi}{4}} \frac{-\sqrt{2}\cos x}{-4}$ $\Rightarrow k = \frac{1}{4}$

35. Since, f(x) is continuous at $x = \frac{\pi}{4}$.

$\therefore$ $f\left(\frac{\pi}{4}\right) = \lim\limits_{x\to\frac{\pi}{4}} f(x)$

$\Rightarrow a = \lim\limits_{x\to\frac{\pi}{4}} \frac{\tan x - \cot x}{x - \frac{\pi}{4}}$

Applying L'Hospital rule on R.H.S., we get

$a = \lim\limits_{x\to\frac{\pi}{4}} \frac{\sec^2 x + \operatorname{cosec}^2 x}{1}$

$\Rightarrow a = \left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 = 4$

36. For f(x) to be continuous at $x = 0$,

$\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^+} f(x) = f(0)$

$\Rightarrow f(0) = \lim\limits_{x\to0^-} f(x)$

$\Rightarrow k = \lim\limits_{h\to0} f(0-h) = \lim\limits_{h\to0} \frac{\cos\frac{\pi}{2}[0-h]}{[0-h]}$

$\Rightarrow k = \lim\limits_{h\to0} \frac{\cos\frac{\pi}{2}[-h]}{[-h]}$

$\Rightarrow k = \lim\limits_{h\to0} \frac{\cos\left(-\frac{\pi}{2}\right)}{-1}$

$\Rightarrow k = 0$

37. If f(x) is continuous from right at $x = 2$, then $f(2) = \lim\limits_{x\to2^+} f(x)$

$\Rightarrow k = \lim\limits_{h\to0} f(2+h)$

$\Rightarrow k = \lim\limits_{h\to0} \left[(2+h)^2 + e^{\frac{1}{2-(2+h)}}\right]^{-1}$

$\Rightarrow k = \lim\limits_{h\to0} \left[4 + h^2 + 4h + e^{\frac{-1}{h}}\right]^{-1}$

$\Rightarrow k = \left(4 + 0 + 0 + e^{-\infty}\right)^{-1}$

$\Rightarrow k = \frac{1}{4}$

38. Since, f(x) is continuous at $x = 0$.

∴ $f(0) = \lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}\frac{\log_e(1+x^2\tan x)}{\sin x^3}$$

$$= \lim_{x\to 0}\left(\frac{\log_e(1+x^2\tan x)}{x^2\tan x}\cdot\frac{x^2\tan x}{\sin x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\log_e(1+x^2\tan x)}{x^2\tan x}\cdot\frac{x^3}{\sin x^3}\cdot\frac{\tan x}{x}\right)$$

∴ f(0) = 1

39. Since, f(x) is continuous at $x = 0$.

∴ $f(0) = \lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{e^{2x}-1}\right)$$

$$= \lim_{x\to 0}\frac{e^{2x}-1-2x}{x(e^{2x}-1)}$$

Applying L'Hospital rule on R.H.S., we get

$$f(0) = \lim_{x\to 0}\frac{2e^{2x}-2}{x\left(2e^{2x}\right)+1\left(e^{2x}-1\right)}$$

Applying L'Hospital rule on R.H.S., we get

$$f(0) = \lim_{x\to 0}\frac{4e^{2x}}{2x\left(2e^{2x}\right)+e^{2x}\left(2\right)+2e^{2x}}$$

$$\Rightarrow f(0) = \frac{4}{2+2} = 1$$

40. Since, f(x) is continuous at $x = 0$.

∴ $f(0) = \lim_{x\to 0} f(x)$

$$\Rightarrow \left(\frac{k}{16}\right)\log\left(\frac{10}{3}\right)\cdot\log 2 = \lim_{x\to 0}\frac{20^x+3^x-6^x-10^x}{1-\cos 8x}$$

$$= \lim_{x\to 0}\frac{\left(10^x-3^x\right)\left(2^x-1\right)}{2\sin^2 4x}$$

$$= \lim_{x\to 0}\frac{\left(\frac{10^x-1}{x}-\frac{3^x-1}{x}\right)\cdot\left(\frac{2^x-1}{x}\right)}{2\times\frac{\sin^2 4x}{16x^2}\times 16}$$

$$= \frac{(\log 10-\log 3)(\log 2)}{32}$$

∴ $\left(\frac{k}{16}\right)\log\left(\frac{10}{3}\right).\log 2 = \frac{1}{32}\log\left(\frac{10}{3}\right)\log 2$

$$\Rightarrow \frac{k}{16} = \frac{1}{32}$$

$$\Rightarrow k = \frac{1}{2}$$

$$\Rightarrow k = 3^{\log_3\left(\frac{1}{2}\right)} \quad \ldots\left[\because a^{\log_a x} = x\right]$$

41. $\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h)$

$$= \lim_{h\to 0}\frac{2-h-2}{|2-h-2|}+a$$

$$= \lim_{h\to 0}\left(-\frac{h}{h}+a\right) = a-1$$

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h)$$

$$= \lim_{h\to 0}\frac{2+h-2}{|2+h-2|}+b = b+1$$

and f(2) = a + b

Since, f(x) is continuous at $x = 2$,

∴ $\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x)$

⇒ a – 1 = a + b = b + 1

⇒ b = – 1 and a = 1

42. Given, $f(x) = |x| + |x-1|$

∴ $f(x) = \begin{cases} -x-(x-1), & \text{if } x<0 \\ x-(x-1), & \text{if } 0\le x<1 \\ x+(x-1), & \text{if } x\ge 1\end{cases}$

∴ $f(x) = \begin{cases} -2x+1, & \text{if } x<0 \\ 1, & \text{if } 0\le x<1 \\ 2x-1, & \text{if } x\ge 1\end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(-2x+1) = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} 1 = 1$$

f(0) = 1

∴ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

∴ f(x) is continuous at $x = 0$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} 1 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(2x-1) = 1$$

f (1) = 2(1) – 1 = 1

∴ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

∴ f(x) is continuous at $x = 1$.

43. $\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{x^2-4x+3}{x^2-1} = \lim_{x\to 1}\frac{(x-3)}{(x+1)} = -1$

$f(1) = 2$

$\therefore$ $\lim_{x\to 1} f(x) \neq f(1)$

$\therefore$ $f(x)$ is discontinuous at $x = 1$.

44. $\lim_{x\to a^-} f(x) = -1$

$\lim_{x\to a^+} f(x) = 1$

$\therefore$ $f(x)$ is discontinuous at $x = a$.

45. $|x|$ is continuous at $x = 0$ and $\frac{|x|}{x}$ is discontinuous at $x = 0$.

$\therefore$ $f(x) = |x| + \frac{|x|}{x}$ is discontinuous at $x = 0$.

46. $f(x) = \frac{2x^2+7}{x^2(x+3)-1(x+3)} = \frac{2x^2+7}{(x^2-1)(x+3)}$

$= \frac{2x^2+7}{(x-1)(x+1)(x+3)}$

$\therefore$ the points of discontinuity are $x = 1$, $x = -1$ and $x = -3$ only.

47. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 2 - x^2 = 1$

$f(1) = 1 - 1 = 0$

$\lim_{x\to 1} f(x) \neq f(1)$

$\therefore$ The function is discontinuous at $x = 1$

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5} x - 10 = -5$

$f(5) = 2(5) = 10$

$\lim_{x\to 5} f(x) \neq f(5)$

$\therefore$ The function is discontinuous at $x = 5$

$\lim_{x\to 3^+} f(x) = x - 10 = -7$

$f(3) = 2 - 3^2 = -7$

$\lim_{x\to 3} f(x) = f(3)$

$\therefore$ The function is continuous at $x = 3$

48. Since, $f(x)$ is not defined at $x = 0, 1, -1$ and at all other points $f(x)$ is continuous.

$\therefore$ the given function is discontinuous at 3 points.

49. Given, $f(x) = [x]$, $x \in (-3.5, 100)$

As we know greatest integer function is discontinuous on integer values.

In given interval, the integer values are $(-3, -2, -1, 0, \ldots, 99)$.

$\therefore$ the total number of integers are 103.

50. Since, $f(x)$ is continuous at every point of its domain.

$\therefore$ it is continuous at $x = 1$.

$\therefore$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\Rightarrow \lim_{x\to 1} (5x - 4) = \lim_{x\to 1} (4x^2 + 3bx)$

$\Rightarrow 1 = 4 + 3b$

$\Rightarrow b = -1$

51. Since, $f(x)$ is continuous for all x.

$\therefore$ $f(x)$ is continuous at $x = 2$.

$\therefore$ $f(2) = \lim_{x\to 2} f(x)$

$\Rightarrow k = \lim_{x\to 2}\frac{x^3+x^2-16x+20}{(x-2)^2}$

$= \lim_{x\to 2}\frac{(x-2)(x^2+3x-10)}{(x-2)^2}$

$= \lim_{x\to 2}\frac{(x-2)^2(x+5)}{(x-2)^2}$

$= 7$

52. Since, $f(x) = [x]$ is continuous at every non integer points.

$\therefore$ option (C) is the correct answer.

53. Let $g(x) = |x|$ and $h(x) = \sin x$.

Then, $f(x) = (hog)(x)$ for all $x \in R$.

As both g and h are continuous functions on R.

$\therefore$ $f(x)$ is also continuous for all $x \in R$.

54. Since, $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$.

$\therefore$ it is continuous at $x = \frac{\pi}{4}$.

$\therefore$ $f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}} f(x) = \lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{4x-\pi}$

Applying L'Hospital rule on R.H.S., we get

$$f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}} \frac{-\sec^2 x}{4}$$

$$\Rightarrow f\left(\frac{\pi}{4}\right) = \frac{-2}{4} = \frac{-1}{2}$$

55. Since, f(x) is continuous at each point of its domain.

$\therefore$ it is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}\left(\frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}\right)$$

Applying L'Hospital rule on R.H.S., we get

$$f(0) = \lim_{x\to 0}\frac{\left(2 - \frac{1}{\sqrt{1-x^2}}\right)}{\left(2 + \frac{1}{1+x^2}\right)}$$

$$= \frac{2-1}{2+1} = \frac{1}{3}$$

56. The given function is defined only in the interval $[1,\infty)$. For $x > 2$, $y = 3x - 2$ which is a straight line, hence continuous. Also, the given function is continuous at $x = 2$.

$\therefore$ option (C) is the correct answer.

57. $\lim_{x\to 1^-} f(x) = 0$, $\lim_{x\to 1^+} f(x) = 0$ and

$f(1) = 0$

$\therefore$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

$\therefore$ f(x) is continuous at $x = 1$.

$\lim_{x\to 2^-} f(x) = 0$ and $\lim_{x\to 2^+} f(x) = 1$

$\therefore$ $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

$\therefore$ f(x) is not continuous at $x = 2$.

58. Since, f(x) is continuous over $[-\pi, \pi]$.

$\therefore$ it is continuous at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

$\therefore$ $\lim_{x\to \frac{-\pi}{2}^-} f(x) = \lim_{x\to \frac{-\pi}{2}^+} f(x)$

$$\Rightarrow \lim_{x\to\frac{-\pi}{2}}(-2\sin x) = \lim_{x\to\frac{-\pi}{2}}(\alpha\sin x + \beta)$$

$\Rightarrow -2(-1) = \alpha(-1) + \beta$

$\Rightarrow -\alpha + \beta = 2$ ....(i)

Also, $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} f(x)$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}(\alpha\sin x + \beta) = \lim_{x\to\frac{\pi}{2}}(\cos x)$$

$\Rightarrow \alpha(1) + \beta = 0$

$\Rightarrow \alpha + \beta = 0$ ....(ii)

From (i) and (ii), we get

$\alpha = -1, \beta = 1$

59. For f(x) to be continuous at $x = \frac{-\pi}{2}$,

$$\lim_{x\to\frac{-\pi}{2}^+} f(x) = \lim_{x\to\frac{-\pi}{2}^-} f(x) = f\left(\frac{-\pi}{2}\right)$$

$\therefore$ $\lim_{x\to\frac{-\pi}{2}} A\sin x + B = -2\sin\left(\frac{-\pi}{2}\right)$

$\Rightarrow -A + B = 2$

$\Rightarrow A - B = -2$ …(i)

For f(x) to be continuous at $x = \frac{\pi}{2}$,

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^-} f(x) = f\left(\frac{\pi}{2}\right)$$

$\therefore$ $\lim_{x\to\frac{\pi}{2}} A\sin x + B = \cos\frac{\pi}{2}$

$\Rightarrow A + B = 0$ …(ii)

On solving (i) and (ii), we get

$A = -1$, $B = 1$

60. Since, f(x) is continuous for all x in R.

$\therefore$ f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim_{x\to 0^-} f(x)$

$$\Rightarrow q = \lim_{x\to 0}\frac{\sin(p+1)x + \sin x}{x}$$

$$\Rightarrow q = \lim_{x\to 0}\left[(p+1)\times\frac{\sin(p+1)x}{(p+1)x} + \frac{\sin x}{x}\right]$$

$\Rightarrow q = (p+1) + 1$

$\Rightarrow q = p + 2$

The values of p and q in option (C) satisfies this condition.

61. Since, f is continuous at every point in R.

$\therefore$ f is continuous at $x = 2n$.

$\therefore$ $\lim\limits_{x \to (2n)^-} f(x) = \lim\limits_{x \to (2n)^+} f(x) = f(2n)$

$\Rightarrow \lim\limits_{x \to (2n)^-} (b_n + \cos \pi x) = \lim\limits_{x \to (2n)^+} (a_n + \sin \pi x)$

$\Rightarrow b_n + \cos 2n\pi = a_n + \sin 2n\pi$

$\Rightarrow b_n + 1 = a_n \Rightarrow a_n - b_n = 1$

So, option (C) is correct.

Also, f is continuous at $x = 2n + 1$.

$\lim\limits_{x \to (2n+1)^-} f(x) = \lim\limits_{x \to (2n+1)^+} f(x) = f(2n + 1)$

$\Rightarrow \lim\limits_{x \to (2n+1)^-} (a_n + \sin \pi x) = \lim\limits_{x \to (2n+1)^+} (b_{n+1} + \cos \pi x)$

$\Rightarrow a_n + \sin(2n + 1)\pi = b_{n+1} + \cos(2n + 1)\pi$

....[$\because$ $f(x) = b_{n+1} + \cos \pi x, x \in (2n + 1, 2n + 2)$]

$\Rightarrow a_n = b_{n+1} - 1$

$\Rightarrow a_n - b_{n+1} = -1$

Replacing n by n − 1, we get

$a_{n-1} - b_n = -1$

So, options (A) and (D) are correct.

Hence, option (B) does not hold.

63. For f(x) to be continuous at $x = 0$,

$$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{(1+x)^{\frac{1}{2}} - (1+x)^{\frac{1}{3}}}{x}$$

$$= \lim_{x \to 0} \frac{\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + ...\right) - \left(1 + \frac{1}{3}x - \frac{1}{9}x^2 + ...\right)}{x}$$

$$= \lim_{x \to 0} \frac{\left(\frac{1}{2} - \frac{1}{3}\right)x + \left(\frac{1}{9} - \frac{1}{8}\right)x^2 + ...}{x}$$

$$= \lim_{x \to 0} \left[\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{9} - \frac{1}{8}\right)x + ...\right]$$

$\therefore$ $f(0) = \dfrac{1}{6}$

Evaluation Test

1. $$f(0) = \lim_{x \to 0} \frac{1 - \cos(1 - \cos x)}{x^4}$$

$$= \lim_{x \to 0} \frac{2\sin^2\left[\dfrac{2\sin^2\left(\frac{x}{2}\right)}{2}\right]}{x^4}$$

$$= \lim_{x \to 0} \frac{2\sin^2\left[\sin^2\left(\frac{x}{2}\right)\right]\left[\sin^2\left(\frac{x}{2}\right)\right]^2}{x^4\left[\sin^2\left(\frac{x}{2}\right)\right]^2}$$

$$= 2\lim_{x \to 0} \frac{\sin^4\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^4 2^4} = \frac{1}{2^3} = \frac{1}{8}$$

2. f is continuous at $x = 0$.

$\therefore$ $$f(0) = \lim_{x \to 0}\left[\frac{\log(1 + x^2) - \log(1 - x^2)}{\sec x - \cos x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\log(1 + x^2) - \log(1 - x^2)}{\left[\dfrac{1 - \cos^2 x}{\cos x}\right]}\right]$$

$$= \lim_{x \to 0}\left[\frac{\cos x\left[\log\left(1 + x^2\right) - \log\left(1 - x^2\right)\right]}{\sin^2 x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\cos x\left[\dfrac{\log\left(1 + x^2\right)}{x^2} + \dfrac{\log\left(1 - x^2\right)}{-x^2}\right]}{\dfrac{\left(\sin^2 x\right)}{x^2}}\right]$$

$$= (\cos 0)\left[\frac{1 + 1}{1^2}\right] = 2$$

3. For f(x) to be continuous at x = 0, we must have $f(0)=\lim_{x\to0}f(x)$

$$=\lim_{x\to0}\frac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$$

$$=\lim_{x\to0}\frac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$$

$$\times\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}\times\frac{\sqrt{a^2-ax+x^2}+\sqrt{a^2+ax+x^2}}{\sqrt{a^2-ax+x^2}+\sqrt{a^2+ax+x^2}}$$

$$=\lim_{x\to0}\frac{\left[\left(a^2-ax+x^2\right)-\left(a^2+ax+x^2\right)\right]\left[\sqrt{a+x}+\sqrt{a-x}\right]}{\left[(a+x)-(a-x)\right]\left[\sqrt{a^2-ax+x^2}+\sqrt{a^2+ax+x^2}\right]}$$

$$=\lim_{x\to0}\frac{-2ax\left(\sqrt{a+x}+\sqrt{a-x}\right)}{2x\left(\sqrt{a^2-ax+x^2}+\sqrt{a^2+ax+x^2}\right)}$$

$$=\frac{-a\left(\sqrt{a}+\sqrt{a}\right)}{\sqrt{a^2}+\sqrt{a^2}}$$

$\therefore$ f(0) = $-\sqrt{a}$

4. $$\lim_{x\to0}\frac{5^x.2^x+7^x-7^x.2^x-5^x}{2\sin^2\frac{x}{2}}$$

$$=\lim_{x\to0}\frac{5^x(2^x-1)-7^x(2^x-1)}{2\sin^2\frac{x}{2}}$$

$$=\lim_{x\to0}\frac{1}{2}\left(\frac{2^x-1}{x}\right)\left(\frac{5^x-1}{x}-\frac{7^x-1}{x}\right)\frac{1}{\frac{\sin^2 x/2}{x^2/4}\times\frac{1}{4}}$$

$$=2(\log 2)\left(\log\frac{5}{7}\right)$$

It is discontinuous at x = 0 and it is removable.

5. $$a=\lim_{x\to0}\frac{\sin^3\sqrt{x}\log(1+3x)}{\left(\tan^{-1}\sqrt{x}\right)^2\left(e^{5\sqrt{x}}-1\right)x}$$

$$=\lim_{x\to0}\frac{\frac{\sin^3\sqrt{x}}{\left(\sqrt{x}\right)^3}\cdot\left(\sqrt{x}\right)^3\cdot\frac{\log(1+3x)}{3x}\cdot3x}{\frac{\left(\tan^{-1}\sqrt{x}\right)^2}{\left(\sqrt{x}\right)^2}\cdot\left(\sqrt{x}\right)^2\cdot\frac{e^{5\sqrt{x}}-1}{5\sqrt{x}}\cdot5\sqrt{x}\cdot x}$$

$$=\frac{(1)^3\left(\sqrt{x}\right)^3.(1)3x}{(1)^2\left(\sqrt{x}\right)^2(1)5\sqrt{x}.x}$$

$$=\frac{3}{5}$$

6. For f to be continuous at x = 2,

$$f(2)=\lim_{x\to0}(x-1)^{\frac{1}{(2-x)}}$$

$$=\lim_{x\to0}\left(1+(x-2)\right)^{\frac{-1}{(x-2)}}=e^{-1}$$

7. Given function is continuous at ($-\infty$, 6).

$\therefore$ at x = 1 and x = 3, function is continuous.

If the function f(x) is continuous at x = 1, then

$$\lim_{x\to1^-}f(x)=\lim_{x\to1^+}f(x)$$

$$\Rightarrow 1+\sin\frac{\pi}{2}=a+b$$

$\therefore$ a + b = 2 .....(i)

If the function is continuous at x = 3, then

$$\lim_{x\to3^-}f(x)=\lim_{x\to3^+}f(x)$$

$$\Rightarrow 3a+b=6\tan\frac{3\pi}{12}$$

$\therefore$ 3a + b = 6 .....(ii)

From (i) and (ii), a = 2, b = 0

8. Since, x and $|x|$ are continuous for all x.

$\therefore$ $x+|x|$ is continuous for $x\in(-\infty,\infty)$.

9. For f(x) to be continuous at x = 0, we must have

$$\lim_{x\to0^-}f(x)=f(0)=\lim_{x\to0^+}f(x)$$

$$\lim_{x\to0^+}f(x)=\lim_{x\to0^+}e^{\tan2x/\tan3x}$$

$$=\lim_{x\to0^+}e^{\left(\frac{\tan2x}{2x}\times2x\right)\Big/\left(\frac{\tan3x}{3x}\times3x\right)}$$

$$=e^{\frac{2}{3}}$$

$$f(0)=\lim_{x\to0^+}f(x)$$

$$\Rightarrow b=e^{\frac{2}{3}}$$

$$\lim_{x\to0^-}f(x)=\lim_{x\to0^-}(1+|\sin x|)^{a/|\sin x|}$$

$$=e^{\lim_{x\to0}\left(|\sin x|\times\frac{a}{|\sin x|}\right)}=e^a$$

$$f(0)=\lim_{x\to0^-}f(x)$$

$$\Rightarrow b=e^a\Rightarrow e^{\frac{2}{3}}=e^a$$

$$\Rightarrow a=\frac{2}{3}$$

10. Given, $f(x) = [x]^2 - [x^2]$
$-1 < x < 0$, $f(x) = (-1)^2 - 0 = 1$
$x = 0$, $f(x) = 0^2 - 0 = 0$
$0 < x < 1$, $f(x) = 0^2 - 0 = 0$
$x = 1$, $f(x) = 1^2 - 1 = 0$
$1 < x < \sqrt{2}$, $f(x) = 1^2 - 1 = 0$
$x = \sqrt{2}$, $f(x) = 1^2 - 2 = -1$
$\sqrt{2} < x < \sqrt{3}$, $f(x) = 1^2 - 2 = -1$
$x = \sqrt{3}$, $f(x) = 1^2 - 3 = -2$
$\sqrt{3} < x < 2$, $f(x) = 1^2 - 3 = -2$
$x = 2$, $f(x) = 2^2 - 4 = 0$
$2 < x < \sqrt{5}$, $f(x) = 2^2 - 4 = 0$
$x = \sqrt{5}$, $f(x) = 2^2 - 5 = -1$
Hence, the given function is discontinuous at all integers except 1.

Textbook Chapter No.

02 Differentiation

Hints

Classical Thinking

1. $f'(2^-) = \lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h}$

$= \lim_{h\to 0} \frac{(2+h+1)-3}{h}$

$= \lim_{h\to 0} \frac{h}{h} = 1$

$f'(2^+) = \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$

$= \lim_{h\to 0} \frac{2(2+h)-1-3}{h}$

$= \lim_{h\to 0} \frac{2h}{h} = 2$

$\therefore$ $f'(2^-) \neq f'(2^+)$

$\therefore$ $f'(2)$ does not exist.

2. $f'(3^-) = \lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h}$

$= \lim_{h\to 0} \frac{(3+h+2)-5}{h} = \lim_{h\to 0} \frac{h}{h} = 1$

$f'(3^+) = \lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h}$

$= \lim_{h\to 0} \frac{8-(3+h)-5}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$

$\therefore$ $f'(3^-) \neq f'(3^+)$

$\therefore$ $f'(3)$ does not exist.

3. $f'(0^+) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|-0}{h}$

$= \lim_{h\to 0} \frac{h}{h} = 1$

$f'(0^-) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|-0}{h}$

$= \lim_{h\to 0} \frac{-h}{h} = -1$

$\therefore$ $f'(0^+) \neq f'(0^-)$

$\therefore$ $f'(0)$ does not exist.

4. $f'(1^-) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1} \frac{px^2+1-p-1}{x-1} = \lim_{x\to 1} \frac{p(x^2-1)}{x-1}$

$= p\lim_{x\to 1}(x+1)$

$= 2p$

$f'(1^+) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x+p-p-1}{x-1}$

$= \lim_{x\to 1} \frac{x-1}{x-1}$

$= 1$

Since, $f(x)$ is differentiable at $x = 1$.

$\therefore$ $2p = 1 \Rightarrow p = \frac{1}{2}$

5. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1} x = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(2x-1) = 1$

$f(1) = 1$

$\therefore$ $f(x)$ is continuous at $x = 1$.

$f'(1^-) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{1+h-1}{h}$

$= \lim_{h\to 0} \frac{h}{h} = 1$

$f'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \frac{2(1+h)-1-1}{h}$

$= \lim_{h\to 0} \frac{2h}{h} = 2$

$\therefore$ $f'(1^-) \neq f(1^+)$

$\therefore$ $f(x)$ is not differentiable at $x = 1$.

6. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2}(x+1) = 3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(5-x) = 3$

$f(2) = 1 + 2 = 3$

$\therefore$ $f(x)$ is continuous at $x = 2$.

$$f'(2^-) = \lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{1+(2+h)-3}{h}$$
$$= \lim_{h\to 0}\frac{h}{h} = 1$$
$$f'(2^+) = \lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{5-(2+h)-3}{h}$$
$$= \lim_{h\to 0}\frac{-h}{h} = -1$$

$\therefore$ $f'(2^-) \neq f'(2^+)$

$\therefore$ f(x) is not differentiable at $x = 2$.

7. $f(0) = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x} = 0$$

$\therefore$ f(x) is continuous at $x = 0$.

$$f(0^-) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}$$
$$= \lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h-0}$$
$$= \lim_{h\to 0}\frac{-h\sin\left(\frac{1}{-h}\right)-0}{-h}$$
$$= -\lim_{h\to 0}\sin\left(\frac{1}{h}\right)$$

= (a number which oscillates between − 1 and 1)

$\therefore$ f(0^-) does not exist.

$\therefore$ f(x) is not differentiable at $x = 0$.

8. $$\frac{d}{dx}[\sin(2x+3)] = \cos(2x+3).\frac{d}{dx}(2x+3)$$
$$= 2\cos(2x+3)$$

9. $$y = e^{\sqrt{x}} \Rightarrow \frac{dy}{dx} = e^{\sqrt{x}}.\frac{d}{dx}\left(\sqrt{x}\right) \Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

10. $$\frac{d}{dx}(e^{x^3}) = e^{x^3}.\frac{d}{dx}(x^3) = 3x^2.e^{x^3}$$

11. Let $y = (\log x)^4$

$$\therefore \quad \frac{dy}{dx} = 4(\log x)^3\cdot\frac{d}{dx}(\log x)$$
$$= \frac{4(\log x)^3}{x}$$

12. $$\frac{d}{dx}[\log(\log x)] = \frac{1}{\log x}.\frac{d}{dx}(\log x)$$
$$= \frac{1}{\log x}.\frac{1}{x} = (x\log x)^{-1}$$

13. $$y = \log_{10}|x| = \frac{\log_e|x|}{\log_e 10}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\log_e 10}.\frac{1}{|x|}.\frac{|x|}{x} = \frac{1}{x\log_e 10}$$

14. $y = f(ax^2 + b)$

$$\therefore \quad \frac{dy}{dx} = f'(ax^2+b).\frac{d}{dx}(ax^2+b) = 2ax\, f'(ax^2+b)$$

15. $y = (4x^3 - 5x^2 + 1)^4$

$$\therefore \quad \frac{dy}{dx} = 4(4x^3-5x^2+1)^3\cdot\frac{d}{dx}(4x^3-5x^2+1)$$
$$= 4(4x^3-5x^2+1)^3(12x^2-10x)$$

16. $$\frac{d}{dx}(x^2+\cos x)^4 = 4(x^2+\cos x)^3.\frac{d}{dx}(x^2+\cos x)$$
$$= 4(x^2+\cos x)^3(2x-\sin x)$$

17. $$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= \frac{2}{(u+1)^2}\cdot\frac{1}{2\sqrt{x}}$$
$$= \frac{1}{\left(\sqrt{x}+1\right)^2}\cdot\frac{1}{\sqrt{x}}$$
$$= \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}$$

18. $y = \log\left(\tan\sqrt{x}\right)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\tan\sqrt{x}}.\frac{d}{dx}\left(\tan\sqrt{x}\right)$$
$$= \frac{1}{\tan\sqrt{x}}.\sec^2\sqrt{x}.\frac{1}{2\sqrt{x}}$$
$$= \frac{\sec^2\sqrt{x}}{2\sqrt{x}\tan\sqrt{x}}$$

19. $y = \log(\sec x + \tan x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec x+\tan x}\cdot\frac{d}{dx}(\sec x+\tan x)$$
$$= \frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}$$
$$= \sec x$$

20. $y = \log(\log(\log x^3))$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\log(\log x^3)} \cdot \frac{d}{dx}\left[\log(\log x^3)\right]$

$= \frac{1}{\log(\log x^3)} \cdot \frac{1}{\log x^3} \cdot \frac{d}{dx}(\log x^3)$

$= \frac{1}{\log(\log x^3)} \cdot \frac{1}{3\log x} \cdot \frac{1}{x^3} \cdot 3x^2$

$= \frac{1}{x \log x \log(\log x^3)}$

21. Derivative exists if $1 - x^2 > 0$ i.e., $1 > x^2$
i.e., $x^2 < 1$ i.e., $|x| < 1$ i.e., $-1 < x < 1$

22. $\frac{d}{dx}\left[\tan^{-1}(\sqrt{x})\right] = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}}$

$= \frac{1}{2\sqrt{x}(1+x)}$

23. $y = \cos^{-1}\left(\frac{1}{x^3}\right)$

$\therefore \quad y = \sec^{-1}(x^3)$

$\therefore \quad \frac{dy}{dx} = \frac{1}{x^3\sqrt{\left(x^3\right)^2 - 1}} \cdot 3x^2 = \frac{3}{x\sqrt{x^6 - 1}}$

24. Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$\therefore \quad y = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right) = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$

$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

25. Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore \quad y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\Rightarrow y = \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{2}{1+x^2}$

26. Let $y = \operatorname{cosec}^{-1}\left(\frac{1+x^2}{2x}\right)$

$= \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore \quad y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta = 2\tan^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{2}{1+x^2}$

27. Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore \quad y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta)$

$= \sin^{-1}\left(\sin\left(\frac{\pi}{2} - 2\theta\right)\right)$

$= \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\tan^{-1}x$

$\therefore \quad \frac{dy}{dx} = -\frac{2}{1+x^2}$

28. Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$\therefore \quad y = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right)$

$= \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$

$= \sec^{-1}(\sec 2\theta)$

$= 2\theta = 2\cos^{-1}x$

$\therefore \quad \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}, \; x \neq \pm 1$

29. Let $y = e^{x \sin x}$
Taking logarithm on both sides, we get
$\log y = x \sin x$
Differentiating both sides w.r.t.x, we get

$\frac{1}{y} \cdot \frac{dy}{dx} = \sin x + x\cos x$

$\therefore \quad \frac{dy}{dx} = e^{x \sin x}(\sin x + x\cos x)$

30. Let $y = x^x$
Taking logarithm on both sides, we get
$\log y = x \log x$
Differentiating both sides w.r.t. x, we get

$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$

$\therefore \quad \frac{dy}{dx} = x^x(1 + \log x)$

$= x^x(\log e + \log x) = x^x \log(ex)$

31. $y = x^{\log x}$

Taking logarithm on both sides, we get

$\log y = \log x \ \log x$

$= (\log x)^2$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \log x \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y}{x} \log x$$

$$\Rightarrow \frac{dy}{dx} = 2 \frac{x^{\log x}}{x} \cdot \log x = 2x^{\log x - 1} \cdot \log x$$

32. $y = x^2 + x^{\log x}$

$$\therefore \frac{dy}{dx} = 2x + \frac{d}{dx}(x^{\log x})$$

$$\Rightarrow \frac{dy}{dx} = 2x + \frac{2}{x} \log x \, (x^{\log x})$$

33. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Differentiating both sides w.r.t. x, we get

$$\frac{2}{3} x^{\frac{2}{3}-1} + \frac{2}{3} y^{\frac{2}{3}-1} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2}{3} x^{\frac{-1}{3}} + \frac{2}{3} y^{\frac{-1}{3}} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow y^{\frac{-1}{3}} \cdot \frac{dy}{dx} = -x^{\frac{-1}{3}} \Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

34. $x^3 + y^3 - 3\,axy = 0$

Differentiating w.r.t. x, we get

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} - 3a\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow 3(x^2 - ay) + 3\frac{dy}{dx}(y^2 - ax) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

35. $x^3 + 8xy + y^3 = 64$

Differentiating both sides w.r.t. x, we get

$$3x^2 + 8\left(y + x\frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3x^2 + 8y}{8x + 3y^2}$$

36. $y = \cos(x + y)$

$$\therefore \frac{dy}{dx} = -\sin(x + y) \cdot \left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}[1 + \sin(x + y)] = -\sin(x + y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin(x + y)}{1 + \sin(x + y)}$$

37. $\sin^2 x + 2\cos y + xy = 0$

Differentiating w.r.t. x, we get

$$2\sin x \cos x - 2\sin y \frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x - 2\sin y) = -y - \sin 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sin 2x}{2\sin y - x}$$

38. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Differentiating w.r.t. x, we get

$$2ax + 2h\left(y + x\frac{dy}{dx}\right) + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(2hx + 2by + 2f) = -(2ax + 2hy + 2g)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}$$

39. $\sqrt{x} + \sqrt{y} = 1$

Differentiating both sides w.r.t.x, we get

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4}, \frac{1}{4}\right)} = -1$$

40. $x = a\cos\theta$ and $y = b\sin\theta$

$$\therefore \frac{dx}{d\theta} = -a\sin\theta \text{ and } \frac{dy}{d\theta} = b\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = \left(-\frac{b}{a}\right)\cot\theta$$

41. Let $y = 5^x$ and $z = \log_5 x$

$$\therefore \frac{dy}{dx} = 5^x \log 5 \text{ and } \frac{dz}{dx} = \frac{1}{x\log 5}$$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{5^x \log 5}{\frac{1}{x\log 5}} = x.5^x (\log 5)^2$$

42. $x = \dfrac{1}{1-t^2}$ and $y = 1 + t^2$

$\therefore \dfrac{dx}{dt} = \dfrac{2t}{(1-t^2)^2}$ and $\dfrac{dy}{dt} = 2t$

$\therefore \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2t}{\frac{2t}{(1-t^2)^2}} = (1-t^2)^2$

43. Let $y = \sin x^2$ and $z = x^2$

$\therefore \dfrac{dy}{dx} = \cos x^2.(2x) = 2x \cos x^2$

and $\dfrac{dz}{dx} = 2x$

$\therefore \dfrac{dy}{dz} = \dfrac{\frac{dy}{dx}}{\frac{dz}{dx}} = \cos x^2$

44. Let $y = e^{x^3}$ and $z = \log x$

$\therefore \dfrac{dy}{dx} = e^{x^3}.3x^2 = 3x^2 e^{x^3}$ and $\dfrac{dz}{dx} = \dfrac{1}{x}$

$\therefore \dfrac{dy}{dz} = \dfrac{\frac{dy}{dx}}{\frac{dz}{dx}} = \dfrac{3x^2 e^{x^3}}{\left(\frac{1}{x}\right)} = 3x^3 e^{x^3}$

45. Let $y = a^{\sin^{-1} x}$ and $z = \sin^{-1} x$

$\therefore y = a^z$

$\therefore \dfrac{dy}{dz} = a^z \log a = a^{\sin^{-1} x} \log a$

46. $x = a \sec^2 \theta$ and $y = b \tan^2 \theta$

$\therefore \dfrac{dx}{d\theta} = 2a \sec^2 \theta \,.\, \tan \theta$

and $\dfrac{dy}{d\theta} = 2b \tan \theta \,.\, \sec^2 \theta$

$\therefore \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{b}{a}$

47. $x = a^2 (\sin \theta + \operatorname{cosec} \theta)$ …(i)

$y = a^2(\sin \theta - \operatorname{cosec} \theta)$ …(ii)

Squaring (i) and (ii) and subtracting, we get

$x^2 - y^2 = 4a^4$

Differentiating w.r.t. x, we get

$2x - 2y \dfrac{dy}{dx} = 0 \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y}$

48. $y = \log(ax + b)$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{ax+b} \times a$

$\therefore \dfrac{d^2y}{dx^2} = \dfrac{-a^2}{(ax+b)^2}$

49. $y = \log(\sin x)$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\sin x} \cdot \cos x = \cot x$

$\therefore \dfrac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$

50. $\sqrt{xy} = 1$

$\Rightarrow xy = 1$

$\Rightarrow y = \dfrac{1}{x}$

$\therefore \dfrac{dy}{dx} = \dfrac{-1}{x^2}$

$\therefore \dfrac{d^2y}{dx^2} = \dfrac{2}{x^3}$

51. $y = \sin mx$ ….(i)

$\therefore \dfrac{dy}{dx} = m \cos mx$

$\therefore \dfrac{d^2y}{dx^2} = -m^2 \sin mx$

$\Rightarrow \dfrac{d^2y}{dx^2} + m^2 y = 0$ ….[From (i)]

52. $y = 2 \sin x + 3 \cos x$

$\therefore \dfrac{dy}{dx} = 2 \cos x - 3 \sin x$

$\therefore \dfrac{d^2y}{dx^2} = -2 \sin x - 3 \cos x$

$\therefore y + \dfrac{d^2y}{dx^2} = 2 \sin x + 3 \cos x - 2 \sin x - 3 \cos x$

$\Rightarrow y + \dfrac{d^2y}{dx^2} = 0$

53. $x = a \cos nt - b \sin nt$ ….(i)

$\therefore \dfrac{dx}{dt} = -na \sin nt - nb \cos nt$

$\therefore \dfrac{d^2x}{dt^2} = -n^2 a \cos nt + n^2 b \sin nt$

$= -n^2 (a \cos nt - b \sin nt)$

$= -n^2 x$ …[From (i)]

54. $y = \text{a} \sin(\text{m}x) + \text{b}\cos(\text{m}x)$ ....(i)

$\therefore \quad \frac{\text{d}y}{\text{d}x} = \text{am}\cos(\text{m}x) - \text{bm}\sin(\text{m}x)$

$\therefore \quad \frac{\text{d}^2y}{\text{d}x^2} = -\text{am}^2\sin(\text{m}x) - \text{bm}^2\cos(\text{m}x)$

$= -\text{m}^2[\text{a}\sin(\text{m}x) + \text{b}\cos(\text{m}x)]$

$= -\text{m}^2 y$ ....[From (i)]

55. $y = \text{a} + \text{b}x^2$

$\therefore \quad \frac{\text{d}y}{\text{d}x} = 2\text{b}x$ ....(i)

$\therefore \quad \frac{\text{d}^2y}{\text{d}x^2} = 2\text{b}$

$\Rightarrow x\frac{\text{d}^2y}{\text{d}x^2} = 2\text{b}x = \frac{\text{d}y}{\text{d}x}$ ....[From (i)]

56. $\text{f}(x) = \text{be}^{\text{a}x} + \text{ae}^{\text{b}x}$

$\therefore \quad \text{f}'(x) = \text{abe}^{\text{a}x} + \text{abe}^{\text{b}x}$

$\therefore \quad \text{f}''(x) = \text{a}^2\text{be}^{\text{a}x} + \text{ab}^2\text{e}^{\text{b}x}$

$\therefore \quad \text{f}''(0) = \text{a}^2\text{b} + \text{ab}^2 = \text{ab}(\text{a}+\text{b})$

Critical Thinking

1. $\text{f}(x) = \begin{cases} -(x-3), & x<3 \\ (x-3), & x \geq 3 \end{cases}$

$\text{f}'(3^-) = -1$ and $\text{f}'(3^+) = 1$

$\therefore \quad \text{f}'(3^-) \neq \text{f}'(3^+)$

$\therefore \quad \text{f}'(3)$ does not exist.

2. $\lim\limits_{x\to 2^-}\text{f}(x) = \lim\limits_{\text{h}\to 0}\text{f}(2-\text{h}) = \lim\limits_{\text{h}\to 0}|2-\text{h}-2| = 0$

$\lim\limits_{x\to 2^+}\text{f}(x) = \lim\limits_{\text{h}\to 0}\text{f}(2+\text{h}) = \lim\limits_{\text{h}\to 0}|2+\text{h}-2| = 0$

$\therefore \quad \text{f}(2) = 0$

$\therefore \quad \text{f}(x)$ is continuous at $x = 2$.

$\text{f}'(2^-) = \lim\limits_{\text{h}\to 0^-}\frac{\text{f}(2+\text{h})-\text{f}(2)}{\text{h}}$

$= \lim\limits_{\text{h}\to 0}\frac{-(2+\text{h}-2)-0}{\text{h}}$

$= -1$

$\text{f}'(2^+) = \lim\limits_{\text{h}\to 0^+}\frac{\text{f}(2+\text{h})-\text{f}(2)}{\text{h}}$

$= \lim\limits_{\text{h}\to 0}\frac{2+\text{h}-2-0}{\text{h}} = 1$

$\therefore \quad \text{f}'(2^-) \neq \text{f}'(2^+)$

$\therefore \quad \text{f}(x)$ is not differentiable at $x = 2$.

3. $\text{f}'(1^-) = \lim\limits_{x\to 1^-}\frac{\text{f}(x)-\text{f}(1)}{x-1}$

$= \lim\limits_{x\to 1^-}\frac{2x^2+3x+4-9}{x-1}$

$= \lim\limits_{x\to 1^-}\frac{(x-1)(2x+5)}{x-1}$

$= \lim\limits_{x\to 1^-}(2x+5)$

$= 7$

$\text{f}'(1^+) = \lim\limits_{x\to 1^+}\frac{\text{f}(x)-\text{f}(1)}{x-1}$

$= \lim\limits_{x\to 1^+}\frac{\text{k}x+9-\text{k}-9}{x-1}$

$= \lim\limits_{x\to 1^+}\frac{\text{k}(x-1)}{x-1}$

$= \text{k}$

Since, f(x) is differentiable at $x = 1$.

$\therefore \quad \text{k} = 7$

4. $\text{f}(x) = \frac{x}{1+|x|}$

$\therefore \quad \text{f}(x) = \begin{cases} \frac{x}{1-x}, & x<0 \\ \frac{x}{1+x}, & x\geq 0 \end{cases}$

$\therefore \quad \text{f}'(x) = \begin{cases} \frac{x}{(1-x)^2}, & x<0 \\ \frac{x}{(1+x)^2}, & x\geq 0 \end{cases}$

$\therefore \quad$ f(x) is differentiable at $(-\infty, \infty)$.

5. Applying L'Hospital rule, we get

$\lim\limits_{x\to 2}\frac{2x^2-4\text{f}'(x)}{x-2} = \lim\limits_{x\to 2}\frac{4x-4\text{f}''(x)}{1}$

$= 8 - 4\text{f}''(2) = 8 - 4(1) = 4$

6. Since, f ′(a) exists.

$\therefore \quad \lim\limits_{x\to \text{a}}\frac{\text{f}(x)-\text{f}(\text{a})}{x-\text{a}} = \text{f}'(\text{a})$ ....(i)

Now, $\lim\limits_{x\to \text{a}}\frac{x\text{f}(\text{a})-\text{af}(x)}{x-\text{a}}$

$= \lim\limits_{x\to \text{a}}\frac{(x-\text{a})\text{f}(\text{a})-\text{a}(\text{f}(x)-\text{f}(\text{a}))}{x-\text{a}}$

$= \lim\limits_{x\to \text{a}}\text{f}(\text{a}) - \text{a}\lim\limits_{x\to \text{a}}\left\{\frac{\text{f}(x)-\text{f}(\text{a})}{x-\text{a}}\right\}$

$= \text{f}(\text{a}) - \text{af}'(\text{a})$ ....[From (i)]

7. If a function f(x) is continuous at $x = \text{a}$, then it may or may not be differentiable at $x = \text{a}$.

$\therefore \quad$ Option (B) is not true.

8. $f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$

$= \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{-h}\right)-0}{-h}$

$= 0$

$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h}-0}{h}$

$= 0$

$\therefore$ $f'(0^-) = f'(0^+)$

$\therefore$ f(x) is derivable at $x = 0$.

9. $f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0} \frac{-x-0}{x}$

$= -1$

$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0} \frac{x^2-0}{x}$

$= 0$

$\therefore$ $f'(0^-) \neq f'(0^+)$

$f'(1^-) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1^-} \frac{x^2-1}{x-1}$

$= 2$

$f'(1^+) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1^+} \frac{x^3-x+1-1}{x-1}$

$= \lim_{x\to 1} \frac{x(x^2-1)}{x-1}$

$= 2$

$\therefore$ $f'(1^-) = f'(1^+)$

$\therefore$ f(x) is differentiable at $x = 1$.

10. f(x) is continuous at $x = 1$.

$\therefore$ $f(1) = \lim_{x\to 1^+} f(x)$

$\Rightarrow a + b = b + a + c$

$\Rightarrow c = 0$

Also, f(x) is differentiable at $x = 1$.

$\therefore$ $Lf'(1) = Rf'(1)$

$\Rightarrow \left[\frac{d}{dx}(ax^2+b)\right]_{x=1} = \left[\frac{d}{dx}(bx^2+ax+c)\right]_{x=1}$

$\Rightarrow [2ax]_{x=1} = [2bx+a]_{x=1}$

$\Rightarrow 2a = 2b + a$

$\Rightarrow a = 2b$

11. $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0} \frac{x^p \cos\frac{1}{x}-0}{x}$

$= \lim_{x\to 0} x^{p-1} \cos\frac{1}{x} = 0$, if $p - 1 > 0$

i.e., if $p > 1$

$\therefore$ f(x) will be differentiable at $x = 0$, if $p > 1$

12. $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0 \end{cases}$

Clearly, f(x) is continuous and differentiable for all non-zero x.

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} e^x = 1$

and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} e^{-x} = 1$

Also at $x = 0$, $f(0) = e^0 = 1$

So, f(x) is continuous for all values of x.

$Lf'(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{e^h-1}{h} = 1$

$Rf'(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{-h}-1}{h} = -1$

So, f(x) is not differentiable at $x = 0$.

$\therefore$ f(x) is continuous every where but not differentiable at $x = 0$.

13. $f(x) = \begin{cases} x e^{-\left(-\frac{1}{x}+\frac{1}{x}\right)} = x, & x < 0 \\ xe^{-2/x}, & x > 0 \\ 0, & x = 0 \end{cases}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} x = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} xe^{-2/x} = 0$ and $f(0) = 0$

$\therefore$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

So, f(x) is continuous at $x = 0$

Also,

$Lf'(0) = 1$ and

$$Rf'(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{he^{-2/h}-0}{h} = \lim_{h\to 0} e^{-2/h} = 0$$

∴ f is not differentiable at $x = 0$.

Thus, f(x) is everywhere continuous but not differentiable at $x = 0$.

14. $$f(x) = \begin{cases} \dfrac{x^2}{x} = x, & x > 0 \\ 0, & x = 0 \\ \dfrac{x^2}{-x} = -x, & x < 0 \end{cases}$$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(-x) = 0, \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x = 0$

and $f(0) = 0$.

So, $f(x)$ is continuous at $x = 0$.

Also, $f(x)$ is continuous for all other values of x.

Hence, $f(x)$ is continuous everywhere.

Here, $Lf'(0) = -1$ and $Rf'(0) = 1$.

∴ f(x) is not differentiable at $x = 0$.

15. $$Lf'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{e^x + ax - b}{x}$$

$$Rf'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0} \frac{b(x-1)^2 - b}{x}$$

$$= \lim_{x\to 0} \frac{b(x^2-2x+1-1)}{x}$$

$$= \lim_{x\to 0} b(x-2) = -2b$$

Since $f'(0)$ exists.

∴ $Lf'(0)$ must exist.

∴ $1 - b = 0 \Rightarrow b = 1$

$Lf'(0) = Rf'(0) = -2$

and $$Lf'(0) = \lim_{x\to 0} \frac{e^x + ax - 1}{x}$$

$$= \lim_{x\to 0}\left(\frac{e^x - 1}{x} + a\right) = 1 + a$$

∴ $1 + a = -2 \Rightarrow a = -3$

∴ $(a, b) = (-3, 1)$

16. $$Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{m(1-h)^2 - m}{-h} = \lim_{h\to 0} \frac{m(1+h^2-2h-1)}{-h}$$

$$= \lim_{h\to 0} m(2-h) = 2m$$

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{2(1+h)-m}{h}$$

For differentiability, $Lf'(1) = Rf'(1)$.

But for any value of m, $Lf'(1) = R\,f'(1)$ is not possible.

17. $$Lf'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0}\left(\frac{-a\sin x + be^{-x} - b}{x}\right)$$

Applying L′ Hospital rule, we get

$$Lf'(0) = \lim_{x\to 0}\left(\frac{-a\cos x - be^{-x}}{1}\right) = -(a+b)$$

$$Rf'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0}\left(\frac{a\sin x + be^x - b}{x}\right)$$

Applying L′ Hospital rule, we get

$$Rf'(0) = \lim_{x\to 0}\left(\frac{a\cos x + be^x}{1}\right) = a + b$$

Since, f(x) is differentiable at $x = 0$.

∴ $Lf'(0) = Rf'(0)$

$\Rightarrow -(a+b) = a + b$

$\Rightarrow a + b = 0$

18. Let $y = \sqrt{\sqrt{x}+1}$

∴ $$\frac{dy}{dx} = \frac{1}{2\sqrt{\sqrt{x}+1}} \cdot \frac{d}{dx}(\sqrt{x}+1)$$

$$= \frac{1}{4\sqrt{x}.\sqrt{\sqrt{x}+1}}$$

$$= \frac{1}{4\sqrt{x(\sqrt{x}+1)}}$$

19. As $x° = \dfrac{\pi x}{180}$ radian.

∴ $$\frac{dy}{dx} = \frac{\pi}{180} \sec x° \tan x°$$

20. $10^{-x\tan x}\left[\frac{d}{dx}(10^{x\tan x})\right]$

$= 10^{-x\tan x}.\ 10^{x\tan x}.\ \log 10\ .\frac{d}{dx}\left(x\tan x\right)$

$= \log 10\ (\tan x + x\sec^2 x)$

21. $y = e^{\frac{x^2}{1+x^2}}$

$\therefore\ \frac{dy}{dx} = e^{\frac{x^2}{1+x^2}}\cdot\frac{d}{dx}\left(\frac{x^2}{1+x^2}\right)$

$= e^{\frac{x^2}{1+x^2}}.\left[\frac{(1+x^2).(2x)-x^2.(0+2x)}{(1+x^2)^2}\right]$

$= \frac{2x\,e^{\frac{x^2}{1+x^2}}}{(1+x^2)^2}$

22. $\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}$

$= \frac{1}{2\sqrt{u}}\times(3-4v)\times 2x$

$= \frac{1}{\sqrt{(3-2v)v}}\times(3-4v)\times x$

$= \frac{1}{\sqrt{(3-2x^2)x^2}}\times(3-4x^2)\times x$

$= \frac{3-4x^2}{\sqrt{3-2x^2}}$

23. $y = (\cos x^2)^2$

$\therefore\ \frac{dy}{dx} = 2\cos x^2.\frac{d}{dx}\left(\cos x^2\right)$

$= 2\cos x^2.(-\sin x^2).\frac{d}{dx}\left(x^2\right)$

$= 2\cos x^2.(-\sin x^2).2x$

$= -2x\ (2\sin x^2\cos x^2) = -2x\sin 2x^2$

24. $y = \frac{\tan x + \cot x}{\tan x - \cot x} = \frac{\tan x + \frac{1}{\tan x}}{\tan x - \frac{1}{\tan x}}$

$= -\frac{1+\tan^2 x}{1-\tan^2 x} = -\sec 2x$

$\therefore\ \frac{dy}{dx} = -\sec 2x\tan 2x.\frac{d}{dx}(2x)$

$= -2\sec 2x\tan 2x$

25. $y = \log\left(\sqrt{x}+\sqrt{x-a}\right)$

$\therefore\ \frac{dy}{dx} = \frac{1}{\sqrt{x}+\sqrt{x-a}}.\frac{d}{dx}\left(\sqrt{x}+\sqrt{x-a}\right)$

$= \frac{1}{\sqrt{x}+\sqrt{x-a}}\left(\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{x-a}}\right)$

$= \frac{1}{2\left(\sqrt{x}+\sqrt{x-a}\right)}\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{x-a}}\right)$

$= \frac{1}{2\left(\sqrt{x}+\sqrt{x-a}\right)}\left(\frac{\sqrt{x-a}+\sqrt{x}}{\sqrt{x}\sqrt{x-a}}\right)$

$= \frac{1}{2\sqrt{x}\sqrt{x-a}}$

26. $\frac{d}{dx}\left[\log\left(\sqrt{\sin\sqrt{e^x}}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\log\left(\sin\sqrt{e^x}\right)\right]$

$= \frac{1}{2}\cdot\frac{1}{\sin\sqrt{e^x}}\cdot\frac{d}{dx}\left(\sin\sqrt{e^x}\right)$

$= \frac{1}{2}\cdot\frac{1}{\sin\sqrt{e^x}}.\cos\sqrt{e^x}.\frac{d}{dx}\left(\sqrt{e^x}\right)$

$= \frac{1}{2}\cot\sqrt{e^x}.\frac{1}{2\sqrt{e^x}}.e^x = \frac{1}{4}e^{\frac{x}{2}}\cot\left(e^{\frac{x}{2}}\right)$

27. $\frac{d}{dx}\left(\frac{e^{ax}}{\sin(bx+c)}\right)$

$= \frac{\sin(bx+c).e^{ax}.\frac{d}{dx}(ax) - e^{ax}.\cos(bx+c).\frac{d}{dx}(bx+c)}{\{\sin(bx+c)\}^2}$

$= \frac{\sin(bx+c).e^{ax}.a - e^{ax}\cos(bx+c).b}{\{\sin(bx+c)\}^2}$

$= \frac{e^{ax}[a\sin(bx+c) - b\cos(bx+c)]}{\sin^2(bx+c)}$

28. $y = \sin\left(\sqrt{\sin x+\cos x}\right)$

$\therefore\ \frac{dy}{dx} = \cos\left(\sqrt{\sin x+\cos x}\right).\frac{d}{dx}\left(\sqrt{\sin x+\cos x}\right)$

$= \cos\left(\sqrt{\sin x+\cos x}\right).\frac{1}{2\sqrt{\sin x+\cos x}}\cdot\frac{d}{dx}(\sin x+\cos x)$

$= \frac{\cos\left(\sqrt{\sin x+\cos x}\right)}{2\sqrt{\sin x+\cos x}}.(\cos x-\sin x)$

29. $\frac{d}{dx}\left(\sqrt{\sec^2 x+\operatorname{cosec}^2 x}\right)=\frac{d}{dx}\left[\sqrt{\left(\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}\right)}\right]$

$=\frac{d}{dx}\left(\sqrt{\frac{1}{\cos^2 x\sin^2 x}}\right)=\frac{d}{dx}\left(\sqrt{\frac{4}{\sin^2 2x}}\right)$

$=\frac{d}{dx}(2\operatorname{cosec}2x)=-2\operatorname{cosec}2x\cot 2x.\frac{d}{dx}(2x)$

$=-4\operatorname{cosec}2x\cot 2x$

30. $y=\left(x\cot^3 x\right)^{\frac{3}{2}}$

$\therefore \quad \frac{dy}{dx}=\frac{3}{2}(x\cot^3 x)^{\frac{1}{2}}.\frac{d}{dx}(x\cot^3 x)$

$=\frac{3}{2}(x\cot^3 x)^{\frac{1}{2}}\left[\cot^3 x.1+x.3\cot^2 x.\frac{d}{dx}(\cot x)\right]$

$=\frac{3}{2}\left(x\cot^3 x\right)^{\frac{1}{2}}[\cot^3 x+3x\cot^2 x(-\operatorname{cosec}^2 x)]$

$=\frac{3}{2}\left(x\cot^3 x\right)^{\frac{1}{2}}(\cot^3 x-3x\cot^2 x\operatorname{cosec}^2 x)$

31. $y=\sqrt{\frac{1+\tan x}{1-\tan x}}=\sqrt{\tan\left(\frac{\pi}{4}+x\right)}$

$\therefore \quad \frac{dy}{dx}=\frac{1}{2\sqrt{\tan\left(\frac{\pi}{4}+x\right)}}\cdot\frac{d}{dx}\left[\tan\left(\frac{\pi}{4}+x\right)\right]$

$=\frac{1}{2}\sqrt{\frac{1-\tan x}{1+\tan x}}.\sec^2\left(\frac{\pi}{4}+x\right)$

32. $y=\log\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$

$\therefore \quad \frac{dy}{dx}=\frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}.\frac{d}{dx}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$

$=\frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}.\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right).\frac{1}{2}$

$=\frac{1}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}=\frac{1}{\sin\left(\frac{\pi}{2}+x\right)}$

$=\frac{1}{\cos x}=\sec x$

33. $\log\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\left(\log\sqrt{\frac{2\sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}}\right)$

$=\log\left(\tan\frac{x}{2}\right)$

$\therefore \quad \frac{dy}{dx}=\frac{1}{\tan\frac{x}{2}}\cdot\frac{d}{dx}\left(\tan\frac{x}{2}\right)$

$=\frac{1}{\tan\frac{x}{2}}.\sec^2\frac{x}{2}.\frac{1}{2}$

$=\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$=\frac{1}{\sin x}$

$=\operatorname{cosec}x$

34. Let $y=\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}\right]$

$\Rightarrow y=\log e^x+\log\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}$

$\Rightarrow y=x+\frac{3}{4}[\log(x-2)-\log(x+2)]$

$\therefore \quad \frac{dy}{dx}=1+\frac{3}{4}\left(\frac{1}{x-2}-\frac{1}{x+2}\right)$

$=1+\frac{3}{x^2-4}=\frac{x^2-1}{x^2-4}$

35. $f(x)=\frac{1}{\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}$

$=\frac{1}{\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}\times\frac{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}}{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}}$

$=\frac{1}{a^2-b^2}\left[\sqrt{x^2+a^2}-\sqrt{x^2+b^2}\right]$

$\therefore \quad f'(x)=\frac{1}{a^2-b^2}\left[\frac{1}{2\sqrt{x^2+a^2}}\cdot 2x-\frac{1}{2\sqrt{x^2+b^2}}\cdot 2x\right]$

$=\frac{x}{a^2-b^2}\left[\frac{1}{\sqrt{x^2+a^2}}-\frac{1}{\sqrt{x^2+b^2}}\right]$

36. $y = \log\left(\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right) = \dfrac{1}{2}\log\left(\dfrac{1+\sin x}{1-\sin x}\right)$

$= \dfrac{1}{2}\log(1+\sin x) - \dfrac{1}{2}\log(1-\sin x)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2}.\dfrac{1}{1+\sin x}.\cos x - \dfrac{1}{2}.\dfrac{1}{1-\sin x}.(-\cos x)$

$= \dfrac{1}{2}\cos x\left(\dfrac{1}{1+\sin x} + \dfrac{1}{1-\sin x}\right)$

$= \dfrac{1}{2}\cos x\left(\dfrac{2}{1-\sin^2 x}\right) = \dfrac{2\cos x}{2\cos^2 x}$

$= \dfrac{1}{\cos x} = \sec x$

37. $y = \dfrac{x}{2}\sqrt{a^2+x^2} + \dfrac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2}\left[\sqrt{a^2+x^2} + x\cdot\dfrac{1}{2\sqrt{a^2+x^2}}.2x\right]$

$+\dfrac{a^2}{2}\cdot\dfrac{1}{x+\sqrt{x^2+a^2}}\left[1+\dfrac{1}{2\sqrt{x^2+a^2}}.2x\right]$

$= \dfrac{1}{2\left(\sqrt{a^2+x^2}\right)}\left(a^2+x^2+x^2\right)$

$+\dfrac{a^2}{2}\cdot\dfrac{1}{x+\sqrt{x^2+a^2}}\cdot\dfrac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$

$= \dfrac{1}{2\left(\sqrt{a^2+x^2}\right)}\left(a^2+2x^2+a^2\right)$

$= \dfrac{2\left(a^2+x^2\right)}{2\sqrt{a^2+x^2}} = \sqrt{a^2+x^2}$

38. $f(x) = \cos(\sin x^2)$

$\therefore \quad f'(x) = -\sin(\sin x^2)\,.\,\dfrac{d}{dx}(\sin x^2)$

$= -\sin(\sin x^2)\,.\,(\cos x^2)\,.(2x)$

$\therefore \quad f'\left(\sqrt{\dfrac{\pi}{2}}\right) = -2\sqrt{\dfrac{\pi}{2}}\sin\left(\sin\dfrac{\pi}{2}\right)\cos\dfrac{\pi}{2}$

$= 0 \qquad \ldots.\left[\because \cos\dfrac{\pi}{2} = 0\right]$

39. Let $y = \dfrac{d}{dx}\left[\log f(e^x+2x)\right]$

$= \dfrac{1}{f(e^x+2x)}.\dfrac{d}{dx}\left[f(e^x+2x)\right]$

$= \dfrac{1}{f(e^x+2x)}.f'(e^x+2x).\dfrac{d}{dx}(e^x+2x)$

$= \dfrac{f'(e^x+2x)(e^x+2)}{f(e^x+2x)}$

$\therefore \quad (y)_{(x=0)} = \dfrac{f'(1).3}{f(1)} = \dfrac{2.3}{3} = 2$

40. $y = \sin^{-1}\left(\dfrac{19}{20}x\right) + \cos^{-1}\left(\dfrac{19}{20}x\right)$

$= \dfrac{\pi}{2} \qquad \ldots.\left[\because \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}\right]$

$\therefore \quad \dfrac{dy}{dx} = 0$

41. $y = \sec^{-1}\left(\dfrac{x+1}{x-1}\right) + \sin^{-1}\left(\dfrac{x-1}{x+1}\right)$

$= \cos^{-1}\left(\dfrac{x-1}{x+1}\right) + \sin^{-1}\left(\dfrac{x-1}{x+1}\right)$

$\therefore \quad y = \dfrac{\pi}{2} \qquad \ldots.\left[\because \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}\right]$

$\therefore \quad \dfrac{dy}{dx} = 0$

42. $y = \cos^{-1}\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \sin^{-1}\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \pi/2$

$\therefore \quad \dfrac{dy}{dx} = 0$

43. $\sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$

$= \sin^{-1}x + \cos^{-1}x \qquad \ldots.\left[\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\right]$

$= \dfrac{\pi}{2}$

$\therefore \quad \dfrac{d}{dx}\left(\sin^{-1}x + \sin^{-1}\sqrt{1-x^2}\right) = \dfrac{d}{dx}\left(\dfrac{\pi}{2}\right) = 0$

44. Let $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$

$= \tan^{-1}\left[\tan\left(\dfrac{\pi}{2}-x\right)\right] + \cot^{-1}\left[\cot\left(\dfrac{\pi}{2}-x\right)\right]$

$= \pi - 2x$

$\therefore \quad \dfrac{dy}{dx} = -2$

45. $\frac{d}{dx}\{\sin(2\cos^{-1}(\sin x))\}$

$= \frac{d}{dx}\left\{\sin\left(2\cos^{-1}\left(\cos\left(\frac{\pi}{2}-x\right)\right)\right)\right\}$

$= \frac{d}{dx}\left\{\sin\left(2\left(\frac{\pi}{2}-x\right)\right)\right\}$

$= \frac{d}{dx}\{\sin(\pi - 2x)\}$

$= -2 . \cos(\pi - 2x)$

$= 2\cos 2x$

46. $y = \tan^{-1}\left(\frac{x^{\frac{1}{3}}+a^{\frac{1}{3}}}{1-x^{\frac{1}{3}}a^{\frac{1}{3}}}\right) = \tan^{-1}\left(x^{\frac{1}{3}}\right) + \tan^{-1}\left(a^{\frac{1}{3}}\right)$

$\therefore \quad \frac{dy}{dx} = \frac{1}{1+\left(x^{\frac{1}{3}}\right)^2}.\frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}(1+x^{\frac{2}{3}})}$

47. $y = \tan^{-1}\left(\frac{\frac{6}{5}+\tan x}{1-\frac{6}{5}(\tan x)}\right)$

$= \tan^{-1}\left(\frac{6}{5}\right) + \tan^{-1}(\tan x)$

$\therefore \quad y = \tan^{-1}\left(\frac{6}{5}\right) + x$

$\therefore \quad \frac{dy}{dx} = 0 + 1 = 1$

48. $y = \tan^{-1}\left(\frac{5x-x}{1+5x.x}\right) + \tan^{-1}\left(\frac{\frac{2}{3}+x}{1-\frac{2}{3}.x}\right)$

$= \tan^{-1} 5x - \tan^{-1} x + \tan^{-1}\frac{2}{3} + \tan^{-1} x$

$= \tan^{-1} 5x + \tan^{-1}\frac{2}{3}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{1+(5x)^2}.5 = \frac{5}{1+25x^2}$

49. $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right]$

$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}\right)\right]$

$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\right)\right]$

$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)\right]$

$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)\right]$

$= \frac{d}{dx}\left[\tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$

$= \frac{d}{dx}\left(\frac{\pi}{4}-\frac{x}{2}\right) = -\frac{1}{2}$

Alternate Method:

Let $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$= \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right]$

$= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) = \frac{\pi}{4} - \frac{x}{2}$

$\therefore \quad \frac{dy}{dx} = \frac{-1}{2}$

50. $y = \tan^{-1}(\sec x - \tan x)$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{1-\sin x}{\cos x}\right)\right]$

$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)\right]$

$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)\right]$

$$= \frac{d}{dx}\left[\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)\right]$$

$$= \frac{d}{dx}\left(\frac{\pi}{4}-\frac{x}{2}\right) = -\frac{1}{2}$$

51. $\frac{d}{dx}\left(\tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

$$= \frac{d}{dx}\left(\tan^{-1}\sqrt{\frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}\right)$$

$$= \frac{d}{dx}\left(\tan^{-1}\left(\cot\frac{x}{2}\right)\right)$$

$$= \frac{d}{dx}\left(\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right)$$

$$= -\frac{1}{2}$$

52. $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

By rationalizing the denominator, we get

$$y = \cot^{-1}\left[\frac{2+2\cos x}{2\sin x}\right]$$

$$= \cot^{-1}\left[\frac{1+\cos x}{\sin x}\right]$$

$$= \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2}$$

$\therefore$ $\frac{dy}{dx} = \frac{1}{2}$

53. Put $\cos\alpha = \frac{5}{\sqrt{41}}$, $\sin\alpha = \frac{4}{\sqrt{41}}$

$\therefore$ $y = \sin^{-1}[\sin(x+\alpha)] = x+\alpha$

$\therefore$ $\frac{dy}{dx} = 1$

54. Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore$ $y = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right] = \frac{\pi}{4}+\theta$$

$$= \frac{\pi}{4}+\tan^{-1}x$$

$\therefore$ $\frac{dy}{dx} = \frac{1}{1+x^2}$

55. Let $y = \tan^{-1}\left(\frac{2}{x^{-1}-x}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore$ $y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$$= \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}x$$

$\therefore$ $\frac{dy}{dx} = \frac{2}{1+x^2}$

56. Let $y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

Put $x = \cot\theta \Rightarrow \theta = \cot^{-1}x$

$\therefore$ $y = \cos^{-1}\left(\frac{\cot^2\theta-1}{\cot^2\theta+1}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$$= \cos^{-1}(\cos 2\theta) = 2\theta = 2\cot^{-1}x$$

$\therefore$ $\frac{dy}{dx} = \frac{-2}{1+x^2}$

57. Let $y = \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$

Put $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$\therefore$ $y = \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta$$

$$= \sin^{-1}\left(\frac{x}{a}\right)$$

$\therefore$ $\frac{dy}{dx} = \frac{1}{a}\cdot\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \frac{1}{\sqrt{a^2-x^2}}$

58. Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore \quad y = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) + \sec^{-1}\left(\dfrac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$= \sin^{-1}(\sin 2\theta) + \sec^{-1}(\sec 2\theta)$

$= 2\theta + 2\theta = 4\theta = 4\tan^{-1}x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{4}{1+x^2}$

59. Put $x = \cos 2\theta \Rightarrow \theta = \dfrac{1}{2}\cos^{-1}x$

$\therefore \quad \sin^{-1}\left(\sqrt{\dfrac{1-x}{2}}\right) = \sin^{-1}\left(\sqrt{\dfrac{1-\cos 2\theta}{2}}\right)$

$= \sin^{-1}\left(\sqrt{\sin^2\theta}\right) = \theta$

$= \dfrac{1}{2}\cos^{-1}x$

$\therefore \quad \dfrac{d}{dx}\left[\sin^{-1}\left(\sqrt{\dfrac{1-x}{2}}\right)\right] = \dfrac{d}{dx}\left(\dfrac{1}{2}\cos^{-1}x\right)$

$= \dfrac{-1}{2\sqrt{1-x^2}}$

60. Put $e^{2x} = \cot\theta \Rightarrow \theta = \cot^{-1}(e^{2x})$

$\therefore \quad y = \tan^{-1}\left(\dfrac{\cot\theta+1}{\cot\theta-1}\right) = \tan^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$

$= \tan^{-1}\left[\tan\left(\dfrac{\pi}{4}+\theta\right)\right]$

$= \dfrac{\pi}{4} + \theta = \dfrac{\pi}{4} + \cot^{-1}(e^{2x})$

$\therefore \quad \dfrac{dy}{dx} = 0 - \dfrac{1}{1+(e^{2x})^2} . e^{2x}.2$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{2e^{2x}}{1+e^{4x}}$

61. Let $y = \sin^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{2}\right)$

Put $x = \cos 2\theta \Rightarrow \theta = \dfrac{1}{2}\cos^{-1}x$

$\therefore \quad y = \sin^{-1}\left[\dfrac{\sqrt{2}}{2}\cos\theta - \dfrac{\sqrt{2}}{2}\sin\theta\right]$

$= \sin^{-1}\left[\dfrac{1}{\sqrt{2}}\cos\theta - \dfrac{1}{\sqrt{2}}\sin\theta\right]$

$= \sin^{-1}\left(\sin\dfrac{\pi}{4}\cos\theta - \cos\dfrac{\pi}{4}\sin\theta\right)$

$= \sin^{-1}\left(\sin\left(\dfrac{\pi}{4}-\theta\right)\right) = \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1-x^2}}$

62. Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Put $x = \cos 2\theta \Rightarrow \theta = \dfrac{1}{2}\cos^{-1}x$

$\therefore \quad y = \tan^{-1}\left(\dfrac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$

$= \tan^{-1}\left(\dfrac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$

$= \tan^{-1}\left(\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right) = \tan^{-1}\left(\dfrac{1-\tan\theta}{1+\tan\theta}\right)$

$= \tan^{-1}\left(\tan\left(\dfrac{\pi}{4}-\theta\right)\right)$

$\therefore \quad y = \dfrac{\pi}{4} - \theta = \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

63. Let $y = \sin^2\left\{\cot^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right)\right\}$

Put $x = \cos\theta$

$\therefore \quad y = \sin^2\left\{\cot^{-1}\left(\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}\right)\right\}$

$= \sin^2\left\{\cot^{-1}\left(\sqrt{\dfrac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right)\right\}$

$= \sin^2\left\{\cot^{-1}\left(\tan\dfrac{\theta}{2}\right)\right\}$

$= \sin^2\left\{\cot^{-1}\left(\cot\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)\right)\right\}$

$= \sin^2\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)$

$\therefore \quad y = \cos^2\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{2} = \dfrac{1+x}{2}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2}$

64. $f(x) = \cot^{-1}(\cos 2x)^{1/2}$

$\therefore$ $f(x) = \cot^{-1}\left(\sqrt{\cos 2x}\right)$

$\therefore$ $f'(x) = \dfrac{-1}{1+\cos 2x}\left[\dfrac{-2\sin 2x}{2\sqrt{\cos 2x}}\right]$

$= \dfrac{\sin 2x}{(1+\cos 2x)\sqrt{\cos 2x}}$

$\therefore$ $f'\left(\dfrac{\pi}{6}\right) = \dfrac{\frac{\sqrt{3}}{2}}{\left(1+\frac{1}{2}\right)\left(\sqrt{\frac{1}{2}}\right)} = \sqrt{\dfrac{2}{3}}$

65. Since, $1 + \sin\theta = \left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2$

and $1 - \sin\theta = \left(\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2}\right)^2$

$\therefore$ $f(x) = \tan^{-1}\left(\dfrac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right) = \tan^{-1}\left(\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$

$= \tan^{-1}\left[\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)\right]$

$\therefore$ $f(x) = \dfrac{\pi}{4} + \dfrac{x}{2}$ $\qquad\therefore$ $f'(x) = \dfrac{1}{2}$

$\therefore$ $f'\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$

66. Put $\log x = \tan\theta \Rightarrow \theta = \tan^{-1}(\log x)$

$\therefore$ $f(x) = \cos^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta = 2\tan^{-1}(\log x)$

$\therefore$ $f'(x) = 2.\dfrac{1}{1+(\log x)^2}.\dfrac{1}{x}$

$\therefore$ $f'(e) = \dfrac{2}{1+(\log e)^2}.\dfrac{1}{e} = \dfrac{2}{1+1^2}.\dfrac{1}{e} = \dfrac{1}{e}$

67. $y = (x^x)^x$

Taking logarithm on both sides, we get

$\log y = x\log x^x = x^2\log x$

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{y}.\dfrac{dy}{dx} = x^2.\dfrac{1}{x} + 2x\log x = x(1 + 2\log x)$

$\Rightarrow \dfrac{dy}{dx} = xy(1 + 2\log x)$

68. $y = x^{x^2}$

Taking logarithm on both sides, we get

$\log y = x^2\log x$

Differentiating both sides w.r.t. x. we get

$\dfrac{1}{y}\cdot\dfrac{dy}{dx} = x^2.\ \dfrac{1}{x} + \log x\ .\ (2x)$

$\therefore$ $\dfrac{dy}{dx} = y(x + 2x\log x)$

$= x^{x^2}(x + 2x\log x) = x^{x^2+1}(1 + 2\log x)$

69. Let $y = x^{4x^3}$

Taking logarithm on both sides, we get

$\log y = 4x^3\ .\ \log x$

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{y}\cdot\dfrac{dy}{dx} = 4x^2 + 12x^2\log x$

$\therefore$ $\dfrac{dy}{dx} = x^{4x^3}\ .\ 4x^2(1 + 3\log x)$

$= 4x^{4x^3+2}(1+3\log x)$

70. $y = \sqrt{\dfrac{1+x}{1-x}}$

Taking logarithm on both sides, we get

$\log y = \dfrac{1}{2}\log(1+x) - \dfrac{1}{2}\log(1-x)$

Differentiating w.r.t. x, we get

$\dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{1}{2}\cdot\dfrac{1}{1+x} - \dfrac{1}{2}\cdot\dfrac{1}{1-x}.(-1)$

$\Rightarrow \dfrac{dy}{dx} = \sqrt{\dfrac{1+x}{1-x}}\cdot\dfrac{1}{(1+x)(1-x)}$

$= \dfrac{1}{(1+x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}}$

71. $y = \dfrac{2(x-\sin x)^{\frac{3}{2}}}{\sqrt{x}}$

Taking logarithm on both sides, we get

$\log y = \log 2 + \dfrac{3}{2}\log(x-\sin x) - \dfrac{1}{2}\log x$

Differentiating w.r.t. x, we get

$\dfrac{1}{y}\cdot\dfrac{dy}{dx} = 0 + \dfrac{3}{2}\cdot\dfrac{1}{x-\sin x}.(1-\cos x) - \dfrac{1}{2x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2(x-\sin x)^{\frac{3}{2}}}{\sqrt{x}}\left(\dfrac{3}{2}.\dfrac{1-\cos x}{x-\sin x} - \dfrac{1}{2x}\right)$

72. $y = \dfrac{e^x \log x}{x^2}$

Taking logarithm on both sides, we get

$\log y = x + \log(\log x) - 2\log x$

Differentiating w.r.t. x, we get

$\dfrac{1}{y}\cdot\dfrac{dy}{dx} = 1 + \dfrac{1}{x\log x} - \dfrac{2}{x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^x \log x}{x^2}\left(\dfrac{x\log x + 1 - 2\log x}{x\log x}\right)$

$= \dfrac{e^x[(x-2)\log x + 1]}{x^3}$

73. Let $y = (\sin x)^{\log x}$

Taking logarithm on both sides, we get

$\log y = \log x \log(\sin x)$

Differentiating both sides w.r.t.x, we get

$\dfrac{1}{y}.\dfrac{dy}{dx} = \log x.\dfrac{1}{\sin x}.\cos x + \log(\sin x).\dfrac{1}{x}$

$\Rightarrow \dfrac{dy}{dx} = (\sin x)^{\log x}\left(\dfrac{1}{x}\log \sin x + \cot x \ \log x\right)$

74. $y = (\tan x)^{\sin x}$

Taking logarithm on both sides, we get

$\log y = \sin x \log(\tan x)$

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{y}\cdot\dfrac{dy}{dx} = \sin x.\dfrac{1}{\tan x}.\sec^2 x + \log(\tan x).\cos x$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \sin x.\dfrac{\cos x}{\sin x}\cdot\dfrac{1}{\cos^2 x} + \cos x \log(\tan x)$

$\Rightarrow \dfrac{dy}{dx} = (\tan x)^{\sin x}[\sec x + \cos x \log(\tan x)]$

75. $x^2 e^y + 2xye^x + 13 = 0$

Differentiating w.r.t. x, we get

$2xe^y + x^2 e^y \dfrac{dy}{dx} + 2\left(xye^x + xe^x\dfrac{dy}{dx} + ye^x\right) = 0$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{2xe^y + 2y(xe^x + e^x)}{x(xe^y + 2e^x)}$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{2xe^{y-x} + 2y(x+1)}{x(xe^{y-x} + 2)}$

76. $\sec\left(\dfrac{x+y}{x-y}\right) = a \Rightarrow \dfrac{x+y}{x-y} = \sec^{-1} a$

Differentiating both sides w.r.t.x, we get

$\dfrac{(x-y)\left(1+\dfrac{dy}{dx}\right) - (x+y)\left(1-\dfrac{dy}{dx}\right)}{(x-y)^2} = 0$

$\Rightarrow (x - y - x - y) + (x - y + x + y)\dfrac{dy}{dx} = 0$

$\Rightarrow 2x\dfrac{dy}{dx} = 2y$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x}$

77. $\cos(x+y) = y\sin x$

Differentiating both sides w.r.t. x, we get

$-\sin(x+y).\left(1+\dfrac{dy}{dx}\right) = y\cos x + \sin x.\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{y\cos x + \sin(x+y)}{\sin(x+y) + \sin x}$

78. $\sin(x+y) + \cos(x+y) = 1$

Differentiating both sides w.r.t. x, we get

$\cos(x+y).\left(1+\dfrac{dy}{dx}\right) - \sin(x+y).\left(1+\dfrac{dy}{dx}\right) = 0$

$\Rightarrow \dfrac{dy}{dx}[\cos(x+y) - \sin(x+y)]$

$= -\cos(x+y) + \sin(x+y)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\sin(x+y) - \cos(x+y)}{\cos(x+y) - \sin(x+y)}$

$\Rightarrow \dfrac{dy}{dx} = -1$

79. $\sin(x+y) = \log(x+y)$

Differentiating both sides w.r.t. x, we get

$\cos(x+y)\left[1+\dfrac{dy}{dx}\right] = \dfrac{1}{x+y}\left(1+\dfrac{dy}{dx}\right)$

$\Rightarrow \cos(x+y)\dfrac{dy}{dx} - \dfrac{1}{x+y}\cdot\dfrac{dy}{dx} = \dfrac{1}{x+y} - \cos(x+y)$

$\Rightarrow -\dfrac{dy}{dx}\left[\dfrac{1}{x+y} - \cos(x+y)\right] = \dfrac{1}{x+y} - \cos(x+y)$

$\Rightarrow \dfrac{dy}{dx} = -1$

80. $3\sin(xy) + 4\cos(xy) = 5$

Differentiating both sides w.r.t. x, we get

$3\cos(xy)\left[y + x\dfrac{dy}{dx}\right] - 4\sin(xy)\left[y + x\dfrac{dy}{dx}\right] = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{4y\sin(xy) - 3y\cos(xy)}{3x\cos(xy) - 4x\sin(xy)}$

$= \dfrac{y[4\sin(xy) - 3\cos(xy)]}{-x[4\sin(xy) - 3\cos(xy)]} = -\dfrac{y}{x}$

81. $x = y\sqrt{1-y^2}$

Differentiating both sides w.r.t. x, we get

$$1 = \frac{dy}{dx}\sqrt{1-y^2} + y.\frac{1}{2\sqrt{1-y^2}}.(-2y).\frac{dy}{dx}$$

$$\Rightarrow\ 1 = \frac{dy}{dx}\sqrt{1-y^2} - \frac{y^2}{\sqrt{1-y^2}}\cdot\frac{dy}{dx}$$

$$\Rightarrow\ 1 = \frac{dy}{dx}\left(\frac{1-y^2-y^2}{\sqrt{1-y^2}}\right)$$

$$\Rightarrow\ 1 = \frac{dy}{dx}\left(\frac{1-2y^2}{\sqrt{1-y^2}}\right)$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1-2y^2}$$

82. $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow x^2(1+y) = y^2(1+x)$

$\Rightarrow (x-y)(x+y+xy) = 0$

$\Rightarrow x+y+xy = 0$ ....(i) [$\because x \neq y$]

Differentiating w.r.t. x, we get

$$1 + \frac{dy}{dx} + x\frac{dy}{dx} + y.1 = 0$$

$$\Rightarrow\ (1+x)\frac{dy}{dx} = -(1+y)$$

$$\Rightarrow\ (1+x)^2\frac{dy}{dx} = -(1+x)(1+y)$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{-(1+y+x+xy)}{(1+x)^2}$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{-1}{(1+x)^2}$$ ....[From (i)]

83. If $y = \sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+...\infty}}}$,

then $\frac{dy}{dx} = \frac{f'(x)}{2y-1}$ $\therefore$ $\frac{dy}{dx} = \frac{1}{x(2y-1)}$

84. If $y = \sqrt{f(x)+y}$, then

$\frac{dy}{dx} = \frac{f'(x)}{2y-1}$ $\therefore$ $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

85. If $y = \sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+....\infty}}}$, then

$$\frac{dy}{dx} = \frac{f'(x)}{2y-1}$$

$$\therefore\ \frac{dy}{dx} = \frac{-\sin x}{2y-1} = \frac{\sin x}{1-2y}$$

86. $y = e^{x+e^{x+e^{x+....\infty}}}$

$\Rightarrow y = e^{x+y}$

Taking logarithm on both sides, we get

$\log y = (x+y)\log e$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y}{1-y}$$

87. $y = \sqrt{x+\sqrt{y+y}}$

$\Rightarrow (y^2-x) = \sqrt{2y}$

$\Rightarrow (y^2-x)^2 = 2y$

Differentiating both sides w.r.t. x, we get

$$2(y^2-x)\left(2y\frac{dy}{dx} - 1\right) = 2\frac{dy}{dx}$$

$$\therefore\ \frac{dy}{dx} = \frac{(y^2-x)}{2y^3-2xy-1}$$

88. If $y = f(x)^{f(x)^{f(x)^{...\infty}}}$, then

$$\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1-y\log f(x)]}$$

$$\therefore\ \frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}$$

$$\Rightarrow\ x(1-y\log x)\frac{dy}{dx} = y^2$$

89. If $y = f(x)^{f(x)^{f(x)^{...\infty}}}$, then

$$\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1-y\log f(x)]}$$

$$\therefore\ \frac{dy}{dx} = \frac{y^2.\frac{1}{2\sqrt{x}}}{\sqrt{x}(1-y\log\sqrt{x})} = \frac{y^2}{2x\left(1-\frac{1}{2}y\log x\right)}$$

$$\Rightarrow\ (2-y\log x)\frac{dy}{dx} = \frac{y^2}{x}$$

90. If $y = f(x)^{f(x)^{f(x)^{...\infty}}}$, then

$$\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1-y\log f(x)]}$$

$$\therefore\ \frac{dy}{dx} = \frac{y^2\cos x}{\sin x(1-y\log\sin x)}$$

$$= \frac{y^2\cot x}{1-y\log\sin x}$$

91. $y = x^2 + \frac{1}{y}$

If $y = \text{f}(x) + \frac{1}{y}$, then $\frac{\text{d}y}{\text{d}x} = \frac{y\text{f}'(x)}{2y - \text{f}(x)}$

$\therefore \quad \frac{\text{d}y}{\text{d}x} = \frac{2xy}{2y - x^2}$

92. $y = xe^{xy}$

Taking logarithm on both sides, we get

$\log y = \log x + \log e^{xy}$

$\Rightarrow \log y = \log x + xy$

Differentiating both sides w.r.t. x, we get

$\frac{1}{y} \cdot \frac{\text{d}y}{\text{d}x} = \frac{1}{x} + x\frac{\text{d}y}{\text{d}x} + y$

$\Rightarrow \frac{\text{d}y}{\text{d}x}\left(\frac{1}{y} - x\right) = \frac{1}{x} + y$

$\Rightarrow \frac{\text{d}y}{\text{d}x} = \frac{(1+xy)y}{(1-xy)x}$

93. $x^y = y^x$

Taking logarithm on both sides, we get

$y \log_e x = x \log_e y$

Differentiating both sides w.r.t. x, we get

$\log_e x\frac{\text{d}y}{\text{d}x} + \frac{y}{x} = \log_e y + x \cdot \frac{1}{y} \cdot \frac{\text{d}y}{\text{d}x}$

$\Rightarrow \frac{\text{d}y}{\text{d}x}\left(\frac{y\log_e x - x}{y}\right) = \frac{x\log_e y - y}{x}$

$\Rightarrow \frac{\text{d}y}{\text{d}x} = \frac{y(x\log_e y - y)}{x(y\log_e x - x)}$

94. $x^y . y^x = 1$

Taking logarithm on both sides, we get

$y \log x + x \log y = 0$

Differentiating w.r.t. x, we get

$\log x . \frac{\text{d}y}{\text{d}x} + y . \frac{1}{x} + x . \frac{1}{y} \cdot \frac{\text{d}y}{\text{d}x} + \log y . 1 = 0$

$\Rightarrow \frac{\text{d}y}{\text{d}x}\left(\log x + \frac{x}{y}\right) + \frac{y}{x} + \log y = 0$

$\Rightarrow \frac{\text{d}y}{\text{d}x} = -\frac{\left(\frac{y}{x} + \log y\right)}{\log x + \frac{x}{y}}$

$\Rightarrow \frac{\text{d}y}{\text{d}x} = -\frac{y}{x}\left(\frac{y + x\log y}{x + y\log x}\right)$

95. $x^{\text{m}}y^{\text{n}} = 2(x+y)^{\text{m}+\text{n}}$

Taking logarithm on both sides, we get

$\text{m} \log x + \text{n} \log y = \log 2 + (\text{m}+\text{n})\log(x+y)$

Differentiating both sides w.r.t. x, we get

$\frac{\text{m}}{x} + \frac{\text{n}}{y} \cdot \frac{\text{d}y}{\text{d}x} = \frac{\text{m}+\text{n}}{x+y}\left(1 + \frac{\text{d}y}{\text{d}x}\right)$

$\Rightarrow \left(\frac{\text{m}+\text{n}}{x+y} - \frac{\text{n}}{y}\right)\frac{\text{d}y}{\text{d}x} = \frac{\text{m}}{x} - \frac{\text{m}+\text{n}}{x+y}$

$\Rightarrow \frac{\text{d}y}{\text{d}x} = \frac{y}{x}$

96. $x^y = 2^{x-y}$

Taking logarithm on both sides, we get

$y \log x = (x - y) \log 2$

Differentiating both sides w.r.t. x, we get

$y . \frac{1}{x} + \log x . \frac{\text{d}y}{\text{d}x} = \log 2\left(1 - \frac{\text{d}y}{\text{d}x}\right)$

$\Rightarrow (\log x + \log 2)\frac{\text{d}y}{\text{d}x} = \log 2 - \frac{y}{x}$

$\Rightarrow [\log(2x)]\frac{\text{d}y}{\text{d}x} = \frac{x\log 2 - y}{x}$

$\Rightarrow \frac{\text{d}y}{\text{d}x} = \frac{x\log 2 - y}{x\log(2x)}$

97. $y = \text{a}^{x^y}$

$\therefore \quad \log y = x^y \log \text{a}$

$\therefore \quad \log(\log y) = y \log x + \log(\log \text{a})$

Differentiating both sides w.r.t.x, we get

$\frac{1}{\log y} \cdot \frac{1}{y} \cdot \frac{\text{d}y}{\text{d}x} = \frac{y}{x} + \frac{\text{d}y}{\text{d}x}\log x$

$\Rightarrow \left(\frac{1}{y\log y} - \log x\right)\frac{\text{d}y}{\text{d}x} = \frac{y}{x}$

$\Rightarrow x(1 - y\log x\log y)\frac{\text{d}y}{\text{d}x} = y^2 \log y$

98. $\log(x+y) = 2xy$ ....(i)

Differentiating both sides w.r.t. x, we get

$\left(\frac{1}{x+y}\right)\left(1 + \frac{\text{d}y}{\text{d}x}\right) = 2\left(x\frac{\text{d}y}{\text{d}x} + y\right)$

$\Rightarrow \frac{\text{d}y}{\text{d}x} = \frac{1 - 2xy - 2y^2}{2x^2 + 2xy - 1}$

Putting $x = 0$ in (i), we get

$y = 1$

$\therefore \quad y'(0) = \frac{1-0-2}{0+0-1} = 1$

99. Let $y = e^x \cos x$ and $z = e^{-x} \sin x$

$\therefore \quad \frac{dy}{dx} = e^x (\cos x - \sin x)$ and

$\frac{dz}{dx} = e^{-x} (\cos x - \sin x)$

$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = e^{2x}$

100. Let $y = \cos^{-1}\left(\sqrt{x}\right)$ and $z = \sqrt{1-x}$

$\therefore \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$ and $\frac{dz}{dx} = \frac{-1}{2\sqrt{1-x}}$

$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{1}{\sqrt{x}}$

101. $x = \frac{e^t + e^{-t}}{2}$ and $y = \frac{e^t - e^{-t}}{2}$

$\therefore \quad \frac{dx}{dt} = \frac{e^t - e^{-t}}{2}$ and $\frac{dy}{dt} = \frac{e^t + e^{-t}}{2}$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{e^t + e^{-t}}{2}}{\frac{e^t - e^{-t}}{2}} = \frac{x}{y}$

102. $x = a(t \cos t - \sin t)$ and $y = a(t \sin t + \cos t)$

$\therefore \quad \frac{dx}{dt} = a(-t \sin t + \cos t - \cos t) = -at \sin t$

and $\frac{dy}{dt} = a(t \cos t + \sin t - \sin t) = at \cos t$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at \cos t}{-at \sin t} = -\cot t$

103. $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$

$\therefore \quad \frac{dx}{d\theta} = -3a \cos^2 \theta \,.\, \sin \theta$

and $\frac{dy}{d\theta} = 3a \sin^2 \theta \,.\, \cos \theta$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\tan \theta$

$\therefore \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2 \theta} = \sqrt{\sec^2 \theta} = |\sec \theta|$

104. $y = \log(1 + \theta)$, $x = \sin^{-1}\theta$

$\therefore \quad \frac{dy}{d\theta} = \frac{1}{1+\theta}$, $\frac{dx}{d\theta} = \frac{1}{\sqrt{1-\theta^2}}$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sqrt{1-\theta^2}}{1+\theta}$

$= \sqrt{\frac{(1+\theta)(1-\theta)}{(1+\theta)^2}} = \sqrt{\frac{1-\theta}{1+\theta}}$

105. Let $y = \sin^{-1} x$ and $z = \cos^{-1}\left(\sqrt{1-x^2}\right)$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$z = \cos^{-1}\left(\sqrt{1-x^2}\right) = \sin^{-1} x$

$\therefore \quad \frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = 1$

106. Let $y = \sin^{-1}\left(\frac{1-x}{1+x}\right)$ and $z = \sqrt{x}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(\frac{1-x}{1+x}\right)^2}} \cdot \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}$

$= \frac{-1}{\sqrt{x}(1+x)}$

and $\frac{dz}{dx} = \frac{1}{2\sqrt{x}}$

$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{-2}{1+x}$

107. Let $y = a^{\sec x}$ and $z = a^{\tan x}$

$\therefore \quad \dfrac{dy}{dx} = a^{\sec x} \log a \sec x \tan x$

and $\dfrac{dz}{dx} = a^{\tan x} \log a \sec^2 x$

$\therefore \quad \dfrac{dy}{dz} = \dfrac{\frac{dy}{dx}}{\frac{dz}{dx}} = \dfrac{a^{\sec x} \log a \sec x \tan x}{a^{\tan x} \log a \sec^2 x}$

$= a^{\sec x - \tan x} . \dfrac{\sin x}{\cos x . \frac{1}{\cos x}} = \sin x \, a^{\sec x - \tan x}$

108. $x = e^{\theta}\left(\theta + \dfrac{1}{\theta}\right)$

$\therefore \quad \dfrac{dx}{d\theta} = e^{\theta}\left(1 - \dfrac{1}{\theta^2}\right) + e^{\theta}\left(\theta + \dfrac{1}{\theta}\right)$

$= e^{\theta}\left(1 + \theta + \dfrac{1}{\theta} - \dfrac{1}{\theta^2}\right)$

$= e^{\theta}\left(\dfrac{\theta^2 + \theta^3 + \theta - 1}{\theta^2}\right)$

$y = e^{-\theta}\left(\theta - \dfrac{1}{\theta}\right)$

$\therefore \quad \dfrac{dy}{d\theta} = e^{-\theta}\left(1 + \dfrac{1}{\theta^2}\right) - e^{-\theta}\left(\theta - \dfrac{1}{\theta}\right)$

$= e^{-\theta}\left(1 + \dfrac{1}{\theta^2} - \theta + \dfrac{1}{\theta}\right)$

$= e^{-\theta}\left(\dfrac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{e^{-2\theta}(1 + \theta^2 - \theta^3 + \theta)}{\theta^2 - 1 + \theta^3 + \theta}$

109. $\dfrac{dx}{dt} = a\left(-\sin t + \dfrac{1}{\tan\frac{t}{2}} \cdot \sec^2 \dfrac{t}{2} \cdot \dfrac{1}{2}\right)$

$= a\left(-\sin t + \dfrac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$

$= a\left(-\sin t + \dfrac{1}{\sin t}\right) = a\left(\dfrac{\cos^2 t}{\sin t}\right)$

$= a \cos t \cot t$

and $\dfrac{dy}{dt} = a \cos t$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1}{\cot t} = \tan t$

110. $x = a(\sin 2\theta + \dfrac{1}{2} \sin 4\theta)$,

$y = b\left[\cos 2\theta - \dfrac{1}{2}(1 + \cos 4\theta)\right]$

$\therefore \quad \dfrac{dx}{d\theta} = 2a(\cos 2\theta + \cos 4\theta) = 2a(2\cos 3\theta \cos\theta)$

and $\dfrac{dy}{d\theta} = 2b(\sin 4\theta - \sin 2\theta) = 2b(2\cos 3\theta \sin\theta)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{b}{a} \tan\theta$

111. $x = t \log t$ and $y = t^t$

$\therefore \quad x = \log t^t = \log y$

Differentiating both sides w.r.t. x, we get

$1 = \dfrac{1}{y} . \dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = y = t^t$

Since, $x = t \log t$

$\therefore \quad x = \log t^t$

$\Rightarrow e^x = t^t$

$\therefore \quad \dfrac{dy}{dx} = e^x$

112. Let $y = \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right)$ and $z = \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right)$

Put $x = \tan\theta$

$\therefore \quad y = \tan^{-1}\left(\dfrac{2\tan\theta}{1 - \tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$

and $z = \sin^{-1}\left(\dfrac{2\tan\theta}{1 + \tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

$\therefore \quad y = z$

$\therefore \quad \dfrac{dy}{dz} = 1$

113. Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $z = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\therefore$ $y = 2\tan^{-1}x$ and $z = 2\tan^{-1}x$

$\therefore$ $\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = 1$

114. Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and $z = \cot^{-1}\left(\frac{1-3x^2}{3x-x^3}\right)$

$\therefore$ $y = 2\tan^{-1}x$ and $z = 3\tan^{-1}x$

$\therefore$ $\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\frac{2}{1+x^2}}{\frac{3}{1+x^2}} = \frac{2}{3}$

115. Put $t = \sin\theta$

$\therefore$ $x = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$
$= \sin^{-1}(\sin 3\theta) = 3\theta$

$y = \cos^{-1}\left(\sqrt{1-\sin^2\theta}\right) = \cos^{-1}(\cos\theta) = \theta$

$\therefore$ $x = 3y$

$\Rightarrow y = \frac{1}{3}x$

$\therefore$ $\frac{dy}{dx} = \frac{1}{3}$

116. $\sin y = \frac{t}{\sqrt{1+t^2}}$

Put $t = \tan\theta$

$\therefore$ $\sin y = \frac{\tan\theta}{\sec\theta} = \sin\theta$

$\therefore$ $y = \theta$

$\therefore$ $\frac{dy}{d\theta} = 1$

$\cos x = \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sec\theta} = \cos\theta$

$\therefore$ $x = \theta$ $\qquad\therefore$ $\frac{dx}{d\theta} = 1$

$\therefore$ $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = 1$

117. Let $y = \tan^{-1}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)$ and $z = \cos^{-1}(x^2)$

Put $x^2 = \cos 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1}x^2$

$\therefore$ $y = \tan^{-1}\left(\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right) = \tan^{-1}(\tan\theta) = \theta$

$\Rightarrow y = \frac{1}{2}\cos^{-1}(x^2)$

$\Rightarrow y = \frac{1}{2}z$

$\therefore$ $\frac{dy}{dz} = \frac{1}{2}$

118. Putting $t = \tan\theta$ in the given equations, we get

$x = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$ and

$y = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$

$\therefore$ $\frac{dx}{d\theta} = -2\sin 2\theta$ and $\frac{dy}{d\theta} = 2\cos 2\theta$

$\therefore$ $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} - \frac{\cos 2\theta}{\sin 2\theta} = -\frac{x}{y}$

119. Put $x = \sin\theta \Rightarrow 2\sin^{-1}x = 2\theta$
$\Rightarrow \sin(2\sin^{-1}x) = \sin 2\theta \Rightarrow y = \sin 2\theta$

$\therefore$ $\frac{dx}{d\theta} = \cos\theta$ and $\frac{dy}{d\theta} = 2\cos 2\theta$

$\therefore$ $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\cos 2\theta}{\cos\theta}$

$= \frac{2(1-2\sin^2\theta)}{\sqrt{1-\sin^2\theta}} = \frac{2-4x^2}{\sqrt{1-x^2}}$

120. Let $y = \tan^{-1}\left[\frac{\sin x}{1+\cos x}\right]$ and $z = \tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]$

$= \tan^{-1}\left[\frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right]$

$= \tan^{-1}\left[\tan\left(\frac{x}{2}\right)\right] = \frac{x}{2}$

$\therefore$ $\frac{dy}{dx} = \frac{1}{2}$

$$z = \tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]$$

$$= \tan^{-1}\left[\frac{(\cos^2 x/2-\sin^2 x/2)}{(\sin x/2+\cos x/2)^2}\right]$$

$$= \tan^{-1}\left[\frac{(\cos x/2-\sin x/2)}{(\cos x/2+\sin x/2)}\right]$$

$$= \tan^{-1}\left(\frac{1-\tan x/2}{1+\tan x/2}\right)$$

$$= \tan^{-1}[\tan(\pi/4 - x/2)] = \pi/4 - x/2$$

$\therefore \quad \frac{dz}{dx} = -\frac{1}{2}$

$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = -1$

121. $x = a\cos^4\theta$ and $y = a\sin^4\theta$

$\therefore \quad \frac{dx}{d\theta} = -4a\cos^3\theta\sin\theta$

and $\frac{dy}{d\theta} = 4a\sin^3\theta\cos\theta$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\sin^2\theta}{\cos^2\theta} = -\tan^2\theta$

$\therefore \quad \left(\frac{dy}{dx}\right)_{\left(\theta=\frac{3\pi}{4}\right)} = -\tan^2\left(\frac{3\pi}{4}\right) = -(-1)^2 = -1$

122. Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ and $z = \sqrt{1+3x}$

$\therefore \quad y = \cos^{-1}(2x^2-1) = 2\cos^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$ and $\frac{dz}{dx} = \frac{3}{2\sqrt{1+3x}}$

$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{-2}{\sqrt{1-x^2}} \times \frac{2\sqrt{1+3x}}{3}$

$\therefore \quad \left(\frac{dy}{dz}\right)_{\left(x=\frac{-1}{3}\right)} = 0$

123. $x = \sin t\cos 2t$ and $y = \cos t\sin 2t$

$\therefore \quad \frac{dx}{dt} = \cos t\cos 2t - 2\sin t\sin 2t$

and $\frac{dy}{dt} = 2\cos t\cos 2t - \sin t\sin 2t$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t\cos 2t - \sin t\sin 2t}{\cos t\cos 2t - 2\sin t\sin 2t}$

$\therefore \quad \left(\frac{dy}{dx}\right)_{\left(t=\frac{\pi}{4}\right)} = \frac{0-\frac{1}{\sqrt{2}}}{0-2\times\frac{1}{\sqrt{2}}} = \frac{1}{2}$

124. Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and

$z = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore \quad y = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2(1+x^2)}$

$z = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$\therefore \quad z = \tan^{-1}\left(\frac{2\sin\theta\cos\theta}{1-2\sin^2\theta}\right) = \tan^{-1}\left(\frac{\sin 2\theta}{\cos 2\theta}\right)$

$= \tan^{-1}(\tan 2\theta) = 2\theta$

$= 2\sin^{-1}x$

$\therefore \quad \frac{dz}{dx} = \frac{2}{\sqrt{1-x^2}}$

$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\sqrt{1-x^2}}{4(1+x^2)}$

$\therefore \quad \left(\frac{dy}{dz}\right)_{x=0} = \frac{1}{4}$

125. $y = \cos^2\frac{3x}{2} - \sin^2\frac{3x}{2}$

$\Rightarrow y = \cos 3x$ ....(i)

$\therefore \quad \frac{dy}{dx} = -3\sin 3x$

$\therefore \quad \frac{d^2y}{dx^2} = -9\cos 3x$

$\Rightarrow \frac{d^2y}{dx^2} = -9y$ ....[From (i)]

126. $x = t^2$ and $y = t^3 + 1$

$\therefore \quad \frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t}{2}$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{3}{2}.\frac{dt}{dx} = \frac{3}{2}.\frac{1}{2t} = \frac{3}{4t}$

127. $\frac{dy}{dt} = 10t^9$ and $\frac{dx}{dt} = 8t^7$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{5t^2}{4}$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{5}{4}\cdot 2t\cdot\frac{dt}{dx} = \frac{5t}{2}\cdot\frac{1}{8t^7} = \frac{5}{16t^6}$

128. $x = \log t$ and $y = \frac{1}{t}$

$\therefore \quad \frac{dx}{dt} = \frac{1}{t}$ and $\frac{dy}{dt} = -\frac{1}{t^2}$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{1}{t}$ ….(i)

$\therefore \quad \frac{d^2y}{dx^2} = -\left(-\frac{1}{t^2}\right)\frac{dt}{dx}$

$= \frac{1}{t^2}\cdot\frac{1}{\frac{dx}{dt}} = \frac{1}{t^2}\cdot\frac{1}{\frac{1}{t}} = \frac{1}{t}$

$\therefore \quad \frac{d^2y}{dx^2} = -\frac{dy}{dx}$ ….[From (i)]

129. $y = 1 - x + \frac{x^2}{(2)!} - \frac{x^3}{(3)!} + ….$

$\Rightarrow y = e^{-x}$ ….(i)

$\therefore \quad \frac{dy}{dx} = e^{-x}(-1)$

$\therefore \quad \frac{d^2y}{dx^2} = (-1)\{e^{-x}.(-1)\} = e^{-x} = y$ ….[From (i)]

130. Consider option (C),

$f(x) = \sin x$

$\Rightarrow f(0) = 0$ and

$f'(x) = \cos x$

$\Rightarrow f'(0) = 1$

Also, $f''(x) = -\sin x = -f(x)$

$\therefore$ option (C) is the correct answer.

131. $e^y(x+1) = 1 \qquad \Rightarrow e^y = \frac{1}{x+1}$

$\Rightarrow y = \log\left(\frac{1}{x+1}\right)$

$\Rightarrow y = -\log(x+1)$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{x+1}$ ....(i)

$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} = \left(\frac{-1}{x+1}\right)^2$

$= \left(\frac{dy}{dx}\right)^2$ .....[From (i)]

132. $y = ax^5 + \frac{b}{x^4}$ ....(i)

$\therefore \quad \frac{dy}{dx} = 5ax^4 - \frac{4b}{x^5}$

$\therefore \quad \frac{d^2y}{dx^2} = 20ax^3 + \frac{20b}{x^6}$

$= \frac{20}{x^2}\left(ax^5 + \frac{b}{x^4}\right) = \frac{20y}{x^2}$ ….[From (i)]

133. $y = ax^{n+1} + bx^{-n}$ ….(i)

$\therefore \quad \frac{dy}{dx} = (n+1)ax^n - nbx^{-n-1}$

$\therefore \quad \frac{d^2y}{dx^2} = n(n+1)ax^{n-1} + n(n+1)bx^{-n-2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{n(n+1)}{x^2}\left(ax^{n+1} + bx^{-n}\right)$

$\Rightarrow x^2\frac{d^2y}{dx^2} = n(n+1)y$ ….[From (i)]

134. $y = a\cos(\log x) + b\sin(\log x)$ ….(i)

$\therefore \quad y' = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$

$\Rightarrow xy' = -a\sin(\log x) + b\cos(\log x)$

Differentiating both sides w.r.t.x, we get

$xy'' + y' = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$

$\Rightarrow x^2y'' + xy' = -[a\cos(\log x) + b\sin(\log x)]$

$\Rightarrow x^2y'' + xy' = -y$ ….[From (i)]

135. $y = a^x.b^{2x-1}$ ….(i)

$\therefore \quad \frac{dy}{dx} = b^{2x-1}.a^x\log a + a^x.\,2b^{2x-1}\log b$

$= a^xb^{2x-1}(\log a + 2\log b)$

$\therefore \quad \frac{d^2y}{dx^2} = a^xb^{2x-1}(\log a + 2\log b)^2$

$= a^xb^{2x-1}(\log ab^2)^2$

$= y(\log ab^2)^2$ ….[From (i)]

136. $y = \log\left(x+\sqrt{x^2+a^2}\right)$

$\therefore \quad \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}}.\left(1+\frac{1}{2\sqrt{x^2+a^2}}.2x\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}} \times \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2+a^2}}$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{-1}{2}\left(x^2+a^2\right)^{-\frac{3}{2}}.2x = \frac{-x}{\left(x^2+a^2\right)^{\frac{3}{2}}}$

137. $y = x^2 + 2x + 3$

$\therefore \quad \frac{dy}{dx} = 2x + 2$

$\Rightarrow \frac{dx}{dy} = \frac{1}{2x+2}$

$\therefore \quad \frac{d^2x}{dy^2} = \frac{-1}{2(x+1)^2}.\frac{dx}{dy} = \frac{-1}{4(x+1)^3}$

138. $y = x + e^x$

$\therefore \quad \frac{dy}{dx} = 1 + e^x$ ....(i)

$\therefore \quad \frac{dx}{dy} = \frac{1}{1+e^x} = (1 + e^x)^{-1}$

$\therefore \quad \frac{d^2x}{dy^2} = -(1 + e^x)^{-2}.\frac{d}{dy}(1 + e^x)$

$= -(1 + e^x)^{-2}.e^x.\frac{dx}{dy}$

$= -\frac{e^x}{(1+e^x)^2}.\frac{1}{1+e^x}$ ....[From (i)]

$= -\frac{e^x}{(1+e^x)^3}$

139. $y = \sin x + e^x$

$\therefore \quad \frac{dy}{dx} = \cos x + e^x$

$\Rightarrow \frac{dx}{dy} = (\cos x + e^x)^{-1}$ ….(i)

$\therefore \quad \frac{d^2x}{dy^2} = -(\cos x + e^x)^{-2}(-\sin x + e^x)\cdot\frac{dx}{dy}$

$= \frac{(\sin x - e^x)}{(\cos x + e^x)^2}\cdot(\cos x + e^x)^{-1}$ ….[From (i)]

$= \frac{\sin x - e^x}{(\cos x + e^x)^3}$

140. $y = e^{2x}$

$\therefore \quad \frac{dy}{dx} = 2e^{2x}$ $\quad\therefore \quad \frac{d^2y}{dx^2} = 4e^{2x}$

Now, $y = e^{2x}$

$\therefore \quad \log y = 2x$

$\therefore \quad x = \frac{1}{2}\log y$ $\quad\therefore \quad \frac{dx}{dy} = \frac{1}{2y}$

$\therefore \quad \frac{d^2x}{dy^2} = \frac{-1}{2y^2} = \frac{-1}{2(e^{2x})^2}$

$\therefore \quad \frac{d^2y}{dx^2} \times \frac{d^2x}{dy^2} = \frac{-2}{e^{2x}} = -2e^{-2x}$

141. Let $y = 2\cos x\cos 3x$

$\Rightarrow y = \cos 4x + \cos 2x$

$\therefore \quad \frac{dy}{dx} = -4\sin 4x - 2\sin 2x$

$\therefore \quad \frac{d^2y}{dx^2} = -16\cos 4x - 4\cos 2x$

$= -4(\cos 2x + 4\cos 4x)$

$= -2^2(\cos 2x + 2^2\cos 4x)$

142. $\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$

$\therefore \quad \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left\{\left(\frac{dy}{dx}\right)^{-1}\right\}$

$\Rightarrow \frac{d^2x}{dy^2} = \frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^{-1}\right\}\frac{dx}{dy}$

$\Rightarrow \frac{d^2x}{dy^2} = -\left(\frac{dy}{dx}\right)^{-2}.\frac{d}{dx}\left(\frac{dy}{dx}\right).\frac{dx}{dy}$

$\Rightarrow \frac{d^2x}{dy^2} = -\left(\frac{dy}{dx}\right)^{-3}\left(\frac{d^2y}{dx^2}\right)$

143. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ….(i)

Differentiating both sides w.r.t.x, we get

$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}$ ….(ii)

$\therefore \quad \dfrac{d^2y}{dx^2} = -\dfrac{b^2}{a^2}\left(\dfrac{y - x\dfrac{dy}{dx}}{y^2}\right)$

$= -\dfrac{b^2}{a^2y^2}\left(y - x\dfrac{dy}{dx}\right)$

$= -\dfrac{b^2}{a^2y^2}\left(y + \dfrac{b^2x^2}{a^2y}\right)$ ….[From (ii)]

$= -\dfrac{b^2}{a^2y^2}\cdot\dfrac{b^2}{y}\left(\dfrac{y^2}{b^2} + \dfrac{x^2}{a^2}\right) = \dfrac{-b^4}{a^2y^3}$ …[From(i)]

144. $x = f(t)$ and $y = g(t)$

$\therefore \quad \dfrac{dx}{dt} = f'(t)$ and $\dfrac{dy}{dt} = g'(t)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{g'(t)}{f'(t)}$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{f'(t).g''(t) - g'(t).f''(t)}{[f'(t)]^2}\cdot\dfrac{dt}{dx}$

$= \dfrac{f'(t).g''(t) - g'(t)f''(t)}{[f'(t)]^3}$

145. $y = \dfrac{\cos x - \sin x}{\cos x + \sin x} = \dfrac{1 - \tan x}{1 + \tan x}$

$\Rightarrow y = \tan\left(\dfrac{\pi}{4} - x\right)$ ....(i)

$\therefore \quad \dfrac{dy}{dx} = -\sec^2\left(\dfrac{\pi}{4} - x\right)$

$\therefore \quad \dfrac{d^2y}{dx^2} = 2\sec^2\left(\dfrac{\pi}{4} - x\right).\tan\left(\dfrac{\pi}{4} - x\right)$

$\therefore \quad \dfrac{\dfrac{d^2y}{dx^2}}{\dfrac{dy}{dx}} = -2\tan\left(\dfrac{\pi}{4} - x\right) = -2y$ ....[From (i)]

146. $y = \cos(\log x)$ ....(i)

$\therefore \quad \dfrac{dy}{dx} = -\sin(\log x).\dfrac{1}{x}$

$\Rightarrow x\dfrac{dy}{dx} = -\sin(\log x)$

Differentiating both sides w.r.t. x, we get

$x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}.1 = -\cos(\log x).\dfrac{1}{x}$

$\Rightarrow x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = -y$ ....[From (i)]

$\Rightarrow x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

147. $y = e^{\tan x}$

$\Rightarrow \log y = \tan x$

Differentiating both sides w.r.t. x, we get

$\dfrac{1}{y}\cdot\dfrac{dy}{dx} = \sec^2x \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{\cos^2x}$

$\Rightarrow \cos^2x\dfrac{dy}{dx} = y$

Differentiating both sides w.r.t. x, we get

$\cos^2x\dfrac{d^2y}{dx^2} - 2\cos x\sin x\dfrac{dy}{dx} = \dfrac{dy}{dx}$

$\Rightarrow \cos^2x\dfrac{d^2y}{dx^2} = (1 + \sin 2x)\dfrac{dy}{dx}$

148. $y = e^{m\cos^{-1}x}$ ....(i)

$\therefore \quad \dfrac{dy}{dx} = e^{m\cos^{-1}x}.m.\dfrac{-1}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\dfrac{dy}{dx} = -my$ ....[From (i)]

$\Rightarrow (1 - x^2)\left(\dfrac{dy}{dx}\right)^2 = m^2y^2$

Differentiating both sides w.r.t. x, we get

$(1 - x^2).2\dfrac{dy}{dx}.\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2.(0 - 2x) = 2m^2y\dfrac{dy}{dx}$

$\Rightarrow (1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = m^2y$

$\Rightarrow (1 - x^2)y_2 - xy_1 - m^2y = 0$

149. $\dfrac{dy}{dx} = \dfrac{2\sin^{-1}x}{\sqrt{1-x^2}} - \dfrac{2\cos^{-1}x}{\sqrt{1-x^2}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2(\sin^{-1}x - \cos^{-1}x)}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\dfrac{dy}{dx} = 2(\sin^{-1}x - \cos^{-1}x)$

Differentiating both sides w.r.t. x, we get

$\sqrt{1-x^2}\cdot\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\cdot\dfrac{1}{2\sqrt{1-x^2}}\cdot(-2x)$

$= 2\left(\dfrac{1}{\sqrt{1-x^2}} - \dfrac{(-1)}{\sqrt{1-x^2}}\right) = \dfrac{4}{\sqrt{1-x^2}}$

$\therefore \quad (1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 4$

150. $y = \cos(\text{m}\sin^{-1}x)$ ....(i)

$\therefore\ y_1 = -\sin(\text{m}\sin^{-1}x)\cdot\frac{\text{m}}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\,y_1 = -\text{m}\sin(\text{m}\sin^{-1}x)$

Differentiating both sides w.r.t. x, we get

$\sqrt{1-x^2}\,y_2 - \frac{xy_1}{\sqrt{1-x^2}} = -\text{m}\cos(\text{m}\sin^{-1}x)\cdot\frac{\text{m}}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)y_2 - xy_1 = -\text{m}^2y$ ....[From (i)]

$\Rightarrow (1-x^2)y_2 - xy_1 + \text{m}^2y = 0$

151. $y^2 = \text{a}x^2 + \text{b}x + \text{c}$

Differentiating both sides w.r.t.x, we get

$2y\frac{\text{d}y}{\text{d}x} = 2\text{a}x + \text{b}$

Differentiating both sides w.r.t.x, we get

$2y\frac{\text{d}^2y}{\text{d}x^2} + \frac{\text{d}y}{\text{d}x}\cdot 2\frac{\text{d}y}{\text{d}x} = 2\text{a}$

Multiplying both the sides by y^2, we get

$y^3\frac{\text{d}^2y}{\text{d}x^2} = \text{a}y^2 - \left(y\frac{\text{d}y}{\text{d}x}\right)^2$

$= \text{a}(\text{a}x^2 + \text{b}x + \text{c}) - \left(\text{a}x + \frac{\text{b}}{2}\right)^2$

$= \text{a}^2x^2 + \text{ab}x + \text{ac} - \text{a}^2x^2 - \frac{\text{b}^2}{4} - \text{ab}x$

$= \text{ac} - \frac{\text{b}^2}{4}$ = a constant

152. $y = \tan^{-1}\left[\frac{\log \text{e}x}{\log\frac{\text{e}}{x}}\right] + \tan^{-1}\left[\frac{8-\log x}{1+8\log x}\right]$

$\Rightarrow y = \tan^{-1}\left[\frac{1+\log x}{1-\log x}\right] + \tan^{-1}\left[\frac{8-\log x}{1+8\log x}\right]$

$\Rightarrow y = \tan^{-1}1 + \tan^{-1}(\log x) + \tan^{-1}8 - \tan^{-1}(\log x)$

$\Rightarrow y = \tan^{-1}1 + \tan^{-1}8$

$\therefore\ \frac{\text{d}y}{\text{d}x} = 0,\qquad \therefore\ \frac{\text{d}^2y}{\text{d}x^2} = 0$

153. $x = \sin \text{t}$ and $y = \sin^3 \text{t}$

$\therefore\ y = x^3$

$\therefore\ \frac{\text{d}y}{\text{d}x} = 3x^2 \qquad \therefore\ \frac{\text{d}^2y}{\text{d}x^2} = 6x$

At $\text{t} = \frac{\pi}{2}$, $x = \sin\frac{\pi}{2} = 1$

$\therefore\ \left(\frac{\text{d}^2y}{\text{d}x^2}\right)_{\text{t}=\frac{\pi}{2}} = \left(\frac{\text{d}^2y}{\text{d}x^2}\right)_{x=1} = 6(1) = 6$

154. $x = \text{a}(1-\cos\theta)$ and $y = \text{a}(\theta + \sin\theta)$

$\therefore\ \frac{\text{d}x}{\text{d}\theta} = \text{a}\sin\theta$ and $\frac{\text{d}y}{\text{d}\theta} = \text{a}(1+\cos\theta)$

$\therefore\ \frac{\text{d}y}{\text{d}x} = \frac{\frac{\text{d}y}{\text{d}\theta}}{\frac{\text{d}x}{\text{d}\theta}} = \frac{\text{a}(1+\cos\theta)}{\text{a}\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \cot\frac{\theta}{2}$

$\therefore\ \frac{\text{d}^2y}{\text{d}x^2} = -\text{cosec}^2\frac{\theta}{2}.\frac{1}{2}.\frac{\text{d}\theta}{\text{d}x}$

$= -\frac{1}{2}\text{cosec}^2\frac{\theta}{2}.\frac{1}{\text{a}\sin\theta}$

$\therefore\ \left(\frac{\text{d}^2y}{\text{d}x^2}\right)_{\theta=\frac{\pi}{2}} = -\frac{1}{2}\left(\sqrt{2}\right)^2.\frac{1}{\text{a}(1)} = -\frac{1}{\text{a}}$

155. Let $y = \text{a}\sin^3\text{t}$ and $x = \text{a}\cos^3\text{t}$

$\therefore\ \frac{\text{d}y}{\text{d}\text{t}} = 3\text{a}\sin^2\text{t}\cos\text{t}$

and $\frac{\text{d}x}{\text{d}\text{t}} = -3\text{a}\cos^2\text{t}\sin\text{t}$ ....(i)

$\therefore\ \frac{\text{d}y}{\text{d}x} = \frac{\text{d}y/\text{d}\text{t}}{\text{d}x/\text{d}\text{t}} = -\tan\text{t}$

$\therefore\ \frac{\text{d}^2y}{\text{d}x^2} = -\sec^2\text{t}.\frac{\text{d}\text{t}}{\text{d}x}$

$= -\sec^2\text{t}.\frac{1}{-3\text{a}\cos^2\text{t}\sin\text{t}}$ ....[From (i)]

$= \frac{1}{3\text{a}\cos^4\text{t}\sin\text{t}}$

$\therefore\ \left(\frac{\text{d}^2y}{\text{d}x^2}\right)_{\text{t}=\frac{\pi}{4}} = \frac{1}{3\text{a}\cos^4\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}$

$= \frac{1}{3\text{a}\left(\frac{1}{\sqrt{2}}\right)^5} = \frac{4\sqrt{2}}{3\text{a}}$

156. $\text{e}^y + xy = \text{e}$

Differentiating both sides w.r.t.x, we get

$\text{e}^y\frac{\text{d}y}{\text{d}x} + y + x\frac{\text{d}y}{\text{d}x} = 0$ ....(i)

Again, differentiating both sides w.r.t.x, we get

$\text{e}^y\frac{\text{d}^2y}{\text{d}x^2} + \text{e}^y\left(\frac{\text{d}y}{\text{d}x}\right)^2 + 2\frac{\text{d}y}{\text{d}x} + x\frac{\text{d}^2y}{\text{d}x^2} = 0$ ....(ii)

Putting $x = 0$ in $\text{e}^y + xy = \text{e}$, we get $y = 1$

Putting $x = 0$, $y = 1$ in (i), we get

$\frac{\text{d}y}{\text{d}x} = -\frac{1}{\text{e}}$

Putting $x = 0, y = 1, \frac{dy}{dx} = -\frac{1}{e}$ in (ii), we get

$$e\frac{d^2y}{dx^2} + e.\frac{1}{e^2} - \frac{2}{e} + 0 = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{e^2}$$

158. $f(-x) = -f(x)$ ....[$\because f(x)$ is an odd function]

$\therefore$ $f(x) = -f(-x)$

Differentiating w.r.t.x, we get

$f'(x) = -[-f'(-x)]$

$\Rightarrow f'(x) = f'(-x)$

$\Rightarrow f'(3) = f'(-3)$

$\Rightarrow f'(-3) = 2$

159. $\left(\frac{y}{x}\right) + \left(\frac{x}{y}\right) = 2$

$\Rightarrow y^2 + x^2 = 2xy$

$\Rightarrow (x-y)^2 = 0$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$

$\therefore$ $\frac{dy}{dx} = 1$

160. $y = e^x.e^{2x}.e^{3x}....e^{nx}$

$\Rightarrow y = e^{x(1+2+3+............+n)}$

$\Rightarrow y = e^{x\left[\frac{n(n+1)}{2}\right]}$

$\Rightarrow \log y = x\left[\frac{n(n+1)}{2}\right]$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}.\frac{dy}{dx} = \frac{n(n+1)}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{n(n+1)y}{2}$$

161. $y = \sqrt{\frac{1+e^x}{1-e^x}} \Rightarrow y^2 = \frac{1+e^x}{1-e^x}$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = \frac{(1-e^x)e^x + (1+e^x)e^x}{(1-e^x)^2} = \frac{2e^x}{(1-e^x)^2}$$

$\therefore$ $\frac{dy}{dx} = \frac{e^x}{(1-e^x)^2}\sqrt{\frac{1-e^x}{1+e^x}}$

$$= \frac{e^x}{(1-e^x)^2}\sqrt{\left(\frac{1-e^x}{1+e^x}\right)\left(\frac{1-e^x}{1-e^x}\right)}$$

$$= \frac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$$

Competitive Thinking

1. $Lf'(0) = \lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0}\frac{1-1}{h} = 0$

$Rf'(0) = \lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0}\frac{1+\sin h-1}{h}$

$= \lim_{h\to 0}\frac{\sin h}{h} = 1$

$\therefore$ $Lf'(0) \neq Rf'(0)$

$\therefore$ $f'(0)$ does not exist.

2. $$f(x) = \begin{cases} \frac{1}{x-1} & , \text{ if } x \neq 1, 2 \\ 2 & , \text{ if } x = 1 \\ 1 & , \text{ if } x = 2 \end{cases}$$

$\therefore$ $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{\frac{1}{x-1}-1}{x-2}$

$= -\lim_{x\to 2}\frac{x-2}{(x-1)(x-2)}$

$= -\lim_{x\to 2}\frac{1}{x-1}$

$= -1$

3. $$f(x) = \begin{cases} \frac{1}{2x-5}, & \text{for } x \neq 1 \\ -\frac{1}{3} & , \text{ for } x = 1 \end{cases}$$

$\therefore$ $f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1}\frac{\frac{1}{2x-5} - \left(-\frac{1}{3}\right)}{x-1}$

$= \lim_{x\to 1}\frac{2x-2}{3(2x-5)(x-1)}$

$= \frac{2}{3}\lim_{x\to 1}\frac{x-1}{(2x-5)(x-1)}$

$= \frac{2}{3}\lim_{x\to 1}\frac{1}{2x-5} = -\frac{2}{9}$

4. $f'(0^-) = \lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0}$

$= \lim\limits_{h \to 0} \dfrac{f(0-h) - f(0)}{-h}$

$= \lim\limits_{h \to 0} \dfrac{\dfrac{h^2 \log(\cosh)}{\log(1+h^2)} - 0}{-h}$

$= \lim\limits_{h \to 0} \dfrac{-\log(\cosh)}{h} \lim\limits_{h \to 0} \dfrac{1}{\dfrac{\log(1+h^2)}{h^2}}$

$= \lim\limits_{h \to 0} \dfrac{-\log(\cosh)}{h} .(1)$

Applying L'Hospital rule, we get

$= \lim\limits_{h \to 0} \dfrac{\sinh}{\cosh}$

$= 0$

$f(0^+) = \lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0}$

$= \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$= \lim\limits_{h \to 0} \dfrac{\dfrac{h^2 \log(\cosh)}{\log(1+h^2)} - 0}{h}$

$= 0$

$\therefore$ $f'(0^-) = f'(0^+)$

$\therefore$ $f(x)$ is differentiable at zero.

5. $Lf'(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1}$

$= \lim\limits_{x \to 1^-} \dfrac{\dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4} - 2}{x - 1}$

$= \lim\limits_{x \to 1^-} \dfrac{x^2 - 6x + 5}{4(x-1)}$

$= \lim\limits_{x \to 1^-} \dfrac{(x-5)(x-1)}{4(x-1)}$

$= \dfrac{1}{4} \lim\limits_{x \to 1^-} (x - 5) = -1$

$Rf'(1) = \lim\limits_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{|x-3| - 2}{x - 1}$

$= \lim\limits_{x \to 1} \dfrac{3 - x - 2}{x - 1} = -1$

$\therefore$ $f'(1) = -1$

$\therefore$ $f(x)$ is differentiable at $x = 1$.

If $f(x)$ is differentiable, it has to be continuous.

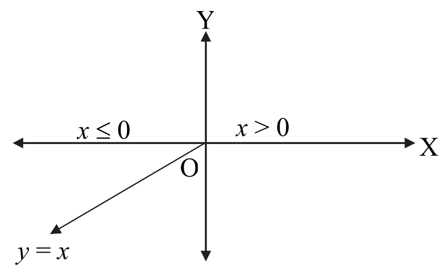
$\therefore$ $f(x)$ is continuous and differentiable at $x = 1$.

6. $\lim\limits_{x \to 0^+} f(x) = 0$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0)$

$\therefore$ The function is continuous at $x = 0$



Since the function has a sharp edge at $x = 0$,

$\therefore$ The function is not differentiable.

7. $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x - 1) = 1 - 1 = 0$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x^3 - 1) = 1 - 1 = 0$

$f(1) = 0$

$\therefore$ $f(x)$ is continuous at $x = 1$.

$Lf'(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{x - 1 - 0}{x - 1} = 1$

$Rf'(1) = \lim\limits_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1}$

$= \lim\limits_{x \to 1^+} \dfrac{x^3 - 1}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{3x^2}{1} = 3$

$\therefore$ $Lf'(1) \neq Rf'(1)$

$\therefore$ $f(x)$ is not differentiable at $x = 1$.

8. Since, $f(x)$ is differentiable at $x = 1$.

$\therefore$ $Lf'(1) = Rf'(1)$

$\Rightarrow \left[\dfrac{d}{dx}(x^2 + bx + c)\right]_{x=1} = \left[\dfrac{d}{dx}(x)\right]_{x=1}$

$\Rightarrow [2x + b]_{x=1} = 1$

$\Rightarrow 2 + b = 1$

$\Rightarrow b = -1$ ....(i)

$f(x)$ is differentiable at $x = 1$.

$\Rightarrow f(x)$ is continuous at $x = 1$.

$\therefore$ $f(1) = \lim\limits_{x \to 1^-} f(x)$

$\Rightarrow 1 = \lim\limits_{x \to 1} \left(x^2 + bx + c\right)$

$\Rightarrow 1 = 1 + b + c$

$\Rightarrow b + c = 0$

$\Rightarrow c = 1$ ….[From (i)]

$\therefore$ $b - c = -1 - 1 = -2$

9. $\lim\limits_{x \to 1} \dfrac{x^2 f(1) - f(x)}{x - 1}$

Applying L' Hospital rule, we get

$\lim\limits_{x \to 1} 2x f(1) - f'(x) = 2f(1) - f'(1)$

10. Applying L'Hospital rule, we get

$\lim\limits_{x \to 2} \dfrac{x f(2) - 2f(x)}{x - 2} = \lim\limits_{x \to 2} \dfrac{f(2) - 2f'(x)}{1}$

$= f(2) - 2f'(2)$

$= 4 - 2(1) = 2$

11. Since, $f(x)$ is differentiable at $x = a$.

$\therefore$ $f'(x)$ exists

Let $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = f'(a)$ ….(i)

Now, $\lim\limits_{x \to a} \dfrac{x^2 f(a) - a^2 f(x)}{x - a}$

$= \lim\limits_{x \to a} \dfrac{x^2 f(a) - a^2 f(a) + a^2 f(a) - a^2 f(x)}{x - a}$

$= \lim\limits_{x \to a} \dfrac{(x^2 - a^2) f(a) - a^2 \{f(x) - f(a)\}}{x - a}$

$= \lim\limits_{x \to a} \left[\dfrac{(x^2 - a^2) f(a)}{x - a} - a^2 \left\{\dfrac{f(x) - f(a)}{x - a}\right\}\right]$

$= \lim\limits_{x \to a} (x + a) f(a) - a^2 \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$= 2a f(a) - a^2 f'(a)$ ....[From (i)]

12. Since, $f(x)$ is differentiable for all x. So, it is everywhere continuous.

$\therefore$ $\lim\limits_{x \to 1} f(x) = f(1)$

$\Rightarrow \lim\limits_{h \to 0} f(1 + h) = f(1)$

$\Rightarrow \lim\limits_{h \to 0} \dfrac{f(1 + h)}{h} \times h = f(1)$

$\Rightarrow \lim\limits_{h \to 0} \dfrac{f(1 + h)}{h} \times \lim\limits_{h \to 0} h = f(1)$

$\Rightarrow 5 \times 0 = f(1)$

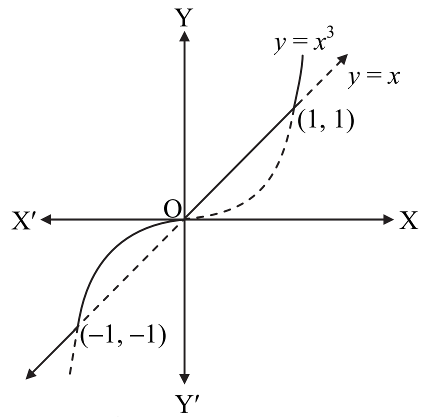
$\Rightarrow f(1) = 0$

Now, $f'(1) = \lim\limits_{h \to 0} \dfrac{f(1 + h) - f(1)}{h}$

$\Rightarrow f'(1) = \lim\limits_{h \to 0} \dfrac{f(1 + h) - 0}{h}$

$\Rightarrow f'(1) = 5$

13. The continuous line shown in the figure below represents the graph of $f(x)$.



$\Rightarrow f(x) = \begin{cases} x, & x \le -1 \\ x^3, & -1 < x \le 0 \\ x, & 0 < x < 1 \\ x^3, & 1 \le x \end{cases}$

Clearly, $f(x)$ is not differentiable at $x = -1, 0, 1$.

14. Let $f(x) = |x - 1| = \begin{cases} x - 1, & x \ge 1 \\ 1 - x, & x < 1 \end{cases}$

$\therefore$ p = left hand derivative of $f(x)$ at $x = 1$

$\Rightarrow p = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{1 - x - 0}{x - 1} = -1$

Now, $\lim\limits_{x \to 1^+} g(x) = p$

$\Rightarrow \lim\limits_{h \to 0} g(1 + h) = -1$

$\Rightarrow \lim\limits_{h \to 0} \dfrac{h^n}{\log \cos^m h} = -1$

$\Rightarrow \lim\limits_{h \to 0} \dfrac{h^n}{m \log \cos h} = -1$

Applying L'Hospital rule on L.H.S., we get

$\dfrac{1}{m} \lim\limits_{h \to 0} \dfrac{n h^{n-1}}{-\tan h} = -1$

$\Rightarrow \dfrac{n}{m} \lim\limits_{h \to 0} \dfrac{h^{n-2}}{\left(\dfrac{\tan h}{h}\right)} = 1$

$\Rightarrow n = 2$ and $\dfrac{n}{m} = 1$

$\Rightarrow m = n = 2$

15. $\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{x^2\left|\cos\dfrac{\pi}{x}\right|-0}{x}$

$= \lim\limits_{x\to 0} x\left|\cos\dfrac{\pi}{x}\right| = 0$

So, f(x) is differentiable at $x = 0$.

Now, $Rf'(2) = \lim\limits_{h\to 0^+}\dfrac{f(2+h)-f(2)}{h}$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2\cos\left(\dfrac{\pi}{2+h}\right)-0}{h}$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2\sin\left(\dfrac{\pi}{2}-\dfrac{\pi}{2+h}\right)}{h}$

$= \lim\limits_{h\to 0}\dfrac{(2+h)^2\sin\left\{\dfrac{\pi h}{2(2+h)}\right\}}{h}$

$= \lim\limits_{h\to 0}\dfrac{\sin\left\{\dfrac{\pi h}{2(2+h)}\right\}}{\dfrac{\pi h}{2(2+h)}}\times\dfrac{(2+h)\pi}{2} = \pi$

Similarly, $Lf'(2) = -\pi$

$\therefore$ $Lf'(2) \neq Rf'(2)$

So, f (x) is not differentiable at $x = 2$.

16. $Lg'(3) = \lim\limits_{x\to 3^-}\dfrac{g(x)-g(3)}{x-3} = \lim\limits_{x\to 3^-}\dfrac{k\sqrt{x+1}-2k}{x-3}$

$= \lim\limits_{x\to 3^-} k\left[\dfrac{x+1-4}{(x-3)\left(\sqrt{x+1}+2\right)}\right]$

$= \lim\limits_{x\to 3^-}\dfrac{k}{\sqrt{x+1}+2} = \dfrac{k}{4}$

$Rg'(3) = \lim\limits_{x\to 3^+}\dfrac{g(x)-g(3)}{x-3} = \lim\limits_{x\to 3^+}\dfrac{mx+2-2k}{x-3}$

Applying L'Hospital rule, we get

$Rg'(3) = m$

Since, $g'(3)$ exists.

$\therefore$ $Rg'(3)$ must exist.

$\therefore$ $3m + 2 - 2k = 0$ ....(i)

Since, g(x) is differentiable.

$\therefore$ $Lg'(3) = Rg'(3)$

$\therefore$ $\dfrac{k}{4} = m \Rightarrow k = 4m$ ....(ii)

Solving (i) and (ii), we get

$m = \dfrac{2}{5}$ and $k = \dfrac{8}{5}$

$\therefore$ $k + m = \dfrac{8}{5}+\dfrac{2}{5} = 2$

17. **Differentiability at $x = \pi$:**

$Ls'(\pi)$

$= \lim\limits_{h\to 0}\dfrac{|\pi-h-\pi|\left(e^{|\pi-h|}-1\right)\sin|\pi-h|-0}{-h}$

$= 0$

$Rs'(\pi)$

$= \lim\limits_{h\to 0}\dfrac{|\pi+h-\pi|\left(e^{|\pi+h|}-1\right)\sin|\pi+h|-0}{h}$

$= 0$

Differentiability at $x = 0$:

$Ls'(0) = \lim\limits_{h\to 0}\dfrac{|-h-\pi|\left(e^{|-h|}-1\right)\sin|-h|-0}{-h}$

$= 0$

$Rs'(0) = \lim\limits_{h\to 0}\dfrac{|h-\pi|\left(e^{|h|}-1\right)\sin|h|-0}{h}$

$= 0$

The function f(x) is differentiable at $x = 0, \pi$.

$\Rightarrow$ Set S is an empty set.

18. $y = \cos(2x + 45)$

$\therefore$ $\dfrac{dy}{dx} = -\sin(2x+45)\cdot\dfrac{d}{dx}(2x+45)$

$= -2\sin(2x+45)$

19. $y = \sqrt{\sin\sqrt{x}}$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{\sin\sqrt{x}}}\cdot\dfrac{d}{dx}\left(\sin\sqrt{x}\right)$

$= \dfrac{1}{2\sqrt{\sin\sqrt{x}}}\cdot\cos\sqrt{x}\cdot\dfrac{1}{2\sqrt{x}}$

$= \dfrac{\cos\sqrt{x}}{4\sqrt{x}\sqrt{\sin\sqrt{x}}}$

20. $\dfrac{d}{dx}\log_{|x|}e = \dfrac{d}{dx}\left(\dfrac{1}{\log|x|}\right)$

$= \dfrac{-1}{\log^2|x|}\times\dfrac{1}{x} = \dfrac{-1}{x\left(\log|x|\right)^2}$

21. $f(x) = \log x$

$\therefore \quad f[\log x] = \log(\log x)$

$\therefore \quad f'[\log x] = \dfrac{1}{\log x} \cdot \dfrac{d}{dx}(\log x)$

$= \dfrac{1}{x \log x}$

22. $y = \log_2(\log_2 x)$

$= \dfrac{\log\left(\dfrac{\log x}{\log 2}\right)}{\log 2}$

$= \dfrac{\log(\log x) - \log(\log 2)}{\log 2}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{\log 2}\left(\dfrac{1}{\log x} \cdot \dfrac{1}{x} - 0\right)$

$= \dfrac{1}{(x \log x)\log 2}$

23. $y = \log\left(\dfrac{1-x^2}{1+x^2}\right)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{\left(\dfrac{1-x^2}{1+x^2}\right)} \cdot \dfrac{(1+x^2)(0-2x)-(1-x^2)(0+2x)}{(1+x^2)^2}$

$= \dfrac{1}{(1-x^2)} \cdot \dfrac{-2x-2x^3-2x+2x^3}{(1+x^2)}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-4x}{1-x^4}$

24. $\dfrac{d}{dx}[\cos(1-x^2)^2] = -\sin(1-x^2)^2 \cdot \dfrac{d}{dx}\left[(1-x^2)^2\right]$

$= -\sin(1-x^2)^2 . 2(1-x^2) . \dfrac{d}{dx}(1-x^2)$

$= -\sin(1-x^2)^2 . \ 2(1-x^2) . (-2x)$

$= 4x(1-x^2)\sin(1-x^2)^2$

25. $\dfrac{d}{dx}\left[e^x \log(1+x^2)\right]$

$= \log(1+x^2) . e^x + e^x . \dfrac{1}{1+x^2} \cdot \dfrac{d}{dx}(1+x^2)$

$= e^x \log(1+x^2) + \dfrac{e^x}{1+x^2} . 2x$

$= e^x\left[\log(1+x^2) + \dfrac{2x}{1+x^2}\right]$

26. $\dfrac{d}{dx}\left(e^x \log \sin 2x\right)$

$= \log \sin 2x . e^x + e^x . \dfrac{1}{\sin 2x} . \dfrac{d}{dx}(\sin 2x)$

$= e^x \log \sin 2x + e^x . \dfrac{1}{\sin 2x} . \cos 2x . \dfrac{d}{dx}(2x)$

$= e^x \log \sin 2x + e^x \cot 2x . 2$

$= e^x(\log \sin 2x + 2\cot 2x)$

27. $\dfrac{d}{dx}\left(e^{\sqrt{1-x^2}} . \tan x\right)$

$= e^{\sqrt{1-x^2}} . \sec^2 x + \tan x . e^{\sqrt{1-x^2}} \cdot \dfrac{d}{dx}\left(\sqrt{1-x^2}\right)$

$= e^{\sqrt{1-x^2}} . \sec^2 x + \tan x . e^{\sqrt{1-x^2}} \cdot \dfrac{1}{2\sqrt{1-x^2}} \cdot (-2x)$

$= e^{\sqrt{1-x^2}}\left[\sec^2 x - \dfrac{x \tan x}{\sqrt{1-x^2}}\right]$

28. $y = \dfrac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{(e^{2x} - e^{-2x})^2}\left[(e^{2x} - e^{-2x}).2(e^{2x} - e^{-2x})\right.$

$\left. - (e^{2x} + e^{-2x}).2(e^{2x} + e^{-2x})\right]$

$= \dfrac{-8}{(e^{2x} - e^{-2x})^2}$

29. $y = \log x . e^{(\tan x + x^2)}$

$\therefore \quad \dfrac{dy}{dx} = e^{(\tan x + x^2)} . \dfrac{1}{x} + \log x \cdot e^{(\tan x + x^2)} . \dfrac{d}{dx}(\tan x + x^2)$

$= e^{(\tan x + x^2)} . \dfrac{1}{x} + \log x . e^{(\tan x + x^2)}(\sec^2 x + 2x)$

$= e^{(\tan x + x^2)}\left[\dfrac{1}{x} + (\sec^2 x + 2x)\log x\right]$

30. $H(x) = G[F(x)]$

$= e^{-e^x}$

$\therefore \quad H'(x) = -e^x . e^{-e^x}$

$\therefore \quad H'(0) = -e^0 . e^{-e^0}$

$= -e^{-1}$

$= -\dfrac{1}{e}$

31. $h(x) = f(g(x))$

$\Rightarrow h(x) = f(\sin^{-1} x) = e^{\sin^{-1} x}$ ....(i)

$\therefore$ $h'(x) = e^{\sin^{-1} x}.\frac{d}{dx}\left(\sin^{-1} x\right) = e^{\sin^{-1} x}.\frac{1}{\sqrt{1-x^2}}$

$\therefore$ $\frac{h'(x)}{h(x)} = \frac{1}{\sqrt{1-x^2}}$ ....[From (i)]

32. At $x = 1$, $f(x)$ is not defined.

For $x \in R - \{1\}$,

$g(x) = f\left[f\left\{f(x)\right\}\right] = f\left[f\left(\frac{1}{1-x}\right)\right] = f\left(\frac{1}{1-\frac{1}{1-x}}\right)$

$= f\left(\frac{x-1}{x}\right) = \frac{1}{1-\frac{x-1}{x}} = x$

$\therefore$ $g'(x) = 1$ for all $x \in R - \{1\}$

33. Let $t = \frac{2x-1}{x^2+1}$. Then, $y = f(t)$

$\therefore$ $\frac{dy}{dx} = f'(t).\frac{dt}{dx} = \sin t^2.\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$

.... [$\because f'(x) = \sin x^2$ (given)]

$= \sin t^2\left[\frac{(x^2+1)(2-0)-(2x-1)(2x+0)}{(x^2+1)^2}\right]$

$= \frac{-2x^2+2x+2}{(x^2+1)^2}.\sin\left(\frac{2x-1}{x^2+1}\right)^2$

34. $f^{-1}(x) = g(x)$

$\Rightarrow x = f\left[g(x)\right]$

Differentiating w.r.t. x, we get

$f'\left[g(x)\right]\cdot g'(x) = 1$

$\Rightarrow \frac{1}{1+\left[g(x)\right]^4}\cdot g'(x) = 1$ ...$\left[\because f'(x) = \frac{1}{1+x^4}\right]$

$\Rightarrow g'(x) = 1 + \left[g(x)\right]^4$

35. $g(x) = \left[f(2f(x)+2)\right]^2$

$\therefore$ $g'(x) = 2\,[f(2f(x)+2)]\,.\,[f(2f(x)+2)]'$

$= 2\,[(2f(x)+2]\,f'[2f(x)+2]\,.\,2f'(x)$

$\therefore$ $g'(0) = 2\,[f(-2+2)]\,f'[2f(0)+2]\,.\,2(1)$

$= 2\,[f(0)]\,(1)\,2$

$= 4\,(-1)$

$= -4$

36. $y = 5x(1-x)^{-\frac{2}{3}} + \cos^2(2x+1)$

$\therefore$ $\frac{dy}{dx} = 5x.\frac{-2}{3}(1-x)^{-\frac{5}{3}}.\frac{d}{dx}(1-x) + 5(1-x)^{-\frac{2}{3}}$

$+ 2\cos(2x+1).\frac{d}{dx}[\cos(2x+1)]$

$= \frac{10x}{3(1-x)^{\frac{5}{3}}} + \frac{5}{(1-x)^{\frac{2}{3}}}$

$-2\,[2\cos(2x+1)\sin(2x+1)]$

$= \frac{5}{(1-x)^{\frac{2}{3}}}\left[\frac{2x}{3(1-x)}+1\right] - 2\sin(4x+2)$

...[$\because 2\sin\theta\cos\theta = \sin 2\theta$]

$= \frac{5(3-x)}{3(1-x)^{\frac{5}{3}}} - 2\sin(4x+2)$

37. $y = f(x^2+2)$

$\therefore$ $\frac{dy}{dx} = f'(x^2+2).(2x)$

$\therefore$ $\left(\frac{dy}{dx}\right)_{x=1} = f'(1^2+2).(2\times1)$

$= f'(3).2 = 5.2 = 10$

38. $f(x) = \log_x(\log x) = \frac{\log(\log x)}{\log x}$

$\therefore$ $f'(x) = \frac{\log x.\frac{1}{\log x}\cdot\frac{d}{dx}(\log x) - \log(\log x).\frac{1}{x}}{(\log x)^2}$

$= \frac{\frac{1}{x}-\frac{1}{x}\log(\log x)}{(\log x)^2}$

$\therefore$ $f'(e) = \frac{\frac{1}{e}-0}{(1)^2} = \frac{1}{e}$

39. $f(x) = \sqrt{1+\cos^2(x^2)}$

$\therefore$ $f'(x) = \frac{1}{2\sqrt{1+\cos^2(x^2)}}.(2\cos x^2).(-\sin x^2).(2x)$

$\therefore$ $f'(x) = \frac{-x\sin 2x^2}{\sqrt{1+\cos^2(x^2)}}$

$\therefore$ $f'\left(\frac{\sqrt{\pi}}{2}\right) = \frac{-\frac{\sqrt{\pi}}{2}.\sin\frac{2\pi}{4}}{\sqrt{1+\cos^2\frac{\pi}{4}}} = \frac{-\frac{\sqrt{\pi}}{2}.1}{\sqrt{\frac{3}{2}}} = -\sqrt{\frac{\pi}{6}}$

40. $f(x) = \frac{\sin^2 x}{1+\cot x} + \frac{\cos^2 x}{1+\tan x}$

$= \frac{\sin^2 x(\sin x)}{\sin x+\cos x} + \frac{\cos^2 x(\cos x)}{\cos x+\sin x}$

$= \frac{\sin^3 x+\cos^3 x}{\sin x+\cos x}$

$= \sin^2 x - \sin x\cos x + \cos^2 x$

...$\left[\because a^3+b^3 = (a+b)(a^2-ab+b^2)\right]$

$= (\sin^2 x+\cos^2 x) - \frac{1}{2}(2\sin x\cos x)$

$= 1 - \frac{1}{2}.\sin 2x$

$\therefore$ $f'(x) = -\cos 2x \Rightarrow f'\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{2}\right) = 0$

41. $\frac{d}{dx}\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{d}{dx}[\tan^{-1}(1) - \tan^{-1}(x)]$

$= 0 - \frac{1}{1+x^2} = \frac{-1}{1+x^2}$

42. $y = \tan^{-1}\left(\frac{\sqrt{a}-\sqrt{x}}{1+\sqrt{ax}}\right) = \tan^{-1}\sqrt{a} - \tan^{-1}\sqrt{x}$

$\therefore$ $\frac{dy}{dx} = 0 - \frac{1}{1+(\sqrt{x})^2}.\frac{d}{dx}(\sqrt{x}) = -\frac{1}{(1+x)}.\frac{1}{2\sqrt{x}}$

43. $y = \tan^{-1}\left(\frac{\sin x+\cos x}{\cos x-\sin x}\right) = \tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+x\right)\right] = \frac{\pi}{4}+x$

$\therefore$ $\frac{dy}{dx} = 1$

44. $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

$= \tan^{-1}\left(\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right)$

$= \tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}(\tan x)$

$\therefore$ $y = \tan^{-1}\left(\frac{a}{b}\right) - x$ $\therefore$ $\frac{dy}{dx} = -1$

45. $y = \sec(\tan^{-1} x)$

$\therefore$ $\frac{dy}{dx} = \sec(\tan^{-1} x)\tan(\tan^{-1} x).\frac{1}{1+x^2}$

$= \sqrt{1+x^2}.\frac{x}{1+x^2}$

....$[\because \tan^{-1} x = \sec^{-1}\sqrt{1+x^2}]$

$= \frac{x}{\sqrt{1+x^2}}$

46. Let $y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right) = \tan^{-1}\left[\frac{6x^{\frac{3}{2}}}{1-\left(3x^{\frac{3}{2}}\right)^2}\right]$

$= \tan^{-1}\left[\frac{2\times 3x^{\frac{3}{2}}}{1-\left(3x^{\frac{3}{2}}\right)^2}\right]$

$= 2\tan^{-1} 3x^{\frac{3}{2}}$

$\therefore$ $\frac{dy}{dx} = \frac{2}{1+\left(3x^{\frac{3}{2}}\right)^2}\cdot 3\times\frac{3}{2}\times x^{\frac{1}{2}} = \frac{9}{1+9x^3}\sqrt{x}$

Comparing with $\sqrt{x}\,g(x)$, we get

$g(x) = \frac{9}{1+9x^3}$

47. $y = e^{m\sin^{-1}x}$ ....(i)

$\therefore$ $\frac{dy}{dx} = e^{m\sin^{-1}x}\cdot\frac{m}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = my$ ....[From (i)]

$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2$

$\therefore$ $A = m^2$

48. Putting $x = \sin A$ and $\sqrt{x} = \sin B$, we get

$y = \sin^{-1}(\sin A\sqrt{1-\sin^2 B} + \sin B\sqrt{1-\sin^2 A})$

$= \sin^{-1}(\sin A\cos B + \sin B\cos A)$

$= \sin^{-1}[\sin(A+B)] = A + B = \sin^{-1} x + \sin^{-1}\sqrt{x}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{x})^2}}.\frac{1}{2\sqrt{x}}$

$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{x-x^2}}$

49. $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right) + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$\therefore \quad y = \tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right) + \sin\left(2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)$

$= \tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right) + \sin\left(2\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right)$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right] + \sin\left[2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right]$

$= \frac{\pi}{4} - \frac{\theta}{2} + \sin\theta$

$= \frac{\pi}{4} - \frac{\cos^{-1}x}{2} + \sin(\cos^{-1}x)$

$= \frac{\pi}{4} - \frac{\cos^{-1}x}{2} + \sin\left(\sin^{-1}\sqrt{1-x^2}\right)$

$= \frac{\pi}{4} - \frac{\cos^{-1}x}{2} + \sqrt{1-x^2}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}} + \frac{(-2x)}{2\sqrt{1-x^2}}$

$= \frac{1-2x}{2\sqrt{1-x^2}}$

50. Put $x^x = \tan\theta \Rightarrow \theta = \tan^{-1}(x^x)$

$\therefore \quad f(x) = \cot^{-1}\left(\frac{\tan^2\theta - 1}{2\tan\theta}\right)$

$= \cot^{-1}(-\cot 2\theta)$

$= \pi - \cot^{-1}(\cot 2\theta)$

$\therefore \quad f(x) = \pi - 2\theta = \pi - 2\tan^{-1}(x^x)$

$\therefore \quad f'(x) = \frac{-2}{1+x^{2x}}.x^x(1+\log x)$

$\therefore \quad f'(1) = \frac{-2}{1+1^2}.1(1+0) = -1$

51. $y = \tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{\frac{3}{2}}}\right) = \tan^{-1}(\sqrt{x}) - \tan^{-1}(x)$

$\therefore \quad y' = \frac{1}{1+x}.\frac{1}{2\sqrt{x}} - \frac{1}{1+x^2}$

$\therefore \quad y'(1) = \frac{1}{2}.\frac{1}{2} - \frac{1}{2} = \frac{-1}{4}$

52. $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore \quad y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$

$\therefore \quad y' = \frac{1}{2(1+x^2)}$

$\therefore \quad y'(1) = \frac{1}{2(1+1^2)} = \frac{1}{4}$

53. $y = \left(1+\frac{1}{x}\right)^x$

Taking logarithm on both sides, we get

$\log y = x\log\left(1+\frac{1}{x}\right)$

Differentiating w.r.t. x, we get

$\frac{1}{y}\cdot\frac{dy}{dx} = \log\left(1+\frac{1}{x}\right) + x\cdot\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)$

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \log\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

$\Rightarrow \frac{dy}{dx} = \left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right) - \frac{1}{1+x}\right]$

54. $y = (\sin x)^{\tan x}$

Taking logarithm on both sides, we get

$\log y = \tan x.\log(\sin x)$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \tan x.\cot x + \log(\sin x).\sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\tan x}\left[1+\sec^2 x\log(\sin x)\right]$$

55. $y = \dfrac{e^{2x}\cos x}{x\sin x}$

Taking logarithm on both sides, we get

$\log y = 2x + \log(\cos x) - \log x - \log(\sin x)$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2+\left(\frac{-\sin x}{\cos x}\right)-\frac{1}{x}-\frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{2x}\cos x}{x\sin x}\left(2-\frac{\sin x}{\cos x}-\frac{1}{x}-\frac{\cos x}{\sin x}\right)$$

$$= e^{2x}\left(\frac{2}{x}\cot x-\frac{1}{x}-\frac{1}{x^2}\cot x-\frac{\cot^2 x}{x}\right)$$

$$= \frac{e^{2x}}{x^2}\left[2x\cot x-\cot x-x(1+\cot^2 x)\right]$$

$$= \frac{e^{2x}}{x^2}[(2x-1)\cot x - x\,\mathrm{cosec}^2 x]$$

56. $y = \{f(x)\}^{\phi(x)}$

Taking logarithm on both sides, we get

$\log y = \phi(x)\log\{f(x)\}$

$\Rightarrow y = e^{\phi(x)\log f(x)}$

$$\therefore \frac{dy}{dx} = e^{\phi(x)\log f(x)}\cdot\frac{d}{dx}[\phi(x)\log f(x)]$$

$$= e^{\phi(x)\log f(x)}\left\{\phi(x)\cdot\frac{f'(x)}{f(x)}+\log f(x).\phi'(x)\right\}$$

57. $y = (x\log x)^{\log(\log x)}$

Taking logarithm on both sides, we get

$\log y = \log(\log x)[\log x + \log(\log x)]$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x\log x}\left[\log x+\log(\log x)\right] + \log(\log x)\left(\frac{1}{x}+\frac{1}{x\log x}\right)$$

$$\Rightarrow \frac{dy}{dx} = (x\log x)^{\log(\log x)}\left\{\frac{1}{x\log x}\left[\log x+\log(\log x)\right] + \log(\log x)\left(\frac{1}{x}+\frac{1}{x\log x}\right)\right\}$$

58. $y = \left[(\tan x)^{\tan x}\right]^{\tan x}$

Taking logarithm on both sides, we get

$\log y = \tan x\log(\tan x)^{\tan x}$

$\Rightarrow \log y = (\tan x)^2\log(\tan x)$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = (\tan x)^2\cdot\frac{1}{\tan x}\cdot\sec^2 x + \log(\tan x).2\tan x.\sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \left[(\tan x)^{\tan x}\right]^{\tan x}.\tan x\sec^2 x\,[1+2\log(\tan x)]$$

$$\therefore \left(\frac{dy}{dx}\right)_{\left(x=\frac{\pi}{4}\right)} = 1.1.\left(\sqrt{2}\right)^2(1+0) = 2$$

59. $y = 1 + x e^y$ ….(i)

$$\therefore \frac{dy}{dx} = e^y.1 + x.\,e^y.\frac{dy}{dx}$$

$$\Rightarrow (1-xe^y)\frac{dy}{dx} = e^y$$

$$\Rightarrow (2-y)\frac{dy}{dx} = e^y \quad \text{….[From (i)]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^y}{2-y}$$

60. $xy = 1 + \log y$

Differentiating both sides w.r.t.x, we get

$$x.\frac{dy}{dx} + y.1 = \frac{1}{y}\cdot\frac{dy}{dx}$$

$$\Rightarrow (xy-1)\frac{dy}{dx} + y^2 = 0$$

$\therefore$ k $= xy - 1$

61. $\tan^{-1}(x^2+y^2) = \alpha$

$\Rightarrow x^2+y^2 = \tan\alpha$

Differentiating both sides w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

62. $y = e^{\sin^{-1}(t^2-1)}$ and $x = e^{\sec^{-1}\left(\frac{1}{t^2-1}\right)} = e^{\cos^{-1}(t^2-1)}$

$\therefore xy = e^{\frac{\pi}{2}}$ ….$\left[\because \sin^{-1}x+\cos^{-1}x = \dfrac{\pi}{2}\right]$

Differentiating both sides w.r.t. x, we get

$$x\frac{dy}{dx} + y.1 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

63. $2x^2 - 3xy + y^2 + x + 2y - 8 = 0$

Differentiating w.r.t. x, we get

$$4x - 3\left(x\cdot\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} + 1 + 2\frac{dy}{dx} = 0$$

$$\Rightarrow (-3x + 2y + 2)\frac{dy}{dx} + 4x - 3y + 1 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3y - 4x - 1}{2y - 3x + 2}$$

64. $y \sec x + \tan x + x^2 y = 0$

Differentiating w.r.t. x, we get

$$\sec x.\frac{dy}{dx} + y.\sec x \tan x + \sec^2 x + y.2x + x^2.\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2xy + \sec^2 x + y\sec x\tan x}{x^2 + \sec x}$$

65. If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + ...\infty}}}$,

then $\frac{dy}{dx} = \frac{f'(x)}{2y - 1}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2y - 1}$

66. $x^y = e^{x-y}$

Taking logarithm on both sides, we get

$y \log x = x - y$

$$\Rightarrow y = \frac{x}{1 + \log x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(1+\log x).1 - x\left(0 + \frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{\log x}{(1+\log x)^2}$$

67. $x^p y^q = (x + y)^{p+q}$

Taking logarithm on both sides, we get

p $\log x$ + q $\log y$ = (p + q)$\log(x + y)$

Differentiating both sides w.r.t.x, we get

$$\frac{p}{x} + \frac{q}{y}\cdot\frac{dy}{dx} = \frac{p+q}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

68. $y^y = x \sin y$

Taking logarithm on both sides, we get

$y \log y = \log x + \log (\sin y)$

Differentiating both sides w.r.t. x, we get

$$y.\frac{1}{y}.\frac{dy}{dx} + \log y.\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\sin y}\cdot\cos y\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(1 + \log y - \cot y) = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x(1 + \log y - \cot y)}$$

69. $\log_{10}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = 2$

$$\Rightarrow \frac{x^2 - y^2}{x^2 + y^2} = 10^2$$

$\Rightarrow x^2 - y^2 = 100x^2 + 100y^2$

$\Rightarrow 99x^2 + 101y^2 = 0$

Differentiating w.r.t. x, we get

$$99(2x) + 101\left(2y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{99x}{101y}$$

70. $\log_{10}\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = 2$

$$\Rightarrow \frac{x^3 - y^3}{x^3 + y^3} = 10^2$$

$\Rightarrow x^3 - y^3 = 100x^3 + 100y^3$

$\Rightarrow 99x^3 = -101y^3$ ...(i)

Differentating w.r.t. x, we get

$$99(3x^2) = -101(3y^2)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-99x^2}{101y^2}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{101y^3}{x}\right) \times \frac{1}{101y^2}$$ ...[From (i)]

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

71. $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \log a$

$\therefore \quad \frac{x^2 - y^2}{x^2 + y^2} = \cos(\log a)$

Differentiating both sides w.r.t. x, we get

$$\frac{(x^2+y^2)\left(2x-2y\frac{dy}{dx}\right)-(x^2-y^2)\left(2x+2y\frac{dy}{dx}\right)}{(x^2+y^2)^2}=0$$

$$\Rightarrow (x^2+y^2)\left(2x-2y\frac{dy}{dx}\right)-(x^2-y^2)\left(2x+2y\frac{dy}{dx}\right)=0$$

$$\Rightarrow 4xy^2-4x^2y\frac{dy}{dx}=0$$

$$\Rightarrow 4xy^2=4x^2y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}$$

72. $\sin y = x\sin(a+y)$

$$\Rightarrow x=\frac{\sin y}{\sin(a+y)}$$

Differentiating both sides w.r.t.x, we get

$$1=\frac{\sin(a+y).\cos y\frac{dy}{dx}-\sin y.\cos(a+y)\frac{dy}{dx}}{\sin^2(a+y)}$$

$$\Rightarrow 1=\frac{\frac{dy}{dx}.\sin(a+y-y)}{\sin^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$$

73. $\cos y = x\cos(a+y)$

$$\Rightarrow x=\frac{\cos y}{\cos(a+y)}$$

Differetiating both sides w.r.t. x, we get

$$1=\frac{-\cos(a+y)\sin y\frac{dy}{dx}+\cos y\sin(a+y)\frac{dy}{dx}}{\cos^2(a+y)}$$

$$\Rightarrow 1=\frac{\frac{dy}{dx}\sin(a+y-y)}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$

74. $\sin(xy)+\dfrac{x}{y}=x^2-y$

Differentiating both sides w.r.t. x, we get

$$\cos(xy)\left[y+x\frac{dy}{dx}\right]+x\left(-\frac{1}{y^2}\right)\frac{dy}{dx}+\frac{1}{y}=2x-\frac{dy}{dx}$$

$$\Rightarrow\left[x\cos(xy)-\frac{x}{y^2}+1\right]\frac{dy}{dx}=2x-\frac{1}{y}-y\cos(xy)$$

$$\Rightarrow \frac{dy}{dx}=\frac{y[2xy-y^2\cos(xy)-1]}{xy^2\cos(xy)+y^2-x}$$

75. $y\sqrt{x^2+1}=\log\left(\sqrt{x^2+1}-x\right)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}.\sqrt{x^2+1}+y.\frac{1}{2\sqrt{x^2+1}}.2x$$

$$=\frac{1}{\sqrt{x^2+1}-x}\times\left(\frac{1}{2\sqrt{x^2+1}}.2x-1\right)$$

$$\Rightarrow \sqrt{x^2+1}\cdot\frac{dy}{dx}+\frac{xy}{\sqrt{x^2+1}}$$

$$=\frac{1}{\sqrt{x^2+1}-x}\times\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$\Rightarrow (x^2+1)\frac{dy}{dx}+xy=\sqrt{x^2+1}\cdot\frac{-1}{\sqrt{x^2+1}}$$

$$\Rightarrow (x^2+1)\frac{dy}{dx}+xy+1=0$$

76. $xe^{xy}=y+\sin^2 x$ ...(i)

When $x=0$, $y=0$

Differentiating (i) w.r.t. x, we get

$$e^{xy}+xe^{xy}\left(x\frac{dy}{dx}+y\right)=\frac{dy}{dx}+2\sin x\cos x$$

Putting $x=0, y=0$, we get

$$\frac{dy}{dx}=1$$

77. $2^x+2^y=2^{x+y}$

Differentiating both sides w.r.t. x, we get

$$2^x(\log 2)+2^y(\log 2)\frac{dy}{dx}=2^{(x+y)}.(\log 2)\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow 2^x+2^y\frac{dy}{dx}=2^{x+y}+2^{x+y}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}(2^y-2^{x+y})=2^{x+y}-2^x$$

$$\Rightarrow \frac{dy}{dx}=\frac{2^{x+y}-2^x}{2^y-2^{x+y}}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=y=1}=\frac{2^2-2}{2-2^2}=\frac{2}{-2}=-1$$

78. $\sin y + e^{-x\cos y} = e$

Differentiating both sides w.r.t. x, we get

$$\cos y\frac{dy}{dx} + e^{-x\cos y}\left\{(-x)\left(-\sin y\frac{dy}{dx}\right) + \cos y(-1)\right\} = 0$$

$$\Rightarrow \cos y\frac{dy}{dx} + x\sin y\ e^{-x\cos y}\frac{dy}{dx} - \cos y\ e^{-x\cos y} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos y\ e^{-x\cos y}}{\cos y + x\sin y\ e^{-x\cos y}}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(1,\pi)} = \frac{\cos\pi\ e^{-\cos\pi}}{\cos\pi + \sin\pi\ e^{-\cos\pi}} = \frac{(-1)e}{-1+0} = e$$

79. $x^{2x} - 2x^x \cot y - 1 = 0$ ....(i)

Putting $x = 1$ in (i), we get

$$1 - 2\cot y - 1 = 0 \Rightarrow \cot y = 0 \Rightarrow y = \frac{\pi}{2}$$

Differentiating (i) w.r.t. x, we get

$$2x^{2x}(1 + \log x) - 2x^x(1 + \log x)\cot y + 2x^x \operatorname{cosec}^2 y.\frac{dy}{dx} = 0$$

Putting $x = 1$ and $y = \frac{\pi}{2}$, we get

$$2 - 0 + 2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -1$$

80. Let $y = x^6$ and $z = x^3$

$$\therefore \frac{dy}{dx} = 6x^5 \text{ and } \frac{dz}{dx} = 3x^2$$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{6x^5}{3x^2} = 2x^3$$

81. Let $y = \sin x$ and $z = \cos x$

$$\therefore \frac{dy}{dx} = \cos x \text{ and } \frac{dz}{dx} = -\sin x$$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\cos x}{-\sin x} = -\cot x$$

82. Let $y = \sin^2 x$ and $z = \cos^2 x$

$$\therefore \frac{dy}{dx} = \sin 2x \text{ and } \frac{dz}{dx} = -\sin 2x$$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = -1$$

83. Let $y = \cos^3 x$ and $z = \sin^3 x$

$$\therefore \frac{dy}{dx} = -3\cos^2 x \sin x \text{ and } \frac{dz}{dx} = 3\sin^2 x\cos x$$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{-\cos x}{\sin x} = -\cot x$$

84. Let $y = \log_{10} x$ and $z = x^2$

$$\therefore \frac{dy}{dx} = \frac{1}{x\log_e 10} \text{ and } \frac{dz}{dx} = 2x$$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{1}{2x^2\log_e 10} = \frac{1}{2x^2}\log_{10} e$$

85. Let $y = \log_{10} x$ and $z = \log_x 10$

$$\therefore \frac{dy}{dx} = \frac{1}{x\log 10}$$

$$\text{and } \frac{dz}{dx} = \log 10.\left[-\frac{1}{(\log x)^2}.\frac{1}{x}\right] = -\frac{\log 10}{x(\log x)^2}$$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\frac{1}{x\log 10}}{-\frac{\log 10}{x(\log x)^2}} = -\frac{(\log x)^2}{(\log 10)^2} = -(\log_{10} x)^2$$

86. $x = a\cos^3\theta$ and $y = a\sin^3\theta$

$$\therefore \frac{dx}{d\theta} = -3a\cos^2\theta.\sin\theta$$

$$\text{and } \frac{dy}{d\theta} = 3a\sin^2\theta.\cos\theta$$

$$\therefore \frac{dy}{dx} = -\tan\theta$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2\theta = \sec^2\theta$$

87. $x = \log(1 + t^2)$ and $y = t - \tan^{-1} t$

$$\therefore \frac{dx}{dt} = \frac{2t}{1+t^2} \text{ and } \frac{dy}{dt} = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t}{2}$$

Since, $x = \log(1 + t^2)$

$$\therefore t = (e^x - 1)^{1/2}$$

$$\therefore \frac{dy}{dx} = \frac{(e^x - 1)^{1/2}}{2}$$

88. $x = a(t - \sin t)$ and $y = a(1 - \cos t)$

$\therefore \quad \frac{dx}{dt} = a(1 - \cos t)$ and $\frac{dy}{dt} = a \sin t$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\sin t}{a(1-\cos t)} = \frac{2a\sin\frac{t}{2}\cos\frac{t}{2}}{2a\sin^2\frac{t}{2}}$$

$$= \cot\frac{t}{2}$$

89. $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$

$\therefore \quad \frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$ and

$\frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}$$

$$= \frac{2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}}{2\cos\frac{3\theta}{2}\sin\frac{\theta}{2}}$$

$$= \tan\frac{3\theta}{2}$$

90. $\sin x = \frac{2t}{1+t^2}$, $\tan y = \frac{2t}{1-t^2}$

Putting $t = \tan\theta$ in both equations, we get

$\sin x = \frac{2\tan\theta}{1+\tan^2\theta}$

$\Rightarrow \sin x = \sin 2\theta$

$\Rightarrow x = 2\theta$

$\therefore \quad \frac{dx}{d\theta} = 2$

$\tan y = \frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \tan y = \tan 2\theta$

$\Rightarrow y = 2\theta$

$\therefore \quad \frac{dy}{d\theta} = 2$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = 1$$

91. Let $y = (\log x)^x$ and $z = \log x$

$\therefore \quad \log y = x\log(\log x)$

Differentiating both sides w.r.t.x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log(\log x) + \frac{1}{\log x}$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$$

$z = \log x$

$\therefore \quad \frac{dz}{dx} = \frac{1}{x}$

$$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = x(\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$$

92. Let $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$ and $z = \sin^{-1} x$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$$\therefore \quad y = \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$= \frac{\sin^{-1}x}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}} \text{ and } \frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{1}{2}$$

93. Let $u = \cos^{-1}(2x^2 - 1)$ and $v = \cos^{-1} x$

Putting $x = \cos\theta$ in both equations, we get

$u = \cos^{-1}(2\cos^2\theta - 1)$

$u = \cos^{-1}(\cos 2\theta)$

$= 2\theta$

$v = \cos^{-1}(\cos\theta)$

$= \theta$

$\therefore \quad \frac{du}{d\theta} = 2$ and $\frac{dv}{d\theta} = 1$

$$\frac{du}{dv} = \frac{\left(\frac{du}{d\theta}\right)}{\left(\frac{dv}{d\theta}\right)} = 2$$

94. Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ and $z = \tan^{-1} x$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$\therefore \quad y = \tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right)$

$= \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\left(\tan\dfrac{\theta}{2}\right) = \dfrac{\theta}{2} = \dfrac{\tan^{-1} x}{2}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2(1+x^2)}$ and $\dfrac{dz}{dx} = \dfrac{1}{1+x^2}$

$\therefore \quad \dfrac{dy}{dz} = \dfrac{\frac{dy}{dx}}{\frac{dz}{dx}} = \dfrac{1}{2}$

95. Let $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

and $z = \sin^{-1}(3x - 4x^3)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

$\therefore \quad y = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$ and

$z = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$\Rightarrow y = \sin^{-1}(\sin 2\theta)$ and $z = \sin^{-1}(\sin 3\theta)$

$\Rightarrow y = 2\theta = 2\sin^{-1}x$ and $z = 3\theta = 3\sin^{-1}x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-x^2}}$ and $\dfrac{dz}{dx} = \dfrac{3}{\sqrt{1-x^2}}$

$\therefore \quad \dfrac{dy}{dz} = \dfrac{\frac{dy}{dx}}{\frac{dz}{dx}} = \dfrac{2}{3}$

96. Let $y = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

and $z = \sin^{-1}(3x - 4x^3)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

$\therefore \quad y = \tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$

$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1} x$ and

$z = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1} x$

$\therefore \quad \dfrac{dy}{dz} = \dfrac{\frac{dy}{dx}}{\frac{dz}{dx}} = \dfrac{\frac{1}{\sqrt{1-x^2}}}{\frac{3}{\sqrt{1-x^2}}} = \dfrac{1}{3}$

97. $f(x) = x^{\tan^{-1} x}$

$\therefore \quad \log f(x) = \tan^{-1}x \log x$

$\therefore \quad \dfrac{1}{f(x)} f'(x) = \dfrac{\log x}{1+x^2} + \dfrac{\tan^{-1} x}{x}$

$\Rightarrow f'(x) = x^{\tan^{-1} x}\left[\dfrac{\log x}{1+x^2} + \dfrac{\tan^{-1} x}{x}\right]$

$g(x) = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$

$\therefore \quad g(x) = \cos^{-1}(2x^2 - 1)$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$\therefore \quad g(x) = \cos^{-1}(2\cos^2\theta - 1)$

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta$

$\therefore \quad g(x) = 2\cos^{-1}x$

$\therefore \quad g'(x) = \dfrac{-2}{\sqrt{1-x^2}}$

Now,

$$\frac{f'(x)}{g'(x)} = \frac{x^{\tan^{-1} x}\left[\frac{\log x}{1+x^2} + \frac{\tan^{-1} x}{x}\right]}{\frac{-2}{\sqrt{1-x^2}}}$$

$$= -\frac{1}{2}\sqrt{1-x^2}\, x^{\tan^{-1} x}\left[\frac{\log x}{1+x^2} + \frac{\tan^{-1} x}{x}\right]$$

98. $x = ct$ and $y = \dfrac{c}{t}$

$\therefore \quad \dfrac{dx}{dt} = c$ and $\dfrac{dy}{dt} = \dfrac{-c}{t^2}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{-c}{t^2}}{c} = \dfrac{-1}{t^2}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(t=2)} = \dfrac{-1}{2^2} = \dfrac{-1}{4}$

99. $y = a\sin^3\theta$ and $x = a\cos^3\theta$

$\therefore \quad \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$ and $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{\theta=\frac{\pi}{3}} = -\tan\dfrac{\pi}{3} = -\sqrt{3}$

100. $x = e^{\theta}(\sin\theta - \cos\theta)$

$\therefore \quad \dfrac{dx}{d\theta} = e^{\theta}(\cos\theta + \sin\theta) + e^{\theta}(\sin\theta - \cos\theta)$

$= 2e^{\theta}\sin\theta$

$y = e^{\theta}(\sin\theta + \cos\theta)$

$\therefore \quad \dfrac{dy}{d\theta} = e^{\theta}(\cos\theta - \sin\theta) + e^{\theta}(\sin\theta + \cos\theta)$

$= 2e^{\theta}\cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2e^{\theta}\cos\theta}{2e^{\theta}\sin\theta} = \cot\theta$$

$\therefore \quad \left.\dfrac{dy}{dx}\right|_{\left(\theta=\frac{\pi}{4}\right)} = 1$

101. Let $y = \log(\sec\theta + \tan\theta)$ and $z = \sec\theta$

$\therefore \quad \dfrac{dy}{d\theta} = \dfrac{1}{\sec\theta + \tan\theta}\cdot(\sec\theta\tan\theta + \sec^2\theta) = \sec\theta$

and $\dfrac{dz}{d\theta} = \sec\theta\tan\theta$

$$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{d\theta}}{\frac{dz}{d\theta}} = \frac{\sec\theta}{\sec\theta\tan\theta} = \frac{1}{\tan\theta} = \cot\theta$$

$\therefore \quad \left(\dfrac{dy}{dz}\right)_{\theta=\frac{\pi}{4}} = \cot\dfrac{\pi}{4} = 1$

102. Let $y = \sec^{-1}\left(\dfrac{1}{2x^2 - 1}\right)$ and $z = \sqrt{1 - x^2}$

$\therefore \quad y = \cos^{-1}(2x^2 - 1)$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$\therefore \quad y = \cos^{-1}(2\cos^2\theta - 1)$

$= \cos^{-1}(\cos 2\theta) = 2\theta = 2\cos^{-1}x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-2}{\sqrt{1 - x^2}}$

and $\dfrac{dz}{dx} = \dfrac{-2x}{2\sqrt{1 - x^2}} = \dfrac{-x}{\sqrt{1 - x^2}}$

$$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{2}{x}$$

$\therefore \quad \left(\dfrac{dy}{dz}\right)_{\left(x=\frac{1}{2}\right)} = 4$

103. Let $y = f(\tan x)$ and $z = g(\sec x)$

$\therefore \quad \dfrac{dy}{dx} = f'(\tan x).\sec^2 x$

and $\dfrac{dz}{dx} = g'(\sec x).\sec x \tan x$

$$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{f'(\tan x)}{g'(\sec x)}.\text{cosec}\, x$$

$$\therefore \quad \left(\frac{dy}{dz}\right)_{\left(x=\frac{\pi}{4}\right)} = \frac{f'(1)}{g'\left(\sqrt{2}\right)}.\sqrt{2} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

104. $y = A\sin 5x$ ...(i)

$\therefore \quad \dfrac{dy}{dx} = 5A\cos 5x$

$\therefore \quad \dfrac{d^2y}{dx^2} = -25A\sin 5x$

$\Rightarrow \dfrac{d^2y}{dx^2} = -25y$ ...[From (i)]

105. $x = A\cos 4t + B\sin 4t$

$\therefore \quad \dfrac{dx}{dt} = -4A\sin 4t + 4B\cos 4t$

$\therefore \quad \dfrac{d^2x}{dt^2} = -16A\cos 4t - 16B\sin 4t$

$= -16(A\cos 4t + B\sin 4t)$

$= -16x$

106. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$

Differentiating w.r.t x, we get

$2b^2x + 2a^2y\dfrac{dy}{dx} = 0$

$\Rightarrow 2a^2y\dfrac{dy}{dx} = -2b^2x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-b^2}{a^2}\left(\dfrac{x}{y}\right)$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\left[\frac{y - x\frac{dy}{dx}}{y^2}\right]$$

$$= \frac{-b^2}{a^2y^2}\left[y - x\left(\frac{-b^2x}{a^2y}\right)\right]$$

$$= \frac{-b^2}{a^2y^2}\left[\frac{a^2y^2 + b^2x^2}{a^2y}\right]$$

$$= \frac{-b^2}{a^2y^2} \times \frac{a^2b^2}{a^2y}$$

$$= \frac{-b^4}{a^2y^3}$$

107. $y = \log(\log x)$

$\therefore \quad \frac{dy}{dx} = \frac{1}{x\log x}$

$\therefore \quad \frac{d^2y}{dx} = \frac{-1}{(x\log x)^2}[1 + \log x]$

108. Let $y = \frac{e^x + 1}{e^x} = 1 + \frac{1}{e^x} = 1 + e^{-x}$

$\therefore \quad \frac{dy}{dx} = -e^{-x}$

$\therefore \quad \frac{d^2y}{dx^2} = e^{-x} = \frac{1}{e^x}$

109. $y = (\tan^{-1}x)^2$

$\therefore \quad \frac{dy}{dx} = \frac{2\tan^{-1}x}{1+x^2}$

$\Rightarrow \frac{dy}{dx}(1 + x^2) = 2\tan^{-1}x$

$\therefore \quad \frac{dy}{dx}(2x) + (1 + x^2)\frac{d^2y}{dx^2} = \frac{2}{1+x^2}$

$\Rightarrow (x^2 + 1)^2\frac{d^2y}{dx^2} + 2x(x^2 + 1)\frac{dy}{dx} = 2$

110. $y = (\sin^{-1}x)^2$

$\therefore \quad \frac{dy}{dx} = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$ ….(i)

$\therefore \quad \frac{d^2y}{dx^2} = 2\left[\frac{1 + x.\sin^{-1}x.(1-x^2)^{-1/2}}{1-x^2}\right]$

$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} = 2\left[1 + x.\sin^{-1}x.(1-x^2)^{-1/2}\right]$

$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$ ….[From (i)]

111. $y = \frac{(\sin^{-1}x)^2}{2}$ $\therefore$ $\frac{dy}{dx} = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = \sin^{-1}x$

$\Rightarrow \sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{-x}{\sqrt{1-x^2}}\right) = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1 - x^2)y_2 - xy_1 = 1$

112. $\sqrt{y} = \cos^{-1}x \Rightarrow y = (\cos^{-1}x)^2$

$\therefore \quad \frac{dy}{dx} = -\frac{2\cos^{-1}x}{\sqrt{1-x^2}}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2 - \frac{2x\cos^{-1}x}{\sqrt{1-x^2}}}{1-x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2 + x\frac{dy}{dx}}{1-x^2}$$

$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$

113. $\sqrt{r} = a.e^{\theta(\cot\alpha)} \Rightarrow r = a^2.e^{2\theta(\cot\alpha)}$

$\therefore \quad \frac{dr}{d\theta} = a^2.e^{2\theta(\cot\alpha)}.2\cot\alpha$

$\Rightarrow \frac{dr}{d\theta} = 2a^2\cot\alpha.e^{2\theta(\cot\alpha)}$

$\therefore \quad \frac{d^2r}{d\theta^2} = 4a^2\cot^2\alpha.e^{2\theta(\cot\alpha)}$

$\therefore \quad \frac{d^2r}{d\theta^2} - 4r\cot^2\alpha$

$= 4a^2\cot^2\alpha.e^{2\theta(\cot\alpha)} - 4a^2\cot^2\alpha.e^{2\theta(\cot\alpha)} = 0$

114. $y = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{\sqrt{a^2-b^2}}\frac{1}{1 + \left(\frac{a-b}{a+b}\right)\tan^2\frac{x}{2}} \times \sqrt{\frac{a-b}{a+b}}\sec^2\frac{x}{2}\left(\frac{1}{2}\right)$$

$$= \frac{1}{a+b} \times \frac{\sec^2\frac{x}{2}}{1 + \left(\frac{a-b}{a+b}\right)\tan^2\frac{x}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 \frac{x}{2}}{(a+b)+(a-b)\tan^2 \frac{x}{2}}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{\left[(a+b)+(a-b)\tan^2\frac{x}{2}\right]\left(\sec\frac{x}{2}\sec\frac{x}{2}\tan\frac{x}{2}\right) - \sec^2\frac{x}{2}\left[(a-b)\tan\frac{x}{2}\sec^2\frac{x}{2}\right]}{\left[(a+b)+(a-b)\tan^2\frac{x}{2}\right]^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\theta=\frac{\pi}{4}\right)} = \frac{(a+b+a-b)\left(\sqrt{2}\times\sqrt{2}\times 1\right) - \left(\sqrt{2}\right)^2\left[(a-b)\left(\sqrt{2}\right)^2\right]}{(a+b+a-b)^2}$$

$$= \frac{4a-4(a-b)}{4a^2}$$

$$= \frac{4b}{4a^2} = \frac{b}{a^2}$$

115. Here, $\frac{dx}{ds} = 1, \frac{dy}{ds} = 2$ ....(i)

and $\frac{d^2x}{ds^2} = 0, \frac{d^2y}{ds^2} = 0$ ....(ii)

Now, $u = x^2 + y^2$

$$\therefore \quad \frac{du}{ds} = 2x.\frac{dx}{ds} + 2y.\frac{dy}{ds}$$

$$\therefore \quad \frac{d^2u}{ds^2} = 2\left(\frac{dx}{ds}\right)^2 + 2x\left(\frac{d^2x}{ds^2}\right) + 2\left(\frac{dy}{ds}\right)^2 + 2y\left(\frac{d^2y}{ds^2}\right)$$

From (i) and (ii), we get

$$\frac{d^2u}{ds^2} = 2(1) + 0 + 2(4) + 0 = 10$$

116. $x = at^2$

$$\therefore \quad \frac{dx}{dt} = 2at$$

$y = 2at$

$$\therefore \quad \frac{dy}{dt} = 2a \quad \text{...(i)}$$

$$\therefore \quad \frac{dx}{dy} = \frac{2at}{2a} \qquad \Rightarrow \frac{dx}{dy} = t$$

$$\therefore \quad \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}(t)$$

$$\therefore \quad \frac{d^2x}{dy^2} = \frac{dt}{dy}$$

$$\therefore \quad \frac{d^2x}{dy^2} = \frac{1}{2a} \quad \text{...[From (i)]}$$

117. $x = f(t)$ and $y = g(t)$

$$\therefore \quad \frac{dx}{dt} = f'(t) \text{ and } \frac{dy}{dt} = g'(t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{f'(t).g''(t) - g'(t).f''(t)}{[f'(t)]^2}\cdot\frac{dt}{dx}$$

$$= \frac{f'(t).g''(t) - g'(t).f''(t)}{[f'(t)]^2}\cdot\frac{1}{f'(x)}$$

$$= \frac{f'(t).g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

118. $y = (x + \sqrt{1+x^2})^n$ ….(i)

$$\therefore \quad \frac{dy}{dx} = n(x + \sqrt{1+x^2})^{n-1}.\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{n\left(x+\sqrt{1+x^2}\right)^n}{\sqrt{1+x^2}}$$

$$\Rightarrow \left(\sqrt{1+x^2}\right)\frac{dy}{dx} = n(x + \sqrt{1+x^2})^n$$

Again, differentiating both sides w.r.t. x, we get

$$\sqrt{1+x^2}.\frac{d^2y}{dx^2} + \frac{dy}{dx}.\left(\frac{x}{\sqrt{1+x^2}}\right) = n^2(x + \sqrt{1+x^2})^{n-1}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow (1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2(x + \sqrt{1+x^2})^n$$

$$\Rightarrow (1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y \quad \text{….[From (i)]}$$

119. $x^2y^3 = (x + y)^5$

Taking logarithm on both sides,we get

$2\log x + 3\log y = 5\log(x + y)$

Differentiating both sides w.r.t. x, we get

$$\frac{2}{x} + \frac{3}{y}\cdot\frac{dy}{dx} = \frac{5}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{3}{y} - \frac{5}{x+y}\right) = \frac{5}{x+y} - \frac{2}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \quad \text{….(i)}$$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{x\dfrac{dy}{dx} - y}{x^2}$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{x\left(\dfrac{y}{x}\right) - y}{x^2}$ ….[From (i)]

$\Rightarrow \dfrac{d^2y}{dx^2} = 0$

120. $x = \sin t$ and $y = \sin pt$

$\therefore \quad \dfrac{dx}{dt} = \cos t$

and $\dfrac{dy}{dt} = p\cos pt$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{p\cos pt}{\cos t}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{p\sqrt{1-y^2}}{\sqrt{1-x^2}}$ ….(i)

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{p\sqrt{1-x^2}\left(\frac{-2y}{2\sqrt{1-y^2}}\right)\frac{dy}{dx} - p\sqrt{1-y^2}\left(\frac{-2x}{2\sqrt{1-x^2}}\right)}{\left(\sqrt{1-x^2}\right)^2}$$

$\Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} = -py\dfrac{\sqrt{1-x^2}}{\sqrt{1-y^2}}\cdot\dfrac{dy}{dx} + px\dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} = -p^2y + x\dfrac{dy}{dx}$ ….[From (i)]

$\Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2y = 0$

121. $x = \cos\theta$ and $y = \sin 5\theta$

$\therefore \quad \dfrac{dx}{d\theta} = -\sin\theta$ and $\dfrac{dy}{d\theta} = 5\cos 5\theta$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\dfrac{5\cos 5\theta}{\sin\theta}$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{5\sqrt{1-y^2}}{\sqrt{1-x^2}}$ ….(i)

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{-5\sqrt{1-x^2}\left(\frac{-2y}{2\sqrt{1-y^2}}\right)\frac{dy}{dx} + 5\sqrt{1-y^2}\left(\frac{-2x}{2\sqrt{1-x^2}}\right)}{\left(\sqrt{1-x^2}\right)^2}$$

$\Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} = \dfrac{5y\sqrt{1-x^2}}{\sqrt{1-y^2}}\cdot\dfrac{dy}{dx} - \dfrac{5x\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} = -25y + x\dfrac{dy}{dx}$ ...[From (i)]

$\Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = -25y$

122. $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}} - \dfrac{e^{-\sqrt{x}}}{2\sqrt{x}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}\left(e^{\sqrt{x}} - e^{-\sqrt{x}}\right)$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{1}{2\sqrt{x}}\left(\dfrac{e^{\sqrt{x}}}{2\sqrt{x}} + \dfrac{e^{-\sqrt{x}}}{2\sqrt{x}}\right) + \dfrac{(e^{\sqrt{x}} - e^{-\sqrt{x}})}{2}\left(\dfrac{-1}{2x^{3/2}}\right)$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{4x} - \dfrac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{4x\sqrt{x}}$

$\therefore \quad x\dfrac{d^2y}{dx^2} + \dfrac{1}{2}.\dfrac{dy}{dx} = x\left(\dfrac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{4x} - \dfrac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{4x\sqrt{x}}\right)$

$+ \dfrac{1}{2}\left(\dfrac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2\sqrt{x}}\right)$

$= \dfrac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{4} - \dfrac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{4\sqrt{x}} + \dfrac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{4\sqrt{x}}$

$= \dfrac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{4}$

$\therefore \quad x\dfrac{d^2y}{dx^2} + \dfrac{1}{2}.\dfrac{dy}{dx} = \dfrac{1}{4}y$ ....[From (i)]

123. $x = 2at^2$ and $y = at^4$

$\therefore \quad \dfrac{dx}{dt} = 4at$ and $\dfrac{dy}{dt} = 4at^3$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = t^2$

$\therefore \quad \dfrac{d^2y}{dx^2} = 2t.\dfrac{dt}{dx} = 2t.\dfrac{1}{4at} = \dfrac{1}{2a}$

$\therefore \quad \left(\dfrac{d^2y}{dx^2}\right)_{(t=2)} = \dfrac{1}{2a}$

124. $x = a \sin\theta$ and $y = b\cos\theta$

$\therefore \quad \frac{dx}{d\theta} = a\cos\theta$ and $\frac{dy}{d\theta} = -b\sin\theta$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b}{a}\tan\theta$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{-b}{a}\sec^2\theta \cdot \frac{d\theta}{dx} = \frac{-b}{a^2}\sec^3\theta$

$\therefore \quad \left(\frac{d^2y}{dx^2}\right)_{\left(\theta=\frac{\pi}{4}\right)} = \frac{-b}{a^2}\sec^3\frac{\pi}{4}$

$= -2\sqrt{2}\frac{b}{a^2}$

125. $y = x^3 \log\log_e(1 + x)$

$\therefore \quad y' = 3x^2 \log\log_e(1 + x) + \frac{x^3}{\log_e(1+x)} \cdot \frac{1}{1+x}$

$\therefore \quad y'' = 6x\log\log_e(1 + x) + \frac{3x^2}{\log_e(1+x)} \cdot \frac{1}{(1+x)}$

$+\left[\frac{(1+x)\log_e(1+x).3x^2 - x^3\left[(1+x).\frac{1}{1+x} + \log_e(1+x)\right]}{(1+x)^2\left[\log_e(1+x)\right]^2}\right]$

$\therefore \quad y''(0) = 0$

126. At (1, 1), $1 = e^t \sin t$ and $1 = e^t\cos t$

$\therefore \quad \tan t = 1 \Rightarrow t = \frac{\pi}{4}$

Now, $x = e^t\sin t$ and $y = e^t\cos t$

$\therefore \quad \frac{dx}{dt} = e^t(\sin t + \cos t)$ and $\frac{dy}{dt} = e^t(\cos t - \sin t)$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t - \sin t}{\cos t + \sin t}$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\cos t - \sin t}{\cos t + \sin t}\right)\frac{dt}{dx}$

$= \left[\frac{(\cos t + \sin t)(-\sin t - \cos t) - (\cos t - \sin t)(-\sin t + \cos t)}{(\cos t + \sin t)^2}\right]\frac{dt}{dx}$

$= \frac{-2}{(\cos t + \sin t)^2} \cdot \frac{1}{e^t(\sin t + \cos t)}$

$= \frac{-2}{(e^t\sin t + e^t\cos t)} \cdot \frac{1}{(\cos t + \sin t)^2}$

$= \frac{-2}{x+y} \cdot \frac{1}{(\cos t + \sin t)^2}$

$\therefore \quad \left(\frac{d^2y}{dx^2}\right)_{(1,1)} = \frac{-2}{1+1} \cdot \frac{1}{\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)^2} = \frac{-1}{2}$

127. At $\left(\frac{3}{2}\sqrt{2}, 2\sqrt{2}\right)$

$\cos t = \frac{1}{\sqrt{2}}$ and $\sin t = \frac{1}{\sqrt{2}}$

$\therefore \quad \tan t = 1 \Rightarrow t = \frac{\pi}{4}$

Now, $x = 3\cos t$ and $y = 4\sin t$

$\therefore \quad \frac{dx}{dt} = -3\sin t$ and $\frac{dy}{dt} = 4\cos t$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{4}{3}\cot t$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{4}{3}\text{cosec}^2 t\,\frac{dt}{dx} = \frac{4}{3}\text{cosec}^2 t \times -\frac{1}{3\sin t}$

$\therefore \quad \left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3}{2}\sqrt{2}, 2\sqrt{2}\right)} = \frac{4}{3}\text{cosec}^2(\pi/4) \times \frac{-1}{3\sin(\pi/4)}$

$= \frac{-8\sqrt{2}}{9}$

128. $f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}$

$\therefore \quad f'(x) = \frac{(x^2 + ax + 1)(2x - a) - (x^2 - ax + 1)(2x + a)}{(x^2 + ax + 1)^2}$

$\Rightarrow f'(x) = \frac{2a(x^2 - 1)}{\left(x^2 + ax + 1\right)^2}$

$\therefore \quad f''(x)$

$= \frac{4ax(x^2 + ax + 1)^2 - 4a(x^2 - 1)(2x + a)(x^2 + ax + 1)}{(x^2 + ax + 1)^4}$

$\Rightarrow f''(x) = \frac{4a\left[x(x^2 + ax + 1) - (x^2 - 1)(2x + a)\right]}{\left(x^2 + ax + 1\right)^3}$

$\therefore \quad f'(1) = 0,\ f''(1) = \frac{4a}{(2+a)^2}$ and $f''(-1) = -\frac{4a}{(2-a)^2}$

$\therefore \quad (2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

129. $y = \dfrac{a^{\cos^{-1}x}}{1+a^{\cos^{-1}x}}$ and $z = a^{\cos^{-1}x} \Rightarrow y = \dfrac{z}{1+z}$

$\therefore \quad \dfrac{dy}{dz} = \dfrac{(1+z)1 - z(1)}{(1+z)^2} = \dfrac{1}{(1+z)^2} = \dfrac{1}{(1+a^{\cos^{-1}x})^2}$

130. Let $f(x) = \cos^{-1}\left(\sin\sqrt{\dfrac{1+x}{2}}\right) + x^x$

$= \cos^{-1}\left[\cos\left(\dfrac{\pi}{2} - \sqrt{\dfrac{1+x}{2}}\right)\right] + x^x$

$= \dfrac{\pi}{2} - \sqrt{\dfrac{1+x}{2}} + x^x$

$\therefore \quad f'(x) = -\dfrac{1}{\sqrt{2}}.\dfrac{1}{2\sqrt{1+x}} + x^x(1 + \log x)$

$\therefore \quad f'(1) = -\dfrac{1}{4} + 1 = \dfrac{3}{4}$

131. Since, $g(x)$ is the inverse of $f(x)$.

$\therefore \quad fog(x) = x$

$\Rightarrow \dfrac{d}{dx}[fog(x)] = \dfrac{d}{dx}(x)$

$\Rightarrow f'[g(x)].g'(x) = 1$

$\Rightarrow \dfrac{1}{1+[g(x)]^3}.g'(x) = 1$

....$\left[\because f'(x) = \dfrac{1}{1+x^3}(\text{given})\right]$

$\Rightarrow g'(x) = 1 + [g(x)]^3$

132. $f(x) = \tan^{-1}x$

$\therefore \quad f'(x) = \dfrac{1}{1+x^2}$

$\therefore \quad f''(x) = \dfrac{-1}{(1+x^2)^2}.2x$

Since, $f'(x) + f''(x) = 0$

$\therefore \quad \dfrac{1}{1+x^2} - \dfrac{2x}{(1+x^2)^2} = 0$

$\Rightarrow 1 + x^2 - 2x = 0$

$\Rightarrow x = 1$

133. $x = a\left(t - \dfrac{1}{t}\right)$ ....(i)

and $y = a\left(t + \dfrac{1}{t}\right)$ ....(ii)

Squaring (i) and (ii) and subtracting, we get

$x^2 - y^2 = a^2(-4) \Rightarrow y^2 - x^2 = 4a^2$

Differentiating both sides w.r.t. x, we get

$2y\dfrac{dy}{dx} - 2x = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y}$

134. $2y = \sin^{-1}(x + 5y)$

$\Rightarrow \sin 2y = x + 5y$

Differentiating both sides w.r.t. x, we get

$2\cos 2y\left(\dfrac{dy}{dx}\right) = 1 + 5\left(\dfrac{dy}{dx}\right)$

$\Rightarrow \dfrac{dy}{dx}(2\cos 2y - 5) = 1$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\cos 2y - 5}$

Now, $\dfrac{dx}{dy} = \dfrac{1}{(dy/dx)}$

$\Rightarrow \dfrac{dx}{dy} = 2\cos 2y - 5$

135. $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

Putting $x = 0$ and $y = 0$, we get

$f(0) = f(0) + f(0) \quad \Rightarrow f(0) = 0$

Now, $f'(0) = \lim\limits_{h\to 0}\dfrac{f(0+h) - f(0)}{h}$

$\Rightarrow f'(0) = \lim\limits_{h\to 0}\dfrac{f(h)}{h}$ ....(i)

$\therefore \quad f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0}\dfrac{f(x) + f(h) - f(x)}{h}$

$\Rightarrow f'(x) = \lim\limits_{h\to 0}\dfrac{f(h)}{h} = f'(0)$ ....[From (i)]

$\Rightarrow f(x) = xf'(0) + c$

But, $f(0) = 0$

$\therefore$ $c = 0$

Hence, $f(x) = xf'(0)$ for all $x \in R$

Clearly, $f(x)$ is everywhere continuous and differentiable and $f'(x)$ is constant for all $x \in R$.

Hence, option (D) is incorrect.

136. $x^2 + y^2 = t + \frac{2}{t}$

Squaring on both sides, we get

$x^4 + y^4 + 2x^2y^2 = t^2 + \frac{4}{t^2} + 4$

$\Rightarrow \left(t^2 + \frac{4}{t^2}\right) + 2x^2y^2 = t^2 + \frac{4}{t^2} + 4$

$\Rightarrow x^2y^2 = 2$ ....(i)

Differentiating both sides w.r.t. x, we get

$x^2.2y\frac{dy}{dx} + y^2.2x = 0$

$\Rightarrow x^2 y\frac{dy}{dx} = -xy^2$

$\Rightarrow x^3 y\frac{dy}{dx} = -x^2y^2$

$\Rightarrow x^3y\frac{dy}{dx} = -2$ ....[From (i)]

137. $\frac{d}{dx} f_n(x) = \frac{d}{dx} e^{f_{n-1}(x)}$

Let n = 3

$\therefore$ $\frac{d}{dx} f_3(x) = \frac{d}{dx} e^{f_2(x)}$

$= e^{f_2(x)} \frac{d}{dx} f_2(x)$

$= e^{f_2(x)} \frac{d}{dx} e^{f_1(x)}$

$= e^{f_2(x)} e^{f_1(x)} \frac{d}{dx} f_1(x)$

$= e^{f_2(x)} e^{f_1(x)} \frac{d}{dx} e^x$

$= e^{f_2(x)} e^{f_1(x)} e^x$

$\frac{d}{dx} f_3(x) = f_3(x)\, f_2(x)\, f_1(x)$

Similarly,

$\frac{d}{dx} f_n(x) = f_n(x)\, f_{n-1}(x) \ldots f_1(x)$

138. $f(x) = f(-x)$ ....[$\because$ $f(x)$ is an even function]

$\therefore$ $f'(x) = -f'(-x)$

$\therefore$ $f'(0) = -f'(0)$

$\therefore$ $2f'(0) = 0$

$\therefore$ $f'(0) = 0$

Evaluation Test

1. $y = \frac{x-1}{4} + \frac{(x-1)^3}{12} + \frac{(x-1)^5}{20} + \frac{(x-1)^7}{28} + \ldots$

$= \frac{1}{4}\left[(x-1) + \frac{(x-1)^3}{3} + \frac{(x-1)^5}{5} + \frac{(x-1)^7}{7} + \ldots\right]$

Now, $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$\therefore$ $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$

$\therefore$ $\log\left(\frac{1+x}{1-x}\right) = \log(1 + x) - \log(1 - x)$

$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right]$

$\therefore$ $x - 1 + \frac{(x-1)^3}{3} + \frac{(x-1)^5}{5} + \frac{(x-1)^7}{7} + \ldots$

$= \frac{1}{2}\log\left[\frac{1+x-1}{1-(x-1)}\right]$

$= \frac{1}{2}\log\left(\frac{x}{2-x}\right)$

$\therefore$ $y = \frac{1}{8}\log\left(\frac{x}{2-x}\right)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{8}\left(\dfrac{2-x}{x}\right)\left[\dfrac{(2-x)(1)-x(-1)}{(2-x)^2}\right]$

$= \dfrac{1}{8}\left(\dfrac{2-x}{x}\right)\left[\dfrac{2-x+x}{(2-x)^2}\right] = \dfrac{1}{4x(2-x)}$

2. $y = (\cos x + i \sin x)(\cos 3x + i \sin 3x)$
$\quad \ldots(\cos(2n-1)x + i\sin(2n-1)x)$

Since, $\cos\theta + i\sin\theta = e^{i\theta}$

$\therefore \quad y = e^{ix} \cdot e^{i3x} \cdot e^{i5x} \ldots e^{i(2n-1)x}$

$= e^{ix[1+3+5+\ldots+(2n-1)]}$

$= e^{in^2x}$

$\therefore \quad \dfrac{dy}{dx} = in^2 e^{in^2x}$

$\therefore \quad \dfrac{d^2y}{dx^2} = i^2 n^4 e^{in^2x} = -n^4 y$

3. $y = f\left(\dfrac{3x+\pi}{5x+4}\right)$

$\therefore \quad \dfrac{dy}{dx} = f'\left(\dfrac{3x+\pi}{5x+4}\right)\cdot\dfrac{d}{dx}\left(\dfrac{3x+\pi}{5x+4}\right)$

$= f'\left(\dfrac{3x+\pi}{5x+4}\right)\left[\dfrac{(5x+4)3-5(3x+\pi)}{(5x+4)^2}\right]$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{x=0} = f'\left(\dfrac{\pi}{4}\right)\left[\dfrac{12-5\pi}{16}\right]$

$= \tan^2\left(\dfrac{\pi}{4}\right)\left(\dfrac{12-5\pi}{16}\right)$

$= (1)^2\left(\dfrac{12-5\pi}{16}\right)$

$= \dfrac{12-5\pi}{16}$

4. $y = |\cos x| + |\sin x|$

Since, $\dfrac{d}{dx}|x| = \dfrac{|x|}{x}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{|\cos x|}{\cos x}\cdot\dfrac{d}{dx}(\cos x) + \dfrac{|\sin x|}{\sin x}\cdot\dfrac{d}{dx}(\sin x)$

$= \dfrac{|\cos x|}{\cos x}(-\sin x) + \dfrac{|\sin x|}{\sin x}\cos x$

When $x = \dfrac{2\pi}{3}$, $\cos x = \cos\dfrac{2\pi}{3} = \dfrac{-1}{2}$, $|\cos x| = \dfrac{1}{2}$

and $\sin x = \sin\dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2}$, $|\sin x| = \dfrac{\sqrt{3}}{2}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{x=\frac{2\pi}{3}} = -1\left(\dfrac{-\sqrt{3}}{2}\right) + 1\left(\dfrac{-1}{2}\right)$

$= \dfrac{\sqrt{3}-1}{2}$

5. $y = \left(1+\dfrac{1}{x}\right)\left(1+\dfrac{2}{x}\right)\ldots\left(1+\dfrac{n}{x}\right)$

$\therefore \quad \dfrac{dy}{dx} = \left(-\dfrac{1}{x^2}\right)\left(1+\dfrac{2}{x}\right)\left(1+\dfrac{3}{x}\right)\ldots\left(1+\dfrac{n}{x}\right)$

$+\left(1+\dfrac{1}{x}\right)\left(-\dfrac{2}{x^2}\right)\left(1+\dfrac{3}{x}\right)\ldots\left(1+\dfrac{n}{x}\right)$

$+\left(1+\dfrac{1}{x}\right)\left(1+\dfrac{2}{x}\right)\left(-\dfrac{3}{x^2}\right)\ldots\left(1+\dfrac{n}{x}\right) + \ldots$

When $x = -1$, $1 + \dfrac{1}{x} = 1 + \dfrac{1}{(-1)} = 1 - 1 = 0$

$\therefore$ Except 1st term all terms are 0.

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(x=-1)} = (-1)(-1)(-2)\ldots(1-n)$

$= (-1)^n (n-1)!$

6. $f(x) = \begin{cases} \dfrac{x}{1+x}, & x \geq 0 \\ \dfrac{x}{1-x}, & x < 0 \end{cases}$

$\therefore \quad Lf'(0) = \lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{\frac{x}{1-x}-0}{x} = 1$

$Rf'(0) = \lim\limits_{x\to 0^+}\dfrac{\frac{x}{1+x}-0}{x-0} = 1$

$\therefore$ $f(x)$ is differentiable at $x = 0$ and $f'(0) = 1$.

7. $f(x) = \sin(\log x)$

$\therefore \quad f'(x) = \cos(\log x)\cdot\dfrac{1}{x}$

$y = f\left(\dfrac{2x+3}{3-2x}\right)$

$\therefore \quad \frac{dy}{dx} = f'\left(\frac{2x+3}{3-2x}\right)\cdot\frac{d}{dx}\left(\frac{2x+3}{3-2x}\right)$

$= \cos\left(\log\left(\frac{2x+3}{3-2x}\right)\right)$

$\cdot\left[\frac{(3-2x)(2)-(-2)(2x+3)}{(3-2x)^2}\right]\cdot\left(\frac{3-2x}{2x+3}\right)$

$= \cos\left(\log\left(\frac{2x+3}{3-2x}\right)\right)\left[\frac{6-4x+4x+6}{3-2x}\right]$

$\cdot\frac{1}{2x+3}$

$= \frac{12}{9-4x^2}\cos\left\{\log\left(\frac{2x+3}{3-2x}\right)\right\}$

8. $\frac{d}{dx}\left[a\tan^{-1}x + b\log\left(\frac{x-1}{x+1}\right)\right] = \frac{1}{x^4-1}$

$\therefore \quad a\tan^{-1}x + b\log\left(\frac{x-1}{x+1}\right)$

$= \int\frac{1}{x^4-1}$

$= \int\frac{1}{(x^2-1)(x^2+1)}$

$= \frac{1}{2}\int\left(\frac{1}{x^2-1} - \frac{1}{x^2+1}\right)dx$

$= \frac{1}{2}\cdot\frac{1}{2}\log\left(\frac{x-1}{x+1}\right) - \frac{1}{2}\tan^{-1}x$

$\therefore \quad a = -\frac{1}{2},\ b = \frac{1}{4}$

$\therefore \quad a - 2b = -\frac{1}{2} - 2\left(\frac{1}{4}\right) = -\frac{1}{2} - \frac{1}{2} = -1$

9. $f(x) = \cos x\cos 2x\cos 4x\cos 8x\cos 16x$

$= \frac{1}{32}\times\frac{16}{\sin x}(2\sin x\cos x\cos 2x\cos 4x$

$\cos 8x\cos 16x)$

$= \frac{1}{32}\times\frac{16}{\sin x}(\sin 2x\cos 2x\cos 4x\cos 8x$

$\cos 16x)$

$= \frac{1}{32}\times\frac{8}{\sin x}(\sin 4x\cos 4x\cos 8x$

$\cos 16x)$

$= \frac{1}{32}\times\frac{4}{\sin x}(\sin 8x\cos 8x\cos 16x)$

$= \frac{1}{32}\times\frac{2}{\sin x}\sin 16x\cos 16x$

$= \frac{\sin 32x}{32\sin x}$

$\therefore \quad f'(x) = \frac{1}{32}\left[\frac{\sin x\cdot 32\cos 32x - \sin 32x\cos x}{\sin^2 x}\right]$

$\therefore \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{32}\frac{\left[\frac{1}{\sqrt{2}}\cdot 32(1) - 0\right]}{\left(\frac{1}{\sqrt{2}}\right)^2}$

$= \frac{1}{32}\times\frac{1}{\sqrt{2}}\times 32\times 2 = \frac{2}{\sqrt{2}} = \sqrt{2}$

10. $1 + x^4 + x^8 = 1 + 2x^4 + x^8 - x^4$

$= (1 + x^4)^2 - x^4$

$= (1 + x^4 + x^2)(1 + x^4 - x^2)$

$\therefore \quad \frac{1+x^4+x^8}{1+x^2+x^4} = 1 - x^2 + x^4$

$\therefore \quad \frac{d}{dx}\left(\frac{1+x^4+x^8}{1+x^2+x^4}\right) = \frac{d}{dx}(1 - x^2 + x^4)$

$= 4x^3 - 2x = ax^3 + bx$

$\therefore \quad a = 4,\ b = -2$

11. $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$

Let $y^{\frac{1}{5}} = a$

$\therefore \quad y^{-\frac{1}{5}} = \frac{1}{a}, \qquad \therefore \quad a + \frac{1}{a} = 2x$

$\therefore \quad a^2 - 2ax + 1 = 0$

$\therefore \quad a = \frac{2x + \sqrt{4x^2-4}}{2}$

$\therefore \quad y^{\frac{1}{5}} = x + \sqrt{x^2-1}$

$\therefore \quad y = \left(x + \sqrt{x^2-1}\right)^5$

$\therefore \quad \frac{dy}{dx} = 5\left(x + \sqrt{x^2-1}\right)^4\left(1 + \frac{1}{2\sqrt{x^2-1}}2x\right)$

$\therefore \quad \sqrt{x^2-1}\frac{dy}{dx} = 5\left(x + \sqrt{x^2-1}\right)^4\left(x + \sqrt{x^2-1}\right)$

$= 5y$

$\therefore \quad (x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 25y^2$

$\therefore \quad (x^2 - 1)\times\frac{2dy}{dx}\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(2x) = 25\times 2y\frac{dy}{dx}$

Dividing both sides by $2\dfrac{dy}{dx}$, we get

$$(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=25y$$

$\therefore$ k = 25

12. $\sqrt{x^2+y^2} = ae^{\tan^{-1}\left(\frac{y}{x}\right)}$ ….(i)

Diff. w.r.t.x, we get

$$\frac{1}{2\sqrt{x^2+y^2}}\left(2x+2y\frac{dy}{dx}\right)$$

$$= ae^{\tan^{-1}\left(\frac{y}{x}\right)}\cdot\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)$$

$\therefore$ $$\frac{1}{\sqrt{x^2+y^2}}\left(x+y\frac{dy}{dx}\right)=\sqrt{x^2+y^2}\left(\frac{x\frac{dy}{dx}-y}{x^2+y^2}\right)$$

….$\left[\because ae^{\tan^{-1}\frac{y}{x}}=\sqrt{x^2+y^2}\right]$

$\therefore$ $x+y\dfrac{dy}{dx}=x\dfrac{dy}{dx}-y$ ….(ii)

Diff. w.r.t.x, we get

$$1+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=x\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{dy}{dx}$$

$\therefore$ $1+y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=x\dfrac{d^2y}{dx^2}$

$\therefore$ $(y-x)\dfrac{d^2y}{dx^2}=-\left(1+\left(\dfrac{dy}{dx}\right)^2\right)$

From (i), when $x = 0$, $y=ae^{\frac{\pi}{2}}$

From (ii), when $x = 0$, $\dfrac{dy}{dx}=-1$

$\therefore$ $ae^{\frac{\pi}{2}}\dfrac{d^2y}{dx^2}=-2$

$\therefore$ $\left(\dfrac{d^2y}{dx^2}\right)_{x=0}=-\dfrac{2}{a}e^{-\frac{\pi}{2}}$

13. $f(x)=\begin{vmatrix} f & g & h\\ f' & g' & h'\\ f'' & g'' & h''\end{vmatrix}$

$\therefore$ $f'(x)=\begin{vmatrix} f' & g' & h'\\ f' & g' & h'\\ f'' & g'' & h''\end{vmatrix}+\begin{vmatrix} f & g & h\\ f'' & g'' & h''\\ f'' & g'' & h''\end{vmatrix}$

$+\begin{vmatrix} f & g & h\\ f' & g' & h'\\ f''' & g''' & h'''\end{vmatrix}$

= 0 + 0 + 0

….[$\because$ f, g, h are polynomials of 2[nd] degree,

$f''' = g''' = h''' = 0$]

= 0

14. $\begin{vmatrix} y & y_1 & y_2\\ y_3 & y_4 & y_5\\ y_6 & y_7 & y_8\end{vmatrix}=\begin{vmatrix} \cos ax & -a\sin ax & -a^2\cos ax\\ a^3\sin ax & a^4\cos ax & -a^5\sin ax\\ -a^6\cos ax & a^7\sin ax & a^8\cos ax\end{vmatrix}$

$= -a^2\times 0$ ….[$\because C_1 \equiv C_3$]

= 0

15. $y = \sin\left[\cos^{-1}\{\sin(\cos^{-1}x)\}\right]$

$=\sin\left[\cos^{-1}\left\{\sin\left(\dfrac{\pi}{2}-\sin^{-1}x\right)\right\}\right]$

$= \sin[\cos^{-1}(\cos(\sin^{-1}x)]$

$= \sin(\sin^{-1}x) = x$

$\therefore$ $\dfrac{dy}{dx}=1$

$\therefore$ $\left(\dfrac{dy}{dx}\right)_{x=\frac{1}{2}}=1$

16. $8\,f(x)+6f\left(\dfrac{1}{x}\right)=x+5$ ….(i)

Replacing x by $\dfrac{1}{x}$, we get

$8f\left(\dfrac{1}{x}\right)+6f(x)=\dfrac{1}{x}+5$

$\therefore$ $6f(x)+8f\left(\dfrac{1}{x}\right)=\dfrac{1}{x}+5$ ….(ii)

(i) × 8 – (ii) × 6 gives

$64\,f(x)-36\,f(x)=8x+40-\dfrac{6}{x}-30$

$\therefore$ $28\,f(x)=8x-\dfrac{6}{x}+10$

Given, $y = x^2 f(x) = \frac{x^2}{28}\left(8x - \frac{6}{x} + 10\right)$

$\therefore \quad y = \frac{1}{28}(8x^3 - 6x + 10x^2)$

$\therefore \quad \frac{dy}{dx} = \frac{1}{28}(24x^2 - 6 + 20x)$

$\therefore \quad \left(\frac{dy}{dx}\right)_{x=-1} = \frac{1}{28}(24 - 6 - 20) = -\frac{2}{28} = -\frac{1}{14}$

17. $f(x^3) = x^5$

Diff. w.r.t. x, we get

$f'(x^3) \,.\, 3x^2 = 5x^4$

$\therefore \quad f'(x^3) = \frac{5}{3}x^2$

$\therefore \quad f'(27) = f'(3^3) = \frac{5}{3}(3)^2 = 15$

18. Since, $g(x)$ is the inverse of $f(x)$.

$\therefore \quad f[g(x)] = x$

$\Rightarrow f'(g(x))g'(x) = 1$

$\Rightarrow f'(g(1))g'(1) = 1$

$\Rightarrow g'(1) = \frac{1}{f'(g(1))}$ ….(i)

$f(x) = x^3 + e^{x/2}$

$\therefore \quad f(0) = 1$

$\Rightarrow 0 = f^{-1}(1)$

$\Rightarrow g(1) = 0$ ….[$\because g(x) = f^{-1}(x)$(given)]

From (i), we get

$g'(1) = \frac{1}{f'(0)}$

Now, $f(x) = x^3 + e^{x/2}$

$\Rightarrow f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$

$\Rightarrow f'(0) = \frac{1}{2}$

$\therefore \quad g'(1) = \frac{1}{1/2} = 2$

19. $y = f(x^3)$

$\therefore \quad \frac{dy}{dx} = f'(x^3).3x^2 = 3x^2 \tan(x^3)$

$z = g(x^5)$

$\therefore \quad \frac{dz}{dx} = g'(x^5).5x^4 = 5x^4 \sec(x^5)$

$\therefore \quad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{3x^2 \tan x^3}{5x^4 \sec x^5} = \frac{3\tan x^3}{5x^2 \sec x^5}$

20. $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$

Put $x^3 = \sin\alpha$ and $y^3 = \sin\beta$

$\therefore \quad \sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a^3(\sin\alpha - \sin\beta)$

$\therefore \quad \cos\alpha + \cos\beta = a^3(\sin\alpha - \sin\beta)$

$\therefore \quad 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$= a^3.2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

$\therefore \quad \cot\left(\frac{\alpha-\beta}{2}\right) = a^3$

$\therefore \quad \alpha - \beta = 2\cot^{-1} a^3$

$\therefore \quad \sin^{-1} x^3 - \sin^{-1} y^3 = \text{constant}$

Diff. w.r.t. x, we get

$\frac{3x^2}{\sqrt{1-x^6}} - \frac{3y^2}{\sqrt{1-y^6}} \cdot \frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$

21. Let $f(x) = px^2 + qx + r$

$\therefore \quad f(1) = f(-1) \Rightarrow p + q + r = p - q + r \Rightarrow q = 0$

$\therefore \quad f(x) = px^2 + r$

$\Rightarrow f'(x) = 2px$

$\Rightarrow f'(a) = 2ap$, $f'(b) = 2bp$ and $f'(c) = 2cp$

Since, a, b, c are in A.P.

$\therefore \quad$ 2ap, 2bp, 2cp are in A.P.

$\Rightarrow f'(a), f'(b), f'(c)$ are in A.P.

22. $\frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$

and $\frac{dy}{d\theta} = n\sec^{n-1}\theta.\sec\theta\tan\theta - n\cos^{n-1}\theta.(-\sin\theta)$

$= n\sec^n\theta\tan\theta + n\cos^{n-1}\theta\sin\theta$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{n\sec^n\theta\tan\theta + n\cos^{n-1}\theta\sin\theta}{\sec\theta\tan\theta + \sin\theta}$

Dividing N^r and D^r by $\tan\theta$, we get

$\frac{dy}{dx} = \frac{n(\sec^n\theta + \cos^n\theta)}{\sec\theta + \cos\theta}$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{n^2(\sec^n\theta+\cos^n\theta)^2}{(\sec\theta+\cos\theta)^2}$$

$$= \frac{n^2[(\sec^n\theta-\cos^n\theta)^2+4\sec^n\theta\cos^n\theta]}{(\sec\theta-\cos\theta)^2+4\sec\theta.\cos\theta}$$

$$= \frac{n^2(y^2+4)}{x^2+4}$$

$$\therefore \quad (x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$

23. $f(x) = \begin{vmatrix} x & \sin x & \cos x \\ x^2 & \tan x & -x^3 \\ 2x & \sin 2x & 5x \end{vmatrix}$

$$\therefore \quad f'(x) = \begin{vmatrix} 1 & \sin x & \cos x \\ 2x & \tan x & -x^3 \\ 2 & \sin 2x & 5x \end{vmatrix}$$

$$+\begin{vmatrix} x & \cos x & \cos x \\ x^2 & \sec^2 x & -x^3 \\ 2x & 2\cos 2x & 5x \end{vmatrix} + \begin{vmatrix} x & \sin x & -\sin x \\ x^2 & \tan x & -3x^2 \\ 2x & \sin 2x & 5 \end{vmatrix}$$

$$\therefore \quad \frac{f'(x)}{x} = \begin{vmatrix} 1 & \sin x & \cos x \\ 2 & \frac{\tan x}{x} & -x^2 \\ 2 & \sin 2x & 5x \end{vmatrix}$$

$$+\begin{vmatrix} 1 & \cos x & \cos x \\ x & \sec^2 x & -x^3 \\ 2 & 2\cos 2x & 5x \end{vmatrix} + \begin{vmatrix} 1 & \sin x & -\sin x \\ x & \tan x & -3x^2 \\ 2 & \sin 2x & 5 \end{vmatrix}$$

$$\therefore \quad \lim_{x\to 0}\frac{f'(x)}{x} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 5 \end{vmatrix}$$

$$= -2-2+0 = -4$$

24. Since, g is the inverse of f.

$\therefore$ $f[g(x)] = x$

Diff. w.r.t.x, we get

$f'(g(x))\, g'(x) = 1$

$$\therefore \quad g'(x) = \frac{1}{f'(g(x))} = 1 + [g(x)]^5$$

25. $y = \dfrac{\sin x}{\sin x\sin 2x} + \dfrac{\sin x}{\sin 2x\sin 3x} + \ldots + \dfrac{\sin x}{\sin nx\sin(n+1)x}$

$$= \frac{\sin(2x-x)}{\sin x\sin 2x} + \frac{\sin(3x-2x)}{\sin 2x\sin 3x} + \ldots + \frac{\sin((n+1)x-nx)}{\sin nx\sin(n+1)x}$$

$$= \frac{\sin 2x\cos x}{\sin x\sin 2x} - \frac{\cos 2x\sin x}{\sin x\sin 2x} + \frac{\sin 3x\cos 2x}{\sin 2x\sin 3x}$$

$$-\frac{\cos 3x\sin 2x}{\sin 2x\sin 3x} + \ldots + \frac{\sin(n+1)x\cos nx}{\sin nx\sin(n+1)x} - \frac{\cos(n+1)x\sin nx}{\sin nx\sin(n+1)x}$$

$$= \cot x - \cot 2x + \cot 2x - \cot 3x + \ldots + \cot nx - \cot(n+1)x$$

$\therefore$ $y = \cot x - \cot(n+1)x$

$$\therefore \quad \frac{dy}{dx} = -\text{cosec}^2 x - [-\text{cosec}^2(n+1)x](n+1)$$

$$= (n+1)\,\text{cosec}^2(n+1)x - \text{cosec}^2 x$$

26. If $|r| < 1$, $a + ar + ar^2 + \ldots + \infty = \dfrac{a}{1-r}$

$$\therefore \quad \sin^2 x + \sin^4 x + \sin^6 x + \ldots = \frac{\sin^2 x}{1-\sin^2 x}$$

$$= \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$$

$\therefore$ $y = e^{\tan^2 x}$

$$\therefore \quad \frac{dy}{dx} = e^{\tan^2 x}.2\tan x\sec^2 x = 2e^{\tan^2 x}\tan x\sec^2 x$$

27. $y = \tan^{-1}\dfrac{1}{1+x+x^2} + \tan^{-1}\dfrac{1}{x^2+3x+3}$

$$+\tan^{-1}\frac{1}{x^2+5x+7} + \ldots \text{ to n terms}$$

$$= \tan^{-1}\frac{1}{1+(1+x)x} + \tan^{-1}\frac{1}{1+(x+2)(x+1)}$$

$$+\tan^{-1}\frac{1}{1+(x+3)(x+2)} + \ldots \text{ to n terms}$$

$$= \tan^{-1}\left[\frac{(x+1)-x}{1+(x+1)x}\right] + \tan^{-1}\left[\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}\right]$$

$$+\tan^{-1}\left[\frac{(x+3)-(x+2)}{1+(x+3)(x+2)}\right] + \ldots \text{ to n terms}$$

$$= \tan^{-1}(x+1) - \tan^{-1}x + \tan^{-1}(x+2) - \tan^{-1}(x+1) + \tan^{-1}(x+3) - \tan^{-1}(x+2) + \ldots + \tan^{-1}(x+n) - \tan^{-1}(x+(n-1))$$

$\therefore$ $y = \tan^{-1}(x+n) - \tan^{-1}x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{1+n^2} - 1 = \frac{1-1-n^2}{1+n^2} = -\frac{n^2}{1+n^2}$$

28. $y = a\sin(bx + c)$

$\therefore$ $y_1 = a\cos(bx + c).b = ab\sin\left(\frac{\pi}{2} + bx + c\right)$

$\therefore$ $y_2 = -ab\sin(bx + c).b = ab^2\sin(\pi + bx + c)$

$y_3 = -ab^2\cos(bx + c).b = ab^3\sin\left(\frac{3\pi}{2} + bx + c\right)$

$y_4 = -ab^3(-\sin(bx + c).b) = ab^4\sin(2\pi + bx + c)$

$= ab^4\sin\left(\frac{4\pi}{2} + bx + c\right)$

In general, $y_n = ab^n\sin\left(\frac{n\pi}{2} + bx + c\right)$

29. $f(x) = x^n$

$f'(x) = nx^{n-1}$

$f''(x) = n(n-1)x^{n-2}$

$f'''(x) = n(n-1)(n-2)x^{n-3}$

$\therefore$ $f(1) = 1^n = 1 = {}^nC_0$

$\frac{f'(1)}{1!} = \frac{n(1)^{n-1}}{1} = n = {}^nC_1$

$\frac{f''(1)}{2!} = \frac{n(n-1)(1)^{n-2}}{2!} = \frac{n(n-1)}{2!} = {}^nC_2$

$\frac{f'''(1)}{3!} = \frac{n(n-1)(n-2)(1)^{n-3}}{3!} = \frac{n(n-1)(n-2)}{3!}$

$= {}^nC_3$

$\therefore$ $f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + (-1)^n\frac{f^n(1)}{n!}$

$= {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\,{}^nC_n$

$= 0$

30. $p = a^2\cos^2\theta + b^2\sin^2\theta$

$\therefore$ $\frac{dp}{d\theta} = a^2.2\cos\theta(-\sin\theta) + b^2.2\sin\theta\cos\theta$

$= (b^2 - a^2)\sin 2\theta$

$\therefore$ $\frac{d^2p}{d\theta^2} = 2(b^2 - a^2)\cos 2\theta$

$= 2(b^2 - a^2)(\cos^2\theta - \sin^2\theta)$

$\therefore$ $4p + \frac{d^2p}{d\theta^2} = 4a^2\cos^2\theta + 4b^2\sin^2\theta$

$+ 2(b^2 - a^2)(\cos^2\theta - \sin^2\theta)$

$= \cos^2\theta(4a^2 + 2b^2 - 2a^2)$

$+ \sin^2\theta(4b^2 - 2b^2 + 2a^2)$

$= \cos^2\theta(2a^2 + 2b^2) + \sin^2\theta(2a^2 + 2b^2)$

$= (2a^2 + 2b^2)(\cos^2\theta + \sin^2\theta)$

$= 2a^2 + 2b^2$

$= 2(a^2 + b^2)$

$= 2c^2$ ….[$\because a^2 + b^2 = c^2$ (given)]

Textbook Chapter No.

03 Applications of Derivatives

Hints

Classical Thinking

1. $x = 3t^2 + 1, y = t^3 - 1$

$\therefore \quad \dfrac{dx}{dt} = 6t, \dfrac{dy}{dt} = 3t^2$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3t^2}{6t} = \dfrac{t}{2}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{t=1} = \dfrac{1}{2}$

2. $y = x^3 - x$

$\therefore \quad \dfrac{dy}{dx} = 3x^2 - 1$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{x=2} = 3\,(2)^2 - 1 = 11$

$\therefore$ slope of normal at $x = 2$ $= -\dfrac{1}{\left(\frac{dy}{dx}\right)_{x=2}} = -\dfrac{1}{11}$

3. If the tangent is perpendicular to X-axis, then $\theta = 90°$

$\therefore \quad \cot\theta = 0$

$\Rightarrow \dfrac{1}{\tan\theta} = 0 \Rightarrow \dfrac{dx}{dy} = 0$

4. $y = x^3 - 3x^2 - 9x + 5 \Rightarrow \dfrac{dy}{dx} = 3x^2 - 6x - 9$

Since, the tangent is parallel to X-axis.

$\therefore \quad \dfrac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 6x - 9 = 0 \Rightarrow x = -1, 3$

5. $x^2 = -4y$

Differentiating both sides w.r.t. x, we get

$2x = -4\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-x}{2}$

$\therefore$ m = Slope of the tangent at $(-4, -4)$

$= \left(\dfrac{dy}{dx}\right)_{(-4,-4)} = 2$

$\therefore$ equation of the tangent at $(-4, -4)$ is

$y - y_1 = m\,(x - x_1)$

$\Rightarrow y + 4 = 2(x + 4)$

$\Rightarrow 2x - y + 4 = 0$

6. $\sqrt{x} + \sqrt{y} = a$

Differentiating both sides w.r.t.x, we get

$\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}} \cdot \dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}$

At $\left(\dfrac{a^2}{4}, \dfrac{a^2}{4}\right)$, $\dfrac{dy}{dx} = -\sqrt{\dfrac{\frac{a^2}{4}}{\frac{a^2}{4}}} = -1$

$\therefore$ Equation of the tangent at $\left(\dfrac{a^2}{4}, \dfrac{a^2}{4}\right)$ is

$y - \dfrac{a^2}{4} = -1\left(x - \dfrac{a^2}{4}\right)$

$\Rightarrow x + y = \dfrac{a^2}{2}$

7. $y = x^2 - 2x + 1$

$\therefore \quad \dfrac{dy}{dx} = 2x - 2$

$\therefore$ m = slope of the normal at (0,1)

$= \dfrac{-1}{\left(\frac{dy}{dx}\right)_{(0,1)}} = \dfrac{-1}{2(0) - 2} = \dfrac{1}{2}$

$\therefore$ Equation of the normal at (0,1) is

$y - y_1 = m\,(x - x_1)$

$\Rightarrow y - 1 = \dfrac{1}{2}(x - 0)$

$\Rightarrow x - 2y + 2 = 0$

8. $y = \sin\dfrac{\pi x}{2} \Rightarrow \dfrac{dy}{dx} = \dfrac{\pi}{2}\cos\dfrac{\pi x}{2} \Rightarrow \left(\dfrac{dy}{dx}\right)_{(1,1)} = 0$

$\therefore$ Equation of the normal at (1,1) is $x = 1$

9. At $x = \frac{\pi}{4}$, $y = \frac{2}{\sqrt{2}} = \sqrt{2}$

$y = 2 \sin x$

$\therefore \frac{dy}{dx} = 2 \cos x$

$\therefore \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \sqrt{2}$

$\therefore$ Equation of the tangent at $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is

$y - \sqrt{2} = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$

10. At $x = \frac{\pi}{2}$,

$y = 4 + \cos^2 \frac{\pi}{2} = 4$

$y = 4 + \cos^2 x$

$\therefore \frac{dy}{dx} = 2\cos x(-\sin x)$

$\therefore \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = 2\cos\frac{\pi}{2}\left(-\sin\frac{\pi}{2}\right) = 0$

$\therefore$ Equation of the tangent at $\left(\frac{\pi}{2}, 4\right)$ is

$y - 4 = 0\left(x - \frac{\pi}{2}\right)$

$\therefore y - 4 = 0 \Rightarrow y = 4$

11. At $x = \frac{\pi}{2}$,

$y = \frac{\pi}{2} - \sin\frac{\pi}{2}\cos\frac{\pi}{2} = \frac{\pi}{2}$

$y = x - \sin x \cos x$

$\therefore \frac{dy}{dx} = 1 - \cos x \cos x - \sin x\,(-\sin x)$

$= 1 - \cos^2 x + \sin^2 x$

$\therefore \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = 1 - \cos^2\frac{\pi}{2} + \sin^2\frac{\pi}{2} = 2$

$\therefore$ Equation of the tangent at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is

$y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{2}\right)$

$\Rightarrow y = 2x - \frac{\pi}{2}$

12. At $x = \frac{\pi}{3}$, $y = 2\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{2}$

$y = 2 \sin x + \sin 2x$

$\therefore \frac{dy}{dx} = 2\cos x + 2\cos 2x$

$\therefore \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{3}} = 2\cos\frac{\pi}{3} + 2\cos\frac{2\pi}{3} = 0$

$\therefore$ Equation of the tangent at $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{2}\right)$ is

$y - \frac{3\sqrt{3}}{2} = 0\left(x - \frac{\pi}{3}\right) \Rightarrow 2y = 3\sqrt{3}$

13. At $x = \frac{\pi}{4}$, $y = \frac{2}{\sqrt{2}} = \sqrt{2}$

$y = 2 \cos x$

$\therefore \frac{dy}{dx} = -2.\sin x$

$\therefore \left(\frac{dy}{dx}\right)_{x=\pi/4} = -\sqrt{2}$

$\therefore$ Equation of the normal at $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is

$y - \sqrt{2} = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$

14. $s = 3t^2 + 2t - 5$

$\therefore \frac{ds}{dt} = 6t + 2$

$\therefore$ Acceleration $= \frac{d^2s}{dt^2} = 6$

15. $s = 2t^2 - 3t + 1$

$\therefore v = \frac{ds}{dt} = 4t - 3 \qquad \therefore \frac{d^2s}{dt^2} = 4$

16. $\frac{ds}{dt}$ = velocity $= 45 + 22t - 3t^2$

When particle will come to rest, then v = 0

$\Rightarrow 3t^2 - 22t - 45 = 0 \Rightarrow t = 9 \quad \ldots\left[\because t \neq -\frac{5}{3}\right]$

17. Given, $s = a \sin t + b \cos 2t$

$\therefore \frac{ds}{dt} = a \cos t - 2b \sin 2t$

$\therefore \frac{d^2s}{dt^2} = -a \sin t - 4b \cos 2t$

At t = 0, $\frac{d^2s}{dt^2} = -a \sin 0^\circ - 4b \cos 0^\circ = -4b$

18. $s = 2t^3 - 9t^2 + 12t$

$\Rightarrow \frac{ds}{dt} = 6t^2 - 18t + 12$

$\Rightarrow \frac{d^2s}{dt^2} = 12t - 18 = \text{acceleration}$

When acceleration of the particle will be zero,
$12t - 18 = 0$

$\Rightarrow t = \frac{3}{2}$ sec

Hence, the acceleration of the particle will be zero after $\frac{3}{2}$ sec.

19. $s = \frac{1}{2}gt^2 \Rightarrow \frac{ds}{dt} = gt \Rightarrow \frac{d^2s}{dt^2} = g$

$\therefore$ the acceleration of the stone is uniform.

20. $\frac{dr}{dt} = 3$

$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\Rightarrow \left(\frac{dA}{dt}\right)_{r=10} = 2\pi \times 10 \times 3 = 60\pi$ cm^2/sec

21. $A = s^2$

$\therefore \frac{dA}{dt} = 2s\frac{ds}{dt}$

$\therefore \frac{dA}{dt} = 2 \times 10 \times 0.5 = 2 \times 5 = 10$ cm^2/sec

22. $V = 5x - \frac{x^2}{6}$

$\Rightarrow \frac{dV}{dt} = 5\frac{dx}{dt} - \frac{x}{3}.\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{\frac{dV}{dt}}{\left(5 - \frac{x}{3}\right)}$

$\Rightarrow \left(\frac{dx}{dt}\right)_{x=2} = \frac{5}{5 - \frac{2}{3}} = \frac{15}{13}$ cm/sec

23. Let $f(x) = \sqrt{x}$

$\therefore f'(x) = \frac{1}{2\sqrt{x}}$

Here, a = 25 and h = 0.2

$\therefore f(a) = f(25) = \sqrt{25} = 5$

and $f'(a) = f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$

$\therefore f(a + h) \approx f(a) + h f'(a)$

$\approx 5 + (0.2)\left(\frac{1}{10}\right)$

$\approx 5 + 0.02$

$\therefore \sqrt{25.2} \approx 5.02$

24. Let $f(x) = x^{\frac{1}{3}}$

$\therefore f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

Here, a = 27 and h = 2

$\therefore f(a + h) \approx f(a) + h f'(a)$

$\approx (27)^{\frac{1}{3}} + 2\left[\frac{1}{3(27)^{\frac{2}{3}}}\right]$

$\approx 3 + 2\left(\frac{1}{27}\right)$

$\approx 3 + 0.07407$

$\therefore (29)^{\frac{1}{3}} \approx 3.07407$

25. If Rolle's theorem is true for any function $f(x)$ in [a,b].
Then f(a) = f(b)
Only option (B) satisfies this condition.

26. According to Lagrange's mean value theorem, in interval [a, b] for $f(x)$,

$f'(c) = \frac{f(b) - f(a)}{b - a}$, where $a < c < b$

$\therefore a < x_1 < b$

27. $f(x) = 2 - 3x$

$\therefore f'(x) = -3 < 0$

$\therefore f(x)$ is a decreasing function.

28. $f(x) = x^2 \Rightarrow f'(x) = 2x$
For increasing function,
$f'(x) > 0$
$\Rightarrow 2x > 0$
$\Rightarrow x \in (0, \infty)$

29. $f(x) = ax + b$

$\therefore f'(x) = a$

For $f(x)$ to be decreasing,
$f'(x) < 0$
$\Rightarrow a < 0$

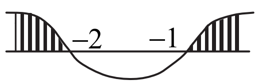
30. $f(x) = 5^{-x}$

$\therefore f'(x) = -5^{-x}\log_e 5 = -\frac{\log_e 5}{5^x}$

$\Rightarrow f'(x) < 0$ for all x
i.e., $f(x)$ is decreasing for all x.

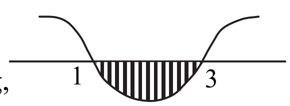
32. Let $f(x) = x^4 - 4x \Rightarrow f'(x) = 4x^3 - 4$
For $f(x)$ to be decreasing, $f'(x) < 0$
$\Rightarrow 4x^3 - 4 < 0 \Rightarrow x^3 < 1$
$\Rightarrow x \in (-\infty, 1)$

33. $f(x) = 4x^4 - 2x + 1$
$\therefore$ $f'(x) = 16x^3 - 2$
For $f(x)$ to be increasing,
$f'(x) > 0$
$\Rightarrow 16x^3 - 2 > 0$
$\Rightarrow x^3 > \dfrac{1}{8}$
$\Rightarrow x > \dfrac{1}{2}$

34. $f(x) = 2x^3 + 9x^2 + 12x + 20$
$\therefore$ $f'(x) = 6x^2 + 18x + 12$


For $f(x)$ to be increasing,
$f'(x) > 0$
$\Rightarrow x^2 + 3x + 2 > 0$
$\Rightarrow (x + 2)(x + 1) > 0$
$\Rightarrow x \in (-\infty, -2) \cup (-1, \infty)$

35. $f(x) = 2x^3 - 3x^2 - 12x + 12$
$\therefore$ $f'(x) = 6x^2 - 6x - 12$
For $f(x)$ to be increasing,
$f'(x) > 0$
$\Rightarrow x^2 - x - 2 > 0$
$\Rightarrow (x - 2)(x + 1) > 0$
$\Rightarrow x \in (-\infty, -1) \cup (2, \infty)$


36. $f(x) = x^3 - 6x^2 + 9x + 3$
$\therefore$ $f'(x) = 3x^2 - 12x + 9$


For $f(x)$ to be decreasing,
$f'(x) < 0$
$\Rightarrow 3(x^2 - 4x + 3) < 0$
$\Rightarrow (x - 3)(x - 1) < 0$
$\Rightarrow x \in (1, 3)$

37. Let $f(x) = 2x^3 - 6x + 5$
$\therefore$ $f'(x) = 6x^2 - 6$
For $f(x)$ to be increasing, $f'(x) > 0$
$\Rightarrow 6x^2 - 6 > 0 \Rightarrow (x - 1)(x + 1) > 0$
$\Rightarrow x > 1$ or $x < -1$

38. Let $f(x) = \dfrac{1}{1+x^2}$
$\therefore$ $f'(x) = -\dfrac{2x}{(1+x^2)^2}$
For $f(x)$ to be decreasing,
$f'(x) < 0 \Rightarrow -\dfrac{2x}{(1+x^2)^2} < 0$
$\Rightarrow x > 0 \Rightarrow x \in (0, \infty)$

39. Let $f(x) = \log(\sin x) \Rightarrow f'(x) = \cot x$
$\therefore$ the given function is increasing in the interval $\left(0, \dfrac{\pi}{2}\right)$.

40. $f(x) = 2x^3 - 3x^2 - 36x + 7$
$\therefore$ $f'(x) = 6x^2 - 6x - 36$
For decreasing function, $f'(x) < 0$
$\Rightarrow x^2 - x - 6 < 0$
$\Rightarrow (x - 3)(x + 2) < 0$
$\Rightarrow x \in (-2, 3)$

41. $f(x) = 2x^3 - 3x^2 - 12x + 5$
$\therefore$ $f'(x) = 6x^2 - 6x - 12$
For maximum or minimum,
$f'(x) = 0$
$\Rightarrow x^2 - x - 2 = 0$
$\Rightarrow (x - 2)(x + 1) = 0$
$\Rightarrow x = 2, -1$
Now, $f''(x) = 12x - 6$
$\therefore$ $f''(2) = 18 > 0$
$\therefore$ $f(x)$ is minimum at $x = 2$.

43. $f(x) = 7 - 20x + 11x^2$
$\therefore$ $f'(x) = -20 + 22x$
For maximum or minimum,
$f'(x) = 0 \Rightarrow -20 + 22x = 0$
$\Rightarrow x = 10/11$
Now, $f''(x) = 22 > 0$
$\therefore$ $f(x)$ is minimum at $x = \dfrac{10}{11}$.
$\therefore$ $[f(x)]_{\min} = f\left(\dfrac{10}{11}\right)$
$$= 7 - \frac{200}{11} + \frac{100 \times 11}{121} = -\frac{23}{11}$$

44. Let $f(x) = 2x^2 + x - 1$
$\therefore$ $f'(x) = 4x + 1$
For maximum or minimum,
$f'(x) = 0 \Rightarrow x = -\dfrac{1}{4}$
Now, $f''(x) = 4 > 0$
$\therefore$ $f(x)$ is minimum at $x = \dfrac{-1}{4}$.
$\therefore$ $[f(x)]_{\min} = \left[f\left(-\dfrac{1}{4}\right)\right] = \dfrac{2}{16} - \dfrac{1}{4} - 1 = \dfrac{-9}{8}$

45. $f(x) = 2x^3 - 3x^2 - 12x + 4$

$\therefore$ $f'(x) = 6x^2 - 6x - 12$

For maximum or minimum,

$f'(x) = 0 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x = 2, -1$

Now, $f''(x) = 12x - 6$

$\therefore$ $f''(2) = 18 > 0$ and $f''(-1) = -18 < 0$

$\therefore$ the given function has one maximum and one minimum.

46. $y = 1 - \cos x$

$\therefore$ $y' = \sin x$

For maximum or minimum,

$y' = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0, \pi$

Now, $y'' = \cos x$

$\Rightarrow y''(0) = 1 > 0$ and $y''(\pi) = -1 < 0$

$\therefore$ y is maximum when $x = \pi$.

Critical Thinking

1. $xy = 15$

$\therefore$ $y = \dfrac{15}{x}$

$\therefore$ $y' = -\dfrac{15}{x^2}$

At (3, 5), $y' = -\dfrac{15}{9}$

$\therefore$ Slope of normal at (3,5) = $\dfrac{9}{15}$

$\therefore$ $\theta = \tan^{-1}\left(\dfrac{9}{15}\right)$

2. $x^2 = 2y$

Differentiating both sides w.r.t. x, we get

$2x = \dfrac{2dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = x$

$\therefore$ $\left(\dfrac{dy}{dx}\right)_{\left(1, \frac{1}{2}\right)} = 1$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = 45^\circ$ ....[$\because$ $\tan 45^\circ = 1$]

3. $x^3 - 8a^2y = 0$ ....(i)

Differentiating w.r.t. x, we get

$3x^2 - 8a^2\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2}{8a^2}$

$\therefore$ Slope of the normal = $-\dfrac{1}{\frac{dy}{dx}} = -\dfrac{1}{\frac{3x^2}{8a^2}} = -\dfrac{8a^2}{3x^2}$

According to the given condition,

$\dfrac{-8a^2}{3x^2} = \dfrac{-2}{3}$

$\Rightarrow 4a^2 = x^2$

$\Rightarrow x = 2a$

From (i), $8a^3 - 8a^2y = 0 \Rightarrow y = a$

$\therefore$ the required point is (2a, a).

4. $x^2 = 3 - 2y$ ....(i)

Differentiating both sides w.r.t. x, we get

$2x = -2\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = -x$

$\therefore$ Slope of the tangent = $-x$

Slope of the given line is -1.

Since, the tangent is parallel to the given line.

$\therefore$ $-x = -1 \Rightarrow x = 1$

From (i), $y = 1$

$\therefore$ the required point is (1, 1).

5. $y = 6x - x^2$ ....(i)

$\therefore$ $\dfrac{dy}{dx} = 6 - 2x$

Slope of the given line is 2.

Since, the tangent is parallel to the given line.

$\therefore$ $6 - 2x = 2 \Rightarrow x = 2$

From (i), $y = 8$

$\therefore$ the point of tangency will be (2, 8).

6. Let the coordinates of P be (x_1, y_1).

Then, $y_1 = 2x_1^2 - x_1 + 1$ ....(i)

Now, $y = 2x^2 - x + 1$

$\therefore$ $\dfrac{dy}{dx} = 4x - 1$

$\therefore$ $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = 4x_1 - 1$

Slope of the given line is 3.

Since, the tangent is parallel to the given line.

$\therefore$ slope of the tangent = 3

$\Rightarrow 4x_1 - 1 = 3$

$\Rightarrow x_1 = 1$

From (i), $y_1 = 2$

$\therefore$ the coordinates of P are (1, 2).

7. $y = x\log x$ ....(i)

$\therefore$ $\frac{dy}{dx} = 1 + \log x$

Slope of the normal $= -\frac{1}{\left(\frac{dy}{dx}\right)} = \frac{-1}{1+\log x}$

Slope of the given line is 1.

Since, the normal is parallel to the given line.

$\therefore$ $\frac{-1}{1+\log x} = 1$

$\Rightarrow \log x = -2$

$\Rightarrow x = e^{-2}$

From (i), $y = -2e^{-2}$

$\therefore$ Co-ordinates of the point are $(e^{-2}, -2e^{-2})$.

8. $y = (x-3)^2$

$\therefore$ $y' = 2(x-3)$

Since, the tangent is parallel to the line joining (3, 0) and (4, 1).

$\therefore$ $2(x-3) = \frac{1-0}{4-3}$

$\Rightarrow 2x - 6 = 1 \Rightarrow x = \frac{7}{2}$

When $x = \frac{7}{2}$,

$y = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}$

$\therefore$ the required point is $\left(\frac{7}{2}, \frac{1}{4}\right)$.

9. $y = x^2 - 4x + 5$ ....(i)

$\therefore$ $\frac{dy}{dx} = 2x - 4$

Slope of the given line $= -\frac{1}{2}$

Since, the tangent is perpendicular to the given line.

$\therefore$ $(2x-4)\left(-\frac{1}{2}\right) = -1$

$\Rightarrow 2x - 4 = 2$

$\Rightarrow x = 3$

From (i), $y = 2$

$\therefore$ the required point is (3, 2).

10. $x^2 + y^2 - 2x - 3 = 0$ ....(i)

Differentiating w.r.t. x, we get

$2x + 2y\frac{dy}{dx} - 2 = 0$

$\therefore$ $\frac{dy}{dx} = \frac{1-x}{y}$

Since, the tangent is parallel to X-axis.

$\therefore$ $\frac{dy}{dx} = 0$

$\Rightarrow \frac{1-x}{y} = 0 \Rightarrow x = 1$

From (i),

$y = \pm 2$

11. $y^3 + 3x^2 - 12y = 0$ ....(i)

Differentiating w.r.t.x, we get

$\frac{dy}{dx} = -\frac{6x}{3y^2 - 12}$

Since, the tangent is parallel to Y-axis.

$\therefore$ $\frac{dx}{dy} = 0$

$\Rightarrow 3y^2 - 12 = 0$

$\Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

$\therefore$ $y = 2$ ....$[\because y \neq -2]$

From (i),

$x = \pm\frac{4}{\sqrt{3}}$

12. $y = ax^2 + bx$

$\therefore$ $\frac{dy}{dx} = 2ax + b \Rightarrow \left(\frac{dy}{dx}\right)_{(2,-8)} = 4a + b$

Since, the tangent is parallel to X-axis.

$\therefore$ $\left(\frac{dy}{dx}\right)_{(2,-8)} = 0 \Rightarrow b = -4a$ ....(i)

Also, the point (2, –8) lies on the curve $y = ax^2 + bx$.

$\therefore$ $-8 = 4a + 2b$ ....(ii)

From (i) and (ii), we get a = 2, b = –8

13. $y = ax^2 - 6x + b$

$\therefore$ $\frac{dy}{dx} = 2ax - 6$

$\therefore$ $\left(\frac{dy}{dx}\right)_{\left(x=\frac{3}{2}\right)} = 3a - 6$

Since, the tangent is parallel to X-axis at $x = \frac{3}{2}$.

$\therefore$ $\left(\frac{dy}{dx}\right)_{\left(x=\frac{3}{2}\right)} = 0$

$\Rightarrow 3a - 6 = 0 \Rightarrow a = 2$

Now, the given curve passes through (0, 2).

$\therefore$ $2 = 0 - 0 + b$

$\Rightarrow b = 2$

14. At t = 2, $x = \frac{1}{2}$

and $y = 2 - \frac{1}{2} = \frac{3}{2}$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1+\frac{1}{t^2}}{\frac{-1}{t^2}} = \frac{t^2+1}{-1}$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(t=2)} = -5$

$\therefore$ Equation of the normal at $\left(\frac{1}{2}, \frac{3}{2}\right)$ is

$y - \frac{3}{2} = \frac{1}{5}\left(x - \frac{1}{2}\right)$

$\therefore$ $x - 5y + 7 = 0$

15. At $\theta = \frac{\pi}{6}$,

$x = a \sec \frac{\pi}{6} = \frac{2a}{\sqrt{3}}$ and $y = a \tan \frac{\pi}{6} = \frac{a}{\sqrt{3}}$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \sec^2 \theta}{a \sec\theta \tan\theta} = \frac{1}{\sin\theta} = \operatorname{cosec}\theta$

$\therefore$ $\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{6}} = \operatorname{cosec}\frac{\pi}{6} = 2$

$\therefore$ Equation of the tangent at $\left(\frac{2a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$ is

$y - \frac{a}{\sqrt{3}} = 2\left(x - \frac{2a}{\sqrt{3}}\right)$

$\Rightarrow 2x - y = \sqrt{3}\,a$

16. $y = x^3 + 2x^2 - 4x - 43$

$\therefore$ $\frac{dy}{dx} = 3x^2 + 4x - 4$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(-2,5)} = 3(-2)^2 + 4(-2) - 4 = 0$

$\therefore$ equation of the tangent at (–2, 5) is

$y - 5 = 0.(x + 2)$

i.e., $y = 5$ (parallel to X-axis)

Normal is perpendicular to X-axis and passes through (–2,5).

$\therefore$ equation of the normal is

$x = -2$, i.e., $x + 2 = 0$

17. $y^2 = 5x - 1$

$\therefore$ $\frac{dy}{dx} = \frac{5}{2y}$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(1,-2)} = \frac{-5}{4}$

$\therefore$ Equation of the normal at (1, –2) is

$y - (-2) = \frac{4}{5}(x - 1)$

$\therefore$ $4x - 5y - 14 = 0$ ….(i)

As the normal is of the form $ax - 5y + b = 0$, comparing this with (i), we get

a = 4 and b = –14

18. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Differentiating both sides w.r.t. x, we get

$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

At $(a \sin^3\theta, a \cos^3\theta)$,

$\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta$

$\therefore$ slope of the normal is $\tan\theta$.

$\therefore$ equation of the normal at $(a \sin^3\theta, a \cos^3\theta)$ is

$y - a\cos^3\theta = \tan\theta\,(x - a\sin^3\theta)$

$\Rightarrow y\cos\theta - a\cos^4\theta = x\sin\theta - a\sin^4\theta$

$\Rightarrow x\sin\theta - y\cos\theta = a\sin^4\theta - a\cos^4\theta$

19. Let (x_1, y_1) be a point on the curve $y = x + \frac{4}{x^2}$.

Since, the tangent is parallel to X-axis.

$\therefore$ $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0 \Rightarrow 1 - \frac{8}{x_1^3} = 0 \Rightarrow x_1 = 2$

Now, $y_1 = x_1 + \frac{4}{x_1^2}$

$\Rightarrow y_1 = 2 + \frac{4}{2^2}$

$\Rightarrow y_1 = 3$

$\therefore$ equation of the tangent at (2, 3) is

$y - 3 = 0 \Rightarrow y = 3$

20. Since, the given curve crosses the X-axis, $y = 0$

$\therefore$ $0 = 2 - x \Rightarrow x = 2$

$\therefore$ the given curve crosses the X-axis at (2, 0).

Now, $(1 + x^2)y = 2 - x$

Differentiating both sides w.r.t. x, we get

$(1 + x^2)\dfrac{dy}{dx} + 2xy = -1$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-1-2xy}{1+x^2}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(2,0)} = -\dfrac{1}{5}$

$\therefore$ equation of the tangent at (2, 0) is

$y - 0 = -\dfrac{1}{5}(x - 2)$

$\Rightarrow x + 5y = 2$

21. Since, the given curve crosses the Y-axis, $x = 0$

$\therefore \quad y = be^0 \Rightarrow y = b$

$\therefore$ the given curve crosses the Y-axis at (0, b).

Now, $y = be^{-\frac{x}{a}}$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{b}{a}e^{-\frac{x}{a}}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(0,b)} = -\dfrac{b}{a}$

$\therefore$ the equation of the tangent at (0, b) is

$y - b = -\dfrac{b}{a}(x - 0)$

$\Rightarrow \dfrac{x}{a} + \dfrac{y}{b} = 1$

22. $y = e^{2x}$

$\therefore \quad \dfrac{dy}{dx} = 2e^{2x}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(0,1)} = 2$

$\therefore$ equation of the tangent at (0, 1) is

$y - 1 = 2(x - 0)$

$\Rightarrow y = 2x + 1$

This tangent meets X-axis, $\therefore y = 0$

$\therefore \quad 0 = 2x + 1 \Rightarrow x = -\dfrac{1}{2}$

$\therefore$ the required point is $\left(-\dfrac{1}{2}, 0\right)$.

23. Let the required point be (x_1, y_1).

$\therefore \quad y_1 = be^{-\frac{x_1}{a}}$ ....(i)

Now, $y = be^{-\frac{x}{a}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-b}{a}\cdot e^{-\frac{x}{a}}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \dfrac{-b}{a}\cdot e^{-\frac{x_1}{a}} = \dfrac{-y_1}{a}$ ....[From (i)]

$\therefore$ equation of the tangent at (x_1, y_1) is

$y - y_1 = \dfrac{-y_1}{a}(x - x_1)$

$\Rightarrow \dfrac{x}{a} + \dfrac{y}{y_1} = \dfrac{x_1}{a} + 1$

Comparing this equation with $\dfrac{x}{a} + \dfrac{y}{b} = 1$, we get

$y_1 = b$ and $1 + \dfrac{x_1}{a} = 1 \Rightarrow x_1 = 0$

$\therefore$ the required point is (0, b).

24. When $x = 0$,

$y = (1 + 0)^y + \sin^{-1}(0) \Rightarrow y = 1$

Now, $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$

$\therefore \quad \dfrac{dy}{dx} = (1+x)^y\left\{\dfrac{dy}{dx}\log(1+x) + \dfrac{y}{1+x}\right\} + \dfrac{\sin 2x}{\sqrt{1-\sin^4 x}}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(0,1)} = 1$

$\therefore$ the equation of the normal at (0, 1) is

$y - 1 = -1(x - 0) \Rightarrow x + y = 1$

25. Let (x_1, y_1) be the point on the curve $y = 2x^2 + 7$, where the tangent is parallel to the line $4x - y + 3 = 0$.

Then, $y_1 = 2{x_1}^2 + 7$ ....(i)

Now, $y = 2x^2 + 7$

$\therefore \quad \dfrac{dy}{dx} = 4x$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = 4x_1$

Slope of the given line is 4.

Since, the tangent is parallel to the given line.

$\therefore$ slope of the tangent = 4

$\Rightarrow 4x_1 = 4$

$\Rightarrow x_1 = 1$

From (i), $y_1 = 9$

$\therefore$ the coordinates of the point are (1, 9).

$\therefore$ Equation of the tangent at (1, 9) is

$y - 9 = 4(x - 1)$

$\Rightarrow 4x - y + 5 = 0$

26. $8y = (x-2)^2$

Differentiating both sides w.r.t.x, we get

$\frac{dy}{dx} = \frac{x-2}{4}$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(-6,8)} = \frac{-6-2}{4} = -2$ ….(i)

$y = x + \frac{3}{x}$

$\therefore$ $\frac{dy}{dx} = 1 - \frac{3}{x^2}$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(1,4)} = 1 - \frac{3}{1^2} = -2$ ….(ii)

From (i) and (ii),

T_1 is parallel to T_2

27. $xy = 1$

$\therefore$ $y = \frac{1}{x}$

$\therefore$ $y' = \frac{-1}{x^2}$

$\therefore$ Slope of the normal $= x^2$

Slope of the line $ax + by + c = 0$ is $\frac{-a}{b}$.

Since, the line $ax + by + c = 0$ is a normal to the curve $xy = 1$.

$\therefore$ $x^2 = -\frac{a}{b}$

For this condition to hold true, either $a < 0$, $b > 0$ or $b < 0$, $a > 0$

28. $\frac{dy}{dx} = 1 - 2x + 3x^2$

$\therefore$ $\frac{dy}{dx} = 3x^2 - 2x + 1 = 3\left(x^2 - \frac{2}{3}x + \frac{1}{3}\right)$

$= 3\left(x^2 - \frac{2}{3}x + \frac{1}{9} - \frac{1}{9} + \frac{1}{3}\right)$

$= 3\left[\left(x - \frac{1}{3}\right)^2 + \frac{2}{9}\right]$

$= 3\left(x - \frac{1}{3}\right)^2 + \frac{2}{3} > 0$

Slope of the given line is $-\frac{l}{m}$.

The slope will be positive only if l and m have opposite signs.

$\therefore$ option (D) is the correct answer.

29. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

Differentiating both sides w.r.t.x, we get

$\frac{1}{a^n} nx^{n-1} + \frac{1}{b^n} ny^{n-1}y' = 0$

$\therefore$ $y' = \frac{-b^n}{a^n} \cdot \frac{x^{n-1}}{y^{n-1}}$

At (a, b),

$y' = \frac{-b^n}{a^n} \cdot \frac{a^{n-1}}{b^{n-1}} = \frac{-b}{a}$, which is independent of n.

30. Since, (x_1, y_1) lies on the curve $xy = a^2$.

$\therefore$ $x_1y_1 = a^2$ ....(i)

Now, $xy = a^2$

Differentiating w.r.t. x, we get

$x\frac{dy}{dx} + y = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{y_1}{x_1}$

$\therefore$ equation of the tangent at (x_1, y_1) is

$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$

$\Rightarrow xy_1 + yx_1 = 2x_1y_1$

$\Rightarrow xy_1 + yx_1 = 2a^2$ ....[From (i)]

This tangent meets the coordinate axes at $\left(\frac{2a^2}{y_1}, 0\right)$ and $\left(0, \frac{2a^2}{x_1}\right)$.

$\therefore$ required area$= \frac{1}{2}\left(\frac{2a^2}{y_1}\right)\left(\frac{2a^2}{x_1}\right)$

$= \frac{2a^4}{x_1y_1}$

$= 2a^2$ ....[From (i)]

31. $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = 2 = m_1$(say)

$6y = 7 - x^3 \Rightarrow 6.\frac{dy}{dx} = -3x^2$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{1}{2} = m_2$(say)

Since, $m_1m_2 = -1$

$\therefore$ the angle of intersection is $\frac{\pi}{2}$.

32. $y = x^2$

$\therefore \quad \dfrac{dy}{dx} = 2x$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(1,1)} = 2 = m_1 \text{(say)}$

and $x = y^2$

$\therefore \quad 1 = 2y\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2y}$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(1,1)} = \dfrac{1}{2} = m_2 \text{(say)}$

$\therefore$ angle of intersection is $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$= \left|\dfrac{2 - \frac{1}{2}}{1 + 2\left(\frac{1}{2}\right)}\right| = \dfrac{3}{4}$$

$\therefore \quad \theta = \tan^{-1}\left(\dfrac{3}{4}\right)$

33. The point of intersection of the given curves is (0, 1).

Now, $y = a^x$

$\therefore \quad \dfrac{dy}{dx} = a^x \log a$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(0,1)} = \log a = m_1 \text{ (say)}$

Also, $y = b^x$

$\therefore \quad \dfrac{dy}{dx} = b^x \log b$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(0,1)} = \log b = m_2 \text{ (say)}$

$\therefore \quad \tan\alpha = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \dfrac{\log a - \log b}{1 + \log a \log b}$

34. $s = ae^t + \dfrac{b}{e^t}$

$\therefore \quad \dfrac{ds}{dt} = \text{velocity} = ae^t - \dfrac{b}{e^t}$

$\therefore \quad \dfrac{d^2s}{dt^2} = \text{acceleration} = ae^t + \dfrac{b}{e^t} = s$

35. $\dfrac{dS}{dt} = \text{velocity} = 15 + 12t - 3t^2$

When particle comes to rest, v = 0

$\Rightarrow 3t^2 - 12t - 15 = 0$

$\Rightarrow t = 5$ sec

36. $s = \sqrt{t} \Rightarrow \dfrac{ds}{dt} = \dfrac{1}{2\sqrt{t}}$

and $\dfrac{d^2s}{dt^2} = -\dfrac{1}{4t^{\frac{3}{2}}}$

$$= -\dfrac{1}{4}\left(\dfrac{2ds}{dt}\right)^3 = -2\left(\dfrac{ds}{dt}\right)^3$$

Hence, acceleration $\propto$ (velocity)^3.

37. $s = \sqrt{at^2 + bt + c}$

$\therefore \quad v = \dfrac{ds}{dt} = \dfrac{1}{2}\dfrac{2at + b}{\sqrt{at^2 + bt + c}}$

$$= \dfrac{2at + b}{2s}$$

acceleration $= \dfrac{d^2s}{dt^2} = \dfrac{dv}{dt}$

$$= \dfrac{2s(2a) - (2at + b)\cdot 2\dfrac{ds}{dt}}{4s^2}$$

$$= \dfrac{4as - 2(2at + b)\dfrac{(2at + b)}{2s}}{4s^2}$$

$$= \dfrac{4as^2 - (2at + b)^2}{4s^3}$$

$$= \dfrac{4a(at^2 + bt + c) - (4a^2t^2 + 4abt + b^2)}{4s^3}$$

$$= \dfrac{4ac - b^2}{4s^3}$$

$\therefore$ acceleration varies as $\dfrac{1}{s^3}$

38. Area of a circle is $A = \pi R^2$ and $\dfrac{dR}{dt} = 0.2$

$\therefore \quad \dfrac{dA}{dt} = 2\pi R\dfrac{dR}{dt} = 1.2\pi cm^2$

39. Let a be each side and A be the area of the square at any time t. Then,

$A = a^2$

$\Rightarrow \dfrac{dA}{dt} = 2a\dfrac{da}{dt}$

$= 2(2)(4)$

....$\left[\because \dfrac{da}{dt} = 4\,\text{cm/sec and } a = 2\,\text{cm(given)}\right]$

$= 16$ cm^2/sec

40. Radius of balloon = r = $\frac{3}{4}(2x+3)$

$\Rightarrow \frac{dr}{dx} = \frac{3}{2}$

$V = \frac{4}{3}\pi r^3$

$\therefore \frac{dV}{dx} = 4\pi\left(\frac{3}{4}\right)^2 (2x+3)^2 . \frac{3}{2}$

$= \frac{27\pi}{8}(2x+3)^2$

41. Given, $\frac{dr}{dt} = 2$ cm/sec, where r be the radius of circle and t be the time.

Now, area of circle is given by $A = \pi r^2$

$\therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\Rightarrow \frac{dA}{dt} = 2\pi . 20 .2$

$\therefore \frac{dA}{dt} = 80\ \pi\ \text{cm}^2/\text{sec}$

$\therefore$ the rate of change of area of circle with respect to time is 80 π cm^2/sec.

42. Let r be the radius and V be the volume of the spherical balloon at any time t. Then,

$V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\therefore \left(\frac{dV}{dt}\right)_{(r=15)} = 4\pi \times (15)^2 \times \left(\frac{dr}{dt}\right)_{(r=15)}$

$\Rightarrow 30 = 900\pi\left(\frac{dr}{dt}\right)_{(r=15)}$

....$\left[\because \frac{dV}{dt} = 30\,\text{ft}^3/\text{min (given)}\right]$

$\Rightarrow \left(\frac{dr}{dt}\right)_{(r=15)} = \frac{1}{30\pi}$ ft / min

43. Let velocity V = 5 cm/sec

(Increasing the rate/sec is called the velocity)

$\frac{da}{dt} = 5$ ….(i)

But if a is edge of a cube, then $V = a^3$

$\therefore \frac{dV}{dt} = 3a^2\frac{da}{dt} = 3a^2 . 5$

$= 15a^2 = 15 \times (12)^2$ …[$\because$ edge a = 12 cm]

$= 2160$ cm^3/sec

44. $\frac{da}{dt} = 60$cm/sec where a is edge and t is time.

$V = a^3$

$\therefore \frac{dV}{dt} = 3a^2\frac{da}{dt}$

$= 3a^2 \times 60 = 180a^2$

$= 180 \times (90)^2$

$= 1458000$ cm^3/sec.

45. $V = \frac{4}{3}\pi(x+10)^3$, where x is thickness of ice.

$\therefore \frac{dV}{dt} = 4\pi(10+x)^2\frac{dx}{dt}$

But, $\frac{dV}{dt} = 50$

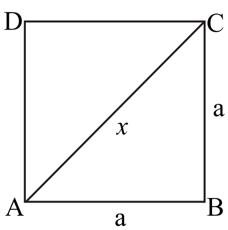
$\therefore 50 = 4\pi\ (10+x)^2 \frac{dx}{dt}$

At $x = 5$, $\frac{dx}{dt} = \frac{50}{4\pi(10+5)^2}$

$= \frac{50}{4\pi(225)}$

$= \frac{1}{18\pi}$ cm/ min

46. $\frac{dx}{dt} = 0.5$ cm/sec



$\therefore$ Area = $\frac{x^2}{2}$

$\therefore \frac{dA}{dt} = \frac{2x}{2}\cdot\frac{dx}{dt}$

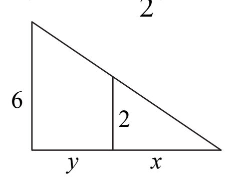
$\therefore \left[\frac{dA}{dt}\right]_{A=400} = \frac{1}{2}\sqrt{800}$ ….$\left[\begin{array}{l}\because A = 400\,\text{cm}^2 \\ \therefore x = \sqrt{800}\,\text{cm}\end{array}\right]$

$= 10\sqrt{2}$ cm^2/sec

47. From the figure,

$\frac{x}{2} = \frac{x+y}{6}$

$\Rightarrow 4x = 2y \Rightarrow x = \frac{1}{2}y$

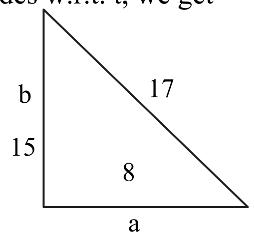


$\therefore \frac{dx}{dt} = \frac{1}{2}\frac{dy}{dt} = \frac{5}{2}$ metre/hour

48. When a = 8, b = 15, $a^2 + b^2 = 17^2$

Differentiating both sides w.r.t. t, we get

$2a \dfrac{da}{dt} + 2b \dfrac{db}{dt} = 0$



$\therefore$ $8(1) + 15 \dfrac{db}{dt} = 0$

$\therefore$ $\dfrac{db}{dt} = \dfrac{-8}{15}$ m/sec

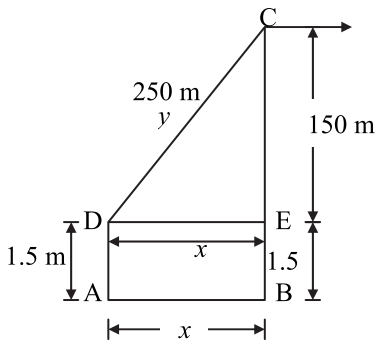
$\therefore$ the upper end is coming down at the rate of $\dfrac{8}{15}$ m /sec.

49. Let the position of the kite at time t be at C.

$\therefore$ BC = 151.5 m

Let AD be the boy who is flying the kite.

$\therefore$ CE = BC – BE = 151.5 – 1.5 = 150 m



In right angled Δ CDE,

$y^2 = x^2 + (150)^2$

Differentiating both sides w.r.t. t, we get

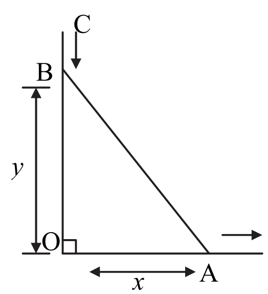
$2y \dfrac{dy}{dt} = 2x \dfrac{dx}{dt}$

$\Rightarrow \dfrac{dy}{dt} = \dfrac{x}{y}.(10)$ .... $\left[\because \dfrac{dx}{dt} = 10\right]$

$= \dfrac{10\sqrt{y^2 - (150)^2}}{y}$

$= \dfrac{10\sqrt{(250)^2 - (150)^2}}{250} = \dfrac{10 \times 200}{250} = 8$ m/s

50.



Let OC be the wall. Let AB be the position of the ladder at any time t such that OA = x and OB = y.

Length of the ladder AB = 20 ft.

In right angled Δ AOB,

$x^2 + y^2 = (20)^2$

Differentiating both sides w.r.t. t, we get

$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

$\therefore$ $\dfrac{dy}{dt} = \dfrac{-x}{y} \cdot \dfrac{dx}{dt} = \dfrac{-x}{\sqrt{400 - x^2}} \cdot \dfrac{dx}{dt}$

$\therefore$ $\left(\dfrac{dy}{dt}\right)_{x=16} = -\dfrac{16}{\sqrt{400 - (16)^2}} \cdot \dfrac{dx}{dt} = -\dfrac{4}{3}.\dfrac{dx}{dt}$

Negative sign indicates, that when x increases with time, y decreases.

Hence, the upper end is moving $\dfrac{4}{3}$ times as fast as the lower end.

51. $f(x) = x^3 - 3x + 5$

$\therefore$ $f'(x) = 3x^2 - 3$

Here, a = 2 and h = – 0.01

$\therefore$ $f(a + h) \approx f(a) + hf'(a)$

$\approx 7 + (-0.01)(9)$

$\therefore$ $f(1.99) \approx 7 - 0.09 \approx 6.91$

52. Let $f(x) = \dfrac{1}{\sqrt{x}}$

$\therefore$ $f'(x) = \dfrac{-1}{2}x^{\frac{-3}{2}} = \dfrac{-1}{2x^{\frac{3}{2}}}$

Here, a = 25 and h = 0.1

$\therefore$ $f(a + h) \approx f(a) + hf'(a)$

$\approx \dfrac{1}{5} - \dfrac{(0.1)}{2 \times 125} \approx \dfrac{1}{5} - \dfrac{0.1}{250}$

$\approx \dfrac{1}{5}\left(1 - \dfrac{1}{500}\right) \approx \dfrac{1}{5}\left(\dfrac{499}{500}\right) \approx \dfrac{1}{5}\left(\dfrac{998}{1000}\right)$

$\therefore$ $\dfrac{1}{\sqrt{25.1}} \approx \dfrac{1}{5} \times 0.998 \approx 0.1996$

53. Let $f(x) = \dfrac{1}{x^2}$

$\therefore$ $f'(x) = -2x^{-3} = \dfrac{-2}{x^3}$

Here, a = 2 and h = 0.002

$\therefore$ $f(a + h) \approx f(a) + hf'(a)$

$\approx \dfrac{1}{4} + (0.002)\left(\dfrac{-2}{8}\right) \approx \dfrac{1}{4} - \dfrac{0.002}{4}$

$\therefore$ $\dfrac{1}{(2.002)^2} \approx \dfrac{0.998}{4} \approx 0.2495$

54. Let $f(x) = x^{\frac{1}{4}}$

$\therefore$ $f'(x) = \frac{1}{4}x^{\frac{-3}{4}} = \frac{1}{4x^{\frac{3}{4}}}$

Here, a = 81 and h = –1

$\therefore$ $f(a + h) \approx f(a) + h f'(a)$

$$\approx (81)^{\frac{1}{4}} + (-1)\left[\frac{1}{4(81)^{\frac{3}{4}}}\right]$$

$$\approx 3 - \frac{1}{108}$$

$$\approx 3 - 0.009259$$

$\therefore$ $(80)^{\frac{1}{4}} \approx 2.9907$

55. Let $f(x) = \cot^{-1}x$

$\therefore$ $f'(x) = \frac{-1}{1+x^2}$

Here, a = 1 and h = 0.001

$\therefore$ $f(a + h) \approx f(a) + hf'(a)$

$$\approx \frac{\pi}{4} + 0.001\left(\frac{-1}{2}\right)$$

$$\approx \frac{3.14}{4} - 0.0005$$

$\therefore$ $\cot^{-1}(1.001) \approx 0.785 - 0.0005 \approx 0.7845$

56. Let $f(x) = \tan^{-1} x$

$\therefore$ $f'(x) = \frac{1}{1+x^2}$

Here, a = 1 and h = – 0.001

$\therefore$ $f(a + h) \approx f(a) + hf'(a)$

$\therefore$ $\tan^{-1}(0.999) \approx \frac{\pi}{4} + \frac{1}{1+1}(-0.001)$

$$\approx \frac{\pi}{4} - \frac{0.001}{2}$$

$$\approx \frac{\pi}{4} - 0.0005$$

57. Let $f(x) = \cos x$

$\therefore$ $f'(x) = -\sin x$

Here, a = 90°

and $h = 30' = \left(\frac{1}{2}\right)^{\circ} = \left(\frac{1}{2}\times 0.0175\right)^{c}$

$= 0.00875$

$f(a) = f(90°) = \cos 90° = 0$

$f'(a) = f'(90°) = -\sin 90° = -1$

$\therefore$ $f(a + h) \approx f(a) + h f'(a)$

$\therefore$ $\cos(90°\ 30') \approx 0 + (0.00875) \times (-1) \approx -0.00875$

58. Let $f(x) = \sin x$

$\therefore$ $f'(x) = \cos x$

Here, a = 30° and

$h = 1° = 0.0175^c$

$\therefore$ $f(a + h) \approx f(a) + h f'(a)$

$$\approx \frac{1}{2} + 0.0175 \times 0.8660$$

$$\approx 0.5 + 0.01515$$

$\therefore$ $\sin(31°) \approx 0.51515$

≈ 0.5152

59. Let $f(x) = \tan x$

$\therefore$ $f'(x) = \sec^2 x$

Here, $a = 45° = \left(\frac{\pi}{4}\right)^c$ and $h = 1° = 0.0175^c$

$f(a + h) \approx f(a) + hf'(a)$

$$\approx \tan(a) + h\sec^2 a$$

$$\approx \tan(a) + h\frac{1}{\cos^2 a}$$

$$\approx \tan\left(\frac{\pi}{4}\right) + (0.0175)\frac{1}{\left(1/\sqrt{2}\right)^2}$$

$$\approx 1 + 0.035$$

$\therefore$ $\tan 46^{\circ} \approx 1.035$

60. Let $f(x) = \log_e x$

$\therefore$ $f'(x) = \frac{1}{x}$

Here, a = 9 and h = 0.01

$\therefore$ $f(a + h) \approx f(a) + hf'(a)$

$$\approx f(9) + (0.01)f'(9)$$

$$\approx \log_e 3^2 + \frac{0.01}{9}$$

$$\approx 2\log_e 3 + \frac{0.01}{9}$$

$$\approx 2.1972 + 0.0011$$

$$\approx 2.1983$$

61. Consider option (B),

$f(x) = x^2$ is a polynomial function.

$\therefore$ $f(x)$ is continuous and differentiable in the given interval.

Also, $f(1) = f(-1) = 1$

So, Rolle's theorem is applicable to

$f(x) = x^2$ on $[-1, 1]$.

62. $f(x) = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

$f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$

$f'(0^+) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1$

$\therefore$ f(x) is not differentiable at $x = 0$.

63. $f(x) = e^{-2x} \sin 2x$

$\Rightarrow f'(x) = 2e^{-2x} (\cos 2x - \sin 2x)$

Now, $f'(c) = 0$

$\Rightarrow \cos 2c - \sin 2c = 0$

$\Rightarrow \tan 2c = 1 \Rightarrow 2c = \frac{\pi}{4} \Rightarrow c = \frac{\pi}{8}$

64. $f(x) = x^3 - 6x^2 + ax + b$

$\Rightarrow f'(x) = 3x^2 - 12x + a$

Now, $f'(c) = 0 \Rightarrow f'\left(2+\frac{1}{\sqrt{3}}\right) = 0$

$\Rightarrow 3\left(2+\frac{1}{\sqrt{3}}\right)^2 - 12\left(2+\frac{1}{\sqrt{3}}\right) + a = 0$

$\Rightarrow 3\left(4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right) - 12\left(2+\frac{1}{\sqrt{3}}\right) + a = 0$

$\Rightarrow 12 + 1 + 4\sqrt{3} - 24 - 4\sqrt{3} + a = 0$

$\Rightarrow a = 11$

65. $f(x) = x(x+3)e^{-\left(\frac{1}{2}\right)x}$

$\therefore$ $f'(x) = (x^2+3x)e^{-\left(\frac{1}{2}\right)x}.\left(-\frac{1}{2}\right) + (2x+3)e^{-\left(\frac{1}{2}\right)x}$

$= e^{-\left(\frac{1}{2}\right)x}\left\{-\frac{1}{2}(x^2+3x) + 2x + 3\right\}$

$= -\frac{1}{2}e^{-\left(\frac{x}{2}\right)}(x^2 - x - 6)$

Since, f(x) satisfies all the conditions of Rolle's theorem. So, there exists c $\in$ (–3, 0) such that

$f'(c) = 0$

$\Rightarrow c^2 - c - 6 = 0$

$\Rightarrow c = 3, -2$

But $c = -2 \in [-3, 0]$

$\therefore$ $c = -2$

66. Here, f(x) is continuous and differentiable on (0, 1) for $\alpha > 0$

Also, $f(0) = f(1) = 0$

For f(x) to be continuous at $x = 0$,

$\lim_{x\to 0^+} f(x) = f(0) \Rightarrow \lim_{x\to 0^+} x^\alpha \log x = 0$

$\Rightarrow \lim_{x\to 0^+} \frac{\log x}{x^{-\alpha}} = 0$

Applying L'Hospital rule, we get

$\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{\alpha}{x^{\alpha+1}}} = 0 \Rightarrow \lim_{x\to 0^+} -\frac{x^\alpha}{\alpha} = 0,$

which is possible only when $\alpha > 0$

$\therefore$ option (D) is the correct answer.

67. $f(x) = \log_e x$

$\therefore$ $f(1) = \log_e 1 = 0,$

$f(3) = \log_e 3$ and $f'(x) = \frac{1}{x}$

By Lagrange's mean value theorem,

$f'(c) = \frac{f(3)-f(1)}{3-1}$

$\Rightarrow \frac{1}{c} = \frac{\log_e 3 - 0}{2} \Rightarrow c = \frac{2}{\log_e 3} \Rightarrow c = 2\log_3 e$

68. $f(x) = x + \frac{1}{x}$

$\therefore$ $f(3) = \frac{10}{3}$, $f(1) = 2$ and $f'(x) = 1 - \frac{1}{x^2}$

By Lagrange's mean value theorem,

$f'(c) = \frac{f(3)-f(1)}{3-1}$

$\Rightarrow 1 - \frac{1}{c^2} = \frac{\frac{10}{3}-2}{2} \Rightarrow 1 - \frac{1}{c^2} = \frac{2}{3}$

$\Rightarrow c^2 = 3 \Rightarrow c = \sqrt{3}$

69. $f(x) = \frac{1}{x}$

$\Rightarrow f(a) = \frac{1}{a}$, $f(b) = \frac{1}{b}$ and $f'(x) = -\frac{1}{x^2}$

Given, $f(b) - f(a) = (b-a) f'(x_1)$

$\Rightarrow \frac{1}{b} - \frac{1}{a} = (b-a)\left(-\frac{1}{x_1^2}\right)$

$\Rightarrow \frac{a-b}{ab} = \frac{(a-b)}{x_1^2}$

$\Rightarrow x_1^2 = ab \Rightarrow x_1 = \sqrt{ab}$

70. $f(x) = x(x-1)^2 = x^3 - 2x^2 + x$

$\therefore$ $f(0) = 0$, $f(2) = 2$ and $f'(x) = 3x^2 - 4x + 1$

By mean value theorem,

$$f'(c) = \frac{f(2)-f(0)}{2-0}$$

$$\Rightarrow 3c^2 - 4c + 1 = \frac{2-0}{2-0} = 1$$

$$\Rightarrow 3c^2 - 4c = 0$$

$$\Rightarrow c(3c-4) = 0 \Rightarrow c = 0,\ c = \frac{4}{3}$$

71. $f(x) = x(x-1)(x-2)$

$\therefore$ $f(a) = f(0) = 0$, $f(b) = f\left(\frac{1}{2}\right) = \frac{3}{8}$ and

$$f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$$

$$\Rightarrow f'(c) = (c-1)(c-2) + c(c-2) + c(c-1)$$

$$\Rightarrow f'(c) = c^2 - 3c + 2 + c^2 - 2c + c^2 - c$$

$$\Rightarrow f'(c) = 3c^2 - 6c + 2$$

Given, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{\frac{3}{8}-0}{\frac{1}{2}-0} = \frac{3}{4}$$

$$\Rightarrow 3c^2 - 6c + \frac{5}{4} = 0$$

$$\therefore\ c = \frac{6 \pm \sqrt{36-15}}{2\times 3} = \frac{6\pm\sqrt{21}}{6} = 1 \pm \frac{\sqrt{21}}{6}$$

72. $f(x) = 1 - x^3 - x^5 \Rightarrow f'(x) = -3x^2 - 5x^4$

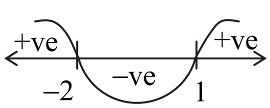
$\Rightarrow$ $f'(x) < 0$ for all values of x.

73. $f(x) = 2x^3 + 3x^2 - 12x + 5$

$\therefore$ $f'(x) = 6x^2 + 6x - 12$

$= 6(x^2 + x - 2)$

$= 6(x+2)(x-1)$



Increasing at $(-\infty, -2) \cup (1, \infty) = l_1$

Decreasing at $(-2, 1) = l_2$

74. $f(x) = \dfrac{x}{1+|x|}$

$$\therefore\ f'(x) = \frac{1+|x| - x\frac{|x|}{x}}{(1+|x|)^2} = \frac{1}{(1+|x|)^2} > 0$$

$\therefore$ the given function is increasing.

75. $f(x) = \dfrac{\log x}{x}$

$$\therefore\ f'(x) = \frac{1-\log x}{x^2} < 0$$

$\Rightarrow 1 - \log x < 0$

$\Rightarrow 1 < \log x$

$\Rightarrow \log x > 1$

$\Rightarrow x > e$

76. $\dfrac{d}{dx}(f(x)) = \dfrac{-2-x}{(x+1)^2}$

For $x > 0$,

$\dfrac{d}{dx}(f(x)) < 0$

77. $f'(x) = 3x^2 + 3x + 3 = 3(x^2 + x + 1)$

$$= 3\left[\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right] \geq \frac{9}{4} > 0$$

$\therefore$ $f(x)$ is an increasing function.

78. $f'(x) = 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2-x)$

Since, f is increasing, $f'(x) > 0$

$\Rightarrow xe^{-x}(2-x) > 0 \Rightarrow x(2-x) > 0$

$\Rightarrow x > 0, 2 - x > 0$ or $x < 0, 2 - x < 0$

$\Rightarrow x > 0, 2 > x$ or $x < 0, 2 < x$

$\Rightarrow 0 < x < 2$ or $2 < x < 0$ (Not possible)

$\Rightarrow 0 < x < 2 \Rightarrow x \in (0, 2)$

79. $f(x) = e^{ax} + e^{-ax}$

$\therefore$ $f'(x) = a(e^{ax} - e^{-ax}) < 0$

But, $a < 0$

$\therefore$ $e^{ax} - e^{-ax} > 0$

$\Rightarrow e^{ax} > e^{-ax}$

$\Rightarrow ax > -ax$

$\Rightarrow 2ax > 0$

$\therefore$ $ax > 0$, then $x < 0$ ....[$\because$ $a < 0$]

80. $f'(x) = 3kx^2 - 18x + 9$

$= 3(kx^2 - 6x + 3)$

Since, $f(x)$ is increasing on R $\therefore$ $f'(x) > 0$

$\therefore$ $kx^2 - 6x + 3 > 0\ \forall\ x \in R$

$\Rightarrow k > 0$ and $36 - 12k < 0$

$\Rightarrow k > 3$

....[$\because$ $ax^2 + bx + c > 0\ \forall\ x \in R$

$\Rightarrow a > 0$ and $b^2 - 4ac < 0$]

Hence, $f(x)$ is increasing on R if $k > 3$.

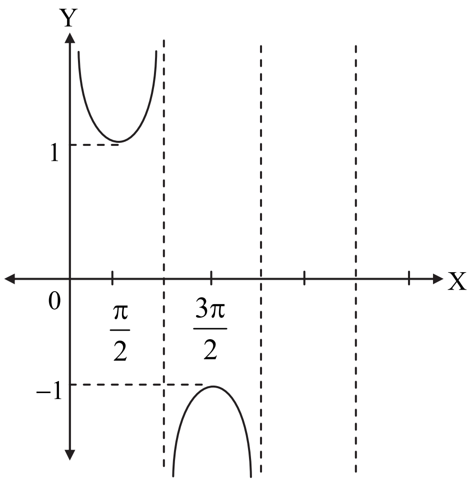
81. Since, f(x) is increasing for all x.

$\therefore$ $f'(x) > 0$ for all x

$\Rightarrow \dfrac{K-2}{(\sin x+\cos x)^2} > 0$ for all x

$\Rightarrow K - 2 > 0 \Rightarrow K > 2$

82. The graph of cosec x is opposite in interval $\left(\dfrac{\pi}{2},\dfrac{3\pi}{2}\right)$



At the point $x = \pi$, cosec x is not defined and $x \in \left(\dfrac{\pi}{2},\dfrac{3\pi}{2}\right)$

$\therefore$ equation is neither increasing nor decreasing.

Also $\dfrac{d}{dx}(\tan x) = \sec^2 x > 0$ which is a increasing function.

Also $y = x^2$ is a parabola, which is increasing

Also $y = |x - 1|$ is a V-shaped upward curve, which is always increasing.

$\therefore$ option (A) is the correct answer.

83. Let $f(x) = x + \dfrac{1}{x}$

$\therefore$ $f'(x) = 1 - \dfrac{1}{x^2} \le 0 \Rightarrow 1 \le \dfrac{1}{x^2} \Rightarrow x^2 \le 1$

$\Rightarrow x \in [-1, 1]$

84. Since $f'(x) = \dfrac{3}{(x+1)^2}$ is greater than '0' in interval $(-\infty, \infty)$, therefore $f(x) = \dfrac{x-2}{x+1}$ is increasing in interval $(-\infty, \infty)$ or R.

85. Let $f(x) = \sin x - bx + c$

$\therefore$ $f'(x) = \cos x - b > 0 \Rightarrow \cos x > b \Rightarrow b < -1$

86. $f(x) = x^4 - \dfrac{x^3}{3} \Rightarrow f'(x) = 4x^3 - x^2$

For f(x) to be increasing, $4x^3 - x^2 > 0$

$\Rightarrow x^2(4x - 1) > 0$

$\therefore$ the function is increasing for $x > \dfrac{1}{4}$

Similarly, decreasing for $x < \dfrac{1}{4}$

87. $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\therefore$ $f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$

$= 6(x - 2)(x - 3)$

To be monotonic decreasing, $f'(x) < 0$

$\Rightarrow (x - 2)(x - 3) < 0 \Rightarrow x \in (2, 3)$

88. As $f(x) = \sin 2x \Rightarrow f'(x) = 2\cos 2x$

Here, $f'(x) > 0$ in $\left(0,\dfrac{\pi}{4}\right)$ and $f'(x) < 0$ in $\left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right)$

89. $f(x) = x + \cos x \Rightarrow f'(x) = 1 - \sin x$

$f'(x) > 0$ for all values of x.

$\therefore$ f(x) is always increasing.

90. $f(x) = \dfrac{\log x}{x}$

$\therefore$ $f'(x) = \dfrac{1}{x^2} - \dfrac{\log x}{x^2} = \dfrac{1-\log x}{x^2}$

For f(x) to be increasing, $f'(x) > 0$

$\Rightarrow 1 - \log x > 0 \Rightarrow 1 > \log x \Rightarrow e > x$

$\therefore$ f(x) is increasing in the interval (0, e).

91. $f(x) = 1 - e^{-\frac{x^2}{2}}$

$\Rightarrow f'(x) = -e^{-\frac{x^2}{2}}(-x) = xe^{-\frac{x^2}{2}}$

For f(x) to be increasing, $f'(x) > 0$

$\Rightarrow xe^{-\frac{x^2}{2}} > 0$

$\Rightarrow$ f(x) is decreasing for $x < 0$ and increasing for $x > 0$.

92. $f(x) = \dfrac{a\sin x + b\cos x}{c\sin x + d\cos x}$

f(x) will be decreasing, if $f'(x) < 0$

$\therefore$ $\dfrac{1}{(c\sin x + d\cos x)^2}\big[(c\sin x + d\cos x)(a\cos x - b\sin x)$

$-(a\sin x + b\cos x)(c\cos x - d\sin x)\big] < 0$

$\Rightarrow ac\sin x\cos x - bc\sin^2 x + ad\cos^2 x$

$-bd\sin x\cos x - ac\sin x\cos x + ad\sin^2 x$

$-bc\cos^2 x + bd\sin x\cos x < 0$

$\Rightarrow ad(\sin^2 x + \cos^2 x) - bc(\sin^2 x + \cos^2 x) < 0$

$\Rightarrow ad - bc < 0$

93. $f(x) = x^4 - 62x^2 + ax + 9$ ....(i)

$\therefore$ $f'(x) = 4x^3 - 124x + a$

For maximum or minimum, $f'(x) = 0$

$\Rightarrow 4x^3 - 124x + a = 0$

Since, $x = 1$ is a root of (i).

$\therefore$ $f'(1) = 4 - 124 + a = 0 \therefore a = 120$

94. $f(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$

$\therefore$ $f'(x) = 2x - (\alpha + \beta)$

For maximum or minimum, $f'(x) = 0$

$\Rightarrow x = \dfrac{\alpha+\beta}{2}$

Now $f''(x) = 2 > 0$

$\therefore$ $f(x) = f\left(\dfrac{\alpha+\beta}{2}\right)$

$= \left(\dfrac{\alpha+\beta}{2} - \alpha\right)\left(\dfrac{\alpha+\beta}{2} - \beta\right)$

$= \left(\dfrac{\beta-\alpha}{2}\right)\left(\dfrac{\alpha-\beta}{2}\right) = -\dfrac{(\alpha-\beta)^2}{4}$

95. $y = xe^x$

$\therefore$ $y' = xe^x + e^x = e^x(x + 1) = 0$

$\therefore$ $x = -1$

$y'' = xe^x + e^x + e^x$

At $x = -1$,

$y'' = -e^{-1} + e^{-1} + e^{-1} = \dfrac{1}{e} > 0$

$\therefore$ Minimum at $x = -1$.

96. $f(x) = x^5 - 5x^4 + 5x^3 - 10$

$\therefore$ $f'(x) = 5x^4 - 20x^3 + 15x^2$

For maximum or minimum, $f'(x) = 0$

$\Rightarrow 5x^2(x^2 - 4x + 3) = 0$

$\Rightarrow x^2(x - 3)(x - 1) = 0$

$\Rightarrow x = 0, x = 3, x = 1$

$f''(x) = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$

$f''(0) = 0$

$f''(3) =$ Positive (Minima)

$f''(1) =$ Negative (Maxima)

$\therefore$ $(p, q) = (1, 3)$

97. $\dfrac{dy}{dx} = \dfrac{a}{x} + 2bx + 1 \Rightarrow \left(\dfrac{dy}{dx}\right)_{x=1} = a + 2b + 1 = 0$

$\Rightarrow a = -2b - 1$ and $\left(\dfrac{dy}{dx}\right)_{x=2} = \dfrac{a}{2} + 4b + 1 = 0$

$\Rightarrow \dfrac{-2b-1}{2} + 4b + 1 = 0$

$\Rightarrow -b + 4b + \dfrac{1}{2} = 0$

$\Rightarrow 3b = \dfrac{-1}{2} \Rightarrow b = \dfrac{-1}{6}$ and $a = \dfrac{1}{3} - 1 = \dfrac{-2}{3}$

98. $f(x) = 3x^4 - 4x^3$

$\therefore$ $f'(x) = 12x^3 - 12x^2$

$\therefore$ $x^2(x - 1) = 0 \Rightarrow x = 1, 0$

Now $f''(x) = 36x^2 - 24x$

$f''(1) = 12 > 0$

$f''(0) = 0$

$f(1) = 3 - 4 = -1$

$f(-1) = 3 + 4 = 7$

$f(2) = 48 - 32 = 16$

$\therefore$ Maximum at 2 and Minimum at 1 and Maximum value is 16 and Minimum value is -1.

99. Let $f(x) = x^3 - 12x^2 + 36x + 17$

$\therefore$ $f'(x) = 3x^2 - 24x + 36 = 0$ at $x = 2, 6$

Again $f''(x) = 6x - 24$ is $-$ve at $x = 2$

So that $f(6) = 17$, $f(2) = 49$

At the end points, $f(1) = 42$, $f(10) = 177$

So that $f(x)$ has its maximum value as 177

100. $x + y = 16 \Rightarrow y = 16 - x$

$\Rightarrow x^2 + y^2 = x^2 + (16 - x)^2$

Let $z = x^2 + (16 - x)^2$

$\Rightarrow z' = 4x - 32$

To be minimum of z, $z'' > 0$,

Therefore $4x - 32 = 0 \Rightarrow x = 8, y = 8$

101. $f(x) = (x-1)^{\frac{1}{3}}(x - 2)$

$\therefore$ $f'(x) = (x - 1)^{\frac{1}{3}} . 1 + (x - 2). \dfrac{1}{3}(x-1)^{\frac{-2}{3}}$

$= \dfrac{4x-5}{3(x-1)^{\frac{2}{3}}}$

For maxima or minima, $f'(x) = 0$

$\therefore$ $\dfrac{4x-5}{3(x-1)^{\frac{2}{3}}} = 0$

$\therefore$ $x = \dfrac{5}{4}$

$\therefore$ $f(1) = (1 - 1)^{\frac{1}{3}}(1 - 2) = 0$

$f\left(\dfrac{5}{4}\right) = \left(\dfrac{5}{4} - 1\right)^{\frac{1}{3}}\left(\dfrac{5}{4} - 2\right) = \dfrac{-3}{4^{\frac{4}{3}}}$, $f(9) = 14$

$\therefore$ absolute minimum occurs at $x = \dfrac{5}{4}$ and min. value $= \dfrac{-3}{4^{\frac{4}{3}}}$

Absolute maximum occurs at $x = 9$ and max. value = 14.

102. Let $f(x) = x\sqrt{1-x^2}$

$\Rightarrow f'(x) = \dfrac{1-2x^2}{\sqrt{1-x^2}} = 0 \Rightarrow x = \pm\dfrac{1}{\sqrt{2}}$

But as $x > 0$, we have $x = \dfrac{1}{\sqrt{2}}$

Now,

$$f''(x) = \frac{\sqrt{1-x^2}(-4x)-(1-2x^2)\dfrac{-x}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$= \frac{2x^3-3x}{(1-x^2)^{3/2}}$$

$\Rightarrow f''\left(\dfrac{1}{\sqrt{2}}\right) = -\text{ve}$

$\therefore$ f(x) is maximum at $x = \dfrac{1}{\sqrt{2}}$

103. Let x and y be two natural numbers such that $x + y = 10$ and the product is xy.

$xy = x(10 - x) = 10x - x^2 = f(x)$

$\therefore$ $f'(x) = 10 - 2x$

$\therefore$ $f''(x) = -2$

Roots of $f'(x) = 0$,

i.e., $10 - 2x = 0$ i.e., $x = 5$

$f'(5) = 10 - 10 = 0$

$\therefore$ f is maximum when $x = 5$, $y = 5$

$\therefore$ The product is maximum if $x = 5$, $y = 5$

104. $2(x + y) = 24$

$\therefore$ $x + y = 12$

$\therefore$ $x = 12 - y$

$f(x) = xy = x(12 - x) = 12x - x^2$

$\therefore$ $f'(x) = 12 - 2x = 0$

$\therefore$ $x = 6$ At $x = 6$, $y = 6$

$\therefore$ max area is 36 m^2.

105. Let $x + y = 3$

According to the given condition,

$f(x) = x^2 \times (3 - x) = 3x^2 - x^3$ ....(i)

$\therefore$ $f'(x) = 6x - 3x^2 = 0$

$\therefore$ $3x(x - 2) = 0$

$\therefore$ $x = 0, x = 2$

Now $f''(x)$ is max at $x = 2$

Its maximum value is 4 ....[From (i)]

106. Let one number be $(100 - x)$ and then another is x. Therefore $f(x) = 2(100 - x) + x^2$

$= x^2 - 2x + 200$

$\therefore$ $f'(x) = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$

Here $f''(x) = 2 > 0$

Therefore function is minimum at $x = 1$.

So the numbers are 99 and 1.

107. According to the given condition,

$2x + 2y = 100 \Rightarrow x + y = 50$ ....(i)

Let the area of rectangle be A.

$\therefore$ $A = xy \Rightarrow y = \dfrac{A}{x}$

Put in (i), we have $x + \dfrac{A}{x} = 50 \Rightarrow A = 50x - x^2$

$\Rightarrow \dfrac{dA}{dx} = 50 - 2x$

For maximum area, $\dfrac{dA}{dx} = 0$

$\therefore$ $50 - 2x = 0 \Rightarrow x = 25$ and $y = 25$

Hence, adjacent sides are 25 and 25 cm.

108. Let the number be x, then the function

$f(x) = \dfrac{x}{x^2+16}$

On differentiating with respect to x, we get

$$\Rightarrow f'(x) = \frac{(x^2+16).1-x(2x)}{(x^2+16)^2}$$

$$= \frac{x^2+16-2x^2}{(x^2+16)^2}$$

$$= \frac{16-x^2}{(x^2+16)^2}$$

Put $f'(x) = 0$ for maxima or minima

$f'(x) = 0 \Rightarrow 16 - x^2 = 0 \Rightarrow x = 4, -4$

Again differentiating

$$f''(x) = \frac{(x^2+16)^2(-2x)-(16-x^2)2(x^2+16)2x}{(x^2+16)^4}$$

At $x = 4$, $f''(x) < 0$ and at $x = -4$, $f''(x) > 0$

$\therefore$ Least value of $f(x) = \dfrac{-4}{16+16} = -\dfrac{1}{8}$

109. Let $y = x^{2x} \Rightarrow \log y = 2x.\log x$, $(x > 0)$

Differentiating, $\dfrac{dy}{dx} = 2x^{2x}(1 + \log x)$;

$\therefore$ $\dfrac{dy}{dx} = 0$

$\Rightarrow \log x = -1 \Rightarrow x = e^{-1} = \dfrac{1}{e}$

$\therefore$ Stationary point is $x = \dfrac{1}{e}$

110. $x + y = 8$ $\therefore$ $y = 8 - x$

Now $f(x) = xy = x(8 - x) = 8x - x^2$

$\therefore$ $f'(x) = 8 - 2x$

For maximum value of f(x), $f'(x) = 0$

$\therefore$ $x = 4$ and $y = 4$

So, maximum value of $xy = 4 \times 4 = 16$

111. $f(x) = 2x^3 - 21x^2 + 36x - 30$
$\Rightarrow f'(x) = 6x^2 - 42x + 36$
$\therefore$ $f'(x) = 0 \Rightarrow x = 6, 1$ and $f''(x) = 12x - 42$
Here $f''(1) = -30 < 0$ and $f''(6) = 30 > 0$
$\therefore$ $f(x)$ has maxima at $x = 1$ and minima at $x = 6$.

112. $f(x) = \cos x + \cos(\sqrt{2}x)$
$\therefore$ $f'(x) = -\sin x - \sqrt{2}\sin(\sqrt{2}x) = 0$
$\therefore$ $x = 0$ is the only solution.
$f''(x) = -\cos x - 2 \cos(\sqrt{2}x) < 0$ at $x = 0$
Hence, maxima occurs at $x = 0$.

113. Let $f(x) = x^3 - 18x^2 + 96x$
$\therefore$ $f'(x) = 3x^2 - 36x + 96$
For maximum or minimum, $f'(x) = 0$
$\Rightarrow x^2 - 12x + 32 = 0 \Rightarrow (x - 4)(x - 8) = 0$
$\Rightarrow x = 4, 8$
Now, $f''(x) = 6x - 36$
At $x = 4$, $f''(x) = 24 - 36 = -12 < 0$
At $x = 4$, $f(x)$ will be maximum
and $[f(4)]_{max.} = 64 - 288 + 384 = 160$
At $x = 8$, $\dfrac{d^2y}{dx^2} = 48 - 36 = 12 > 0$
At $x = 8$, $f(x)$ will be minimum and
$[f(8)]_{min.} = 128$

114. Let PQ = a and PR = b, then $\Delta = \dfrac{1}{2}ab\sin\theta$
$\because$ $-1 \le \sin\theta \le 1$
Since, area is maximum when $\sin\theta = 1$
$\Rightarrow \theta = \dfrac{\pi}{2}$

115. Here $f(x) = |\sin 4x + 3|$
We know that minimum value of $\sin x$ is -1 and maximum is 1.
Hence minimum $|\sin 4x + 3| = |-1 + 3| = 2$ and maximum $|\sin 4x + 3| = |1 + 3| = 4$

116. $f(x) = |px - 9| + r|x|, x \in (-\infty, \infty)$
Where $p > 0$, $q > 0$ and $r > 0$ can assume its minimum value only at one point, if $p = q = r$.

117. $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$
$\therefore$ $f'(x) = 12x^3 - 24x^2 + 24x - 48$
$= 12(x^3 - 2x^2 + 2x - 4) = 12[(x - 2)(x^2 + 2)]$
For maximum or minimum of $f(x)$, $f'(x) = 0$
$\Rightarrow 12[(x - 2)(x^2 + 2)] = 0$
$\Rightarrow x = 2$.
Now, $f''(x) = 12(3x^2 - 4x + 2)$
$\therefore$ $f''(2) = 12(12 - 8 + 2) = 72 > 0$
$\therefore$ f has minimum at $x = 2$ and the minimum value of f at $x = 2$ is
$f(2) = 48 - 64 + 48 - 96 + 25 = -39$

118. $f(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$
$\therefore$ $f'(x) = 2x - (\alpha + \beta)$
For maximum or minimum of $f(x)$, $f'(x) = 0$
$\Rightarrow 2x - (\alpha + \beta) = 0$
Now, $f''(x) = 2 > 0$
$\therefore$ f has minimum at $x = \dfrac{\alpha + \beta}{2}$
and the minimum value of f at $x = \dfrac{\alpha + \beta}{2}$ is
$\left(\dfrac{\alpha + \beta}{2} - \alpha\right)\left(\dfrac{\alpha + \beta}{2} - \beta\right)$
$= \left(\dfrac{\beta - \alpha}{2}\right)\left(\dfrac{\alpha - \beta}{2}\right) = -\dfrac{(\alpha - \beta)^2}{4}$

119. Let $x + y = 20 \Rightarrow y = 20 - x$ ....(i)
and $x^3y^2 = z$
$\Rightarrow z = x^3(20 - x)^2 \Rightarrow z = 400x^3 + x^5 - 40x^4$
$\therefore$ $\dfrac{dz}{dx} = 1200x^2 + 5x^4 - 160x^3$
For maximum or minimum,
$\dfrac{dz}{dx} = 0$
$\Rightarrow 1200x^2 + 5x^4 - 160x^3 = 0$
$\Rightarrow x = 12, 20$
$\dfrac{d^2z}{dx^2} = 2400x + 20x^3 - 480x^2$
$\therefore$ $\left(\dfrac{d^2z}{dx^2}\right)_{x=12} = -5760 < 0$
$\therefore$ z is maximum at $x = 12$.
From (i), $y = 20 - 12 = 8$
$\therefore$ the parts of 20 are 12 and 8.

120. Let $y = \sin^p x . \cos^q x$
$\therefore$ $\dfrac{dy}{dx} = p\sin^{p-1} x . \cos x . \cos^q x + q\cos^{q-1} x . (-\sin x)\sin^p x$
$\therefore$ $\dfrac{dy}{dx} = p\sin^{p-1} x . \cos^{q+1} x - q\cos^{q-1} x . \sin^{p+1} x$
For maximum or minimum,
$\dfrac{dy}{dx} = 0$
$\therefore$ $\tan^2 x = \dfrac{p}{q} \Rightarrow \tan x = \pm\sqrt{\dfrac{p}{q}}$
$\therefore$ Point of maxima $x = \tan^{-1}\sqrt{\dfrac{p}{q}}$

121. $4x + 2\pi r = a$

$A = x^2 + \pi r^2 = \dfrac{1}{16}(a - 2\pi r)^2 + \pi r^2$

$\therefore \quad \dfrac{dA}{dr} = 0$ gives $r = \dfrac{a}{2(\pi + 4)}$, thus $\dfrac{d^2A}{dr^2} > 0$

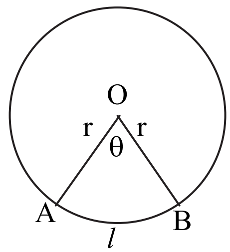
and hence minimum,

$\therefore \quad 4x = a - 2\pi r = a - \dfrac{a\pi}{\pi + 4} = \dfrac{4a}{\pi + 4}$

$\therefore \quad x = \dfrac{a}{\pi + 4}$

$\therefore \quad A = x^2 + \pi r^2 = \dfrac{a^2}{4(\pi + 4)}$

122.



Let OAB be a given sector of a circle of a radius r cm such that arc AB = l cm, and $\angle AOB = \theta$ radians.

Then

$2r + l = 20$ ....(i)

$\dfrac{l}{r} = \theta$ ....(ii)

$A = \dfrac{1}{2}r^2\theta$ ....(iii)

From (i), (ii) ,(iii), we get

$A = \dfrac{1}{2}r^2 \times \dfrac{l}{r} = \dfrac{1}{2}r\,l = \dfrac{1}{2}r(20 - 2r)$

$\Rightarrow A = 10\,r - r^2$ ....(iv)

Now, $\dfrac{dA}{dr} = 10 - 2r = 0 \Rightarrow r = 5$

$\therefore \quad \dfrac{d^2A}{dr^2} = -2 < 0$

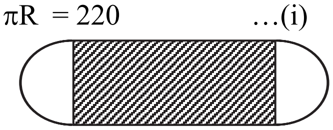
$\therefore$ A is maximum at r = 5

Hence the maximum area

$= 10 \times 5 - 25 = 25\ cm^2$ .... [From (iv)]

123. $2l + 2\pi R = 440$

$\therefore \quad l + \pi R = 220$ ...(i)



Now $f(x) = l\,(2R) = (220 - \pi R)(2R)$

$\therefore \quad f(x) = 440R - 2\pi R^2$

$\therefore \quad f'(x) = 440 - 4\pi R = 0$

$\therefore \quad 0 = 110 - \pi R$

$\therefore \quad 110 = \dfrac{22}{7}R$

$\therefore \quad R = 35 \qquad \therefore \quad 2R = 70$

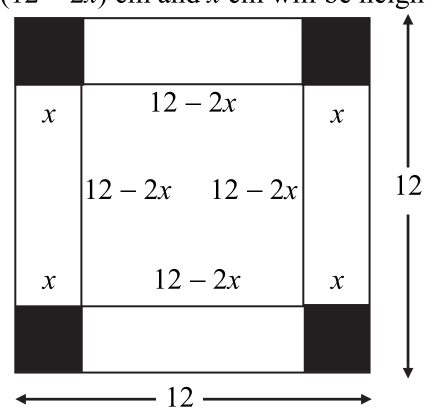
From (i),

$l + \dfrac{22}{7} \times 35 = 220$

$\therefore \quad l + 110 = 220$

$\therefore \quad l = 110$

124. Let the length of side of each square cut out be x sq cm. Then, each side of base of the box is $(12 - 2x)$ cm and x cm will be height of box.



V = Volume of box

$= (12 - 2x)^2 x = 4(36 + x^2 - 12x)x$

$= 4(x^3 - 12x^2 + 36x)$

$\Rightarrow \dfrac{dV}{dx} = 4(3x^2 - 24x + 36)$

$= 12\,(x^2 - 8x + 12)$

and $\dfrac{d^2V}{dx^2} = 4(6x - 24)$

Now, $\dfrac{dV}{dx} = 0 \Rightarrow x^2 - 8x + 12 = 0$

$\Rightarrow (x - 2)(x - 6) = 0 \Rightarrow x = 2$ or $x = 6$

But $x < 6$

$\therefore \quad x = 2$

For $x = 2$, $\dfrac{d^2V}{dx^2} = 4\,(12 - 24) = -48 < 0$

$\therefore$ Volume is maximum when each square of 2 cm length is cut out from each corner.

125. Given equation is $10s = 10ut - 49t^2$

$\Rightarrow s = ut - 4.9t^2$

$\Rightarrow \dfrac{ds}{dt} = u - 9.8t = v$

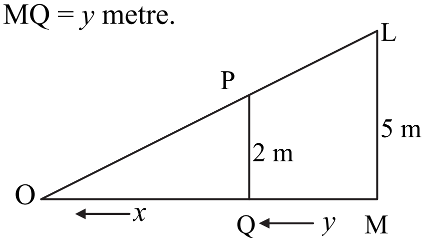
When stone reaches the maximum height, then $v = 0$

$\Rightarrow u - 9.8t = 0 \Rightarrow u = 9.8t$

But time t = 5 sec

So the value of $u = 9.8 \times 5 = 49.0$ m/sec

126. Let L be the lamp and PQ be the man and OQ = x metre be his shadow and let MQ = y metre.



$\frac{dy}{dt}$ = speed of the man = 3 m/s (given)

Since, Δ OPQ and Δ OLM are similar.

∴ $\frac{OM}{OQ} = \frac{LM}{PQ} \Rightarrow \frac{x+y}{x} = \frac{5}{2}$

$\Rightarrow y = \frac{3}{2}x$

∴ $\frac{dy}{dt} = \frac{3}{2}\cdot\frac{dx}{dt}$

$\Rightarrow 3 = \frac{3}{2}\cdot\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = 2$m/s.

127. Let A, P and x be the area, perimeter and length of the side of the square respectively at time t seconds. Then, A = x2 and P = 4x

∴ $P = 4\sqrt{A}$

∴ $\frac{dP}{dt} = 4.\frac{1}{2\sqrt{A}}.\frac{dA}{dt}$

$= \frac{2}{x}.\frac{dA}{dt} = \frac{2}{16}.2 = \frac{1}{4}\text{cm / sec}.$

128. Let

$f(A) = \cos A \cos B = \cos A \cos\left(\frac{\pi}{2} - A\right)$

$= \cos A \sin A$

∴ $f'(A) = \cos^2 A - \sin^2 A = \cos 2A$

For maximum or minimum,

$f'(A) = 0 \Rightarrow \cos 2A = 0$

$\Rightarrow 2A = \frac{\pi}{2} \Rightarrow A = \frac{\pi}{4}$

Now, $f''(A) = -2 \sin 2A$

$= -2 \sin \frac{\pi}{2} = -2 < 0$

∴ f(A) is maximum at $A = \frac{\pi}{4}$.

∴ Maximum value $= \cos\frac{\pi}{4} \sin\frac{\pi}{4} = \frac{1}{2}$

129. Since, f(x) satisfies all the conditions of Rolle's theorem.

∴ f(3) = f(5) = 0

$\Rightarrow x = 3$ and $x = 5$ are the roots of f(x).

$\Rightarrow f(x) = (x-3)(x-5) = x^2 - 8x + 15$

∴ $\int_3^5 f(x)\,dx = \int_3^5 (x^2 - 8x + 15)\,dx$

$= \left[\frac{x^3}{3} - 4x^2 + 15x\right]_3^5$

$= \frac{1}{3}(125-27) - 4(25-9) + 15(5-3)$

$= -\frac{4}{3}$

Competitive Thinking

1. $y = x^2 - \frac{1}{x^2}$

∴ $\frac{dy}{dx} = 2x + \frac{2}{x^3}$

∴ $\left(\frac{dy}{dx}\right)_{(-1,0)} = 2(-1) + \frac{2}{(-1)^3} = -4$

∴ Slope of normal at (–1, 0) $= -\frac{1}{\left(\frac{dy}{dx}\right)_{(-1,0)}} = \frac{1}{4}$

2. For the point (2, –1) on the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$, we have

$t^2 + 3t - 8 = 2$ and $2t^2 - 2t - 5 = -1$

$\Rightarrow (t+5)(t-2) = 0$ and $(t-2)(t+1) = 0$

$\Rightarrow t = 2$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t-2}{2t+3}$

∴ $\left(\frac{dy}{dx}\right)_{(t=2)} = \frac{4(2)-2}{2(2)+3} = \frac{6}{7}$

3. Slope of the normal $= \frac{-1}{\frac{dy}{dx}}$

$\Rightarrow \tan\frac{3\pi}{4} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(3,4)}}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(3,4)} = 1 \Rightarrow f'(3) = 1$

4. $y = ax^3 + bx + 4$

$\therefore \quad \frac{dy}{dx} = 3ax^2 + b$

Slope of tangent at (2, 14) = $\left(\frac{dy}{dx}\right)_{(2,\,14)}$

$\Rightarrow 21 = 3a(2)^2 + b$

$\Rightarrow 21 = 12a + b$ ...(i)

$y = ax^3 + bx + 4$

$\therefore \quad 14 = a\,(8) + b\,(2) + 4$

$\Rightarrow 8a + 2b = 10$ ...(ii)

On solving (i) and (ii), we get

$a = 2, b = –3$

5. $x = t^2 – 1, y = t^2 – t$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-1}{2t}$

Since, the tangent is perpendicular to X-axis.

$\therefore \quad \frac{dx}{dy} = 0 \Rightarrow \frac{2t}{2t-1} = 0 \Rightarrow t = 0$

6. $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$

According to the given condition,

$3x^2 = y$

$\Rightarrow 3x^2 = x^3$ ....[$\because y = x^3$]

$\Rightarrow x = 0, 3$

Thus, the two points are (0, 0) and (3, 27).

7. $y = x^2 – 3x + 2 \Rightarrow \frac{dy}{dx} = 2x – 3$

Slope of the given line = 1

Since, the tangent is perpendicular to the given line.

$\therefore \quad (2x – 3)\,(1) = –1$

$\Rightarrow x = 1$

At $x = 1$, $y = 0$

$\therefore$ the required point is (1, 0).

8. Given equation of curve is $y = \sqrt{x-1}$

Slope of tangent to the curve is

$\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}}$

Slope of line $2x + y – 5 = 0$ is –2

Since the tangent is perpendicular to the given line,

$\left(\frac{1}{2\sqrt{x-1}}\right)(-2) = -1$

$\Rightarrow \sqrt{x-1} = 1 \qquad \Rightarrow x = 2$

$y = \sqrt{x-1} = \sqrt{2-1} = 1$

$\therefore \quad (x, y) = (2, 1)$

9. $y^2 = px^3 + q$ .....(i)

Differentiating both sides w.r.t. x, we get

$2y.\frac{dy}{dx} = 3px^2$

$\Rightarrow \frac{dy}{dx} = \frac{3p}{2}\left(\frac{x^2}{y}\right)$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{3p}{2} \times \frac{4}{3} = 2p$

Since the line touches the curve, their slopes are equal.

$\therefore \quad 2p = 4 \Rightarrow p = 2$

Since, (2,3) lies on $y^2 = px^3 + q$.

$\therefore \quad 9 = 2 \times 8 + q \Rightarrow q = -7$

10. $y^2 = ax^3 + b$ .....(i)

Differentiating both sides w.r.t. x, we get

$2y.\frac{dy}{dx} = 3ax^2$

$\Rightarrow \frac{dy}{dx} = \frac{3a}{2}\left(\frac{x^2}{y}\right)$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{3a}{2} \times \frac{4}{3} = 2a$

Since, the line touches the curve, their slopes are equal.

$\therefore \quad 2a = 4 \Rightarrow a = 2$

Since, (2,3) lies on $y^2 = ax^3 + b$.

$\therefore \quad 9 = 2 \times 8 + b \Rightarrow b = -7$

Now, $7a + 2b = 7(2) + 2(–7) = 0$

11. $y = \frac{1}{x}$ ....(i)

$\therefore \quad \frac{dy}{dx} = \frac{-1}{x^2}$

$\therefore$ Slope of tangent to the curve = $\frac{-1}{x^2}$

Slope of $y = –4x + b$ is –4.

$\therefore \quad \frac{-1}{x^2} = -4 \qquad \Rightarrow x = \pm\frac{1}{2}$

From (i),

$y = \pm 2$

Putting the values of x and y in $y = –4x + b$, we get

$b = \pm 4$

12. $y^2 = 2(x - 3)$ ....(i)
Differentiating w.r.t. x, we get
$2y\frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} = \frac{1}{y}$
∴ Slope of the normal = $\frac{-1}{\frac{dy}{dx}} = -y$
Slope of the given line = 2.
Since, the normal is parallel to the given line.
∴ $y = -2$
From (i), $x = 5$
∴ the required point is (5, –2).

13. Given equation of curve is
$x^2 - 4y^2 = 1$ ...(i)
Slope of tangent to the curve is
$\frac{dy}{dx} = \frac{x}{4y}$
Slope of line is $x = 2y$ is $\frac{1}{2}$
Since, the tangent is parallel to the given line,
$\frac{x}{4y} = \frac{1}{2}$
∴ $x = 2y$
Substituting $x = 2y$ in equation (i), we get
$(2y)^2 - 4y^2 = 1$
∴ tangent is parallel to curve at zero point.

14. $x = a(1 + \cos\theta)$ and $y = a\sin\theta$
∴ $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$
∴ $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\cot\theta$
∴ slope of the normal = $-\frac{1}{\frac{dy}{dx}} = \frac{-1}{-\cot\theta} = \tan\theta$
∴ equation of the normal at θ is
$y - a\sin\theta = \tan\theta[x - a(1 + \cos\theta)]$
Clearly, this line passes through (a, 0).

15. $y^2 = 12x$ ....(i)
Differentiating w.r.t. x, we get
$2y\frac{dy}{dx} = 12 \Rightarrow \frac{dy}{dx} = \frac{6}{y}$
∴ slope of the normal $= -\frac{1}{\frac{dy}{dx}} = -\frac{y}{6}$
Slope of the line $x + y = k$ is –1.
∴ $-\frac{y}{6} = -1 \Rightarrow y = 6$
From (i), $x = 3$
Putting the values of x and y in $x + y = k$, we get k = 9

16. Slope of given line = $-\frac{a}{b}$
$y = \frac{4}{x} \Rightarrow \frac{dy}{dx} = -\frac{4}{x^2}$
$\Rightarrow -\frac{a}{b} = -\frac{4}{x^2}$
$\Rightarrow \frac{a}{b} = \frac{4}{x^2} > 0$
$\Rightarrow a < 0, b < 0$

17. $y^2 = 4ax$
∴ $2y\frac{dy}{dx} = 4a$
$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$
$\Rightarrow \left(\frac{dy}{dx}\right)_{(at^2,\, 2at)} = \frac{2a}{2at} = \frac{1}{t}$
∴ Slope of tangent $(m_1) = \frac{1}{t}$
$x^2 - y^2 = a^2$
$\Rightarrow 2x - 2y\frac{dy}{dx} = 0$
$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$
$\Rightarrow \left(\frac{dy}{dx}\right)_{(a\sec\theta,\, a\tan\theta)} = \frac{a\sec\theta}{a\tan\theta}$
$= \operatorname{cosec}\theta$
∴ Slope of normal $(m_2) = \operatorname{cosec}\theta$
Now, $m_1 \cdot m_2 = -1$
$\Rightarrow \left(\frac{1}{t}\right)(\operatorname{cosec}\theta) = -1$
$\Rightarrow t = -\operatorname{cosec}\theta$

18. $9y^2 = x^3$ ....(i)
Differentiating w.r.t. x, get
$18y\frac{dy}{dx} = 3x^2$
$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$

$\therefore$ slope of the normal $= -\dfrac{6y}{x^2}$

Since, the normal to the given curve makes equal intercepts with the axis.

$\therefore$ $-\dfrac{6y}{x^2} = \pm 1$

$\Rightarrow y = -\dfrac{x^2}{6}$ or $\dfrac{x^2}{6}$

Putting these values in (i), we get

$9\left(\dfrac{x^4}{36}\right) = x^3 \Rightarrow x = 0$ or $x = 4$

$\therefore$ $y = 0$ or $y = -\dfrac{16}{6}$ or $\dfrac{16}{6} = -\dfrac{8}{3}$ or $\dfrac{8}{3}$

$\therefore$ the required points are $\left(4, \dfrac{8}{3}\right)$ or $\left(4, -\dfrac{8}{3}\right)$.

19. $y = \dfrac{2}{3}x^3 + \dfrac{1}{2}x^2$ ....(i)

$\therefore$ $\dfrac{dy}{dx} = 2x^2 + x$

Since, the tangent makes equal angles with the axis.

$\therefore$ $\dfrac{dy}{dx} = \pm 1$

$\Rightarrow 2x^2 + x = \pm 1$

$\Rightarrow 2x^2 + x = 1$ (taking +ve sign)

$\Rightarrow 2x^2 + x - 1 = 0$

$\Rightarrow (2x - 1)(x + 1) = 0$

$\Rightarrow x = \dfrac{1}{2}, -1$

From (i),

when $x = \dfrac{1}{2}$, $y = \dfrac{2}{3}\cdot\dfrac{1}{8} + \dfrac{1}{2}\cdot\dfrac{1}{4} = \dfrac{5}{24}$

and when $x = -1$, $y = \dfrac{2}{3}(-1) + \dfrac{1}{2}\cdot 1 = -\dfrac{1}{6}$

$\therefore$ the required points are $\left(\dfrac{1}{2}, \dfrac{5}{24}\right)$ and $\left(-1, -\dfrac{1}{6}\right)$.

20. At $x = 4$,

$4^2 = 8y \Rightarrow y = 2$

Now, $x^2 = 8y$

Differentiating w.r.t. x, we get

$2x = 8\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{4}$

$\therefore$ $\left(\dfrac{dy}{dx}\right)_{(4,2)} = 1$

$\therefore$ equation of the normal at (4, 2) is

$y - 2 = -1(x - 4) \Rightarrow x + y = 6$

21. At $t = 1$, $x = (1)^2 = 1$ and $y = 2(1) = 2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{2t} = \frac{1}{t}$$

$\therefore$ $\left(\dfrac{dy}{dx}\right)_{t=1} = 1$

$\therefore$ Equation of the normal at (1, 2) is

$y - 2 = -1(x - 1) \Rightarrow x + y - 3 = 0$

22. Centre of circle is (1, –2) and point A(2,1) lie on circle.

$\therefore$ Equation of normal is $y + 2 = \dfrac{1+2}{2-1}(x - 1)$

$\Rightarrow y + 2 = 3(x - 1) \Rightarrow y = 3x - 5$

23. $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2$

Differentiating w.r.t x, we get

$n\left(\dfrac{x}{a}\right)^{n-1}\left(\dfrac{1}{a}\right) + n\left(\dfrac{y}{b}\right)^{n-1}\left(\dfrac{1}{b}\right)\left(\dfrac{dy}{dx}\right) = 0$

$\Rightarrow \dfrac{n}{b}\left(\dfrac{y}{b}\right)^{n-1}\dfrac{dy}{dx} = \dfrac{-n}{a}\left(\dfrac{x}{a}\right)^{n-1}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-b}{a}\left(\dfrac{x}{a}\right)^{n-1}\left(\dfrac{b}{y}\right)^{n-1}$

Slope of tangent at (a, b) $= \left(\dfrac{dy}{dx}\right)_{(a,\,b)}$

$= \dfrac{-b}{a}\left(\dfrac{a}{a}\right)^{n-1}\left(\dfrac{b}{b}\right)^{n-1}$

$= \dfrac{-b}{a}$

Equation of tangent is $y - b = \dfrac{-b}{a}(x - a)$

$\Rightarrow ay - ab = -bx + ab$

$\Rightarrow ay + bx = 2ab$

$\Rightarrow \dfrac{x}{a} + \dfrac{y}{b} = 2$

24. At $\theta = \dfrac{\pi}{4}$,

$x = 2\cos^3\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$ and $y = 3\sin^3\dfrac{\pi}{4} = \dfrac{3}{2\sqrt{2}}$

$x = 2\cos^3\theta$ and $y = 3\sin^3\theta$

$\therefore$ $\dfrac{dx}{d\theta} = -6\cos^2\theta\sin\theta$ and $\dfrac{dy}{d\theta} = 9\sin^2\theta\cos\theta$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\dfrac{3}{2}\tan\theta$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{\left(\theta=\frac{\pi}{4}\right)} = -\dfrac{3}{2}$

$\therefore$ equation of the tangent at $\left(\dfrac{1}{\sqrt{2}}, \dfrac{3}{2\sqrt{2}}\right)$ is

$y - \dfrac{3}{2\sqrt{2}} = -\dfrac{3}{2}\left(x - \dfrac{1}{\sqrt{2}}\right)$

$\Rightarrow 3x + 2y = 3\sqrt{2}$

25. At $x = 0$, $y = e^0 + 0 = 1$

$y = e^{2x} + x^2$

$\therefore \quad \dfrac{dy}{dx} = 2e^{2x} + 2x$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(0,1)} = 2$

Also, $-\left(\dfrac{dx}{dy}\right)_{(0,1)} = -\dfrac{1}{2}$

Equation of normal at (0, 1) is

$(y - 1) = \dfrac{-1}{2}(x - 0)$

$\Rightarrow 2y - 2 = -x \Rightarrow x + 2y - 2 = 0$

$\therefore$ distance between origin and normal

$= \left|\dfrac{0+0-2}{\sqrt{1+4}}\right| = \dfrac{2}{\sqrt{5}}$

26. $x^2 + y^2 - 13 = 0$

$\therefore \quad 2x + 2y\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-x}{y}$

$\therefore$ Slope of tangent at (2, 3) = $\left(\dfrac{dy}{dx}\right)_{(2,\,3)}$

$\Rightarrow m = \dfrac{-2}{3}$

Given equation of circle is $x^2 + y^2 = 13$

$\therefore$ Centre of circle 0 = (0, 0), radius = $\sqrt{13}$ units

Given point M $\left(m, \dfrac{-1}{m}\right) = \left(\dfrac{-2}{3}, \dfrac{3}{2}\right)$

$= (-0.67, 1.5)$

OM < radius

$\therefore$ The point lies inside the circle

27. $y = a(\sin\theta - \theta\cos\theta)$, $x = a(\cos\theta + \theta\sin\theta)$

$\therefore \quad \dfrac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\,\theta\sin\theta$

and $\dfrac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$

$= a\,\theta\cos\theta$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$

$\therefore$ Slope of the normal $= \dfrac{-1}{\tan\theta} = -\cot\theta$

$\therefore$ Equation of the normal is

$y - a\sin\theta + a\,\theta\cos\theta$

$= -\dfrac{\cos\theta}{\sin\theta}(x - a\cos\theta - a\,\theta\sin\theta)$

$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta$

$= -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$

$\Rightarrow x\cos\theta + y\sin\theta = a(\sin^2\theta + \cos^2\theta)$

$\Rightarrow x\cos\theta + y\sin\theta = a$

$\therefore$ Distance from origin $= \left|\dfrac{-a}{\sqrt{\sin^2\theta + \cos^2\theta}}\right|$

$= a =$ constant

28. $y = x^2 - x + 1$

$\Rightarrow \dfrac{dy}{dx} = 2x - 1$

$\left(\dfrac{dy}{dx}\right)_{(x=0)} = -1$, $\left(\dfrac{dy}{dx}\right)_{(x=-1)} = -3$, $\left(\dfrac{dy}{dx}\right)_{\left(x=\frac{5}{2}\right)} = 4$

equation of normal at (0, 1) and having slope 1 is $y - 1 = x - 0$

$\Rightarrow x - y + 1 = 0 \quad$ ...(i)

Equation of normal at (–1, 3) and having slope $\dfrac{1}{3}$ is

$y - 3 = \dfrac{1}{3}(x + 1)$

$\Rightarrow x - 3y + 10 = 0 \quad$ ...(ii)

Equation of normal at $\left(\dfrac{5}{2}, \dfrac{19}{4}\right)$ and having slope $\dfrac{-1}{4}$ is

$y - \dfrac{19}{4} = \dfrac{-1}{4}\left(x - \dfrac{5}{2}\right) \Rightarrow 4y - 19 = -x + \dfrac{5}{2}$

$\Rightarrow 2x + 8y - 43 = 0 \quad$ ...(iii)

Equation (i), (ii) and (iii) are passes through point $\left(\dfrac{7}{2}, \dfrac{9}{2}\right)$.

$\therefore$ they are concurrent

29. Given, $x^2 + 2xy - 3y^2 = 0$ ....(i)

Differentiating w.r.t.x, we get

$$2x + 2\left(x\frac{dy}{dx} + y\right) - 6y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{3y-x} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = 1$$

$\therefore$ equation of the normal at (1, 1) is

$y - 1 = -1(x - 1)$

$\Rightarrow y = 2 - x$

Putting $y = 2 - x$ in (i), we get

$x^2 + 2x(2 - x) - 3(2 - x)^2 = 0$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow x = 1, 3$

$\therefore$ the points of intersection are (1,1) and (3,–1).

$\therefore$ the normal at (1, 1) meets the curve again at (3, –1) which lies in the fourth quadrant.

30. $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$$\therefore \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$\therefore$ Equation of the tangent at (x, y) is

$$Y - y = -\frac{\sqrt{y}}{\sqrt{x}}(X - x)$$

$$\therefore \quad X\sqrt{y} + Y\sqrt{x} = \sqrt{xy}\left(\sqrt{x} + \sqrt{y}\right)$$

$$\therefore \quad X\sqrt{y} + Y\sqrt{x} = \sqrt{xy} \,.\, \sqrt{a}$$

$$\therefore \quad \frac{X}{\sqrt{a}\sqrt{x}} + \frac{Y}{\sqrt{a}\sqrt{y}} = 1$$

Clearly its intercepts on the axes are $\sqrt{a}\sqrt{x}$ and $\sqrt{a}\sqrt{y}$.

Sum of the intercepts

$$= \sqrt{a}\left(\sqrt{x} + \sqrt{y}\right) = \sqrt{a}.\sqrt{a} = a$$

31. Let the coordinates of P be (x_1, y_1).

$$\therefore \quad x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = a^{\frac{2}{3}} \quad ....(i)$$

Now, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Differentiating w.r.t. x, we get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{y_1^{\frac{1}{3}}}{x_1^{\frac{1}{3}}}$$

$\therefore$ equation of the tangent at (x_1, y_1) is

$$y - y_1 = -\frac{y_1^{\frac{1}{3}}}{x_1^{\frac{1}{3}}}(x - x_1)$$

$$\Rightarrow \frac{y - y_1}{y_1^{\frac{1}{3}}} = -\frac{x - x_1}{x_1^{\frac{1}{3}}}$$

$$\Rightarrow x x_1^{-\frac{1}{3}} + y y_1^{-\frac{1}{3}} = x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}}$$

$$\Rightarrow x x_1^{-\frac{1}{3}} + y y_1^{-\frac{1}{3}} = a^{\frac{2}{3}} \quad ....[\text{From (i)}]$$

This tangent meets the coordinate axes at $A\left(a^{\frac{2}{3}} x_1^{\frac{1}{3}}, 0\right)$ and $B\left(0, a^{\frac{2}{3}} y_1^{\frac{1}{3}}\right)$.

$$\therefore \quad AB = \sqrt{a^{\frac{4}{3}} x_1^{\frac{2}{3}} + a^{\frac{4}{3}} y_1^{\frac{2}{3}}} = \sqrt{a^{\frac{4}{3}}\left(x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}}\right)}$$

$$= \sqrt{a^{\frac{4}{3}} \cdot a^{\frac{2}{3}}} \quad ....[\text{From (i)}]$$

$= a$

32. $y = x^2 - 5x + 6$

$$\therefore \quad \frac{dy}{dx} = 2x - 5$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(2,0)} = 2(2) - 5 = -1 = m_1 \text{ (say)}$$

and $\left(\frac{dy}{dx}\right)_{(3,0)} = 2(3) - 5 = 1 = m_2$ (say)

Here, $m_1 m_2 = -1$

$\therefore$ the required angle is $\frac{\pi}{2}$.

33. If $\sin x = \cos x$, then $x = \frac{\pi}{4}$

Now, $y = \sin x$

$$\therefore \quad \frac{dy}{dx} = \cos x$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\left(x = \frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}} = m_1 \text{ (say)}$$

Also, $y = \cos x$

$$\therefore \quad \frac{dy}{dx} = -\sin x$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\left(x = \frac{\pi}{4}\right)} = -\frac{1}{\sqrt{2}} = m_2 \text{ (say)}$$

$\therefore$ angle between the curves is

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 + \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)}\right|$$

$\Rightarrow \tan\theta = 2\sqrt{2}$

$\Rightarrow \theta = \tan^{-1}\left(2\sqrt{2}\right)$

34. $y = e^{x^2}$ ....(i)

$y = e^{x^2}\sin x$ ....(ii)

From (i) and (ii), we get

$e^{x^2} = e^{x^2}\sin x$

$\therefore$ $\sin x = 1 \Rightarrow x = \frac{\pi}{2}$

Slope of tangent to (i) at $x = \frac{\pi}{2}$ is given by

$$\left(\frac{dy}{dx}\right)_{x=\pi/2} = \left[2xe^{x^2}\right]_{x=\frac{\pi}{2}} = \pi e^{\frac{\pi^2}{4}}$$

Slope of tangent to (ii) at $x = \frac{\pi}{2}$ is given by

$$\left(\frac{dy}{dx}\right)_{x=\pi/2} = \left[2xe^{x^2}\sin x + e^{x^2}\cos x\right]_{x=\frac{\pi}{2}} = \pi e^{\frac{\pi^2}{4}}$$

Since both tangents have equal slopes, the angle between them is zero.

35. Let the given curves intersect each other at $P(x_1, y_1)$.

$y^2 = 6x$

Differentiating w.r.t. x, we get

$2y\frac{dy}{dx} = 6 \Rightarrow \left(\frac{dy}{dx}\right)_P = \frac{3}{y_1}$

$9x^2 + by^2 = 16$

Differentiating w.r.t. x, we get

$18x + 2by\frac{dy}{dx} = 0$

$\Rightarrow \left(\frac{dy}{dx}\right)_P = -\frac{9x_1}{by_1}$

Since, the given curves intersect each other at right angles.

$\Rightarrow \left(\frac{3}{y_1}\right)\left(\frac{-9x_1}{by_1}\right) = -1$

$\Rightarrow \frac{27x_1}{by_1^2} = 1$

$\Rightarrow b = \frac{9}{2}$ $\quad\ldots\left[y_1^2 = 6x_1\right]$

36. Acceleration, $\frac{dv}{dt} = 2t$, then acceleration after 3 second $= 2 \times 3 = 6$ cm / sec^2.

37. Motion of a particle $s = 15t - 2t^2$

$\therefore$ velocity $= \frac{ds}{dt} = 15 - 4t$

$\Rightarrow \left(\frac{ds}{dt}\right)_{t=0} = 15$ and $\left(\frac{ds}{dt}\right)_{t=3} = 3$

$\therefore$ average velocity $= \frac{15+3}{2} = 9$ units

38. Velocity, $v^2 = 2 - 3x$

Differentiating both sides w.r.t.t, we get

$2v\frac{dv}{dt} = -3\frac{dx}{dt}$

$\Rightarrow 2v\frac{dv}{dt} = -3v$

$\Rightarrow \frac{dv}{dt} = -\frac{3}{2}$

Hence, the acceleration is uniform.

39. $x = At^2 + Bt + C$

$\therefore$ $v = 2At + B \Rightarrow v^2 = 4A^2t^2 + 4ABt + B^2$

and $4Ax = 4A^2t^2 + 4ABt + 4AC$

$\Rightarrow v^2 - 4Ax = B^2 - 4AC$

$\Rightarrow 4Ax - v^2 = 4AC - B^2$

40. $t = \frac{v^2}{2} \Rightarrow v^2 = 2t$

Differentiating both sides w.r.t.t., we get

$2v\frac{dv}{dt} = 2$

$\Rightarrow \frac{dv}{dt} = \frac{1}{v} = f$

$\Rightarrow \frac{df}{dt} = -\frac{1}{v^2}\cdot\frac{dv}{dt} = -\frac{1}{v^2} \times \frac{1}{v}$

$\Rightarrow -\frac{df}{dt} = \frac{1}{v^3} = f^3$

41. $\frac{d^2t}{dx^2} = \frac{d}{dx}\left(\frac{dt}{dx}\right) = \frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\cdot\frac{dv}{dx}$

Since, $v\frac{dv}{dx} = f \Rightarrow \frac{dv}{dx} = \frac{f}{v}$

$\therefore$ $\frac{d^2t}{dx^2} = -\frac{1}{v^2}\cdot\frac{f}{v} \Rightarrow -v^3\frac{d^2t}{dx^2} = f$

42. $s = \frac{1}{2}vt \Rightarrow 2s = vt \Rightarrow 2\frac{ds}{dt} = v + t.\frac{dv}{dt}$

$\Rightarrow 2\frac{d^2s}{dt^2} = \frac{dv}{dt} + t.\frac{d^2v}{dt^2} + \frac{dv}{dt}$

But $\frac{dv}{dt}$ = acceleration (a)

$\Rightarrow 2a = a + t.\frac{da}{dt} + a \Rightarrow \frac{da}{dt} = 0$ or $t = 0$

But t = 0 is impossible

$\therefore$ $\frac{da}{dt} = 0$ i.e., a is constant.

44. $s = 6 + 48t - t^3$

$\therefore$ $v = \frac{ds}{dt} = 0 + 48 - 3t^2$

When direction of motion reverses, v = 0

$\Rightarrow 48 - 3t^2 = 0 \Rightarrow t = -4, 4$

$\therefore$ $(s)_4 = 6 + 192 - 64 = 134$

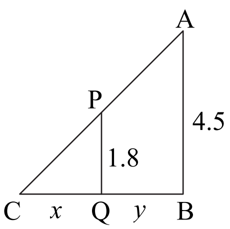
45. $a + bv^2 = x^2$

Differentiating both sides w.r.t.t, we get

$0 + b\left(2v.\frac{dv}{dt}\right) = 2x.\frac{dx}{dt}$

$\Rightarrow v.b\frac{dv}{dt} = x.\frac{dx}{dt} \Rightarrow \frac{dv}{dt} = \frac{x}{b}$ ....$\left[\because \frac{dx}{dt} = v\right]$

46. $\frac{dy}{dt} = 1.2.$



From the figure,

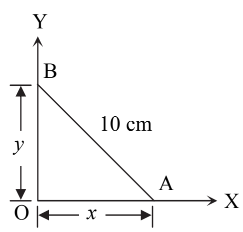
$x = \frac{2}{3}y \Rightarrow \frac{dx}{dt} = \frac{2}{3}.\frac{dy}{dt}$

$\therefore$ Required rate of length of shadow

$= \frac{dx}{dt} = 0.8$ m/s

47. From the figure, $x^2 + y^2 = 100$ .....(i)

$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ .....(ii)



From (i) and (ii),

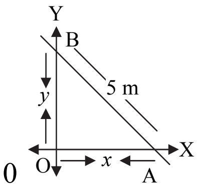
$\frac{dy}{dt} = -\frac{16}{6} = -\frac{8}{3}$ cm / sec

The rate at which the end B is moving is $\frac{8}{3}$ cm / sec.

48. According to the figure, $x^2 + y^2 = 25$ ….(i)

Differentiate (i) w.r.t. t, we get

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ ….(ii)



Here $x = 4$ and $\frac{dx}{dt} = 1.5$

From (i), $4^2 + y^2 = 25 \Rightarrow y = 3$

From (ii), $2(4)(1.5) + 2(3)\frac{dy}{dt} = 0$

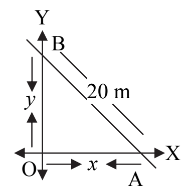
$\Rightarrow \frac{dy}{dt} = -2$ m/sec

Hence, length of the highest point decreases at the rate of 2 m/sec.

49. According to the figure, $x^2 + y^2 = 400$ ….(i)

Differentiate (i) w.r.t. t, we get

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ ….(ii)



Here $x = 12$ and $\frac{dy}{dt} = 2$

From (i), $12^2 + y^2 = 400$

$\Rightarrow y = 16$

From (ii), $2(12)\frac{dx}{dt} + 2(16)(2) = 0$

$\Rightarrow \frac{dx}{dt} = -\frac{8}{3}$

50. Surface area, $S = 4\pi r^2$ and $\frac{dr}{dt} = 2$

$\therefore$ $\frac{dS}{dt} = 4\pi \times 2r\frac{dr}{dt} = 8\pi r \times 2 = 16\pi r$

$\Rightarrow \frac{dS}{dt} \propto r$

51. Given the rate of increasing the radius

$= \frac{dr}{dt} = 3.5$cm/sec and r = 10cm

Area of circle = $A = \pi r^2$

$\therefore$ $\frac{dA}{dt} = 2\pi r.\frac{dr}{dt}$

$\Rightarrow \frac{dA}{dt} = 2\pi \times 10 \times 3.5 \Rightarrow \frac{dA}{dt} = 220$ cm^2/sec

52. If x is the length of each side of an equilateral triangle and A is its area, then

$A = \frac{\sqrt{3}}{4}x^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{4}.2x\frac{dx}{dt}$

Here, $x = 10$ cm and $\frac{dx}{dt} = 2$ cm / sec

$\therefore$ $A = 10\sqrt{3}$ sq. unit/sec

53. $A_1 = x^2$, and $A_2 = y^2$

$\Rightarrow \dfrac{dA_1}{dt} = 2x\dfrac{dx}{dt}$, and $\dfrac{dA_2}{dt} = 2y\dfrac{dy}{dt}$

$\therefore \quad \dfrac{dA_2}{dA_1} = \dfrac{\dfrac{dA_2}{dt}}{\dfrac{dA_1}{dt}} = \dfrac{2y\dfrac{dy}{dt}}{2x\dfrac{dx}{dt}} = \dfrac{y}{x}\left(\dfrac{dy}{dx}\right)$

Given, $y = x + x^2$

$\therefore \quad \dfrac{dy}{dx} = 1 + 2x$

$\therefore \quad \dfrac{dA_2}{dA_1} = \dfrac{y}{x}(1 + 2x)$

$= \dfrac{x + x^2}{x}(1 + 2x)$

$= (1 + x)(1 + 2x) = 2x^2 + 3x + 1$

54. $h = 6$ m, $r = 4$ m $= \dfrac{2}{3}h$

$V = \dfrac{1}{3}\pi r^2 h$

$\Rightarrow V = \dfrac{1}{3} \times \dfrac{4}{9} \times \pi h^3$

$\Rightarrow \dfrac{dV}{dt} = \dfrac{4}{9}\pi h^2\dfrac{dh}{dt}$

But $\dfrac{dV}{dt} = 3$ m^3/min and h = 3 m

$\therefore \quad 3 = \dfrac{4}{9}\pi \times 9 \times \dfrac{dh}{dt}$

$\Rightarrow \dfrac{dh}{dt} = \dfrac{3}{4\pi}$ m/min

55. $V = \dfrac{4}{3}\pi r^3$

$\therefore \quad \dfrac{dV}{dt} = 4\pi r^2 . \dfrac{dr}{dt} \Rightarrow \dfrac{dr}{dt} = \dfrac{1}{4\pi r^2} . \dfrac{dV}{dt}$

$\Rightarrow \dfrac{dr}{dt} = \dfrac{1}{4 \times \pi \times 15 \times 15} \times 900$

$\Rightarrow \dfrac{dr}{dt} = \dfrac{1}{\pi} = \dfrac{7}{22}$

56. Here, $V = \dfrac{4}{3}\pi r^3$ and $S = 4\pi r^2$

$\Rightarrow \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} \Rightarrow \dfrac{dr}{dt} = \dfrac{40}{4\pi r^2} = \dfrac{5}{32\pi}$

$\therefore \quad \dfrac{dS}{dt} = 8\pi r\dfrac{dr}{dt}$

$= 8\pi \times 8 \times \dfrac{5}{32\pi} = 10$ cm^2/min

57. $V = \dfrac{4}{3}\pi r^3$

$\Rightarrow 288\pi = \dfrac{4}{3}\pi r^3$

$\Rightarrow r = 6$ cm

$V = \dfrac{4}{3}\pi r^3$

$\therefore \quad \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$

$\Rightarrow 4\pi = 4\pi r^2\dfrac{dr}{dt}$

$\Rightarrow \dfrac{dr}{dt} = \dfrac{1}{r^2}$

Now, $A = 4\pi r^2$

$\therefore \quad \dfrac{dA}{dt} = 8\pi r\dfrac{dr}{dt}$

$= 8\pi r \times \dfrac{1}{r^2}$

$= \dfrac{8\pi}{r} = \dfrac{8\pi}{6} = \dfrac{4\pi}{3}$ cm^2/sec

58. Volume $= V = \dfrac{4}{3}\pi r^3$

$\Rightarrow \dfrac{dV}{dt} = 4\pi r^2 . \dfrac{dr}{dt}$, at r = 7 cm

$\Rightarrow 35 = 4\pi(7)^2\dfrac{dr}{dt} \Rightarrow \dfrac{dr}{dt} = \dfrac{5}{28\pi}$

Surface area, $S = 4\pi r^2$

$\therefore \quad \dfrac{dS}{dt} = 8\pi r\dfrac{dr}{dt} = 8\pi(7)\left(\dfrac{5}{28\pi}\right) = 10$ cm^2/min

59. $V = \dfrac{4}{3}\pi r^3$

$\therefore \quad \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$ ....(i)

After 49 min, $(4500 - 49 \times 72)\pi = 972\,\pi$m^3

$\therefore \quad 972\pi = \dfrac{4}{3}\pi r^3$

$\therefore \quad r^3 = 3 \times 243 = 3 \times 3^5$

$\therefore \quad r = 9$

Given, $\dfrac{dV}{dt} = 72\pi$

$\therefore \quad 72\pi = 4\pi \times 9 \times 9\left(\dfrac{dr}{dt}\right)$ ....[From (i)]

$\therefore \quad \dfrac{dr}{dt} = \dfrac{2}{9}$

60. Volume of sphere (V) = $\frac{4}{3}\pi r^3$

Surface area of sphere (A) = $4\pi r^2$

$\frac{dV}{dr} = 4\pi r^2$ and $\frac{dA}{dr} = 8\pi r$

$\therefore$ $\left(\frac{dV}{dA}\right) = \frac{\left(\frac{dV}{dr}\right)}{\left(\frac{dA}{dr}\right)} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$

$\therefore$ $\left(\frac{dV}{dA}\right)_{r=4} = \frac{4}{2} = 2\ \text{cm}^3/\text{cm}^2$

61. $W = nw$, $n = 2t^2 + 3$ and $w = t^2 - t + 2$

$\therefore$ $\frac{dW}{dt} = w\frac{dn}{dt} + n\frac{dw}{dt}, \frac{dn}{dt} = 4t, \frac{dW}{dt} = 2t - 1$

At t = 1,

$n = 5, w = 2, \frac{dn}{dt} = 4, \frac{dW}{dt} = 1$

$\therefore$ $\left(\frac{dW}{dt}\right)_{(t=1)} = 2(4) + 5(1) = 13$

62. According to the given condition,

$\frac{dy}{dt} = 8\frac{dx}{dt}$ ….(i)

Given, $6y = x^3 + 2$ ….(ii)

$\Rightarrow 6\left(\frac{dy}{dx}\right) = 3x^2\ \frac{dx}{dt}$

$\Rightarrow 6\left(\frac{8dx}{dt}\right) = 3x^2\ \frac{dx}{dt}$ ….[From (i)]

$\Rightarrow 3x^2 = 48 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Putting $x = 4$ in (ii), we get

$6y = (4)^3 + 2 = 64 + 2$

$\therefore$ $y = 11$

Putting $x = -4$ in (ii), we get

$y = -64 + 2$

$\therefore$ $y = \frac{-62}{6} = \frac{-31}{3}$

$\therefore$ the required points on the curve are (4, 11) and $\left(-4, \frac{-31}{3}\right)$.

63. $f(x) = x^3 + 5x^2 - 7x + 9$

$\therefore$ $f'(x) = 3x^2 + 10x - 7$

Here, a = 1 and h = 0.1

$\therefore$ $f(a) = f(1) = 1^3 + 5(1)^2 - 7(1) + 9 = 8$

and $f'(a) = f'(1) = 3(1)^2 + 10(1) - 7 = 6$

$\therefore$ $f(a + h) \approx f(a) + hf'(a)$

$\approx 8 + 0.1\ (6)$

$\approx 8 + 0.6 \approx 8.6$

64. Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$\therefore$ $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

Here, a = –1, and h = 0.01

$f(a + h) \approx f(a) + h f'(a)$

$\approx (-1)^{\frac{1}{3}} + 0.01 \times \frac{1}{3(-1)^{\frac{2}{3}}}$

$\approx -1 + 0.0033$

≈ -0.9967

65. Let $f(x) = \sqrt[5]{x} = x^{1/5}$

$\therefore$ $f'(x) = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}}$

Here, a = 243 and h = –0.001

$f(a + h) \approx f(a) + h\ f'(a)$

$= (243)^{1/5} - 0.001 \times \frac{1}{5(243)^{4/5}}$

$= 3 - \frac{0.001}{5 \times 81}$

$= 3 - \frac{1}{405000}$

$\therefore$ $f(242.999) = \frac{1214999}{405000}$

66. Let $f(x) = \cos x$

$\therefore$ $f'(x) = -\sin x$

Here, a = 30° and h = 1° = 0.0174

$\therefore$ $f(a + h) \approx f(a) + h\ f'(a)$

$\approx \frac{\sqrt{3}}{2} + 0.0174\left(\frac{-1}{2}\right)$

$\approx \frac{1.73}{2} - \frac{0.0174}{2}$

≈ 0.8563

67. $f(x) = e^x(\sin x - \cos x)$

$\therefore$ $f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x)$

$\therefore$ $f'(x) = 2e^x \sin x$

Now, $f'(c) = 0$

$\Rightarrow 2e^c \sin c = 0$

$\Rightarrow \sin c = 0 = \sin \pi$

$\Rightarrow c = \pi$

68. Here, $f\left(\frac{\pi}{2}\right) = e^0 = 1$ and $f\left(\frac{3\pi}{2}\right) = e^0 = 1$

$\therefore$ $f\left(\frac{\pi}{2}\right) = f\left(\frac{3\pi}{2}\right)$

$\therefore$ Third condition of Rolle's theorem is satisfied by option (A) only.

69. (A) $f(x) = |x|$ is not differentiable at $x = 0$.

(B) $f(x) = \tan x$ is discontinuous at $x = \frac{\pi}{2}$.

(C) $f(x) = 1 + (x-2)^{\frac{2}{3}}$ is not differentiable at $x = 2$.

(D) $f(x) = x(x-2)^2$ is a polynomial function.

$\therefore$ $f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$.

Also, $f(0) = f(2)$

Hence, Rolle's theorem is applicable.

70. $f(x) = e^x$

$\therefore$ $f(0) = e^0 = 1$, $f(1) = e$ and $f'(x) = e^x$

By mean value theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow f'(c) = \frac{e^b - e^a}{b-a}$$

$$\Rightarrow e^c = \frac{e-1}{1-0}$$

$$\Rightarrow c = \log(e-1)$$

71. $f(x) = x^2$

$f(2) = 4$, $f(4) = 16$

$f'(x) = 2x$

$\therefore$ By Lagrange's mean value theorem,

$$f'(c) = \frac{f(4) - f(2)}{4-2}$$

$$\Rightarrow 2c = \frac{16-4}{2}$$

$$\Rightarrow c = 3$$

72. $f(x) = \sqrt{x}$

$\therefore$ $f(a) = f(4) = \sqrt{4} = 2$, $f(b) = f(9) = \sqrt{9} = 3$ and

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Given, $f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{3-2}{9-4} = \dfrac{1}{5}$

$\therefore$ $\dfrac{1}{2\sqrt{c}} = \dfrac{1}{5} \Rightarrow c = \dfrac{25}{4} = 6.25$

73. $f(x) = (x-1)(x-2)$

$\Rightarrow f(x) = x^2 - 3x + 2$

$f(0) = 2$

$$f\left(\frac{1}{2}\right) = \frac{3}{4}$$

$f'(x) = 2x - 3$

By Lagrange's mean value theorem,

$$f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}$$

$$\Rightarrow 2c - 3 = \frac{\frac{3}{4} - 2}{\frac{1}{2}}$$

$$\Rightarrow 2c = \frac{-5}{2} + 3 \Rightarrow c = \frac{1}{4}$$

74. $f(x) = \dfrac{2x+3}{4x-1}$

$f(1) = \dfrac{5}{3}$, $f(2) = 1$

$$f'(x) = \frac{(4x-1)(2)-(2x+3)(4)}{(4x-1)^2} = \frac{-14}{(4x-1)^2}$$

$\therefore$ By Lagrange's mean value theorem,

$$f'(c) = \frac{f(2)-f(1)}{2-1}$$

$$\Rightarrow \frac{-14}{(4c-1)^2} = \frac{1-\frac{5}{3}}{1}$$

$$\Rightarrow (4c-1)^2 = \frac{-14}{\frac{-2}{3}}$$

$$\Rightarrow 16c^2 - 8c + 1 = 21$$

$$\Rightarrow 4c^2 - 2c - 5 = 0$$

$$\Rightarrow c = \frac{1+\sqrt{21}}{4}$$

75. $f(x) = \cos x$

$\therefore$ $f(0) = 1$, $f\left(\dfrac{\pi}{2}\right) = 0$ and $f'(x) = -\sin x$

By mean value theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow -\sin c = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0}$$

$$\Rightarrow -\sin c = \frac{0-1}{\frac{\pi}{2}} = -\frac{2}{\pi}$$

$$\Rightarrow \sin c = \frac{2}{\pi} \Rightarrow c = \sin^{-1}\left(\frac{2}{\pi}\right)$$

76. $g(x) = \dfrac{f(x)}{x+1}$

$\therefore$ $g(0) = \dfrac{f(0)}{0+1} = 12$ and $f(6) = \dfrac{f(6)}{6+1} = \dfrac{-4}{7}$

By mean value theorem,

$g'(c) = \dfrac{g(6)-g(0)}{6-0}$

$= \dfrac{\frac{-4}{7}-12}{6}$

$= \dfrac{-4-84}{7\times 6} = -\dfrac{44}{21}$

77. Consider option (A),

$Lf'\left(\dfrac{1}{2}\right) = -1$ and $Rf'\left(\dfrac{1}{2}\right) = 0$

So, it is not differentiable at $x = \dfrac{1}{2} \in (0, 1)$.

Hence, Lagrange's mean value theorem is not applicable.

78. $y = x^3 = f(x)$

$\therefore$ $f(2) = 8$, $f(-2) = -8$ and $f'(x) = 3x^2$

By mean value theorem,

$f'(x) = \dfrac{f(2)-f(-2)}{2-(-2)}$

$\Rightarrow 3x^2 = \dfrac{8-(-8)}{4}$

$\Rightarrow x^2 = \dfrac{4}{3}$

$\Rightarrow x = \pm\dfrac{2}{\sqrt{3}}$

79. $f(b) = f(2) = 8 - 24a + 10 = 18 - 24a$

$f(a) = f(1) = 1 - 6a + 5 = 6 - 6a$

$f'(x) = 3x^2 - 12ax + 5$

By Lagrange's mean value theorem,

$f'(x) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{18-24a-6+6a}{2-1}$

$\therefore$ $f'(x) = 12 - 18a$

$\therefore$ $3x^2 - 12ax + 5 = 12 - 18a$

At $x = \dfrac{7}{4}$,

$3\left(\dfrac{49}{16}\right) - 12a\left(\dfrac{7}{4}\right) + 5 = 12 - 18a$

$\Rightarrow 3a = \dfrac{147}{16} - 7 \Rightarrow 3a = \dfrac{35}{16} \Rightarrow a = \dfrac{35}{48}$

80. Since, $f(x) = x^3 \Rightarrow f'(x) = 3x^2$, which is non-negative for all real values of x.

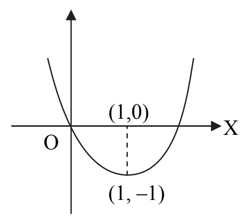
$\therefore$ Option (C) is the correct answer.

81. $f(x) = ax + b \Rightarrow f'(x) = a$

$\therefore$ For strictly increasing, $f'(x) > 0$

$\Rightarrow a > 0$ for all real x.

82. $f(x) = (x-1)^2 - 1$. Hence decreasing in $x < 1$.



Alternate Method:

$f'(x) = 2x - 2 = 2(x-1)$

To be decreasing, $2(x-1) < 0 \Rightarrow (x-1) < 0$

$\Rightarrow x < 1$

83. $y = \tan x - x \Rightarrow \dfrac{dy}{dx} = \sec^2 x - 1 = \tan^2 x \geq 0$

84. $f(x) = \begin{cases} 0, & x = 0 \\ x - 3, & x > 0 \end{cases}$

$\therefore$ $f'(x) = \begin{cases} 0, & x = 0 \\ 1, & x > 0 \end{cases}$

$\therefore$ It is strictly increasing when $x > 0$.

85. $f(x)$ will be monotonically decreasing, if $f'(x) < 0$.

$\Rightarrow$ $f'(x) = -\sin x - 2p < 0$

$\Rightarrow \dfrac{1}{2}\sin x + p > 0$

$\Rightarrow p > \dfrac{1}{2}$ ....[$\because -1 \leq \sin x \leq 1$]

86. Function is monotonically decreasing, when

$f'(x) < 0$

$\Rightarrow 6x^2 - 18x + 12 < 0$

$\Rightarrow x^2 - 3x + 2 < 0$

$\Rightarrow x^2 - 2x - x + 2 < 0$

$\Rightarrow (x-2)(x-1) < 0$

$\Rightarrow 1 < x < 2$

87. $f(x) = x^2 + 2x - 5$

$f'(x) = 2x + 2$

$= 2(x+1)$

For increasing function, $f'(x) > 0$

$\Rightarrow 2(x+1) > 0$

$\Rightarrow x > -1$

$\Rightarrow x \in (-1, \infty)$

88. $f(x) = x^3 - 3x^2 - 24x + 5$
For $f(x)$ to be increasing, $f'(x) > 0$
$\Rightarrow 3x^2 - 6x - 24 > 0$
$\Rightarrow x^2 - 2x - 8 > 0$
$\Rightarrow x^2 - 4x + 2x - 8 > 0$
$\Rightarrow (x + 2)(x - 4) > 0$
$\Rightarrow x \in (-\infty, -2) \cup (4, \infty)$

89. $f(x) = -2x^3 - 9x^2 - 12x + 1$
$\Rightarrow f'(x) = -6x^2 - 18x - 12$
For $f(x)$ to be decreasing, $f'(x) < 0$
$\Rightarrow -6x^2 - 18x - 12 < 0$
$\Rightarrow x^2 + 3x + 2 > 0 \Rightarrow (x + 2)(x + 1) > 0$
$\Rightarrow x < -2$ or $x > -1$
$\Rightarrow x \in (-1, \infty)$ or $(-\infty, -2)$

90. $f(x) = x + \sqrt{1-x}$
$\therefore$ $f'(x) = 1 - \dfrac{1}{2\sqrt{1-x}}$
For $f(x)$ to be decreasing $f'(x) < 0$
$\Rightarrow 1 - \dfrac{1}{2\sqrt{1-x}} < 0$
$\Rightarrow 1 < \dfrac{1}{2\sqrt{1-x}}$
$\Rightarrow 2\sqrt{1-x} < 1$
$\Rightarrow 4(1 - x) < 1$
$\Rightarrow 1 - x < \dfrac{1}{4}$
$\Rightarrow \dfrac{3}{4} < x$
$\therefore$ $x \in \left(\dfrac{3}{4}, 1\right)$

91. $f(x) = \sin^4 x + \cos^4 x$
$\therefore$ $f'(x) = -\sin 4x$
$\because$ $f'(x) > 0$
$\therefore$ $-\sin 4x > 0$
$\therefore$ $\sin 4x < 0$
$\therefore$ $(2n + 1)\pi < 4x < (2n + 2)\pi$
$\Rightarrow \dfrac{(2n+1)\pi}{4} < x < \dfrac{(n+1)\pi}{2}$
For $n = 0, \dfrac{\pi}{4} < x < \dfrac{\pi}{2}$
Now, $\dfrac{\pi}{2} = \dfrac{4\pi}{8} > \dfrac{3\pi}{8}$
$\therefore$ $f(x)$ is increasing in $\left(\dfrac{\pi}{4}, \dfrac{3\pi}{8}\right)$.

92. If $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$, then $f'(x) \le 0$ for all $x \in R$
$\Rightarrow 3(a + 2)x^2 - 6ax + 9a \le 0$ for all $x \in R$
$\Rightarrow (a + 2)x^2 - 2ax + 3a \le 0$ for all $x \in R$
$\Rightarrow a + 2 < 0$ and Discriminant ≤ 0
$\Rightarrow a < -2, -8a^2 - 24a \le 0$
$\Rightarrow a < -2$ and $a(a + 3) \ge 0$
$\Rightarrow a < -2, a \le -3$ or $a \ge 0$
$\Rightarrow a \le -3 \Rightarrow -\infty < a \le -3$

93. $f(x) = \dfrac{x}{x^2+1}$
$\therefore$ $f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$
For $f(x)$ to be increasing
$f'(x) > 0 \Rightarrow \dfrac{1-x^2}{(x^2+1)^2} > 0$
$x^2 + 1 \ne 0 \Rightarrow x^2 \ne -1$
$1 - x^2 > 0$
$\Rightarrow x^2 < 1$
$\Rightarrow x \in (-1, 1)$

94. $f(x) = \log(1 + x) - \dfrac{2x}{2+x}$
$\Rightarrow f'(x) = \dfrac{1}{1+x} - \dfrac{(2+x).(2) - 2x(1)}{(2+x)^2}$
$\Rightarrow f'(x) = \dfrac{x^2}{(x+1)(x+2)^2}$
$\therefore$ $f'(x) > 0$ for all $x > 0$
Hence, $f(x)$ is increasing on $(0, \infty)$.

95. $f(x) = (x + 2)e^{-x}$
$\therefore$ $f'(x) = e^{-x} - e^{-x}(x + 2) = -e^{-x}(x + 1)$
For $f(x)$ to be increasing,
$-e^{-x}(x + 1) > 0 \Rightarrow e^{-x}(x + 1) < 0$
$\Rightarrow (x + 1) < 0$
$\Rightarrow x < -1$
$\therefore$ $x \in (-\infty, 1)$
$\therefore$ the function is increasing in $(-\infty, -1)$.
$\therefore$ For $f(x)$ to be decreasing, $-e^{-x}(x + 1) < 0$
$\Rightarrow e^{-x}(x + 1) > 0$
$\Rightarrow x + 1 > 0$
$\Rightarrow x > -1$
$\Rightarrow x \in (-1, \infty)$
$\therefore$ the function is decreasing in $(-1, \infty)$.

96. $f(x) = 3x^2 - 2x + 1, \Rightarrow f'(x) = 6x - 2 \geq 0 \Rightarrow x \geq \frac{1}{3}$

Option (A) is incorrectly matched.

97. Let $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$

$\therefore$ $f'(x) = \frac{\ln(e+x) \times \frac{1}{\pi + x} - \ln(\pi + x) \times \frac{1}{e+x}}{[\ln(e+x)]^2}$

$= \frac{(e+x)\ln(e+x) - (\pi+x)\ln(\pi+x)}{[\ln(e+x)]^2 \times (e+x)(\pi+x)}$

$\Rightarrow f'(x) < 0$ for all $x \geq 0$ ....[$\because \pi > e$]

$\therefore$ $f(x)$ is decreasing on $[0, \infty)$.

98. $f(x) = x^3 - 10x^2 + 200x - 10$

$\Rightarrow f'(x) = 3x^2 - 20x + 200$

For $f(x)$ to be increasing $f'(x) > 0$

$\Rightarrow 3x^2 - 20x + 200 > 0$

$\Rightarrow 3\left(x^2 - \frac{20}{3}x + \frac{200}{3} + \frac{100}{9} - \frac{100}{9}\right) > 0$

$\Rightarrow 3\left[\left(x - \frac{10}{3}\right)^2 + \frac{500}{9}\right] > 0$

$\Rightarrow 3\left(x - \frac{10}{3}\right)^2 + \frac{500}{3} > 0$

Always increasing throughout real line.

99. $f(x) = x^{\frac{3}{2}}(3x - 10), x \geq 0$

$\therefore$ $f'(x) = \frac{3}{2}x^{\frac{1}{2}}(3x - 10) + x^{\frac{3}{2}}(3)$

$= \frac{15}{2}x^{\frac{1}{2}}(x - 2)$

For $f(x)$ to be increasing,

$f'(x) \geq 0 \Rightarrow \frac{15}{2}x^{\frac{1}{2}}(x-2) \geq 0$

$\Rightarrow x \geq 2 \Rightarrow x \in [2, \infty)$

100. $f(x) = \tan^{-1}(\sin x + \cos x)$

$\Rightarrow f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \times (\cos x - \sin x)$

$= \frac{\sqrt{2}\cos\left(x + \frac{\pi}{4}\right)}{1 + (\sin x + \cos x)^2}$

For $f(x)$ to be increasing, $f'(x) > 0$

$\Rightarrow \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow -\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2}$

$\Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$

$\therefore$ $f(x)$ is an increasing function in $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$.

101. $f(x) = \log(\sin x + \cos x)$

$\Rightarrow f'(x) = \frac{\cos x - \sin x}{\sin x + \cos x} = \frac{1 - \tan x}{1 + \tan x} = \tan\left(\frac{\pi}{4} - x\right)$

For $f(x)$ to be increasing,

$f'(x) > 0$

$\Rightarrow \tan\left(\frac{\pi}{4} - x\right) > 0$

$\Rightarrow 0 < \frac{\pi}{4} - x < \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4} < -x < \frac{\pi}{4}$

$\Rightarrow -\frac{\pi}{4} < x < \frac{\pi}{4}$

$\Rightarrow x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

102. $f(x) = \int e^x(x-1)(x-2)dx$

$\Rightarrow f'(x) = e^x(x - 1)(x - 2)$

For $f(x)$ to be decreasing, $f'(x) < 0$

$\Rightarrow e^x(x - 1)(x - 2) < 0$

$\Rightarrow (x - 1)(x - 2) < 0$

$\Rightarrow x \in (1, 2)$

103. $f(x) = \frac{x}{\sin x}$

$\Rightarrow f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} = \frac{\cos x(\tan x - x)}{\sin^2 x}$

$\therefore$ $f'(x) > 0$ for $0 < x \leq 1$

$\Rightarrow f(x)$ is an increasing function.

Now, $g(x) = \frac{x}{\sin x}$

$\Rightarrow g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}$

$= \frac{\sin x\cos x - x}{\sin^2 x}$

$= \frac{\sin 2x - 2x}{2\sin^2 x}$

$\therefore$ $g'(x) < 0$ for $0 < x \leq 1$.

$\Rightarrow g(x)$ is a decreasing function.

104. $f(x) = \sin x - \cos x$

$\Rightarrow f'(x) = \cos x + \sin x = \sqrt{2}\left[\cos\left(x - \frac{\pi}{4}\right)\right]$

For f(x) to be decreasing, $f'(x) < 0$

$\Rightarrow \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) < 0$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) < 0$

$\Rightarrow \frac{\pi}{2} < x - \frac{\pi}{4} < \frac{3\pi}{2} \Rightarrow \frac{3\pi}{4} < x < \frac{7\pi}{4}$

105. $h(x) = f(x) - (f(x))^2 + (f(x))^3$

$\therefore$ $h'(x) = f'(x) - 2f(x)\, f'(x) + 3(f(x))^2\, f'(x)$

$= f'(x)\,[1 - 2\, f(x) + 3\,(f(x))^2]$

Here, $1 - 2\, f(x) + 3(f(x))^2 > 0$ for all f(x)

$\Rightarrow h'(x) > 0$, if $f'(x) > 0$ and $h'(x) < 0$, if $f'(x)<0$

$\Rightarrow$ h is increasing whenever f is increasing and h is decreasing whenever f is decreasing.

106. $f(x) = [x(x-2)]^2$

$\Rightarrow f(x) = x^2\,(x-2)^2$

$\Rightarrow f'(x) = x^2\{2(x-2)\} + (x-2)^2\,(2x)$

$= 2x(x-2)\ \{x + (x-2)\}$

$= 4x(x-2)(x-1)$

For f(x) to be increasing, $f'(x) > 0$

$\Rightarrow 4x(x-1)(x-2) > 0$

$\Rightarrow x(x-1)(x-2) > 0 \Rightarrow x \in (0, 1) \cup (2, \infty)$

107. $y = \{x(x-3)\}^2$

$\Rightarrow y = x^2\,(x-3)^2$

$\therefore$ $\frac{dy}{dx} = 2x(x-3)^2 + 2(x-3)x^2$

$= 2x(x-3)[x - 3 + x]$

$= 2x(x-3)(2x-3)$

For y to be increasing, $\frac{dy}{dx} > 0$

$\Rightarrow 2x(x-3)\,(2x-3) > 0$

$\Rightarrow x(x-3)(2x-3) > 0 \Rightarrow x \in \left(0, \frac{3}{2}\right)$

108. $f(a) = 2a^2 - 3a + 10$

$\Rightarrow f'(a) = 4a - 3 \Rightarrow f''(a) = 4 > 0$

For minimum value of f (a),

$f'(a) = 0 \Rightarrow a = \frac{3}{4}$

$\therefore$ f(a) is minimum at $a = \frac{3}{4}$.

$\therefore$ $[f(a)]_{\min} = f\left(\frac{3}{4}\right) = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 10 = \frac{71}{8}$

109. $f(x) = a \sin x + \frac{1}{3}\sin 3x$

$\Rightarrow f'(x) = a\cos x + \frac{1}{3}\cdot 3\cos 3x$

$\Rightarrow f'(x) = a\cos x + \cos 3x$

Now, $f'\left(\frac{\pi}{3}\right) = 0$

$\Rightarrow a\cos\frac{\pi}{3} + \cos\pi = \frac{a}{2} - 1 = 0$

$\Rightarrow a = 2$

110. Clearly, it has a maximum at $x = 1$.

112. $y = x^3 - 3x^2 + 5$

$f(x) = x^3 - 3x^2 + 5$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

$f'(x) = 0$ at $x = 0$, $x = 2$

$f''(0) < 0$, $f''(2) > 0$

$\therefore$ f (x) is maximum at $x = 0$

113. Let $f(x) = 2x^3 - 15x^2 + 36x + 4$

$\therefore$ $f'(x) = 6x^2 - 30x + 36 = 0$ at $x = 3, 2$

$\therefore$ $f''(x) = 12x - 30$ is –ve at $x = 2$

$\therefore$ maximum value of f(x) attained at $x = 2$

114. $f'(x) = 6x^2 - 6x - 12$

$f'(x) = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1, 2$

Here $f(4) = 128 - 48 - 48 + 5 = 37$

$f(-1) = -2 - 3 + 12 + 5 = 12$

$f(2) = 16 - 12 - 24 + 5 = -15$

$f(-2) = -16 - 12 + 24 + 5 = 1$

$\therefore$ the maximum value of function is 37 at $x = 4$.

115. Given $f(x) = x(1-x)^2$, $f(x) = x^3 - 2x^2 + x$

$\therefore$ $f'(x) = 3x^2 - 4x + 1$

Put $f'(x) = 0$ i.e., $3x^2 - 4x + 1 = 0$

$\Rightarrow 3x^2 - 3x - x + 1 = 0 \Rightarrow x = 1, 1/3$

$f''(x) = 6x - 4$

$\therefore$ $f''(1) = 2 > 0$ and $f''(1/3) = -2 < 0$

$\therefore$ f(x) is maximum at $x = \frac{1}{3}$.

$\therefore$ Maximum value $= f\left(\frac{1}{3}\right) = \frac{4}{27}$

116. Let $f(x) = x^2 + \frac{250}{x}$

$\Rightarrow f'(x) = 2x - \frac{250}{x^2}$

$\Rightarrow f''(x) = 2 + \frac{500}{x^3}$

For maximum or minimum of f(x),
$f'(x) = 0 \Rightarrow 2x^3 - 250 = 0$
$\Rightarrow x^3 = 125 \Rightarrow x = 5$
$\therefore$ $f''(5) = 2 + \frac{500}{125} = 6 > 0$
$\therefore$ f has minimum at $x = 5$ and minimum value of f at $x = 5$ is f(5) = 25 + 50 = 75

117. $f(x) = x \log x$
$f'(x) = 1 + \log x$
for minimum, $f'(x) = 0 \Rightarrow \log x = -1 \Rightarrow x = \frac{1}{e}$
$f''(x) = \frac{1}{x}$ $\qquad$ $f''(e) = \frac{1}{e} > 0$
$\therefore$ f(x) is minimum at $x = \frac{1}{e}$
$\therefore$ $f\left(\frac{1}{e}\right) = \frac{1}{e}\log\left(\frac{1}{e}\right) = -\frac{1}{e}$

118. Let $f(x) = \frac{\log x}{x} \Rightarrow f'(x) = \frac{1}{x^2} - \frac{\log x}{x^2}$
For maximum or minimum value of f(x),
$f'(x) = 0$
$\Rightarrow \frac{1 - \log_e x}{x^2} = 0$
$\therefore$ $\log_e x = 1$ or $x = e$, which lie in $(0, \infty)$.
For $x = e$, $\frac{d^2y}{dx^2} = -\frac{1}{e^3}$, which is –ve.
$\therefore$ y is maximum at $x = e$
and its maximum value $= \frac{\log e}{e} = \frac{1}{e}$.

119. $x + y = 32 \Rightarrow y = 32 - x$
$\Rightarrow x^2 + y^2 = x^2 + (32 - x)^2$
Let $z = x^2 + (32 - x)^2$
$\Rightarrow z' = 2x + 2(32 - x)(-1) = 4x - 64$
Now, $z'' = 4 > 0$
$\therefore$ at $x = 16$ and $y = 32 - x = 32 - 16 = 16$
$x^2 + y^2 = 32$ have minimum value
$\therefore$ minimum value $= x^2 + y^2 = (16)^2 + (16)^2 = 512$

120. Let $f(x) = x^{25}(1 - x)^{75}$
$\therefore$ $f'(x) = x^{25}(75)(1 - x)^{74}(-1) + 25x^{24}(1 - x)^{75}$
For maximum value of f(x), $f'(x) = 0$
$\Rightarrow -75x^{25}(1 - x)^{74} + 25x^{24}(1 - x)^{75} = 0$
$\Rightarrow 25x^{24}(1 - x)^{74}(1 - 4x) = 0$
$\Rightarrow$ $x = 0$ or $x = 1$ or $x = \frac{1}{4}$
At $x = \frac{1}{4}$, $f'\left(\frac{1}{4} - h\right) > 0$ and $f'\left(\frac{1}{4} + h\right) < 0$
$\therefore$ f(x) has maximum value at $x = \frac{1}{4}$.

121. $h(x) = \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}}$
$= \frac{\left(x - \frac{1}{x}\right)^2 + 2}{\left(x - \frac{1}{x}\right)}$
$= \left(x - \frac{1}{x}\right) + \frac{2}{\left(x - \frac{1}{x}\right)}$
When $x - \frac{1}{x} < 0$, $\left(x - \frac{1}{x}\right) + \frac{2}{\left(x - \frac{1}{x}\right)} \le -2\sqrt{2}$
When $x - \frac{1}{x} > 0$, $\left(x - \frac{1}{x}\right) + \frac{2}{\left(x - \frac{1}{x}\right)} \ge 2\sqrt{2}$
The local minimum value of h(x) is $2\sqrt{2}$.

122. $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$
$\Rightarrow f'(x) = 6x^2 - 18ax + 12a^2$
$\Rightarrow f''(x) = 12x - 18a$
For maximum or minimum of f(x), $f'(x) = 0$
$\Rightarrow 6x^2 - 18ax + 12a^2 = 0$
$\Rightarrow x^2 - 3ax + 2a^2 = 0$
$\Rightarrow x = a$ or $x = 2a$
At $x = a$, f has maximum $(5a^3 + 1)$
and at $x = 2a$, f has minimum $(4a^3 + 1)$
Since, $p^3 = q$
$\therefore$ $a^3 = 2a \Rightarrow a = \sqrt{2}$ or $a = 0$
But $a > 0$
$\therefore$ $a = \sqrt{2}$

123. $f(x) = x^2 + 2bx + 2c^2$
$f'(x) = 2x + 2b = 0$, at $x = -b$
$f''(x) = 2 > 0$
$\therefore$ f(x) is minimum at $x = -b$
$\therefore$ $f(-b) = b^2 - 2b^2 + 2c^2 = 2c^2 - b^2$
$g(x) = -x^2 - 2cx + b^2$
$g'(x) = -2x - 2c = 0$ at $x = -c$
$g''(x) = -2 < 0$
$\therefore$ g(x) is maximum at $x = -c$
$\therefore$ $g(-c) = -c^2 + 2c^2 + b^2 = b^2 + c^2$
Given, minimum value of f(x) > maximum of g(x)
$\Rightarrow 2c^2 - b^2 > b^2 + c^2$
$\Rightarrow c^2 > 2b^2$

124. $f(x) = x^2 + e^x$

$f'(x) = 2x + e^x$

$f''(x) = 2 + e^x$

$f'''(x) = e^x$

$f''''(x) = e^x$

$\Rightarrow f_3 = f_4 \Rightarrow n = 3$

125. Let x and y be the lengths of two adjacent sides of the rectangle.

Then, its perimeter is $2(x + y) = 36$

$\Rightarrow x + y = 18 \Rightarrow y = 18 - x$ ….(i)

Area of rectangle,

$A = xy = x(18 - x) = 18x - x^2$

$\therefore \quad \frac{dA}{dx} = 18 - 2x$

For maximum or minimum,

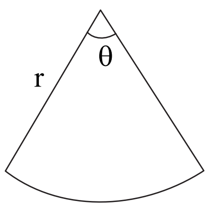
$\frac{dA}{dx} = 0 \Rightarrow 18 - 2x = 0 \Rightarrow x = 9$

From (i), $y = 18 - 9 = 9$

126. Total length of wire = $r + r + r\theta$

$\Rightarrow 20 = 2r + r\theta$

$\Rightarrow \theta = \frac{20-2r}{r}$



$A = \frac{1}{2}r^2\theta$

$= \frac{1}{2}r^2\left(\frac{20-2r}{r}\right) = 10r - r^2$

$\therefore \quad \frac{dA}{dr} = 10 - 2r$

For maximum area, $\frac{dA}{dr} = 0$

$\Rightarrow 0 = 10 - 2r \Rightarrow 10 = 2r \Rightarrow r = 5$ m

$\therefore \quad$ Area $= \frac{1}{2}r(20 - 2r)$

$= \frac{1}{2} \times 5 \times (20 - 10) = 25$ sq.m.

127. Let $x + y = 4 \Rightarrow y = 4 - x$

$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}$

$f(x) = \frac{4}{xy} = \frac{4}{x(4-x)} = \frac{4}{4x-x^2}$

$\therefore \quad f'(x) = \frac{-4}{(4x-x^2)^2}.(4-2x)$

For maximum or minimum of $f(x)$,

$f'(x) = 0 \Rightarrow 4 - 2x = 0$

$\therefore \quad x = 2$ and $y = 2$

$\therefore \quad \min\left(\frac{1}{x}+\frac{1}{y}\right) = \frac{1}{2}+\frac{1}{2} = 1$

128. Let x and y be the lengths of two adjacent sides of the rectangle.

Then, its perimeter is $P = 2(x + y)$ ….(i)

$\Rightarrow y = \frac{P-2x}{2}$

Area of rectangle, $A = xy$

$= x\left(\frac{P-2x}{2}\right) = \frac{Px-2x^2}{2}$

$\therefore \quad \frac{dA}{dx} = \frac{P-4x}{2}$ and $\frac{d^2A}{dx^2} = -2$

For maximum or minimum,

$\frac{dA}{dx} = 0$

$\Rightarrow \frac{P-4x}{2} = 0$

$\Rightarrow P = 4x$

$\Rightarrow 2x + 2y = 4x$ ….[From (i)]

$\Rightarrow x = y$

$\therefore \quad \left(\frac{d^2A}{dx^2}\right)_{x=y} = -2 < 0$

Hence, the area of a rectangle will be maximum when rectangle is a square.

129. $p(t) = 1000 + \frac{1000t}{100+t^2}$

$\therefore \quad \frac{dp}{dt} = \frac{(100+t^2)1000 - 1000t.2t}{(100+t^2)^2}$

$= \frac{1000(100-t^2)}{(100+t^2)^2}$

For extremum,

$\frac{dp}{dt} = 0 \Rightarrow t = 10$

Now $\left.\frac{dp}{dt}\right|_{t<10} > 0$ and $\left.\frac{dp}{dt}\right|_{t>10} < 0$

$\therefore \quad$ At $t = 10$, $\frac{dp}{dt}$ change from positive to negative.

It is a critical point.

$\therefore \quad$ p is maximum at $t = 10$.

$\therefore \quad p_{max} = p(10)$

$= 1000 + \frac{1000.10}{100+10^2} = 1050$

130. $f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - \frac{1}{x^2}$

$\Rightarrow f'(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1, -1$

But it is given that x is positive

$\therefore \quad$ at $x = 1$, $f(x) = 1 + \frac{1}{1} = 2$

131. $f(x) = x + \sin x \Rightarrow f'(x) = 1 + \cos x$
Now, $f'(x) = 0 \Rightarrow 1 + \cos x = 0 \Rightarrow \cos x = -1$
$\Rightarrow x = \pi$
Now, $f''(x) = -\sin x$, $f''(\pi) = 0$
$f'''(x) = -\cos x$
$\therefore$ $f'''(\pi) = 1 \neq 0$
$\therefore$ Neither maximum nor minimum.

132. Let α, β be the roots of the equation
$x^2 - (a - 2)x - a + 1 = 0$,
then $\alpha + \beta = a - 2$, $\alpha\beta = -a + 1$
$\therefore$ $z = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= (a - 2)^2 + 2(a - 1) = a^2 - 2a + 2$
$\therefore$ $\dfrac{dz}{da} = 2a - 2 = 0 \Rightarrow a = 1$
$\dfrac{d^2z}{da^2} = 2 > 0$, so z has minima at a = 1
So $\alpha^2 + \beta^2$ has least value for a = 1. This is because we have only one stationary value at which we have minima. Hence a = 1.

133. $f(x) = \dfrac{x^2 - 1}{x^2 + 1} = \dfrac{x^2 + 1 - 2}{x^2 + 1} = 1 - \dfrac{2}{x^2 + 1}$
$\therefore$ $f(x) < 1 \ \forall x$ and $f(x) \geq -1$ as $\dfrac{2}{x^2 + 1} \leq 2$
$\therefore$ $-1 \leq f(x) < 1$
$\therefore$ $f(x)$ has minimum value -1.

134. Let $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$
$\Rightarrow \dfrac{dy}{dx} = \dfrac{(x^2 + x + 1)(2x - 1) - (x^2 - x + 1)(2x + 1)}{(x^2 + x + 1)^2}$
$\Rightarrow \dfrac{dy}{dx} = \dfrac{2x^2 - 2}{(x^2 + x + 1)^2} = 0$
$\Rightarrow 2x^2 - 2 = 0 \Rightarrow x = -1, +1$
$\dfrac{d^2y}{dx^2} = \dfrac{4(-x^3 + 3x + 1)}{x^2 + x + 1}$
At $x = -1$, $\dfrac{d^2y}{dx^2} < 0$ the function will occupy maximum value,
$\therefore$ $f(-1) = 3$ and at $x = 1$, $\dfrac{d^2y}{dx^2} > 0$ the function will occupy minimum value.
$\therefore$ $f(1) = \dfrac{1}{3}$

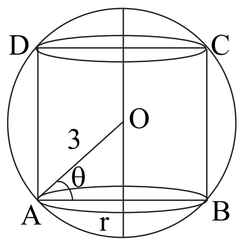
135. Let $f(x) = \exp(2 + \sqrt{3}\cos x + \sin x)$
$\Rightarrow f'(x) = \exp(2 + \sqrt{3}\cos x + \sin x)$
$\times (-\sqrt{3}\sin x + \cos x)$
For maximum or minimum of $f(x)$, $f'(x) = 0$
$\Rightarrow \exp(2+\sqrt{3}\cos x + \sin x)(-\sqrt{3}\sin x + \cos x) = 0$
$\Rightarrow -\sqrt{3}\sin x + \cos x = 0$
$\Rightarrow \dfrac{\sqrt{3}}{2}\sin x - \dfrac{1}{2}\cos x = 0$
$\Rightarrow \sin\left(x - \dfrac{\pi}{6}\right) = 0$
$\Rightarrow x = \dfrac{\pi}{6}$
At $x = \dfrac{\pi}{6}$, $f''(x)$ is negative
$\therefore$ f has maximum at $x = \dfrac{\pi}{6}$ and maximum value of f at $x = \dfrac{\pi}{6}$ is
$f\left(\dfrac{\pi}{6}\right) = \exp\left(2 + \sqrt{3}\left(\dfrac{\sqrt{3}}{2}\right) + \dfrac{1}{2}\right) = \exp(4)$

136. Let $y = x^x \Rightarrow \log y = x.\log x$, $(x > 0)$
Differentiating, $\dfrac{dy}{dx} = x^x(1 + \log x)$;
$\therefore$ $\dfrac{dy}{dx} = 0$
$\Rightarrow \log x = -1 \Rightarrow x = e^{-1} = \dfrac{1}{e}$
$\therefore$ Stationary point is $x = \dfrac{1}{e}$

137. Let $f(x) = \left(\dfrac{1}{x}\right)^x$
$\Rightarrow f(x) = x^{-x}$
$\Rightarrow f'(x) = -x^{-x}(1 + \log x)$
$\Rightarrow f''(x) = x^{-x}(1 + \log x)^2 - x^{-x-1}$
For maximum or minimum of $f(x)$, $f'(x) = 0$
$\Rightarrow -x^{-x}(1 + \log x) = 0$
$\Rightarrow 1 + \log x = 0$
$\Rightarrow \log x = -1 = \log\dfrac{1}{e}$
$\Rightarrow x = \dfrac{1}{e}$
$\therefore$ $f''\left(\dfrac{1}{e}\right) = \left(\dfrac{1}{e}\right)^{\frac{-1}{e}}\left(1 + \log\dfrac{1}{e}\right)^2 - \left(\dfrac{1}{e}\right)^{-\frac{1}{e}-1}$
$= e^{\frac{1}{e}}(1 - 1)^2 - e^{\frac{1}{e}+1}$
$= -e^{\frac{1}{e}+1} < 0$

$\therefore$ f has maximum at $x = \frac{1}{e}$ and maximum value of f at $x = \frac{1}{e}$ is $f\left(\frac{1}{e}\right) = (e)^{1/e}$

138. Let r be the radius and h be the height, then from the figure, $r^2 + \left(\frac{h}{2}\right)^2 = 3^2$



$\therefore$ $h^2 = 4(9 - r^2)$

$\Rightarrow r^2 = 36 - h^2$

Now, $V = \pi r^2 h$

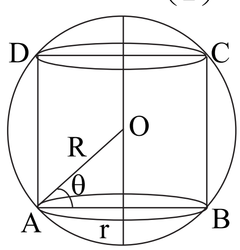
$\Rightarrow V = \pi(36 - h^2)h$

$\therefore$ $\frac{dV}{dh} = \pi(36 - 3h^2)$

for max or min, $\frac{dV}{dh} = 0$

$\Rightarrow \pi(36 - 3h^2) = 0 \Rightarrow h^2 = 12 \Rightarrow h = 2\sqrt{3}$

139. Let r be the radius and h be the height, then from the figure, $r^2 + \left(\frac{h}{2}\right)^2 = R^2$



$\therefore$ $h^2 = 4(R^2 - r^2)$

Now, $V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$

$\therefore$ $\frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + 2\pi r^2 . \frac{(-2r)}{2\sqrt{R^2 - r^2}}$

For max. or min., $\frac{dV}{dr} = 0$

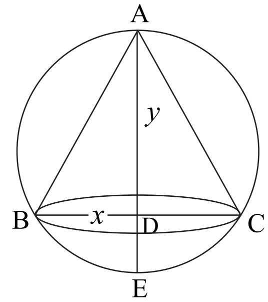
$\Rightarrow 4\pi r\sqrt{R^2 - r^2} = \frac{2\pi r^3}{\sqrt{R^2 - r^2}} \Rightarrow 2(R^2 - r^2) = r^2$

$\Rightarrow 2R^2 = 3r^2 \Rightarrow r = \sqrt{\frac{2}{3}}R \Rightarrow \frac{d^2V}{dr^2} .= -ve$

$\therefore$ V is max., when $r = \sqrt{\frac{2}{3}}R$.

140. Let diameter of sphere be AE = 2r

Let radius of cone be x and height be y.



$\therefore$ AD = y

Since, $BD^2 = AD.DE$

$\Rightarrow x^2 = y(2r - y)$

Volume of cone $V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi y(2r - y)y$

$= \frac{1}{3}\pi(2ry^2 - y^3)$

$\Rightarrow \frac{dV}{dy} = \frac{1}{3}\pi(4ry - 3y^2)$

Now $\frac{dV}{dy} = 0$

$\Rightarrow \frac{1}{3}\pi(4ry - 3y^2) = 0 \Rightarrow y(4r - 3y) = 0$

$\Rightarrow y = \frac{4}{3}r, 0$

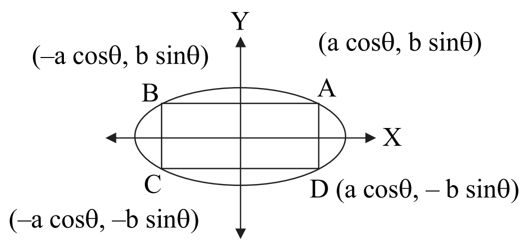
Now $\frac{d^2V}{dy^2} = \frac{1}{3}\pi(4r - 6y)$

$\Rightarrow \left(\frac{d^2V}{dy^2}\right)_{y=\frac{4}{3}r} = \frac{1}{3}\pi\left(4r - 6\times\frac{4}{3}r\right) < 0$

So, volume of cone is maximum at $y = \frac{4}{3}r$.

$\therefore$ $\frac{\text{Height of Cone}}{\text{Diameter of Sphere}} = \frac{y}{2r} = \frac{2}{3}$

141.

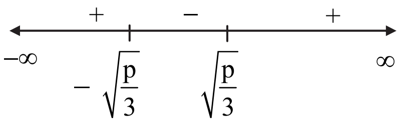


Area of rectangle ABCD

$= (2a\cos\theta)(2b\sin\theta) = 2ab\sin 2\theta$

Hence, area of greatest rectangle is equal to 2ab, when $\sin 2\theta = 1$.

142. Let $f(x) = x^3 - px + q$. Then,
$f'(x) = 3x^2 - p$
$= 3\left(x - \sqrt{\frac{p}{3}}\right)\left(x + \sqrt{\frac{p}{3}}\right)$
The signs of $f'(x)$ for different values of x are as shown below:



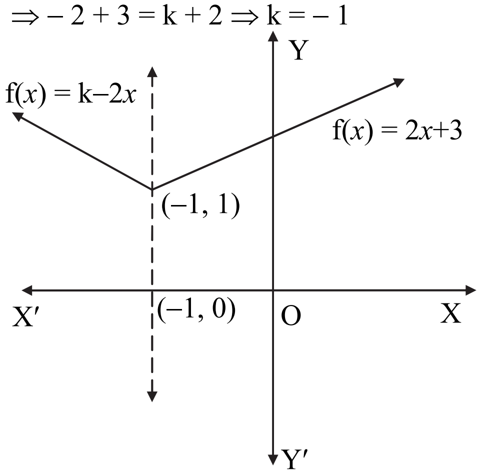
Since, $f'(x)$ changes its sign form positive to negative in the neighbourhood of $-\sqrt{\frac{p}{3}}$.

So, $-\sqrt{\frac{p}{3}}$ is a point of local maximum.

Similarly, $x = \sqrt{\frac{p}{3}}$ is a point of local minimum.

143. For any $x \in [0, 1]$, we have
$x^2 \le x \le 1$
$\Rightarrow x^2 e^{x^2} \le x e^{x^2} \le e^{x^2}$
$\Rightarrow e^{-x^2} + x^2 e^{x^2} \le e^{-x^2} + x e^{x^2} \le e^{-x^2} + e^{x^2}$
$\Rightarrow h(x) \le g(x) \le f(x)$
Now, $f(x) = e^{x^2} + e^{-x^2}$
$\Rightarrow f'(x) = 2x\left(e^{x^2} - e^{-x^2}\right) > 0$ for all $x \in (0, 1]$
$\Rightarrow f(x)$ is increasing on (0, 1]
$\Rightarrow f(1)$ is the maximum value of $f(x)$ on [0, 1]
$\Rightarrow a = e + e^{-1}$
Also, $f(1) = g(1) = h(1) = e + e^{-1}$
$\therefore \quad a = b = c = e + e^{-1}$

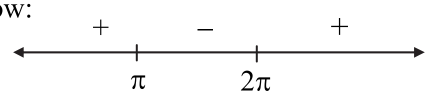
144. If $f(x)$ has a local minimum at $x = -1$, then
$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^-} f(x)$
$\Rightarrow \lim\limits_{x \to -1^+} (2x + 3) = \lim\limits_{x \to -1^-} (k - 2x)$
$\Rightarrow -2 + 3 = k + 2 \Rightarrow k = -1$



145. $f(x) = \int_0^x \sqrt{t} \sin t \, dt \Rightarrow f'(x) = \sqrt{x} \sin x$

For local maximum or minimum of $f(x)$,
$f'(x) = 0 \Rightarrow \sqrt{x} \sin x = 0$
$\Rightarrow x = \pi, 2\pi \qquad \ldots\left[\because x \in \left(0, \frac{5\pi}{2}\right)\right]$

The changes in signs of $f'(x)$ in the neighbourhoods of π and 2π are as shown below:

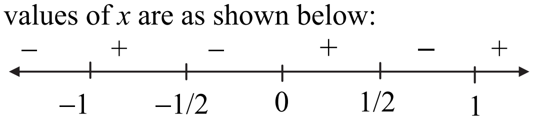


Clearly, $f'(x)$ changes its sign from positive to negative in the neighbourhood of $x = \pi$ and negative to positive in the neighbourhood of $x = 2\pi$. Thus, $f(x)$ has a local maximum at $x = \pi$ and a local minimum at $x = 2\pi$.

146. $f(x) = |x| + |x^2 - 1| = \begin{cases} -x + x^2 - 1, & x \le -1 \\ -x - x^2 + 1, & -1 \le x < 0 \\ x - x^2 + 1, & 0 \le x < 1 \\ x + x^2 - 1, & x \ge 1 \end{cases}$

$\therefore \quad f'(x) = \begin{cases} 2x - 1, & x < -1 \\ -2x - 1, & -1 < x < 0 \\ -2x + 1, & 0 < x < 1 \\ 2x + 1, & x > 1 \end{cases}$

Here, $f(x)$ is not differentiable at $x = -1, 0, 1$.
The changes in signs of $f'(x)$ for different values of x are as shown below:



So, $f'(x)$ changes its sign at 5 points.
Hence, total number of points of local maximum or local minimum of $f(x)$ is 5.

147. $f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$

and $g(x) = \int_0^x f(t) dt$, $x \in [1, 3]$

$\therefore \quad g'(x) = f(x) = \begin{cases} 2 - e^{x-1} & 1 < x \le 2 \\ x - e & 2 < x \le 3 \end{cases}$

Now, $g'(x) = 0 \Rightarrow x = 1 + \log_e 2$ and $x = e$
Also, $g'(x) > 0$ for $x \in (1, 1 + \log_e 2)$
and $g'(x) < 0$ for $x \in (1 + \log_e 2, 2)$.

So, g(x) attains a local maximum at $x = 1 + \log_e 2$.
Similarly,
$g'(x) < 0$ for $2 < x < e$
and $g'(x) > 0$ for $e < x < 3$
So, g(x) attains a local minimum at $x = e$.
We have,

$$f'(x) = \begin{cases} e^x & , \quad 0 < x < 1 \\ -e^{x-1} & , \quad 1 < x < 2 \\ 1 & , \quad 2 < x < 3 \end{cases}$$

Clearly, $f'(x) > 0$ for $x \in (0, 1)$
$f'(x) < 0$ for $x \in (1, 2)$
$f'(x) > 0$ for $x \in (2, 3)$
So, f(x) attains local maximum at $x = 1$ and local minimum at $x = 2$.
Hence, option (C) is incorrect.

148. $f(x) = \begin{cases} (2+x)^3 & , \quad -3 < x \le -1 \\ x^{2/3} & , \quad -1 < x < 2 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} 3(2+x)^2 & , \quad -3 < x < -1 \\ \frac{2}{3}x^{-1/3} & , \quad -1 < x < 2 \end{cases}$$

Clearly, $f'(x)$ changes its sign from positive to negative as x passes through $x = -1$ from left to right.
So, f(x) attains a local maximum at $x = -1$.
Here, $f'(x) > 0$ for all $x \in (-3, -1)$ and $f'(x) < 0$ for $x \in (-1, 0)$.
Also, $f'(x) > 0$ for $x \in (0, 2)$.
But, $f'(0)$ does not exist.
So, f(x) attains a local minimum at $x = 0$
Hence, the total number of local maxima and local minima is 2.

149. $f(x) = (1 + b^2)x^2 + 2bx + 1$
$\Rightarrow f'(x) = 2(1 + b^2)\,x + 2b$
$\Rightarrow f''(x) = 2(1 + b^2) > 0$
For minimum value of f(x),
$f'(x) = 0$
$\Rightarrow 2(1 + b^2)\,x + 2b = 0$
$\Rightarrow x = -\dfrac{b}{1+b^2}$

$\therefore$ f(x) is minimum at $x = -\dfrac{b}{1+b^2}$

$\therefore$ Minimum value of f(x) $= \dfrac{1}{1+b^2}$

$\therefore$ $m(b) = \dfrac{1}{1+b^2}$

Since, $\dfrac{1}{1+b^2} \le 1$ and $\dfrac{1}{1+b^2} > 0\ \forall\ b \in R$
$\therefore$ $0 < m(b) \le 1$
$\therefore$ range of m (b) is (0, 1].

150. $P(x) = x^4 + ax^3 + bx^2 + cx + d$
$\Rightarrow P'(x) = 4x^3 + 3ax^2 + 2bx + c$ ....(i)
Since, $x = 0$ is the only real root of $P'(x) = 0$.
$\therefore$ $P'(0) = 0 \Rightarrow c = 0$
Putting c = 0 in (i), we get
$P'(x) = x(4x^2 + 3ax + 2b)$
Since, $x = 0$ is the only real root of $P'(x) = 0$.
$\therefore$ $4x^2 + 3ax + 2b = 0$ has no real root.
$\Rightarrow 9a^2 - 32b < 0$
Given, $P(-1) < P(1)$
$\Rightarrow 1 - a + b - c + d < 1 + a + b + c + d$
$\Rightarrow a > 0$
But, $9a^2 - 32b < 0$. Therefore, $b > 0$
$\therefore$ $P'(x) = x(4x^2 + 3ax + 2b) > 0$ for all $x \in (0, 1]$
$\Rightarrow$ P(x) is increasing in (0, 1]
$\Rightarrow$ P(1) is the maximum value of P(x).
Also, $P'(x) = x(4x^2 + 3ax + 2b) < 0$ for all $x \in [-1, 0)$
....[$\because 4x^2 + 3ax + 2b > 0$ for all x]
$\Rightarrow$ P(x) is decreasing in $[-1, 0)$.
$\Rightarrow$ P($-$1) is not the minimum value of P.

151. $f(x) = \ln\{g(x)\}$
$\Rightarrow g(x) = e^{f(x)}$
$\Rightarrow g'(x) = e^{f(x)}\, f'(x)$
For local maximum of g(x),
$g'(x) = 0$
$\Rightarrow e^{f(x)}\, f'(x) = 0$
$\Rightarrow f'(x) = 0$
$\Rightarrow 2010(x - 2009)\,(x - 2010)^2\,(x - 2011)^3 \times (x - 2012)^4 = 0$
$\Rightarrow x = 2009, 2010, 2011, 2012$
$\therefore$ $f'(x)$ changes its sign from positive to negative in the neighbourhood of $x = 2009$.
$\therefore$ g(x) has a local maximum at $x = 2009$ only.

152. According to the given condition,
$\dfrac{dy}{dx} = 0$
$\Rightarrow 12 - 3x^2 = 0$
$\Rightarrow x = \pm 2$
When $x = 2$, $y = 16$
When $x = -2$, $y = -16$
$\therefore$ the required points are (2, 16) and ($-$2, $-$16).

153. $v = \frac{dx}{dt} = 4t^3 - 3kt^2$

$\therefore$ $\frac{dv}{dt} = 12t^2 - 6kt$

At $t = 2$, $\frac{dv}{dt} = 0$

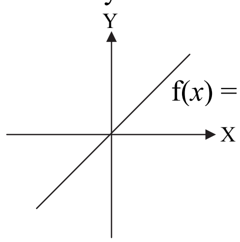
$\Rightarrow 48 - 12k = 0 \Rightarrow k = 4$

154. Since, $f(x)$ satisfies the conditions of Rolle's theorem.

$\therefore$ $f(2) = f(1)$

Now, $\int_1^2 f'(x)dx = [f(x)]_1^2 = f(2) - f(1) = 0$

155. It is always increasing.



156. $f(x) = x^3 + bx^2 + cx + d$

$\therefore$ $f'(x) = 3x^2 + 2bx + c$

Now its discriminant $= 4(b^2 - 3c)$

$\Rightarrow 4(b^2 - c) - 8c < 0$, as $b^2 < c$ and $c > 0$

$\Rightarrow f'(x) > 0$ for all $x \in R$

$\Rightarrow$ f is strictly increasing on R.

157. Since $x = 1$ and $x = 3$ are extreme points of $p(x)$.

$\therefore$ $p'(1) = 0$ and $p'(3) = 0$

$\therefore$ $(x - 1)$ and $(x - 3)$ are the factors of $p'(x)$.

$\therefore$ $p'(x) = k(x - 1)(x - 3) = k(x^2 - 4x + 3)$

$\Rightarrow p(x) = k\left(\frac{x^3}{3} - 2x^2 + 3x\right) + c$

Given, $p(1) = 6$ and $p(3) = 2$

$\Rightarrow 6 = k\left(\frac{1}{3} - 2 + 3\right) + c$ and 2

$= k(9-18+9) + c$

$\Rightarrow 6 = \frac{4k}{3} + c$ and $c = 2 \Rightarrow k = 3$

$\therefore$ $p'(x) = 3(x^2 - 4x + 3)$

$\therefore$ $p'(0) = 9$

158. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

Given, $\lim\limits_{x\to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$

$\Rightarrow \lim\limits_{x\to 0}\frac{f(x)}{x^2} = 3 - 1 = 2$

$\Rightarrow \lim\limits_{x\to 0}\frac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}{x^2} = 2$

$\therefore$ $a_0 = 0, a_1 = 0, a_2 = 2$

$\therefore$ $f(x) = 2x^2 + a_3x^3 + a_4x^4$

$\therefore$ $f'(x) = 4x + 3a_3x^2 + 4a_4x^3 = x(4 + 3a_3x + 4a_4x^2)$

Given, $f'(1) = 0$ and $f'(2) = 0$

$\Rightarrow 4 + 3a_3 + 4a_4 = 0$ ....(i)

and $4 + 6a_3 + 16a_4 = 0$ ....(ii)

Solving (i) and (ii), we get

$a_4 = \frac{1}{2}$, $a_3 = -2$

$\therefore$ $f(x) = 2x^2 - 2x^3 + \frac{x^4}{2}$

$\therefore$ $f(2) = 8 - 16 + 8 = 0$

159. tan A. tan B is maximum if $A = B = \frac{\pi}{6}$

$\therefore$ Maximum of tanA.tanB $= \frac{1}{3}$

160. According to the given condition,

$4x + 2\pi r = 2$

$\Rightarrow 2x + \pi r = 1$ ....(i)

$A = x^2 + \pi r^2 = \left(\frac{1 - \pi r}{2}\right)^2 + \pi r^2$

$\therefore$ $\frac{dA}{dr} = 2\left(\frac{1 - \pi r}{2}\right)\left(-\frac{\pi}{2}\right) + 2\pi r$

For maximum or minimum, $\frac{dA}{dr} = 0$

$\Rightarrow \pi(1 - \pi r) = 4\pi r$

$\Rightarrow 1 = 4r + \pi r$ ...(ii)

From (i) and (ii), we get

$2x + \pi r = 4r + \pi r$

$\Rightarrow x = 2r$

161. $f(x) = \tan^{-1}\sqrt{\frac{1 + \sin x}{1 - \sin x}}$

$= \tan^{-1}\sqrt{\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right) = \frac{\pi}{4} + \frac{x}{2}$

$\Rightarrow f'(x) = \frac{1}{2}$ and at $x = \frac{\pi}{6}$, $f(x) = \frac{\pi}{3}$

$\therefore$ equation of the normal at $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ is

$y - \frac{\pi}{3} = -2\left(x - \frac{\pi}{6}\right) \Rightarrow y + 2x = \frac{2\pi}{3}$

Only option (A) satisfies this equation.

Evaluation Test

1. Let $f(x) = ax^4 + bx^3 + cx^2 + dx$

$\therefore$ $f(0) = 0$

and $f(3) = a.3^4 + b.3^3 + c.3^2 + d.3$

$= 81a + 27b + 9c + 3d$

$= 3(27a + 9b + 3c + d)$

$= 3 \times 0$

$\therefore$ $f(0) = f(3) = 0$

$f(x)$ is a polynomial function, it is continuous and differentiable.

Now, $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

By Rolle's theorem, there exist at least 1 root of the equation $f'(x) = 0$ in between 0 and 3.

2. The equation of the curve is $y = x^2 + bx + c$.

$\therefore$ $\frac{dy}{dx} = 2x + b$ ....(i)

Since, the curve touches the line $y = x$ at (1,1).

$\therefore$ $[2x + b]_{(1,1)} = 1$

$\therefore$ $2(1) + b = 1$

$\Rightarrow b = -1$

Substituting the value of b in equation (i), we get

$\frac{dy}{dx} = 2x - 1$

Since, gradient is negative.

$\therefore$ $\frac{dy}{dx} < 0$

$\Rightarrow 2x - 1 < 0$

$\Rightarrow 2x < 1$

$\Rightarrow x < \frac{1}{2}$

3. The equation of the parabola is $y^2 = 8x$.

$\therefore$ $2y\frac{dy}{dx} = 8$

$\therefore$ $\frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y} = m_1$

Slope of given line, $m_2 = 3$

Since, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

$\therefore$ $\tan\frac{\pi}{4} = \left|\frac{\frac{4}{y} - 3}{1 + \frac{4}{y}\cdot 3}\right|$

$\Rightarrow 1 = \left|\frac{4 - 3y}{y + 12}\right|$

$\therefore$ $y = -2$ or $y = 8$

Putting $y = -2$ in the equation of the curve, we get

$x = \frac{1}{2}$

$\therefore$ the point of contact is $\left(\frac{1}{2}, -2\right)$.

4. $f(x) = \tan^{-1}x - \frac{1}{2}\log x$

$\therefore$ $f'(x) = \frac{1}{1+x^2} - \frac{1}{2x} = -\frac{(x-1)^2}{2x(1+x^2)}$

Now, $f'(x) = 0 \Rightarrow x = 1$

$f(1) = \tan^{-1} 1 - \frac{1}{2}\log 1 = \frac{\pi}{4} = \frac{3.14}{4} = 0.785$

Since, we are finding maxima on an interval $\left(\frac{1}{\sqrt{3}}, \sqrt{3}\right)$. We have to find the value of $f(x)$ at $\left(\frac{1}{\sqrt{3}}\right)$ and $\left(\sqrt{3}\right)$

$f\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}\frac{1}{\sqrt{3}} + \frac{1}{4}\log 3 = \frac{\pi}{6} + \frac{1}{4}\log 3$

$= \frac{3.14}{6} + \frac{1}{4}\log 3 = 0.52 + \frac{1}{4} \times 1.0986$

$= 0.52 + 0.2746 = 0.7946$

$f(\sqrt{3}) = \tan^{-1}\left(\sqrt{3}\right) - \frac{1}{4}\log 3 = \frac{\pi}{3} - \frac{1}{4}\log 3$

$= \frac{3.14}{3} - 0.2746$

$= 1.04 - 0.2746$

$= 0.7654$

$\therefore$ the greatest value of $f(x)$ is $\frac{\pi}{6} + \frac{1}{4}\log 3$.

5. $\alpha + \beta = \frac{\pi}{2}$

$\therefore$ $\cos\beta = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$

Let $y = \cos\alpha\cos\beta = \cos\alpha\sin\alpha = \frac{1}{2}\sin 2\alpha$

$\therefore$ $\frac{dy}{d\alpha} = \frac{1}{2}\cos 2\alpha.2 = \cos 2\alpha$

Now, $\frac{dy}{d\alpha} = 0 \Rightarrow \cos 2\alpha = 0 \Rightarrow 2\alpha = \frac{\pi}{2}$

$\Rightarrow \alpha = \frac{\pi}{4}$

Also, $\frac{d^2y}{d\alpha^2} = -2\sin 2\alpha = -2 < 0$

$\therefore$ y is maximum when $\alpha = \frac{\pi}{4}$

$\therefore$ it is maximum at $\beta = \frac{\pi}{4}$

6. Let $P(x_1, y_1)$ be the point on the curve at which tangent is drawn.
The equation of the curve is $xy = c^2$.

$\therefore$ $x\frac{dy}{dx} + y(1) = 0$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{y_1}{x_1}$

$\therefore$ The equation of the tangent is

$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$

$\therefore$ $yx_1 - x_1y_1 = -xy_1 + x_1y_1$

$\therefore$ $xy_1 + yx_1 = 2x_1y_1$

$\therefore$ $\frac{x}{2x_1} + \frac{y}{2y_1} = 1$

$\therefore$ The tangent meets the X-axis in the point $A(2x_1, 0)$ and the Y-axis in the point $B(0, 2y_1)$

$\therefore$ P is the mid point of AB

$\therefore$ The ratio is 1 : 1

7. Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$

$\therefore$ $f'(x) = ax^2 + bx + c$

Now, $f(1) = \frac{a}{3} + \frac{b}{2} + c + d = \frac{2a + 3b + 6c + 6d}{6}$

$\Rightarrow f(1) = \frac{0 + 6d}{6} = d$ ....[$\because 2a + 3b + 6c = 0$]

Also, $f(0) = d$

$\therefore$ $f(0) = f(1)$

$f(x)$ is a polynomial function, it is continuous and differentiable.

$\therefore$ There exists at least one value of x in (0, 1) at which $f'(x) = ax^2 + bx + c = 0$

$\therefore$ one root of the equation $ax^2 + bx + c = 0$ has value between 0 and 1.

8. $f(x) = \sin x(1 + \cos x)$
$= \sin x + \sin x \cos x$

$\therefore$ $f(x) = \sin x + \frac{1}{2}\sin 2x$

$\therefore$ $f'(x) = \cos x + \cos 2x = 2\cos\frac{x}{2}\cos\frac{3x}{2}$

$\therefore$ $f'(x) = 0 \Rightarrow \cos\frac{x}{2} = 0$ or $\cos\frac{3x}{2} = 0$

$\therefore$ $\frac{x}{2} = \frac{\pi}{2}$ or $\frac{3x}{2} = \frac{\pi}{2}$

$\therefore$ $x = \pi$ or $x = \frac{\pi}{3}$

$f''(x) = -\sin x - 2\sin 2x < 0$, only when $x = \frac{\pi}{3}$

$\therefore$ The maximum value of function is at $\frac{\pi}{3}$

$\therefore$ $f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{\sqrt{3}}{2} \times \frac{3}{2} = \frac{3\sqrt{3}}{4}$

9. $f(x) = \sin^4 x + \cos^4 x$
$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$\therefore$ $f(x) = 1 - \frac{1}{2}\sin^2 2x$

$\therefore$ $f'(x) = -\frac{1}{2}(2\sin 2x\cos 2x) \times 2$

$f'(x) = -2\sin 2x\cos 2x$

Now, $f'(x) = 0$

$\Rightarrow \sin 2x = 0$ or $\cos 2x = 0$

$\Rightarrow x = 0$ or $x = \frac{\pi}{4}$

Since, $f'(x) = -2\sin 2x\cos 2x$

$\Rightarrow f'(x) = -\sin 4x$

$\therefore$ $f''(x) = -4\cos 4x$

For $x = 0$, $f''(x) = -4 < 0$

For $x = \frac{\pi}{4}$ $f''(x) = 4 > 0$

$\therefore$ At $x = \frac{\pi}{4}$, $f(x)$ is minimum

$\therefore$ Minimum value of $f(x) = 1 - \frac{1}{2}(1) = 1 - \frac{1}{2} = \frac{1}{2}$

10. $2^{(x^2-3)^3+27}$ is minimum when $(x^2-3)^3+27$ is minimum.

Since, $(x^2 - 3)^3 + 27$
$= x^6 - 9x^4 + 27x^2$
$= x^2(x^4 - 9x^2 + 27)$
$= x^2\left[\left(x^2 - \frac{9}{2}\right)^2 + \frac{27}{4}\right] \geq 0$, for all x

$\therefore$ Minimum value of $(x^2 - 3)^3 + 27$ is 0.

$\therefore$ Minimum value of $2^{(x^2-3)^3+27} = 2^0 = 1$

11. $f(x) = 3\cos|x| - 6ax + b$

$= 3\cos x - 6ax + b$ ....[$\because \cos(-x) = \cos x$]

$\therefore$ $f'(x) = -3\sin x - 6a$

The function $f(x)$ is increasing for all $x \in R$.

$\therefore$ $f'(x) > 0$

$\therefore$ $-3\sin x - 6a > 0$

$\Rightarrow 6a < -3\sin x$

$\Rightarrow a < -\frac{1}{2}\sin x$

$\Rightarrow a < -\frac{1}{2}$

12. Let $f(x) = a^2\sec^2 x + b^2\text{cosec}^2 x$

$\therefore$ $f'(x) = a^2.2\sec x\sec x\tan x + b^2.2\,\text{cosec}\,x(-\text{cosec}\,x\cot x)$

$= 2a^2\sec^2 x\tan x - 2b^2\text{cosec}^2 x\cot x$

Now, $f'(x) = 0$

$\Rightarrow 2a^2\sec^2 x\tan x - 2b^2\text{cosec}^2 x\cot x = 0$

$$2a^2.\frac{1}{\cos^2 x}\cdot\frac{\sin x}{\cos x} = 2b^2\frac{1}{\sin^2 x}\cdot\frac{\cos x}{\sin x}$$

$\therefore$ $\frac{\sin^4 x}{\cos^4 x} = \frac{b^2}{a^2}$

$\therefore$ $\tan^4 x = \frac{b^2}{a^2}$

$\therefore$ $\tan^2 x = \frac{b}{a}$ and $\cot^2 x = \frac{a}{b}$

Also,

$$f''(x) = 2a^2\left[\sec^2 x.\sec^2 x + \tan x.2\sec x\sec x\tan x\right]$$
$$-2b^2\left[\begin{array}{l}\text{cosec}^2 x(-\text{cosec}^2 x)\\ \quad + \cot x.2\text{cosec}\,x(-\text{cosec}\,x\cot x)\end{array}\right]$$
$$= 2a^2\left[\sec^4 x + 2\sec^2 x\tan^2 x\right] + 2b^2\left[\text{cosec}^4 x + 2\text{cosec}^2 x\cot^2 x\right]$$

$\therefore$ $f''(x) > 0$ for all x.

$\therefore$ $f(x)$ is minimum when $\tan^2 x = \frac{b}{a}$

$\therefore$ Minimum value of $f(x) = a^2(1 + \tan^2 x) + b^2(1 + \cot^2 x)$

$$= a^2\left(1 + \frac{b}{a}\right) + b^2\left(1 + \frac{a}{b}\right)$$
$$= a^2\left(\frac{a+b}{a}\right) + b^2\left(\frac{a+b}{b}\right)$$
$$= a(a + b) + b(a + b) = (a + b)^2$$

13. $y = \frac{ax+b}{(x-4)(x-1)} = \frac{ax+b}{x^2-5x+4}$

$\therefore$ $\frac{dy}{dx} = \frac{(x^2-5x+4)a-(ax+b)(2x-5)}{(x^2-5x+4)^2}$

For extreme (i.e., maximum or minimum)

$\frac{dy}{dx} = 0$

$\therefore$ $a(x^2 - 5x + 4) - (ax + b)(2x - 5) = 0$

Since, y has an extreme at P(2, –1)

$\therefore$ $x = 2$ satisfies above equation

$a(4 - 10 + 4) - (2a + b)(-1) = 0$

$\Rightarrow -2a + 2a + b = 0$

$\Rightarrow b = 0$

$x = 2, y = -1$ satisfies the equation of the curve

$\therefore$ $-1 = \frac{a(2)+b}{4-10+4}$

$\therefore$ $-1 = \frac{2a+0}{-2} = -a$

$\therefore$ $a = 1$ $\quad\therefore$ $a = 1, b = 0$

14. Let $f(x) = x\tan x$

$\therefore$ $f'(x) = x\sec^2 x + \tan x$

$\therefore$ $f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ $f(x)$ is increasing in the interval $\left(0, \frac{\pi}{2}\right)$

Since, $0 < \alpha < \beta < \frac{\pi}{2}$

$\therefore$ $f(\alpha) < f(\beta)$

$\therefore$ $\alpha\tan\alpha < \beta\tan\beta$

$\Rightarrow \frac{\alpha}{\beta} < \frac{\tan\beta}{\tan\alpha}$

15. The point of intersection of the given curves is (0, 1).

Now, $y = 3^x$

$\therefore$ $\frac{dy}{dx} = 3^x\log 3$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(0,1)} = \log 3 = m_1$ (say)

Also, $y = 5^x$

$\therefore$ $\frac{dy}{dx} = 5^x\log 5$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(0,1)} = \log 5 = m_2$ (say)

$\therefore$ $\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \frac{\log 3 - \log 5}{1 + \log 3\log 5}$

16. Let $f(x) = ax^2 + bx + c$

$\therefore$ $f'(x) = 2ax + b$

since, α and β are roots of the equation $ax^2 + bx + c = 0$

$\therefore$ $f(\alpha) = f(\beta) = 0$

$\therefore$ $f(x)$ being a polynomial function in x, it is continuous and differentiable.

$\therefore$ There exists k in (α, β) such that $f'(k) = 0$

$\therefore$ $2ak + b = 0$, $\quad\therefore\quad k = -\dfrac{b}{2a}$

But $k \in [\alpha, \beta]$

$\therefore$ $\alpha < k < \beta$

$\therefore$ $\alpha < -\dfrac{b}{2a} < \beta$

17. $f(x) = \tan^{-1}(\sin x + \cos x)$

$\therefore$ $f'(x) = \dfrac{1}{1+(\sin x+\cos x)^2}\times(\cos x-\sin x)$

$$= \frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+(\sin x+\cos x)^2}$$

For $f(x)$ to be increasing, $f'(x) > 0$

$\Rightarrow \sqrt{2}\cos\left(x+\dfrac{\pi}{4}\right) > 0$

$\Rightarrow \cos\left(x+\dfrac{\pi}{4}\right) > 0$

$\Rightarrow -\dfrac{\pi}{2} < x + \dfrac{\pi}{4} < \dfrac{\pi}{2} \Rightarrow -\dfrac{3\pi}{4} < x < \dfrac{\pi}{4}$

$\therefore$ $f(x)$ is an increasing function in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$.

18. $f(x) = x^3 - 12ax^2 + 36a^2x - 4$

Diff. w.r.t. x, we get

$f'(x) = 3x^2 - 12a(2x) + 36a^2(1)$

$\quad = 3x^2 - 24ax + 36a^2$

Now, $f'(x) = 0 \Rightarrow 3x^2 - 24ax + 36a^2 = 0$

$\therefore$ $x^2 - 8ax + 12a^2 = 0$

$\therefore$ $(x - 2a)(x - 6a) = 0$

$\therefore$ $x = 2a$ or $x = 6a$

Also, $f''(x) = 6x - 24a$

$[f''(x)]_{x=2a} = 12a - 24a = -12a < 0$

$[f''(x)]_{x=6a} = 36a - 24a = 12a > 0$

$\therefore$ Maxima at $p = 2a$ and minima at $q = 6a$

$3p = q^2$ ....(given)

$\therefore$ $3 \times 2a = (6a)^2$

$\therefore$ $6a = 36a^2$

$\therefore$ $a = \dfrac{1}{6}$

19. The functions e^{-x}, $\sin x$, $\cos x$ are continuous and differentiable in their respective domains.

$\therefore$ $f(x)$ is continuous and differentiable

Also $f\left(\dfrac{\pi}{4}\right) = 0 = f\left(\dfrac{5\pi}{4}\right)$

Now,

$f'(x) = -e^{-x}(\sin x - \cos x) + e^{-x}(\cos x + \sin x)$

$\quad = e^{-x}(-\sin x + \cos x + \cos x + \sin x)$

$\quad = 2e^{-x}\cos x$

Also, $f'(x) = 0 \Rightarrow \cos x = 0$

$\therefore$ $x = \dfrac{\pi}{2} \in \left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

20. $ay^2 = x^3$ ....(i)

Diff. w.r.t.x, we get

$2ay\dfrac{dy}{dx} = 3x^2$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2}{2ay}$

$\therefore$ slope of the normal $= -\dfrac{2ay}{3x^2}$

Since, the normal to the given curve makes equal intercepts with the axis.

$\therefore$ $-\dfrac{2ay}{3x^2} = -1$

$\Rightarrow y = \dfrac{3x^2}{2a}$

Substituting $y = \dfrac{3x^2}{2a}$ in (i) and solving, we get the point $\left(\dfrac{4a}{9}, \dfrac{8a}{27}\right)$.

Textbook Chapter No.

04 Integration

Hints

Classical Thinking

1. $\int 7e^{7x+5}\,dx = 7\cdot\dfrac{e^{7x+5}}{7} + c = e^{7x+5} + c$

2. $\int (a^x - a^{2x})\,dx$

$= \dfrac{a^x}{\log a} - \dfrac{a^{2x}}{\log a}\cdot\dfrac{1}{2} + c$

$= \dfrac{1}{\log a}\left(a^x - \dfrac{a^{2x}}{2}\right) + c$

3. $\int \dfrac{2^x + 3^x}{5^x}\,dx = \int\left(\dfrac{2^x}{5^x} + \dfrac{3^x}{5^x}\right)dx$

$= \int\left(\left(\dfrac{2}{5}\right)^x + \left(\dfrac{3}{5}\right)^x\right)dx$

$= \dfrac{\left(\frac{2}{5}\right)^x}{\log\left(\frac{2}{5}\right)} + \dfrac{\left(\frac{3}{5}\right)^x}{\log\left(\frac{3}{5}\right)} + c$

4. $\int \dfrac{1}{(x-5)^2}\,dx = \dfrac{(x-5)^{-2+1}}{-2+1} + c$

$= \dfrac{(x-5)^{-1}}{-1} + c = -\dfrac{1}{(x-5)} + c$

5. $\int \dfrac{dx}{\sqrt{1-x}} = \int (1-x)^{-1/2}\,dx$

$= \dfrac{(1-x)^{\frac{-1}{2}+1}}{(-1)\left(-\frac{1}{2}+1\right)} + c$

$= -2\sqrt{1-x} + c$

6. $\int \dfrac{x^2-1}{x^3}\,dx = \int \dfrac{1}{x} - \int x^{-3}\,dx$

$= \log x + \dfrac{1}{2x^2} + c$

7. $\int \dfrac{3x^3 - 2\sqrt{x}}{x}\,dx = 3\int x^2 dx - 2\int x^{\frac{-1}{2}}dx$

$= x^3 - 4\sqrt{x} + c$

8. $\int \dfrac{ax^{-2} + bx^{-1} + c}{x^{-3}}\,dx = \int (ax + bx^2 + cx^3)\,dx$

$= \dfrac{1}{2}ax^2 + \dfrac{1}{3}bx^3 + \dfrac{1}{4}cx^4 + k$

9. $\int\left(x + \dfrac{1}{x}\right)^3 dx = \int\left(x^3 + 3x + \dfrac{3}{x} + \dfrac{1}{x^3}\right)dx$

$= \dfrac{x^4}{4} + \dfrac{3x^2}{2} + 3\log x - \dfrac{1}{2x^2} + c$

10. $\int \dfrac{1}{x^2}(2x+1)^3\,dx = \int \dfrac{(8x^3 + 12x^2 + 6x + 1)}{x^2}\,dx$

$= \int\left(8x + 12 + \dfrac{6}{x} + \dfrac{1}{x^2}\right)dx$

$= 4x^2 + 12x + 6\log x - \dfrac{1}{x} + c$

11. $\int \dfrac{3x^3 - 2\sqrt{x}}{x}\,dx = 3\int x^2 dx - 2\int x^{-1/2}dx$

$= x^3 - 4\sqrt{x} + c$

12. $\int \dfrac{(\sqrt{x} + \sqrt[3]{x^2})^2}{x}\,dx = \int \dfrac{(x^{\frac{1}{2}} + x^{\frac{2}{3}})^2}{x}\,dx$

$= \int \dfrac{1}{x}\left(x + 2x^{\frac{1}{2}}x^{\frac{2}{3}} + x^{\frac{4}{3}}\right)dx$

$= \int\left(1 + 2x^{\frac{1}{6}} + x^{\frac{1}{3}}\right)dx = x + \dfrac{2x^{\frac{7}{6}}}{\frac{7}{6}} + \dfrac{x^{\frac{4}{3}}}{\frac{4}{3}} + c$

$= x + \dfrac{12}{7}x^{\frac{7}{6}} + \dfrac{3}{4}x^{\frac{4}{3}} + c$

13. $\dfrac{(1+x)^2}{x(1+x^2)} = \dfrac{(1+x^2) + 2x}{x(1+x^2)}$

$= \dfrac{1}{x} + 2\cdot\dfrac{1}{1+x^2}$

$\therefore \int \dfrac{(1+x)^2}{x(1+x^2)}\,dx = \int \dfrac{1}{x}dx + \int \dfrac{2}{(1+x^2)}dx$

$= \log x + 2\tan^{-1}x + c$

15. $\int \dfrac{\sin^2 x - \cos^2 x}{\sin^2 x\cos^2 x}\,dx = \int \sec^2 x\,dx - \int \operatorname{cosec}^2 x\,dx$

$= \tan x + \cot x + c$

16. $\int \sqrt{1+\cos x}\ dx$

$= \int \sqrt{2\cos^2 \frac{x}{2}}\ dx$

$= \sqrt{2} \int \cos\left(\frac{x}{2}\right) dx$

$= 2\sqrt{2} \sin\left(\frac{x}{2}\right) + c$

17. $\int \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2 dx$

$= \int\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right)dx$

$= \int (1 - \sin x)\,dx = x + \cos x + c$

18. $f(x) = \int f'(x)dx$

$= \int\left(x^2 + 5\right)dx$

$= \frac{x^3}{3} + 5x + c$

$\therefore$ $f(0) = \frac{0}{3} + 0 + c$

$\Rightarrow c = -1$

$\therefore$ $f(x) = \frac{x^3}{3} + 5x - 1$

19. $f(x) = \int f'(x)\ dx = \int\left(\frac{1}{x} + x\right) dx$

$= \log x + \frac{x^2}{2} + c$

$\therefore$ $f(1) = \log 1 + \frac{1^2}{2} + c$

$\Rightarrow \frac{5}{2} = 0 + \frac{1}{2} + c \Rightarrow c = 2$

$\therefore$ $f(x) = \log x + \frac{x^2}{2} + 2$

20. Put $\cos x = t \Rightarrow -\sin x\ dx = dt$

$\therefore$ $\int \sin x \cos^4 x\ dx = \int t^4\ (-dt) = -\frac{t^5}{5} + c$

$= -\frac{\cos^5 x}{5} + c$

21. Put $(1 + \log x) = t \Rightarrow \frac{1}{x} dx = dt$

$\therefore$ $\int \frac{(1+\log x)^2}{x} dx = \int t^2 dt = \frac{t^3}{3} + c$

$= \frac{(1+\log x)^3}{3} + c$

22. Put $1 + x^2 = t \Rightarrow x\ dx = \frac{dt}{2}$

$\therefore$ $\int x\sqrt{1+x^2}\ dx = \frac{1}{2}\int t^{1/2} dt$

$= \frac{1}{2} \times \frac{t^{3/2}}{3/2} = \frac{1}{3}(1 + x^2)^{3/2} + c$

23. Put $t = \tan^{-1} x \Rightarrow dt = \frac{1}{1+x^2} dx$

$\therefore$ $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^t dt = e^t + c = e^{\tan^{-1} x} + c$

24. Put $t = 1 + \tan\ x \Rightarrow dt = \sec^2 x\ dx$

$\therefore$ $\int \frac{\sec^2 x}{1+\tan x} dx = \int \frac{1}{t} dt = \log|t| + c$

$= \log|1 + \tan x| + c$

25. Put $\log \sin x = t$

$\Rightarrow \cot x\ dx = dt$

$\therefore$ $\int \frac{\cot x}{\log \sin x} dx = \int \frac{dt}{t} = \log t + c$

$= \log(\log \sin x) + c$

26. Put $(1 + \sin^2 x) = t \Rightarrow \sin 2x\ dx = dt$

$\therefore$ $\int \frac{\sin 2x}{1+\sin^2 x} dx = \int \frac{1}{t} dt = \log t + c$

$= \log(1 + \sin^2 x) + c$

27. Let $I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\ dx$

Put $e^x + e^{-x} = t \Rightarrow (e^x - e^{-x})\ dx = dt$

$\therefore$ $I = \int \frac{dt}{t} = \log|t| + c = \log\left|e^x + e^{-x}\right| + c$

28. Put $\cos^{-1}x = t \Rightarrow -\frac{1}{\sqrt{1-x^2}} dx = dt$

$\therefore$ $\int \frac{1}{\cos^{-1} x\sqrt{1-x^2}} dx = -\int \frac{1}{t} dt = -\log|t| + c$

$= -\log|\cos^{-1}x| + c$

29. Put $x + \cos^2 x = t$

$\Rightarrow [1 + 2\cos x\,(-\sin x)]\ dx = dt$

$\Rightarrow (1 - \sin 2x)\ dx = dt$

$\therefore$ $\int \frac{1-\sin 2x}{x + \cos^2 x}\ dx = \int \frac{dt}{t} = \log|t| + c$

$= \log|x + \cos^2 x| + c$

30. Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt$

$\therefore$ $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = 2\int \cos t\, dt$

$= 2\sin t + c$

$= 2\sin\sqrt{x} + c$

31. Put $e^{\sqrt{x}} = t \Rightarrow \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2\ dt$

$\therefore \int \frac{e^{\sqrt{x}} \cos\left(e^{\sqrt{x}}\right)}{\sqrt{x}} dx$

$= 2\int \cos t\, dt$

$= 2 \sin t + c = 2 \sin\left(e^{\sqrt{x}}\right) + c$

32. Put $a^x = t \Rightarrow a^x\, dx = \frac{dt}{\log a}$

$\therefore \int \frac{1}{\log a}(a^x \cos a^x)\, dx = \int \frac{1}{(\log a)^2} \cos t\, dt$

$= \frac{1}{(\log a)^2} \sin t + c$

$= \frac{1}{(\log a)^2} \sin a^x + c$

33. Put $\log x = t \Rightarrow \frac{1}{x}\ dx = dt$

$\therefore \int \frac{\tan(\log x)}{x} dx = \int \tan(t)\ dt = \log|\sec(t)| + c$

$= \log|\sec(\log x)| + c$

34. Put $\log x = t$

$\Rightarrow \frac{1}{x}\ dx = dt$

$\therefore \int \frac{\sec^2(\log x)}{x}\ dx = \int \sec^2 t\ dt = \tan t + c$

$= \tan(\log x) + c$

35. Let $I = \int \frac{\log(\log x)}{x \log x} dx$

Put $\log(\log x) = t \Rightarrow \frac{1}{x \log x} dx = dt$

$\therefore I = \int t\,dt = \frac{t^2}{2} + c = \frac{[\log(\log x)]^2}{2} + c$

36. Let $I = \int \sec x \cdot \log(\sec x + \tan x) dx$

Put $\log(\sec x + \tan x) = t$

$\Rightarrow \sec x\ dx = dt$

$\therefore I = \int t\ dt = \frac{t^2}{2} + c = \frac{1}{2}[\log(\sec x + \tan x)]^2 + c$

37. $\int \frac{dx}{\sqrt{1-16x^2}} = \int \frac{dx}{\sqrt{1-(4x)^2}}$

$= \frac{1}{4}\sin^{-1}(4x) + c$

38. $\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$

$= \sin^{-1}(x - 1) + c$

39. $\int \frac{dx}{x^2 - 2x + 2}$

$= \int \frac{dx}{x^2 - 2x + 1 + 2 - 1}$

$= \int \frac{dx}{(x-1)^2 + 1}$

$= \tan^{-1}\left(\frac{x-1}{1}\right) + c = \tan^{-1}(x - 1) + c$

40. $\int \frac{dx}{x^2 + 4x + 13} = \int \frac{dx}{(x+2)^2 + 3^2}$

$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c$

41. Let $I = \int \frac{1}{9x^2 - 25} dx = \frac{1}{9}\int \frac{dx}{x^2 - \left(\frac{5}{3}\right)^2}$

$= \frac{1}{9} . \frac{1}{2\left(\frac{5}{3}\right)} . \log\left|\frac{x - \frac{5}{3}}{x + \frac{5}{3}}\right| + c$

$= \frac{1}{30}\log\left|\frac{3x-5}{3x+5}\right| + c$

42. Integrating by parts, taking x as the first function.

$\therefore \int x \cos x dx = x\int \cos x dx - \int\left[\frac{d}{dx}(x)\int \cos x dx\right]dx$

$= x \sin x - \int 1.\sin x\, dx = x \sin x - \int \sin x\, dx$

$= x \sin x + \cos x + c$

43. $\int x e^x dx = x\int e^x dx - \int\left[\frac{d}{dx}(x)\int e^x dx\right]dx$

$= x.e^x - \int 1.e^x dx$

$= xe^x - e^x + c = e^x(x - 1) + c$

44. $\int \frac{x}{\cos^2 x} dx = \int x \sec^2 x\, dx$

$= x \tan x - \int 1.\tan x dx$

$= x \tan x + \log|\cos x| + c$

45. $\int x \sin 2x\,dx$

$= x\int \sin 2x\,dx - \int\left[\frac{d}{dx}(x)\int \sin 2x\,dx\right]dx$

$= x\left(-\frac{\cos 2x}{2}\right) - \int 1.\left(-\frac{\cos 2x}{2}\right)dx$

$= -\frac{x\cos 2x}{2} + \frac{1}{2}.\frac{\sin 2x}{2} + c$

$= \frac{\sin 2x}{4} - \frac{x\cos 2x}{2} + c$

46. $\int x^2 \log x\,dx$

$= \log x\,.\,\frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}dx$

$= \frac{1}{3}x^3.\log x - \frac{1}{3}\left[\frac{x^3}{3}\right] + c$

$= \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + c$

47. $\int (x-1)\,e^{-x}\,dx = \frac{(x-1)e^{-x}}{-1} - \int \frac{e^{-x}}{-1}dx$

$= -xe^{-x} + e^{-x} - e^{-x} + c$

$= -xe^{-x} + c$

48. Let $I = \int e^x \sin x\,dx$

$= \sin x.e^x - \int \cos x.e^x dx$

$= e^x \sin x - \cos x.e^x + \int(-\sin x).e^x dx$

$= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

$\therefore$ $I = e^x \sin x - e^x \cos x - I$

$\therefore$ $2I = e^x(\sin x - \cos x) + c$

$\therefore$ $I = \frac{e^x}{2}(\sin x - \cos x) + c$

49. $\int e^x(\sin x + \cos x)\,dx = e^x \sin x + c$

....$[\because \int e^x[f(x)+f'(x)]dx = e^x f(x) + c]$

50. $\int e^x(\sec x + \sec x\tan x)\,dx = e^x \sec x + c$

....$\left[\because \int e^x\left[f(x)+f'(x)\right]dx = e^x f(x) + c\right]$

51. $\int \frac{dx}{x - x^2} = \int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx$

$= \log x - \log(1-x) + c$

52. $\int \frac{dx}{(x+1)(x+2)} = \int\left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx$

$= \log|x+1| - \log|x+2| + c$

$= \log\left|\frac{x+1}{x+2}\right| + c$

53. Let $\frac{x-1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$

$\therefore$ $x - 1 = A(x-2) + B(x-3)$ ....(i)

Putting $x = 2$ in (i), we get

$B = -1$

Putting $x = 3$ in (i), we get

$A = 2$

$\therefore$ $\int \frac{x-1}{(x-3)(x-2)}dx = \int\left(\frac{2}{x-3} - \frac{1}{x-2}\right)dx$

$= 2\log|x-3| - \log|x-2| + c$

$= \log|(x-3)^2| - \log|x-2| + c$

54. $\int \frac{x}{(x-2)(x-1)}dx = -\int \frac{1}{x-1}dx + \int \frac{2}{x-2}dx$

$= -\log|x-1| + 2\log|x-2| + p$

$= \log\left|\frac{(x-2)^2}{(x-1)}\right| + p$

55. $\int \frac{dx}{x^4 + 5x^2 + 4} = \int \frac{dx}{(x^2+1)(x^2+4)}$

$= \frac{1}{3}\int\left[\frac{1}{x^2+1} - \frac{1}{x^2+4}\right]dx$

$= \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right) + c$

Critical Thinking

1. $\int \frac{(x^3+3x^2+3x+1)}{(x+1)^5}dx = \int \frac{(x+1)^3}{(x+1)^5}dx$

$= \int \frac{1}{(x+1)^2}dx = \int (x+1)^{-2}dx$

$= -\frac{1}{x+1} + c$

2. $\int(1 + 2x + 3x^2 + 4x^3 + ...)dx$

$= \int(1-x)^{-2}dx$

$= (1-x)^{-1} + c$

3. $\int\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)dx = \int e^x dx = e^x + c$

4. Rationalizing the denominator, we get

$$\int \frac{dx}{\sqrt{x+3}-\sqrt{x+2}}$$

$$= \int \frac{\sqrt{x+3}+\sqrt{x+2}}{(\sqrt{x+3}-\sqrt{x+2})(\sqrt{x+3}+\sqrt{x+2})}\ dx$$

$$= \int \frac{\sqrt{x+3}+\sqrt{x+2}}{(x+3)-(x+2)}\ dx = \int \frac{\sqrt{x+3}+\sqrt{x+2}}{x+3-x-2}\ dx$$

$$= \int \{(x+3)^{\frac{1}{2}}+(x+2)^{\frac{1}{2}}\}\ dx$$

$$= \frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{3}\left[(x+3)^{\frac{3}{2}} + (x+2)^{\frac{3}{2}}\right] + c$$

5. $$\int \frac{x-1}{(x+1)^2}dx = \int \frac{x+1-2}{(x+1)^2}dx$$

$$= \int \frac{1}{x+1}dx - \int \frac{2}{(x+1)^2}dx$$

$$= \log|x+1| + \frac{2}{(x+1)} + c$$

6. Since, the degree of the N[r] is more than degree of the D[r], divide the N[r] by D[r].

$$\begin{array}{r|l} & x^2-1 \\ \hline x^2+1 & x^4 \quad +1 \\ & x^4 + x^2 \\ & - \quad - \\ \hline & -x^2+1 \\ & -x^2-1 \\ & + \quad + \\ \hline & 2 \end{array}$$

$\therefore$ $$\int \frac{x^4+1}{x^2+1}\ dx = \int\left(x^2-1+\frac{2}{x^2+1}\right)dx$$

$$= \frac{x^3}{3} - x + 2\tan^{-1}x + c$$

7. $$\int \frac{5(x^6+1)}{x^2+1}dx = \int \frac{5(x^2+1)(x^4-x^2+1)}{(x^2+1)}dx$$

$$= 5\int (x^4-x^2+1)\,dx$$

$$= x^5 - \frac{5}{3}x^3 + 5x + c$$

8. $$\int 2^x\, 3^{x+1} 4^{x+2} dx = 16 \times 3 \int 2^x\, 3^x\, 4^x\, dx$$

$$= 48\int (24)^x\, dx = \frac{48(24)^x}{\log 24} + c$$

$$= \frac{2^x 3^{x+1} 4^{x+2}}{\log 2 + \log 4 + \log 3} + c$$

9. $$f(x) = \frac{1}{1-x}$$

$\therefore$ $$f(f(f(x))) = f\left(f\left(\frac{1}{1-x}\right)\right)$$

$$= f\left(\frac{1}{1-\left(\frac{1}{1-x}\right)}\right)$$

$$= f\left(\frac{1-x}{-x}\right)$$

$$= \frac{1}{1-\left(\frac{1-x}{-x}\right)} = \frac{1}{1+\frac{1-x}{x}} = x$$

$\therefore$ Required integral $= \dfrac{x^2}{2} + c$

10. Since, $a^{\log_a m} = m$

$\therefore$ $$\int 9^{\log_3(\sec x)} dx$$

$$= \int \sec^2 x\ dx \qquad \ldots\left[\begin{array}{l} \because 3^{2\log_3(\sec x)} \\ = 3^{\log_3(\sec x)^2} \\ = (\sec x)^2 \end{array}\right]$$

$$= \tan x + c$$

11. $$\int (e^{a\log x} + e^{x\log a})\ dx = \int (e^{\log_e x^a} + e^{\log_e a^x})\ dx$$

$$= \int (x^a + a^x)\,dx$$

$$= \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + c$$

12. Since, $\sec^2 x \,.\, \text{cosec}^2 x = \sec^2 x + \text{cosec}^2 x$

$\therefore$ $$\int \sec^2 x\,.\text{cosec}^2 x\ dx = \int \sec^2 x\,dx + \int \text{cosec}^2 x\,dx$$

$$= \tan x - \cot x + c$$

13. $$\int (\sin^{-1}x + \cos^{-1}x)\,dx = \int\left(\frac{\pi}{2}\right)dx$$

$$\ldots\left[\because\ \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$$= \frac{\pi x}{2} + c = x(\cos^{-1}x + \sin^{-1}x) + c$$

14. $\int \sin^{-1}(\cos x)dx = \int \left\{\frac{\pi}{2} - \cos^{-1}(\cos x)\right\} dx$

$$= \frac{\pi}{2}x - \frac{x^2}{2} = \frac{\pi x - x^2}{2}$$

15. $\int (\cos x - \sin x)\,dx$

$= \sin x + \cos x + c$

$= \sqrt{2}\left(\frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}}\right) + c$

$= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) + c$

$\therefore \quad \alpha = \frac{\pi}{4}$

16. $\int \sin 3x \; \cos 4x \; dx$

$= \frac{1}{2} \int 2\sin 3x \cos 4x \; dx$

$= \frac{1}{2} \int [\sin(3x + 4x) + \sin(3x - 4x)]\,dx$

$[\because 2 \sin A \cos B = \sin(A + B) + \sin(A - B)]$

$= \frac{1}{2}\left(\int \sin 7x\,dx - \int \sin x\,dx\right)$

$= \frac{1}{2}\left(\frac{-\cos 7x}{7} + \cos x\right) + c$

$= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + c$

17. $\int \frac{dx}{\tan x + \cot x} = \int \frac{dx}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}$

$$= \frac{1}{2}\int 2\sin x \cos x\,dx = \frac{-\cos 2x}{4} + c$$

18. $\int 2\sin x.\cos x \; dx$

$= \int \sin 2x \; dx = -\frac{\cos 2x}{2} + c_1$

$= -\frac{(1 - 2\sin^2 x)}{2} + c_1$

$= -\frac{1}{2} + \sin^2 x + c_1$

$= \sin^2 x + c$, where $c = \frac{-1}{2} + c_1$

19. $\int \sqrt{1 + \sin 2x} \; dx$

$= \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x} \; dx$

$\ldots.[\because \sin^2 x + \cos^2 x = 1]$

$= \int \sqrt{(\cos x + \sin x)^2} \; dx$

$= \int (\cos x + \sin x)\,dx$

$= \int \cos x\,dx + \int \sin x\,dx$

$= \sin x - \cos x + c$

20. $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx = \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx$

$$= \int dx = x + c$$

21. $\sin^2 2x = (2 \sin x \,.\, \cos x)^2$

$= 4 \sin^2 x \,.\, \cos^2 x$

$\therefore \quad 4 \int \frac{\sin^3 x + \cos^3 x}{\sin^2 2x} dx$

$= 4\int \frac{\sin^3 x + \cos^3 x}{4\sin^2 x.\cos^2 x} \; dx$

$= \int \frac{\sin x}{\cos^2 x} dx + \int \frac{\cos x}{\sin^2 x} \; dx$

$= \int \tan x \,.\, \sec x \; dx + \int \cot x \,.\, \operatorname{cosec} x \; dx$

$= \sec x - \operatorname{cosec} x + c$

22. $\int \frac{dx}{\cos 2x + \sin^2 x} = \int \frac{dx}{\cos^2 x - \sin^2 x + \sin^2 x}$

$= \int \frac{dx}{\cos^2 x} = \int \sec^2 x\,dx = \tan x + c$

23. $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx = \int \frac{2(\cos^2 x - \cos^2 \alpha)}{\cos x - \cos \alpha} dx$

$= 2\int (\cos x + \cos \alpha)\,dx$

$= 2(\sin x + x \cos \alpha) + c$

24. Since, $1 + \cos 2x = 2 \cos^2 x$

$\therefore \quad \int \sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos 8x}}} \; dx$

$= \int \sqrt{2 + \sqrt{2 + 2\cos 4x}} \; dx$

$= \int \sqrt{2 + 2\cos 2x} \; dx$

$= \int 2\cos x \; dx = 2 \sin x + c$

25. $\sin^4 x - \cos^4 x = (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$

$= -(\cos^2 x - \sin^2 x)(1)$

$= -\cos 2x$

$\therefore \quad \int (\sin^4 x - \cos^4 x)\,dx = -\int \cos 2x \; dx$

$= -\frac{\sin 2x}{2} + c$

$= -\sin x \cos x + c$

26. $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$

$= \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x} dx$

$= \int (\sin^4 x - \cos^4 x) dx$

$= \int (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) dx$

$= \int (\sin^2 x - \cos^2 x) dx = \int -\cos 2x \, dx$

$= -\frac{\sin 2x}{2} + c$

27. $\int \tan^{-1}\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right) dx = \int \tan^{-1}\left(\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right) dx$

$= \int \tan^{-1}(\tan x) dx$

$= \int x \, dx = \frac{x^2}{2} + c$

28. $\int \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right) dx$

$= \int \tan^{-1}\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right) dx$

$= \int \tan^{-1}\left(\tan\frac{x}{2}\right) dx$

$= \int \frac{x}{2} dx$

$= \frac{1}{2} \cdot \frac{x^2}{2} + c$

$= \frac{x^2}{4} + c$

29. $\int \frac{\cos 4x + 1}{\cot x - \tan x} dx = \int \frac{(\cos 4x + 1)(\sin x \cos x)}{\cos^2 x - \sin^2 x} dx$

$= \int \frac{2\cos^2 2x(\sin x \cos x)}{\cos 2x} dx$

$= \int \cos 2x \, (2 \sin x \cos x) dx$

$= \int \cos 2x \sin 2x \, dx$

$= \frac{1}{2} \int \sin 4x \, dx$

$= \frac{-1}{8} \cos 4x + c$

$\therefore$ A $= \frac{-1}{8}$

30. $\int \cos\frac{x}{16} \cos\frac{x}{8} \cos\frac{x}{4} \sin\frac{x}{16} dx$

$= \int \cos\frac{x}{16} \sin\frac{x}{16} \cos\frac{x}{8} \cos\frac{x}{4} dx$

$= \frac{1}{2} \int \sin\frac{x}{8} \cos\frac{x}{8} \cos\frac{x}{4} dx$

....[$\because \sin 2\theta = 2 \sin\theta \cos\theta$]

$= \frac{1}{4} \int \sin\frac{x}{4} \cos\frac{x}{4} dx$

$= \frac{1}{8} \int \sin\frac{x}{2} dx$

$= \left(\frac{2}{8}\right)\left(-\cos\frac{x}{2}\right) + c$

$= \frac{-1}{4} \cos\frac{x}{2} + c$

31. $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x\cos^2 x} dx$

$= \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x\cos^2 x} dx$

....$\left[\begin{array}{l}\because a^3 + b^3 = (a+b)^3 - 3ab(a+b) \\ \text{and} \sin^6 x = (\sin^2 x)^3\end{array}\right]$

$= \int \frac{1 - 3\sin^2 x\cos^2 x(1)}{\sin^2 x\cos^2 x} dx$

$= \int \left(\frac{1}{\sin^2 x\cos^2 x} - 3\right) dx$

$= \int \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x} - 3\right) dx$

$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} - 3\right) dx$

$= \int \sec^2 x \, dx + \int \text{cosec}^2 x \, dx - 3\int dx$

$= \tan x - \cot x - 3x + c$

32. $\int \frac{\cot x \tan x}{\sec^2 x - 1} dx = \int \frac{1}{\tan^2 x} dx$

$= \int \cot^2 x \, dx$

$= \int (\text{cosec}^2 x - 1) \, dx$

$= -\cot x - x + c$

33. $\int \frac{1 + \cos^2 x}{\sin^2 x} dx = \int (\text{cosec}^2 x + \cot^2 x) dx$

$= \int (2\text{cosec}^2 x - 1) dx$

....[$\because 1 + \cot^2\theta = \text{cosec}^2\theta$]

$= -2 \cot x - x + c$

34. $2\int \frac{1+\cos 4x}{1-\cos 4x}\, dx = 2\int \frac{\cos^2 2x}{\sin^2 2x} dx$

$= 2 \int \cot^2 2x\, dx$

$= 2 \int (\operatorname{cosec}^2 2x - 1)\, dx$

$= 2\left(-\frac{\cot 2x}{2}\right) - 2x + c$

$= -\cot 2x - 2x + c$

35. $\int \left(\frac{1+\tan x}{1-\tan x}\right)^2 dx = \int \left(\tan\left(\frac{\pi}{4}+x\right)\right)^2 dx$

$= \int \left(\sec^2\left(\frac{\pi}{4}+x\right) - 1\right) dx$

$= \tan\left(\frac{\pi}{4}+x\right) - x + c$

36. $\int (\sec x + \tan x)^2\, dx$

$= \int (\sec^2 x + \tan^2 x + 2\sec x \tan x)\, dx$

$= \int \left(2\sec^2 x - 1 + 2\sec x \tan x\right) dx$

$= 2\tan x + 2\sec x - x + c$

$= 2(\sec x + \tan x) - x + c$

37. $\int \frac{\tan x}{(\sec x + \tan x)} dx$

$= \int \frac{\tan x(\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)} dx$

$= \int \frac{\tan x(\sec x - \tan x)}{(\sec^2 x - \tan^2 x)} dx$

$= \int (\sec x \tan x - \tan^2 x)\, dx$

$= \int \sec x \tan x\, dx - \int (\sec^2 x - 1)\, dx$

$= \int \sec x \tan x\, dx - \int \sec^2 x\, dx + \int 1\, dx$

$= \sec x - \tan x + x + c$

38. Put $t = 3x - 5 \Rightarrow dt = 3dx$

$\therefore \int \tan(3x-5)\sec(3x-5)\, dx = \frac{1}{3}\int \tan t . \sec t\, dt$

$= \frac{\sec t}{3} + c = \frac{\sec(3x-5)}{3} + c$

39. Put $f(x) = t \Rightarrow f'(x)\, dx = dt$

$\therefore \int \frac{f'(x)}{[f(x)]^2} dx = \int \frac{1}{t^2} dt = -\frac{1}{t} + c = -\frac{1}{f(x)} + c$

40. Put $x^{10} + 10^x = t \Rightarrow (10x^9 + 10^x \log_e 10)dx = dt$

$\therefore \int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx = \int \frac{1}{t} dt = \log t + c$

$= \log(x^{10} + 10^x) + c$

41. Put $x^2 - 4x + 3 = t$

$\Rightarrow (2x-4)dx = dt \Rightarrow (x-2)dx = \frac{1}{2}dt$

$\therefore \int \frac{x-2}{x^2-4x+3} dx = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log t + c$

$= \frac{1}{2}\log(x^2 - 4x + 3) + c$

$= \log\left(\sqrt{x^2-4x+3}\right) + c$

42. Put $5x^7 = t$

$\Rightarrow 35x^6\, dx = dt$

$\Rightarrow x^6\, dx = \frac{dt}{35}$

$\therefore \int x^6 \sin(5x^7)\, dx = \int \sin t \cdot \frac{dt}{35}$

$= -\frac{\cos t}{35} = \frac{-\cos(5x^7)}{35}$

$\therefore k = -\frac{1}{7}$

43. $\int \frac{\sin x}{\sin(x-\alpha)} dx = \int \frac{\sin(x-\alpha+\alpha)}{\sin(x-\alpha)} dx$

$= \int \frac{\{\sin(x-\alpha)\cos\alpha + \cos(x-\alpha)\sin\alpha\}}{\sin(x-\alpha)} dx$

$= \int \cos\alpha\, dx + \int \sin\alpha \cot(x-\alpha)\, dx$

$= x\cos\alpha + \sin\alpha . \log|\sin(x-\alpha)| + c$

44. $\int \frac{\cos(x+\alpha)}{\cos x} dx$

$= \int \left[\frac{\cos x \cos\alpha - \sin x \sin\alpha}{\cos x}\right] dx$

$= \int (\cos\alpha - \sin\alpha \tan x)\, dx$

$= (\cos\alpha)\, x - \sin\alpha \log|\sec x| + c$

45. $\int \frac{1}{\sqrt{1+\cos x}} dx = \int \frac{dx}{\sqrt{2\cos^2\left(\frac{x}{2}\right)}} = \frac{1}{\sqrt{2}}\int \sec\frac{x}{2} dx$

$= \frac{1}{\sqrt{2}}\left\{\log\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right|\right\} . \frac{1}{1/2} + c$

$= \sqrt{2}\log\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + c$

46. $\int \frac{dx}{4\cos^3 2x - 3\cos 2x} = \int \frac{dx}{\cos 6x} = \int \sec 6x\, dx$

$= \frac{1}{6}\log|\sec 6x + \tan 6x| + c$

47. $\int \frac{dx}{\sin x+\sqrt{3}\cos x} = \frac{1}{2}\int \frac{dx}{\frac{\sin x}{2}+\frac{\sqrt{3}}{2}\cos x}$

$= \frac{1}{2}\int \frac{dx}{\sin x\cos\frac{\pi}{3}+\sin\frac{\pi}{3}\cos x}$

$= \frac{1}{2}\int \frac{dx}{\sin\left(x+\frac{\pi}{3}\right)}$

$= \frac{1}{2}\int \operatorname{cosec}\left(x+\frac{\pi}{3}\right)dx$

$= \frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right|+c$

48. $\int \frac{\sin 2x}{\sin 3x\sin 5x}dx = \int \frac{\sin(5x-3x)}{\sin 3x.\sin 5x}dx$

$= \int \frac{\sin 5x\cos 3x-\cos 5x\,\sin 3x}{\sin 3x\sin 5x}dx$

$= \int \cot 3x\,dx - \int \cot 5x\,dx$

$= \frac{1}{3}\log|\sin 3x| - \frac{1}{5}\log|\sin 5x| + c$

49. Let $I = \int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}dx$

$= \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x}{\sqrt{1-x^2}}dx$

In 2nd integral, put $1-x^2 = t \Rightarrow -2x\,dx = dt$

$\therefore$ $I = \int \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2}\int \frac{dt}{t^{\frac{1}{2}}}$

$= \sin^{-1}x - \sqrt{t} + c$

$= \sin^{-1}x - \sqrt{1-x^2} + c$

50. Put $t = x + \log x \Rightarrow dt = \left(1+\frac{1}{x}\right)dx$

$\therefore$ $\int \frac{(x+1)(x+\log x)^2}{x}dx = \int t^2\,dt = \frac{t^3}{3}+c$

$= \frac{1}{3}(x+\log x)^3 + c$

51. Put $\log(\log x) = t$

$\Rightarrow \frac{1}{x\log x}dx = dt$

$\therefore$ $\int \frac{dx}{x\log x\log(\log x)} = \int \frac{dt}{t} = \log|t| + c$

$= \log|\log(\log x)| + c$

52. Let $I = \int \frac{1}{x^3}[\log x^x]^2\,dx = \int \frac{1}{x^3}[x\log x]^2\,dx$

$= \int \frac{1}{x}(\log x)^2\,dx$

Put $\log x = t \Rightarrow \frac{1}{x}\,dx = dt$

$\therefore$ $I = \int t^2\,dt = \frac{t^3}{3} + c = \frac{(\log x)^3}{3} + c$

53. $\frac{dI}{dy} = 3^{\cos y}\sin y$

$\therefore$ $I = \int 3^{\cos y}\,\sin y\cdot dy$

Put $\cos y = t$

$\Rightarrow -\sin y\,dy = dt$ $\Rightarrow \sin y\,dy = -dt$

$\therefore$ $I = -\int 3^t\,dt = \frac{-3^t}{\log 3} + c = \frac{-3^{\cos y}}{\log 3} + c$

54. Put $a^x = t$

$\Rightarrow a^x\log a\,dx = dt \Rightarrow a^x\,dx = \frac{1}{\log a}\,dt$

$\therefore$ $\int a^{a^x}.a^x\,dx = \frac{1}{\log a}\int a^t dt = \frac{1}{\log a}.a^t.\frac{1}{\log a}+c$

$= \frac{a^{a^x}}{(\log a)^2}+c$

55. Put $2e^{-x}+5 = t \Rightarrow -2e^{-x}\,dx = dt$

$\therefore$ $\int e^{-x}\operatorname{cosec}^2\left(2e^{-x}+5\right)dx = -\frac{1}{2}\int \operatorname{cosec}^2 t\,dt$

$= \frac{1}{2}\cot t + c$

$= \frac{1}{2}\cot(2e^{-x}+5) + c$

56. Let $I = \int \frac{dx}{1+e^x} = \int \frac{e^{-x}}{1+e^{-x}}dx$

Put $1+e^{-x} = t \Rightarrow e^{-x}\,dx = -dt$

$\therefore$ $I = -\int \frac{dt}{t} = -\log|t| + c = -\log|1+e^{-x}| + c$

57. Let $I = \int \frac{1}{\left(e^{2x}+e^{-2x}\right)^2}dx$

$= \int \frac{e^{4x}}{\left(e^{4x}+1\right)^2}dx$

Put $e^{4x}+1 = t \Rightarrow 4\,e^{4x}\,dx = dt$

$\therefore$ $I = \frac{1}{4}\int \frac{1}{t^2}dt$

$= \frac{1}{4}\left(\frac{-1}{t}\right) + c = \frac{-1}{4\left(e^{4x}+1\right)} + c$

58. Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$\therefore$ $$\int \frac{\log x}{(1+\log x)^2}dx = \int \frac{t}{(1+t)^2}e^t\, dt$$
$$= \int e^t\left[\frac{t+1-1}{(1+t)^2}\right]dt$$
$$= \int e^t\left[\frac{1}{1+t} - \frac{1}{(1+t)^2}\right]dt$$
$$= \frac{e^t}{1+t} + c$$
$$= \frac{x}{1+\log x} + c$$

59. Put $x\, e^x = t$

$\Rightarrow (e^x + xe^x)\, dx = dt \qquad \Rightarrow e^x\,(1+x)\, dx = dt$

$\therefore$ $$\int \frac{e^x(1+x)}{\sin(xe^x)}\, dx = \int \frac{dt}{\sin t} = \int \operatorname{cosec} t\, dt$$
$$= \log\left|\tan\left(\frac{t}{2}\right)\right| + c$$
$$= \log\left|\tan\left(\frac{xe^x}{2}\right)\right| + c$$

60. Put $\log\left(\tan\frac{x}{2}\right) = t$

$$\Rightarrow \frac{1}{\tan\frac{x}{2}}.\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt \qquad \Rightarrow \operatorname{cosec} x\, dx = dt$$

$\therefore$ $$\int \frac{\operatorname{cosec} x}{\log\left(\tan\frac{x}{2}\right)}dx = \int \frac{1}{t}dt = \log|t| + c$$
$$= \log\left|\log\left(\tan\frac{x}{2}\right)\right| + c$$

61. Put $\tan^{-1}(x^3) = t$

$$\Rightarrow \frac{1}{1+\left(x^3\right)^2}.3x^2dx = dt \Rightarrow \frac{x^2}{1+x^6}\, dx = \frac{dt}{3}$$

$\therefore$ $$\int \frac{x^2\tan^{-1}\left(x^3\right)}{1+x^6}\, dx = \frac{1}{3}\int t\, dt = \frac{1}{3}.\frac{t^2}{2} + c$$
$$= \frac{\left(\tan^{-1}x^3\right)^2}{6} + c$$

62. Put $\tan\sqrt{x} = t \Rightarrow \dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}dx = dt$

$\therefore$ $$\int \frac{1}{\sqrt{x}}\tan^4\sqrt{x}.\sec^2\sqrt{x}\, dx = 2\int t^4\, dt$$
$$= \frac{2t^5}{5} + c = \frac{2}{5}\tan^5\sqrt{x} + c$$

63. Put $e^x = t \Rightarrow e^x\, dx = dt$

$\therefore$ $$\int e^x\tan^2(e^x)\, dx = \int \tan^2 t\, dt = \int (\sec^2 t - 1)\, dt$$
$$= \tan t - t + c$$
$$= \tan(e^x) - e^x + c$$

64. Let $$I = \int \frac{x^2+1}{x(x^2-1)}dx = \int \frac{1+\frac{1}{x^2}}{x - \frac{1}{x}}dx$$

Put $x - \dfrac{1}{x} = t \Rightarrow \left(1 + \dfrac{1}{x^2}\right)dx = dt$

$\therefore$ $$I = \int \frac{dt}{t} = \log t + c = \log\left(x - \frac{1}{x}\right) + c$$
$$= \log\left(\frac{x^2-1}{x}\right) + c$$

65. Let $$I = \int \frac{(x^4 - x)^{\frac{1}{4}}dx}{x^5}$$
$$= \int x\cdot\frac{\left(1 - \frac{1}{x^3}\right)^{\frac{1}{4}}}{x^5}\, dx = \int \frac{\left(1 - \frac{1}{x^3}\right)^{\frac{1}{4}}}{x^4}\, dx$$

Put $1 - \dfrac{1}{x^3} = t \Rightarrow \dfrac{3}{x^4}\, dx = dt$

$\therefore$ $$I = \int t^{\frac{1}{4}}.\frac{dt}{3} = \frac{t^{\frac{5}{4}}}{\frac{5}{4}}.\frac{1}{3} + c$$
$$= \frac{4}{15}\, t^{\frac{5}{4}} + c = \frac{4}{15}\left(1 - \frac{1}{x^3}\right)^{\frac{5}{4}} + c$$

66. Let $$I = \int \frac{dx}{x(x^7+1)} = \int \frac{dx}{x^8\left(1 + \frac{1}{x^7}\right)}$$

Put $1 + \dfrac{1}{x^7} = t \Rightarrow \dfrac{-7}{x^8}dx = dt$

$\therefore$ $$I = \frac{-1}{7}\int \frac{dt}{t} = \frac{-1}{7}\log|t| + c$$
$$= -\frac{1}{7}\log\left|\frac{x^7+1}{x^7}\right| + c$$
$$= \frac{1}{7}\log\left|\frac{x^7}{x^7+1}\right| + c$$

67. $\int \frac{1}{x(x^n+1)}\,dx = \int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)}$

$= \frac{-1}{n}\int \frac{-nx^{-n-1}}{(1+x^{-n})}\,dx$

$= \frac{-1}{n}\log\left|1+\frac{1}{x^n}\right| + c$

$= \frac{-1}{n}\log\left|\frac{x^n+1}{x^n}\right| + c$

$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + c$

68. Let I = $\int \frac{1}{\cos^2 x(1-\tan x)^2}dx = \int \frac{\sec^2 x\,dx}{(\tan x-1)^2}$

Put $\tan x - 1 = t \Rightarrow \sec^2 x\,dx = dt$

$\therefore$ I = $\int \frac{1}{t^2}dt = -\frac{1}{t} + c = \frac{-1}{\tan x - 1} + c = \frac{1}{1-\tan x} + c$

69. Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

$\therefore \int \frac{1}{x^2\sqrt{1+x^2}}dx = \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sec\theta}$

$= \int \text{cosec}\,\theta\cot\theta\,d\theta = -\text{cosec}\,\theta + c$

$= \frac{-\sqrt{\tan^2\theta+1}}{\tan\theta} + c = \frac{-\sqrt{x^2+1}}{x} + c$

70. $\int \frac{x^3}{\sqrt{1+x^4}}dx = \frac{1}{4}\int \frac{4x^3}{\sqrt{1+x^4}}dx$

$= \frac{1}{4}\int \frac{dt}{t^{1/2}}$

….[Put $1 + x^4 = t \Rightarrow 4x^3\,dx = dt$]

$= \frac{1}{4}\cdot\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{1}{2}\sqrt{t} + c = \frac{1}{2}\sqrt{1+x^4} + c$

71. Let I = $\int \frac{\sin 2x}{\sin^2 x + 2\cos^2 x}dx = \int \frac{2\sin x\cos x}{1+\cos^2 x}dx$

Put $1 + \cos^2 x = t \Rightarrow -2\sin x\cos x\,dx = dt$

$\therefore$ I = $\int -\left(\frac{dt}{t}\right) = -\log|t| + c$

$= -\log|1 + \cos^2 x| + c$

72. Let I = $\int \frac{\cos 2x}{(\cos x+\sin x)^2}dx$

$= \int \frac{(\cos x-\sin x)(\cos x+\sin x)}{(\cos x+\sin x)^2}dx$

$= \int \frac{\cos x-\sin x}{\cos x+\sin x}dx$

Put $t = \sin x + \cos x$

$\Rightarrow dt = (\cos x - \sin x)dx$

$\therefore$ I = $\int \frac{1}{t}dt = \log|t| + c = \log|\sin x + \cos x| + c$

73. Put $3\sin^2 x + 5\cos^2 x = t$

$\Rightarrow (3 \times 2\sin x\cos x - 5 \times 2\sin x\cos x)\,dx = dt$

$\Rightarrow -4\sin x\cos x\,dx = dt$

$\Rightarrow \sin x\cos x\,dx = \frac{dt}{-4}$

$\therefore \int \frac{\sin x\cos x}{3\sin^2 x+5\cos^2 x}dx = \int \frac{dt}{(-4)t}$

$= -\frac{1}{4}\int \frac{1}{t}dt = -\frac{1}{4}\log|t| + c$

$= -\frac{1}{4}\log|3\sin^2 x + 5\cos^2 x| + c$

74. Let I = $\int \frac{\cos x + x\sin x}{x^2 + x\cos x}dx = \int \frac{\frac{\cos x + x\sin x}{x^2}}{1+\frac{\cos x}{x}}dx$

Put $1 + \frac{\cos x}{x} = t$

$\Rightarrow \frac{-(x\sin x+\cos x)}{x^2}dx = dt$

$\therefore$ I = $-\int \frac{dt}{t} = -\log|t| + c$

$= -\log\left|\frac{x+\cos x}{x}\right| + c$

$= \log\left|\frac{x}{x+\cos x}\right| + c$

75. Let I = $\int \frac{\log\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$

Put $\log(x + \sqrt{1+x^2}) = t$

$\Rightarrow \frac{1+\frac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}dx = dt \Rightarrow \frac{dx}{\sqrt{1+x^2}} = dt$

$\therefore$ I = $\int t\,dt = \frac{t^2}{2} + c = \frac{\left[\log\left(x+\sqrt{1+x^2}\right)\right]^2}{2} + c$

76. Put $(x^x)^x = t \Rightarrow \log(x^x)^x = \log t$

$\Rightarrow x^2 \log x = \log t$

$\Rightarrow (2x \log x + x)dx = \frac{1}{t}.dt$

$\Rightarrow (2\log x + 1)x\,(x^x)^x dx = dt$

$\therefore \int x\left(x^x\right)^x (2\log x + 1)dx = \int dt = t + c = (x^x)^x + c$

77. $1 + 2\tan x(\sec x + \tan x)$

$= 1 + 2\tan x . \sec x + 2\tan^2 x$

$= (1 + \tan^2 x) + 2\sec x . \tan x + \tan^2 x$

$= \sec^2 x + 2\sec x . \tan x + \tan^2 x$

$= (\sec x + \tan x)^2$

$\therefore \int \sqrt{1 + 2\tan x(\sec x + \tan x)dx}$

$= \int(\sec x + \tan x)\ dx$

$= \int \frac{1+\sin x}{\cos x}\ dx$

$= \int \frac{1-\sin^2 x}{\cos x(1-\sin x)}\ dx$

$= -\int \frac{(-\cos x)}{1-\sin x}\ dx = -\log|1 - \sin x| + c$

78. Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$\therefore \int \frac{dx}{x\sqrt{1-(\log x)^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + c$

$= \sin^{-1}(\log x) + c$

79. Put $t = \cos x \Rightarrow dt = -\sin x\,dx$

$\therefore \int \frac{\sin x}{\sqrt{4-\cos^2 x}}\ dx = -\int \frac{dt}{\sqrt{2^2 - t^2}}$

$= -\sin^{-1}\left(\frac{t}{2}\right) + c = -\sin^{-1}\left(\frac{\cos x}{2}\right) + c$

80. $\int \frac{\sec x\,dx}{\sqrt{\cos 2x}} = \int \frac{\sec x}{\sqrt{\cos^2 x - \sin^2 x}}dx$

$= \int \frac{\sec^2 x\,dx}{\sqrt{1-\tan^2 x}}$

….[Multiplying N^r and D^r by $\sec x$]

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\therefore \int \frac{\sec x\,dx}{\sqrt{\cos 2x}} = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + c$

$= \sin^{-1}(\tan x) + c$

81. Put $2x = \sin\theta \Rightarrow 2dx = \cos\theta\,d\theta$

$\therefore \int \frac{2dx}{\sqrt{1-4x^2}} = \int \frac{\cos\theta\ d\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta}{\cos\theta}d\theta$

$= \int d\theta = \theta + c$

$= \sin^{-1}(2x) + c$

82. $\int \frac{dx}{\sqrt{2-3x-x^2}} = \int \frac{dx}{\sqrt{\left(\frac{17}{4}\right) - \left(x+\frac{3}{2}\right)^2}}$

$= \int \frac{dx}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x+\frac{3}{2}\right)^2}}$

$= \sin^{-1}\left[\frac{\left(x+\frac{3}{2}\right)}{\left(\frac{\sqrt{17}}{2}\right)}\right] + c$

$= \sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right) + c$

83. Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$\therefore \int \cos x\sqrt{4-\sin^2 x}\,dx = \int \sqrt{4-t^2}dt$

$= \int \sqrt{(2)^2 - t^2}dt = \frac{t}{2}\sqrt{4-t^2} + \frac{4}{2}\sin^{-1}\left(\frac{t}{2}\right) + c$

$= \frac{1}{2}\sin x\sqrt{4-\sin^2 x} + 2\sin^{-1}\left(\frac{1}{2}\sin x\right) + c$

84. Let $I = \int \frac{3x^2}{\sqrt{9-16x^6}}dx = \int \frac{3x^2}{\sqrt{(3)^2 - (4x^3)^2}}dx$

Put $4x^3 = t$

$\Rightarrow 12x^2\,dx = dt$

$\therefore I = \frac{1}{4}\int \frac{dt}{\sqrt{(3)^2 - t^2}}$

$= \frac{1}{4}\sin^{-1}\left(\frac{t}{3}\right) + c$

$= \frac{1}{4}\sin^{-1}\left(\frac{4x^3}{3}\right) + c$

85. $\int \sqrt{\frac{a-x}{a+x}}dx = \int \frac{a-x}{\sqrt{a^2-x^2}}dx$

$= \int\left(\frac{a}{\sqrt{a^2-x^2}} - \frac{x}{\sqrt{a^2-x^2}}\right)dx$

$= a\int \frac{1}{\sqrt{a^2-x^2}}dx + \frac{1}{2}\int \frac{-2x}{\sqrt{a^2-x^2}}dx$

$= a.\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}.2\sqrt{a^2-x^2} + c$

$= a\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2-x^2} + c$

86. Let $I = \int \frac{2^x}{\sqrt{1-(2^x)^2}} dx$

Put $2^x = t \Rightarrow 2^x\, dx = \frac{dt}{\log 2}$

$\therefore \quad I = \frac{1}{\log 2}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\log 2}\sin^{-1} t + c$

$= \frac{1}{\log 2}\ \sin^{-1} 2^x + c$

$\therefore \quad K = \frac{1}{\log 2}$

87. $\int \frac{\sqrt{x}}{1+x} dx = \int \frac{\sqrt{x}.\sqrt{x}}{\sqrt{x}(1+x)} dx$

$= \int \frac{x+1}{\sqrt{x}(x+1)} dx - \int \frac{1}{\sqrt{x}(x+1)} dx$

$= \int \frac{1}{\sqrt{x}} dx - \int \frac{1}{\sqrt{x}\left[1+\left(\sqrt{x}\right)^2\right]} dx$

$= 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + c$

$= 2(\sqrt{x} - \tan^{-1}\sqrt{x}) + c$

88. Let $I = \int \frac{e^{\log\left(1+\frac{1}{x^2}\right)}}{x^2+\frac{1}{x^2}} dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2} dx$

$\ldots.\left[\because e^{\log a} = a\right]$

Put $x - \frac{1}{x} = t$

$\Rightarrow \left(1+\frac{1}{x^2}\right) dx = dt$

$\therefore \quad I = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\ \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$

$= \frac{1}{\sqrt{2}}\ \tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + c$

$= \frac{1}{\sqrt{2}}\ \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + c$

89. Put $x^2 = t \Rightarrow x dx = \frac{1}{2}\ dt$

$\therefore \quad \int \frac{x}{x^4+x^2+1}\ dx = \frac{1}{2}\int \frac{dt}{t^2+t+1}$

$= \frac{1}{2}\int \frac{dt}{t^2+t+\frac{1}{4}+\frac{3}{4}}$

$= \frac{1}{2}\int \frac{dt}{(t+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{1}{2}\cdot\frac{1}{(\sqrt{3}/2)}\ \tan^{-1}\left(\frac{t+1/2}{\sqrt{3}/2}\right) + c$

$= \frac{1}{\sqrt{3}}\ \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + c$

$= \frac{1}{\sqrt{3}}\ \tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$

90. Let $I = \int \frac{1}{1+\sin^2 x} dx = \int \frac{dx}{2\sin^2 x + \cos^2 x}$

$= \int \frac{\sec^2 x\, dx}{2\tan^2 x + 1}$

$= \frac{1}{2}\int \frac{\sec^2 x\, dx}{\tan^2 x + \frac{1}{2}}$

Put $\tan x = t \Rightarrow \sec^2 x\ dx = dt$

$\therefore \quad I = \frac{1}{2}\int \frac{dt}{t^2+\frac{1}{2}} = \frac{1}{2}.\frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\frac{t}{\frac{1}{\sqrt{2}}} + c$

$= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + c$

91. $\int \frac{1}{1+\cos^2 x} dx = \int \frac{\sec^2 x}{\sec^2 x + 1} dx = \int \frac{\sec^2 x}{\tan^2 x + 2} dx$

Put $\tan x = t \Rightarrow \sec^2 x\ dx = dt$

$\therefore \quad \int \frac{1}{1+\cos^2 x} dx = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right) + c$

92. Put $\cos x = t$

$\Rightarrow -\sin x\, dx = dt$

$$\therefore \int \frac{\sin x}{3+4\cos^2 x} dx = \int \frac{-dt}{3+4t^2} = \frac{-1}{4} \int \frac{dt}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{4.\frac{\sqrt{3}}{2}} . \tan^{-1} \frac{t}{\left(\frac{\sqrt{3}}{2}\right)} + c$$

$$= \frac{-1}{2\sqrt{3}} \tan^{-1}\left(\frac{2t}{\sqrt{3}}\right) + c$$

$$= \frac{-1}{2\sqrt{3}} \tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right) + c$$

93. Let $I = \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

Dividing N^r and D^r by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{a^2 \tan^2 x + b^2} dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\therefore I = \int \frac{dt}{a^2 t^2 + b^2} = \frac{1}{a^2} \int \frac{dt}{t^2 + \frac{b^2}{a^2}}$$

$$= \frac{1}{a^2} \cdot \frac{1}{\left(\frac{b}{a}\right)} \tan^{-1}\left(\frac{t}{\left(\frac{b}{a}\right)}\right) + c$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{a}{b}\tan x\right) + c$$

94. Let $I = \int \frac{dx}{4\sin^2 x + 5\cos^2 x}$

Dividing N^r and D^r by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x\, dx}{4\tan^2 x + 5}$$

$$= \frac{1}{4}\int \frac{\sec^2 x\, dx}{\tan^2 x + \frac{5}{4}}$$

Put $\tan x = t$

$\Rightarrow \sec^2 x\, dx = dt$

$$\therefore I = \frac{1}{4}\int \frac{dt}{t^2 + \left(\frac{\sqrt{5}}{2}\right)^2} = \frac{1}{4} \cdot \frac{1}{\frac{\sqrt{5}}{2}} \tan^{-1}\left(\frac{2t}{\sqrt{5}}\right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + c$$

95. Let $I = \int \frac{dx}{2\sin^2 x - 3\cos^2 x + 7}$

Dividing N^r and D^r by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x\, dx}{2\tan^2 x - 3 + 7\sec^2 x}$$

$$= \int \frac{\sec^2 x dx}{2\tan^2 x - 3 + 7(1+\tan^2 x)}$$

$$= \int \frac{\sec^2 x dx}{4 + 9\tan^2 x}$$

Put $t = \tan x$

$\Rightarrow dt = \sec^2 x\, dx$

$$\therefore I = \int \frac{dt}{2^2 + (3t)^2} = \frac{1}{6} \tan^{-1}\left(\frac{3t}{2}\right) + c$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{3\tan x}{2}\right) + c$$

96. $\int \frac{dx}{2+\cos x}$

$$= \int \frac{dx}{2\sin^2\left(\frac{x}{2}\right) + 2\cos^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}$$

$$= \int \frac{dx}{\sin^2\left(\frac{x}{2}\right) + 3\cos^2\left(\frac{x}{2}\right)} = \int \frac{\sec^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right) + 3} dx$$

Put $\tan\left(\frac{x}{2}\right) = t$

$\Rightarrow \sec^2\left(\frac{x}{2}\right) dx = 2dt$

$$\therefore \int \frac{dx}{2+\cos x} = 2\int \frac{dt}{t^2+3} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right) + c$$

97. Let $I = \int \frac{dx}{2\sin x + \cos x + 3}$

Put $t = \tan\left(\frac{x}{2}\right)$

$$\therefore dx = \frac{2dt}{1+t^2} \text{ and } \cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}$$

$$\therefore \quad I = \int \frac{\frac{2dt}{1+t^2}}{2\left(\frac{2t}{1+t^2}\right)+\left(\frac{1-t^2}{1+t^2}\right)+3}$$

$$= 2\int \frac{dt}{4t+1-t^2+3+3t^2}$$

$$= 2\int \frac{dt}{2t^2+4t+4} = \int \frac{dt}{t^2+2t+2}$$

$$= \int \frac{dt}{t^2+2t+1+1} = \int \frac{dt}{(t+1)^2+1^2}$$

$$= \tan^{-1}\left(\frac{t+1}{1}\right) + c = \tan^{-1}\left(\tan\left(\frac{x}{2}\right)+1\right) + c$$

98. Put $x^2 = t$

$\Rightarrow 2x\,dx = dt \qquad \Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

$$\therefore \quad \int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{dt}{2t\sqrt{t^2-1}} = \frac{1}{2}\sec^{-1} t + c$$

$$= \frac{1}{2}\sec^{-1} x^2 + c$$

99. Let $I = \int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}dx$

Dividing N^r and D^r by x^2, we get

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}}\,dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-2}}dx$$

Put $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$

$$\therefore \quad I = \int \frac{dt}{t\sqrt{t^2-2}} = \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\left[\frac{\left(x+\frac{1}{x}\right)}{\sqrt{2}}\right] + c$$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{x\sqrt{2}}\right) + c$$

100. Put $t = \tan x \Rightarrow dt = \sec^2 x\,dx$

$$\therefore \quad \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int \frac{1}{\sqrt{t^2+2^2}}dt$$

$$= \log\left|t+\sqrt{t^2+4}\right| + c$$

$$= \log\left|\tan x+\sqrt{\tan^2 x+4}\right| + c$$

101. $\int \sqrt{x^2-8x+7}\,dx = \int \sqrt{(x-4)^2-(3)^2}\,dx$

$$= \frac{(x-4)}{2}\sqrt{x^2-8x+7}$$

$$- \frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + c$$

102. Put $x^2 = t$

$\Rightarrow 2x\,dx = dt$

$\Rightarrow x\,dx = \frac{dt}{2}$

$$\therefore \quad \int \frac{x}{\sqrt{x^4-4}}\,dx = \frac{1}{2}\int \frac{dt}{\sqrt{t^2-2^2}}$$

$$= \frac{1}{2}\log|t+\sqrt{t^2-4}| + c$$

$$= \frac{1}{2}\log|x^2+\sqrt{x^4-4}| + c$$

103. Let $I = \int \frac{e^x}{\sqrt{e^{2x}+4e^x+13}}\,dx$

Put $e^x = t \Rightarrow e^x\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{\sqrt{t^2+4t+13}}$$

$$= \int \frac{dt}{\sqrt{(t+2)^2+3^2}}$$

$$= \log\left|t+2+\sqrt{(t+2)^2+3^2}\right| + c$$

$$= \log\left|e^x+2+\sqrt{e^{2x}+4e^x+13}\right| + c$$

104. Let $I = \int \frac{1}{\sqrt{\operatorname{cosec}^2 x+\cot^2 x}}dx$

$$= \int \frac{\sin x}{\sqrt{1+\cos^2 x}}\,dx$$

Put $t = \cos x \Rightarrow dt = -\sin x\,dx$

$$\therefore \quad I = -\int \frac{dt}{\sqrt{1+t^2}} = -\log\left(t+\sqrt{1+t^2}\right) + c$$

$$= -\log\left(\cos x+\sqrt{1+\cos^2 x}\right) + c$$

105. $\int \frac{dx}{2x^2+x-1} = \frac{1}{2}\int \frac{dx}{x^2+\frac{1}{2}x-\frac{1}{2}}$

$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2}$

$= \frac{1}{2}.\frac{1}{2.\left(\frac{3}{4}\right)}\log\left|\frac{\left(x+\frac{1}{4}\right)-\frac{3}{4}}{\left(x+\frac{1}{4}\right)+\frac{3}{4}}\right|+c$

$= \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right|+c$

106. Put $\log x = t$

$\Rightarrow \frac{1}{x}\, dx = dt$

$\therefore$ $\int \frac{dx}{x[(\log x)^2+4\log x-1]} = \int \frac{dt}{t^2+4t-1}$

$= \int \frac{dt}{(t+2)^2-(\sqrt{5})^2}$

$= \frac{1}{2\sqrt{5}}\log\left|\frac{t+2-\sqrt{5}}{t+2+\sqrt{5}}\right|+c$

$= \frac{1}{2\sqrt{5}}\log\left|\frac{\log x+2-\sqrt{5}}{\log x+2+\sqrt{5}}\right|+c$

107. Let $I = \int \frac{1}{(x^2-1)\sqrt{x^2+1}}\,dx$

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\, dt$

$\therefore$ $I = \int \frac{-\frac{1}{t^2}}{\left(\frac{1}{t^2}-1\right)\sqrt{\frac{1}{t^2}+1}}\,dt$

$= -\int \frac{t}{\left(1-t^2\right)\sqrt{1+t^2}}\,dt$

Put $\sqrt{1+t^2} = u \Rightarrow 1+t^2 = u^2 \Rightarrow t\,dt = u\,du$

$\therefore$ $I = -\int \frac{u}{\left[1-(u^2-1)\right]u}\,du$

$= -\int \frac{du}{2-u^2} = \int \frac{du}{u^2-\left(\sqrt{2}\right)^2}$

$= \frac{1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|+c$

$= \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{1+t^2}-\sqrt{2}}{\sqrt{1+t^2}+\sqrt{2}}\right|+c$

$= \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{1+\left(\frac{1}{x}\right)^2}-\sqrt{2}}{\sqrt{1+\left(\frac{1}{x}\right)^2}+\sqrt{2}}\right|+c$

$= \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{x^2+1}-\sqrt{2}\,x}{\sqrt{x^2+1}+\sqrt{2}\,x}\right|+c$

108. Let $I = \int \frac{2x+1}{x^4+2x^3+x^2-1}\,dx$

$= \int \frac{2x+1}{[x(x+1)]^2-1}\,dx$

Put $t = x(x+1)$

$\Rightarrow dt = (2x+1)\,dx$

$\therefore$ $I = \int \frac{dt}{t^2-1}$

$= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|+c$

$= \frac{1}{2}\log\left|\frac{x^2+x-1}{x^2+x+1}\right|+c$

$= -\frac{1}{2}\log\left|\frac{x^2+x+1}{x^2+x-1}\right|+c$

$\therefore$ $A = -\frac{1}{2}$

109. $\int \frac{1}{\sin x\sqrt{\sin x.\cos x}}\,dx = \int \frac{1}{\sin x.\sqrt{\sin^2 x.\frac{\cos x}{\sin x}}}\,dx$

$= \int \frac{1}{\sin x.\sin x.\sqrt{\frac{\cos x}{\sin x}}}\,dx$

$= \int \frac{1}{\sqrt{\cot x}} \times \operatorname{cosec}^2 x\,dx$

Put $t = \cot x$

$\Rightarrow -dt = \operatorname{cosec}^2 x\,dx$

$\therefore$ $\int \frac{1}{\sin x\sqrt{\sin x.\cos x}}\,dx = \int \frac{-dt}{\sqrt{t}} = -\int t^{-\frac{1}{2}}\,dt$

$= -2t^{\frac{1}{2}}+c$

$= -2\sqrt{\cot x}+c$

110. Let $I = \int \sin^3 x \sqrt{\cos x}\ dx$

$= \int (1-\cos^2 x)\sqrt{\cos x}\ \sin x\, dx$

Put $t = \cos x$

$\Rightarrow dt = -\sin x\, dx$

$\Rightarrow -dt = \sin x\, dx$

$\therefore\quad I = \int (1-t^2)\sqrt{t}\ (-dt)$

$= -\int \sqrt{t}\ dt + \int t^{\frac{5}{2}}\ dt = -\dfrac{t^{3/2}}{3/2} + \dfrac{t^{\frac{7}{2}}}{\frac{7}{2}} + c$

$= \dfrac{2}{7}\left(\sqrt{\cos x}\right)^7 - \dfrac{2}{3}\left(\sqrt{\cos x}\right)^3 + c$

111. Let $I = \int \dfrac{dx}{\sqrt{x}+\sqrt[3]{x}}$

Put $x^{\frac{1}{6}} = t$

$\Rightarrow x = t^6$

$\Rightarrow dx = 6t^5 dt$

$\therefore\quad I = \int \dfrac{6t^5}{t^3+t^2}\ dt$

$= \int \dfrac{6t^5}{t^2(t+1)}\ dt = 6\int \dfrac{t^3}{t+1}\ dt$

$= 6\int \dfrac{t^3+1-1}{t+1}\ dt = 6\int \left(t^2 - t + 1 - \dfrac{1}{t+1}\right) dt$

$= 6\left[\dfrac{t^3}{3} - \dfrac{t^2}{2} + t - \log|t+1|\right] + c$

$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\log|\sqrt[6]{x}+1| + c$

112. Let $I = \int \dfrac{x^5 dx}{\sqrt{1+x^3}} = \int \dfrac{x^3.x^2}{\sqrt{1+x^3}}\, dx$

Put $1 + x^3 = t^2$

$\Rightarrow 3x^2\, dx = 2t\, dt$

$\therefore\quad I = \int \dfrac{(t^2-1)}{t}\left(\dfrac{2}{3}\right) t\, dt$

$= \dfrac{2}{3}\int (t^2-1)\ dt = \dfrac{2}{3}\left(\dfrac{t^3}{3} - t\right) + c$

$= \dfrac{2}{9}t(t^2-3) + c$

$= \dfrac{2}{9}\sqrt{1+x^3}\ (1+x^3-3) + c$

$= \dfrac{2}{9}\sqrt{1+x^3}\ (x^3-2) + c$

113. Let $I = \int \sec^6 x\, dx = \int \sec^4 x.\sec^2 x\ dx$

$= \int (1+\tan^2 x)^2 \sec^2 x\ dx$

Put $t = \tan x$

$\Rightarrow dt = \sec^2 x\, dx$

$\therefore\quad I = \int (1+t^2)^2\ dt = \int (1+2t^2+t^4)\, dt$

$= t + \dfrac{2t^3}{3} + \dfrac{t^5}{5} + c$

$= \tan x + \dfrac{2}{3}\tan^3 x + \dfrac{1}{5}\tan^5 x + c$

114. Let $I = \int \sec^{\frac{2}{3}} x \operatorname{cosec}^{\frac{4}{3}} x\, dx = \int \dfrac{1}{\cos^{\frac{2}{3}} x \sin^{\frac{4}{3}} x}\, dx$

Dividing N^r and D^r by $\cos^{\frac{4}{3}} x$, we get

$I = \int \dfrac{\sec^2 x}{\tan^{\frac{4}{3}} x}\, dx$

Put $\tan x = t$

$\Rightarrow \sec^2 x\, dx = dt$

$\therefore\quad I = \int \dfrac{dt}{t^{4/3}} = \int t^{\frac{-4}{3}}\ dt = \dfrac{t^{\frac{-1}{3}}}{\frac{-1}{3}} + c$

$= -3\ (\tan x)^{\frac{-1}{3}} + c$

115. Let $I = \int \tan^4 x\ dx = \int \tan^2 x(\sec^2 x - 1) dx$

$= \int (\tan^2 x \sec^2 x - \tan^2 x)\ dx$

$= \int (\tan^2 x.\sec^2 x - \sec^2 x + 1)\ dx$

$= \int (\tan^2 x - 1)\ \sec^2 x dx + \int 1 dx$

In 1st integral,

Put $t = \tan x$

$\Rightarrow dt = \sec^2 x dx$

$\therefore\quad I = \dfrac{t^3}{3} - t + x + c$

$\therefore\quad I = \dfrac{\tan^3 x}{3} - \tan x + x + c$

$\therefore\quad A = \dfrac{1}{3}, B = -1, f(x) = x + c$

116. Let $I = \int \dfrac{\sin^3 2x}{\cos^5 2x}\, dx$

$= \int \dfrac{\sin^3 2x}{\cos^3 2x}.\dfrac{1}{\cos^2 2x}\, dx$

$= \int \tan^3 2x.\sec^2 2x\, dx$

Put $\tan 2x = t \Rightarrow 2\sec^2 2x\, dx = dt$

$\therefore \quad I = \int t^3 \cdot \frac{dt}{2} = \frac{1}{2}.\frac{t^4}{4} + c$

$= \frac{1}{8}(\tan^4 2x) + c$

117. $\int \log x \; dx = \int \log x .1\, dx$

$= \log x.\, x - \int \frac{1}{x}\cdot x dx$

$= x \log x - x + c = x (\log x - 1) + c$

$= x (\log x - \log e) + c = x \log\left(\frac{x}{e}\right) + c$

118. $\int xe^{2x}dx = x.\frac{e^{2x}}{2} - \int 1.\frac{e^{2x}}{2}dx$

$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$

$= e^{2x}\left(\frac{2x-1}{4}\right) + c$

$\therefore \quad f(x) = \frac{2x-1}{4}$

119. $\int x^2 \; e^{3x}\, dx = x^2 \cdot \frac{e^{3x}}{3} - \int 2x \cdot \frac{e^{3x}}{3}\, dx$

$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left[x.\frac{e^{3x}}{3} - \int 1.\frac{e^{3x}}{3}dx\right]$

$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left(\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right) + c$

$= \frac{1}{3}\, x^2 e^{3x} - \frac{2}{9}\, xe^{3x} + \frac{2}{27}\, e^{3x} + c$

120. $\int x^3 \log x\, dx = \log x.\frac{x^4}{4} - \int \frac{1}{x}.\frac{x^4}{4}dx$

$= \frac{x^4}{4}\log x - \int \frac{x^3}{4}dx$

$= \frac{x^4}{4}\log x - \frac{x^4}{16} + c$

$= \frac{1}{16}(4x^4 \log x - x^4) + c$

121. $\int \frac{\log x}{x^3}dx = \int \log x.\, x^{-3}dx$

$= \log x \cdot \frac{x^{-2}}{-2} - \int\left(\frac{1}{x}\cdot\frac{x^{-2}}{-2}\right)dx$

$= -\frac{\log x}{2x^2} + \frac{1}{2}\int x^{-3}\, dx$

$= -\frac{\log x}{2x^2} + \frac{1}{2}.\frac{x^{-2}}{-2} + c$

$= -\frac{\log x}{2x^2} - \frac{1}{4x^2} + c$

$= -\frac{1}{4x^2}(2\log x + 1) + c$

122. $\int x^n \log x\, dx = \log x.\frac{x^{n+1}}{n+1} - \int \frac{1}{x}\cdot\frac{x^{n+1}}{n+1}\, dx$

$= \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2} + c$

$= \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + c$

123. $\int\left[f(x) + x\, f'(x)\right]\, dx$

$= \int f(x)dx + \int xf'(x)dx$

$= \int f(x)\; dx + x\,.\, f(x) - \int f(x)\; dx + c$

$= x\, f(x) + c$

124. $\int [f(x)\, g''(x) - f''(x)\, g(x)]\, dx$

$= f(x)\, g'(x) - \int f'(x)\, g'(x)\, dx - g(x)\, f'(x) + \int f'(x)\, g'(x)\, dx$

$= f(x)\, g'(x) - g(x)\, f'(x) + c$

125. $I_5 + 5I_4 = \int x^5 e^x\, dx + 5\int x^4\,.\, e^x\, dx$

$= x^5\, e^x - 5\int x^4\, e^x\, dx + 5\int x^4\,.\, e^x\, dx + c$

$= x^5\, e^x + c$

126. Let $I = \int \tan^{-1} x.\, 1\, dx$

$= \tan^{-1} x.x - \int \frac{1}{1+x^2}\cdot x\, dx$

$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$

$= x \tan^{-1} x - \frac{1}{2}\log|1 + x^2| + c$

127. $\int x\tan^{-1} x dx = (\tan^{-1} x)\left(\frac{x^2}{2}\right) - \int \frac{1}{1+x^2}\left(\frac{x^2}{2}\right)dx$

$= \frac{1}{2}\, x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\, dx$

$= \frac{1}{2}\, x^2 \tan^{-1} x - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right) dx$

$= \frac{1}{2}\, x^2 \tan^{-1} x - \frac{1}{2}\, (x - \tan^{-1}x) + c$

$= \frac{1}{2}\, (x^2 + 1)\tan^{-1} x - \frac{1}{2}\, x + c$

128. $I = \int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

$\therefore$ $I = 2\int \tan^{-1} x dx$

$= 2\left(\tan^{-1} x . x - \int \frac{1}{1+x^2} . x dx\right)$

$= 2\left(x \tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2} dx\right)$

$\therefore$ $I = 2x \tan^{-1} x - \log(1+x^2) + c$

$\therefore$ $I - 2x \tan^{-1} x = -\log(1+x^2) + c$

129. $\int x^3 \tan^{-1} x\, dx = \tan^{-1} x . \frac{x^4}{4} - \frac{1}{4}\int \frac{x^4}{x^2+1} dx$

$= \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\int \frac{x^4-1+1}{x^2+1} dx$

$= \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\int \left[(x^2-1) + \frac{1}{x^2+1}\right] dx$

$= \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\left[\frac{x^3}{3} - x + \tan^{-1} x\right] + c$

$= \frac{1}{4}\left[(x^4-1)\tan^{-1} x - \frac{x^3}{3} + x\right] + c$

130. $\int x \log\left(1+\frac{1}{x}\right) dx$

$= \log\left(1+\frac{1}{x}\right) . \frac{x^2}{2} + \frac{1}{2}\int \frac{x}{x+1} dx$

$= \frac{x^2}{2}\log\left(1+\frac{1}{x}\right) + \frac{1}{2}x - \frac{1}{2}\log(x+1) + c$

$= \left(\frac{x^2-1}{2}\right)\log(x+1) - \frac{x^2}{2}\log x + \frac{1}{2}x + c$

131. $\int \log(x^2+x)dx = \int \log[x(x+1)]dx$

$= \int \log x . 1 dx + \int \log(x+1) . 1 dx$

$= \log x . x - \int \frac{1}{x} . x dx + \log(x+1) . x - \int \frac{1}{x+1} . x dx$

$= x \log x - \int dx + x\log(x+1) - \int\left(\frac{x+1-1}{x+1}\right) dx$

$= x \log x - x + x \log(x+1) - \int\left(1 - \frac{1}{x+1}\right) dx$

$= x \log x - x + x \log(x+1) - x + \log|x+1| + c$

$= x[\log x + \log(x+1)] - 2x + \log|x+1| + c$

$= x \log(x^2+x) - 2x + \log|x+1| + c$

$\therefore$ $A = -2x + \log|x+1| + c$

132. Put $\sin^{-1} x = t \Rightarrow x = \sin t \Rightarrow dx = \cos t\, dt$

$\therefore$ $\int \sin^{-1} x dx = \int t \cos t dt$

$= t \sin t - \int 1 . \sin t dt$

$= t \sin t + \cos t + c$

$= t \sin t + \sqrt{1-\sin^2 t} + c$

$= x \sin^{-1} x + \sqrt{1-x^2} + c$

133. Put $x = t^2$

$\Rightarrow dx = 2t\, dt$

$\therefore$ $\int \sin\sqrt{x}\, dx = \int \sin t\,(2t)\, dt$

$= 2\int t \sin t\, dt$

$= 2\left[t(-\cos t) - \int (1)(-\cos t)dt\right]$

$= 2\left(-t\cos t + \int \cos t\, dt\right)$

$= -2t \cos t + 2 \sin t + c$

$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + c$

134. Put $x = t^2 \Rightarrow dx = 2tdt$

$\therefore$ $\int \sqrt{x} . e^{\sqrt{x}} dx = 2\int t^2 . e^t dt$

$= 2(t^2 . e^t - 2te^t + 2e^t) + c$

$= 2\left(x . e^{\sqrt{x}} - 2\sqrt{x}\, e^{\sqrt{x}} + 2\, e^{\sqrt{x}}\right) + c$

$= e^{\sqrt{x}}\left(2x - 4\sqrt{x} + 4\right) + c$

135. Put $x^2 = t$

$\Rightarrow 2x\, dx = dt$

$\therefore$ $\int x^5 e^{x^2} dx = \frac{1}{2}\int t^2 e^t dt$

$= \frac{1}{2}\left[t^2 e^t - 2\int te^t dt\right]$

$= \frac{t^2 e^t}{2} - \left(te^t - e^t\right) + c$

$= \frac{1}{2}x^4 e^{x^2} - x^2 e^{x^2} + e^{x^2} + c$

136. Let $I = \int \sin(\log x)\, dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$\therefore$ $I = \int \sin t . e^t dt = \sin t . e^t - \int \cos t . e^t dt$

$= \sin t . e^t - \left[\cos t . e^t + \int \sin t . e^t dt\right]$

$\therefore$ $I = \sin t . e^t - \cos t . e^t - I + c_1$

$\Rightarrow 2I = \sin t . e^t - \cos t . e^t + c_1$

$\Rightarrow I = \frac{1}{2} x [\sin(\log x) - \cos(\log x)] + c,$

where $c = \frac{c_1}{2}$

137. $\int \sin x \log(\sec x + \tan x)\,dx$

$= \log(\sec x + \tan x).(-\cos x) - \int \sec x.(-\cos x)\,dx$

$\ldots\left[\because \frac{d}{dx}[\log(\sec x + \tan x)] = \sec x\right]$

$= -\cos x \log(\sec x + \tan x) + \int 1\,dx$

$= x - \cos x \log(\sec x + \tan x) + c$

138. Put $x = \sin\theta$

$\Rightarrow dx = \cos\theta\,d\theta$

$\therefore \int \sin^{-1}(3x - 4x^3)\,dx$

$= \int \sin^{-1}(\sin 3\theta)\cos\theta\,d\theta$

$= \int 3\theta\cos\theta\,d\theta = 3\left(\theta\sin\theta - \int \sin\theta\,d\theta\right)$

$= 3(\theta\sin\theta + \cos\theta) + c$

$= 3\left(x\sin^{-1}x + \sqrt{1-x^2}\right) + c$

139. Put $\sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$

$\therefore \int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}dx = \int t\sec^2 t\,dt$

$= t\tan t - \int 1.\tan t\,dt$

$= t\tan t + \log(\cos t) + c$

$= t\cdot\frac{\sin t}{\sqrt{1-\sin^2 t}} + \log\left(\sqrt{1-\sin^2 t}\right) + c$

$= \frac{x}{\sqrt{1-x^2}}\sin^{-1}x + \frac{1}{2}\log(1-x^2) + c$

140. Put $\tan^{-1}x = \theta \Rightarrow x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

$\therefore \int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx = \int \frac{\theta\tan\theta\sec^2\theta}{(1+\tan^2\theta)^{3/2}}d\theta$

$= \int \theta\sin\theta\,d\theta = -\theta\cos\theta + \sin\theta + c$

$= \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} - \theta.\frac{1}{\sqrt{1+\tan^2\theta}} + c$

$= \frac{x}{\sqrt{1+x^2}} - \tan^{-1}x\frac{1}{\sqrt{1+x^2}} + c$

$= \frac{x - \tan^{-1}x}{\sqrt{1+x^2}} + c$

141. $\int \sec^3\theta\,d\theta = \int \sec\theta.\sec^2\theta\,d\theta$

$= \sec\theta\tan\theta - \int(\sec\theta\tan\theta.\tan\theta)\,d\theta$

$= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta$

$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\,d\theta$

$= \sec\theta\tan\theta - \int(\sec^3\theta - \sec\theta)\,d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$

$\therefore 2\int \sec^3\theta\,d\theta = \sec\theta\tan\theta + \int \sec\theta\,d\theta$

$= \sec\theta\tan\theta + \log|\sec\theta + \tan\theta| + c$

$\therefore \int \sec^3\theta\,d\theta = \frac{1}{2}[\sec\theta\tan\theta + \log|\sec\theta + \tan\theta|] + c$

142. $\int \frac{e^x(x-1)}{x^2}dx = \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \frac{e^x}{x} + c$

$\ldots\left[\because \int e^x[f(x) + f'(x)]dx = e^x f(x) + c\right]$

143. $\int e^x\left(x^5 + 5x^4 + 1\right)dx$

$= \int e^x\left(x^5 + 5x^4\right)dx + \int e^x dx$

$= e^x.x^5 + e^x + c$

144. $\int e^x\left[\tan x - \log(\cos x)\right]dx$

$= \int e^x\left[\tan x + \log(\sec x)\right]dx$

$= e^x\log(\sec x) + c$

$\ldots\left[\because \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c\right]$

145. $\int e^x(1 + \tan x + \tan^2 x)dx = \int e^x(\tan x + \sec^2 x)dx$

$= e^x\tan x + c$

$\ldots\left[\because \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c\right]$

146. $\int e^x(1 - \cot x + \cot^2 x)\,dx$

$= \int e^x(-\cot x + \operatorname{cosec}^2 x)\,dx$

$= e^x(-\cot x) + c = -e^x\cot x + c$

147. $\int \frac{xe^x}{(x+1)^2}dx = \int e^x\frac{(x+1-1)}{(x+1)^2}dx$

$= \int e^x\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]dx$

$= \frac{e^x}{x+1} + c$

$\ldots\left[\because \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c\right]$

148. $\int \frac{(x+3)e^x}{(x+4)^2}dx = \int \frac{(x+4-1)}{(x+4)^2}e^x dx$

$= \int e^x\left(\frac{1}{x+4} - \frac{1}{(x+4)^2}\right)dx$

$= \frac{e^x}{x+4} + c$

149. $\int \frac{e^x(x^2+1)}{(x+1)^2}dx = \int \frac{e^x(x^2-1+2)}{(x+1)^2}dx$

$= \int e^x\left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right]dx$

$= e^x\left(\frac{x-1}{x+1}\right) + c$

150. $\int\left(\frac{2+\sin 2x}{1+\cos 2x}\right) e^x\, dx$

$= \int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x\, dx$

$= \int(\sec^2 x + \tan x)e^x\, dx$

$= e^x \tan x + c$

151. $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx = \int e^x\left[\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\left(\frac{x}{2}\right)}\right]dx$

$= \int e^x\left[\frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right) - \cot\left(\frac{x}{2}\right)\right]dx$

$= -e^x\cot\left(\frac{x}{2}\right) + c$

152. $\int e^{2x}(2\cos x - \sin x)\,dx = e^{2x}\cos x + c$

....$\left[\because \int e^{mx}\left[mf(x) + f'(x)\right]dx = e^{mx}f(x) + c\right]$

153. Let I = $\int \log x(\log x + 2)\, dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$\therefore$ I = $\int t(t+2)e^t dt = \int e^t\,(t^2 + 2t)\,dt$

$= e^t.\, t^2 + c = x(\log x)^2 + c$

154. Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$\therefore \int\left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx = \int\left(\frac{1}{t} - \frac{1}{t^2}\right)e^t dt$

$= \frac{e^t}{t} + c$

$= \frac{x}{\log x} + c$

155. Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$\therefore \int \frac{\log x}{(1+\log x)^2}dx = \int \frac{t}{(1+t)^2}e^t\, dt$

$= \int e^t\left[\frac{t+1-1}{(1+t)^2}\right]dt$

$= \int e^t\left[\frac{1}{1+t} - \frac{1}{(1+t)^2}\right]dt$

$= \frac{e^t}{1+t} + c$

$= \frac{x}{1+\log x} + c$

156. Let $I = \int\left\{\log(\log x) + \frac{1}{(\log x)^2}\right\}dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$\therefore \quad I = \int e^t\left(\log t + \frac{1}{t^2}\right)dt$

$= \int e^t\left(\log t + \frac{1}{t}\right)dt + \int e^t\left(-\frac{1}{t} + \frac{1}{t^2}\right)dt$

$= e^t\log t + e^t\left(-\frac{1}{t}\right) + c = x\left(\log(\log x) - \frac{1}{\log x}\right) + c$

$\therefore$ f $(x) = \log(\log x)$ and g $(x) = \frac{1}{\log x}$

157. $\int \frac{dx}{x^2 - x^3} = \int \frac{(1-x)\,dx}{x^2(1-x)} + \int \frac{x\,dx}{x^2(1-x)}$

$= \int \frac{1}{x^2}dx + \int \frac{dx}{x(1-x)}$

$= -\frac{1}{x} + \int \frac{dx}{x} + \int \frac{dx}{1-x}$

$= -\frac{1}{x} + \log|x| - \log|1-x| + c$

$= \log\left|\frac{x}{1-x}\right| - \frac{1}{x} + c$

158. $\int \frac{x^2+x-1}{x^2+x-6}dx = \int\left[1 + \frac{5}{x^2+x-6}\right]dx$

$= \int\left[1 + \frac{5}{(x+3)(x-2)}\right]dx$

$= \int dx + \int \frac{dx}{x-2} - \int \frac{dx}{x+3}$

$= x + \log|x-2| - \log|x+3| + c$

159. $\int \frac{x}{x^4-1}dx = \int \frac{x}{(x^2-1)(x^2+1)}dx$

$= \frac{1}{2}\int\left(\frac{x}{x^2-1}-\frac{x}{x^2+1}\right)dx$

$= \frac{1}{4}\int\left(\frac{2x}{x^2-1}-\frac{2x}{x^2+1}\right)dx$

$= \frac{1}{4}\log|x^2-1| - \frac{1}{4}\log|x^2+1| + c$

$= \frac{1}{4}\log\left|\frac{x^2-1}{x^2+1}\right| + c$

160. $\int \frac{x^2}{(x^2+2)(x^2+3)}dx = \int\left[\frac{3}{x^2+3}-\frac{2}{x^2+2}\right]dx$

$= \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$

$= \sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$

161. $\int \frac{dx}{(x^2-1)(1-2x)} = \int \frac{-1}{(1-x)(1+x)(1-2x)}dx$

Let $\frac{-1}{(1-x)(1+x)(1-2x)} = \frac{A}{1-x}+\frac{B}{1+x}+\frac{C}{1-2x}$

$\Rightarrow -1 = A(1+x)(1-2x) + B(1-x)(1-2x) + C(1-x)(1+x)$ ….(i)

Putting $x = -1$ in (i), we get $B = -\frac{1}{6}$

Putting $x = 1$ in (i), we get $A = \frac{1}{2}$

Putting $x = \frac{1}{2}$ in (i), we get $C = -\frac{4}{3}$

$\therefore \int \frac{dx}{(x^2-1)(1-2x)}$

$= \frac{1}{2}\int\frac{1}{1-x}dx - \frac{1}{6}\int\frac{1}{1+x}dx - \frac{4}{3}\int\frac{1}{1-2x}dx$

$= -\frac{1}{2}\log|1-x| - \frac{1}{6}\log|1+x| + \frac{2}{3}\log|1-2x| + c$

162. $\int \frac{1}{x-x^3}dx = \int \frac{1}{x(1+x)(1-x)}dx$

$= \frac{1}{2}\int\left(\frac{2}{x}-\frac{1}{1+x}+\frac{1}{1-x}\right)dx$

$= \frac{1}{2}\left(2\log|x| - \log|1+x| - \log|1-x|\right) + c$

$= \frac{1}{2}\left(\log|x^2| - \log|1-x^2|\right) + c = \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + c$

163. Put $e^x = t \Rightarrow e^x dx = dt$

$\therefore \int \frac{e^x}{(1+e^x)(2+e^x)}dx = \int \frac{dt}{(1+t)(2+t)}$

$= \int\left(\frac{1}{1+t}-\frac{1}{2+t}\right)dt$

$= \log|1+t| - \log|2+t| + c$

$= \log|1+e^x| - \log|2+e^x| + c$

$= \log\left|\frac{1+e^x}{2+e^x}\right| + c$

164. $\int \frac{dx}{e^x+1-2e^{-x}} = \int \frac{e^x}{e^{2x}+e^x-2}dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$\therefore \int \frac{dx}{e^x+1-2e^{-x}} = \int \frac{dt}{t^2+t-2}$

$= \int \frac{dt}{(t+2)(t-1)} = \frac{1}{3}\int\left(\frac{1}{t-1}-\frac{1}{t+2}\right)dt$

$= \frac{1}{3}\log|t-1| - \frac{1}{3}\log|t+2| + c$

$= \frac{1}{3}\log|e^x-1| - \frac{1}{3}\log|e^x+2| + c$

165. Let $I = \int \frac{a}{b+ce^x}dx = \int \frac{ae^x}{be^x+ce^{2x}}dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$\therefore I = a\int\frac{dt}{bt+ct^2}$

$= a\int\frac{dt}{t(ct+b)}$

$= -\frac{a}{b}\int\left(\frac{c}{ct+b}-\frac{1}{t}\right)dt$

$= -\frac{a}{b}\log|ct+b| + \frac{a}{b}\log|t| + c$

$= \frac{a}{b}\log\left|\frac{t}{ct+b}\right| + c$

$= \frac{a}{b}\log\left|\frac{e^x}{b+ce^x}\right| + c$

166. Put $\sin x = t$

$\Rightarrow \cos x\, dx = dt$

$\therefore \int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx = \int \frac{dt}{(t+1)(t+2)}$

$= \int\frac{1}{t+1}dt - \int\frac{1}{t+2}dt$

$= \log|t+1| - \log|t+2| + c$

$= \log\left|\dfrac{t+1}{t+2}\right| + c$

$= \log\left|\dfrac{\sin x+1}{\sin x+2}\right| + c$

167. $\int \dfrac{x^3-1}{x^3+x}dx = \int \dfrac{x^3}{x(x^2+1)}dx - \int \dfrac{1}{x(x^2+1)}dx$

$= \int \dfrac{x^2}{x^2+1}dx - \int\left(\dfrac{1}{x} - \dfrac{x}{x^2+1}\right)dx$

$= \int\left(1-\dfrac{1}{x^2+1}\right)dx - \int\dfrac{1}{x}dx + \dfrac{1}{2}\int\dfrac{2x}{x^2+1}dx$

$= x - \tan^{-1}x - \log|x| + \dfrac{1}{2}\log|x^2+1| + c$

$= x - \tan^{-1}x - \log|x| + \log\left|\sqrt{x^2+1}\right| + c$

168. Let $\dfrac{2x+7}{(x-4)^2} = \dfrac{A}{x-4} + \dfrac{B}{(x-4)^2}$

$\Rightarrow 2x + 7 = A(x-4) + B = Ax + (-4A + B)$

$\therefore$ A = 2 and $-4A + B = 7$

$\therefore$ B = 7 + 4A = 7 + 8 = 15

$\therefore$ $\int \dfrac{2x+7}{(x-4)^2}dx = \int\left(\dfrac{2}{x-4} + \dfrac{15}{(x-4)^2}\right)dx$

$= 2\log|x-4| - \dfrac{15}{(x-4)} + c$

169. Let $\dfrac{x^2+1}{(x-2)^2(x+3)} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{x+3}$

$\Rightarrow x^2+1 = A(x-2)(x+3) + B(x+3) + C(x-2)^2$ ….(i)

Putting $x = 2$ in (i), we get B = 1

Putting $x = -3$ in (i), we get C = $\dfrac{2}{5}$

Putting $x = 3$ in (i), we get

$6A + 6B + C = 10 \Rightarrow A = \dfrac{3}{5}$

$\therefore$ $\int \dfrac{x^2+1}{(x-2)^2(x+3)}dx$

$= \dfrac{3}{5}\int\dfrac{1}{x-2}dx + \int\dfrac{1}{(x-2)^2}dx + \dfrac{2}{5}\int\dfrac{1}{x+3}dx$

$= \dfrac{3}{5}\log|x-2| - \dfrac{1}{x-2} + \dfrac{2}{5}\log|x+3| + c$

170. Let $\dfrac{1}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$\therefore$ $1 = A(x^2+1) + (Bx+C)(x-1)$ ….(i)

Putting $x = 1$ in (i), we get

$A = \dfrac{1}{2}$

Putting $x = 0$ in (i), we get

$A - C = 1 \Rightarrow C = -\dfrac{1}{2}$

Comparing the coefficient of x^2, we get

$A + B = 0 \Rightarrow B = -\dfrac{1}{2}$

$\therefore$ $\int\dfrac{1}{(x-1)(x^2+1)}dx = \int\left[\dfrac{1}{2(x-1)} - \dfrac{x+1}{2(x^2+1)}\right]dx$

$= \dfrac{1}{2}\int\dfrac{1}{x-1}dx - \dfrac{1}{4}\int\dfrac{2x}{x^2+1}dx - \dfrac{1}{2}\int\dfrac{1}{x^2+1}dx$

$= \dfrac{1}{2}\log(x-1) - \dfrac{1}{4}\log(x^2+1) - \dfrac{1}{2}\tan^{-1}x + c$

171. $\int\dfrac{dx}{1+x+x^2+x^3} = \int\dfrac{dx}{(1+x)(1+x^2)}$

$= \dfrac{1}{2}\int\dfrac{1}{1+x^2}dx + \dfrac{1}{2}\int\dfrac{1}{1+x}dx - \dfrac{1}{2}\int\dfrac{x}{1+x^2}dx$

$= \dfrac{1}{2}\tan^{-1}x + \log\left|\sqrt{1+x}\right| - \dfrac{1}{2}\log\left|\sqrt{1+x^2}\right| + c$

172. $\int\dfrac{x^4}{(x-1)(x^2+1)}dx$

$= \int\dfrac{x^4-1}{(x-1)(x^2+1)}dx + \int\dfrac{1}{(x-1)(x^2+1)}dx$

$= \int\dfrac{(x+1)(x-1)(x^2+1)}{(x-1)(x^2+1)}dx + \int\dfrac{dx}{(x-1)(x^2+1)}$

$= \int(x+1)dx + \int\left[\dfrac{1}{2(x-1)} - \dfrac{x+1}{2(x^2+1)}\right]dx$

$= \int x\,dx + \int dx + \dfrac{1}{2}\int\dfrac{1}{x-1}dx$

$-\dfrac{1}{4}\int\dfrac{2x}{x^2+1}dx - \dfrac{1}{2}\int\dfrac{1}{x^2+1}dx$

$= \dfrac{x^2}{2} + x + \dfrac{1}{2}\log|x-1| - \dfrac{1}{4}\log|x^2+1| - \dfrac{1}{2}\tan^{-1}x + c$

173. $\int \frac{dx}{f(x)} = \log [f(x)]^2 + c$

Differentiating on both sides, we get

$$\frac{1}{f(x)} = \frac{2f(x)f'(x)}{[f(x)]^2}$$

$\Rightarrow f'(x) = \frac{1}{2}$

$\therefore \quad f(x) = \int f'(x)dx = \frac{x}{2} + \alpha$

174. $|A| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = a\,b\,c$

Let I = $\int |A|\, dx = \int 7^x 7^{7^x} 7^{7^{7^x}}\, dx$

Put $7^{7^{7^x}} = t$

$\Rightarrow 7^{7^{7^x}} (\log 7)^3\; 7^{7^x}\, 7^x\, dx = dt$

$\Rightarrow 7^{7^x}\, 7^x\, dx = \frac{dt}{7^{7^{7^x}}(\log 7)^3} = \frac{dt}{t(\log 7)^3}$

$\therefore \quad I = \frac{1}{(\log 7)^3}\int dt = \frac{t}{(\log 7)^3} + c$

$= \frac{7^{7^{7^x}}}{(\log 7)^3} + c$

175. Put $x^2 = t \Rightarrow 2x\, dx = dt$

$\therefore \quad \int x^3 \cos x^2 dx$

$= \frac{1}{2}\int t\cos t\, dt = \frac{1}{2}\left(t\sin t - \int \sin t\, dt\right)$

$= \frac{1}{2}(t\sin t + \cos t) + c$

$= \frac{1}{2}\left(x^2 \sin x^2 + \cos x^2\right) + c$

176. Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$\therefore \quad \tan^{-1}\sqrt{\frac{1-x}{1+x}} = \tan^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)$

$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right)$

$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$

$= \frac{1}{2}\cos^{-1} x$

$\therefore \quad \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\, dx = \frac{1}{2}\int (\cos^{-1} x.1dx)$

$= \frac{1}{2}\left[\cos^{-1} x.x + \int \frac{1}{\sqrt{1-x^2}}.x\,dx\right]$

$= \frac{1}{2}\left(x\cos^{-1} x - \sqrt{1-x^2}\right) + c$

177. Let I $= \int x\sin x\sec^3 x\, dx$

$= \int x\tan x.\sec^2 x\, dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore \quad I = \int \tan^{-1} t.t\, dt$

$= \tan^{-1} t.\frac{t^2}{2} - \int \frac{1}{1+t^2}.\frac{t^2}{2}dt$

$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}\int\left(\frac{t^2+1-1}{1+t^2}\right)dt$

$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}\int\left(1 - \frac{1}{1+t^2}\right)dt$

$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}t + \frac{1}{2}\tan^{-1} t + c$

$= \frac{x\tan^2 x}{2} - \frac{1}{2}\tan x + \frac{1}{2}x + c$

$= \frac{x(\sec^2 x - 1)}{2} - \frac{1}{2}\tan x + \frac{1}{2}x + c$

$= \frac{1}{2}(x\sec^2 x - \tan x) + c$

178. $\int \log(x+1)dx = \int \log(x+1).1dx$

$= \log(x+1).x - \int \frac{x}{x+1}dx$

$= x\log(x+1) - \int \frac{x+1-1}{x+1}dx$

$= x\log(x+1) - \int\left(1 - \frac{1}{x+1}\right)dx$

$= x\log(x+1) - x + \log(x+1) + c$

$= (x+1)\log(x+1) - x + c$

179. $\int \frac{1}{\cos x(1+\cos x)}dx$

$= \int \frac{1+\cos x-\cos x}{\cos x(1+\cos x)}dx$

$= \int \frac{dx}{\cos x} - \int \frac{dx}{1+\cos x}$

$$= \int \sec x\,dx - \int \frac{dx}{2\cos^2\frac{x}{2}}$$

$$= \int \sec x\,dx - \frac{1}{2}\int \sec^2\frac{x}{2}dx$$

$$= \log|\sec x + \tan x| - \frac{1}{2}\left(\tan\frac{x}{2}\right)\cdot 2 + c$$

$$= \log|\sec x + \tan x| - \tan\frac{x}{2} + c$$

Competitive Thinking

1. Rationalizing the denominator, we get

$$\int \frac{dx}{\sqrt{x}+\sqrt{x-2}} = \frac{1}{2}\int(\sqrt{x}-\sqrt{x-2})\,dx$$

$$= \frac{1}{2}\left[\frac{x^{3/2}}{3/2} - \frac{(x-2)^{3/2}}{3/2}\right] + c$$

$$= \frac{1}{3}\left\{x^{3/2} - (x-2)^{3/2}\right\} + c$$

2. Let $f(x) = e^x$

$\therefore$ $\int [f(x)]^2\,dx = \int (e^x)^2\,dx = \frac{e^{2x}}{2} = \frac{1}{2}[f(x)]^2$

3. $\int e^{x\log a}\cdot e^x\,dx = \int e^{\log a^x}\cdot e^x\,dx = \int a^x e^x\,dx$

$$= \int (ae)^x\,dx = \frac{(ae)^x}{\log(ae)} + c$$

4. $\int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}dx = \int \frac{x^5 - x^4}{x^3 - x^2}dx$

$$= \int \frac{x^4(x-1)}{x^2(x-1)}dx = \int x^2 dx = \frac{x^3}{3} + c$$

5. $\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}dx = \int \frac{x^6 - x^5}{x^4 - x^3}dx$

$$= \int \frac{x^5(x-1)}{x^3(x-1)}dx$$

$$= \int x^2 dx = \frac{x^3}{3} + c$$

6. $\int x^{51}(\tan^{-1}x + \cot^{-1}x)\,dx = \int x^{51}.\frac{\pi}{2}dx$

$\ldots\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$

$$= \frac{\pi}{2}\cdot\frac{x^{52}}{52} + c$$

$$= \frac{x^{52}}{52}(\tan^{-1}x + \cot^{-1}x) + c$$

7. $\int \frac{x+1}{\sqrt{x}}\,dx = \int\left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)dx$

$$= \int (x^{\frac{1}{2}} + x^{-\frac{1}{2}})\,dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

8. $\int \frac{x+\sqrt{1-x^2}}{x\sqrt{1-x^2}}dx = \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{1}{x}dx$

$$= \sin^{-1}x + \log x + c$$

9. $\int \frac{\sin^2 x}{1+\cos x}dx = \int \frac{1-\cos^2 x}{1+\cos x}dx$

$$= \int (1-\cos x)\,dx$$

$$= x - \sin x + C$$

10. $\int \frac{dx}{1-\sin x} = \int \frac{(1+\sin x)}{1-\sin^2 x}dx = \int \frac{1+\sin x}{\cos^2 x}dx$

$$= \int \sec^2 x\,dx + \int \tan x.\sec x\,dx = \tan x + \sec x + c$$

11. $\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{(\cos^2 x + \sin^2 x)}{\cos^2 x\sin^2 x}dx$

$$= \int\left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}\right)dx$$

$$= \int \text{cosec}^2 x\,dx + \int \sec^2 x\,dx$$

$$= -\cot x + \tan x + c$$

12. $\int \frac{dx}{\sin x\cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x\cos^2 x}dx$

$$= \int \frac{\sin x}{\cos^2 x}dx + \int \frac{1}{\sin x}dx$$

$$= \int \sec x\tan x\,dx + \int \text{cosec}\,x\,dx$$

$$= \sec x + \log|\text{cosec}\,x - \cot x| + c$$

13. $\int \frac{1}{1+\cos 8x}dx = \int \frac{1}{2\cos^2 4x}dx$

$$= \frac{1}{2}\int \sec^2 4x\,dx$$

$$= \frac{\tan 4x}{8} + c$$

14. Let I = $\int \frac{\sin 2x}{\sin 5x \sin 3x}\, dx$

$= \int \frac{\sin(5x - 3x)}{\sin 5x \sin 3x}\, dx$

$= \int \frac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x}\, dx$

$= \int (\cot 3x - \cot 5x)\, dx$

$= \frac{1}{3}\log|\sin 3x| - \frac{1}{5}\log|\sin 5x| + c$

15. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx = \int \frac{2(\cos^2 x - \cos^2\theta)}{\cos x - \cos\theta} dx$

$= 2\int (\cos x + \cos\theta) dx$

$= 2(\sin x + x\cos\theta) + c$

16. $\int \frac{\cos x - 1}{\cos x + 1} dx = -\int \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} dx$

$= -\int \tan^2 \frac{x}{2} dx = -\int \left(\sec^2 \frac{x}{2} - 1\right) dx$

$= \int \left(1 - \sec^2 \frac{x}{2}\right) dx = x - 2\tan\frac{x}{2} + c$

17. $\int \frac{x + \sin x}{1 + \cos x} dx = \int \frac{x + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\, dx$

$= \frac{1}{2}\int x\sec^2\frac{x}{2} dx + \int \tan\frac{x}{2} dx$

$= \frac{1}{2}\cdot\frac{x\tan\frac{x}{2}}{\frac{1}{2}} - \frac{1}{2}\int \frac{\tan\frac{x}{2}}{\frac{1}{2}} dx + \int \tan\frac{x}{2} dx$

$= x\tan\frac{x}{2} + c$

18. $\int (1 - \cos x)\operatorname{cosec}^2 x\, dx$

$= \int \operatorname{cosec}^2 x\, dx - \int \cot x \operatorname{cosec} x\, dx$

$= -\cot x + \operatorname{cosec} x + c$

$= \frac{1 - \cos x}{\sin x} + c$

$= \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} + c$

$= \tan\left(\frac{x}{2}\right) + c$

19. $\int \sqrt{1 - \sin 2x}\, dx$

$= \int \sqrt{\cos^2 x + \sin^2 x - 2\sin x\cos x}\, dx$

$= \int \sqrt{(\cos x - \sin x)^2}\, dx$

$= \int (\cos x - \sin x) dx = \sin x + \cos x + c$

20. $\int \sqrt{1 + \sin\frac{x}{2}}\, dx$

$= \int \sqrt{\left(\sin^2\frac{x}{4} + \cos^2\frac{x}{4} + 2\sin\frac{x}{4}\cos\frac{x}{4}\right)}\, dx$

$= \int \left(\sin\frac{x}{4} + \cos\frac{x}{4}\right) dx = 4\left(\sin\frac{x}{4} - \cos\frac{x}{4}\right) + c$

21. $\int \sqrt{2}\sqrt{1 + \sin x}\, dx = \sqrt{2}\int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx$

$= 2\int \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) dx$

$= -4\cos\left(\frac{x}{2} + \frac{\pi}{4}\right) + c$

$\therefore$ a = $\frac{1}{2}$, b = $\frac{\pi}{4}$

22. $\int (\sin 2x - \cos 2x) dx = \frac{1}{\sqrt{2}}\sin(2x - a) + b$

$\Rightarrow -\frac{1}{2}(\sin 2x + \cos 2x) = \frac{1}{\sqrt{2}}\sin(2x - a) + b$

$\Rightarrow -\left[\frac{1}{\sqrt{2}}\sin 2x + \frac{1}{\sqrt{2}}\cos 2x\right] = \sin(2x - a) + b\sqrt{2}$

$\Rightarrow -\sin\left(2x + \frac{\pi}{4}\right) = \sin(2x - a) + b\sqrt{2}$

$\Rightarrow \sin\left(2x + \frac{5\pi}{4}\right) = \sin(2x - a) + b\sqrt{2}$

$\therefore$ b is any constant and a = $\frac{-5\pi}{4}$

23. $\int \frac{dx}{1 + \sin x} = \int \frac{dx}{1 + \cos\left(\frac{\pi}{2} - x\right)}$

$= \int \frac{dx}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}$

$= \frac{1}{2}\int \sec^2\left(\frac{x}{2} - \frac{\pi}{4}\right) dx$

$$= \frac{1}{2}.\frac{\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}{\frac{1}{2}} + c$$

$$= \tan\left(\frac{x}{2}-\frac{\pi}{4}\right)+c$$

$\therefore$ $a = \frac{-\pi}{4}$ and b = arbitrary constant

24. $\int \frac{\cos\alpha}{\sin x\cos(\alpha - x)}dx = \int\frac{\cos[(\alpha - x)+x]}{\sin x\cos(\alpha-x)}dx$

$$= \int\frac{\cos(\alpha-x)\cos x - \sin(\alpha-x)\sin x}{\sin x\cos(\alpha-x)}dx$$

$$= \int[\cot x - \tan(a-x)]dx$$

$$= \log|\sin x| - \log|\cos(\alpha - x)| + c_1$$

$$= -\log\left|\frac{\cos(\alpha-x)}{\sin x}\right| + c_1$$

$$= -\log\left|\frac{\cos\alpha\cos x + \sin\alpha\sin x}{\sin x}\right| + c_1$$

$$= -\log|\cos\alpha(\cot x + \tan\alpha)| + c_1$$

$$= -\log|\cot x + \tan\alpha| - \log|\cos\alpha| + c_1$$

$$= -\log|\cot x + \tan\alpha| + c$$

25. $\int\sec^4 x\tan x\,dx = \int\sec^3 x\sec x\tan x\,dx$

Put $t = \sec x \Rightarrow dt = \sec x\tan x\,dx$

$\therefore$ $\int\sec^4 x\tan x\,dx = \int t^3dt = \frac{t^4}{4}+c = \frac{1}{4}\sec^4 x + c$

26. $I_4 = \int\tan^4 x\,dx$, $I_6 = \int\tan^6 x\,dx$

$I_4 + I_6 = \int(\tan^4 x + \tan^6 x)\,dx$

$$= \int\tan^4 x(1+\tan^2 x)dx$$

$$= \int\tan^4 x\cdot\sec^2 x\,dx$$

$$= \frac{1}{5}\tan^5 x + c$$

Comparing with $a\tan^5 x + bx^5 + c$, we get

$\therefore$ $a = \frac{1}{5}$, $b = 0$

27. $I_4 - \frac{2}{3}I_2 = \int\left(\sec^4 x - \frac{2}{3}\sec^2 x\right)dx$

$$= \int\sec^2 x\left(\sec^2 x - \frac{2}{3}\right)dx$$

$$= \int\sec^2 x\left(\frac{3\sec^2 x - 2}{3}\right)dx$$

$$= \frac{1}{3}\int\sec^2 x(3\tan^2 x + 1)dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\therefore$ $I_4 - \frac{2}{3}I_2 = \frac{1}{3}\int(3t^2+1)dt$

$$= \frac{1}{3}(t^3 + t) + c$$

$$= \frac{1}{3}[t(t^2+1)] + c$$

$$= \frac{1}{3}[\tan x(\tan^2 x + 1)] + c$$

$$= \frac{1}{3}\sec^2 x\tan x + c$$

28. Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$\therefore$ $\int\frac{\cos(\log x)}{x}dx = \int\cos t\,dt$

$$= \sin t + c = \sin(\log x) + c$$

29. Put $x^2 = t \Rightarrow 2x\,dx = dt$

$\int x e^{x^2\log 2}e^{x^2}dx = \frac{1}{2}\int e^t 2^t dt$

$$= \frac{1}{2}\int(2e)^t dt$$

$$= \frac{(2e)^t}{2\log(2e)} + c$$

$$= \frac{2^{x^2}e^{x^2}}{2(\log 2 + 1)} + c$$

30. Put $t = \tan^{-1}x^2 \Rightarrow dt = \frac{1}{1+x^4}.2x\,dx$

$\therefore$ $\int\frac{2x\tan^{-1}x^2}{1+x^4}dx = \int t\,dt = \frac{t^2}{2}+c$

$$= \frac{1}{2}(\tan^{-1}x^2)^2 + c$$

31. Let $I = \int(e^x + e^{-x})^2(e^x - e^{-x})dx$

Put $e^x + e^{-x} = t \Rightarrow (e^x - e^{-x})\,dx = dt$

$\therefore$ $I = \int t^2.dt = \frac{t^3}{3}+c = \frac{1}{3}(e^x + e^{-x})^3 + c$

32. Put $t = x + \log\sec x$

$\Rightarrow dt = (1 + \tan x)dx$

$\therefore$ $\int\frac{1+\tan x}{x+\log\sec x}dx = \int\frac{1}{t}dt = \log t + c$

$$= \log(x + \log\sec x) + c$$

33. Put $1 + \log x = t \Rightarrow \frac{1}{x}dx = dt$

$\therefore \int \frac{1+\log x}{x}dx = \int t\,dt = \frac{t^2}{2} + c = \frac{(1+\log x)^2}{2} + c$

34. Put $a^2 + b^2 \sin^2 x = t$

$\Rightarrow b^2 \sin 2x\,dx = dt$

$\therefore \int \frac{\sin 2x}{a^2 + b^2 \sin^2 x}dx = \frac{1}{b^2}\int \frac{dt}{t}$

$= \frac{1}{b^2}\log t + c$

$= \frac{1}{b^2}\log(a^2 + b^2 \sin^2 x) + c$

35. Put $x^3 = t \Rightarrow 3x^2 dx = dt$

$\therefore \int x^2 \sec x^3 dx = \frac{1}{3}\int \sec t\,dt$

$= \frac{1}{3}\log(\sec t + \tan t) + c$

$= \frac{1}{3}\log\left(\sec x^3 + \tan x^3\right) + c$

36. $\int \frac{e^x(x+1)}{\cos^2(xe^x)}dx = \int e^x(x+1)\sec^2(xe^x)dx$

Put $xe^x = t \Rightarrow (x+1)e^x\,dx = dt$

$\therefore \int \frac{e^x(x+1)}{\cos^2(xe^x)}dx = \int \sec^2 t\,dt = \tan t + c$

$= \tan(xe^x) + c$

37. Let $I = \int \frac{1+x}{x+e^{-x}}dx = \int \frac{e^x(1+x)}{xe^x+1}dx$

Put $xe^x + 1 = t \Rightarrow e^x(1+x)dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + c = \log|1 + xe^x| + c$

38. Put $x + \tan^{-1}x = t$

$\Rightarrow \left(1 + \frac{1}{1+x^2}\right)dx = dt \quad \Rightarrow \frac{2+x^2}{1+x^2}dx = dt$

$\therefore \int\left[\frac{(x^2+2)a^{(x+\tan^{-1}x)}}{x^2+1}\right]dx = \int a^t dt$

$= \frac{a^t}{\log a} + c$

$= \frac{a^{x+\tan^{-1}x}}{\log a} + c$

39. Let $I = \int \frac{1}{4\sqrt{x}+x}dx = \int \frac{1}{\sqrt{x}\left(4+\sqrt{x}\right)}dx$

Put $4 + \sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$\therefore I = \int \frac{2dt}{t} = 2\log t + c$

$= 2\log\left(\sqrt{x}+4\right) + c$

40. Let $I = \int \frac{x\sin x}{x\cos x - \sin x - 1}dx$

Put $x\cos x - \sin x - 1 = t$

$\Rightarrow x \sin x\,dx = -dt$

$\therefore I = -\int \frac{dt}{t} = -\log|t| + c$

$= -\log|x\cos x - \sin x - 1| + c$

41. Let $I = \int \frac{dx}{\sin x\cos x + 3\cos^2 x} = \int \frac{\sec^2 x}{\tan x + 3}dx$

Put $\tan x + 3 = t$

$\Rightarrow \sec^2 x\,dx = dt$

$\therefore I = \int \frac{dt}{t} = \log|t| + c = \log|\tan x + 3| + c$

42. $\int \frac{\cos x}{\sqrt{1+\sin x}}dx = 2\sqrt{1+\sin x} + c$

$\ldots\left[\because \int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c\right]$

$= 2\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} + c$

$= 2\left[\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right] + c$

43. Put $1 + \log\tan\frac{x}{2} = t$

$\Rightarrow \left(\frac{1}{\tan\left(\frac{x}{2}\right)}\cdot\sec^2\frac{x}{2}\cdot\frac{1}{2}\right)dx = dt$

$\Rightarrow \operatorname{cosec} x\,dx = dt$

$\therefore \int \frac{\operatorname{cosec} x}{\cos^2\left(1+\log\tan\frac{x}{2}\right)}dx = \int \frac{dt}{\cos^2 t}$

$= \int \sec^2 t\,dt$

$= \tan t + c$

$= \tan\left(1 + \log\tan\frac{x}{2}\right) + c$

44. Put $x = t^2 \Rightarrow dx = 2t\,dt$

$\therefore \quad \int \frac{\log\sqrt{x}}{3x}dx = \int \frac{\log t}{3t^2}(2tdt)$

$= \frac{2}{3}\int \frac{\log t}{t}dt$

$= \frac{2}{3}\cdot\frac{(\log t)^2}{2} + c$

$= \frac{\left(\log\sqrt{x}\right)^2}{3} + c$

45. Let $I = \int\left[\frac{\log x - 1}{1+(\log x)^2}\right]^2 dx$

Put $\log x = t$

$\therefore \quad x = e^t \Rightarrow dx = e^t\,dt$

$\therefore \quad I = \int\left(\frac{t-1}{1+t^2}\right)^2 e^t dt$

$= \int\left[\frac{1+t^2-2t}{\left(1+t^2\right)^2}\right]e^t\,dt$

$= \int e^t\left[\frac{1}{1+t^2} + \frac{(-2t)}{\left(1+t^2\right)^2}\right]dt$

$= e^t\left(\frac{1}{1+t^2}\right) + c$

$\ldots\left[\because \int e^x\left[f(x)+f'(x)\right]dx = e^x f(x) + c\right]$

$= \frac{x}{1+(\log x)^2} + c$

46. $\int \frac{\sqrt{\tan x}}{\sin x\cos x}dx = \int \frac{\tan x}{\sqrt{\tan x}\sin x\cos x}dx$

$= \int \frac{\sin x\sec x}{\sqrt{\tan x}\sin x\cos x}dx$

$= \int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

Put $t = \tan x \Rightarrow dt = \sec^2 x\,dx$

$\therefore \quad \int \frac{\sqrt{\tan x}}{\sin x\cos x}dx = \int \frac{1}{\sqrt{t}}dt = 2t^{1/2} + c = 2\sqrt{\tan x} + c$

47. Let $I = \int \frac{x^3\sin\left[\tan^{-1}\left(x^4\right)\right]}{1+x^8}dx$

Put $x^4 = t \Rightarrow 4x^3\,dx = dt$

$\therefore \quad I = \frac{1}{4}\int \frac{\sin(\tan^{-1}t)}{1+t^2}dt$

Put $\tan^{-1} t = z \Rightarrow \frac{1}{1+t^2}dt = dz$

$\therefore \quad I = \frac{1}{4}\int \sin z\,dz = \frac{1}{4}(-\cos z) + c$

$= -\frac{1}{4}\cos(\tan^{-1}t) + c = \frac{-1}{4}\cos[\tan^{-1}(x^4)] + c$

48. Let $I = \int \operatorname{cosec}^4 x\,dx = \int \operatorname{cosec}^2 x.\operatorname{cosec}^2 x\,dx$

$= \int \operatorname{cosec}^2 x(1+\cot^2 x)dx$

$= \int \operatorname{cosec}^2 x\,dx + \int \cot^2 x.\operatorname{cosec}^2 x\,dx$

In 2nd integral, put $\cot x = t \Rightarrow -\operatorname{cosec}^2 x\,dx = dt$

$\therefore \quad I = \int \operatorname{cosec}^2 x\,dx - \int t^2 dt$

$= -\cot x - \frac{t^3}{3} + c = -\cot x - \frac{\cot^3 x}{3} + c$

49. Let $I = \int (x+1)(x+2)^7(x+3)dx$

Put $x + 2 = t \Rightarrow dx = dt$

$\therefore \quad I = \int (t-1)t^7(t+1)dt = \int (t^2-1)t^7 dt$

$= \int (t^9 - t^7)dt$

$= \frac{t^{10}}{10} - \frac{t^8}{8} + c$

$= \frac{(x+2)^{10}}{10} - \frac{(x+2)^8}{8} + c$

50. $\int \sec x\,dx = \log(\sec x + \tan x) + c$

$= \log\left(\frac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right) + c$

$= \log\left(\frac{1}{\sec x - \tan x}\right) + c$

$= -\log(\sec x - \tan x) + c$

51. $\int \frac{1-\tan x}{1+\tan x}dx = \int \tan\left(\frac{\pi}{4} - x\right)dx$

$= \frac{-\log\cos\left(\frac{\pi}{4} - x\right)}{-1} + c$

$= \log\cos\left(\frac{\pi}{4} - x\right) + c$

$= \log\sin\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - x\right)\right] + c$

$= \log\sin\left(\frac{\pi}{4} + x\right) + c$

52. $\int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}}$

$= \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx$

$= \frac{1}{\sqrt{2}} \log \tan\left(\frac{\pi}{8} + \frac{x}{2}\right) + c$

53. $\int \frac{1}{\sqrt{1+\sin x}} dx = \int \frac{1}{\sin\frac{x}{2} + \cos\frac{x}{2}} dx$

$= \int \frac{1}{\sqrt{2}\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)} dx$

$= \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(\frac{x}{2} + \frac{\pi}{4}\right) dx$

$= \sqrt{2} \log \tan\left(\frac{\pi}{8} + \frac{x}{4}\right) + c$

54. Let $I = \sqrt{2} \int \frac{\sin x}{\sin\left(x - \frac{\pi}{4}\right)} dx$

Put $x - \frac{\pi}{4} = t \Rightarrow dx = dt$

$\therefore$ $I = \sqrt{2} \int \frac{\sin\left(\frac{\pi}{4} + t\right)}{\sin t} dt = \int \frac{\cos t + \sin t}{\sin t} dt$

$= \int \cot t \, dt + \int dt = \log |\sin t| + t + c_1$

$= x - \frac{\pi}{4} + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + c_1$

$= x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + c$, where $c = c_1 - \frac{\pi}{4}$

55. $\int (1 + 2\tan^2 x + 2\tan x \sec x)^{1/2} dx$

$= \int (\sec^2 x + \tan^2 x + 2\tan x \sec x)^{1/2} dx$

$= \int (\sec x + \tan x) dx$

$= \log(\sec x + \tan x) + \log \sec x + c$

$= \log \sec x(\sec x + \tan x) + c$

56. $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{e^{2x} + 1} dx$

$= \int \frac{dt}{t^2 + 1}$

….[Put $e^x = t \Rightarrow e^x dx = dt$]

$= \tan^{-1}(t) + c$

$= \tan^{-1}(e^x) + c$

57. Let $I = \int \frac{1}{1+e^x} dx = \int \frac{1}{1 + \frac{1}{e^{-x}}} dx$

$= \int \frac{e^{-x}}{1 + e^{-x}} dx$

Put $e^{-x} = t \Rightarrow - e^{-x} dx = dt$

$\therefore$ $I = -\int \frac{1}{1+t} dt$

$= - \log(1 + t) + c$

$= - \log(1 + e^{-x}) + c$

$= - \log\left(\frac{1+e^x}{e^x}\right) + c$

$= \log\left(\frac{e^x}{1+e^x}\right) + c$

58. Let $I = \int \frac{dx}{e^x + e^{-x} + 2}$

$= \int \frac{e^x \, dx}{e^{2x} + 2e^x + 1}$

Put $e^x = t \Rightarrow e^x dx = dt$

$\therefore$ $I = \int \frac{dt}{t^2 + 2t + 1} = \int \frac{dt}{(t+1)^2}$

$= \frac{-1}{t+1} + c = \frac{-1}{e^x + 1} + c$

59. Put $x^2 = t \Rightarrow x dx = \frac{dt}{2}$

$\therefore$ $\int \frac{x}{1+x^4} dx = \frac{1}{2} \int \frac{dt}{1+t^2}$

$= \frac{1}{2} \tan^{-1} t + c$

$= \frac{1}{2} \tan^{-1}(x^2) + c$

60. $\int \frac{1}{(1+x)\sqrt{x}} dx = \int \frac{1}{\left[1 + \left(\sqrt{x}\right)^2\right]\sqrt{x}} dx$

Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt$

$\therefore$ $\int \frac{1}{(1+x)\sqrt{x}} dx = \int \frac{2}{1+t^2} dt$

$= 2\tan^{-1} t + A$

$\therefore$ $\int \frac{1}{(1+x)\sqrt{x}} dx = 2\tan^{-1}\sqrt{x} + A$

$\therefore$ $f(x) = 2\tan^{-1}\sqrt{x}$

61. Let $I = \int \frac{x^2}{1+(x^3)^2}\,dx$

Put $x^3 = t \Rightarrow 3x^2\,dx = dt$

$\therefore$ $I = \frac{1}{3}\int \frac{dt}{1+t^2}$

$= \frac{1}{3}\tan^{-1} t + c$

$= \frac{1}{3}\tan^{-1} x^3 + c$

62. Put $x^4 = t \Rightarrow 4x^3\,dx = dt$

$\therefore$ $\int \frac{x^3 dx}{1+x^8} = \frac{1}{4}\int \frac{dt}{1+t^2} = \frac{1}{4}\tan^{-1} t + c$

$= \frac{1}{4}\tan^{-1}(x^4) + c$

63. $\int \frac{dx}{16x^2+9} = \int \frac{dx}{(4x)^2+3^2}$

$= \frac{1}{12}\tan^{-1}\left(\frac{4x}{3}\right) + c$

64. Let $I = \int \frac{1}{\sqrt{9-16x^2}}\,dx$

$= \int \frac{1}{\sqrt{3^2-(4x)^2}}\,dx$

$= \frac{1}{4}\sin^{-1}\left(\frac{4x}{3}\right) + c$

Comparing with $\alpha\sin^{-1}(\beta x) + c$, we get

$\alpha = \frac{1}{4}$, $\beta = \frac{4}{3}$

$\therefore$ $\alpha + \frac{1}{\beta} = \frac{1}{4} + \frac{3}{4} = 1$

65. Let $I = \int \frac{dx}{\sqrt{16-9x^2}}$

$= \int \frac{1}{\sqrt{4^2-(3x)^2}}\,dx$

$= \frac{1}{3}\sin^{-1}\frac{3x}{4} + C$

Comparing with $A\sin^{-1}(Bx) + C$, we get

$A = \frac{1}{3}$, $B = \frac{3}{4}$

$\therefore$ $A + B = \frac{1}{3} + \frac{3}{4} = \frac{13}{12}$

66. Let $I = \int \frac{dx}{\sqrt{x^{10}-x^2}}$

$= \int \frac{dx}{x\sqrt{x^8-1}}$

$= \int \frac{x^3\,dx}{x^4\sqrt{x^8-1}}$

Put $x^4 = t \Rightarrow 4x^3\,dx = dt$

$\therefore$ $I = \frac{1}{4}\int \frac{dt}{t\sqrt{t^2-1}}$

$= \frac{1}{4}\sec^{-1} t + c$

$= \frac{1}{4}\sec^{-1}(x^4) + c$

67. Let $I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x}dx$

$= \int \frac{2\sin x\cos x}{\sin^4 x + \cos^4 x}dx$

$= \int \frac{2\tan x\sec^2 x}{1+\tan^4 x}dx$

Put $\tan^2 x = t \Rightarrow 2\tan x\sec^2 x dx = dt$

$\therefore$ $I = \int \frac{dt}{1+t^2} = \tan^{-1} t + c = \tan^{-1}(\tan^2 x) + c$

68. Let $I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x}dx$

$= \int \frac{2\sin x\cos x}{\sin^4 x + \cos^4 x}dx$

$= \int \frac{2\tan x\sec^2 x}{1+\tan^4 x}dx$

Put $\tan^2 x = t \Rightarrow 2\tan x\sec^2 x\,dx = dt$

$\therefore$ $I = \int \frac{dt}{1+t^2} = \tan^{-1} t + c = \tan^{-1}(\tan^2 x) + c$

Comparing with $\tan^{-1}[f(x)] + c$, we get

$f(x) = \tan^2 x$

$\therefore$ $f\left(\frac{\pi}{3}\right) = \tan^2\frac{\pi}{3} = \left(\sqrt{3}\right)^2 = 3$

69. Let I

$= \int \frac{\sin^2 x\cos^2 x}{\left(\sin^5 x + \cos^3 x\sin^2 x + \sin^3 x\cos^2 x + \cos^5 x\right)^2}\,dx$

$= \int \frac{\sin^2 x\cos^2 x}{\left[\left(\sin^2 x + \cos^2 x\right)\left(\sin^3 x + \cos^3 x\right)\right]^2}\,dx$

$= \int \frac{\sin^2 x\cos^2 x}{\left(\sin^3 x + \cos^3 x\right)^2}\,dx$

Dividing numerator and denominator by $\cos^6 x$, we get

$$I = \int \frac{\tan^2 x \cdot \sec^2 x}{\left(1+\tan^3 x\right)^2}\, dx$$

Put $1 + \tan^3 x = t \Rightarrow 3\tan^2 x \sec^2 x\, dx = dt$

$\therefore$ $I = \frac{1}{3}\int \frac{dt}{t^2} = -\frac{1}{3t} + c = \frac{-1}{3\left(1+\tan^3 x\right)} + c$

70. Put $a^x = t \Rightarrow a^x \log_e a\, dx = dt$

$\therefore$ $\int \frac{a^x}{\sqrt{1-a^{2x}}} dx = \frac{1}{\log_e a}\int \frac{dt}{\sqrt{1-t^2}}$

$$= \frac{1}{\log_e a}\sin^{-1}(t) + c = \frac{\sin^{-1}(a^x)}{\log_e a} + c$$

71. $\int \frac{1}{\sqrt{8+2x-x^2}} dx = \int \frac{1}{\sqrt{8+1-\left(x^2-2x+1\right)}} dx$

$$= \int \frac{1}{\sqrt{3^2-\left(x-1\right)^2}} dx$$

$$= \sin^{-1}\left(\frac{x-1}{3}\right) + c$$

72. $\int \frac{1}{\sqrt{3-6x-9x^2}} dx = \int \frac{1}{\sqrt{3-\left(9x^2+6x\right)}} dx$

$$= \int \frac{1}{\sqrt{4-\left(9x^2+6x+1\right)}} dx$$

$$= \int \frac{1}{\sqrt{2^2-\left(3x+1\right)^2}} dx$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x+1}{2}\right) + c$$

73. $I = \int \frac{dx}{\sqrt{(1-x)(x-2)}} = \int \frac{dx}{\sqrt{-2+3x-x^2}}$

$$= \int \frac{dx}{\sqrt{-2+\frac{9}{4}-\left(x^2-3x+\frac{9}{4}\right)}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}$$

$$= \sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{1}{2}}\right) + C$$

$\therefore$ $I = \sin^{-1}(2x-3) + C$

74. Let $I = \int \sqrt{\frac{x-5}{x-7}}\, dx$

$$= \int \frac{x-5}{\sqrt{(x-7)(x-5)}}\, dx$$

$$= \int \frac{x-5}{\sqrt{x^2-12x+35}}\, dx$$

$$= \frac{1}{2}\int \frac{2x-10}{\sqrt{x^2-12x+35}}\, dx$$

$$= \frac{1}{2}\int \frac{2x-12+2}{\sqrt{x^2-12x+35}}\, dx$$

$$= \frac{1}{2}\int \frac{2x-12}{\sqrt{x^2-12x+35}}\, dx + \frac{2}{2}\int \frac{dx}{\sqrt{x^2-12x+36-1}}$$

$$= \frac{1}{2}\times 2\sqrt{x^2-12x+35} + \int \frac{dx}{\sqrt{(x-6)^2-1}}$$

$$= \sqrt{x^2-12x+35} + \log\left|(x-6)+\sqrt{x^2-12x+35}\right| + c$$

Comparing with $A\sqrt{x^2-12x+35}$

$+\log\left|(x-6)+\sqrt{x^2-12x+35}\right| + c$, we get

$A = 1$

75. Let $I = \int \sqrt{x^2+2x+5}\, dx$

$$= \int \sqrt{\left(x+1\right)^2+2^2}\, dx$$

$$= \frac{x+1}{2}\sqrt{x^2+2x+5}$$

$$+ 2\log\left|x+1+\sqrt{x^2+2x+5}\right| + c$$

76. Let $I = \int \frac{\sec^8 x}{\operatorname{cosec} x} dx$

$$= \int \frac{\sin x}{\cos^8 x} dx$$

$$= \int \tan x \cdot \sec^7 x\, dx$$

$$= \int \sec^6 x \sec x \tan x\, dx$$

Put $\sec x = t \Rightarrow \sec x \tan x\, dx = dt$

$\therefore$ $I = \int t^6 dt$

$$= \frac{t^7}{7} + c$$

$$= \frac{\sec^7 x}{7} + c$$

77. $\int \frac{dx}{5+4\cos x} = \int \frac{dx}{5+4\left[\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]}$

$= \int \frac{\sec^2\frac{x}{2}}{9+\tan^2\frac{x}{2}}\, dx$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$\therefore \int \frac{dx}{5+4\cos x} = 2\int \frac{dt}{3^2+t^2}$

$= \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+c$

$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$

78. Let $I = \int \frac{dx}{7+5\cos x} = \int \frac{dx}{7+5\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right)}$

$= \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{12+2\tan^2\left(\frac{x}{2}\right)} = \int \frac{\frac{1}{2}\sec^2\left(\frac{x}{2}\right).dx}{6+\tan^2\left(\frac{x}{2}\right)}$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$\therefore I = \int \frac{dt}{t^2+\left(\sqrt{6}\right)^2} = \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{t}{\sqrt{6}}\right)+c$

$= \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{6}}\right)+c$

79. $\int \frac{dx}{1+3\sin^2 x} = \int \frac{dx}{\sin^2 x+\cos^2 x+3\sin^2 x}$

$= \int \frac{dx}{4\sin^2 x+\cos^2 x}$

$= \int \frac{\sec^2 x\, dx}{4\tan^2 x+1}$

$= \frac{1}{4}\int \frac{\sec^2 x\, dx}{\tan^2 x+\frac{1}{4}}$

Put $t = \tan x \Rightarrow dt = \sec^2 x dx$

$\therefore \int \frac{dx}{1+3\sin^2 x} = \frac{1}{4}\int \frac{dt}{t^2+\left(\frac{1}{2}\right)^2} = \frac{1}{4}.2\tan^{-1}(2t)+c$

$= \frac{1}{2}\tan^{-1}(2t)+c$

$= \frac{1}{2}\tan^{-1}(2\tan x)+c$

80. $\int \frac{5\tan x}{\tan x-2}dx = \int \frac{5\sin x}{\sin x-2\cos x}dx$

$= \int \frac{(\sin x-2\cos x)+2(\cos x+2\sin x)}{\sin x-2\cos x}dx$

$= \int dx + 2\int \frac{\cos x+2\sin x}{\sin x-2\cos x}dx$

$= x + 2\log|\sin x - 2\cos x| + k$

$\therefore$ a = 2

81. $\int \frac{\sin x\, dx}{\sin x-\cos x} = \frac{1}{2}\int \frac{2\sin x}{\sin x-\cos x}\, dx$

$= \frac{1}{2}\int \frac{(\sin x-\cos x+\sin x+\cos x)}{\sin x-\cos x}dx$

$= \frac{1}{2}\int \left(1+\frac{\sin x+\cos x}{\sin x-\cos x}\right)dx$

$= \frac{1}{2}[x+\log(\sin x-\cos x)]+c$

82. $\int \frac{4e^x-25}{2e^x-5}\, dx = \int \frac{5(2e^x-5)-3\left(2e^x\right)}{2e^x-5}\, dx$

$= 5\int dx - 3\int \frac{2e^x}{2e^x-5}\, dx$

$= 5x - 3\log|2e^x - 5| + c$

$\therefore$ A = 5 and B = – 3

83. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

$= \frac{1}{\sin(a-b)}\int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}\, dx$

$= \frac{1}{\sin(a-b)}\int \frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}dx$

$= \frac{1}{\sin(a-b)}\left[\int \cot(x-a)dx - \int \cot(x-b)dx\right]$

$= \frac{1}{\sin(a-b)}\left[\log|\sin(x-a)| - \log|\sin(x-b)|\right]+c$

$= \frac{1}{\sin(a-b)}\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+c$

84. Let $I = \int \sqrt{e^x - 1}\, dx$

Put $e^x - 1 = t^2$

$\Rightarrow e^x\, dx = 2t\, dt$

$\Rightarrow dx = \frac{2t}{t^2+1} dt$

$\therefore \quad I = \int t \cdot \frac{2t}{t^2+1} dt = \int \frac{2t^2}{t^2+1} dt$

$= \int \frac{2(t^2+1)-2}{t^2+1} dt$

$= 2\int dt - 2\int \frac{dt}{t^2+1}$

$= 2t - 2\tan^{-1} t + c$

$= 2\left(\sqrt{e^x-1} - \tan^{-1}\sqrt{e^x-1}\right) + c$

85. Let $I = \int \frac{1}{\left(e^x + e^{-x}\right)^2} dx = \int \frac{e^{2x}}{\left(e^{2x}+1\right)^2} dx$

Put $e^{2x} + 1 = t \Rightarrow 2\, e^{2x}\, dx = dt$

$\therefore \quad I = \frac{1}{2}\int \frac{1}{t^2} dt = -\frac{1}{2}.\frac{1}{t} + c$

$= \frac{-1}{2\left(2e^x+1\right)} + c$

86. $\int \frac{1}{\sqrt{1-e^{2x}}} dx = \int \frac{e^{-x}}{\sqrt{e^{-2x}-1}} dx$

Put $e^{-x} = t \Rightarrow -e^{-x}dx = dt$

$\therefore \quad \int \frac{1}{\sqrt{1-e^{2x}}} dx = -\int \frac{1}{\sqrt{t^2-1}} dt$

$= -\log\left[t + \sqrt{t^2-1}\right] + c$

$= -\log\left[e^{-x} + \sqrt{e^{-2x}-1}\right] + c$

$= -\log\left[\frac{1}{e^x} + \frac{\sqrt{1-e^{2x}}}{e^x}\right] + c$

$= -\log\left[1 + \sqrt{1-e^{2x}}\right] + \log e^x + c$

$= x - \log\left[1 + \sqrt{1-e^{2x}}\right] + c$

87. Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\therefore \quad \int \frac{1+x^2}{\sqrt{1-x^2}} dx = \int \frac{1+\sin^2\theta}{\sqrt{1-\sin^2\theta}} .\cos\theta\, d\theta$

$= \int (1+\sin^2\theta) d\theta$

$= \int d\theta + \int \left(\frac{1-\cos 2\theta}{2}\right) d\theta$

$= \theta + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + c$

$= \frac{3}{2}\theta - \frac{\sin\theta\cos\theta}{2} + c$

$= \frac{3}{2}\theta - \frac{\sin\theta\sqrt{1-\sin^2\theta}}{2} + c$

$= \frac{3}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + c$

88. Let $I = \int \frac{1}{x^2(x^4+1)^{\frac{3}{4}}} dx = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}}$

Put $1 + \frac{1}{x^4} = t \Rightarrow \frac{-4}{x^5} dx = dt$

$\therefore \quad I = -\frac{1}{4}\int \frac{dt}{t^{\frac{3}{4}}} = -\frac{1}{4} \times 4t^{\frac{1}{4}} + c = -t^{\frac{1}{4}} + c$

$= -\left(1 + \frac{1}{x^4}\right)^{\frac{1}{4}} + c = -\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$

89. $\int \frac{2x^{12} + 5x^9}{\left(x^5 + x^3 + 1\right)^3} dx$

$= \int \frac{x^{15}\left(\frac{2}{x^3} + \frac{5}{x^6}\right)}{x^{15}\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^3} dx$

Put $1 + \frac{1}{x^2} + \frac{1}{x^5} = t \Rightarrow \left(\frac{-2}{x^3} - \frac{5}{x^6}\right) dx = dt$

$\therefore \quad \int \frac{2x^{12} + 5x^9}{\left(x^5 + x^3 + 1\right)^3} dx = -\int \frac{dt}{t^3}$

$= \frac{1}{2t^2} + C$

$= \frac{1}{2\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^2} + C$

$= \frac{x^{10}}{2\left(x^5 + x^3 + 1\right)^2} + C$

90. Put $\frac{x-1}{x+2} = t \Rightarrow \frac{1}{(x+2)^2}dx = \frac{1}{3}dt$

$\therefore \quad \int \frac{1}{\left[(x-1)^3 (x+2)^5\right]^{1/4}} dx$

$= \int \frac{1}{(x-1)^{3/4}(x+2)^{-3/4}(x+2)^2} dx$

$= \frac{1}{3}\int t^{-3/4} dt = \frac{1}{3}\cdot\frac{t^{\frac{1}{4}}}{\frac{1}{4}} + c$

$= \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$

91. Let I $= \int \frac{(x-2)dx}{\left\{(x-2)^2 (x+3)^7\right\}^{1/3}}$

$= \int \frac{dx}{(x-2)^{-1/3}(x+3)^{7/3}}$

$= \int \frac{dx}{(x-2)^{-1/3}.(x-2)^{7/3}\left(\frac{x+3}{x-2}\right)^{7/3}}$

$\therefore \quad I = \int \frac{dx}{(x-2)^2\left(\frac{x+3}{x-2}\right)^{7/3}}$

Put $\frac{x+3}{x-2} = t \Rightarrow \frac{-5}{(x-2)^2}dx = dt$

$\Rightarrow \frac{dx}{(x-2)^2} = \frac{-1}{5} dt$

$\therefore \quad I = \frac{-1}{5}\int \frac{dt}{t^{7/3}} = \frac{-1}{5}\cdot\frac{t^{-4/3}}{\left(\frac{-4}{3}\right)} + c$

$= \frac{3}{20}\left(\frac{x-2}{x+3}\right)^{\frac{4}{3}} + c$

92. $I = \int \frac{\sin 2x}{(3+4\cos x)^3} dx$

$\Rightarrow I = \int \frac{2\sin x \cos x}{(3+4\cos x)^3} dx$

Put $3 + 4\cos x = t \Rightarrow \cos x = \frac{t-3}{4}$

$\Rightarrow \sin x\, dx = \frac{dt}{(-4)}$

$\therefore \quad I = \int \frac{2\left(\frac{dt}{-4}\right).\left(\frac{t-3}{4}\right)}{(t)^3}$

$= \frac{-1}{8}\int \frac{t-3}{t^3} dt$

$= \frac{-1}{8}\left(\int \frac{dt}{t^2} - 3\int \frac{dt}{t^3}\right)$

$= \frac{-1}{8}\left(\frac{-1}{t} + \frac{3}{2t^2}\right) + C$

$= \left(\frac{1}{8t} - \frac{3}{16t^2}\right) + C$

$= \frac{2t-3}{16t^2} + C = \frac{2(3+4\cos x)-3}{16(3+4\cos x)^2} + C$

$\therefore \quad I = \frac{3+8\cos x}{16(3+4\cos x)^2} + C$

93. Let $I = \int \frac{dx}{\cos x\sqrt{1+\cos 2x+\sin 2x}}$

$= \int \frac{dx}{\cos x\sqrt{2\cos^2 x + 2\sin x \cos x}}$

$= \int \frac{dx}{\cos^2 x\sqrt{2+2\tan x}}$

$= \int \frac{\sec^2 x\, dx}{\sqrt{2+2\tan x}}$

Put $2 + 2\tan x = t \Rightarrow 2\sec^2 x\, dx = dt$

$\therefore \quad I = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$

$= \frac{1}{2}(2)\sqrt{t} + c$

$= \sqrt{t} + c = \sqrt{2+2\tan x} + c$

94. Let $I = \int \frac{\sin\theta + \cos\theta}{\sqrt{2\sin\theta\cos\theta}} d\theta$

$= \int \frac{\sin\theta+\cos\theta}{\sqrt{1-(1-2\sin\theta\cos\theta)}} d\theta$

$= \int \frac{(\sin\theta+\cos\theta)d\theta}{\sqrt{1-(\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta)}}$

$= \int \frac{\sin\theta+\cos\theta}{\sqrt{1-(\sin\theta-\cos\theta)^2}} d\theta$

Put $(\sin\theta - \cos\theta) = t$
$\Rightarrow (\cos\theta + \sin\theta)\, d\theta = dt$

$\therefore \quad I = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) + c = \sin^{-1}(\sin\theta - \cos\theta) + c$

95. Let $I = \int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}$

Put $x = a \tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

$\therefore \quad I = \int \frac{a\sec^2\theta}{(a^2 + a^2\tan^2\theta)^{\frac{3}{2}}} d\theta = \int \frac{a\sec^2\theta}{a^3(\sec^2\theta)^{\frac{3}{2}}} d\theta$

$= \frac{1}{a^2}\int \frac{d\theta}{\sec\theta} = \frac{1}{a^2}\int \cos\theta\, d\theta = \frac{1}{a^2}\sin\theta + c$

$= \frac{1}{a^2}\cdot\frac{\tan\theta}{\sqrt{1+\tan^2\theta}} + c$

$= \frac{x}{a^2(x^2+a^2)^{\frac{1}{2}}} + c$

96. Let $I = \int \frac{1}{a + be^x}dx = \int \frac{e^{-x}}{ae^{-x} + b}dx$

Put $ae^{-x} + b = t \Rightarrow -ae^{-x}\,dx = dt$

$\therefore \quad I = -\frac{1}{a}\int\frac{dt}{t} = -\frac{1}{a}\ \log|t| + c$

$= -\frac{1}{a}\ \log|ae^{-x} + b| + c$

$= -\frac{1}{a}\log\left|\frac{a + be^x}{e^x}\right| + c$

$= \frac{1}{a}\log\left|\frac{e^x}{a + be^x}\right| + c$

97. Put $x^e + e^x = t$

$\Rightarrow (ex^{e-1} + e^x)\,dx = dt$

$\Rightarrow e(x^{e-1} + e^{x-1})dx = dt$

$\Rightarrow (x^{e-1} + e^{x-1})dx = \frac{dt}{e}$

$\therefore \quad \int\frac{x^{e-1} + e^{x-1}}{x^e + e^x}\ dx = \frac{1}{e}\int\frac{dt}{t} = \frac{1}{e}\ \log|t| + c$

$= \frac{1}{e}\ \log|x^e + e^x| + c$

98. Let $I = \int (x^2+1)\sqrt{x+1}\,dx$

Put $\sqrt{x+1} = t \Rightarrow x + 1 = t^2$

$\Rightarrow x = t^2 - 1 \Rightarrow dx = 2t\ dt$

$\therefore \quad I = \int\left((t^2-1)^2 + 1\right)(t)(2tdt)$

$= \int (t^4 - 2t^2 + 2)(2t^2)dt$

$= 2\left[\int t^6dt - 2\int t^4dt + 2\int t^2dt\right]$

$= 2\left[\frac{t^7}{7} - \frac{2t^5}{5} + \frac{2t^3}{3}\right] + c$

$= 2\left[\frac{(x+1)^{7/2}}{7} - \frac{2(x+1)^{5/2}}{5} + \frac{2(x+1)^{3/2}}{3}\right] + c$

99. Let $I = \int\frac{x^3dx}{(x^2+1)^3}$

Put $x^2 + 1 = t \Rightarrow 2x\,dx = dt$

$\therefore \quad I = \frac{1}{2}\int\frac{(t-1)dt}{t^3} = \frac{1}{2}\int\left(t^{-2} - t^{-3}\right)dt$

$= \frac{1}{2}\left[\frac{t^{-1}}{-1} - \frac{t^{-2}}{-2}\right] + K$

$= \frac{1}{2}\left[\frac{1}{2(x^2+1)^2} - \frac{1}{x^2+1}\right] + K$

$= \frac{1}{2}\left[\frac{1 - 2x^2 - 2}{2(x^2+1)^2}\right] + K$

$= \frac{1}{2}\left[\frac{-(1+2x^2)}{2(x^2+1)^2}\right] + K$

$= \frac{-1}{4}\left[\frac{2x^2+1}{(x^2+1)^2}\right] + K$

100. Let $I = \int\frac{x^2-1}{x^4+3x^2+1}\,dx$

$= \int\frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2} + 3}\,dx = \int\frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 + 1}\,dx$

Put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right)dx = dt$

$\therefore \quad I = \int\frac{dt}{1+t^2}$

$= \tan^{-1} t + c$

$= \tan^{-1}\left(x + \frac{1}{x}\right) + c$

101. Let $I = \int\frac{2 - \sin x}{2 + \cos x}dx$

$= \int\frac{2}{2+\cos x}dx - \int\frac{\sin x}{2+\cos x}dx$

$= I_1 - I_2$

$I_1 = \int\frac{2}{2+\cos x}dx$

Put $\tan\left(\frac{x}{2}\right) = t$

$\therefore$ $dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

$\therefore$ $I_1 = 2\int \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$

$= \int \frac{2.2dt}{t^2+3} = 4\int \frac{dt}{t^2+\left(\sqrt{3}\right)^2}$

$= \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c_1$

$= \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{3}}\right) + c_1$

and $I_2 = \int \frac{\sin x}{2+\cos x} dx = -\log(2 + \cos x) + c_2$

$\therefore$ $I = I_1 - I_2$

$= \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{3}}\right) + \log(2 + \cos x) + c$

102. Let $I = \int \frac{x^2 dx}{\sqrt{1-x}}$

Put $1 - x = t^2 \Rightarrow dx = -2tdt$

$\therefore$ $I = -2\int \frac{\left(1-t^2\right)^2 tdt}{t} = -2\int \left(1-t^2\right)^2 dt$

$= -2\int \left(1+t^4-2t^2\right) dt$

$= -2\left[t+\frac{t^5}{5}-\frac{2t^3}{3}\right]$

$= -2t\left[\frac{15+3t^4-10t^2}{15}\right]$

$= \frac{-2}{15}\sqrt{1-x}\ [15 + 3(1 - x)^2 - 10(1 - x)]$

$= \frac{-2}{15}\sqrt{1-x}\ (3x^2 + 4x + 8)$

$\therefore$ $P = \frac{-2}{15}$

103. Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$\therefore$ $f(x) = \int \frac{\tan^2\theta \sec^2\theta d\theta}{\sec^2\theta(1+\sec\theta)}$

$= \int \frac{\tan^2\theta d\theta}{1+\sec\theta} = \int \frac{\sin^2\theta d\theta}{\cos\theta(1+\cos\theta)}$

$= \int \frac{1-\cos^2\theta d\theta}{\cos\theta(1+\cos\theta)}$

$= \int \frac{(1-\cos\theta)d\theta}{\cos\theta} = \int \sec\theta d\theta - \int d\theta$

$= \log(\sec\theta + \tan\theta) - \theta + c$

$\therefore$ $f(x) = \log\left(x+\sqrt{1+x^2}\right) - \tan^{-1} x + c$

$\therefore$ $f(0) = \log\left(0+\sqrt{1+0}\right) - \tan^{-1}(0) + c$

$\Rightarrow 0 = \log 1 - 0 + c \Rightarrow c = 0$

$\therefore$ $f(x) = \log\left(x+\sqrt{1+x^2}\right) - \tan^{-1} x$

$\therefore$ $f(1) = \log\left(1+\sqrt{1+1^2}\right) - \tan^{-1}(1)$

$= \log\left(1+\sqrt{2}\right) - \frac{\pi}{4}$

104. $J - I = \int\left(\frac{e^{3x}}{e^{4x}+e^{2x}+1} - \frac{e^x}{e^{4x}+e^{2x}+1}\right)dx$

$= \int \frac{\left(e^{2x}-1\right)e^x}{e^{4x}+e^{2x}+1} dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$\therefore$ $J - I = \int \frac{t^2-1}{t^4+t^2+1} dt = \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-1} dt$

Put $t + \frac{1}{t} = y$

$\Rightarrow \left(1-\frac{1}{t^2}\right)dt = dy$

$\therefore$ $J - I = \int \frac{dy}{y^2-1^2} = \frac{1}{2}\log\left|\frac{y-1}{y+1}\right| + C$

$= \frac{1}{2}\log\left|\frac{t+\frac{1}{t}-1}{t+\frac{1}{t}+1}\right| + C$

$= \frac{1}{2}\log\left|\frac{t^2-t+1}{t^2+t+1}\right| + C$

$= \frac{1}{2}\log\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right| + C$

105. Let $I = \int \frac{\sec^2 x}{(\sec x+\tan x)^{\frac{9}{2}}} dx$

Put $\sec x + \tan x = t$ ….(i)

$\Rightarrow \sec x(\sec x + \tan x)dx = dt$

$\Rightarrow \sec x\, dx = \frac{1}{t} dt$

Also, $\sec x - \tan x = \frac{1}{t}$ ....(ii)

Adding (i) and (ii), we get

$\sec x = \frac{1}{2}\left(t + \frac{1}{t}\right)$

$\therefore$ $I = \frac{1}{2}\int \frac{\left(t+\frac{1}{t}\right)}{t^{\frac{9}{2}}}\cdot\frac{1}{t}dt = \frac{1}{2}\int\left[\frac{1}{t^{\frac{9}{2}}} + \frac{1}{t^{\frac{13}{2}}}\right]dt$

$= -\frac{1}{7t^{\frac{7}{2}}} - \frac{1}{11t^{\frac{11}{2}}} + K = -\frac{1}{t^{\frac{11}{2}}}\left(\frac{t^2}{7} + \frac{1}{11}\right) + K$

$= \frac{-1}{(\sec x + \tan x)^{\frac{11}{2}}}\left[\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right] + K$

106. $\int f(x)\cdot g(x)dx = \int x\cdot \sin x\, dx$

$= -x\cos x + \int \cos x\, dx$

$= -x \cos x + \sin x + c$

107. $\int x\sin x\, dx = -x\cos x + \int \cos x\, dx$

$= -x\cos x + \sin x + \text{constant}$

$\therefore$ $A = \sin x + \text{constant}$

108. $\int \log_{10} x\, dx = \int \frac{\log x}{\log 10} dx$

$= \frac{1}{\log 10}(x \log x - x) + c$

$= x(\log_{10}x - \log_{10}e) + c$

109. $\int x^2 \sin 2x\, dx = x^2\left(-\frac{\cos 2x}{2}\right) - \int 2x.\left(-\frac{\cos 2x}{2}\right)dx$

$= -\frac{x^2\cos 2x}{2} + \int x\cos 2x\, dx$

$= -\frac{x^2\cos 2x}{2} + \frac{x\sin 2x}{2} - \int\frac{\sin 2x}{2}dx$

$= -\frac{x^2\cos 2x}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + c$

110. $\int x\sin^2 x\, dx = \int x\left(\frac{1-\cos 2x}{2}\right)dx$

$= \frac{1}{2}\left[\int x\, dx - \int x.\cos 2x\, dx\right]$

$= \frac{1}{2}\left[\frac{x^2}{2} - \frac{x\sin 2x}{2} + \int\frac{\sin 2x}{2}dx\right]$

$= \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + c$

111. $\int x\cos^2 x dx = \int x\left(\frac{1+\cos 2x}{2}\right)dx$

$= \frac{1}{2}\left[\int x\, dx + \int x\cos 2x\, dx\right]$

$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x\sin 2x}{2} - \int\frac{\sin 2x}{2}dx\right]$

$= \frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8} + c$

112. $\int \cos^{-1} x\, dx = \cos^{-1} x\cdot x + \int \frac{1}{\sqrt{1-x^2}}\cdot x\, dx$

$= x\cos^{-1} x - \frac{1}{2}\int\frac{-2x}{\sqrt{1-x^2}}dx$

$= x\cos^{-1} x - \sqrt{1-x^2} + c$

113. $\int \sin(\log x)\, dx + \int \cos(\log x)\, dx$

$= x\sin(\log x) - \int\frac{x\cos(\log x)}{x}dx + \int \cos(\log x)\, dx + c$

$= x\sin(\log x) + c$

114. $\int\frac{(\sin x + \cos x)(2 - \sin 2x)}{\sin^2 2x}dx$

$= -\int\frac{(\sin x + \cos x)(\sin 2x - 2)}{\sin^2 2x}dx$

$= -\int\frac{(\sin x + \cos x)(2\sin x\cos x - 2)}{4\sin^2 x\cos^2 x}dx$

$= -\int\frac{2\sin^2 x\cos x - 2\sin x + 2\sin x\cos^2 x - 2\cos x}{4\sin^2 x\cos^2 x}dx$

$= -\int\left[\frac{1}{2\cos x} + \frac{1}{2\sin x} - \frac{1}{2\sin^2 x\cos x} - \frac{1}{2\cos^2 x\sin x}\right]dx$

$= \frac{-1}{2}\left[\int \sec x\, dx + \int \text{cosec}\, x\, dx - \int\frac{\text{cosec}^2 x}{\cos x}dx - \int\frac{\sec^2 x}{\sin x}dx\right]$

$= \frac{-1}{2}\left[\int \sec x\, dx + \int \text{cosec}\, x\, dx - \int\frac{1+\cot^2 x}{\cos x}dx - \int\frac{1+\tan^2 x}{\sin x}dx\right]$

$= \frac{-1}{2}\left[\int \sec x\, dx + \int \operatorname{cosec} x\, dx - \int \sec x\, dx\right.$

$\left. - \int \operatorname{cosec} x \cot x\, dx - \int \operatorname{cosec} x\, dx - \int \sec x \tan x\, dx\right]$

$= \frac{-1}{2}\left[-\int \operatorname{cosec} x \cot x\, dx - \int \sec x \tan x\, dx\right]$

$= \frac{\sec x - \operatorname{cosec} x}{2} + c$

$= \frac{\sin x - \cos x}{2\sin x \cos x} + c = \frac{\sin x - \cos x}{\sin 2x} + c$

115. Let $I = \int e^{\sin x}(x\cos x - \sec x \tan x)\, dx$

$= \int x e^{\sin x} \cos x\, dx - \int e^{\sin x} \sec x \tan x\, dx$

$I_1 = \int x e^{\sin x} \cos x\, dx$

$= x \int e^{\sin x} \cos x\, dx - \int e^{\sin x}\, dx + c_1$

$= x\, e^{\sin x} - \int e^{\sin x}\, dx + c_1$

$I_2 = \int e^{\sin x} \sec x \tan x\, dx$

$= e^{\sin x}\int \sec x \tan x\, dx - \int \sec x \cos x\, e^{\sin x}\, dx + c_2$

$= \sec x\, e^{\sin x} - \int e^{\sin x}\, dx + c_2$

$I = I_1 - I_2$

$= x e^{\sin x} - \int e^{\sin x}\, dx - \sec x\, e^{\sin x} + \int e^{\sin x}\, dx + c$

$= e^{\sin x}(x - \sec x) + c$

116. Let $I = \int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right)$

$= \int e^{\sin x}(x\cos x - \sec x \tan x)\, dx$

$= e^{\sin x}(x - \sec x) + c$

117. Let $I = \int \sin(11x) .\, \sin^9 x\, dx$

$= \int \sin(10x + x)\cdot \sin^9 x\, dx$

$= \int \sin 10x \cos x \sin^9 x\, dx$

$+ \int \cos 10x \sin x \sin^9 x\, dx$

$= \sin 10x \int \cos x \sin^9 x\, dx$

$- \int 10\cos 10x \int \cos x \sin^9 x\, dx$

$+ \int \cos 10x \sin^{10} x\, dx$

$= \frac{\sin 10x \cdot \sin^{10} x}{10} - \frac{10}{10}\int \cos 10x \sin^{10} x\, dx$

$+ \int \cos 10x \sin^{10} x\, dx$

$= \frac{\sin 10x \cdot \sin^{10} x}{10} + c$

118. Let $I = \int \left[e^{2x} f(x) + e^{2x} f'(x)\right] dx$

$= \int e^{2x} f(x)\, dx + \int e^{2x} f'(x)\, dx$

$= f(x) \int e^{2x}\, dx - \int\left[\int e^{2x}\, dx\right] f'(x)\, dx$

$+ g(x) + c$

$= \frac{e^{2x} f(x)}{2} - \int \frac{e^{2x}}{2} f'(x)\, dx + g(x) + c$

$= \frac{e^{2x} f(x)}{2} - \frac{g(x)}{2} + g(x) + c$

$= \frac{e^{2x} f(x)}{2} + \frac{g(x)}{2} + c$

$= \frac{1}{2}[e^{2x} f(x) + g(x)] + c$

119. Let $I = \int e^{\sqrt{x}}\, dx$

Put $\sqrt{x} = t$

$\Rightarrow \frac{1}{2\sqrt{x}}\, dx = dt$

$\Rightarrow dx = 2t dt$

$\therefore \quad I = \int e^t .2t dt = 2(t.e^t - e^t) + A$

$= 2\left(\sqrt{x}.e^{\sqrt{x}} - e^{\sqrt{x}}\right) + A$

$= 2(\sqrt{x} - 1)\, e^{\sqrt{x}} + A$

120. Let $I = \int \cos(\log x)\, dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$\therefore \quad I = \int e^t \cos t\, dt$

$= e^t \cos t - \int e^t(-\sin t)\, dt + c_1$

$= e^t \cos t + \int e^t \sin t\, dt + c_1$

$= e^t \cos t + e^t \sin t - \int e^t \cos t\, dt + c_2$

$= e^t \cos t + e^t \sin t - I + c_2$

$\Rightarrow 2I = e^t(\cos t + \sin t) + c_2$

$\Rightarrow I = \frac{e^t}{2}(\cos t + \sin t) + c$

$= \frac{x}{2}\left[\cos(\log x) + \sin(\log x)\right] + c$

121. Put $\sin^{-1} x = t$

$\Rightarrow \frac{1}{\sqrt{1-x^2}}\, dx = dt$

$\therefore \quad \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int t \sin t\, dt = -t\cos t + \sin t + c$

$= -\sin^{-1} x \cos(\sin^{-1} x) + \sin(\sin^{-1} x) + c$

$= x - \sin^{-1} x.\sqrt{1-x^2} + c$

122. Let $I = \int \cos(\log_e x)\,dx$

Put $\log_e x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$\therefore$ $I = \int \cos t.e^t dt$

$= \cos t.e^t - \int(-\sin t).e^t dt$

$= \cos t.e^t + \left[\sin t.e^t - \int \cos t.e^t dt\right]$

$\therefore$ $I = \cos t.e^t + \sin t.e^t - I + c_1$

$\Rightarrow 2I = \cos t.e^t + \sin t.e^t + c_1$

$\Rightarrow I = \frac{x}{2}[\cos(\log_e x) + \sin(\log_e x)] + c,$

where $c = \frac{c_1}{2}$

123. $\int 32x^3 (\log x)^2 dx$

$= 32 \int x^3 (\log x)^2 dx$

$= 32\left[(\log x)^2.\frac{x^4}{4} - \int 2\log x.\frac{1}{x}.\frac{x^4}{4}dx\right]$

$= 32\left[(\log x)^2.\frac{x^4}{4} - \frac{1}{2}\int x^3 \log x\,dx\right]$

$= 32\left[\frac{(\log x)^2 x^4}{4} - \frac{1}{2}\left(\frac{\log x.x^4}{4} - \int \frac{1}{x}.\frac{x^4}{4}dx\right)\right]$

$= 32\left[\frac{(\log x)^2 x^4}{4} - \frac{1}{2}\left(\frac{x^4 \log x}{4} - \frac{1}{4}.\frac{x^4}{4}\right)\right] + c$

$= 8\left[(\log x)^2 x^4 - \frac{1}{2}\left(x^4 \log x - \frac{x^4}{4}\right)\right] + c$

$= 8x^4\left[(\log x)^2 - \frac{\log x}{2} + \frac{1}{8}\right] + c$

$= x^4[8(\log x)^2 - 4\log x + 1] + c$

124. Let $I = \int x^4 e^{2x}\,dx$

$= \frac{x^4 e^{2x}}{2} - \int 4x^3 \cdot \frac{e^{2x}}{2}dx$

$= \frac{x^4 e^{2x}}{2} - 2\int x^3 e^{2x} dx$

$= \frac{x^4 e^{2x}}{2} - 2\left[\int \frac{x^3 e^{2x}}{2}dx - \int 3x^2 \cdot \frac{e^{2x}}{2}dx\right]$

$= \frac{x^4 e^{2x}}{2} - x^3 e^{2x} + 3\int x^2 e^{2x}\,dx$

$= \frac{x^4 e^{2x}}{2} - x^3 e^{2x} + 3\left[\frac{x^2 e^{2x}}{2} - \int 2x \cdot \frac{e^{2x}}{2}dx\right]$

$= \frac{x^4 e^{2x}}{2} - x^3 e^{2x} + \frac{3x^2 e^{2x}}{2} - \frac{3xe^{2x}}{2} + 3\int \frac{e^{2x}}{2}dx$

$= \frac{x^4 e^{2x}}{2} - x^3 e^{2x} + \frac{3x^2 e^{2x}}{2} - \frac{3xe^{2x}}{2} + \frac{3e^{2x}}{4} + c$

$= \frac{e^{2x}}{4}[2x^4 - 4x^3 + 6x^2 - 6x + 3] + c$

125. $\int x^3 e^{5x} dx = x^3.\frac{e^{5x}}{5} - \int 3x^2.\frac{e^{5x}}{5}dx$

$= \frac{x^3 e^{5x}}{5} - \frac{3}{5}x^2.\frac{e^{5x}}{5} + \frac{3}{5}\int 2x.\frac{e^{5x}}{5}$

$= x^3\frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}x.\frac{e^{5x}}{5} - \frac{6}{25}.\frac{e^{5x}}{25} + c$

$\therefore$ $\int x^3 e^{5x} dx = \frac{e^{5x}}{5^4}\left(5^3 x^3 - 75x^2 + 30x - 6\right) + c$

$\therefore$ $f(x) = 5^3 x^3 - 75x^2 + 30x - 6$

126. $\int \log\left(a^2 + x^2\right)dx$

$= \log\left(a^2 + x^2\right).x - \int \frac{1}{a^2 + x^2}.2x.x dx$

$= x\log\left(a^2 + x^2\right) - 2\int \frac{x^2}{a^2 + x^2}dx$

$= x\log\left(a^2 + x^2\right) - 2\int\left(1 - \frac{a^2}{a^2 + x^2}\right)dx$

$= x\log\left(a^2 + x^2\right) - 2x + 2a^2.\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$= x\log\left(a^2 + x^2\right) - 2x + 2a\tan^{-1}\left(\frac{x}{a}\right) + c$

127. Put $\log x = t$

$\Rightarrow x = e^t$

$\Rightarrow dx = e^t dt$

$\therefore$ $\int (\log x)^5 dx = \int t^5 e^t dt$

$= e^t(t^5 - 5t^4 + 20t^3 - 60t^2 + 120t - 120)$ + constant

$= x[(\log x)^5 - 5(\log x)^4 + 20(\log x)^3 - 60(\log x)^2 + 120\log x - 120]$ + constant

$\therefore$ A = 1, B = –5, C = 20, D = –60, E = 120 and F = –120

$\therefore$ A + B + C + D + E + F = –44

128. Put $x = \sec\theta \Rightarrow dx = \sec\theta \tan\theta\, d\theta$

$\therefore$ $\int \cos^{-1}\left(\frac{1}{x}\right)dx = \int \cos^{-1}\left(\frac{1}{\sec\theta}\right).\sec\theta\tan\theta\,d\theta$

$= \int \cos^{-1}(\cos\theta).\sec\theta\tan\theta\,d\theta$

$= \int \theta.(\sec\theta \tan\theta)\, d\theta$

$= \theta\sec\theta - \int 1.\sec\theta d\theta$

$= \theta\sec\theta - \log|\tan\theta + \sec\theta| + c$

$= \theta\sec\theta - \log\left|\sqrt{(\sec^2\theta - 1)} + \sec\theta\right| + c$

$= x\sec^{-1}x - \log\left|\sqrt{(x^2 - 1)} + x\right| + c$

129. $\int e^x\left[\frac{1 + x\log x}{x}\right]dx = \int e^x\left(\log x + \frac{1}{x}\right)dx$

$= e^x \log x + c$

$....\left[\because \int e^x[f(x) + f'(x)]dx = e^x f(x) + c\right]$

130. $\int e^x \sin x(\sin x + 2\cos x)\,dx$

$= \int e^x(\sin^2 x + 2\sin x\cos x)\,dx$

$= e^x \sin^2 x + c$

131. $\int e^x\left[\frac{2 + \sin 2x}{1 + \cos 2x}\right]dx = \int e^x\left[\frac{2(1 + \sin x\cos x)}{2\cos^2 x}\right]dx$

$= \int e^x(\sec^2 x + \tan x)\,dx$

$= e^x \tan x + c$

$...\left[\because \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c\right]$

132. $\int \frac{(x+3)e^x}{(x+4)^2}\,dx = \int \frac{(x+4-1)}{(x+4)^2}\, e^x\,dx$

$= \int e^x\left(\frac{1}{x+4} - \frac{1}{(x+4)^2}\right)dx$

$= \frac{e^x}{x+4} + c$

133. $\int\left(\frac{x+2}{x+4}\right)^2 e^x dx$

$= \int e^x\left[\frac{x^2 + 4x + 4}{(x+4)^2}\right]dx$

$= \int e^x\left[\frac{x(x+4)}{(x+4)^2} + \frac{4}{(x+4)^2}\right]dx$

$= \int e^x\left[\frac{x}{x+4} + \frac{4}{(x+4)^2}\right]dx = e^x\left(\frac{x}{x+4}\right) + c$

134. $\int \frac{xe^x}{(x+1)^2}\,dx = \int e^x\left[\frac{x+1-1}{(x+1)^2}\right]dx$

$= \int e^x\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]dx$

$= \frac{e^x}{x+1} + c$

$...\left[\because \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c\right]$

135. $\int \frac{e^x(1+\sin x)}{1+\cos x}\,dx = \int \frac{e^x\left[1 + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right]}{2\cos^2\left(\frac{x}{2}\right)}\,dx$

$= \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx$

$= e^x \tan\frac{x}{2} + c$

$....\left[\because \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c\right]$

136. $\int \frac{e^x(x^2\tan^{-1}x + \tan^{-1}x + 1)}{x^2+1}\,dx$

$= \int \frac{e^x[(x^2+1)\tan^{-1}x + 1]}{x^2+1}\,dx$

$= \int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx$

$= e^x \tan^{-1}x + c$

137. Let $I = \int e^{\tan x}(\sec^2 x + \sec^3 x \cdot \sin x)\,dx$

$= \int e^{\tan x}(1 + \tan x)\sec^2 x\,dx$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\therefore$ $I = \int e^t(1+t)\,dt = e^t.t + c = \tan x \cdot e^{\tan x} + c$

138. Let $I = \int e^{\sin x}\left(\frac{\sin x + 1}{\sec x}\right)dx$

$= \int e^{\sin x}(\sin x + 1)\cos x\,dx$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$\therefore$ $I = \int e^t(1+t)\,dx$

$= te^t + c$ $...\left[\because \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x)\right]$

$= \sin x.e^{\sin x} + c$

139. Put $\tan^{-1}x = t \Rightarrow \frac{dx}{1+x^2} = dt$

$\therefore$ $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx = \int e^t\left(\tan t + \sec^2 t\right)dt$

$= e^t \tan t + c$

$= xe^{\tan^{-1}x} + c$

140. Put $\cot^{-1} x = t \Rightarrow \frac{-dx}{1+x^2} = dt$

$\therefore \quad \int e^{\cot^{-1} x}\left(\frac{x^2-x+1}{1+x^2}\right) dx$

$= -\int e^t\left(\cot^2 t - \cot t + 1\right) dt$

$= -\int e^t\left(\text{cosec}^2 t - \cot t\right) dt$

$= \int e^t\left(\cot t - \text{cosec}^2 t\right) dt$

$= e^t \cot t + c$

$= x\, e^{\cot^{-1} x} + c$

141. $\int e^{2x}\left(\frac{1}{x} - \frac{1}{2x^2}\right) dx = \frac{e^{2x}}{2x} + c$

$\ldots\left[\because e^{mx}\left[f(x) + \frac{f'(x)}{m}\right] dx = \frac{e^{mx} f(x)}{m} + c\right]$

142. $\int (1 + x - x^{-1}) e^{x+x^{-1}} dx$

$= \int\left[x e^{x+x^{-1}}\left(1 - \frac{1}{x^2}\right) + e^{x+x^{-1}}\right] dx$

$= x e^{x+x^{-1}} + c$

$\ldots\left[\because \int [x f'(x) + f(x)] dx = x f(x) + c\right]$

143. $\int \frac{2x+3}{x^2-5x+6} dx = \int \frac{2x+3}{(x-3)(x-2)} dx$

$= \int\left(\frac{9}{x-3} - \frac{7}{x-2}\right) dx$

$= 9 \log(x-3) - 7 \log(x-2) + c$

$\therefore \quad$ A = constant

144. $\int\left(\frac{1}{x-3} - \frac{1}{x^2-3x}\right) dx$

$= \int\left[\frac{1}{x-3} - \frac{1}{x(x-3)}\right] dx$

$= \int\left[\frac{1}{x-3} + \frac{1}{3x} - \frac{1}{3(x-3)}\right] dx$

$= \int\left[\frac{2}{3(x-3)} + \frac{1}{3x}\right] dx$

$= \frac{2}{3}\log(x-3) + \frac{1}{3}\log x + c$

$= \frac{2}{3}\log(x-3) + \frac{2}{3}\log\sqrt{x} + c$

$= \frac{2}{3}\log\left[\sqrt{x}(x-3)\right] + c$

145. Let $I = \int \frac{1}{(x^2+4)(x^2+9)} dx$

$\therefore \quad I = \int \frac{1}{5}\left[\frac{1}{x^2+4} - \frac{1}{x^2+9}\right] dx$

$\ldots\left\{\frac{1}{\alpha\beta} = \frac{1}{\beta-\alpha}\left[\frac{1}{\alpha} - \frac{1}{\beta}\right]\right\}$

$= \frac{1}{5}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right] + c$

$= \frac{1}{10}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{15}\tan^{-1}\left(\frac{x}{3}\right) + c$

Comparing with $A \tan^{-1}\left(\frac{x}{2}\right) + B \tan^{-1}\left(\frac{x}{3}\right) + c$, we get

$A = \frac{1}{10}, B = \frac{-1}{15}$

$A - B = \frac{1}{10} + \frac{1}{15} = \frac{1}{6}$

146. $\int \frac{x}{(x^2-a^2)(x^2-b^2)} dx$

$= \frac{1}{a^2-b^2}\left[\int \frac{x}{x^2-a^2} dx - \int \frac{x\,dx}{x^2-b^2}\right]$

$= \frac{1}{2(a^2-b^2)}\left[\log(x^2-a^2) - \log(x^2-b^2)\right] + c$

$= \frac{1}{2(a^2-b^2)}\log\left(\frac{x^2-a^2}{x^2-b^2}\right) + c$

147. Let $\frac{2x^2+1}{(x^2-4)(x^2-1)} = \frac{A}{x^2-4} + \frac{B}{x^2-1}$

$\therefore \quad 2x^2 + 1 = A(x^2-1) + B(x^2-4)$

Comparing the coefficient of x^2 and constant term on both sides, we get

A + B = 2 and –A – 4B = 1

Solving these two equations, we get

A = 3 and B = –1

$\therefore \quad \int \frac{2x^2+1}{(x^2-4)(x^2-1)} dx$

$= \int\left[\frac{3}{x^2-4} - \frac{1}{x^2-1}\right] dx$

$= \frac{3}{2\times 2}\log\left|\frac{x-2}{x+2}\right| - \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$

$= \log\left|\frac{x-2}{x+2}\right|^{\frac{3}{4}} + \log\left|\frac{x+1}{x-1}\right|^{\frac{1}{2}} + c$

$$= \log\left[\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right]+c$$

$\therefore$ $a = \frac{1}{2}$ and $b = \frac{3}{4}$

148. $\int \frac{2x^2+3}{(x^2-1)(x^2-4)}dx = \int \frac{-\frac{5}{3}}{x^2-1}dx + \int \frac{\frac{11}{3}}{x^2-4}dx$

$$= \frac{-5}{3}\cdot\frac{1}{2\times1}\log\left|\frac{x-1}{x+1}\right| + \frac{11}{3}\cdot\frac{1}{2\times2}\log\left|\frac{x-2}{x+2}\right|+c$$

$$= \log\left|\frac{x+1}{x-1}\right|^{\frac{5}{6}} + \log\left|\frac{x-2}{x+2}\right|^{\frac{11}{12}} + c$$

$$= \log\left[\left(\frac{x+1}{x-1}\right)^{\frac{5}{6}}\left(\frac{x-2}{x+2}\right)^{\frac{11}{12}}\right]+c$$

$\therefore$ $a = \frac{11}{12}$ and $b = \frac{5}{6}$

149. Let $I = \int \frac{5x^2+3}{x^2\left(x^2-2\right)}\, dx$

$$= \int \frac{5}{x^2-2}\, dx + \int \frac{3}{x^2\left(x^2-2\right)}dx$$

$$= \frac{5}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + \frac{3}{2}\int\left(\frac{1}{x^2-2}-\frac{1}{x^2}\right)dx$$

$$= \frac{5}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + \frac{3}{2}\left[\frac{1}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + \frac{1}{x}\right]+c$$

$$= \frac{13}{4\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + \frac{3}{2}x + c$$

150. $\int \frac{dx}{x^6+x^4} = \int \frac{(x^2+1)\,dx}{x^4(x^2+1)} - \int \frac{x^2\, dx}{x^4(x^2+1)}$

$$= \int \frac{1}{x^4}dx - \int \frac{dx}{x^2(x^2+1)}$$

$$= -\frac{1}{3x^3} - \int \frac{dx}{x^2} + \int \frac{dx}{x^2+1}$$

$$= \frac{-1}{3x^3} + \frac{1}{x} + \tan^{-1}x + c$$

151. $\int \frac{dx}{e^{2x}-3e^x} = \int \frac{dx}{e^x(e^x-3)}$

$$= -\frac{1}{3}\left[\int \frac{dx}{e^x} - \int \frac{dx}{e^x-3}\right]$$

$$= \frac{1}{3}\int \frac{dx}{e^x-3} - \frac{1}{3}\int \frac{dx}{e^x}$$

$$= \frac{1}{3}\int \frac{e^{-x}}{1-3e^{-x}}dx - \frac{1}{3}\int e^{-x}dx$$

$$= \frac{1}{9}\log(1-3e^{-x}) + \frac{1}{3e^x} + c$$

$$= \frac{1}{9}\log\left(\frac{e^x-3}{e^x}\right) + \frac{1}{3e^x} + c$$

$$= \frac{1}{3e^x} + \frac{1}{9}\log(e^x-3) - \frac{1}{9}\log e^x + c$$

$$= \frac{1}{3e^x} + \frac{1}{9}\log(e^x-3) - \frac{x}{9} + c$$

152. $\int \frac{x}{\left(x^2+1\right)\left(x-1\right)}dx$

$$= \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{x-1}{x^2+1}dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x + D_1$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log\left|x^2+1\right| + \frac{1}{2}\tan^{-1}x + D$$

Comparing with
$A\log|x^2+1| + B\tan^{-1}x + C\log|x-1| + D$,
we get

$A = \frac{-1}{4}$, $B = \frac{1}{2}$, $C = \frac{1}{2}$

$\therefore$ $A + B + C = \frac{-1}{4} + \frac{1}{2} + \frac{1}{2} = \frac{3}{4}$

153. $\int \frac{2x+3}{(x-1)(x^2+1)}\, dx$

$$= \int \frac{5\,dx}{2(x-1)} + \int \frac{-\left(\frac{5}{2}x+\frac{1}{2}\right)}{x^2+1}\, dx$$

$$= \frac{5}{2}\log(x-1) - \frac{5}{2}\int \frac{x\,dx}{1+x^2} - \frac{1}{2}\int \frac{dx}{1+x^2}$$

$$= \frac{5}{2}\log(x-1) - \frac{5}{4}\log(1+x^2) - \frac{1}{2}\tan^{-1}x + A$$

$$= \log(x-1)^{\frac{5}{2}}(1+x^2)^{-\frac{5}{4}} - \frac{1}{2}\tan^{-1}x + A$$

$\therefore$ $a = -\frac{5}{4}$

154. Let $I = \int \frac{1}{x\left(x^4+1\right)} \, dx$

$$= \int \frac{x^3}{x^4\left(x^4+1\right)} \, dx$$

Put $x^4 = t \Rightarrow 4x^3 \, dx = dt$

$\therefore \quad I = \frac{1}{4} \int \frac{dt}{t(1+t)}$

$$= \frac{1}{4} \int \left(\frac{1}{t} - \frac{1}{1+t}\right) dt$$

$$= \frac{1}{4} [\log|t| - \log|1+t|] + c$$

$$= \frac{1}{4} \log\left|\frac{t}{1+t}\right| + c$$

$$= \frac{1}{4} \log\left|\frac{x^4}{1+x^4}\right| + c$$

155. $\int \frac{dx}{x^3+3x^2+2x} = \int \frac{1}{x\left(x^2+3x+2\right)} dx$

$$= \int \frac{1}{x(x+2)(x+1)} dx$$

$$= \int \frac{1}{2x} \, dx + \int \frac{1}{2(x+2)} \, dx - \int \frac{1}{x+1} \, dx$$

$$= \frac{1}{2} \log|x| + \frac{1}{2} \log|x+2| - \log|x+1| + c$$

$$= \frac{1}{2} \log|x(x+2)| - \frac{1}{2} \log(x+1)^2 + c$$

$$= \frac{1}{2} \log\left[\frac{\left|x^2+2x\right|}{(x+1)^2}\right] + c$$

156. Let $I = \int \frac{dx}{\sin x + \sin 2x}$

$$= \int \frac{dx}{\sin x(1+2\cos x)}$$

$$= \int \frac{\sin x \, dx}{\sin^2 x(1+2\cos x)}$$

$$= \int \frac{\sin x \, dx}{(1-\cos x)(1+\cos x)(1+2\cos x)}$$

Put $\cos x = t$

$\Rightarrow -\sin x \, dx = dt$

$\therefore \quad I = -\int \frac{dt}{(1-t)(1+t)(1+2t)}$

$\therefore \quad I = -\int \left[\frac{1}{6(1-t)} - \frac{1}{2(1+t)} + \frac{4}{3(1+2t)}\right] dt$

$$= \frac{1}{6}\log(1-t) + \frac{1}{2}\log(1+t) - \frac{2}{3}\log(1+2t) + c$$

$$= \frac{1}{6}\log(1-\cos x) + \frac{1}{2}\log(1+\cos x)$$

$$- \frac{2}{3}\log(1+2\cos x) + c$$

157. $\int \frac{f(x)}{\log \cos x} dx = -\log(\log \cos x) + c$

Differentiating on both sides, we get

$$\frac{f(x)}{\log \cos x} = \frac{-1}{\log \cos x} \times \frac{1}{\cos x} \times (-\sin x)$$

$$\Rightarrow \frac{f(x)}{\log \cos x} = \frac{\tan x}{\log \cos x} \quad \Rightarrow f(x) = \tan x$$

158. $\int \frac{f(x)}{\log(\sin x)} dx = \log(\log \sin x) + c$

Differentiating on both sides, we get

$$\frac{f(x)}{\log(\sin x)} = \frac{1}{\log(\sin x)} \times \frac{1}{\sin x} \times \cos x$$

$$\Rightarrow \frac{f(x)}{\log(\sin x)} = \frac{\cot x}{\log(\sin x)} \Rightarrow f(x) = \cot x$$

159. $\int f(x) \cos x \, dx = \frac{1}{2} [f(x)]^2 + c$

Differentiating both sides w.r.t. x, we get

$f(x) \cos x = f(x) \, . \, f'(x)$

$\Rightarrow f'(x) = \cos x$

$\Rightarrow \int f'(x) \, dx = \int \cos x \, dx$

$\Rightarrow f(x) = \sin x + c$

$\therefore \quad f\left(\frac{\pi}{2}\right) = 1 + c$

160. $\int f(x) \cdot \cos x \, dx = \frac{1}{2} [f(x)]^2 + c$

Differentiating w.r.t x, we get

$$f(x) \cdot \cos x = \frac{1}{2} \times 2 \, f(x) \, . \, f'(x)$$

$\Rightarrow \cos x = f'(x)$

$\Rightarrow \cos 0 = f'(0)$

$\Rightarrow f'(0) = 1$

161. $f'(x) = 2 - \frac{5}{x^4}$

$\therefore$ $f(x) = \int\left(2 - \frac{5}{x^4}\right)dx = 2x + \frac{5}{3x^3} + c$

$f(1) = 2(1) + \frac{5}{3(1)^3} + c$

$\Rightarrow \frac{14}{3} = 2 + \frac{5}{3} + c \Rightarrow c = 1$

$\therefore$ $f(x) = 2x + \frac{5}{3x^3} + 1$

$f(-1) = 2(-1) + \frac{5}{3(-1)^3} + 1$

$= -2 - \frac{5}{3} + 1 = \frac{-8}{3}$

162. $\int f(x) \sin x \cos x \, dx = \frac{1}{2(b^2 - a^2)} \log f(x) + c$

Differentiating both sides w.r.t. x, we get

$f(x) \sin x \cos x = \frac{1}{2(b^2 - a^2)} \cdot \left[\frac{f'(x)}{f(x)}\right]$

$\Rightarrow [f(x)]^2 \sin 2x = \frac{1}{(b^2 - a^2)} f'(x)$

$\Rightarrow y^2 \sin 2x = \frac{1}{b^2 - a^2}\left(\frac{dy}{dx}\right)$

…[Putting $f(x) = y$]

$\Rightarrow \frac{dy}{y^2} = (b^2 - a^2) \sin 2x \, dx$

$\Rightarrow \int \frac{dy}{y^2} = (b^2 - a^2) \int \sin 2x \, dx \sin 2x \, dx$

$\Rightarrow \frac{-1}{y} = \frac{-(b^2 - a^2)\cos 2x}{2}$

$\Rightarrow y = \frac{2}{(b^2 - a^2)\cos 2x} = f(x)$

163. $\frac{d}{dx}[f(x)] = x \cos x + \sin x$

$\Rightarrow f(x) = \int (x\cos x + \sin x)\, dx = x \sin x + c$

Since, $f(0) = 2 \Rightarrow c = 2$

$\therefore$ $f(x) = x \sin x + 2$

164. $\int \log(x^2 + x)dx = \int \log[x(x+1)]dx$

$= \int \log x.1 dx + \int \log(x+1).1 dx$

$= \log x.x - \int \frac{1}{x}.x\, dx + \log(x+1).x - \int \frac{1}{x+1}.x\, dx$

$= x \log x - \int dx + x\log(x+1) - \int\left(\frac{x+1-1}{x+1}\right)dx$

$= x \log x - x + x \log(x + 1) - \int\left(1 - \frac{1}{x+1}\right)dx$

$= x \log x - x + x \log(x + 1) - x + \log|x + 1| + c$

$= x[\log x + \log(x + 1)] - 2x + \log|x + 1| + c$

$= x \log(x^2 + x) - 2x + \log|x + 1| + c$

$\therefore$ $A = -2x + \log|x + 1| + c$

165. $I_1 = \int \sin^{-1} x\, dx$

Put $\sin^{-1} x = \theta \Rightarrow x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\therefore$ $I_1 = \int \theta\cos\theta d\theta = \theta\sin\theta - \int 1\cdot\sin\theta d\theta$

$= \theta \sin\theta + \cos\theta$

$= x\sin^{-1} x + \sqrt{1 - x^2}$

Now, $I_2 = \int \sin^{-1}\sqrt{1 - x^2}dx = \int \cos^{-1} x dx$

Put $\cos^{-1} x = \phi \Rightarrow x = \cos\phi \Rightarrow dx = -\sin\phi\, d\phi$

$\therefore$ $I_2 = -\int \phi \sin\phi\, d\phi = \phi\cos\phi + \int 1.(-\cos\phi)\, d\phi$

$= \phi\cos\phi - \sin\phi = x\cos^{-1}x - \sqrt{1 - x^2}$

$\therefore$ $I_1 + I_2 = x(\sin^{-1}x + \cos^{-1}x) = \frac{\pi}{2}x$

166. $x = f''(t) \cos t + f'(t) \sin t$

$\therefore$ $\frac{dx}{dt} = -f''(t) \sin t + f'''(t) \cos t + f''(t) \sin t + f'(t) \cos t$

$= f'''(t) \cos t + f'(t) \cos t$

$= \cos t\, [f'''(t) + f'(t)]$

$y = -f''(t) \sin t + f'(t) \cos t$

$\therefore$ $\frac{dy}{dt} = -f'''(t) \sin t - f''(t) \cos t + f''(t) \cos t - f'(t) \sin t$

$= -\sin t\, [f'''(t) + f'(t)]$

$\therefore$ $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = [\sin^2 t + \cos^2 t]\, [f'''(t) + f'(t)]^2$

$= [f'''(t) + f'(t)]^2$

Let $I = \int\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}} dt$

$= \int\left\{[f'''(t) + f'(t)]^2\right\}^{\frac{1}{2}} dt$

$= \int [f'''(t) + f'(t)]\, dt$

$= f''(t) + f(t) + c$

Evaluation Test

1. Let $I = \int \frac{\sqrt{5+x^{10}}}{x^{16}}\,dx$

$= \int \sqrt{\frac{5+x^{10}}{x^{10}}}.\frac{1}{x^{11}}\,dx$

$= \int \sqrt{\frac{5}{x^{10}}+1}.\frac{1}{x^{11}}\,dx$

Put $\frac{5}{x^{10}}+1 = t$

$\therefore$ $5(-10)x^{-11}\,dx = dt$

$\therefore$ $\frac{1}{x^{11}}dx = -\frac{1}{50}dt$

$\therefore$ $I = \int t^{\frac{1}{2}}\left(-\frac{1}{50}\right)dt$

$= -\frac{1}{50}.\frac{t^{3/2}}{3/2} + c$

$= -\frac{1}{75}\left(1+\frac{5}{x^{10}}\right)^{3/2} + c$

2. Multiplying N^r and D^r by $\sin 3x$, we get

$\int \frac{\cos 5x+\cos 4x}{1-2\cos 3x}dx$

$= \int \frac{\sin 3x\cos 5x+\sin 3x\cos 4x}{\sin 3x-2\sin 3x\cos 3x}dx$

$= \int \frac{\sin 3x(\cos 5x+\cos 4x)}{\sin 3x-\sin 6x}dx$

$= \int \frac{\left(2\sin\frac{3x}{2}\cos\frac{3x}{2}\right)\left(2\cos\frac{9x}{2}\cos\frac{x}{2}\right)}{-2\cos\frac{9x}{2}\sin\frac{3x}{2}}dx$

$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2}dx$

$= -\int(\cos 2x+\cos x)dx$

$= -\left[\frac{1}{2}\sin 2x+\sin x\right] + c$

3. Let $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Put $x = a\tan^2 t$

$\therefore$ $dx = 2a\tan t\sec^2 t\,dt$

$\therefore$ $I = \int \sin^{-1}\sqrt{\frac{a\tan^2 t}{a+a\tan^2 t}} \times 2a\tan t\sec^2 t\,dt$

$= \int \sin^{-1}(\sin t)\times 2a\tan t\sec^2 t\,dt$

$= 2a\int t\tan t\sec^2 t\,dt$

$= 2a\left[t\int \tan t\sec^2 t\,dt - \int\left\{\frac{d}{dt}(t)\int \tan t\sec^2 t\,dt\right\}dt\right]$

$= 2a\left[t.\frac{\tan^2 t}{2} - \int 1.\frac{\tan^2 t}{2}dt\right]$

$= a\left[t\tan^2 t - \int(\sec^2 t-1)dt\right]$

$= a\left[t\tan^2 t - \tan t + t\right] + c$, where $t = \tan^{-1}\sqrt{\frac{x}{a}}$

$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + c$

4. Let $I = \int \sqrt{\operatorname{cosec} x - 1}\,dx$

$= \int \sqrt{\frac{1}{\sin x}-1}\,dx$

$= \int \sqrt{\frac{1-\sin x}{\sin x}}\,dx$

$= \int \sqrt{\frac{1-\sin x}{\sin x}\times\frac{1+\sin x}{1+\sin x}}\,dx$

$= \int \frac{\cos x}{\sqrt{\sin^2 x+\sin x}}dx$

Put $\sin x = t$

$\therefore$ $\cos x\,dx = dt$

$\therefore$ $I = \int \frac{1}{\sqrt{t^2+t}}dt$

$= \int \frac{1}{\sqrt{t^2+t+\frac{1}{4}-\frac{1}{4}}}dt$

$= \int \frac{1}{\sqrt{\left(t+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dt$

$= \log\left|t+\frac{1}{2}+\sqrt{t^2+t}\right| + c$, where $t = \sin x$

$= \log\left|\sin x+\frac{1}{2}+\sqrt{\sin^2 x+\sin x}\right| + c$

5. Let $I = \int \sqrt{\tan x}\, dx$

Put $\tan x = t^2$

$\therefore$ $\sec^2 x\, dx = 2t dt$

$\therefore$ $dx = \frac{2t}{1+t^4} dt$

$\therefore$ $I = \int \sqrt{t^2}.\frac{2t}{1+t^4} dt = 2\int \frac{t^2}{1+t^4} dt$

$= \int \frac{t^2+1+t^2-1}{t^4+1} dt = \int \frac{t^2+1}{t^4+1} dt + \int \frac{t^2-1}{t^4+1} dt$

$= I_1 + I_2$ (say) ....(i)

$I_1 = \int \frac{t^2+1}{t^4+1} dt$

$= \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt$

$= \int \frac{1}{\left(t-\frac{1}{t}\right)^2+2}\left(1+\frac{1}{t^2}\right) dt$

$= \int \frac{1}{y^2+2} dy$, where $t - \frac{1}{t} = y$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)$

$I_2 = \int \frac{t^2-1}{t^4+1} dt = \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt$

$= \int \frac{1}{\left(t+\frac{1}{t}\right)^2-2}\left(1-\frac{1}{t^2}\right) dt$

$= \int \frac{1}{m^2-2} dm$, where $t + \frac{1}{t} = m$

$= \frac{1}{2\sqrt{2}}\log\left|\frac{m-\sqrt{2}}{m+\sqrt{2}}\right| = \frac{1}{2\sqrt{2}}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|$

$= \frac{1}{2\sqrt{2}}\log\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right|$

$\therefore$ From (i),

$I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right| + c$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right)$

$+ \frac{1}{2\sqrt{2}}\log\left|\frac{\tan x - \sqrt{2\tan x}+1}{\tan x + \sqrt{2\tan x}+1}\right| + c$

6. Let $I = \int x^{13/2}\left(1+x^{5/2}\right)^{1/2} dx$

$= \int x^5.(1+x^{5/2})^{1/2}.x^{3/2} dx$

Put $1+x^{5/2} = t$

$\therefore$ $\frac{5}{2}x^{3/2}\, dx = dt$, $\therefore$ $x^{3/2} dx = \frac{2}{5} dt$

$\therefore$ $I = \int (t-1)^2.t^{1/2}.\frac{2}{5} dt$

$= \frac{2}{5}\int \left(t^{5/2} - 2t^{3/2} + t^{1/2}\right) dt$

$= \frac{2}{5}\left[\frac{2}{7}t^{7/2} - \frac{4}{5}t^{5/2} + \frac{2}{3}t^{3/2}\right] + c$, where $t = 1+x^{5/2}$

$= \frac{2}{5}\left[\frac{2}{7}\left(1+x^{5/2}\right)^{7/2} - \frac{4}{5}\left(1+x^{5/2}\right)^{5/2}\right.$

$\left. + \frac{2}{3}\left(1+x^{5/2}\right)^{3/2}\right] + c$

7. Let $I = \int \frac{\tan x}{1+\tan x+\tan^2 x} dx$

$= \int \frac{\tan x}{\sec^2 x + \tan x} dx$

$= \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x}+\frac{\sin x}{\cos x}} dx = \int \frac{\sin x\cos x}{1+\sin x\cos x} dx$

$= \int \frac{\frac{1}{2}\sin 2x}{1+\frac{1}{2}\sin 2x} dx = \int \frac{\sin 2x}{2+\sin 2x} dx$

$= \int \frac{2+\sin 2x - 2}{2+\sin 2x} dx = \int \left(1 - \frac{2}{2+\sin 2x}\right) dx$

$= x - I_1$ (say) ....(i)

$I_1 = \int \frac{2}{2+\sin 2x} dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt \Rightarrow dx = \frac{1}{1+t^2}\, dt$

$\sin 2x = \frac{2\tan x}{1+\tan^2 x} = \frac{2t}{1+t^2}$

$\therefore \quad I_1 = \int \frac{2}{2+\frac{2t}{1+t^2}} \times \frac{1}{1+t^2}\, dt$

$= \int \frac{1}{t^2+t+1}\, dt$

$= \int \frac{1}{t^2+t+\frac{1}{4}+\frac{3}{4}}\, dt$

$= \int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\, dt$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c_1$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan x+1}{\sqrt{3}}\right) + c_1$

$\therefore$ From (i),

$I = x - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan x+1}{\sqrt{3}}\right) + c$

$\therefore \quad \sqrt{A} = \sqrt{3}$

$\therefore \quad A = 3$

8. Let $I = \int \log\left(\frac{x-1}{x+1}\right) \cdot \frac{1}{x^2-1}\, dx$

Put $\log\left(\frac{x-1}{x+1}\right) = t$

$\therefore \quad \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx = dt$

$\therefore \quad \frac{1}{x^2-1}\, dx = \frac{1}{2}\, dt$

$\therefore \quad I = \int t.\frac{1}{2}\, dt = \frac{1}{4} t^2 + c$

$= \frac{1}{4}\left[\log\left(\frac{x-1}{x+1}\right)\right]^2 + c$

$\therefore \quad A = \frac{1}{4}$

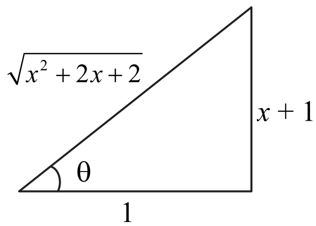
9. Let $I = \int \frac{1}{\left(x^2+2x+2\right)^2}\, dx$

$= \int \frac{1}{\left[(x+1)^2+1\right]^2}\, dx$

Put $x + 1 = \tan\theta$

$\therefore \quad dx = \sec^2\theta\, d\theta$

$\therefore \quad I = \int \frac{\sec^2\theta}{(\tan^2\theta+1)^2}\, d\theta$



$= \int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta$

$= \int \cos^2\theta\, d\theta$

$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$

$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$

$= \frac{1}{2}(\theta + \sin\theta\cos\theta) + c$

$= \frac{1}{2}\left[\tan^{-1}(x+1) + \frac{x+1}{\sqrt{x^2+2x+2}} \cdot \frac{1}{\sqrt{x^2+2x+2}}\right] + c$

$= \frac{1}{2}\left[\tan^{-1}(x+1) + \frac{x+1}{x^2+2x+2}\right] + c$

10. Let $I = \int \frac{1}{\cos^6 x + \sin^6 x}\, dx$

Since, $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$

$\therefore \quad \cos^6 x + \sin^6 x = 1 - 3\sin^2 x \cos^2 x$

....[$\because$ $a + b = \cos^2 x + \sin^2 x = 1$]

$\therefore \quad I = \int \frac{1}{1-3\sin^2 x\cos^2 x}\, dx$

$= \int \frac{1}{1-\frac{3}{4}\sin^2 2x}\, dx$

$= \int \frac{4}{4-3\sin^2 2x}\, dx$

$= \int \frac{4\,\text{cosec}^2\, 2x}{4\,\text{cosec}^2\, 2x - 3}\, dx$

$= \int \frac{4\,\text{cosec}^2\, 2x}{4(1+\cot^2 2x)-3}\, dx$

$= -\int \frac{1}{4\cot^2 2x+1}(-4\,\text{cosec}^2\, 2x)\, dx$

Put $2\cot 2x = t$

$\therefore \quad -4\,\text{cosec}^2\, 2x\, dx = dt$

$\therefore \quad I = -\int \frac{1}{t^2+1}\, dt$

$= -\tan^{-1}(t) + c$

$= -\tan^{-1}(2\cot 2x) + c$

$= -\tan^{-1}\left(\frac{2\cos^2 x - 2\sin^2 x}{2\sin x\cos x}\right) + c$

$= -\tan^{-1}(\cot x - \tan x) + c$

$= \tan^{-1}(\tan x - \cot x) + c$

11. $\int f(x)\sin x\cos x\, dx = \frac{1}{2(b^2-a^2)}\log[f(x)] + c$

$\therefore \quad \frac{d}{dx}\left[\frac{1}{2(b^2-a^2)}\log[f(x)] + c\right] = f(x)\sin x\cos x$

$\therefore \quad \frac{1}{2(b^2-a^2)}\cdot\frac{1}{f(x)}\cdot f'(x) = f(x)\sin x\cos x$

$\therefore \quad \frac{f'(x)}{[f(x)]^2} = 2(b^2-a^2)\sin x\cos x$

Integrating on both sides, we get

$\int \frac{f'(x)}{[f(x)]^2}\, dx = (b^2-a^2)\int 2\sin x\cos x\, dx$

$\therefore \quad -\frac{1}{f(x)} = (b^2-a^2)\int 2\sin x\cos x\, dx$

$= b^2\int 2\sin x\cos x\, dx - a^2\int 2\sin x\cos x\, dx$

$= b^2(-\cos^2 x) - a^2(\sin^2 x)$

$\therefore \quad -\frac{1}{f(x)} = -b^2\cos^2 x - a^2\sin^2 x$

$\therefore \quad f(x) = \frac{1}{a^2\sin^2 x + b^2\cos^2 x}$

12. Let $I = \int e^{\sin\theta}\left[\log(\sin\theta) + \operatorname{cosec}^2\theta\right]\cos\theta\, d\theta$

Put $\sin\theta = t$

$\therefore \quad \cos\theta\, d\theta = dt$

$\therefore \quad I = \int e^t\left[\log t + \frac{1}{t^2}\right]dt$

$= \int e^t\left[\log t - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right]dt$

$= e^t\left(\log t - \frac{1}{t}\right) + c$

$\ldots\left[\because \frac{d}{dt}\left(\log t - \frac{1}{t}\right) = \frac{1}{t} + \frac{1}{t^2}\right]$

$= e^{\sin\theta}\left[\log(\sin\theta) - \frac{1}{\sin\theta}\right] + c$

$= e^{\sin\theta}\left[\log(\sin\theta) - \operatorname{cosec}\theta\right] + c$

13. $\log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) = \log\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$

$= \log\tan\left(\frac{\pi}{4} + \theta\right)$

Since, $\int \sec 2\theta\, d\theta = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \theta\right)$

$\therefore \quad \frac{d}{d\theta}\log\tan\left(\frac{\pi}{4} + \theta\right) = 2\sec 2\theta \quad \ldots(i)$

Integrating the given expression by parts, we get

$I = \log\tan\left(\frac{\pi}{4} + \theta\right)\cdot\frac{1}{2}\sin 2\theta - \int \frac{\sin 2\theta}{2}\cdot 2\sec 2\theta\, d\theta$

….[From (i)]

$= \frac{1}{2}\sin 2\theta\log\tan\left(\frac{\pi}{4} + \theta\right) - \int \tan 2\theta\, d\theta$

$= \frac{1}{2}\sin 2\theta\log\tan\left(\frac{\pi}{4} + \theta\right) - \frac{1}{2}\log(\sec 2\theta) + c$

14. Let $\frac{3x-4}{3x+4} = t$

$\Rightarrow \frac{(3x-4)+(3x+4)}{(3x-4)-(3x+4)} = \frac{t+1}{t-1}$

$\Rightarrow \frac{6x}{-8} = \frac{t+1}{t-1} \Rightarrow x = -\frac{4}{3}\left(\frac{t+1}{t-1}\right)$

$\Rightarrow x + 2 = -\frac{4t+4}{3t-3} + 2$

$= \frac{-4t-4+6t-6}{3t-3} = \frac{2t-10}{3t-3}$

Given, $f\left(\frac{3x-4}{3x+4}\right) = x + 2$

$\therefore \quad f(t) = \frac{2t-10}{3t-3} = \frac{2}{3}\left(\frac{t-5}{t-1}\right)$

$= \frac{2}{3}\left(\frac{t-1-4}{t-1}\right) = \frac{2}{3}\left(1 - \frac{4}{t-1}\right) = \frac{2}{3} - \frac{8}{3(t-1)}$

$\therefore \quad f(x) = \frac{2}{3} - \frac{8}{3(x-1)}$

$\therefore \quad \int f(x)\, dx = \int\left\{\frac{2}{3} - \frac{8}{3(x-1)}\right\}dx$

$= \frac{2}{3}x - \frac{8}{3}\log|x-1| + c$

15. Let $I = \int \log\left(1-\sqrt{x}\right)dx$

$= \int \log\left(1-\sqrt{x}\right)(1)dx$

$= \log(1-\sqrt{x}) \int 1dx - \int\left\{\frac{d}{dx}\log\left(1-\sqrt{x}\right)\int 1dx\right\}dx$

$= \log(1-\sqrt{x}) .x - \int \frac{1}{1-\sqrt{x}}\left(-\frac{1}{2\sqrt{x}}\right)x\ dx$

$= x\log(1-\sqrt{x}) + \frac{1}{2}\int\frac{\sqrt{x}}{1-\sqrt{x}}\ dx$

$= x\log(1-\sqrt{x}) + \frac{1}{2}I_1$ ….(i)

Now, $I_1 = \int\frac{\sqrt{x}}{1-\sqrt{x}}\ dx$

Put $x = t^2$,

$\therefore$ $dx = 2t\ dt$

$\therefore$ $I_1 = \int\frac{\sqrt{t^2}}{1-\sqrt{t^2}}.2t\ dt = 2\int\frac{t^2}{1-t}\ dt$

$= -2\int\frac{1-t^2-1}{1-t}dt$

$= -2\int\left(1+t-\frac{1}{1-t}\right)dt$

$= 2\int\left(\frac{1}{1-t}-1-t\right)dt$

$= 2\left[\frac{1}{-1}\log(1-t)-t-\frac{t^2}{2}\right]+c_1$

$= -2\left[\log(1-\sqrt{x})+\sqrt{x}+\frac{1}{2}x\right]+c_1$

$\therefore$ From (i),

$I = x\log(1-\sqrt{x})$

$- \frac{1}{2}.2\left[\log(1-\sqrt{x})+\sqrt{x}+\frac{1}{2}x\right]+c$

$= (x-1)\log\left(1-\sqrt{x}\right)-\sqrt{x}-\frac{1}{2}x+c$

16. $P(x) = \int\frac{x^3}{x^3-x^2}dx,\ Q(x) = \int\frac{1}{x^3-x^2}dx$

$\therefore$ $P(x)+Q(x) = \int\frac{x^3+1}{x^3-x^2}dx$

$= \int\frac{x^3-x^2+x^2+1}{x^3-x^2}dx$

$= \int\left(1+\frac{x^2+1}{x^3-x^2}\right)dx$

$= x + I$ ….(i)

$I = \int\frac{x^2+1}{x^2(x-1)}\ dx$

Put $\frac{x^2+1}{x^2(x-1)} = \frac{A}{x-1}+\frac{B}{x}+\frac{C}{x^2}$

$\therefore$ $x^2+1 = Ax^2+Bx(x-1)+C(x-1)$ ….(ii)

Putting $x = 0$ in (ii), C = –1

Putting $x = 1$ in (ii), A = 2

Putting $x = -1$ in (ii), B = –1

$\therefore$ $I = \int\left(\frac{2}{x-1}-\frac{1}{x}-\frac{1}{x^2}\right)dx$

$= 2\log|x-1| - \log|x| + \frac{1}{x}$

$\therefore$ From (i),

$P(x)+Q(x) = x + 2\log|x-1| - \log|x| + \frac{1}{x} + c$

$\therefore$ $(P+Q)(2) = P(2)+Q(2)$

$= 2 + 2\log 1 - \log 2 + \frac{1}{2} + c$

$\therefore$ $\frac{5}{2} = \frac{5}{2} - \log 2 + c$ ….$\left[\because\ (P+Q)(2)=\frac{5}{2}\right]$

$\therefore$ $c = \log 2$

$\therefore$ $P(x)+Q(x) = x + 2\log|x-1| - \log|x|$

$+ \frac{1}{x} + \log 2$

$\therefore$ $P(3)+Q(3) = 3 + 2\log 2 - \log 3 + \frac{1}{3} + \log 2$

$= \frac{10}{3} + \log\frac{8}{3}$

17. Let $I = \int\frac{2a\sin x+b\sin 2x}{(b+a\cos x)^3}dx$

$= 2\int\frac{(a+b\cos x)}{(b+a\cos x)^3}\cdot\sin x\ dx$

Put $b + a\cos x = t$

$\therefore$ $-a\sin x\ dx = dt$

$\therefore$ $\sin x\ dx = -\frac{1}{a}dt$

$\therefore$ $I = 2\int\frac{a+b\left(\frac{t-b}{a}\right)}{t^3}\left(-\frac{1}{a}\right)dt$

$= -\frac{2}{a}\int\frac{a^2+bt-b^2}{at^3}dt$

$= -\frac{2}{a^2}\int\left[(a^2-b^2)t^{-3}+bt^{-2}\right]dt$

$$= -\frac{2}{a^2}\left[\frac{a^2-b^2}{-2t^2}+\frac{b}{-t}\right]+c$$

$$= \frac{1}{a^2}\cdot\frac{a^2-b^2}{t^2}+\frac{2b}{a^2t}+c$$

18. Let $I = \int \frac{1}{(x-1)\sqrt{x^2+4}}dx$

Put $x - 1 = \frac{1}{t}$, $\therefore$ $dx = -\frac{1}{t^2}dt$

$\therefore$ $I = \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2+4}}\left(-\frac{1}{t^2}\right)dt$

$$= -\int \frac{1}{\sqrt{\frac{1}{t^2}+\frac{2}{t}+1+4}}\left(\frac{1}{t}\right)dt$$

$$= -\int \frac{1}{\sqrt{1+2t+5t^2}}dt$$

$$= -\frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{t^2+\frac{2}{5}t+\frac{1}{5}}}dt$$

$$= -\frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{t^2+\frac{2}{5}t+\frac{1}{25}+\frac{4}{25}}}dt$$

$$= -\frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{\left(t+\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2}}dt$$

$$= -\frac{1}{\sqrt{5}}\log\left|t+\frac{1}{5}+\sqrt{t^2+\frac{2t}{5}+\frac{1}{5}}\right|+c$$

$$= -\frac{1}{\sqrt{5}}\log\left|\frac{1}{x-1}+\frac{1}{5}+\sqrt{\frac{1}{(x-1)^2}+\frac{2}{5(x-1)}+\frac{1}{5}}\right|+c$$

$$= -\frac{1}{\sqrt{5}}\log\left|\frac{1}{x-1}+\frac{1}{5}+\sqrt{\frac{x^2+4}{5(x-1)^2}}\right|+c$$

19. Let $I = \int \frac{1+x\cos x}{x\left\{1-\left(xe^{\sin x}\right)^2\right\}}dx$

$$= \int \frac{e^{\sin x}(1+x\cos x)}{xe^{\sin x}\left\{1-\left(xe^{\sin x}\right)^2\right\}}dx$$

Put $xe^{\sin x} = t$

$\therefore$ $[xe^{\sin x}\cos x + e^{\sin x}(1)]\,dx = dt$

$\therefore$ $e^{\sin x}(1 + x\cos x)\,dx = dt$

$\therefore$ $I = \int \frac{1}{t(1-t^2)}dt = \int \frac{1}{t(1-t)(1+t)}dt$

Put $\frac{1}{t(1-t)(1+t)} = \frac{A}{t}+\frac{B}{1-t}+\frac{C}{1+t}$

$\therefore$ $1 = A(1 - t)(1 + t) + Bt(1 + t) + Ct(1 - t)$ ….(i)

Putting t = 0 in (i), we get

A = 1

Putting t = 1 in (i), we get

$B = \frac{1}{2}$

Putting t = –1 in (i), we get

$C = -\frac{1}{2}$

$\therefore$ $I = \int\left(\frac{1}{t}+\frac{1/2}{1-t}-\frac{1/2}{1+t}\right)dt$

$$= \log|t| - \frac{1}{2}\log|1-t| - \frac{1}{2}\log|1+t| + c$$

$$= \frac{1}{2}\log\left|\frac{t^2}{1-t^2}\right|+c = \frac{1}{2}\log\left|\frac{x^2e^{2\sin x}}{1-x^2e^{2\sin x}}\right|+c$$

20. Let $I = \int \tan(\sin^{-1}x)dx$

$$= \int \tan\left[\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right]dx$$

$$= \int \frac{x}{\sqrt{1-x^2}}dx$$

Put $1 - x^2 = t$,

$\therefore$ $-2x\,dx = dt$,

$\therefore$ $x\,dx = -\frac{1}{2}dt$

$\therefore$ $I = \int \frac{1}{\sqrt{t}}\left(-\frac{1}{2}\right)dt$

$$= -\int \frac{1}{2\sqrt{t}}dt = -\sqrt{t}+c = -\sqrt{1-x^2}+c$$

21. Let $I = \int \sec^{25/13}x\,\mathrm{cosec}^{27/13}x\,dx$

$$= \int \cos^{-25/13}x\sin^{-27/13}x\,dx$$

Now $-\frac{25}{13}-\frac{27}{13} = -\frac{52}{13} = -4$

Multiplying and dividing by $\cos^4x$, we get

$$I = \int \cos^4x\cos^{-25/13}x\sin^{-27/13}x\sec^4x\,dx$$

$$= \int \tan^{-27/13}x(1+\tan^2x)\sec^2x\,dx$$

Put $\tan x = t$, $\therefore$ $\sec^2 x\,dx = dt$

$\therefore$ $I = \int t^{-27/13}(1+t^2)\,dt$

$= \int\left(t^{-27/13} + t^{-1/13}\right)dt$

$= -\frac{13}{14}t^{-14/13} + \frac{13}{12}t^{12/13} + c$

$= -\frac{13}{14}(\tan x)^{-14/13} + \frac{13}{12}(\tan x)^{12/13} + c$

22. Let $I = \int \frac{1}{x+\sqrt{x^2-x+1}}\,dx$

Put $x + \sqrt{x^2-x+1} = t$

$\therefore$ $\sqrt{x^2-x+1} = t - x$

$\therefore$ $x^2 - x + 1 = t^2 - 2tx + x^2$

$\therefore$ $x = \frac{t^2-1}{2t-1}$

$\therefore$ $\frac{dx}{dt} = \frac{(2t-1).2t-(t^2-1).2}{(2t-1)^2}$

$\therefore$ $dx = \frac{2t^2-2t+2}{(2t-1)^2}\,dt$

$\therefore$ $I = \int \frac{1}{t}\times\frac{2t^2-2t+2}{(2t-1)^2}\,dt$

$= 2\int \frac{t^2-t+1}{t(2t-1)^2}\,dt$

Put $\frac{t^2-t+1}{t(2t-1)^2} = \frac{A}{t} + \frac{B}{2t-1} + \frac{C}{(2t-1)^2}$

$\therefore$ $t^2 - t + 1 = A(2t-1)^2 + Bt(2t-1) + Ct$ ….(i)

Putting t = 0 in (i), we get

A = 1

Putting $t = \frac{1}{2}$ in (i), we get

$C = \frac{3}{2}$

Putting t = 1 in (i), we get

1 = A + B + C

$\Rightarrow B = 1 - 1 - \frac{3}{2} \Rightarrow B = -\frac{3}{2}$

$\therefore$ $I = 2\int\left(\frac{1}{t} - \frac{3}{2(2t-1)} + \frac{3}{2}.\frac{1}{(2t-1)^2}\right)dt$

$= 2\log t - \frac{3}{2}\log(2t-1) - \frac{3}{2}.\frac{1}{2t-1} + c$

$= 2\log t - \frac{3}{2}\log(2t-1) - \frac{1}{2}\left(\frac{3}{2t-1}\right) + c,$

where $t = x + \sqrt{x^2-x+1}$

and $2t - 1 = 2x - 1 + 2\sqrt{x^2-x+1}$

$\therefore$ $P = 2,\ Q = -\frac{3}{2},\ R = -\frac{1}{2}$

23. Let $I = \int \frac{1}{(1+x^2)\sqrt{1-x^2}}\,dx$

Put $x = \frac{1}{t}$,

$\therefore$ $dx = -\frac{1}{t^2}\,dt$

$\therefore$ $I = \int \frac{1}{\left(1+\frac{1}{t^2}\right)\sqrt{1-\frac{1}{t^2}}}\left(-\frac{1}{t^2}\right)dt$

$= -\int \frac{t\,dt}{\left(t^2+1\right)\sqrt{t^2-1}}$

Put $t^2 - 1 = m^2$

$\therefore$ $2t\,dt = 2m\,dm$,

$\therefore$ $t\,dt = m\,dm$

$\therefore$ $I = -\int \frac{m\,dm}{\left(m^2+2\right)\sqrt{m^2}}$

$= -\int \frac{1}{m^2+\left(\sqrt{2}\right)^2}\,dm$

$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{m}{\sqrt{2}}\right) + c$

$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right) + c$

$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{\frac{1}{x^2}-1}}{\sqrt{2}}\right) + c$

$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + c$

$= -\frac{1}{\sqrt{2}}\left[\frac{\pi}{2} - \cot^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)\right] + c$

$= -\frac{1}{\sqrt{2}}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1-x^2}}\right)\right] + c$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1-x^2}}\right) - \frac{\pi}{2\sqrt{2}} + c$

24. Let $I = \int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}} dx$

Dividing N^r and D^r by x^5, we get

$$I = \int \frac{\left(\frac{1}{x^3} - \frac{1}{x^5}\right)}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} dx$$

Put $2 - \frac{2}{x^2} + \frac{1}{x^4} = t \Rightarrow \left(\frac{4}{x^3} - \frac{4}{x^5}\right)dx = dt$

$\therefore$ $I = \frac{1}{4}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\sqrt{t} + c$

$$= \frac{1}{2}\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + c$$

25. $\int \frac{\log x}{(x+1)^2} dx = \int \log x.(x+1)^{-2} dx$

$$= \log x.\frac{(x+1)^{-1}}{-1} - \int \frac{1}{x}.\frac{(x+1)^{-1}}{-1} dx$$

$$= -\frac{\log x}{x+1} + \int \left(\frac{1}{x(x+1)}\right) dx$$

$$= -\frac{\log x}{x+1} + \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= -\frac{\log x}{x+1} + \log |x| - \log |x + 1| + c$$

26. $I_n = \int \sin^n x\, dx$

$= \int \sin^{n-1} x.\sin x\, dx$

$= \sin^{n-1} x \int \sin x\, dx$

$$- \int \left\{\frac{d}{dx}\left(\sin^{n-1} x\right) \int \sin x\, dx\right\} dx$$

$= \sin^{n-1} x(-\cos x)$

$$- \int (n-1)\sin^{n-2} x \cos x(-\cos x) dx$$

$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x\, dx$

$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\left(1 - \sin^2 x\right) dx$

$= -\sin^{n-1} x \cos x + (n-1) \int \left(\sin^{n-2} x - \sin^n x\right) dx$

$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx$

$$-(n-1)\int \sin^n x\, dx$$

$\therefore$ $I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$

$\therefore$ $I_n + (n-1)I_n - (n-1)I_{n-2} = -\sin^{n-1} x \cos x$

$\therefore$ $nI_n - (n-1) I_{n-2} = -\sin^{n-1} x \cos x$

27. $u = -f''(\theta) \sin \theta + f'(\theta) \cos \theta$

$\therefore$ $\frac{du}{d\theta} = -f'''(\theta) \sin \theta - f''(\theta) \cos \theta$

$$+ f''(\theta) \cos \theta - f'(\theta) \sin \theta$$

$= -f'''(\theta) \sin \theta - f'(\theta) \sin \theta$

$v = f''(\theta) \cos \theta + f'(\theta) \sin \theta$

$\therefore$ $\frac{dv}{d\theta} = -f''(\theta) \sin \theta + f'''(\theta) \cos \theta$

$$+ f'(\theta) \cos \theta + f''(\theta) \sin \theta$$

$= f'''(\theta) \cos \theta + f'(\theta) \cos \theta$

$\therefore$ $\left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2 = [-f'''(\theta) \sin \theta - f'(\theta) \sin \theta]^2$

$$+ [f'''(\theta) \cos \theta + f'(\theta) \cos \theta]^2$$

$= [f'''(\theta)]^2 \sin^2\theta + 2 f'''(\theta) f'(\theta) \sin^2 \theta$

$$+ [f'(\theta)]^2 \sin^2 \theta + [f'''(\theta)]^2 \cos^2 \theta$$

$$+ 2f'''(\theta) f'(\theta) \cos^2 \theta + [f'(\theta)]^2 \cos^2 \theta$$

$= [f'''(\theta)]^2 + 2f'''(\theta) f'(\theta) + [f'(\theta)]^2$

$\quad$ ....[$\because \sin^2 \theta + \cos^2 \theta = 1$]

$= [f'''(\theta) + f'(\theta)]^2$

$\therefore$ $\int \left[\left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2\right]^{1/2} d\theta = \int [f'''(\theta) + f'(\theta)] d\theta$

$$= f''(\theta) + f(\theta) + c$$

28. Let $I = \int \frac{x - 1}{(x+1)\sqrt{x^3 + x^2 + x}} dx$

$$= \int \frac{x^2 - 1}{(x+1)^2\sqrt{x^3 + x^2 + x}} dx$$

$$= \int \frac{x^2 - 1}{(x^2 + 2x + 1)\sqrt{x^3 + x^2 + x}} dx$$

$$= \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x} + 2\right)\sqrt{x + \frac{1}{x} + 1}} dx$$

Put $x + \frac{1}{x} + 1 = t^2$

$\therefore$ $\left(1 - \frac{1}{x^2}\right) dx = 2t\, dt$

$\therefore$ $I = \int \frac{2t}{(t^2 + 1)\sqrt{t^2}} dt = 2\int \frac{1}{t^2 + 1} dt = 2 \tan^{-1} t + c$

$$= 2 \tan^{-1}\left(\sqrt{x + \frac{1}{x} + 1}\right) + c$$

29. Let $I = \int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$

$$= \int\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx$$

$$= \int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx$$

$$= \int\frac{\sqrt{2}(\sin x+\cos x)}{\sqrt{2\sin x\cos x}}dx$$

$$= \sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-1+2\sin x\cos x}}dx$$

$$= \sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-\left(1-2\sin x\cos x\right)}}dx$$

$$= \sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-\left(\sin^2 x-2\sin x\cos x+\cos^2 x\right)}}dx$$

$$= \sqrt{2}\int\frac{1}{\sqrt{1-\left(\sin x-\cos x\right)^2}}(\sin x+\cos x)\,dx$$

Put $\sin x - \cos x = t$

$\therefore$ $(\cos x + \sin x)dx = dt$

$\therefore$ $I = \sqrt{2}\int\frac{1}{\sqrt{1-t^2}}dt$

$$= \sqrt{2}\sin^{-1}(t)+c$$

$$= \sqrt{2}\sin^{-1}(\sin x - \cos x) + c$$

Textbook Chapter No.

05 Definite Integrals

Hints

Classical Thinking

1. $\int_1^e \frac{1}{x} dx = [\log x]_1^e = \log_e e - \log 1 = 1$

2. $\int_1^3 (x-1)(x-2)(x-3)\, dx$

$= \int_1^3 (x^3 - 6x^2 + 11x - 6)\, dx$

$= \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{11x^2}{2} - 6x\right]_1^3 = 0$

3. $\int_0^1 (1-x)^9\, dx = \left[\frac{-(1-x)^{10}}{10}\right]_0^1 = \frac{1}{10}$

4. $\int_0^1 e^{2\log x}\, dx = \int_0^1 e^{\log x^2}\, dx$

$= \int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

5. $\int_0^{\pi/3} \cos 3x\, dx = \left[\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{3}} = 0$

6. $\int_{\pi/4}^{\pi/2} \operatorname{cosec}^2 x\, dx = [-\cot x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\left[\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right] = 1$

7. $\int_0^{2\pi} (\sin x + \cos x)\, dx = [-\cos x + \sin x]_0^{2\pi} = 0$

8. $\frac{1}{2}\int_0^{\pi/8} \sec^2 2x dx = \frac{1}{4}[\tan 2x]_0^{\frac{\pi}{8}} = \frac{1}{4}(1) = \frac{1}{4}$

9. $\int_{\pi/6}^{\pi/4} \operatorname{cosec} 2x dx = \frac{1}{2}[\log\tan x]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\log\tan\frac{\pi}{4} - \log\tan\frac{\pi}{6}\right] = \frac{1}{2}\log\sqrt{3}$

10. Put $1 + \log x = t \Rightarrow \frac{1}{x}dx = dt$

When $x = 1$, $t = 1$ and when $x = e$, $t = 2$

$\therefore \int_1^e \frac{1+\log x}{x}\, dx = \int_1^2 t\, dt = \left[\frac{t^2}{2}\right]_1^2 = \frac{3}{2}$

11. Put $t = -\frac{1}{x} \Rightarrow dt = \frac{1}{x^2}\, dx$

When $x = 1$, $t = -1$ and when $x = 2$, $t = \frac{-1}{2}$

$\therefore \int_1^2 \frac{1}{x^2} e^{-\frac{1}{x}}\, dx = \int_{-1}^{-1/2} e^t\, dt = [e^t]_{-1}^{-1/2}$

$= e^{\frac{-1}{2}} - e^{-1} = \frac{\sqrt{e}-1}{e}$

12. Put $\log x = t \Rightarrow \frac{1}{x}\, dx = dt$

When $x = 1$, $t = 0$ and when $x = 2$, $t = \log 2$

$\therefore \int_1^2 \frac{\cos(\log x)}{x}\, dx = \int_0^{\log 2} \cos t\, dt$

$= [\sin t]_0^{\log 2} = \sin(\log 2)$

13. $\int_1^{\sqrt{3}} \frac{1}{1+x^2}\, dx = [\tan^{-1}x]_1^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

14. Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

When $x = 0$, $t = 0$ and when $x = 1$, $t = \frac{\pi}{4}$

$\therefore \int_0^1 \frac{\tan^{-1}x}{1+x^2}dx = \int_0^{\pi/4} t\, dt = \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{32}$

15. $\int_0^1 \frac{dx}{x^2 - 2x + 2} = \int_0^1 \frac{dx}{(x-1)^2 + 1}$

$= \left[\tan^{-1}(x-1)\right]_0^1$

$= 0 - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$

16. $\int_1^2 \log x dx = [x\log x - x]_1^2$

$= 2\log 2 - 2 + 1$

$= \log 4 - 1 = \log 4 - \log e = \log\left(\frac{4}{e}\right)$

17. $\int_1^2 e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \left[\frac{1}{x}e^x\right]_1^2 = \frac{e^2}{2} - e$

18. $\int_2^3 \frac{dx}{x^2 - x} = \int_2^3 \frac{dx}{x(x-1)} = \int_2^3 \left[\frac{1}{x-1} - \frac{1}{x}\right] dx$

$= \int_2^3 \frac{1}{(x-1)} dx - \int_2^3 \frac{1}{x} dx$

$= [\log(x-1)]_2^3 - [\log x]_2^3$

$= (\log 2 - \log 1) - (\log 3 - \log 2) = 2\log 2 - \log 3$

$= \log\left(\frac{4}{3}\right)$

19. Put $x = a - t \Rightarrow dx = -dt$

When $x = 0$, $t = a$ and when $x = a$, $t = 0$

$\therefore \quad \int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt$

$= \int_0^a f(a-t)\,dt \quad \ldots\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$

$= \int_0^a f(a-x)\,dx \quad \ldots\left[\because \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right]$

20. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx \quad \ldots(i)$

$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$

$\ldots\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx \quad \ldots(ii)$

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\pi/2}$

$\therefore \quad 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

21. Let $I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \ldots(i)$

$\therefore \quad I = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$

$\ldots\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$\therefore \quad I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \ldots(ii)$

Adding (i) and (ii), we get

$2I = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

22. Let $I = \int_0^{\pi} x \sin x\,dx \quad \ldots(i)$

$\therefore \quad I = \int_0^{\pi} (\pi - x)\sin(\pi - x)\,dx$

$\ldots\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$\therefore \quad I = \int_0^{\pi} (\pi - x)\sin x\,dx \quad \ldots(ii)$

Adding (i) and (ii), we get

$2I = \pi\int_0^{\pi} \sin x\,dx = \pi[-\cos x]_0^{\pi} = 2\pi$

$\Rightarrow I = \pi$

23. Let $I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx \quad \ldots(i)$

$\therefore \quad I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}} dx \quad \ldots(ii)$

$\ldots\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$

Adding (i) and (ii), we get

$2I = \int_2^3 dx = [x]_2^3 = 3 - 2 = 1$

$\Rightarrow I = \frac{1}{2}$

24. Let $f(x) = x^{17}\cos^4 x$

$\therefore \quad f(-x) = (-x)^{17}\{\cos(-x)\}^4 = -f(x)$

$\therefore \quad f(x)$ is an odd function.

$\therefore \quad \int_{-1}^{1} x^{17}\cos^4 x\,dx = 0$

25. Since, $\sin^{11} x$ is an odd function.

$\therefore \quad \int_{-1}^{1} \sin^{11} x\,dx = 0$

26. Since, $3\sin x + \sin^3 x$ is an odd function.

$\therefore \quad \int_{-\pi/2}^{\pi/2} \left(3\sin x + \sin^3 x\right) dx = 0$

Critical Thinking

1. $\int_{-2}^{2}(ax^3+bx+c)dx = \left[\frac{ax^4}{4}+\frac{bx^2}{2}+cx\right]_{-2}^{2} = 4c$

Hence, the value depends on c.

2. $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$= \int_0^1 \frac{(\sqrt{1+x}+\sqrt{x})}{(\sqrt{1+x}-\sqrt{x})(\sqrt{1+x}+\sqrt{x})}dx$

$= \int_0^1 \frac{(\sqrt{1+x}+\sqrt{x})}{1+x-x}dx = \int_0^1 \sqrt{1+x}\,dx + \int_0^1 \sqrt{x}dx$

$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$

$= \frac{4\sqrt{2}}{3} - \frac{2}{3} + \frac{2}{3} - 0 = \frac{4\sqrt{2}}{3}$

3. $I + J = \int_0^{\pi/4}(\sin^2 x + \cos^2 x)dx = \int_0^{\pi/4} dx = \frac{\pi}{4}$

$\therefore$ $I = \frac{\pi}{4} - J$

4. $\int_0^{\pi/4}\tan^2 x dx = \int_0^{\pi/4}(\sec^2 x - 1)\,dx$

$= [\tan x]_0^{\frac{\pi}{4}} - [x]_0^{\frac{\pi}{4}}$

$= 1 - \frac{\pi}{4}$

5. $\int_0^{\pi}\frac{dx}{1+\sin x} = \int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$

$= \int_0^{\pi}(\sec^2 x - \sec x\tan x)\,dx$

$= [\tan x - \sec x]_0^{\pi} = \tan\pi - \sec\pi + 1$

$= 0 + 1 + 1 = 2$

6. $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$

$= \int_{\pi/4}^{3\pi/4}\frac{1-\cos x}{1-\cos^2 x}dx = \int_{\pi/4}^{3\pi/4}\frac{1-\cos x}{\sin^2 x}dx$

$= \int_{\pi/4}^{3\pi/4}(\operatorname{cosec}^2 x - \cot x\operatorname{cosec} x)\,dx$

$= [-\cot x + \operatorname{cosec} x]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 2$

7. $I = \int_0^{\pi/2}\frac{(\sin x+\cos x)^2}{\sqrt{1+\sin 2x}}dx$

$= \int_0^{\pi/2}\frac{(\sin x+\cos x)^2}{\sqrt{(\sin x+\cos x)^2}}dx$

$= \int_0^{\pi/2}(\sin x+\cos x)\,dx = [-\cos x+\sin x]_0^{\pi/2}$

$= 2$

8. $\int_0^{2\pi}\sqrt{1+\sin\frac{x}{2}}\,dx = \int_0^{2\pi}\sqrt{\left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)^2}dx$

$= \int_0^{2\pi}\left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)dx$

....$\left[\because x\in(0,2\pi), \therefore \left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)>0\right]$

$= 4\left[-\cos\frac{x}{4}+\sin\frac{x}{4}\right]_0^{2\pi}$

$= 4(0 + 1 + 1 - 0)$

$= 8$

9. $\int_{-1}^{3}\left\{\tan^{-1}\left(\frac{x}{x^2+1}\right)+\tan^{-1}\left(\frac{x^2+1}{x}\right)\right\}dx$

$= \int_{-1}^{3}\left\{\tan^{-1}\left(\frac{x}{x^2+1}\right)+\cot^{-1}\left(\frac{x}{x^2+1}\right)\right\}dx$

$= \int_{-1}^{3}\frac{\pi}{2}dx$ ....$\left[\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$

$= \frac{\pi}{2}[x]_{-1}^{3} = 2\pi$

10. Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{4}$, $t = 1$

$\therefore$ $\int_0^{\pi/4}\tan^6 x\sec^2 x dx = \int_0^1 t^6 dt = \frac{1}{7}[t^7]_0^1 = \frac{1}{7}$

11. Let $I = \int_{\pi/4}^{\pi/2}\cos\theta\,\frac{1}{\sin^2\theta}d\theta$

Put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

When $\theta = \frac{\pi}{4}$, $t = \frac{1}{\sqrt{2}}$ and when $\theta = \frac{\pi}{2}$, $t = 1$

$\therefore$ $I = \int_{1/\sqrt{2}}^{1}\frac{1}{t^2}dt = \left[\frac{-1}{t}\right]_{1/\sqrt{2}}^{1}$

$= \sqrt{2} - 1$

12. Let $I = \int_0^{\pi/6} \frac{\sin x}{\cos^3 x} dx = \int_0^{\frac{\pi}{6}} \tan x \sec^2 x \, dx$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

When $x = 0$, t = 0 and when $x = \frac{\pi}{6}$, $t = \frac{1}{\sqrt{3}}$

$\therefore \quad I = \int_0^{\frac{1}{\sqrt{3}}} t \, dt = \left[\frac{t^2}{2}\right]_0^{\frac{1}{\sqrt{3}}} = \frac{1}{6}$

13. Let $I = \int_0^{\pi/4} \sec^7 \theta.\sin^3 \theta d\theta = \int_0^{\pi/4} \tan^3\theta \sec^4 \theta d\theta$

Put $\tan\theta = t \Rightarrow \sec^2 \theta \, d\theta = dt$

When $\theta = 0$, t = 0 and when $\theta = \frac{\pi}{4}$, t = 1

$\therefore \quad I = \int_0^1 t^3 (1 + t^2) \, dt = \left[\frac{t^4}{4} + \frac{t^6}{6}\right]_0^1 = \frac{5}{12}$

14. Put $x^3 = t \Rightarrow x^2 \, dx = \frac{dt}{3}$

When $x = 0$, t = 0 and when x = a, t = a^3

$\therefore \quad \int_0^a x^2 \sin x^3 \, dx = \frac{1}{3}\int_0^{a^3} \sin t \, dt = -\frac{1}{3}[\cos t]_0^{a^3}$

$= -\frac{1}{3}(\cos a^3 - 1) = \frac{1}{3}(1 - \cos a^3)$

15. Put $\sqrt{x} = t \Rightarrow \frac{1}{\sqrt{x}} dx = 2 \, dt$

When $x = 0$, t = 0 and when $x = 2$, t = $\sqrt{2}$

$\therefore \quad \int_0^2 \frac{3^{\sqrt{x}}}{\sqrt{x}} dx = 2\int_0^{\sqrt{2}} 3^t dt = 2\left[\frac{3^t}{\log 3}\right]_0^{\sqrt{2}} = \frac{2}{\log 3}(3^{\sqrt{2}} - 1)$

16. Let $I = \int_0^{\pi/2} \sin x \sin 2x \, dx = 2 \int_0^{\pi/2} \sin^2 x \cos x dx$

Put $\sin x = t \Rightarrow \cos x \, dx = dt$

When $x = 0$, t = 0 and when $x = \frac{\pi}{2}$, t = 1

$\therefore \quad I = 2 \int_0^1 t^2 dt = \frac{2}{3}[t^3]_0^1 = \frac{2}{3}$

17. Let $I = \int_1^{e^2} \frac{dx}{x(1 + \log x)^2}$

Put $(1 + \log x) = t \Rightarrow \frac{1}{x} \, dx = dt$

When $x = 1$, t = 1 and when $x = e^2$, t = 3

$\therefore \quad I = \int_1^3 \frac{dt}{t^2} = \left[\frac{-1}{t}\right]_1^3 = -\left(\frac{1}{3} - 1\right) = \frac{2}{3}$

18. Let $I = \int_0^{\pi/4} \sec x \log (\sec x + \tan x) \, dx$

Put $\log(\sec x + \tan x) = t \Rightarrow \sec x \, dx = dt$

$\therefore \quad I = \int_0^{\log(\sqrt{2}+1)} t \, dt = \left[\frac{t^2}{2}\right]_0^{\log(\sqrt{2}+1)} = \frac{[\log(\sqrt{2}+1)]^2}{2}$

19. $\int_0^{-\pi/4} \frac{1 + \tan x}{1 - \tan x} dx = \int_0^{\frac{-\pi}{4}} \tan\left(\frac{\pi}{4} + x\right) dx$

$= \left[\log\left\{\sec\left(\frac{\pi}{4} + x\right)\right\}\right]_0^{-\pi/4}$

$= -\frac{1}{2}\log 2$

20. Since, $\sin\theta$ is positive in interval $(0, \pi)$.

$\therefore \quad \int_0^{\pi} |\sin^3\theta| d\theta = \int_0^{\pi} \sin^3 \theta d\theta = \int_0^{\pi} \sin \theta(1 - \cos^2\theta)d\theta$

$= \int_0^{\pi} \sin\theta \, d\theta + \int_0^{\pi} (-\sin\theta) \cos^2 \theta d\theta$

$= [-\cos\theta]_0^{\pi} + \left[\frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{4}{3}$

21. Let $I = \int_0^{\pi/8} \cos^3 4\theta \, d\theta = \int_0^{\pi/8} \cos^2 4\theta.\cos 4\theta d\theta$

$= \int_0^{\pi/8} (1 - \sin^2 4\theta) \cos 4\theta d\theta$

Put $\sin 4\theta = t \Rightarrow \cos 4\theta d\theta = \frac{dt}{4}$

When $\theta = 0$, t = 0 and when $\theta = \frac{\pi}{8}$, t = 1

$\therefore \quad I = \frac{1}{4}\int_0^1 (1 - t^2) \, dt = \frac{1}{4}\left[t - \frac{t^3}{3}\right]_0^1 = \frac{1}{6}$

22. Let $I = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1 + \cos x}}{(1 - \cos x)^{\frac{5}{2}}} dx$

$= \int_{\pi/3}^{\pi/2} \frac{\sqrt{1 + \cos x}}{(1 - \cos x)^{\frac{5}{2}}} \times \frac{\sqrt{1 - \cos x}}{\sqrt{1 - \cos x}} dx$

$= \int_{\pi/3}^{\pi/2} \frac{\sin x}{(1 - \cos x)^3} dx$

Put $1 - \cos x = t$

$\Rightarrow \sin x \, dx = dt$

$\therefore \quad I = \int_{1/2}^1 \frac{dt}{t^3} = \left[\frac{t^{-2}}{-2}\right]_{1/2}^1 = \frac{3}{2}$

23. Put $\sin^2 x = t \Rightarrow 2\sin x\cos x\,dx = dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = 1$

$$\therefore \int_0^{\pi/2}\frac{\sin x\cos x}{1+\sin^4 x}dx = \frac{1}{2}\int_0^1\frac{1}{1+t^2}dt$$

$$= \frac{1}{2}[\tan^{-1}t]_0^1 = \frac{\pi}{8}$$

24. Let $I = \int_0^{\pi/4}\frac{4\sin 2\theta\,d\theta}{\sin^4\theta+\cos^4\theta} = 4\int_0^{\pi/4}\frac{2\sin\theta\cos\theta\,d\theta}{\sin^4\theta+\cos^4\theta}$

$$= 4\int_0^{\pi/4}\frac{2\tan\theta\sec^2\theta\,d\theta}{\tan^4\theta+1}$$

Put $\tan^2\theta = t \Rightarrow 2\tan\theta\sec^2\theta\,d\theta = dt$

$$\therefore I = 4\int_0^1\frac{dt}{t^2+1} = 4[\tan^{-1}t]_0^1 = 4\left[\frac{\pi}{4}-0\right] = \pi$$

25. $k\int_0^1 x f(3x)dx = \int_0^3 t f(t)dt$ ....(i)

Put $3x = t \Rightarrow dx = \frac{dt}{3}$

$$\therefore k\int_0^1 xf(3x)dx = k\int_0^3\frac{t}{3}\cdot f(t)\cdot\frac{dt}{3} = \frac{k}{9}\int_0^3 tf(t)dt$$

From (i),

$$\frac{k}{9}\int_0^3 t f(t)dt = \int_0^3 t f(t)dt$$

$$\Rightarrow \frac{k}{9} = 1 \Rightarrow k = 9$$

26. $\int_0^{2/3}\frac{dx}{4+9x^2} = \frac{1}{9}\int_0^{\frac{2}{3}}\frac{dx}{\left(\frac{2}{3}\right)^2+x^2}$

$$= \frac{1}{9}\times\frac{1}{\frac{2}{3}}\left[\tan^{-1}\frac{x}{\frac{2}{3}}\right]_0^{\frac{2}{3}}$$

$$= \frac{1}{6}\times\frac{\pi}{4} = \frac{\pi}{24}$$

27. $\int_0^k\frac{dx}{2+8x^2} = \frac{\pi}{16}$

$$\Rightarrow \frac{1}{2}\int_0^k\frac{1}{1^2+(2x)^2}dx = \frac{\pi}{16}$$

$$\Rightarrow \frac{1}{2}\left[\frac{\tan^{-1}(2x)}{2}\right]_0^k = \frac{\pi}{16}$$

$$\Rightarrow \frac{1}{4}\left(\tan^{-1}2k\right) = \frac{\pi}{16}$$

$$\Rightarrow \tan^{-1}2k = \frac{\pi}{4} \Rightarrow k = \frac{1}{2}$$

28. $\int_0^1\frac{dx}{x^2-x+1} = \int_0^1\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)\right] = \frac{2}{\sqrt{3}}.\frac{\pi}{3} = \frac{2\pi}{3\sqrt{3}}$$

29. Let $I = \int_0^1\frac{dx}{e^x+e^{-x}} = \int_0^1\frac{e^x}{1+e^{2x}}dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$$\therefore I = \int_1^e\frac{dt}{1+t^2} = [\tan^{-1}t]_1^e = \tan^{-1}e - \tan^{-1}1$$

$$= \tan^{-1}\left(\frac{e-1}{e+1}\right)$$

$$\ldots\left[\because \tan^{-1}x-\tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

30. $\int_{1/4}^{1/2}\frac{dx}{\sqrt{x-x^2}} = \int_{1/4}^{1/2}\frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}$

$$= \left[\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)\right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= [\sin^{-1}(2x-1)]_{1/4}^{1/2} = \frac{\pi}{6}$$

31. $\int_3^5\frac{x^2}{x^2-4}dx = \int_3^5\left(1+\frac{4}{x^2-4}\right)dx$

$$= \left[x+\frac{4}{2(2)}\log\left|\frac{x-2}{x+2}\right|\right]_3^5$$

$$= 2+\log_e\left(\frac{15}{7}\right)$$

32. Let $I = \int_0^1 \frac{1}{[ax+b(1-x)]^2}\,dx$

$= \int_0^1 \frac{1}{[(a-b)x+b]^2}\,dx$

Put $(a-b)x + b = t \Rightarrow (a-b)\,dx = dt$

When $x = 0$, t = b and when $x = 1$, t = a

$\therefore \quad I = \frac{1}{a-b}\int_b^a \frac{1}{t^2}dt$

$= \frac{1}{(a-b)}\left[-\frac{1}{t}\right]_b^a$

$= \frac{1}{(a-b)}\left(\frac{a-b}{ab}\right)$

$\therefore \quad I = \frac{1}{ab}$

33. Put $a^2 + x^2 = t \Rightarrow 2x\,dx = dt$

When $x = 0$, $t = a^2$ and when $x = a$, $t = 2a^2$

$\therefore \quad \int_0^a \frac{x\,dx}{\sqrt{a^2+x^2}} = \frac{1}{2}\int_{a^2}^{2a^2}\frac{1}{\sqrt{t}}dt$

$= \left[\sqrt{t}\right]_{a^2}^{2a^2} = \left(2a^2\right)^{\frac{1}{2}} - \left(a^2\right)^{\frac{1}{2}} = a(\sqrt{2}-1)$

34. Put $1+e^{-x} = t \Rightarrow -e^{-x}dx = dt$

When $x = 0$, t = 2 and when $x = 1$, $t = 1+\frac{1}{e}$

$\therefore \quad \int_0^1 \frac{e^{-2x}}{1+e^{-x}}dx = \int_2^{1+\frac{1}{e}}\frac{(t-1)(-dt)}{t} = \int_2^{1+\frac{1}{e}}\left(\frac{1}{t}-1\right)dt$

$= \left[\log t - t\right]_2^{1+\frac{1}{e}}$

$= \log\left(1+\frac{1}{e}\right) - \left(1+\frac{1}{e}\right) - \log 2 + 2$

$= \log\left(\frac{e+1}{2e}\right) - \frac{1}{e} + 1$

35. Let $I = \int_0^{\pi/2}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Dividing N^r and D^r by $\cos^2 x$, we get

$I = \int_0^{\pi/2}\frac{\sec^2 x}{a^2+b^2\tan^2 x}dx$

Put $b\tan x = t \Rightarrow b\sec^2 x\,dx = dt$

When $x = 0$, t = 0 and when $x = \frac{\pi}{2}$, $t = \infty$

$\therefore \quad I = \int_0^\infty \frac{\frac{dt}{b}}{a^2+t^2} = \frac{1}{b}\left[\frac{1}{a}\tan^{-1}\left(\frac{t}{a}\right)\right]_0^\infty$

$= \frac{1}{ab}(\tan^{-1}\infty - \tan^{-1}0)$

$= \frac{1}{ab}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{2ab}$

36. Put $e^x - 1 = t^2 \Rightarrow e^x\,dx = 2t\,dt$

When $x = 0$, t = 0 and when $x = \log 5$, t = 2

$\therefore \quad \int_0^{\log 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}dx = \int_0^2 \frac{2t^2}{t^2+4}dt$

$= 2\int_0^2\left(1-\frac{4}{t^2+4}\right)dt$

$= 2\left[t - 4.\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$

$= 2\left(2-2.\frac{\pi}{4}\right) = 4-\pi$

37. Put $x = 2\cos\theta \Rightarrow dx = -2\sin\theta\,d\theta$

$\therefore \quad \int_0^2\sqrt{\frac{2+x}{2-x}}dx = -2\int_{\pi/2}^0\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\sin\theta\,d\theta$

$= -4\int_{\pi/2}^0\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$

$= -2\int_{\pi/2}^0(1+\cos\theta)\,d\theta$

$= -2\left[\theta+\sin\theta\right]_{\frac{\pi}{2}}^0$

$= 2\left(\frac{\pi}{2}+1\right) = \pi+2$

38. Since, $\int_a^b\sqrt{\frac{x-a}{b-x}}dx = \frac{\pi}{2}(b-a)$

$\therefore \quad \int_3^4\sqrt{\frac{x-3}{4-x}}dx = \frac{\pi}{2}(4-3) = \frac{\pi}{2}$

39. Since, $\int_a^b\sqrt{(x-a)(b-x)}\,dx = \frac{\pi}{8}(b-a)^2$

$\therefore \quad \int_3^7\sqrt{(x-3)(7-x)}\,dx = \frac{\pi}{8}(7-3)^2$

$= \frac{\pi}{8}\times 16 = 2\pi$

40. Put $x^2 + 1 = t \Rightarrow 2x\,dx = dt$

When $x = 0$, t = 1 and when $x = 2$, t = 5

$$\therefore \int_0^2 \frac{x^3}{(x^2+1)^{\frac{3}{2}}}dx = \frac{1}{2}\int_1^5 \frac{(t-1)}{t^{\frac{3}{2}}}dt$$

$$= \frac{1}{2}\int_1^5\left(t^{\frac{-1}{2}} - t^{\frac{-3}{2}}\right)dt$$

$$= \frac{1}{2}\left[2\sqrt{t} + 2\frac{1}{\sqrt{t}}\right]_1^5$$

$$= \frac{1}{2}\left[2\sqrt{5} + \frac{2}{\sqrt{5}} - 2 - 2\right]$$

$$= \sqrt{5} + \frac{1}{\sqrt{5}} - 2 = \frac{6-2\sqrt{5}}{\sqrt{5}}$$

41. Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\,d\theta$

$$\therefore \int_0^a \frac{x^4}{(a^2+x^2)^4}dx = \int_0^{\frac{\pi}{4}} \frac{a^4\tan^4\theta . a\sec^2\theta}{a^8\sec^8\theta}d\theta$$

$$= \frac{1}{a^3}\int_0^{\frac{\pi}{4}}\sin^4\theta\ \cos^2\theta\ d\theta$$

$$= \frac{1}{a^3}\int_0^{\frac{\pi}{4}}(\sin^4\theta - \sin^6\theta)\,d\theta$$

$$= \frac{1}{a^3}\int_0^{\frac{\pi}{4}}\left[\frac{(1-\cos 2\theta)^2}{4} - \frac{(1-\cos 2\theta)^3}{8}\right]d\theta$$

$$= \frac{1}{8a^3}\int_0^{\frac{\pi}{4}}(1+\cos 2\theta)(1-\cos 2\theta)^2\,d\theta$$

$$= \frac{1}{8a^3}\int_0^{\frac{\pi}{4}}(1-\cos 2\theta - \cos^2 2\theta + \cos^3 2\theta)\,d\theta$$

$$= \frac{1}{8a^3}\int_0^{\frac{\pi}{4}}\frac{1}{4}\left[2 - \cos 2\theta - 2\cos 4\theta + \cos 6\theta\right]d\theta$$

$$\ldots\left[\because \cos^2 A = \frac{1+\cos 2A}{2} \text{ and } \cos^3 A = \frac{\cos 3A + 3\cos A}{4}\right]$$

$$= \frac{1}{32a^3}\left[2\theta - \frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{2} + \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{16a^3}\left(\frac{\pi}{4} - \frac{1}{3}\right)$$

42. $$\int_0^{\pi}\frac{dx}{1 - 2a\cos x + a^2}$$

$$= \int_0^{\pi}\frac{dx}{(1+a^2)\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) - 2a\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$= \int_0^{\pi}\frac{dx}{(1-a)^2\cos^2\frac{x}{2} + (1+a)^2\sin^2\frac{x}{2}}$$

$$= \int_0^{\pi}\frac{\sec^2\frac{x}{2}}{(1-a)^2 + (1+a^2)\tan^2\frac{x}{2}}\,dx$$

$$= \frac{2}{(1+a)^2}\int_0^{\infty}\frac{dt}{\left\{\frac{(1-a)}{(1+a)}\right\}^2 + t^2}$$

$$\ldots\left[\text{Put } t = \tan\frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}dx\right]$$

$$= \frac{2}{(1+a)^2}\cdot\frac{(1+a)}{(1-a)}\left[\tan^{-1}\left(\frac{1+a}{1-a}.t\right)\right]_0^{\infty}$$

$$= \frac{2}{(1-a^2)}(\tan^{-1}\infty - \tan^{-1}0) = \frac{\pi}{1-a^2}$$

43. $$\int_0^1 x^2 e^x dx = [x^2.e^x]_0^1 - \int_0^1 2xe^x dx$$

$$= e - 2\ [xe^x - e^x]_0^1$$

$$= e - 2\ [e - e - (0 - 1)] = e - 2$$

44. Let $I = \int_{\frac{-\pi}{4}}^{\frac{\pi}{2}} e^{-x}\sin x\,dx$

$$\therefore\ I = \left[-e^{-x}\sin x - e^{-x}\cos x\right]_{-\pi/4}^{\pi/2} - \int_{\frac{-\pi}{4}}^{\frac{\pi}{2}} e^{-x}\sin x\,dx$$

$$\Rightarrow 2I = \left[e^{-x}(-\sin x - \cos x)\right]_{-\pi/4}^{\pi/2}$$

$$\Rightarrow I = \frac{1}{2}\left[e^{\frac{-\pi}{2}}(-1-0) - \left\{e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right\}\right]$$

$$\Rightarrow I = -\frac{1}{2}e^{-\frac{\pi}{2}}$$

45. $$\int_0^1 \tan^{-1}x\,dx = \left[(\tan^{-1}x).x\right]_0^1 - \int_0^1\frac{1}{1+x^2}\cdot x\,dx$$

$$= \left[x\tan^{-1}x - \frac{1}{2}\log\left|1+x^2\right|\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2$$

46. $\int_0^1 \cos^{-1} x \, dx = \left[x\cos^{-1} x - \sqrt{1-x^2}\right]_0^1 = 1$

47. Put $x = t^2 \Rightarrow dx = 2t\, dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi^2}{4}$, $t = \frac{\pi}{2}$

$\therefore \int_0^{\frac{\pi^2}{4}} \sin\sqrt{x}\, dx = 2\int_0^{\frac{\pi}{2}} t\sin t\, dt$

$= 2\left[-t\cos t + \sin t\right]_0^{\pi/2} = 2$

48. Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{4}$

$\therefore \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \theta\sec^2\theta\, d\theta$

$= 2[\theta\tan\theta]_0^{\pi/4} - 2\int_0^{\frac{\pi}{4}} \tan\theta\, d\theta$

$= \frac{\pi}{2} + 2[\log\cos x]_0^{\pi/4}$

$= \frac{\pi}{2} - 2\log\sqrt{2}$

49. $\int_0^{\frac{\pi}{2}} \frac{x+\sin x}{1+\cos x}dx = \int_0^{\frac{\pi}{2}}\left[\frac{x + 2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right]dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left[x\sec^2\frac{x}{2} + 2\tan\frac{x}{2}\right]dx$

$= \frac{1}{2}\left[2x\tan\frac{x}{2}\right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}}\tan\frac{x}{2}dx + \int_0^{\frac{\pi}{2}}\tan\frac{x}{2}dx$

$= \left[x\tan\frac{x}{2}\right]_0^{\pi/2} = \frac{\pi}{2}\tan\frac{\pi}{4} = \frac{\pi}{2}$

50. $\int_0^{\frac{\pi}{6}} (2+3x^2)\cos 3x\, dx$

$= \left[(2+3x^2)\cdot\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} 6x\cdot\frac{\sin 3x}{3}dx$

$= \frac{2}{3} + \frac{\pi^2}{36} + \left[\frac{2x\cos 3x}{3}\right]_0^{\frac{\pi}{6}} - \frac{2}{3}\int_0^{\frac{\pi}{6}}\cos 3x\, dx$

$= \frac{2}{3} + \frac{\pi^2}{36} + 0 - \frac{2}{9}[\sin 3x]_0^{\pi/6}$

$= \frac{2}{3} + \frac{\pi^2}{36} - \frac{2}{9} = \frac{1}{36}(\pi^2 + 16)$

51. $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx$

$= \int_2^e \frac{1}{\log x}dx - \int_2^e \frac{1}{(\log x)^2}dx$

$= \left[\frac{x}{\log x}\right]_2^e - \int_2^e\left\{-\frac{1}{x(\log x)^2}\right\}x\, dx - \int_2^e \frac{1}{(\log x)^2}dx$

$= \left[\frac{x}{\log x}\right]_2^e = e - \frac{2}{\log 2}$

$\therefore \alpha = e, \beta = -2$

52. $\int_1^e \frac{e^x}{x}(1 + x\log x)\, dx = \int_1^e e^x\left(\frac{1}{x} + \log x\right)dx$

$= \left[e^x . \log x\right]_1^e = e^e$

53. $\int_{\pi/4}^{\pi/2} e^x (\log \sin x + \cot x)\, dx$

$= [e^x \log\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= e^{\frac{\pi}{2}}\log\sin\frac{\pi}{2} - e^{\frac{\pi}{4}}\log\sin\frac{\pi}{4} = \frac{1}{2}e^{\frac{\pi}{4}}\log 2$

54. $\int_0^1 \frac{e^x(x-1)}{(x+1)^3}dx = \int_0^1 \frac{e^x(x+1-2)}{(x+1)^3}dx$

$= \int_0^1 e^x\left[\frac{1}{(1+x)^2} + \frac{-2}{(1+x)^3}\right]dx$

$= \left[\frac{e^x}{(1+x)^2}\right]_0^1 = \frac{e}{4} - 1$

55. $\phi(x) = \frac{1}{x(x^4+1)} = \frac{1}{x} - \frac{x^3}{x^4+1}$

$\therefore \int_1^2 \phi(x)\, dx = \int_1^2\left(\frac{1}{x} - \frac{x^3}{x^4+1}\right)dx$

$= [\log x]_1^2 - \left[\frac{1}{4}\log(x^4+1)\right]_1^2$

$= \frac{1}{4}\log\frac{32}{17}$

56. Put $\sin x = t \Rightarrow \cos x\,dx = dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = 1$

$\therefore \int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)} dx = \int_0^1 \frac{dt}{(1+t)(2+t)}$

$= \int_0^1 \left(\frac{1}{1+t} - \frac{1}{2+t}\right) dt$

$= [\log(1+t) - \log(2+t)]_0^1$

$= \log\left(\frac{2}{3}\right) - \log\left(\frac{1}{2}\right) = \log\left(\frac{4}{3}\right)$

57. Put $1 + \tan x = t \Rightarrow \sec^2 x\,dx = dt$

When $x = 0$, $t = 1$ and when $x = \frac{\pi}{4}$, $t = 2$

$\therefore \int_0^{\pi/4} \frac{\sec^2 x}{(1+\tan x)(2+\tan x)} dx$

$= \int_1^2 \frac{dt}{t(1+t)} = \int_1^2 \frac{dt}{t} - \int_1^2 \frac{dt}{1+t}$

$= [\log t - \log(1+t)]_1^2$

$= \log_e 2 - \log_e 3 + \log_e 2 = \log_e \left(\frac{4}{3}\right)$

58. Let $I = \int_0^{\pi/4} \frac{\sec x}{1+2\sin^2 x} dx$

$= \int_0^{\pi/4} \frac{\cos x}{\cos^2 x(1+2\sin^2 x)} dx$

$= \int_0^{\pi/4} \frac{\cos x}{(1-\sin^2 x)(1+2\sin^2 x)} dx$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$\therefore I = \int_0^{1/\sqrt{2}} \frac{1}{(1-t^2)(1+2t^2)} dt$

$= \frac{1}{3}\int_0^{1/\sqrt{2}} \left(\frac{1}{1-t^2} + \frac{2}{1+2t^2}\right) dt$

$= \frac{1}{3}\left[\frac{1}{2.1}\log\left(\frac{1+t}{1-t}\right) + \frac{2}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\,t\right)\right]_0^{\frac{1}{\sqrt{2}}}$

$= \frac{1}{3}\left[\frac{1}{2}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) + \sqrt{2}\tan^{-1} 1\right]$

$= \frac{1}{3}\left[\frac{1}{2}\log(\sqrt{2}+1)^2 + \sqrt{2}.\frac{\pi}{4}\right]$

$= \frac{1}{3}\left[\log(\sqrt{2}+1) + \frac{\pi}{2\sqrt{2}}\right]$

59. $\int_{\frac{1}{e}}^{\tan x} \frac{t}{1+t^2} dt + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)}$

$= \frac{1}{2}\int_{\frac{1}{e}}^{\tan x} \frac{2t}{1+t^2} dt + \int_{\frac{1}{e}}^{\cot x} \left(\frac{1}{t} - \frac{t}{1+t^2}\right) dt$

$= \frac{1}{2}\left[\log(1+t^2)\right]_{1/e}^{\tan x} + \left[\log t - \frac{1}{2}\log(1+t^2)\right]_{1/e}^{\cot x}$

$= \frac{1}{2}\left[\log(\sec^2 x) - \log\left(1+\frac{1}{e^2}\right)\right] + \log(\cot x)$

$- \log\left(\frac{1}{e}\right) - \frac{1}{2}\left[\log(\operatorname{cosec}^2 x) - \log\left(1+\frac{1}{e^2}\right)\right]$

$= -\log\left(\frac{1}{e}\right) = \log e = 1$

60. $\int_1^4 f(x)\,dx = \int_1^2 (4x+3)\,dx + \int_2^4 (3x+5)\,dx$

$= \left[2x^2 + 3x\right]_1^2 + \left[\frac{3x^2}{2} + 5x\right]_2^4 = 37$

61. $\int_{-1}^2 |x|\,dx = \int_{-1}^0 (-x)\,dx + \int_0^2 x\,dx$

$= -\left[\frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^2}{2}\right]_0^2$

$= -\left(0 - \frac{1}{2}\right) + 2$

$= 2 + \frac{1}{2} = \frac{5}{2}$

62. $\int_0^3 |2-x|\,dx = \int_0^2 (2-x)dx + \int_2^3 -(2-x)dx$

$= \left[2x - \frac{x^2}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_2^3$

$= (4-2) - \left[6 - \frac{9}{2} - (4-2)\right]$

$= 2 - \left(4 - \frac{9}{2}\right) = \frac{5}{2}$

63. $\int_{-4}^4 |x+2|\,dx = -\int_{-4}^{-2} (x+2)\,dx + \int_{-2}^4 (x+2)\,dx$

$= \left[\frac{-x^2}{2} - 2x\right]_{-4}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^4 = 20$

64. Since, $\sin x$ is positive in the interval $(0, \pi)$ and negative in the interval $(\pi, 2\pi)$.

$\therefore \int_0^{2\pi} |\sin x| dx = \int_0^{\pi} \sin x \, dx + \int_{\pi}^{2\pi} (-\sin x) dx$

$= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi}$

$= 1 + 1 + 1 + 1 = 4$

65. $\int_0^{2\pi} (\sin x + |\sin x|) dx = \int_0^{\pi} 2\sin x \, dx + \int_{\pi}^{2\pi} 0.dx$

$= 2[-\cos x]_0^{\pi} + 0$

$= -2(\cos\pi - \cos 0)$

$= -2(-1-1) = 4$

66. $\int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}} \, dx = \int_0^{\pi} |\cos x| dx$

$= \int_0^{\pi/2} \cos x \, dx - \int_{\pi/2}^{\pi} \cos x \, dx$

$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$

$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin\pi - \sin\frac{\pi}{2}\right] = 1 + 1 = 2$

67. $\int_0^2 x^2[x]dx = \int_0^1 x^2[x]dx + \int_1^2 x^2[x]dx$

$= \int_0^1 x^2(0)dx + \int_1^2 x^2(1)dx$

$= 0 + \left[\frac{x^3}{3}\right]_1^2 = \frac{7}{3}$

68. Let $I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx$ ....(i)

$\therefore I = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-x\right) - \sin\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)} dx$

....$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\therefore I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\cos x \sin x} dx$ ....(ii)

Adding (i) and (ii), we get

$2I = 0 \Rightarrow I = 0$

69. $\int_0^{\frac{\pi}{2}} \log\tan x \, dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin x}{\cos x}\right) dx$

$= \int_0^{\frac{\pi}{2}} \log\sin x \, dx - \int_0^{\frac{\pi}{2}} \log\cos x \, dx$

$= \int_0^{\frac{\pi}{2}} \log\sin x \, dx - \int_0^{\frac{\pi}{2}} \log\sin x \, dx$

....$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= 0$

70. Let $I = \int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)} dx$ .....(i)

$\therefore I = \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)} dx$ .....(ii)

Adding (i) and (ii), we get

$2I = \int_0^{2a} dx = 2a$

$\Rightarrow I = a$

71. Let $I = \int_0^{\pi/2} \frac{1000^{\sin x}}{1000^{\sin x} + 1000^{\cos x}} dx$ ....(i)

$\therefore I = \int_0^{\pi/2} \frac{1000^{\sin\left(\frac{\pi}{2}-x\right)}}{1000^{\sin\left(\frac{\pi}{2}-x\right)} + 1000^{\cos\left(\frac{\pi}{2}-x\right)}} dx$

....$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\therefore I = \int_0^{\pi/2} \frac{1000^{\cos x}}{1000^{\cos x} + 1000^{\sin x}} dx$ ....(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

72. Let $I = \int_0^{\frac{\pi}{2}} \frac{e^{x^2}}{e^{x^2} + e^{\left(\frac{\pi}{2}-x\right)^2}} dx$ ….(i)

$\therefore I = \int_0^{\frac{\pi}{2}} \frac{e^{\left(\frac{\pi}{2}-x\right)^2}}{e^{\left(\frac{\pi}{2}-x\right)^2} + e^{x^2}} dx$ ….(ii)

....$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

73. Let $I = \int_0^{\pi/2} \frac{\sin^{\frac{3}{2}} x\,dx}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x}$ ….(i)

$= \int_0^{\pi/2} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}{\cos^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) + \sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}\,dx$

$= \int_0^{\pi/2} \frac{\cos^{\frac{3}{2}} x\,dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$ ....(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

Alternate Method:

$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx = \frac{\pi}{4}$

74. Let $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \tan^3 x}$

$= \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^3 x + \cos^3 x}\,dx$ ....(i)

$\therefore$ $I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos^3 x + \sin^3 x}\,dx$ ....(ii)

....$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\pi/2}$

$\therefore$ $2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

75. Let $I = \int_0^{\pi} e^{\cos^2 x} \cos^5 3x\,dx$

$= \int_0^{\pi} e^{\cos^2(\pi - x)} \cos^5 3(\pi - x)\,dx$

….$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$

$\therefore$ $I = -\int_0^{\pi} e^{\cos^2 x} \cos^5 3x\,dx = -I$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

76. Let $I = \int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx$

$= \int_0^{\pi/2} \log\left(\frac{4 + 3\cos x}{4 + 3\sin x}\right) dx$

….$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$

$\therefore$ $I = -\int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx = -I$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

77. Let $I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \tan\theta}$ ….(i)

$= \int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \tan\left(\frac{\pi}{2} - \theta\right)}$

….$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$

$\therefore$ $I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \cot\theta}$ ….(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} \left(\frac{1}{1 + \tan\theta} + \frac{1}{1 + \cot\theta}\right) d\theta$

$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{1 + \tan\theta} + \frac{\tan\theta}{\tan\theta + 1}\right) d\theta$

$= \int_0^{\frac{\pi}{2}} d\theta = [\theta]_0^{\pi/2}$

$\therefore$ $2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

78. Let $I = \int_0^{\pi} x \sin^3 x\,dx$ …..(i)

$= \int_0^{\pi} (\pi - x) \sin^3 x\,dx$ …..(ii)

....$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \sin^3 x\, dx = \frac{\pi}{4}\int_0^{\pi} (3\sin x - \sin 3x)\, dx$$

$$= \frac{\pi}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\pi}$$

$$= \frac{\pi}{4}\left[3 - \frac{1}{3} + 3 - \frac{1}{3}\right]$$

$$= \frac{4\pi}{3}$$

$\therefore$ $I = \frac{2\pi}{3}$

79. Let $I = \int_0^{\frac{\pi}{2}} \log\sin x\, dx$

$$= \int_0^{\frac{\pi}{4}} (\log\sin x + \log\cos x)\, dx$$

$$\ldots\left[\because \int_0^{2a} f(x)\,dx = \int_0^{a}[f(x) + f(2a - x)]\,dx\right]$$

$$= \int_0^{\frac{\pi}{4}} \log\sin x\cos x\, dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \log\sin 2x\, dx - \int_0^{\frac{\pi}{4}} \log 2 dx$$

In 1st integral, put $2x = t \Rightarrow 2dx = dt$

$\therefore$ $I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\sin t dt - \frac{\pi}{4}\log 2$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\sin x\, dx - \frac{\pi}{4}\log 2$$

$$\ldots\left[\because \int_a^{b} f(x)dx = \int_a^{b} f(t)dt\right]$$

$\therefore$ $I = \frac{1}{2}I - \frac{\pi}{4}\log 2$

$\therefore$ $I = \frac{-\pi}{2}\log 2$

80. Let $I = \int_0^{\pi} x\log\sin x\, dx$ ....(i)

$\therefore$ $I = \int_0^{\pi} (\pi - x)\log\sin x\, dx$ ....(ii)

$$\ldots\left[\because \int_0^{a} f(x)dx = \int_0^{a} f(a - x)dx\right]$$

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \log\sin x\, dx = 2\pi\int_0^{\frac{\pi}{2}} \log\sin x\, dx$$

$\therefore$ $2I = 2\pi\left(-\frac{\pi}{2}\log 2\right)$

$$\ldots\left[\because \int_0^{\frac{\pi}{2}} \log\sin x\, dx = -\frac{\pi}{2}\log 2\right]$$

$$\Rightarrow I = \pi\left(\frac{\pi}{2}\log\frac{1}{2}\right) = \frac{\pi^2}{2}\log\frac{1}{2}$$

81. Let $I = \int_0^{\frac{\pi}{4}} \log(1 + \tan\theta)d\theta$

$$= \int_0^{\frac{\pi}{4}} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right]d\theta$$

$$\ldots\left[\because \int_0^{a} f(x)dx = \int_0^{a} f(a - x)dx\right]$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log 2d\theta - \int_0^{\frac{\pi}{4}} \log(1 + \tan\theta)\, d\theta$$

$\therefore$ $2I = \int_0^{\frac{\pi}{4}} \log 2d\theta \Rightarrow I = \frac{\log 2}{2}[\theta]_0^{\pi/4} = \frac{\pi}{8}\log 2$

82. $\int_0^{1} \tan^{-1}\left(\frac{2x - 1}{1 + x - x^2}\right)dx$

$$= \int_0^{1} \tan^{-1}\left(\frac{x + (x - 1)}{1 - x(x - 1)}\right)dx$$

$$= \int_0^{1} (\tan^{-1}x + \tan^{-1}(x - 1))\, dx$$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}(x-1)\,dx$$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}(1-x-1)\,dx$$

$$\ldots\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 \tan^{-1} x\,dx - \int_0^1 \tan^{-1} x\,dx = 0$$

83. Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx$ ....(i)

$\therefore$ $I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x}dx$ ....(ii)

$$\ldots\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{\tan x}{\sec x + \tan x}dx$$

$\therefore$ $I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\sin x}dx$

$$= \frac{\pi}{2}\left[\int_0^{\pi} 1dx - \int_0^{\pi}\frac{dx}{1+\sin x}\right]$$

On solving, we get

$$I = \frac{\pi}{2}(\pi - 2) = \pi\left(\frac{\pi}{2} - 1\right)$$

84. Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\cot x}}$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad \ldots.(i)$$

$\therefore$ $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ ....(ii)

$$\ldots\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

Adding (i) and (ii), we get

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx$$

$\therefore$ $I = \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{12}$

85. Let $I = \int_a^b x f(x)\,dx$

$$= \int_a^b (a+b-x)f(a+b-x)\,dx$$

$\therefore$ $I = \int_a^b (a+b)f(x)dx - \int_a^b x f(x)\,dx$

$$\ldots[\because f(a+b-x) = f(x)(\text{given})]$$

$$\Rightarrow 2I = (a+b)\int_a^b f(x)\,dx \Rightarrow I = \frac{a+b}{2}\int_a^b f(x)\,dx$$

86. Let $I = \int_{0.5}^{1.5} x f(x)\,dx = \int_{0.5}^{1.5} (2-x)f(2-x)dx$

$$\ldots\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$$= \int_{0.5}^{1.5} (2-x)f(x)\,dx$$

$$\ldots[\because f(x) = f(2-x)\ (\text{given})]$$

$\therefore$ $I = 2\int_{0.5}^{1.5} f(x)\,dx - I$

$$\Rightarrow 2I = 2\int_{0.5}^{1.5} f(x)dx \Rightarrow I = \int_{0.5}^{1.5} f(x)dx$$

87. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

Let $I_1 = \int_a^{2a} f(x)\,dx$

Put $x = 2a - t \Rightarrow dx = -\,dt$

$\therefore$ $I_1 = -\int_a^0 f(2a-t)\,dt$

$$= \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

$\therefore$ $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

$$= 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x)$$

88. Let $I = \int_0^{\pi} \sin^2 x\, dx = 2\int_0^{\pi/2} \sin^2 x\, dx$

$\ldots\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$

$\therefore \quad I = 2 \times \frac{1}{2} \times \frac{\pi}{2}$

$= \frac{\pi}{2}$

89. $\int_0^{\pi} |\cos x|\,dx = 2\int_0^{\frac{\pi}{2}} |\cos x|\,dx$

$\ldots\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x)=f(x)\right]$

$= 2[\sin x]_0^{\pi/2} = 2$

90. $\int_0^{2\pi} \cos^{99} x\, dx = 2\int_0^{\pi} \cos^{99} x\, dx$

$\ldots\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$

Let $I_1 = \int_0^{\pi} \cos^{99} x\, dx$

$\Rightarrow I_1 = -\int_0^{\pi} \cos^{99} x\, dx$

$\ldots\left[\because \int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx\right]$

$\Rightarrow I_1 = -I_1 \Rightarrow 2I_1 = 0 \Rightarrow I_1 = 0$

$\therefore \quad \int_0^{2\pi} \cos^{99} x\, dx = 2(0) = 0$

91. $\int_0^{\pi} \log \sin^2 x\, dx = \int_0^{\pi} 2\log \sin x\,dx = 2\int_0^{2\frac{\pi}{2}} \log \sin x\,dx$

$= 2\int_0^{\frac{\pi}{2}} [\log \sin x + \log \sin(\pi - x)]\,dx$

$\ldots\left[\because \int_0^{2a} f(x)\,dx = \int_0^{a} [f(x) + f(2a-x)]\,dx\right]$

$= 4\int_0^{\frac{\pi}{2}} \log \sin x\,dx$

$= 4 \times \left(-\frac{\pi}{2}\log 2\right) = -2\pi \log_e 2 = 2\pi \log_e \left(\frac{1}{2}\right)$

92. Let $I = \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x}\,dx$ $\quad\ldots(i)$

$\therefore \quad I = \int_0^{\pi} \frac{\pi - x}{a^2\cos^2 x + b^2\sin^2 x}\,dx$ $\quad\ldots(ii)$

$\ldots\left[\because \int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx\right]$

Adding (i) and (ii), we get

$2I = \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$\therefore \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$= 2.\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$\ldots\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$

$= \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\,dx$

Put $b\tan x = t \Rightarrow b\sec^2 x\,dx = dt$

$\therefore \quad I = \frac{\pi}{b}\int_0^{\infty} \frac{dt}{a^2 + t^2} = \frac{\pi}{b}.\frac{1}{a}\left[\tan^{-1}\frac{t}{a}\right]_0^{\infty}$

$= \frac{\pi}{ab}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{2ab}$

93. $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_0^{1} f(x)\,dx$

In 1st integral, put $x = -t \Rightarrow dx = -dt$

$\therefore \quad \int_{-1}^{0} f(x)\,dx = -\int_1^{0} f(-t)\,dt$

$= \int_0^{1} f(-t)\,dt$

$= \int_0^{1} f(-x)\,dx$

$\therefore \quad \int_{-1}^{1} f(x)\,dx = \int_0^{1} f(-x)\,dx + \int_0^{1} f(x)\,dx$

$= 0$, if $f(-x) = -f(x)$

94. Since, $\int_{-a}^{a} f(x)dx = 0$, if $f(-x) = -f(x)$

$\therefore$ $\int_{-1}^{1} f(x)dx = 0$

$\Rightarrow \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx = 0$

$\Rightarrow \int_{-1}^{0} f(x)dx = -5$

$\Rightarrow \int_{-1}^{0} f(t)\,dt = -5$

95. Let $f(x) = x|x|$

$\therefore$ $f(-x) = -x|-x| = -x|x| = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ $\int_{-1}^{1} x|x|dx = 0$

96. Since, $|\sin x|$ is an even function.

$\therefore$ $I = 2\int_{0}^{\frac{\pi}{2}} |\sin x|dx = 2\int_{0}^{\frac{\pi}{2}} \sin x\,dx = 2[-\cos x]_0^{\pi/2}$

$= 2(-0 + 1) = 2$

97. Since, $\frac{1}{x+x^3}$ is an odd function.

$\therefore$ $\int_{-a}^{a} \frac{dx}{x+x^3} = 0$

98. Let $f(x) = \sin x\, f(\cos x)$

$\therefore$ $f(-x) = -\sin x\, f(\cos x) = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ $\int_{-a}^{a} f(x)\,dx = 0$

99. Let $f(x) = \frac{\sin x}{1+\cos^2 x} e^{-\cos^2 x}$

$\therefore$ $f(-x) = -\frac{\sin x}{1+\cos^2 x} e^{-\cos^2 x} = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ $\int_{-\pi/2}^{\pi/2} f(x)dx = 0$

100. Let $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

$\therefore$ $f(-x) = \sqrt{1-x+x^2} - \sqrt{1+x+x^2} = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ $\int_{-1}^{1} f(x)\,dx = 0$

101. Let $f(x) = \left(e^{x^3} + e^{-x^3}\right)\left(e^x - e^{-x}\right)$

$\therefore$ $f(-x) = \left(e^{-x^3} + e^{x^3}\right)\left(e^{-x} - e^{x}\right)$

$= -\left(e^{x^3} + e^{-x^3}\right)\left(e^x - e^{-x}\right) = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ $\int_{-1}^{1} \left(e^{x^3} + e^{-x^3}\right)\left(e^x - e^{-x}\right)dx = 0$

102. Let $f(x) = \log\left(\frac{1+x}{1-x}\right)$

$\therefore$ $f(-x) = \log\left(\frac{1+x}{1-x}\right)^{-1} = -\log\left(\frac{1+x}{1-x}\right) = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ $\int_{-1}^{1} \log\left(\frac{1+x}{1-x}\right) dx = 0$

103. Let $f(x) = \cos x \log\left(\frac{1-x}{1+x}\right)$

$\therefore$ $f(-x) = \cos x \log\left(\frac{1-x}{1+x}\right)^{-1}$

$= -\cos x \log\left(\frac{1-x}{1+x}\right) = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ $\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \log\left(\frac{1-x}{1+x}\right)dx = 0$

104. Let $f(\theta) = \log(\sec\theta - \tan\theta)$

$\therefore$ $f(-\theta) = \log(\sec\theta + \tan\theta)$

$= \log\left(\frac{1}{\sec\theta - \tan\theta}\right)$

$= -\log(\sec\theta - \tan\theta) = -f(\theta)$

$\therefore$ $f(\theta)$ is an odd function.

$\therefore$ $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\sec\theta - \tan\theta)d\theta = 0$

105. Let $f(x) = \log\left(\sqrt{1+x^2} + x\right)$

$\therefore$ $f(-x) = \log\left(\sqrt{1+x^2} - x\right)$

$= \log\left(\sqrt{1+x^2} - x\right) \cdot \frac{\left(\sqrt{1+x^2} + x\right)}{\left(\sqrt{1+x^2} + x\right)}$

$$= \log\left(\frac{1+x^2-x^2}{\sqrt{1+x^2}+x}\right)$$

$$= \log 1 - \log\left(\sqrt{1+x^2}+x\right)$$

$$= -\log\left(\sqrt{1+x^2}+x\right) = -f(x)$$

$\therefore$ f(x) is an odd function.

$\therefore$ $\int_{-1}^{1} \log\left(\sqrt{1+x^2}+x\right) dx = 0$

106. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx = 2\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$

....[$\because$ $\sin^2 x$ is an even function]

Since, $\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{(n-1)(n-3)....1}{n(n-2)....2}\cdot\frac{\pi}{2}$, if n is even

$\therefore$ $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx = 2\left(\frac{2-1}{2}\cdot\frac{\pi}{2}\right) = \frac{\pi}{2}$

107. Let I = $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^{-4} x\, dx = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \text{cosec}^4 x\, dx$

$$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \text{cosec}^2 x(1+\cot^2 x)\, dx$$

Put $\cot x = t \Rightarrow \text{cosec}^2 x\, dx = -dt$

$\therefore$ $I = -\int_{-1}^{1} (1+t^2)\, dt$

$$= -2\int_0^1 (1+t^2)\, dt = -2\left[t+\frac{t^3}{3}\right]_0^1$$

$$= -2\left(1+\frac{1}{3}\right) = -\frac{8}{3}$$

108. $\int_{-1}^{1} \frac{1+x^3}{9-x^2} dx = \int_{-1}^{1} \frac{1}{9-x^2} dx + \int_{-1}^{1} \frac{x^3}{9-x^2} dx$

$$= 2\int_0^1 \frac{1}{9-x^2} dx + 0$$

....$\left[\begin{array}{l}\because \frac{1}{9-x^2} \text{ is an even function and} \\ \frac{x^3}{9-x^2} \text{ is an odd function.}\end{array}\right]$

$$= 2\left[\frac{1}{2\times 3}\log\left|\frac{3+x}{3-x}\right|\right]_0^1 = \frac{1}{3}(\log 2 - \log 1) = \frac{1}{3}\log 2$$

109. Let I $= \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$

$$= \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x} dx$$

Since, $\frac{2x}{1+\cos^2 x}$ is an odd function

and $\frac{2x\sin x}{1+\cos^2 x}$ is an even function.

$\therefore$ $I = 0 + 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x} dx$

$\Rightarrow I = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$ .....(i)

$\Rightarrow I = 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x} dx$ .....(ii)

....$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

Adding (i) and (ii), we get

$$2I = 4\int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x} dx \Rightarrow I = 2\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\therefore$ $I = 2\pi\int_1^{-1} \frac{-dt}{1+t^2}$

$$\Rightarrow I = -2\pi\left[\tan^{-1} t\right]_1^{-1} = -2\pi\left(\frac{-\pi}{4}-\frac{\pi}{4}\right) = \pi^2$$

110. Let I $= \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\therefore$ $I = \int_1^0 \frac{-dt}{1+t^2} = \int_0^1 \frac{dt}{1+t^2} = [\tan^{-1} t]_0^1 = \frac{\pi}{4}$

111. Let I $= \int_0^{\pi/2} \sqrt{\cos\theta}\, \sin^3\theta\, d\theta$

Put $t = \cos\theta \Rightarrow dt = -\sin\theta\, d\theta$

$\therefore$ $I = -\int_1^0 t^{\frac{1}{2}}(1-t^2)\, dt = \int_0^1 (t^{\frac{1}{2}} - t^{\frac{5}{2}})\, dt$

$$= \left[\frac{2}{3}t^{\frac{3}{2}} - \frac{2}{7}t^{\frac{7}{2}}\right]_0^1 = \frac{8}{21}$$

112. $\int_0^1 \frac{dx}{x+\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sin\theta+\cos\theta}$

....[Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$]

$= \frac{\pi}{4}$

113. Since, $\sqrt{1+x^2} > x$, for all $x \in (1,2)$

$\Rightarrow \frac{1}{\sqrt{1+x^2}} < \frac{1}{x}$, for all $x \in (1,2)$

$\Rightarrow \int_1^2 \frac{dx}{\sqrt{1+x^2}} < \int_1^2 \frac{dx}{x}$

$\Rightarrow I_1 < I_2$

114. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

When $x = 0$, $t = 1$ and when $x = \frac{\pi}{2}$, $t = 0$

$\therefore$ $I = -\int_1^0 \frac{t}{t^2+3t+2} dt$

$= \int_0^1 \frac{t}{(t+2)(t+1)} dt$ ....$\left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$

$= \int_0^1 \left(\frac{2}{t+2} - \frac{1}{t+1}\right) dt$

$= \left[2\log(t+2) - \log(t+1)\right]_0^1$

$= 2\log 3 - \log 2 - 2\log 2$

$= 2\log 3 - 3\log 2 = \log 9 - \log 8 = \log\left(\frac{9}{8}\right)$

115. Put $x + 1 = t^2 \Rightarrow dx = 2t\, dt$

When $x = 3$, $t = 2$ and when $x = 8$, $t = 3$

$\therefore$ $\int_3^8 \frac{2-3x}{x\sqrt{1+x}} dx = 2\int_2^3 \frac{2-3(t^2-1)}{t^2-1} dt$

$= 2\int_2^3 \left(\frac{2}{t^2-1} - 3\right) dt$

$= 2\left[2 \cdot \frac{1}{2\times 1} \log\left(\frac{t-1}{t+1}\right) - 3t\right]_2^3$

$= 2\left(\log\frac{1}{2} - \log\frac{1}{3} - 3\right)$

$= 2\left(\log\frac{3}{2} - 3\log e\right) = 2\left(\log\frac{3}{2} - \log e^3\right)$

$= 2\log\left(\frac{3}{2e^3}\right)$

Competitive Thinking

1. $\int_0^1 \sqrt{x}\, dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$

2. $\int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx$

$= \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$

3. $\int_0^{\frac{\pi}{2}} \log \sec x\, dx = \int_0^{\frac{\pi}{2}} \log\frac{1}{\cos x} dx$

$= -\int_0^{\frac{\pi}{2}} \log\cos x\, dx$

$= -\frac{\pi}{2}\log\frac{1}{2}$

$= \frac{\pi}{2}\log 2$

4. $\int_0^1 \tan^{-1}\left(\frac{1-x}{1+x}\right) dx = \int_0^1 \tan^{-1} 1\, dx - \int_0^1 \tan^{-1} x\, dx$

$= \left(\tan^{-1} 1\right)[x]_0^1 - p$

$= \frac{\pi}{4} - p$

5. $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$

$= \int_{-\pi/4}^{\pi/4} \frac{dx}{2\cos^2 x} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$

$= \frac{1}{2}\left[\tan x\right]_{-\pi/4}^{\pi/4} = \frac{1}{2}[1-(-1)]$

$= 1$

6. $\int_0^{\frac{\pi}{2}} \frac{\cos 2x}{\cos x + \sin x} dx$

$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x} dx$

$= \int_0^{\frac{\pi}{2}} (\cos x - \sin x)\, dx$

$= \left[\sin x + \cos x\right]_0^{\pi/2} = 0$

7. $\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{(n-1)(n-3)....1}{n(n-2)....2}\cdot\frac{\pi}{2}$, if n is even.

$\therefore \int_0^{\frac{\pi}{2}} \cos^6 x\,dx = \frac{(6-1)(6-3)(6-5)}{6(6-2)(6-4)}\cdot\frac{\pi}{2}$

$= \frac{5\pi}{32}$

8. Let $I = \int_{-1}^{0} \frac{dx}{x^2+2x+2}$

$= \int_{-1}^{0} \frac{dx}{x^2+2x+1+1}$

$= \int_{-1}^{0} \frac{dx}{(x+1)^2+1}$

$= \left[\tan^{-1}(x+1)\right]_{-1}^{0}$

$= \frac{\pi}{4}$

9. $\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx = \left[\frac{x}{2}\sqrt{2-x^2} + \frac{2}{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\sqrt{2}}$

$= \sin^{-1} 1$

$= \frac{\pi}{2}$

10. Let $I = \int_0^1 \frac{dx}{e^x+e^{-x}} = \int_0^1 \frac{e^x}{(e^x)^2+1}\,dx$

Put $e^x = t \Rightarrow e^x\,dx = dt$

$\therefore I = \int_1^e \frac{dt}{1+t^2}$

$= \left[\tan^{-1} t\right]_1^e$

$= \tan^{-1} e - \frac{\pi}{4}$

11. $3a\int_0^1 \left(\frac{ax-1}{a-1}\right)^2 dx = \frac{3a}{(a-1)^2}\int_0^1 (ax-1)^2\,dx$

$= \frac{3a\left[(ax-1)^3\right]_0^1}{3a(a-1)^2}$

$= \frac{1}{(a-1)^2}\left[(a-1)^3+1\right]$

$= (a-1)+(a-1)^{-2}$

12. $\int_0^{\frac{\pi}{4}} \sin(x-[x])\,d(x-[x])$

$= \int_0^{\frac{\pi}{4}} \sin(x-0)\,d(x-0)$

$= \int_0^{\frac{\pi}{4}} \sin x\,dx$

$= \left[-\cos x\right]_0^{\pi/4} = -\cos\frac{\pi}{4}+\cos 0 = 1-\frac{1}{\sqrt{2}}$

13. $L(x) = \int_1^x \frac{1}{t}dt = \left[\log t\right]_1^x = \log x - \log 1 = \log x$

$\therefore$ $L(xy) = \log(xy) = \log x + \log y = L(x) + L(y)$

14. Given, $\int_a^b \{f(x)-3x\}\,dx = a^2-b^2$

$\Rightarrow \int_a^b f(x)\,dx - 3\int_a^b x\,dx = a^2-b^2$

$\Rightarrow \int_a^b f(x)\,dx - \frac{3}{2}(b^2-a^2) = a^2-b^2$

$\Rightarrow \int_a^b f(x)\,dx = \frac{1}{2}(b^2-a^2)$

$\therefore$ $f(x) = x$ ....$\left[\because \int_a^b x\,dx = \frac{1}{2}(b^2-a^2)\right]$

$\therefore f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$

15. $I = \int_0^1 x(1-x)^n\,dx$

$\therefore -I = \int_0^1 -x(1-x)^n\,dx$

$= \int_0^1 (1-x-1)(1-x)^n\,dx$

$= \int_0^1 (1-x)^{n+1}\,dx - \int_0^1 (1-x)^n\,dx$

$= \left[\frac{(1-x)^{n+2}}{-(n+2)}\right]_0^1 - \left[\frac{(1-x)^{n+1}}{-(n+1)}\right]_0^1$

$\therefore -I = \frac{1}{n+2} - \frac{1}{n+1}$

$\therefore I = \frac{1}{n+1} - \frac{1}{n+2}$

16. $\int_0^{\frac{\pi}{4}}(\cos x-\sin x)dx+\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}(\sin x-\cos x)dx$

$+\int_{2\pi}^{\frac{\pi}{4}}(\cos x-\sin x)dx$

$=[\sin x+\cos x]_0^{\pi/4}-[\cos x+\sin x]_{\pi/4}^{5\pi/4}$

$+[\sin x+\cos x]_{2\pi}^{\pi/4}$

$=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right)-\left[-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\right]$

$+\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right)$

$=(\sqrt{2}-1)-(-\sqrt{2}-\sqrt{2})+(\sqrt{2}-1)$

$=4\sqrt{2}-2$

17. $\frac{d}{dx}f(x)=\frac{e^{\sin x}}{x}$

$\Rightarrow f(x)=\int\frac{e^{\sin x}}{x}dx$

Let $I=\int_1^4\frac{3e^{\sin x^3}}{x}dx$

$=\int_1^4\frac{3x^2\ e^{\sin x^3}}{x^3}dx$

Put $x^3=t\Rightarrow 3x^2\ dx=dt$

$\therefore$ $I=\int_1^{64}\frac{e^{\sin t}}{t}dt$

$=[f(t)]_1^{64}$

$=f(64)-f(1)$

$\Rightarrow k=64$

18. Let $I=\int_1^3\frac{\sin 2x}{x}dx$

Put $2x=t\Rightarrow 2dx=dt\Rightarrow dx=\frac{dt}{2}$

$\therefore$ $I=\int_2^6\frac{\sin t}{\frac{t}{2}}\frac{dt}{2}$

$=\int_2^6\frac{\sin t}{t}dt=[F(t)]_2^6$

$=F(6)-F(2)$

19. Put $\frac{1}{x}=t\Rightarrow-\frac{1}{x^2}\ dx=dt$

When $x=\frac{1}{\pi}$, $t=\pi$ and when $x=\frac{2}{\pi}$, $t=\frac{\pi}{2}$

$\therefore$ $\int_{1/\pi}^{2/\pi}\frac{\sin\left(\frac{1}{x}\right)}{x^2}dx=-\int_{\pi}^{\pi/2}\sin t\,dt=[\cos t]_{\pi}^{\pi/2}$

$=0-(-1)=1$

20. Let $I=\int_0^{\frac{\pi}{4}}\frac{\sqrt{\tan x}}{\sin x\cos x}dx$

$=\int_0^{\frac{\pi}{4}}\frac{\tan x}{\sin x\cos x\sqrt{\tan x}}dx$

$=\int_0^{\frac{\pi}{4}}\frac{\sec^2 x}{\sqrt{\tan x}}dx$

Put $\tan x=t\Rightarrow\sec^2 x\ dx=dt$

When $x=0$, $t=0$ and when $x=\frac{\pi}{4}$, $t=1$

$\therefore$ $I=\int_0^1\frac{dt}{\sqrt{t}}=\left[2\sqrt{t}\right]_0^1=2$

21. $I_8+I_6=\int_0^{\frac{\pi}{4}}(\tan^8\theta+\tan^6\theta)\,d\theta$

$=\int_0^{\frac{\pi}{4}}\tan^6\theta\sec^2\theta\,d\theta$

Put $\tan\theta=t\Rightarrow\sec^2\theta\ d\theta=dt$

When $\theta=0$, $t=0$ and when $\theta=\frac{\pi}{4}$, $t=1$

$\therefore$ $I_8+I_6=\int_0^1 t^6\ dt=\left[\frac{t^7}{7}\right]_0^1=\frac{1}{7}$

22. Let $I=\int_1^2[fg(x)]^{-1}f'[g(x)]g'(x)dx$

Put $f[g(x)]=z\Rightarrow f'[g(x)]g'(x)dx=dz$

When $x=1$, $z=f[g(1)]$

and when $x=2$, $z=f[g(2)]$

$\therefore$ $I=\int_{f[g(1)]}^{f[g(2)]}\frac{1}{z}dz=[\log z]_{f[g(1)]}^{f[g(2)]}$

$=\log f[g(2)]-\log f[g(1)]$

$=0$ ….[$\because$ $g(1)=g(2)$ (given)]

23. $\int_0^k \frac{dx}{2+18x^2} = \frac{1}{18}\int_0^k \frac{dx}{x^2+\frac{1}{9}}$

$\Rightarrow \frac{\pi}{24} = \frac{1}{18}\int_0^k \frac{dx}{x^2+\left(\frac{1}{3}\right)^2}$

$= \frac{1}{18}.\frac{1}{\left(\frac{1}{3}\right)}\left[\tan^{-1}\frac{x}{\left(\frac{1}{3}\right)}\right]_0^k$

$= \frac{1}{6}\left[\tan^{-1}3x\right]_0^k$

$\Rightarrow \frac{\pi}{24} = \frac{1}{6}\left(\tan^{-1}3k - 0\right)$

$\Rightarrow \frac{\pi}{4} = \tan^{-1} 3k$

$\Rightarrow \tan\frac{\pi}{4} = 3k$

$\Rightarrow 3k = 1$

$\Rightarrow k = \frac{1}{3}$

24. $\int_{-1}^0 \frac{dx}{x^2+2x+2} = \int_{-1}^0 \frac{dx}{(x+1)^2+1}$

$= \left[\tan^{-1}(x+1)\right]_{-1}^0 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4}$

25. $\int_0^1 \frac{x^2}{1+x^2}dx = \int_0^1\left(1-\frac{1}{1+x^2}\right)dx$

$= \left[x-\tan^{-1}x\right]_0^1$

$= 1 - \frac{\pi}{4}$

26. $\int_0^1 \frac{x^4+1}{x^2+1}dx = \int_0^1 \frac{x^4-1}{x^2+1}dx + 2\int_0^1\frac{dx}{x^2+1}$

$= \int_0^1 (x^2-1)\,dx + 2\int_0^1 \frac{dx}{x^2+1}$

$= \left[\frac{x^3}{3}-x\right]_0^1 + \left[2\tan^{-1}x\right]_0^1$

$= -\frac{2}{3} + \frac{\pi}{2}$

$= \frac{3\pi-4}{6}$

27. $\int_0^3 \frac{3x+1}{x^2+9}dx = \frac{3}{2}\int_0^3 \frac{2x}{x^2+9}dx + \int_0^3\frac{dx}{x^2+9}$

$= \left[\frac{3}{2}\log(x^2+9)+\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$

$= \frac{3}{2}(\log 18 - \log 9) + \frac{1}{3}\left(\frac{\pi}{4}\right)$

$= \frac{3}{2}\log 2 + \frac{\pi}{12} = \log(2\sqrt{2}) + \frac{\pi}{12}$

28. $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$

$= \int_0^1\left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)dx$

$= \left[\frac{x^7}{7}-\frac{2x^6}{3}+x^5-\frac{4}{3}x^3+4x-4\tan^{-1}x\right]_0^1$

$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4}\right)$

$= \frac{22}{7} - \pi$

29. $\int_0^1 \sqrt{\frac{1-x}{1+x}}dx$

$= \int_0^1\sqrt{\frac{1-x}{1+x}.\frac{1-x}{1-x}}dx = \int_0^1 \frac{1-x}{\sqrt{1-x^2}}dx$

$= \int_0^1\frac{dx}{\sqrt{1-x^2}} - \int_0^1\frac{x}{\sqrt{1-x^2}}dx$

$= [\sin^{-1}x]_0^1 + [\sqrt{1-x^2}]_0^1$

$= \frac{\pi}{2} - 1$

30. $\int_{-1}^1 \sqrt{\frac{1-x}{1+x}}dx = \int_{-1}^1\sqrt{\frac{1-x}{1+x}.\frac{1-x}{1-x}}dx$

$= \int_{-1}^1 \frac{1-x}{\sqrt{1-x^2}}dx$

$= \int_{-1}^1\frac{dx}{\sqrt{1-x^2}} - \int_{-1}^1\frac{x}{\sqrt{1-x^2}}dx$

$= \left[\sin^{-1}x\right]_{-1}^1 + \left[\sqrt{1-x^2}\right]_{-1}^1$

$= \sin^{-1}1 - \sin^{-1}(-1) + 0$

$= 2.\frac{\pi}{2} = \pi$

31. Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{a^2 \cdot \sin^2 x + b^2 \cdot \cos^2 x} dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x\left(a^2 \tan^2 x + b^2\right)} dx$

$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{b^2 + a^2 \tan^2 x} dx$

Put $a \tan x = t \Rightarrow a \sec^2 x\, dx = dt$

$\therefore \quad I = \frac{1}{a}\int_0^{\infty} \frac{dt}{b^2 + t^2}$

$= \frac{1}{ab}\left[\tan^{-1}\left(\frac{t}{b}\right)\right]_0^{\infty} = \frac{\pi}{2ab}$

32. Let $I = \int_0^{1/2} \frac{dx}{\left(1 + x^2\right)\sqrt{1 - x^2}}$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\therefore \quad I = \int_0^{\pi/6} \frac{\cos\theta\, d\theta}{\left(1 + \sin^2\theta\right)\cos\theta}$

$= \int_0^{\pi/6} \frac{\sec^2\theta}{\sec^2\theta + \tan^2\theta} d\theta$

$= \int_0^{\pi/6} \frac{\sec^2\theta}{1 + 2\tan^2\theta} d\theta$

$= \int_0^{\pi/6} \frac{\sec^2\theta}{1 + \left(\sqrt{2}\tan\theta\right)^2} d\theta$

$= \frac{1}{\sqrt{2}}\int_0^{\pi/6} \frac{\sqrt{2}\sec^2\theta}{1 + \left(\sqrt{2}\tan\theta\right)^2} d\theta$

$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2}\tan\theta\right)\right]_0^{\pi/6}$

$= \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{\frac{2}{3}}$

33. Put $\tan\frac{x}{2} = t$

$\therefore \quad dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \int_0^{1} \frac{2dt}{3+t^2} = \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_0^1$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

34. Put $\tan\frac{x}{2} = t$

$\therefore \quad dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

$\therefore \quad \int_0^{\pi} \frac{dx}{5+4\cos x} = \int_0^{\infty} \frac{2dt}{9+t^2}$

$= \left[\frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)\right]_0^{\infty}$

$= \frac{2}{3}\left(\tan^{-1}\infty - 0\right)$

$= \frac{2}{3}.\frac{\pi}{2} = \frac{\pi}{3}$

35. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x+\sin x}dx$

$= \int_0^{\frac{\pi}{2}} \frac{\cos^2(x/2) - \sin^2(x/2)}{2\cos^2(x/2) + 2\sin(x/2)\cos(x/2)}dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\tan^2(x/2)}{1+\tan(x/2)}dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[1 - \tan\left(\frac{x}{2}\right)\right]dx$

$= \frac{1}{2}\left[x + 2\log\left|\cos\left(\frac{x}{2}\right)\right|\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} = \frac{\pi}{4} - \frac{1}{2}\log 2$

36. $\int_0^{1} \frac{dx}{x^2 + 2x\cos\alpha + 1} = \int_0^{1} \frac{dx}{(x+\cos\alpha)^2 + 1 - \cos^2\alpha}$

$= \int_0^{1} \frac{dx}{(x+\cos\alpha)^2 + \sin^2\alpha}$

$= \left[\frac{1}{\sin\alpha}\tan^{-1}\left(\frac{x+\cos\alpha}{\sin\alpha}\right)\right]_0^1$

$= \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\cot\frac{\alpha}{2}\right) - \tan^{-1}\left(\cot\alpha\right)\right]$

$= \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right) - \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \alpha\right)\right)\right]$

$= \frac{\alpha}{2}\,(\sin\alpha)^{-1}$

37. Let $I = \int_0^{\pi/4} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$$

$$= \sqrt{2} \int_0^{\pi/4} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}} dx$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\, dx = dt$

$\therefore$ $$I = \sqrt{2} \int_{-1}^{0} \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\, [\sin^{-1} t]_{-1}^{0}$$

$$= \sqrt{2}\left[0 - \left(\frac{-\pi}{2}\right)\right] = \frac{\pi}{\sqrt{2}}$$

38. $$\int_{\log 2}^{a} \frac{e^x}{\sqrt{e^x - 1}} dx = 2$$

Put $e^x - 1 = t \Rightarrow e^x dx = dt$

$\therefore$ $$\int_1^{e^a - 1} \frac{dt}{\sqrt{t}} = 2$$

$\Rightarrow \left[2\sqrt{t}\right]_1^{e^a - 1} = 2$

$\Rightarrow \sqrt{e^a - 1} - 1 = 1$

$\Rightarrow \sqrt{e^a - 1} = 2$

$\Rightarrow e^a - 1 = 4$

$\Rightarrow e^a = 5$

$\Rightarrow a = \log 5$

39. $$\int_{\log 2}^{x} \frac{du}{(e^u - 1)^{1/2}} = \frac{\pi}{6}$$

$$\Rightarrow \int_{\log 2}^{x} \frac{e^u}{e^u (e^u - 1)^{1/2}} du = \frac{\pi}{6}$$

Put $e^u - 1 = t^2 \Rightarrow e^u\, du = 2t\, dt$

When $u = \log 2$, $t = 1$

and when $u = x$, $t = \sqrt{e^x - 1}$

$\therefore$ $$\int_1^{\sqrt{e^x - 1}} \frac{2}{1+t^2} dt = \frac{\pi}{6}$$

$$\Rightarrow 2\left[\tan^{-1} t\right]_1^{\sqrt{e^x - 1}} = \frac{\pi}{6}$$

$$\Rightarrow \tan^{-1}\left(\sqrt{e^x - 1}\right) - \frac{\pi}{4} = \frac{\pi}{12}$$

$$\Rightarrow \sqrt{e^x - 1} = \tan\frac{\pi}{3} \Rightarrow \sqrt{e^x - 1} = \sqrt{3} \Rightarrow e^x = 4$$

40. $$\frac{\pi^2}{\log 3} \int_{\frac{7}{6}}^{\frac{5}{6}} \sec(\pi x)\, dx$$

$$= \frac{\pi^2}{\log 3} \times \frac{1}{\pi} \left[\log|\sec \pi x + \tan \pi x|\right]_{7/6}^{5/6}$$

$$= \frac{\pi}{\log 3}\left[\log\left|\sec\frac{5\pi}{6} + \tan\frac{5\pi}{6}\right| - \log\left|\sec\frac{7\pi}{6} + \tan\frac{7\pi}{6}\right|\right]$$

$$= \frac{\pi}{\log 3}\left[\log\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) - \log\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{\pi}{\log 3}\left[\log\sqrt{3} - \log\left(\frac{1}{\sqrt{3}}\right)\right] = \frac{\pi}{\log 3}(\log 3) = \pi$$

41. Put $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$\therefore$ $$\int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right) dx$$

$$= -\int_{\frac{\pi}{2}}^{0} \sin\left(2\tan^{-1}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\right).\sin\theta\, d\theta$$

$$= -\int_{\frac{\pi}{2}}^{0} \sin\left[2\tan^{-1}\left(\cot\frac{\theta}{2}\right)\right].\sin\theta\, d\theta$$

$$= -\int_{\frac{\pi}{2}}^{0} \sin\left[2\tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right\}\right].\sin\theta\, d\theta$$

$$= -\int_{\frac{\pi}{2}}^{0} \sin\left[2\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right].\sin\theta\, d\theta$$

$$= -\int_{\frac{\pi}{2}}^{0} \sin(\pi - \theta).\sin\theta\, d\theta$$

$$= -\int_{\frac{\pi}{2}}^{0} \sin\theta.\sin\theta\, d\theta$$

$$= -\int_{\frac{\pi}{2}}^{0} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= -\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{0}$$

$$= \frac{\pi}{4}$$

42. $\int_0^{\frac{\pi}{4}} x\sec^2 x\, dx = [x\tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\, dx$

$= \left[\frac{\pi}{4} - 0\right] - [\log|\sec x|]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} - \log\left|\sec\frac{\pi}{4}\right| + \log|\sec 0|$

$= \frac{\pi}{4} - \log\sqrt{2} + \log 1$

$= \frac{\pi}{4} - \log\sqrt{2}$

43. Let I $= \int_0^{\frac{\pi}{2}} e^x \sin x\, dx$

$= [\sin x.e^x]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \cos x.e^x\, dx$

$\therefore$ I $= [e^x \sin x]_0^{\pi/2} - [\cos x.e^x]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \sin x.e^x\, dx$

$\Rightarrow 2I = [e^x(\sin x - \cos x)]_0^{\pi/2}$

$\Rightarrow 2I = e^{\pi/2} + 1$

$\Rightarrow I = \frac{e^{\pi/2}+1}{2}$

44. Let I $= \int_a^b \frac{1}{x}\log x\, dx$

$\Rightarrow I = [\log x \log x]_a^b - \int_a^b \frac{1}{x}\log x dx$

$\Rightarrow 2I = [(\log x)^2]_a^b$

$\Rightarrow I = \frac{1}{2}[(\log b)^2 - (\log a)^2]$

$= \frac{1}{2}[(\log b + \log a)(\log b - \log a)]$

$= \frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)$

45. Let I $= \int_0^1 x\tan^{-1} x\, dx$

$= \tan^{-1} x \int_0^1 x\, dx - \int_0^1 \frac{1}{1+x^2}\cdot\frac{x^2}{2}\, dx$

$= \left[\frac{x^2}{2}\tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2}\, dx$

$= \left(\frac{\pi}{4}\cdot\frac{1}{2} - 0\right) - \frac{1}{2}[x - \tan^{-1} x]_0^1$

$= \frac{\pi}{8} - \frac{1}{2}\left[(1-0) - \left(\frac{\pi}{4} - 0\right)\right]$

$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}$

46. Let I $= \int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}}\, dx$

Put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}\, dx = dt$

When $x = 0$, t = 0 and when $x = \frac{1}{\sqrt{2}}$, t $= \frac{\pi}{4}$

$\therefore$ I $= \int_0^{\pi/4} t.\sec^2 t dt = \frac{\pi}{4} - \frac{1}{2}\log 2$

47. Put $t = \sin^{-1} x \Rightarrow dt = \frac{1}{\sqrt{1-x^2}}\, dx$

$\therefore \int_0^{1/2} \frac{x\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/6} t \sin t\, dt$

$= [-t\cos t + \sin t]_0^{\frac{\pi}{6}}$

$= -\frac{\pi}{6}.\frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{1}{2} - \frac{\sqrt{3}\pi}{12}$

48. Let I $= \int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x}\, dx$

$= \int_{\pi/4}^{3\pi/4} \frac{x\sec x}{\sec x + \tan x}\, dx$

Let $I_1 = \int \frac{\sec x}{\sec x + \tan x}\, dx$

Put $\frac{1}{\sec x + \tan x} = t$

$\Rightarrow -\frac{(\sec x\tan x + \sec^2 x)}{(\sec x + \tan x)^2}\, dx = dt$

$\therefore I_1 = -\int \frac{-\sec x(\sec x + \tan x)}{(\sec x + \tan x)^2}\, dx$

$= -\int dt$

$= -t + c$

$= \frac{-1}{\sec x + \tan x} + c$

$$\therefore \quad I = \left[\frac{-x}{\sec x + \tan x}\right]_{\pi/4}^{3\pi/4} - \int_{\pi/4}^{3\pi/4} \frac{-1}{\sec x + \tan x}\,dx$$

$$\ldots\left[\because \int_a^b (uv)\,dx = \left[u\int v dx\right]_a^b - \int_a^b \left[\frac{du}{dx}\int v\,dx\right]dx\right]$$

$$= \left(\frac{\frac{-3\pi}{4}}{-\sqrt{2}-1}\right) - \left(\frac{\frac{-\pi}{4}}{\sqrt{2}+1}\right) + \int_{\pi/4}^{3\pi/4} \frac{\cos x}{1+\sin x}\,dx$$

$$= \frac{\pi}{1+\sqrt{2}} + \left[\log\left|1+\sin x\right|\right]_{\pi/4}^{3\pi/4}$$

$$= \frac{\pi}{1+\sqrt{2}} + \log\left|1+\frac{1}{\sqrt{2}}\right| - \log\left|1+\frac{1}{\sqrt{2}}\right|$$

$$= \frac{\pi}{1+\sqrt{2}}$$

$$= \pi\left(\sqrt{2}-1\right)$$

49. $$F(t) = \int_0^t f(t-y)g(y)\,dy$$

$$= \int_0^t e^{t-y} y\,dy = e^t \int_0^t e^{-y}\, y dy$$

$$= -e^t[ye^{-y} + e^{-y}]_0^t$$

$$= -e^t(te^{-t} + e^{-t} - 0 - 1)$$

$$= e^t - (1+t)$$

50. Let $I = \int_0^{2\pi} e^{\frac{x}{2}}.\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx$

Put $\frac{x}{2} = t$

$\Rightarrow dx = 2dt$

$$\therefore \quad I = 2\int_0^{\pi} e^t \sin\left(t+\frac{\pi}{4}\right)dt$$

$$= 2\left[\frac{e^t}{\sqrt{1+1}}\sin\left(t+\frac{\pi}{4}-\tan^{-1}\frac{1}{1}\right)\right]_0^{\pi}$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx - \tan^{-1}\frac{b}{a}\right)+c\right]$$

$$= \frac{2}{\sqrt{2}}\left[e^t \sin t\right]_0^{\pi}$$

$$= \frac{2}{\sqrt{2}}[0] = 0$$

51. $$I_{10} = \int_0^{\frac{\pi}{2}} x^{10}\sin x\,dx$$

$$= \left[-x^{10}\cos x\right]_0^{\pi/2} - 10\int_0^{\frac{\pi}{2}} x^9(-\cos x)dx$$

$$= \left[-\left(\frac{\pi}{2}\right)^{10}\cos\frac{\pi}{2} + 0\right]$$

$$+10\left[\left[x^9 \sin x\right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} 9x^8(\sin x)dx\right]$$

$$= 10\left[x^9\sin x\right]_0^{\pi/2} - 90\int_0^{\frac{\pi}{2}} x^8\sin x\,dx$$

$$\therefore \quad I_{10} = 10\left(\frac{\pi}{2}\right)^9 \sin\frac{\pi}{2} - 90I_8$$

$$\Rightarrow I_{10} + 90I_8 = 10\left(\frac{\pi}{2}\right)^9$$

52. $$\int_0^1 x\log\left(1+\frac{x}{2}\right)dx$$

$$= \left[\log\left(1+\frac{x}{2}\right).\frac{x^2}{2}\right]_0^1 - \int_0^1\left[\frac{1}{1+\frac{x}{2}}.\frac{1}{2}.\frac{x^2}{2}\right]dx$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\int_0^1 \frac{x^2}{x+2}\,dx$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\int_0^1\left[x - \frac{2x}{x+2}\right]dx$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \int_0^1 \frac{x}{x+2}\,dx$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{4} + \int_0^1\left[1 - \frac{2}{x+2}\right]dx$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{4} + \left[x - 2\log(x+2)\right]_0^1$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{4} + 1 - 2\log 3 + 2\log 2$$

$$= \frac{3}{4} + \frac{3}{2}\log\frac{2}{3}$$

$$\therefore \quad a = \frac{3}{4},\ b = \frac{3}{2}$$

53. $I(m, n) = \int_0^1 t^m (1-t)^n \, dt$

$\therefore$ $I(m+1, n-1) = \int_0^1 t^{m+1}(1-t)^{n-1} \, dt$

$\Rightarrow I(m+1, n-1)$

$= \left[-\frac{t^{m+1}(1-t)^n}{n} \right]_0^1 + \frac{m+1}{n} \int_0^1 t^m (1-t)^n \, dt$

$\Rightarrow I(m+1, n-1) = 0 + \frac{m+1}{n} I(m, n)$

$\Rightarrow I(m, n) = \frac{n}{m+1} I(m+1, n-1)$

54. Let $I_1 = \int_0^1 (1-x^{50})^{100} \, dx$ and $I_2 = \int_0^1 (1-x^{50})^{101} \, dx$

Now, $I_2 = \int_0^1 (1-x^{50})^{101} .1 dx$

$= \left[(1-x^{50})^{101} \cdot x\right]_0^1 + 5050 \int_0^1 (1-x^{50})^{100} .x^{49} .x \, dx$

$= -5050 \int_0^1 \left(1-x^{50}\right)^{100} \left\{\left(1-x^{50}\right)-1\right\} dx$

$= -5050 \int_0^1 (1-x^{50})^{101} \, dx + 5050 \int_0^1 (1-x^{50})^{100} \, dx$

$\therefore$ $I_2 = -5050 \, I_2 + 5050 \, I_1$

$\Rightarrow \frac{5050 I_1}{I_2} = 5051$

55. $\int_5^{10} \frac{1}{(x-1)(x-2)} dx = \int_5^{10} \left(\frac{1}{x-2} - \frac{1}{x-1} \right) dx$

$= \left[\log(x-2) - \log(x-1)\right]_5^{10}$

$= \log 8 - \log 9 - (\log 3 - \log 4)$

$= \log \frac{8}{9} - \log \frac{3}{4} = \log\left(\frac{8}{9} \times \frac{4}{3}\right)$

$= \log\left(\frac{32}{27}\right)$

56. $\int_1^3 \frac{dx}{x(1+x^2)} = \int_1^3 \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx$

$= \int_1^3 \frac{1}{x} dx - \frac{1}{2} \int_1^3 \frac{2x}{1+x^2} dx$

$= \left[\log x\right]_1^3 - \frac{1}{2}\left[\log\left(1+x^2\right)\right]_1^3$

$= \log 3 - \log 1 - \frac{1}{2} (\log 10 - \log 2)$

$= \log 3 - \frac{1}{2} \log 5$

$= \frac{1}{2} \log 3^2 - \frac{1}{2} \log 5$

$= \frac{1}{2} (\log 9 - \log 5) = \frac{1}{2} \log\left(\frac{9}{5}\right)$

57. $\int_2^3 \frac{x+1}{x^2(x-1)} dx = \int_2^3 \left(-\frac{1}{x^2} - \frac{2}{x} + \frac{2}{x-1} \right) dx$

$= \left[\frac{1}{x}\right]_2^3 - 2[\log x]_2^3 + 2[\log(x-1)]_2^3$

$= \frac{1}{3} - \frac{1}{2} - 2\log\frac{3}{2} + 2\log 2$

$= \log\frac{16}{9} - \frac{1}{6}$

58. Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

$= \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\left[1 - (\sin x - \cos x)^2\right]} dx$

Put $\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x) dx = dt$

When $x = 0$, $t = -1$ and when $x = \frac{\pi}{4}$, $t = 0$

$\therefore$ $I = \int_{-1}^0 \frac{dt}{9 + 16(1-t^2)}$

$= \int_{-1}^0 \frac{1}{25 - 16t^2} \, dt$

$= \int_{-1}^0 \frac{1}{(5-4t)(5+4t)} \, dt$

$= \int_{-1}^0 \left[\frac{\frac{1}{10}}{5-4t} + \frac{\frac{1}{10}}{5+4t} \right] dt$

$= \frac{1}{10} \left[\frac{-1}{4} \log(5-4t) + \frac{1}{4} \log(5+4t) \right]_{-1}^0$

$= \frac{1}{40} (\log 9 - \log 1)$

$= \frac{1}{20} \log 3$

59. $\int_0^2 \frac{\log(x^2+2)}{(x+2)^2}dx$

$= -\left[\frac{\log(x^2+2)}{x+2}\right]_0^2 + \int_0^2 \frac{2x}{(x^2+2)(x+2)}dx$

$= -\frac{1}{4}\log 6 + \frac{1}{2}\log 2 + \int_0^2 \left\{\frac{-2}{3(x+2)} + \frac{\frac{2}{3}x+\frac{2}{3}}{x^2+2}\right\}dx$

$= -\frac{1}{4}\log 3 - \frac{1}{4}\log 2 + \frac{1}{2}\log 2$

$+\left[-\frac{2}{3}\log(x+2)+\frac{1}{3}\log(x^2+2)+\frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^2$

$= \frac{1}{4}\log 2 - \frac{1}{4}\log 3$

$+ \left(-\frac{2}{3}\log 2+\frac{1}{3}\log 3+\frac{\sqrt{2}}{3}\tan^{-1}\sqrt{2}\right)$

$= \frac{\sqrt{2}}{3}\tan^{-1}\sqrt{2} - \frac{5}{12}\log 2 + \frac{1}{12}\log 3$

60. Put $\sin^{-1}\left(\frac{x}{2}\right) = t \Rightarrow x = 2\sin t \Rightarrow dx = 2\cos t dt$

$\therefore \quad \int_0^1 \frac{\sin^{-1}\left(\frac{x}{2}\right)}{x}dx = \int_0^{\frac{\pi}{6}} \frac{t}{(2\sin t)}(2\cos t\, dt)$

$= \int_0^{\frac{\pi}{6}} \frac{t}{\tan t}dt = \int_0^{\frac{\pi}{6}} \frac{x}{\tan x}dx$

$\ldots.\left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$

61. $I_1 = \int_e^{e^2} \frac{dx}{\log x}$

Put $\log x = t$

$\Rightarrow dx = x\, dt = e^t\, dt$

When $x = e$, $t = 1$ and when $x = e^2$, $t = 2$

$\therefore \quad I_1 = \int_1^2 \frac{e^t}{t}dt$

$= \int_1^2 \frac{e^x}{x}dx \quad \ldots.\left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$

$\therefore \quad I_1 = I_2$

62. $\int_0^3 (3ax^2+2bx+c)\,dx = \int_1^3 (3ax^2+2bx+c)dx$

$\Rightarrow \int_0^1 (3ax^2+2bx+c)dx + \int_1^3 (3ax^2+2bx+c)dx$

$= \int_1^3 (3ax^2+2bx+c)dx$

$\Rightarrow \int_0^1 (3ax^2+2bx+c)dx = 0$

$\Rightarrow \left[\frac{3ax^3}{3}+\frac{2bx^2}{2}+cx\right]_0^1 = 0 \Rightarrow a+b+c=0$

63. $\int_2^4 (3-f(x))\,dx = 7$

$\Rightarrow \int_2^4 3\,dx - \int_2^4 f(x)\,dx = 7$

$\Rightarrow 3[x]_2^4 - 7 = \int_2^4 f(x)\,dx$

$\Rightarrow \int_2^4 f(x)dx = 3\,(4-2) - 7$

$= -1$

$\int_{-1}^4 f(x)\,dx = \int_{-1}^2 f(x)\,dx + \int_2^4 f(x)\,dx$

$\ldots\left[\begin{array}{l}\because \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \\ \text{if } a < c < b\end{array}\right]$

$\Rightarrow 4 = \int_{-1}^2 f(x)\,dx - 1$

$\Rightarrow \int_{-1}^2 f(x)\,dx = 5$

64. $g(x+\pi) = \int_0^{x+\pi} \cos^4 t\, dt$

$= \int_0^{\pi} \cos^4 t\, dt + \int_{\pi}^{x+\pi} \cos^4 t\, dt$

In 2[nd] integral, put $t = u + \pi \Rightarrow dt = du$

$\therefore \quad \int_{\pi}^{x+\pi} \cos^4 t\, dt = \int_0^x \cos^4(\pi+u)\,du$

$= \int_0^x \cos^4 u\, du = g(x)$

$\therefore \quad g(x+\pi) = g(x) + g(\pi)$

65. Since, $|x-1| = -(x-1)$, if $x-1<0$ i.e., $x<1$

$= x-1$, if $x-1 \geq 0$ i.e., $x \geq 1$

$\therefore \int_0^2 f(x)dx = \int_0^2 |x-1|dx$

$= \int_0^1 (1-x)dx + \int_1^2 (x-1)dx$

$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2$

$= \left(1 - \frac{1}{2}\right) + (2-2) - \left(\frac{1}{2} - 1\right) = 1$

66. $\int_{-5}^{5} |x+2|\ dx = -\int_{-5}^{-2} (x+2)\,dx + \int_{-2}^{5} (x+2)\,dx$

$= \left[\frac{-x^2}{2} - 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$

$= 29$

67. $f(x) = \int_{-1}^{x} |t|\,dt$

$= \int_{-1}^{0} (-t)\,dt + \int_0^x t\,dt$

$= \left[\frac{-t^2}{2}\right]_{-1}^{0} + \left[\frac{t^2}{2}\right]_0^x = \frac{1}{2} + \frac{x^2}{2} = \frac{1}{2}(1+x^2)$

68. $\int_0^1 |3x^2-1|\,dx = \int_0^{\frac{1}{\sqrt{3}}} (1-3x^2)dx + \int_{\frac{1}{\sqrt{3}}}^{1} (3x^2-1)dx$

$= [x - x^3]_0^{1/\sqrt{3}} + [x^3 - x]_{1/\sqrt{3}}^{1}$

$= \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} - \frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{4}{3\sqrt{3}}$

69. $\int_{-2}^{2} \left|1 - x^2\right| dx$

$= \int_{-2}^{-1} \left|1-x^2\right| dx + \int_{-1}^{1} \left|1-x^2\right| dx + \int_1^2 \left|1-x^2\right| dx$

$= -\int_{-2}^{-1} (1-x^2)\,dx + \int_{-1}^{1} (1-x^2)\,dx - \int_1^2 (1-x^2)\,dx$

$= -\left[x - \frac{x^3}{3}\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} - \left[x - \frac{x^3}{3}\right]_1^2$

$= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$

70. Since, $\left|x - \frac{1}{2}\right| = -\left(x - \frac{1}{2}\right)$, if $x < \frac{1}{2}$

$= x - \frac{1}{2}$, if $x \geq \frac{1}{2}$

$\therefore \int_0^1 x\left|x - \frac{1}{2}\right| dx$

$= -\int_0^{\frac{1}{2}} x\left(x - \frac{1}{2}\right)dx + \int_{\frac{1}{2}}^{1} x\left(x - \frac{1}{2}\right)dx$

$= \int_0^{\frac{1}{2}} \left(\frac{1}{2}x - x^2\right)dx + \int_{\frac{1}{2}}^{1} \left(x^2 - \frac{1}{2}x\right)dx$

$= \left[\frac{x^2}{4} - \frac{x^3}{3}\right]_0^{1/2} + \left[\frac{x^3}{3} - \frac{x^2}{4}\right]_{1/2}^{1}$

$= \left(\frac{1}{16} - \frac{1}{24}\right) + \left(\frac{1}{3} - \frac{1}{4} + \frac{1}{16} - \frac{1}{24}\right) = \frac{1}{8}$

71. Let $I = \int_0^{100\pi} |\cos x|\ dx$

$= 200 \int_0^{\frac{\pi}{2}} |\cos x|\ dx$

$\ldots\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)dx, \text{ if } f(2a - x) = f(x)\right]$

Since $\cos x$ is positive in the interval $\left(0, \frac{\pi}{2}\right)$

$\therefore I = 200 \int_0^{\frac{\pi}{2}} \cos x\,dx$

$= 200\ [\sin x]_0^{\frac{\pi}{2}}$

$= 200$

72. $\int_0^{\pi/2} |\sin x - \cos x|\,dx$

$= -\int_0^{\pi/4} (\sin x - \cos x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$

$= -[-\cos x - \sin x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$

$= 2(\sqrt{2} - 1)$

73. $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$

$= \int_0^{\pi}\left|2\sin\frac{x}{2}-1\right|dx$

$= \int_0^{\frac{\pi}{3}}\left|2\sin\frac{x}{2}-1\right|dx + \int_{\frac{\pi}{3}}^{\pi}\left|2\sin\frac{x}{2}-1\right|dx$

$= \int_0^{\frac{\pi}{3}}\left(1-2\sin\frac{x}{2}\right)dx + \int_{\frac{\pi}{3}}^{\pi}\left(2\sin\frac{x}{2}-1\right)dx$

$= \left[x+4\cos\frac{x}{2}\right]_0^{\frac{\pi}{3}} + \left[-4\cos\frac{x}{2}-x\right]_{\frac{\pi}{3}}^{\pi}$

$= \frac{\pi}{3}+4\left(\frac{\sqrt{3}}{2}-1\right)+\left[-4\left(0-\frac{\sqrt{3}}{2}\right)-\left(\pi-\frac{\pi}{3}\right)\right]$

$= 4\sqrt{3}-4-\frac{\pi}{3}$

74. Since, $|\log x| = -\log x$, if $\frac{1}{e} < x < 1$

$= \log x$, if $1 < x < e$

$\therefore$ $I = \int_{1/e}^{1}\frac{(-\log x)}{x^2}dx + \int_1^{e}\frac{\log x}{x^2}dx$

$= -\left[-\frac{\log x}{x}-\frac{1}{x}\right]_{1/e}^{1} + \left[-\frac{\log x}{x}-\frac{1}{x}\right]_1^{e}$

$= \left[0+1-\left(\frac{\log\frac{1}{e}}{\frac{1}{e}}+\frac{1}{\frac{1}{e}}\right)\right]-\left[\frac{\log e}{e}+\frac{1}{e}-(0+1)\right]$

$= 1-(-e+e)-\left(\frac{2}{e}-1\right)$

$= 2-\frac{2}{e} = 2\left(1-\frac{1}{e}\right)$

75. Let $I = \int_0^3 [x]\,dx$

$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx$

$= [x]_1^2 + 2[x]_2^3$

$= (2-1)+2(3-2)$

$= 3$

76. $\int_{-1}^{1}(x-[x])\,dx = \int_{-1}^{0}(x-[x])dx + \int_0^1 (x-[x])dx$

$= \int_{-1}^{0}(x+1)dx + \int_0^1 (x-0)dx$

$= \left[\frac{(x+1)^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1$

$= \frac{1}{2}+\frac{1}{2} = 1$

77. $\int_0^{1.5}[x^2]\,dx = \int_0^1 [x^2]\,dx + \int_1^{\sqrt{2}}[x^2]\,dx + \int_{\sqrt{2}}^{1.5}[x^2]\,dx$

$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{1.5} 2\,dx$

$= \sqrt{2}-1+3-2\sqrt{2} = 2-\sqrt{2}$

78. $\int_0^9 [\sqrt{x}+2]\,dx$

$= \int_0^1 [\sqrt{x}+2]\,dx + \int_1^4 [\sqrt{x}+2]\,dx + \int_4^9 [\sqrt{x}+2]\,dx$

$= \int_0^1 2\,dx + \int_1^4 3\,dx + \int_4^9 4\,dx$

$= 2+(12-3)+(36-16) = 2+9+20 = 31$

79. $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}[2\sin x]dx = \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}}[2\sin x]dx + \int_{\frac{5\pi}{6}}^{\pi}[2\sin x]dx$

$+ \int_{\pi}^{\frac{7\pi}{6}}[2\sin x]dx + \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}}[2\sin x]dx$

$= \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} 1.dx + \int_{\frac{5\pi}{6}}^{\pi} 0\,dx + \int_{\pi}^{\frac{7\pi}{6}}(-1)\,dx + \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}}(-2)\,dx$

$= \left(\frac{5\pi}{6}-\frac{\pi}{2}\right)-\left(\frac{7\pi}{6}-\pi\right)-2\left(\frac{3\pi}{2}-\frac{7\pi}{6}\right) = -\frac{\pi}{2}$

80. Let $I = \int_0^{\frac{\pi}{2}}(x-[\cos x])\,dx$

$= \int_0^{\frac{\pi}{2}} x\,dx - \int_0^{\frac{\pi}{2}}[\cos x]dx$

$= \left[\frac{x^2}{2}\right]_0^{\pi/2} - 0 = \frac{\pi^2}{8}$

81. $\int_1^a [x]f'(x)\,dx$

$= \int_1^2 1.f'(x)\,dx + \int_2^3 2.f'(x)\,dx + \ldots + \int_{[a]}^a [a]f'(x)\,dx$

$= [f(2) - f(1)] + 2[f(3) - f(2)] + \ldots + [a][f(a) - f([a])]$

$= [a]\ f(a) - \{f(1) + f(2) + \ldots + f([a])\}$

82. $\int_0^2 (|x-2| + [x])\,dx = \int_0^2 |x-2|\,dx + \int_0^2 [x]\,dx$

$= -\int_0^2 (x-2)\,dx + \int_0^1 [x]\,dx + \int_1^2 [x]\,dx$

$= \int_0^2 (2-x)\,dx + \int_0^1 0\cdot dx + \int_1^2 1\cdot dx$

$= \left[2x - \frac{x^2}{2}\right]_0^2 + [x]_1^2$

$= (4-2) + (2-1) = 3$

83. $\int_{-2}^2 |[x]|\,dx$

$= \int_{-2}^{-1} |[x]|\,dx + \int_{-1}^0 |[x]|\,dx + \int_0^1 |[x]|\,dx + \int_1^2 |[x]|\,dx$

$= \int_{-2}^{-1} |-2|\,dx + \int_{-1}^0 |-1|\,dx + \int_0^1 |0|\,dx + \int_1^2 |1|\,dx$

$= 2\int_{-2}^{-1} dx + \int_{-1}^0 dx + \int_1^2 dx$

$= 2[x]_{-2}^{-1} + [x]_{-1}^0 + [x]_1^2$

$= 2(-1+2) + (0+1) + (2-1)$

$= 2 + 1 + 1 = 4$

84. $\int_1^5 [|x-3|]\,dx$

$= \int_1^3 [-(x-3)]\,dx + \int_3^5 [(x-3)]\,dx$

$= \int_1^2 [-(x-3)]\,dx + \int_2^3 [-(x-3)]\,dx + \int_3^4 [x-3]\,dx + \int_4^5 [x-3]\,dx$

$= \int_1^2 1.dx + \int_2^3 0.dx + \int_3^4 0.dx + \int_4^5 1.dx$

$= [x]_1^2 + [x]_4^5 = (2-1) + (5-4) = 2$

85. Let $I = \int_0^{11} \frac{(11-x)^2}{x^2 + (11-x)^2}\,dx$ ….(i)

$= \int_0^{11} \frac{x^2}{(11-x)^2 + x^2}\,dx$ ….(ii)

…. $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

Adding (i) and (ii), we get

$2I = \int_0^{11} dx$

$\Rightarrow 2I = [x]_0^{11}$

$\Rightarrow I = \frac{11}{2}$

86. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ …(i)

$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\,dx$

… $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$\therefore$ $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx$ …(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}$

87. Since, $\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx = \frac{\pi}{4}$

$\therefore$ $\int_0^{\frac{\pi}{2}} \frac{\sin^{1000} x}{\sin^{1000} x + \cos^{1000} x}\,dx = \frac{\pi}{4}$

88. Let $I = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\cot^7 x + \tan^7 x}\, dx$ …(i)

$= \int_0^{\frac{\pi}{2}} \frac{\tan^7\left(\frac{\pi}{2} - x\right)}{\cot^7\left(\frac{\pi}{2} - x\right) + \tan^7\left(\frac{\pi}{2} - x\right)}\, dx$

…$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x) dx\right]$

$\therefore$ $I = \int_0^{\frac{\pi}{2}} \frac{\cot^7 x}{\tan^7 x + \cot^7 x}\, dx$ …(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} dx$

$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}} \Rightarrow I = \frac{\pi}{4}$

89. Let $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$ .....(i)

$= \int_0^{\pi/2} \frac{\sqrt{\cot\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cot\left(\frac{\pi}{2} - x\right)} + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}} dx$

....$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx\right]$

$= \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$ .....(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

90. Let $I = \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt[n]{\sec x}}{\sqrt[n]{\sec x} + \sqrt[n]{\operatorname{cosec} x}}\right) dx$ ….(i)

$\therefore$ $I = \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt[n]{\operatorname{cosec} x}}{\sqrt[n]{\operatorname{cosec} x} + \sqrt[n]{\sec x}}\right) dx$ ….(ii)

….$\left[\because \int_0^a f(x) dx = \int_0^a (a-x) dx\right]$

Adding (i) and (ii), we get

$2\, I = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

91. Let $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sqrt{\tan x}}$ ….(i)

$= \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sqrt{\cot x}}$ ….(ii)

….$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx\right]$

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} \left(\frac{1}{1 + \sqrt{\tan x}} + \frac{1}{1 + \sqrt{\cot x}}\right) dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

92. Let $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)^{\sqrt{2018}}}$ …(i)

$\therefore$ $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\cot x)^{\sqrt{2018}}}$ …(ii)

…$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} \left[\frac{1}{1 + (\tan x)^{\sqrt{2018}}} + \frac{1}{1 + (\cot x)^{\sqrt{2018}}}\right] dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{1 + (\tan x)^{\sqrt{2018}}} + \frac{1}{1 + \left(\frac{1}{\tan x}\right)^{\sqrt{2018}}}\, dx$

$= \int_0^{\frac{\pi}{2}} dx$

$\Rightarrow I = \frac{\pi}{4}$

93. Let $I = \int_0^{\frac{\pi}{2}} \frac{2^{\sin x}}{2^{\sin x} + 2^{\cos x}}\, dx$ …(i)

$\therefore$ $I = \int_0^{\frac{\pi}{2}} \frac{2^{\cos x}}{2^{\sin x} + 2^{\cos x}}\, dx$ …(ii)

…$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{2^{\sin x}+2^{\cos x}}{2^{\sin x}+2^{\cos x}}\,dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$\therefore \quad I = \frac{\pi}{4}$

94. Let $I = \int_0^{\frac{\pi}{2}} \frac{2008^{\sin x}}{2008^{\sin x}+2008^{\cos x}}\,dx \quad$ ….(i)

$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{2008^{\cos x}}{2008^{\cos x}+2008^{\sin x}}\,dx \quad$ ….(ii)

$$\text{....}\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

95. Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin x+\cos x}\,dx \quad$ …(i)

$I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos x+\sin x}\,dx \quad$ …(ii)

$$\text{…}\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x+\cos^3 x}{\sin x+\cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\sin^2 x - \sin x\cos x + \cos^2 x\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(1-\sin x\cos x\right)dx$$

$$= \int_0^{\frac{\pi}{2}} 1\,dx - \int_0^{\frac{\pi}{2}} \sin x\cos x\,dx$$

$$= [x]_0^{\pi/2} - \left[\frac{\sin^2 x}{2}\right]_0^{\pi/2} = \frac{\pi}{2}-\frac{1}{2}$$

$\therefore \quad 2I = \frac{\pi-1}{2} \Rightarrow I = \frac{\pi-1}{4}$

96. Let $I = \int_0^{\frac{\pi}{2}} \log(\cot x)\,dx \quad$ ….(i)

$= \int_0^{\frac{\pi}{2}} \log(\tan x)\,dx \quad$ ….(ii)

$$\text{….}\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \log(\cot x\tan x)\,dx$$

$\therefore \quad 2I = \int_0^{\frac{\pi}{2}} \log 1\,dx$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

97. $\int_0^{\frac{\pi}{2}} \log(\operatorname{cosec} x)\,dx = \int_0^{\frac{\pi}{2}} \log(\sec x)\,dx$

$$\text{….}\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{\cos x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} [\log 1 - \log(\cos x)]\,dx$$

$$= -\int_0^{\frac{\pi}{2}} \log(\cos x)\,dx$$

$$= \frac{\pi}{2}\log 2$$

98. Let $I = \int_0^{\pi/2} \sin 2x \log\tan x\,dx$

$$= \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2}-x\right)\log\tan\left(\frac{\pi}{2}-x\right)dx$$

$$\text{….}\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \sin 2x \log\cot x\,dx$$

$$= -\int_0^{\pi/2} \sin 2x \log\tan x\,dx$$

$\therefore \quad I = -I \Rightarrow 2I = 0 \Rightarrow I = 0$

99. Let $I = \int_0^{\frac{\alpha}{3}} \frac{f(x)}{f(x)+f\left(\frac{\alpha-3x}{3}\right)} dx$

$\Rightarrow I = \int_0^{\frac{\alpha}{3}} \frac{f(x)}{f(x)+f\left(\frac{\alpha}{3}-x\right)} dx$ ...(i)

$\Rightarrow I = \int_0^{\frac{\alpha}{3}} \frac{f\left(\frac{\alpha}{3}-x\right)}{f\left(\frac{\alpha}{3}-x\right)+f(x)} dx$ ...(ii)

$\ldots\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\alpha}{3}} dx = [x]_0^{\alpha/3} = \frac{\alpha}{3}$

$\Rightarrow I = \frac{\alpha}{6}$

100. $I = \int_0^a \frac{dx}{1+f(x)}$ ...(i)

$= \int_0^a \frac{dx}{1+f(a-x)}$ $\ldots\left[\int_0^a f(x) = \int_0^a f(a-x)\, dx\right]$

$= \int_0^a \frac{dx}{1+\frac{1}{f(x)}}$ $\ldots[\because f(x)f(a-x)=1]$

$\therefore$ $I = \int_0^a \frac{f(x)}{1+f(x)} dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_0^a dx = [x]_0^a$

$\Rightarrow I = \frac{a}{2}$

101. Let $I = \int_0^{\pi} \frac{x}{1+\sin x} dx$ .... (i)

$\therefore$ $I = \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx$ ....(ii)

$\ldots\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

Adding (i) and (ii), we get

$2I = \int_0^{\pi} \frac{\pi}{1+\sin x} dx$

$= \pi\int_0^{\pi} \frac{1-\sin x}{(1+\sin x)(1-\sin x)} dx$

$= \pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx = \pi\int_0^{\pi}(\sec^2 x - \sec x\tan x)dx$

$= \pi[\tan x - \sec x]_0^{\pi}$

$\therefore$ $2I = \pi[0-(-1)-(0-1)] = 2\pi$

$\Rightarrow I = \pi$

102. Let $I = \int_0^{\pi} \frac{x\,dx}{4\cos^2 x + 9\sin^2 x}$ ...(i)

$\therefore$ $I = \int_0^{\pi} \frac{(\pi - x)dx}{4\cos^2(\pi-x) + 9\sin^2(\pi-x)}$

$\ldots\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\therefore$ $I = \int_0^{\pi} \frac{(\pi - x)dx}{4\cos^2 x + 9\sin^2 x}$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi} \frac{\pi\,dx}{4\cos^2 x + 9\sin^2 x}$

$\therefore$ $I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{4\cos^2 x + 9\sin^2 x}$

$= 2\left(\frac{\pi}{2}\right)\int_0^{\frac{\pi}{2}} \frac{dx}{4\cos^2 x + 9\sin^2 x}$

$\ldots\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx\right]$

$= \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{4+9\tan^2 x} dx$

$= \frac{\pi}{9}\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\frac{4}{9}+\tan^2 x} dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore$ $I = \frac{\pi}{9}\int_0^{\infty} \frac{dt}{\frac{4}{9}+t^2}$

$= \frac{\pi}{9} \times \frac{3}{2}\left[\tan^{-1}\frac{3t}{2}\right]_0^{\infty} = \frac{\pi}{6}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{12}$

103. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx$ ....(i)

$\therefore$ $I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \cos x} dx$ ....(ii)

....$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

Adding (i) and (ii), we get

$2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \cos x} dx$

$\Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$

Put $\cos x = t \Rightarrow \sin x \, dx = -dt$

$\therefore$ $I = -\frac{\pi}{2} \int_1^{-1} \frac{dt}{1+t^2} = -\frac{\pi}{2}\left[\tan^{-1} t\right]_1^{-1}$

$= \left(-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right) = \frac{\pi^2}{4}$

104. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx$ .....(i)

$\therefore$ $I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\cos x \sin x}{\sin^4 x + \cos^4 x} dx$ .....(ii)

....$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

Adding (i) and (ii), we get

$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x \sin x}{\cos^4 x + \sin^4 x} dx$

$\Rightarrow I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$

Put $\tan^2 x = t \Rightarrow \tan x \sec^2 x \, dx = \frac{dt}{2}$

$\therefore$ $I = \frac{\pi}{8} \int_0^{\infty} \frac{dt}{1+t^2}$

$= \frac{\pi}{8}[\tan^{-1} t]_0^{\infty} = \frac{\pi^2}{16}$

105. Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$\therefore$ $I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} . \sec^2\theta \, d\theta$

$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} . \sec^2\theta \, d\theta$

$\therefore$ $I = \int_0^{\pi/4} \log(1+\tan\theta) \, d\theta$ …(i)

$= \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta$

$= \int_0^{\pi/4} \log\left[1 + \frac{1-\tan\theta}{1+\tan\theta}\right] d\theta$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$

$\therefore$ $I = \int_0^{\pi/4} \log 2 \, d\theta - \int_0^{\pi/4} \log(1+\tan\theta) \, d\theta$

$\Rightarrow I = \int_0^{\pi/4} \log 2 \, d\theta - I$ …[From (i)]

$\Rightarrow 2I = \int_0^{\pi/4} \log 2 \, d\theta$

$= \frac{\pi}{4} \log 2$

$\Rightarrow I = \frac{\pi}{8} \log 2$

106. Put $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$\therefore$ $\int_0^1 \frac{8\log(1+x)}{1+x^2} dx = \int_0^{\frac{\pi}{4}} \frac{8\log(1+\tan\theta)}{1+\tan^2\theta}\left(\sec^2\theta d\theta\right)$

$= 8\int_0^{\frac{\pi}{4}} \log(1+\tan\theta) d\theta = 8\left(\frac{\pi}{8}\log 2\right) = \pi \log 2$

107. Let $I = \int_0^{\pi} [\cot x] dx$ ....(i)

$\Rightarrow I = \int_0^{\pi} [\cot(\pi - x)] dx$

....$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\Rightarrow I = \int_0^{\pi} [-\cot x] dx$ ....(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi} \{[\cot x] + [-\cot x]\} dx$

$\Rightarrow 2I = \int_0^{\pi} -1 dx$ ….$[\because [x] + [-x] = -1, \text{ if } x \notin Z]$

$\Rightarrow 2I = -\pi \Rightarrow I = -\frac{\pi}{2}$

108. Let $I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$ ....(i)

$\therefore$ $I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx$ ....(ii)

....$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

Adding (i) and (ii), we get

$2I = \int_2^8 dx = [x]_2^8 = 8-2 = 6 \Rightarrow I = \frac{6}{2} = 3$

109. Let $I = \int_{2016}^{2017} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4033-x}}dx$ ...(i)

$= \int_{2016}^{2017} \frac{\sqrt{4033-x}}{\sqrt{4033-x}+\sqrt{x}}dx$ ...(ii)

...$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$

Adding (i) and (ii), we get

$2I = \int_{2016}^{2017} dx = [x]_{2016}^{2017} = 1$

$\therefore$ $I = \frac{1}{2}$

110. Let $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ ...(i)

$\therefore$ $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos(\pi-x)}$

...$\left[\because \int_a^b f(x)dx = \int_a^b f(b+a-x)dx\right]$

$\therefore$ $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1-\cos x}$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{2}{1-\cos^2 x}dx$

$\therefore$ $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \operatorname{cosec}^2 x\,dx$

$= -[\cot x]_{\pi/4}^{3\pi/4} = 2$

111. Let $I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)}dx$ ....(i)

$\therefore$ $I = \int_2^4 \frac{\log(6-x)^2}{\log(6-x)^2 + \log x^2}dx$

....(ii)$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

Adding (i) and (ii), we get

$2I = \int_2^4 1\,dx = [x]_2^4 = 4-2 = 2$

$\therefore$ $I = 1$

112. Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{e^{\sin x}+1} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{e^{\sin x}\left(1+e^{-\sin x}\right)}$

$\therefore$ $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-\sin x}}{1+e^{-\sin x}}dx$ ...(i)

Also, $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{\sin x}+1}dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{\sin(-x)}+1}dx$

...$\left[\because \int_a^b f(x)dx = \int_a^b f(b+a-x)\,dx\right]$

$\therefore$ $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{-\sin x}+1}dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx$

$\Rightarrow 2I = [x]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \Rightarrow I = \frac{\pi}{2}$

113. Let $I = \int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x\sin x^2}{\sin x^2 + \sin(\log 6 - x^2)}dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$\therefore$ $I = \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin t}{\sin t + \sin(\log 6 - t)}dt$ ...(i)

$\therefore \quad I = \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t} dt$ ....(ii)

....$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

Adding (i) and (ii), we get

$2I = \frac{1}{2}\int_{\log 2}^{\log 3} dt = \frac{1}{2}(\log 3 - \log 2) = \frac{1}{2}\log\left(\frac{3}{2}\right)$

$\Rightarrow I = \frac{1}{4}\log\left(\frac{3}{2}\right)$

114. $I = \int_{1/2014}^{2014} \frac{\tan^{-1} x}{x} dx$ …(i)

Put $x = \frac{1}{t} \Rightarrow dx = \frac{-1}{t^2}dt$

$\therefore \quad I = \int_{2014}^{1/2014} \frac{\tan^{-1}\left(\frac{1}{t}\right)}{\frac{1}{t}}\left(\frac{-1}{t^2}\right) dt$

$= \int_{2014}^{1/2014} \frac{-\cot^{-1} t}{t} dt = \int_{1/2014}^{2014} \frac{\cot^{-1} t}{t} dt$

$\therefore \quad I = \int_{1/2014}^{2014} \frac{\cot^{-1} x}{x} dx$ …(ii)

Adding (i) and (ii), we get

$2I = \int_{1/2014}^{2014} \frac{\tan^{-1} x + \cot^{-1} x}{x} dx$

$= \frac{\pi}{2}\int_{1/2014}^{2014} \frac{dx}{x} = \frac{\pi}{2}\left[\log x\right]_{1/2014}^{2014}$

$= \frac{\pi}{2}\left[\log 2014 - \log\frac{1}{2014}\right]$

$= \frac{\pi}{2} \times 2 \log 2014$

$\Rightarrow I = \frac{\pi}{2}\log 2014$

115. $I = \int_{\pi/2}^{5\pi/2} \frac{e^{\tan^{-1}(\sin x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}} dx$

$= \int_{\pi/2}^{0} \frac{e^{\tan^{-1}(\sin x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}} dx$

$+ \int_{0}^{5\pi/2} \frac{e^{\tan^{-1}(\sin x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}} dx$

$\therefore \quad I = \int_{0}^{5\pi/2} \frac{e^{\tan^{-1}(\sin x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}} dx$

$- \int_{0}^{\pi/2} \frac{e^{\tan^{-1}(\sin x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}} dx$ ...(i)

$= \int_{0}^{5\pi/2} \frac{e^{\tan^{-1}\sin\left(\frac{5\pi}{2} - x\right)}}{e^{\tan^{-1}\sin\left(\frac{5\pi}{2} - x\right)} + e^{\tan^{-1}\cos\left(\frac{5\pi}{2} - x\right)}} dx$

$- \int_{0}^{\pi/2} \frac{e^{\tan^{-1}\sin\left(\frac{\pi}{2} - x\right)}}{e^{\tan^{-1}\sin\left(\frac{\pi}{2} - x\right)} + e^{\tan^{-1}\cos\left(\frac{\pi}{2} - x\right)}} dx$

$\therefore \quad I = \int_{0}^{5\pi/2} \frac{e^{\tan^{-1}(\cos x)}}{e^{\tan^{-1}(\cos x)} + e^{\tan^{-1}(\sin x)}} dx$

$- \int_{0}^{\pi/2} \frac{e^{\tan^{-1}(\cos x)}}{e^{\tan^{-1}(\cos x)} + e^{\tan^{-1}(\sin x)}} dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_{0}^{5\pi/2} \frac{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}} dx$

$- \int_{0}^{\pi/2} \frac{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}} dx$

$\Rightarrow 2I = \frac{5\pi}{2} - \frac{\pi}{2} = 2\pi$

$\Rightarrow I = \pi$

116. Let $I = \int_e^{\pi} x f(x) dx$

$= \int_e^{\pi} (e + \pi - x) f(e + \pi - x) dx$

....$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

$= \int_e^{\pi} (e + \pi - x) f(x) dx$

....[$\because$ $f(x) = f(\pi + e - x)$ (given)]

$\therefore \quad I = \int_e^{\pi} (e + \pi) f(x) dx - I$

$\Rightarrow 2I = (e + \pi) \int_e^{\pi} f(x) dx \Rightarrow 2I = (e + \pi).\frac{2}{e + \pi}$

$\Rightarrow I = 1$

117. Let $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\phi}{1+\sin\phi} d\phi$ ....(i)

$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\pi-\phi}{1+\sin(\pi-\phi)} d\phi$

$....\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$

$\therefore$ $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\pi-\phi}{1+\sin\phi} d\phi$ ....(ii)

Adding (i) and (ii), we get

$2I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\pi}{1+\sin\phi}\, d\phi$

On solving, we get

$2I = 2\pi\left(\sqrt{2}-1\right)$

$\therefore$ $I = \pi(\sqrt{2}-1) = \pi\tan\frac{\pi}{8}$

118. $I_1 = \int_a^{\pi-a} x f(\sin x)dx$

$= \int_a^{\pi-a} (\pi-x) f(\sin(\pi-x))\,dx$

$....\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx\right]$

$= \int_a^{\pi-a} (\pi-x) f(\sin x)\,dx$

$\therefore$ $I_1 = \int_a^{\pi-a} \pi f(\sin x)\,dx - I_1$

$\Rightarrow 2I_1 = \pi I_2 \Rightarrow I_2 = \frac{2}{\pi} I_1$

119. $f(x) = \frac{e^x}{1+e^x}$

$\therefore$ $f(a) + f(-a) = \frac{e^a}{1+e^a} + \frac{e^{-a}}{1+e^{-a}}$

$\Rightarrow f(a) + f(-a) = \frac{e^a}{1+e^a} + \frac{1}{1+e^a} = 1$

Using $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, we have

$I_1 = \int_{f(-a)}^{f(a)} (1-x)g\left((1-x)x\right)dx$

$....[\because f(a) + f(-a) = 1]$

$\Rightarrow I_1 = \int_{f(-a)}^{f(a)} g\left((1-x)x\right)dx - \int_{f(-a)}^{f(a)} xg\left((1-x)x\right)dx$

$\Rightarrow I_1 = I_2 - I_1 \Rightarrow 2I_1 = I_2 \Rightarrow \frac{I_2}{I_1} = 2$

120. Let $f(x) = e^{\cos^2 x}\cos^3(2n+1)x$

$\therefore$ $f(\pi - x) = e^{\cos^2(\pi-x)}\cos^3\left[(2n+1)(\pi-x)\right]$

$= e^{\cos^2 x}\cos^3\left[(2n+1)\pi-(2n+1)x\right]$

$= -e^{\cos^2 x}\cos^3(2n+1)x = -f(x)$

Since, $\int_0^{2a} f(x)\,dx = 0$, if $f(2a-x) = -f(x)$

$\therefore$ $\int_0^{\pi} e^{\cos^2 x}\cos^3(2n+1)x\,dx = 0$

121. Let $I = \int_0^{\pi} x f(\sin x)dx$ ....(i)

$\therefore$ $I = \int_0^{\pi} (\pi-x) f(\sin x)\,dx$ ....(ii)

$....\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

Adding (i) and (ii), we get

$2I = \pi\int_0^{\pi} f(\sin x)dx$

$\Rightarrow 2I = 2\pi\int_0^{\frac{\pi}{2}} f(\sin x)dx$ $....\left[\begin{array}{l}\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \\ \text{if } f(2a-x) = f(x)\end{array}\right]$

$\Rightarrow I = \pi\int_0^{\frac{\pi}{2}} f(\sin x)dx$

$\Rightarrow I = \pi\int_0^{\frac{\pi}{2}} f\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx$

$....\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \pi\int_0^{\frac{\pi}{2}} f(\cos x)dx$

122. $I_1 = \int_0^{\frac{\pi}{2}} f(\sin 2x)\sin x\, dx$

$= \int_0^{\frac{\pi}{4}} \left[f(\sin 2x)\sin x + f\left\{\sin 2\left(\frac{\pi}{2} - x\right)\right\}\sin\left(\frac{\pi}{2} - x\right)\right] dx$

....$\left[\because \int_0^{2a} f(x)\,dx = \int_0^{a} [f(x) + f(2a - x)]\,dx\right]$

$= \int_0^{\frac{\pi}{4}} \left[f(\sin 2x)\sin x + f\{\sin(\pi - 2x)\}\cos x\right] dx$

$\therefore \quad I_1 = \int_0^{\frac{\pi}{4}} [f(\sin 2x)\sin x + f(\sin 2x)\cos x]\, dx$ ....(i)

$I_2 = \int_0^{\frac{\pi}{4}} f(\cos 2x)\cos x\, dx$

$= \int_0^{\frac{\pi}{4}} f\left[\cos 2\left(\frac{\pi}{4} - x\right)\right].\cos\left(\frac{\pi}{4} - x\right) dx$

....$\left[\because \int_0^{a} f(x)\,dx = \int_0^{a} f(a - x)\,dx\right]$

$= \int_0^{\frac{\pi}{4}} f\left[\cos\left(\frac{\pi}{2} - 2x\right)\right].\left(\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x\right) dx$

$= \int_0^{\frac{\pi}{4}} f(\sin 2x)\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right) dx$

$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} [f(\sin 2x)\cos x + f(\sin 2x)\sin x]\, dx$

$\therefore \quad I_2 = \frac{1}{\sqrt{2}} I_1$ ....[From (i)]

$\therefore \quad \frac{I_1}{I_2} = \sqrt{2}$

123. $I = \int_0^{100\pi} \sqrt{(1 - \cos 2x)}\, dx = \int_0^{100\pi} \sqrt{2\sin^2 x}\, dx$

$= \sqrt{2} \int_0^{100\pi} \sin x\, dx$

$= 100\sqrt{2} \int_0^{\pi} \sin x\, dx$

...$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)dx, \text{ if } f(2a - x) = f(x)\right]$

$= 100\sqrt{2}\,[-\cos x]_0^{\pi} \quad 200\sqrt{2}$

124. Let $f(x) = x\,|x|$

$f(-x) = -x\,|-x| = -x\,|x| = -f(x)$

$\therefore$ $f(x)$ is an odd function

$\therefore \quad \int_{-1}^{1} x|x|\, dx = 0$

Let $I = \int_0^{\frac{\pi}{2}} \left[1 + \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right)\right] dx$ ...(i)

$= \int_0^{\frac{\pi}{2}} \left[1 + \log\left(\frac{4 + 3\sin\left(\frac{\pi}{2} - x\right)}{4 + 3\cos\left(\frac{\pi}{2} - x\right)}\right)\right] dx$

...$\left[\because \int_0^{a} f(x)\,dx = \int_0^{a} f(a - x)\,dx\right]$

$\therefore \quad I = \int_0^{\frac{\pi}{2}} \left[1 + \log\left(\frac{4 + 3\cos x}{4 + 3\sin x}\right)\right] dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} \left[1 + \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) + 1 + \log\left(\frac{4 + 3\cos x}{4 + 3\sin x}\right)\right] dx$

$= \int_0^{\frac{\pi}{2}} \left[2 + \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) - \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right)\right] dx$

$= \int_0^{\frac{\pi}{2}} 2\, dx = 2\,[x]_0^{\frac{\pi}{2}}$

$\therefore \quad 2I = 2\left(\frac{\pi}{2}\right)$

$\therefore \quad I = \frac{\pi}{2}$

125. Since, $\sin^3 x \cos^2 x$ is an odd function.

$\therefore \quad \int_{-1}^{1} \sin^3 x \cos^2 x dx = 0$

126. Since, $\sin^{103} x \cos^{101} x$ is an odd function.

$\therefore \quad \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^{103} x \cos^{101} x dx = 0$

127. Since, $x\cos x + \sin x$ is an odd function.

$\therefore \quad \int_{-2}^{2} (x\cos x + \sin x)\, dx + \int_{-2}^{2} dx = 0 + [x]_{-2}^{2} = 4$

128. Let $f(x) = \log\left(\frac{9-x}{9+x}\right)$

$\therefore$ $f(-x) = \log\left(\frac{9-x}{9+x}\right)^{-1}$

$= -\log\left(\frac{9-x}{9+x}\right) = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ $\int_{-4}^{4} \log\left(\frac{9-x}{9+x}\right)dx = 0$

129. Let $f(x) = \log\left(\frac{2-\sin x}{2+\sin x}\right)$

$\therefore$ $f(-x) = \log\left(\frac{2-\sin(-x)}{2+\sin(-x)}\right)$

$= -\log\left(\frac{2-\sin x}{2+\sin x}\right) = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{2-\sin x}{2+\sin x}\right)dx = 0$

130. Let $I = \int_{-2}^{2} (px^2 + qx + s)dx$

$= \int_{-2}^{2} (px^2 + s)dx + q\int_{-2}^{2} x dx$

$= 2\int_{0}^{2} (px^2 + s)dx + 0 = \frac{4}{3}(4p + 3s)$

Thus, to find the numerical value of I, it is necessary to know the values of p and s.

131. $\int_{-2}^{3} f(x)\ dx = \int_{-2}^{2} f(x)\ dx + \int_{2}^{3} f(x)\ dx$

$= \int_{-2}^{2} e^{\cos x} \sin x\ dx + \int_{2}^{3} 2dx$

Since, $e^{\cos x} \sin x$ is an odd function.

$\therefore$ $\int_{-2}^{3} f(x)\ dx = 0 + 2\ (3 - 2) = 2$

132. Let $I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]\ dx$

$= \int_{-2}^{0} [(x+1)^3 + 2 + (x+1)\cos(x+1)]\ dx$

Put $x + 1 = t \Rightarrow dx = dt$

$\therefore$ $I = \int_{-1}^{1} (t^3 + 2 + t\cos t)\ dt$

Since, t^3 and t cos t are odd functions.

$\therefore$ $I = \int_{-1}^{1} 2dt = [2t]_{-1}^{1} = 4$

133. $g(x) = \frac{f(-x) - f(x)}{x^2 + 3}$

$= \frac{4x^2 + 3x + 1 - 4x^2 + 3x - 1}{x^2 + 3} = \frac{6x}{x^2 + 3}$

$\therefore$ $g(-x) = -\frac{6x}{x^2 + 3} = -g(x)$

$\therefore$ $g(x)$ is an odd function.

$\therefore$ $\int_{-2}^{2} g(x)\, dx = 0$

134. Let $I = \int_{-2}^{2} |x|\ dx$

$= 2\int_{0}^{2} x\ dx$ ...$\left[\because \int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\, dx \text{ if } f(x) \text{ is an even function}\right]$

$= 2\left[\frac{x^2}{2}\right]_0^2 = 2\left(\frac{4}{2}\right) = 4$

135. Let $I = \int_{-2}^{2} |x\cos \pi x|\ dx$

$= 2\int_{0}^{2} |x\cos \pi x|\ dx$

...$\left[\because \int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\, dx \text{ if } f(x) \text{ is an even function}\right]$

$= 2\left[\int_{0}^{1/2} x\cos\pi x\ dx - \int_{1/2}^{3/2} x\cos\pi x\ dx + \int_{3/2}^{2} x\cos\pi x\ dx\right]$

Since $\int x\cos\pi x = \frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}$

$\therefore$ $I = 2\left[\left(\frac{1}{2\pi} - \frac{1}{\pi^2}\right) - \left(\frac{-3}{2\pi} - \frac{1}{2\pi}\right) + \left(\frac{1}{\pi^2} + \frac{3}{2\pi}\right)\right]$

$= 2\left[\frac{1}{2\pi} - \frac{1}{\pi^2} + \frac{3}{2\pi} + \frac{1}{2\pi} + \frac{1}{\pi^2} + \frac{3}{2\pi}\right]$

$= 2\left(\frac{8}{2\pi}\right) = \frac{8}{\pi}$

136. Since, $x\tan^{-1}x$ is an even function.

$\therefore \quad \int_{-1}^{1} x\tan^{-1}x\,dx = 2\int_{0}^{1} x\ \tan^{-1}x\,dx$

$= \left[2\tan^{-1}x\cdot\frac{x^2}{2}\right]_0^1 - 2\int_0^1 \frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$

$= [x^2\tan^{-1}x]_0^1 - \int_0^1 \frac{x^2+1-1}{1+x^2}\,dx$

$= [x^2\tan^{-1}x]_0^1 - [x]_0^1 + [\tan^{-1}x]_0^1$

$= \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1$

137. Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,dx$

Since, $\frac{\sin^4 x}{\sin^4 x+\cos^4 x}$ is an even function.

$\therefore \quad I = 2\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}\,dx = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

138. Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}\,dx \quad \ldots(i)$

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^{-x}}\,dx \quad \ldots(ii)$

$\ldots\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$

Adding (i) and (ii), we get

$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,dx$

$\Rightarrow 2I = 2\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

$\ldots\left[\begin{array}{l}\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \\ \text{if } f(x) \text{ is an even function}\end{array}\right]$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}$

139. Let $I = \int_{-1}^{1} \frac{\sin x - x^2}{3-|x|}\,dx$

$= \int_{-1}^{1} \frac{\sin x}{3-|x|}\,dx - \int_{-1}^{1} \frac{x^2}{3-|x|}\,dx$

Since, $\frac{\sin x}{3-|x|}$ is an odd function and $\frac{x^2}{3-|x|}$ is an even function.

$\therefore \quad I = 0 - 2\int_0^1 \frac{x^2}{3-|x|}\,dx = 2\int_0^1 \frac{-x^2}{3-|x|}\,dx$

140. $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2\,dx$

$= \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx - 2\cos ax\,\sin bx)\,dx$

$= \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx)\,dx - 2\int_{-\pi}^{\pi} \cos ax \sin bx\,dx$

$= 2\int_0^{\pi} (\cos^2 ax + \sin^2 bx)\,dx - 0$

$\ldots\left[\begin{array}{l}\because \cos ax \sin bx \text{ is an odd function and} \\ (\cos^2 ax + \sin^2 bx) \text{ is an even function}\end{array}\right]$

$= 2\int_0^{\pi} \left(\frac{1+\cos 2ax}{2} + \frac{1-\cos 2bx}{2}\right) dx$

$= \int_0^{\pi} (2 + \cos 2ax - \cos 2bx)\,dx = 2\pi$

141. Let $I = \int_{\frac{-3\pi}{2}}^{\frac{-\pi}{2}} \left[(x+\pi)^3 + \cos^2(x+3\pi)\right] dx$

Put $x + \pi = t \Rightarrow dx = dt$

$\therefore \quad I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left[t^3 + \cos^2(2\pi + t)\right] dt$

$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} t^3\,dt + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt$

Since, t^3 is an odd function and $\cos^2 t$ is an even function.

$\therefore \quad I = 0 + 2\int_0^{\frac{\pi}{2}} \cos^2 t\,dt = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$

142. Let I $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{x^2 + \log\left(\frac{\pi+x}{\pi-x}\right)\right\}\cos x\, dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x\, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{\pi+x}{\pi-x}\right)\cos x\, dx$

$= 2\int_0^{\frac{\pi}{2}} x^2 \cos x\, dx + 0$

$$\ldots\left[\begin{array}{l}\because \log\left(\frac{\pi+x}{\pi-x}\right)\cos x \text{ is an odd function} \\ \text{and } x^2\cos x \text{ is an even function.}\end{array}\right]$$

$= 2\left[x^2 \sin x + 2x\cos x - 2\sin x\right]_0^{\pi/2}$

$= 2\left(\frac{\pi^2}{4} - 2\right) = \frac{\pi^2}{2} - 4$

143. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x+\frac{\pi}{4}}{2-\cos 2x} dx$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x}{2-\cos 2x} dx + \frac{\pi}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{2-\cos 2x}$

$= 0 + 2.\frac{\pi}{4}\int_0^{\frac{\pi}{4}} \frac{dx}{2-\cos 2x}$

$$\ldots\left[\begin{array}{l}\because \frac{x}{2-\cos 2x} \text{ is an odd function} \\ \text{and } \frac{1}{2-\cos 2x} \text{ is an even function}\end{array}\right]$$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \frac{dx}{2-\cos 2x}$

$= \frac{\pi}{2}\int_0^1 \frac{1}{2-\left(\frac{1-t^2}{1+t^2}\right)} . \frac{dt}{1+t^2}$ ....[Put $\tan x = t$]

$= \frac{\pi}{2}\int_0^1 \frac{dt}{1+3t^2} = \frac{\pi}{2}.\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}t\right)\right]_0^1$

$= \frac{\pi}{2\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\right) - 0\right]$

$= \frac{\pi}{2\sqrt{3}}.\frac{\pi}{3} = \frac{\pi^2}{6\sqrt{3}}$

144. $p'(x) = p'(1-x)$

Integrating on both sides, we get

$p(x) = -p(1-x) + c$ ....(i)

$\therefore$ $p(0) = -p(1-0) + c$

$\Rightarrow 1 = -41 + c \Rightarrow c = 42$

$\therefore$ $p(x) + p(1-x) = 42$ ....(ii)[From (i)]

Let I $= \int_0^1 p(x)dx$

$= \int_0^1 p(1-x)dx$ ....$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^1 [42 - p(x)]dx$ ....[From (ii)]

$\therefore$ $I = 42\int_0^1 dx - I$

$\Rightarrow 2I = 42 \Rightarrow I = 21$

145. $A(x) = \begin{vmatrix} 1 & 2 & 3 \\ x+1 & 2x+1 & 3x+1 \\ x^2+1 & 2x^2+1 & 3x^2+1 \end{vmatrix}$

Applying $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$, we get

$A(x) = \begin{vmatrix} 1 & 0 & 0 \\ x+1 & -1 & -2 \\ x^2+1 & -1 & -2 \end{vmatrix} = 1(2-2) - 0 + 0 = 0$

$\therefore$ $\int_0^1 A(x)\,dx = 0$

146. Let I $= \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$

Putting $t = \sin^2 u$ in the first integral and $t = \cos^2 v$ in the second integral, we get

$I = \int_0^x u\sin 2u\,du - \int_{\frac{\pi}{2}}^x v\sin 2v\,dv$

$= \int_0^{\frac{\pi}{2}} u\sin 2u\,du + \int_{\frac{\pi}{2}}^x u\sin 2u\,du - \int_{\frac{\pi}{2}}^x v\sin 2v\,dv$

$= \int_0^{\frac{\pi}{2}} u\sin 2u\,du$ ....$\left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$

$= \left[\frac{-u\cos 2u}{2}\right]_0^{\pi/2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2u\,du$

$= \left[\frac{-u\cos 2u}{2}\right]_0^{\pi/2} + \frac{1}{4}\left[\sin 2u\right]_0^{\pi/2} = \frac{\pi}{4}$

147. $N = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{(x+1)^2} dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{(x+1)^2} dx$

$= \frac{1}{2}\left[\left[\sin 2x\left(-\frac{1}{x+1}\right)\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{2\cos 2x}{(x+1)} dx\right]$

$= -\frac{2}{\pi+4} + \int_0^{\frac{\pi}{4}} \frac{\cos 2x}{x+1} dx = \frac{-2}{\pi+4} + I_2$

In I_2, put $2x = t \Rightarrow dx = \frac{dt}{2}$

$\therefore \quad I_2 = \int_0^{\frac{\pi}{2}} \frac{\cos t}{t+2} dt = \int_0^{\frac{\pi}{2}} \frac{\cos x}{x+2} dx = M$

$\therefore \quad N = -\frac{2}{\pi+4} + M \qquad \therefore \quad M - N = \frac{2}{\pi+4}$

148. $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$ ....(i)

Replacing x by $\frac{1}{x}$ in (i), we get

$af\left(\frac{1}{x}\right) + bf\ (x) = x - 5$ ....(ii)

Eliminating $f\left(\frac{1}{x}\right)$ from (i) and (ii), we get

$(a^2 - b^2)f(x) = \frac{a}{x} - bx - 5a + 5b$

$\therefore \quad (a^2 - b^2)\int_1^2 f\ (x)dx$

$= \left[a \log|x| - \frac{b}{2}x^2 - 5(a-b)x\right]_1^2$

$= a\log 2 - 2b - 10(a - b) - a\log 1 + \frac{b}{2} + 5(a - b)$

$= a \log 2 - 5a + \frac{7}{2}b$

$\therefore \quad \int_1^2 f(x)dx = \frac{1}{(a^2-b^2)}\left[a\log 2 - 5a + \frac{7}{2}b\right]$

149. Let $I = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$= \frac{2}{\pi}\int_{-\pi}^{\pi} \frac{\sin\left(\frac{9x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx = \frac{4}{\pi}\int_0^{\pi} \frac{\sin\left(\frac{9x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx$

....[$\because$ f(x) is an even function]

Put $\frac{x}{2} = \theta \Rightarrow dx = 2d\theta$

$\therefore \quad I = \frac{8}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin 9\theta}{\sin\theta} d\theta$

$= \frac{8}{\pi}\int_0^{\frac{\pi}{2}} \frac{(\sin 9\theta - \sin 7\theta) + (\sin 7\theta - \sin 5\theta) + (\sin 5\theta - \sin 3\theta) + (\sin 3\theta - \sin\theta) + \sin\theta}{\sin\theta} d\theta$

$= \frac{8}{\pi}\int_0^{\frac{\pi}{2}} (2\cos 8\theta + 2\cos 6\theta + 2\cos 4\theta + 2\cos 2\theta + 1)d\theta$

$= \frac{8}{\pi} \times \frac{\pi}{2} = 4$

Evaluation Test

1. Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Put $x = \tan t \Rightarrow dx = \sec^2 t\ dt$

When $x = 0$, t = 0 and when $x = 1$, $t = \frac{\pi}{4}$

$\therefore \quad I = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan t)}{1+\tan^2 t} \cdot \sec^2 t\, dt = \int_0^{\frac{\pi}{4}} \log(1+\tan t)\, dt$

$= \int_0^{\frac{\pi}{4}} \log\left[1 + \tan\left(\frac{\pi}{4} - t\right)\right] dt$

$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1-\tan t}{1+\tan t}\right) dt = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan t}\right) dt$

$= \int_0^{\frac{\pi}{4}} [\log 2 - \log(1+\tan t)] dt$

$\therefore \quad I = \int_0^{\frac{\pi}{4}} (\log 2) dt - I$

$\therefore \quad 2I = \log 2\,[t]_0^{\pi/4} = \frac{\pi}{4}\log 2$

$\therefore \quad I = \frac{\pi}{8}\log 2$

2. $\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx$

$= 0 + \int_1^2 1^2\,dx + \int_2^3 2^2\,dx + \int_3^4 3^2\,dx + \int_4^5 4^2\,dx$

$= 1(2-1) + 4(3-2) + 9(4-3) + 16(5-4)$

$= 1 + 4 + 9 + 16 = 30$

3. Let $I = \int_0^3 \frac{1}{1+2^{f(x)}}dx$ …. (i)

$= \int_0^3 \frac{1}{1+2^{f(3-x)}}dx$ …. $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^3 \frac{1}{1+2^{-f(x)}}dx$ ….[$\because$ $f(x) + f(3-x) = 0$ (given)]

$\therefore$ $I = \int_0^3 \frac{2^{f(x)}}{2^{f(x)}+1}dx$ …. (ii)

Adding (i) and (ii), we get

$2I = \int_0^3 \frac{1}{1+2^{f(x)}}dx + \int_0^3 \frac{2^{f(x)}}{2^{f(x)}+1}dx$

$= \int_0^3 \frac{1+2^{f(x)}}{1+2^{f(x)}}dx = \int_0^3 1\,dx = 3$

$\therefore$ $I = \frac{3}{2}$

4. Let $I = \int_0^\infty \frac{x\log x}{(1+x^2)^2}dx$

$= \int_0^1 \frac{x\log x}{(1+x^2)^2}dx + \int_1^\infty \frac{x\log x}{(1+x^2)^2}dx$

$= I_1 + I_2$ (Say) ….(i)

$I_2 = \int_0^\infty \frac{x\log x}{(1+x^2)^2}dx$

Put $x = \frac{1}{y}$

$\therefore$ $dx = -\frac{1}{y^2}dy$

When $x = 1$, $y = 1$ and when $x \to \infty$, $y \to 0$

$\therefore$ $I_2 = \int_1^0 \frac{\frac{1}{y}\log\left(\frac{1}{y}\right)}{\left(1+\frac{1}{y^2}\right)^2}\left(-\frac{1}{y^2}\right)dy$

$= \int_1^0 \frac{y\log y}{(1+y^2)^2}dy$ …. $\left[\because \log\left(\frac{1}{y}\right) = -\log y\right]$

$= -\int_0^1 \frac{y\log y}{(1+y^2)^2}dy$

$= -\int_0^1 \frac{x\log x}{(1+x^2)^2}dx$

$\therefore$ $I_2 = -I_1$

$\therefore$ From (i), $I = I_1 + I_2 = 0$

5. $\int_0^n [x]dx$

$= \int_0^1 [x]dx + \int_1^2 [x]dx + \int_2^3 [x]dx + \ldots + \int_{n-1}^n [x]dx$

$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \ldots + \int_{n-1}^n (n-1)dx$

$= 0 + [x]_1^2 + 2[x]_2^3 + \ldots. + (n-1)\,[x]_{n-1}^n$

$= (2-1) + 2(3-2) + \ldots. + (n-1)(n-n+1)$

$= 1 + 2(1) + \ldots. + (n-1)(1)$

$= 1 + 2 + \ldots. + (n-1)$

$= \frac{(n-1)n}{2} = \frac{n(n-1)}{2}$

6. $I_1 = \int_0^{\pi/2} \frac{\sin^2 x}{\sin^2 x}dx = \frac{\pi}{2}$

$I_2 = \int_0^{\pi/2} \frac{\sin^2 2x}{\sin^2 x}dx = \int_0^{\pi/2} \frac{(2\sin x\cos x)^2}{\sin^2 x}dx$

$= \int_0^{\pi/2} 4\cos^2 x\,dx = 4 \times \frac{\pi}{4} = \pi$

$I_3 = \int_0^{\pi/2} \frac{\sin^2 3x}{\sin^2 x}dx$

$= \int_0^{\pi/2} \frac{(3\sin x - 4\sin^3 x)^2}{\sin^2 x}dx$

$= \int_0^{\pi/2} (9 - 24\sin^2 x + 16\sin^4 x)dx$

$= \frac{9\pi}{2} - 24.\frac{\pi}{4} + 16.\frac{3.1}{4.2}.\frac{\pi}{2} = \frac{3\pi}{2}$

$\therefore$ $I_1 + I_3 = \frac{\pi}{2} + \frac{3\pi}{2} = 2\pi = 2I_2$

$\therefore$ I_1, I_2, I_3 are in A.P.

7. Let $I = \int_{\alpha}^{\beta} \frac{1}{\sqrt{(x-\alpha)(\beta-x)}} dx$

Put $x = \alpha \sin^2 t + \beta \cos^2 t$

$\therefore$ $dx = (\alpha.2 \sin t \cos t + \beta.2 \cos t (-\sin t))dt$

$= 2(\alpha - \beta) \sin t \cos t\, dt$

When $x = \alpha$, $\alpha = \alpha \sin^2 t + \beta(1 - \sin^2 t)$

$\alpha = \beta + (\alpha - \beta) \sin^2 t$

$\therefore$ $\sin t = 1$, $\therefore$ $t = \frac{\pi}{2}$

When $x = \beta$,

$\beta = \alpha(1 - \cos^2 t) + \beta \cos^2 t$

$= \alpha + (\beta - \alpha)\cos^2 t$

$\therefore$ $\cos t = 1$, $t = 0$

$(x - \alpha)(\beta - x) = (\alpha \sin^2 t + \beta \cos^2 t - \alpha)(\beta - \alpha \sin^2 t - \beta \cos^2 t)$

$= [\beta \cos^2 t - \alpha(1 - \sin^2)][\beta(1 - \cos^2 t) - \alpha \sin^2 t]$

$= (\beta - \alpha) \cos^2 t\, (\beta - \alpha) \sin^2 t$

Since, $\beta > \alpha$

$\therefore$ $\sqrt{(x-\alpha)(\beta-x)} = (\beta - \alpha) \sin t \cos t$

$\therefore$ $I = 2\int_{\frac{\pi}{2}}^{0} \frac{(\alpha-\beta)\sin t \cos t}{(\beta-\alpha)\sin t \cos t} dt$

$= 2\int_{\frac{\pi}{2}}^{0} (-1)\, dt$

$= 2\int_{0}^{\frac{\pi}{2}} 1\, dt$

$= 2[t]_0^{\pi/2}$

$= 2\left(\frac{\pi}{2} - 0\right)$

$= \pi$

8. Let $h(x) = x^3 f(x) = x^3\left(\frac{e^x + 1}{e^x - 1}\right)$

$\therefore$ $h(-x) = (-x)^3\left(\frac{e^{-x} + 1}{e^{-x} - 1}\right) = -x^3\left(\frac{1 + e^x}{1 - e^x}\right)$

$= x^3\left(\frac{e^x + 1}{e^x - 1}\right)$

$\therefore$ $h(-x) = h(x)$

$\therefore$ $h(x)$ is an even function.

$\therefore$ $\int_{-1}^{1} t^3 f(t)dt = \int_{-1}^{1} h(t)dt = 2\int_{0}^{1} h(t)dt$

$= 2\int_{0}^{1} t^3 f(t)dt$

$= 2\int_{0}^{1} x^3 f(x)dx$

$= 2\alpha$ ....$\left[\because \int_{0}^{1} x^3 f(x)dx = \alpha\right]$

9. $f(m, n) = \int_{0}^{1} (\log x)^m x^{n-1} dx$

$= \left[(\log x)^m \int x^{n-1} dx\right]_0^1 - \int_{0}^{1}\left\{\frac{d}{dx}(\log x)^m \int x^{n-1} dx\right\}dx$

$= \left[(\log x)^m \cdot \frac{x^n}{n}\right]_0^1 - \int_{0}^{1} m(\log x)^{m-1} \cdot \frac{1}{x} \cdot \frac{x^n}{n} dx$

$= 0 - 0 - \frac{m}{n}\int_{0}^{1} (\log x)^{m-1} \cdot x^{n-1} dx$ ....[$\because$ log 1 = 0]

$= -\frac{m}{n} f(m-1, n)$

10. $\phi(x) = \int_{\frac{7\pi}{6}}^{x} (4\sin t + 3\cos t)dt$

$\therefore$ $\phi'(x) = 4 \sin x + 3 \cos x$

If $x \in \left[\frac{7\pi}{6}, \frac{4\pi}{3}\right]$,

then x is in the third quadrant.

$\therefore$ $\sin x$ and $\cos x$ are both negative.

$\therefore$ $\phi'(x) = 4 \sin x + 3\cos x < 0$

$\therefore$ $\phi(x)$ is decreasing on the interval $\left[\frac{7\pi}{6}, \frac{4\pi}{3}\right]$

$\therefore$ Minimum (least) value of $\phi(x)$ on $\left[\frac{7\pi}{6}, \frac{4\pi}{3}\right]$

is $\phi\left(\frac{4\pi}{3}\right) = \int_{7\pi/6}^{4\pi/3} (4\sin t + 3\cos t)dt$

$= [-4\cos t + 3\sin t]_{7\pi/6}^{4\pi/3}$

$= -4\left(\cos\frac{4\pi}{3} - \cos\frac{7\pi}{6}\right) + 3\left(\sin\frac{4\pi}{3} - \sin\frac{7\pi}{6}\right)$

$= -4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) + 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\right)$

$= \frac{7\left(1 - \sqrt{3}\right)}{2}$

11. Let $I = \int_a^b f(x)dx$

Put $x = (b - a)t + a$

$\therefore$ $dx = (b - a)dt$

When $x = a$, $t = 0$ and when $x = b$, $t = 1$

$\therefore$ $I = \int_0^1 f[(b-a)t + a](b-a)dt$

$= (b-a)\int_0^1 f[(b-a)t+a]dt$

$= (b-a)\int_0^1 f[(b-a)x+a]dx$

$\therefore$ $\lambda = b - a$

12. If $0 \le x < 1$, then $0 \le x^2 < 1$, $\therefore$ $[x^2] = 0$

If $1 \le x < \sqrt{2}$, then $1 \le x^2 < 2$, $\therefore$ $[x^2] = 1$

If $\sqrt{2} \le x \le 1.5$, then $2 \le x^2 \le 2.25$, $\therefore$ $[x^2] = 2$

$\therefore$ $\int_0^{1.5}[x^2]dx = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}}[x^2]dx + \int_{\sqrt{2}}^{1.5}[x^2]dx$

$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{1.5} 2\,dx$

$= 0 + [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{1.5}$

$= \sqrt{2} - 1 + 2(1.5 - \sqrt{2})$

$= \sqrt{2} - 1 + 3 - 2\sqrt{2}$

$= 2 - \sqrt{2}$

13. $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ ….(given)

$\therefore$ $f(x) = -\frac{1}{x^2} f\left(\frac{1}{x}\right)$

Let $I = \int_{\cos\theta}^{\sec\theta} f(x)dx = \int_{\cos\theta}^{\sec\theta} -\frac{1}{x^2} f\left(\frac{1}{x}\right)dx$

Put $\frac{1}{x} = t$, $\therefore -\frac{1}{x^2}dx = dt$

When $x = \cos\theta$, $t = \sec\theta$

and when $x = \sec\theta$, $t = \cos\theta$

$\therefore$ $I = \int_{\sec\theta}^{\cos\theta} f(t)dt = -\int_{\cos\theta}^{\sec\theta} f(t)dt = -\int_{\cos\theta}^{\sec\theta} f(x)dx = -I$

$\therefore$ $I + I = 0$

$\therefore$ $2I = 0$

$\therefore$ $I = 0$

14. $\lim\limits_{n\to\infty}\frac{1}{n}\left[1 + \sqrt{\frac{n}{n+1}} + \sqrt{\frac{n}{n+2}} + \sqrt{\frac{n}{n+3}} + \ldots + \sqrt{\frac{n}{n+3(n-1)}}\right]$

$= \lim\limits_{n\to\infty}\frac{1}{n}\left[\frac{1}{\sqrt{1+\frac{0}{n}}} + \frac{1}{\sqrt{1+\frac{1}{n}}} + \frac{1}{\sqrt{1+\frac{2}{n}}} + \ldots + \frac{1}{\sqrt{1+\frac{3(n-1)}{n}}}\right]$

$= \lim\limits_{n\to\infty}\frac{1}{n}\sum_{r=0}^{3(n-1)}\frac{1}{\sqrt{1+\frac{r}{n}}}$

$= \int_0^3 \frac{1}{\sqrt{1+x}}dx$

$= \left[2\sqrt{1+x}\right]_0^3$

$= 2\left(\sqrt{1+3} - \sqrt{1+0}\right)$

$= 2(2 - 1) = 2(1) = 2$

15. $\lim\limits_{n\to\infty}\sum_{r=1}^{4n}\frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$

$= \lim\limits_{n\to\infty}\sum_{r=1}^{4n}\frac{1}{n}\cdot\frac{1}{\sqrt{\frac{r}{n}}}\cdot\frac{1}{\frac{\left(3\sqrt{r}+4\sqrt{n}\right)^2}{n}}$

$= \lim\limits_{n\to\infty}\sum_{r=1}^{4n}\frac{1}{n}\cdot\frac{1}{\sqrt{\frac{r}{n}}}\cdot\frac{1}{\left(3\sqrt{\frac{r}{n}}+4\right)^2}$

$= \int_0^4 \frac{1}{\sqrt{x}\left(3\sqrt{x}+4\right)^2}dx$

Put $3\sqrt{x} + 4 = t$

$\therefore$ $3.\frac{1}{2\sqrt{x}}dx = dt$

$\therefore$ $\frac{1}{\sqrt{x}}dx = \frac{2}{3}dt$

When $x = 0$, $t = 4$ and when $x = 4$, $t = 10$

$\therefore$ $I = \int_4^{10}\frac{1}{t^2}\cdot\frac{2}{3}dt = -\frac{2}{3}\left[\frac{1}{t}\right]_4^{10}$

$= -\frac{2}{3}\left(\frac{1}{10} - \frac{1}{4}\right) = -\frac{2}{3}\left(\frac{2-5}{20}\right)$

$= -\frac{2}{3}\left(-\frac{3}{20}\right) = \frac{1}{10}$

16. $f'(x) = f(x) \Rightarrow \dfrac{f'(x)}{f(x)} = 1$

Integrating on both sides, we get

$\log f(x) = x + \log c \Rightarrow f(x) = ce^x$

$\therefore$ $f(0) = c \Rightarrow c = 1$

$\therefore$ $f(x) = e^x$

Now, $f(x) + g(x) = x^2$

$\Rightarrow g(x) = x^2 - e^x$

$\therefore$ $\int_0^1 f(x)g(x)\,dx = \int_0^1 e^x(x^2 - e^x)\,dx$

$= \int_0^1 x^2e^x - \int_0^1 e^{2x}dx$

$= \left[x^2e^x - 2xe^x + 2e^x - \dfrac{e^{2x}}{2}\right]_0^1$

$= e - \dfrac{1}{2}e^2 - \dfrac{3}{2}$

17. Let $I = \int_0^{100\pi} (|\sin^3 x| + |\cos^3 x|)\,dx$

$= \int_0^{200\times\frac{\pi}{2}} (|\sin^3 x| + |\cos^3 x|)\,dx$

$= 200\int_0^{\frac{\pi}{2}} (|\sin^3 x| + |\cos^3 x|)\,dx$

$\ldots\left[\because\ |\sin^3 x| + |\cos^3 x| \text{ is a periodic function with period } \dfrac{\pi}{2}\right]$

$\therefore$ $I = 200\int_0^{\frac{\pi}{2}} (\sin^3 x + \cos^3 x)\,dx$

$= 200\left[\int_0^{\frac{\pi}{2}} \sin^3 x\,dx + \int_0^{\frac{\pi}{2}} \cos^3 x\,dx\right]$

$= 200[I_1 + I_2]$ (Say) $\quad\ldots$(i)

Where $I_1 = \int_0^{\frac{\pi}{2}} \sin^3 x\,dx$

$= -\int_0^{\frac{\pi}{2}} (1 - \cos^2 x)(-\sin x)\,dx$

Put $\cos x = t$, $\therefore$ $-\sin x\,dx = dt$

When $x = 0$, $t = \cos 0 = 1$ and when $x = \dfrac{\pi}{2}$,

$t = \cos\dfrac{\pi}{2} = 0$

$\therefore$ $I_1 = -\int_1^0 (1 - t^2)\,dt$

$= \int_0^1 (1 - t^2)\,dt$

$= \left[t - \dfrac{t^3}{3}\right]_0^1 = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$I_2 = \int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos^3\left(\dfrac{\pi}{2} - x\right)dx$

$= \int_0^{\frac{\pi}{2}} \sin^3 x\,dx = I_1 = \dfrac{2}{3}$

$\therefore$ From (i),

$I = 200\left(\dfrac{2}{3} + \dfrac{2}{3}\right) = \dfrac{800}{3}$

18. $I_1 = \int_0^{\frac{\pi}{2}} \dfrac{\sin x - \cos x}{1 - \sin x\cos x}\,dx$

$= \int_0^{\frac{\pi}{2}} \dfrac{\cos x - \sin x}{1 - \sin x\cos x}\,dx$

$\ldots\left[\because \int_0^{\frac{\pi}{2}} f(x)\,dx = \int_0^{\frac{\pi}{2}} f\left(\dfrac{\pi}{2} - x\right)dx\right]$

$= -\int_0^{\frac{\pi}{2}} \dfrac{\sin x - \cos x}{1 - \sin x\cos x}\,dx = -I_1$

$\therefore$ $2I_1 = 0 \Rightarrow I_1 = 0$

$\therefore$ $I_2 = \int_0^{2\pi} \cos^6 x\,dx$

$= 2\int_0^{\pi} \cos^6 x\,dx$

$\ldots[\because \cos^6 x$ is a periodic f^n with period $\pi]$

$= 2\int_0^{\frac{\pi}{2}} (\cos^6 x + \cos^6(\pi - x))\,dx$

$\ldots\left[\because \int_0^{2a} f(x)dx = \int_0^{a} [f(x) + f(2a - x)]dx\right]$

$= 2.2\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$

$= 4.\dfrac{(6-1)(6-3)(6-5)}{6(6-2)(6-4)}.\dfrac{\pi}{2} = \dfrac{5\pi}{8}$

$\therefore$ $I_2 \neq 0$

$$I_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x\,dx = 0$$

....[$\because$ $\sin^3 x$ is an odd function]

$$I_4 = \int_0^1 \log\left(\frac{1-x}{x}\right)dx$$

$$= \int_0^1 \log\left(\frac{1-1+x}{1-x}\right)dx$$

....$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$= \int_0^1 \log\left(\frac{x}{1-x}\right)dx$$

$$= -\int_0^1 \log\left(\frac{1-x}{x}\right)dx$$

$\therefore$ $I_4 = -I_4$

$\therefore$ $2I_4 = 0$

$\therefore$ $I_4 = 0$

$\therefore$ $I_1 = I_3 = I_4 = 0$, but $I_2 \neq 0$

19. Let $I = \int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}dx$

$$= \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}(2\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)}dx$$

$$= 2\pi\int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}dx - I$$

$\therefore$ $I = \pi \times 2\int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}dx$

$\left(\because \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} \text{ is a periodic } f^n \text{ with period } \pi\right)$

$$= 4\pi\int_0^{\frac{\pi}{2}} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}dx$$

....$\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x)\right]$

$$= 4\pi \times \frac{\pi}{4}$$

$$= \pi^2$$

20. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\{a|\sin x| + \frac{b\sin x}{1+\cos x} + c\right\}dx = 0$

i.e., $I_1 + I_2 + I_3 = 0$

$$I_1 = a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sin x|\,dx$$

$$= a\left[\int_{-\frac{\pi}{4}}^{0} |\sin x|\,dx + \int_0^{\frac{\pi}{4}} |\sin x|\,dx\right]$$

$$= a\left[\int_{-\frac{\pi}{4}}^{0} (-\sin x)\,dx + \int_0^{\frac{\pi}{4}} \sin x\,dx\right]$$

$$= a\left[\left[\cos x\right]_{-\pi/4}^{0} - \left[\cos x\right]_0^{\pi/4}\right]$$

$$= a\left(1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1\right) = a\left(2-\sqrt{2}\right)$$

$$I_2 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{b\sin x}{1+\cos x}dx$$

$$= \left[-b\log|1+\cos x|\right]_{-\pi/4}^{\pi/4}$$

$= 0$ ....$\left[\because \cos\left(\frac{-\pi}{4}\right) = \cos\frac{\pi}{4}\right]$

$$I_3 = c\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1\,dx = c\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{c\pi}{2}$$

$\therefore$ $I_1 + I_2 + I_3 = 0$ becomes

$a\left(2-\sqrt{2}\right) + \frac{c\pi}{2} = 0$ ....[$\because$ $I_2 = 0$]

$\therefore$ The given equation is a relation between a and c.

21. Let $I = \int_0^{\sqrt{\log\left(\frac{\pi}{2}\right)}} \cos\left(e^{x^2}\right)\cdot 2xe^{x^2}\,dx$

Put $e^{x^2} = t \Rightarrow 2xe^{x^2}dx = dt$

When $x = 0$, $t = e^0 = 1$

When $x = \sqrt{\log\frac{\pi}{2}}$, $t = e^{\log \pi/2} = \frac{\pi}{2}$

$\therefore$ $I = \int_1^{\frac{\pi}{2}} \cos t\,dt = \left[\sin t\right]_1^{\pi/2}$

$$= \sin\frac{\pi}{2} - \sin 1 = 1 - \sin 1$$

22. $f(x) = \int_0^x \sin^6 t dt$

$\therefore$ $f(x+\pi) = \int_0^{x+\pi} \sin^6 t dt = \int_0^{\pi} \sin^6 t dt + \int_{\pi}^{x+\pi} \sin^6 t dt$

$= f(\pi) + \int_0^x \sin^6(u+\pi) du$, where $t = u + \pi$

$= f(\pi) + \int_0^x \sin^6 u du$

$= f(\pi) + \int_0^x \sin^6 t dt = f(\pi) + f(x)$

$\therefore$ $f(\pi + x) = f(\pi) + f(x)$

23. Let $f(x) = ax^2 + bx + c$

$\therefore$ $f'(x) = 2ax + b$

$f''(x) = 2a$

$f(0) = c = 3$

$f'(0) = b = -7$

$f''(0) = 2a = 8$

$\therefore$ $a = 4$

$\therefore$ $f(x) = 4x^2 - 7x + 3$

$$\int_1^2 f(x) dx = \int_1^2 (4x^2 - 7x + 3) dx$$

$$= \left[\frac{4x^3}{3} - \frac{7x^2}{2} + 3x\right]_1^2$$

$$= \frac{32}{3} - 14 + 6 - \left(\frac{4}{3} - \frac{7}{2} + 3\right)$$

$$= \frac{32 - 42 + 18}{3} - \left(\frac{8 - 21 + 18}{6}\right)$$

$$= \frac{8}{3} - \frac{5}{6} = \frac{16 - 5}{6} = \frac{11}{6}$$

24. $\int_0^1 f(x) dx = \frac{2A}{\pi}$

$\Rightarrow \int_0^1 \left[A \sin\left(\frac{\pi x}{2}\right) + B\right] dx = \frac{2A}{\pi}$

$\Rightarrow \left[-\frac{2A}{\pi}\cos\left(\frac{\pi x}{2}\right) + Bx\right]_0^1 = \frac{2A}{\pi}$

$\Rightarrow \frac{2A}{\pi} + B = \frac{2A}{\pi} \Rightarrow B = 0$

Now, $f(x) = A \sin\left(\frac{\pi x}{2}\right) + B$

$\Rightarrow f'(x) = A \cos\left(\frac{\pi x}{2}\right) . \frac{\pi}{2}$

$\Rightarrow f'\left(\frac{1}{2}\right) = \frac{\pi A}{2} . \frac{1}{\sqrt{2}}$

$\Rightarrow \sqrt{2} = \frac{\pi A}{2\sqrt{2}} \Rightarrow A = \frac{4}{\pi}$

25. $f(x) = \begin{vmatrix} \sin x + \sin 2x + \sin 3x & \sin 2x & \sin 3x \\ 3 + 4\sin x & 3 & 4\sin x \\ 1 + \sin x & \sin x & 1 \end{vmatrix}$

$= \begin{vmatrix} \sin x & \sin 2x & \sin 3x \\ 0 & 3 & 4\sin x \\ 0 & \sin x & 1 \end{vmatrix}$ $(C_1 \to C_1 - C_2 - C_3)$

$= \sin x(3 - 4\sin^2 x) = 3\sin x - 4\sin^3 x = \sin 3x$

$\therefore$ $\int_0^{\frac{\pi}{2}} f(x) dx = \int_0^{\frac{\pi}{2}} \sin 3x dx$

$$= -\frac{1}{3}[\cos 3x]_0^{\pi/2}$$

$$= -\frac{1}{3}\left[\cos\frac{3\pi}{2} - \cos 0\right]$$

$$= -\frac{1}{3}(0 - 1) = \frac{1}{3}$$

26. $\int_0^{\frac{\pi}{2}} \left\{a^2\left(\frac{\cos 3t}{4} + \frac{3}{4}\cos t\right) + a\sin t - 20\cos t\right\} dt$

$= \left[a^2\left(\frac{1}{12}\sin 3t + \frac{3}{4}\sin t\right) - a\cos t - 20\sin t\right]_0^{\pi/2}$

$= a^2\left(\frac{1}{12}(-1) + \frac{3}{4}(1) - 0\right) - a(0 - 1) - 20(1 - 0)$

$= a^2\left(\frac{3}{4} - \frac{1}{12}\right) + a - 20$

$= \frac{2a^2}{3} + a - 20$

$\therefore$ from the given condition,

$\frac{2a^2}{3} + a - 20 \le \frac{-a^2}{3}$

$\therefore$ $\frac{3a^2}{3} + a - 20 \le 0$

$\therefore$ $a^2 + a - 20 \le 0$

$\therefore$ $(a + 5)(a - 4) \le 0$

$\therefore$ $-5 \le a \le 4$

The positive integer values of a satisfying the above inequality are 1, 2, 3, 4.

$\therefore$ There are 4 such values.

27. Since, $-1 \le \sin x \le 1 \Rightarrow -2 \le 2\sin x \le 2$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [2\sin x]dx = \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} [2\sin x]dx + \int_{\frac{5\pi}{6}}^{\pi} [2\sin x]dx + \int_{\pi}^{\frac{7\pi}{6}} [2\sin x]dx + \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} [2\sin x]dx$$

$$= \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (1)dx + \int_{\frac{5\pi}{6}}^{\pi} (0)dx + \int_{\pi}^{\frac{7\pi}{6}} (-1)dx + \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} (-2)dx$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{2}\right) + 0 - \left(\frac{7\pi}{6} - \pi\right) - 2\left(\frac{3\pi}{2} - \frac{7\pi}{6}\right)$$

$$= \frac{2\pi}{6} - \frac{\pi}{6} - \frac{4\pi}{6} = -\frac{\pi}{2}$$

28. Applying $R_1 \to R_1 - \sec x R_3$, we get

$$f(x) = \begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \operatorname{cosec} x - \cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

$= (\sec^2 x + \cot x \operatorname{cosec} x - \cos x)(\cos^4 x - \cos^2 x)$
$= (\sec^2 x + \cot x \operatorname{cosec} x - \cos x)(-\cos^2 x \sin^2 x)$
$= -\sin^2 x - \cos^3 x + \cos^3 x \sin^2 x$
$= -\sin^2 x - \cos^3 x (1 - \sin^2 x)$
$= -\sin^2 x - \cos^5 x$

$\therefore$ $$\int_0^{\frac{\pi}{2}} f(x)dx = -\int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^5 x)dx$$

$$= -\left(\frac{1}{2} \times \frac{\pi}{2} + \frac{4.2}{5.3.1}\right) = -\frac{\pi}{4} - \frac{8}{15}$$

29. $$\frac{1}{\sqrt{a}} \int_1^a \left(\frac{3}{2}\sqrt{x} + 1 - \frac{1}{\sqrt{x}}\right)dx < 4$$

$\therefore$ $$\frac{1}{\sqrt{a}} \left[\frac{3}{2} \cdot \frac{x^{3/2}}{\frac{3}{2}} + x - 2\sqrt{x}\right]_1^a < 4$$

$\therefore$ $$\frac{1}{\sqrt{a}} \left[a\sqrt{a} - 1 + a - 1 - 2\sqrt{a} + 2\right] < 4$$

$\therefore$ $a + \sqrt{a} - 2 < 4$

$\therefore$ $a + \sqrt{a} - 6 < 0$

$\therefore$ $\left(\sqrt{a} + 3\right)\left(\sqrt{a} - 2\right) < 0$

$\therefore$ $-3 < \sqrt{a} < 2$

But $\sqrt{a}$ cannot be negative and according to the problem, $a \ne 0$

$\therefore$ $0 < \sqrt{a} < 2$

$\therefore$ $0 < a < 4$

30. Let $$I = \int_1^4 \frac{3e^{\sin x^3}}{x} dx = \int_1^4 \frac{3x^2 e^{\sin x^3}}{x^3} dx$$

Put $x^3 = t \Rightarrow 3x^2 dx = dt$

$\therefore$ $$I = \int_1^{64} \frac{e^{\sin t}}{t} dt$$

$$= \int_1^{64} \frac{e^{\sin x}}{x} dx$$

$$= [f(x)]_1^{64} \quad \ldots \left[\because \frac{d}{dx}[f(x)] = \frac{e^{\sin x}}{x}\right]$$

$= f(64) - f(1)$

$\therefore$ $k = 64$

Textbook Chapter No.

06 Applications of Definite Integral

Hints

Classical Thinking

1. Required area $= \int_1^4 x^3\, dx = \left[\frac{x^4}{4}\right]_1^4 = \frac{255}{4}$ sq. units

2. Required area $= \int_1^4 y\, dx = c\int_1^4 \frac{1}{x}\, dx$

$= 2c \log 2$ sq. units.

3. Required area $= \int_0^4 \sqrt{3x+4}\, dx$

$$= \left[\frac{(3x+4)^{\frac{3}{2}}}{3\left(\frac{3}{2}\right)}\right]_0^4$$

$$= \frac{2}{9} \times 56 = \frac{112}{9} \text{ sq.units}$$

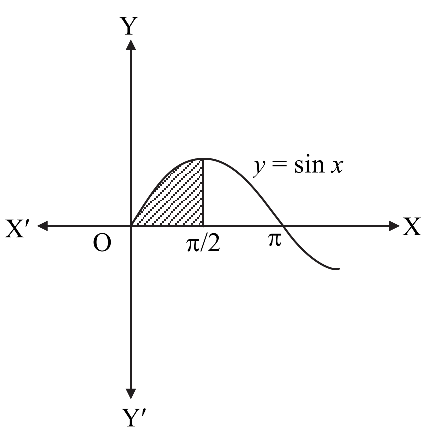
4. Required area $= \int_2^4 \left(1+\frac{8}{x^2}\right) dx = \left[x - \frac{8}{x}\right]_2^4$

$= (4 - 2) - (2 - 4)$
$= 2 + 2 = 4$

5. Required area $= \int_1^2 y\, dx = \int_1^2 \log x\, dx$

$= \left[x\log x - x\right]_1^2$
$= 2\log 2 - 1$
$= (\log 4 - 1)$ sq. units

6.



Required area $= \int_0^{\pi/2} \sin x\, dx$

$= [-\cos x]_0^{\pi/2}$

$= -\left(\cos\frac{\pi}{2} - \cos 0\right)$

$= 1$ sq. unit

7. Required area $= \int_0^2 (4x - x^2)\, dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^2$

$= 8 - \frac{8}{3}$

$= \frac{16}{3}$

8. Required area $= \int_0^{\frac{\pi}{2}} (2x + \sin x)\, dx$

$= \left[x^2 - \cos x\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{\pi^2}{4} - \cos\frac{\pi}{2}\right) - (0 - \cos 0)$

$= \frac{\pi^2}{4} - 0 -(0 - 1)$

$= \frac{\pi^2}{4} + 1$

9. Required area $= \int_0^a y\, dx = \int_0^a x e^{x^2}\, dx$

Put $x^2 = t \Rightarrow x dx = \frac{dt}{2}$

$\therefore$ required area $= \frac{1}{2}\int_0^{a^2} e^t dt$

$= \frac{1}{2}\left[e^t\right]_0^{a^2}$

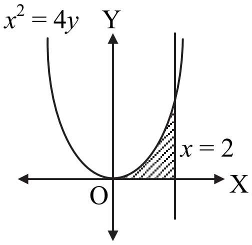
$= \frac{e^{a^2} - 1}{2}$ sq.units

10. Required area $= \int_0^{\pi/2} \sin^2 x.dx$

$$= \int_0^{\pi/2} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}[x]_0^{\pi/2} - \frac{1}{4}[\sin 2x]_0^{\pi/2} = \frac{\pi}{4}$$

11. Required area $= \int_0^2 \frac{x^2}{4} dx$



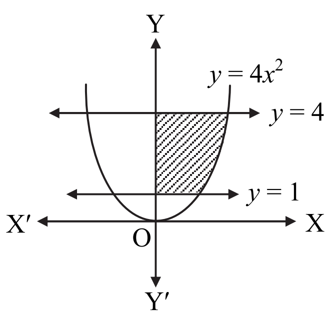
$$= \frac{1}{12}\left[x^3\right]_0^2$$

$$= \frac{8}{12}$$

$$= \frac{2}{3}$$

Critical Thinking

1.

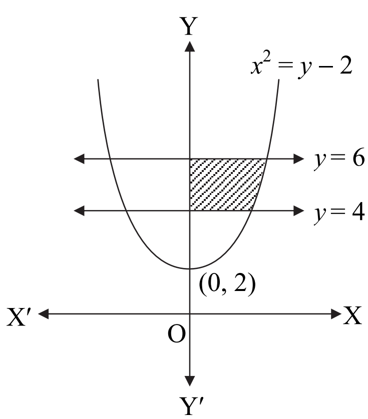


Required area $= \int_1^4 x\, dy = \int_1^4 \frac{\sqrt{y}}{2} dy$

$$= \frac{1}{2}.\frac{2}{3}\left[y^{3/2}\right]_1^4$$

$$= \frac{7}{3} \text{ sq. units}$$

2.



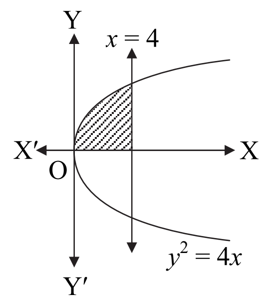
$x^2 = y - 2 \Rightarrow x = \sqrt{y-2}$

$\therefore$ required area $= \int_4^6 \sqrt{y-2}\, dy = \left[\frac{(y-2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^6$

$$= \frac{2}{3}\left[4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left(8 - 2\sqrt{2}\right)$$

3. Required area $= \int_0^4 2\sqrt{x}\, dx$



$$= 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = \frac{4}{3}\left[4^{\frac{3}{2}} - 0\right]$$

$$= \frac{4}{3}(8) = \frac{32}{3}$$

4. $A_1 = \int_0^{\pi/3} \cos x\ dx = \frac{\sqrt{3}}{2}$

$A_2 = \int_0^{\pi/3} \cos 2x\ dx = \frac{\sqrt{3}}{4}$

$\therefore$ $A_1 : A_2 = 2 : 1$

5. $xy - 3x - 2y - 10 = 0$

$\Rightarrow y(x-2) = 3x + 10$

$\Rightarrow y = \frac{3x+10}{x-2}$

$\therefore$ Required area $= \int_3^4 y\, dx = \int_3^4 \frac{3x+10}{x-2} dx$

$$= [3x + 16\log(x-2)]_3^4$$

$$= 3 + 16 \log 2 \text{ sq. units}$$

6. According to the given condition,

$\int_1^3 (3x^2 - 4x + k)\, dx = 20$

$\Rightarrow \left[x^3 - 2x^2 + kx\right]_1^3 = 20$

$\Rightarrow (27 - 18 + 3k) - (1 - 2 + k) = 20$

$\Rightarrow 9 + 3k + 1 - k = 20 \Rightarrow 2k = 10$

$\Rightarrow k = 5$

7. For X-axis, $y = 0$

$\therefore$ $4 + 3x - x^2 = 0$

$\Rightarrow (x+1)(x-4) = 0 \qquad \Rightarrow x = -1$ or $x = 4$

$\therefore$ Required area $= \int_{-1}^4 (4 + 3x - x^2)\, dx = \frac{125}{6}$

8. For Y-axis, $x = 0$

$\therefore$ $y^2 - y = 0$

$\Rightarrow y(y-1) = 0$

$\Rightarrow y = 0$ or $y = 1$

$\therefore$ required area $= \int_0^1 (y^2 - y)\,dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_0^1$

$= \frac{1}{3} - \frac{1}{2} = \left|\frac{-1}{6}\right| = \frac{1}{6}$ sq. units

9. For X-axis, $y = 0$

$\therefore$ $4x - x^2 = 0$

$\Rightarrow x(4-x) = 0 \quad \Rightarrow x = 0, 4$

Required area $= \int_0^4 (4x - x^2)\,dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$

$= 32 - \frac{64}{3} = \frac{32}{3}$ sq. units

10. According to the given condition,

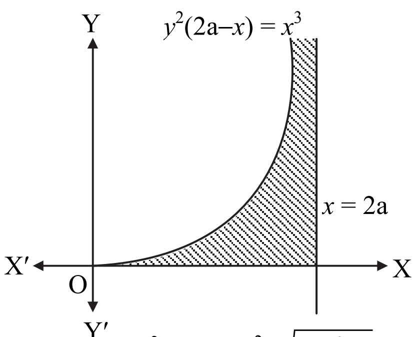
$\int_1^b f(x)\,dx = (b-1)\sin(3b+4)$

Differentiating w.r.t.b, we get

f(b).1 = 3(b – 1) cos(3b + 4) + sin(3b + 4)

$\therefore$ $f(x) = 3(x-1)\cos(3x+4) + \sin(3x+4)$

11.



Required area $= \int_0^{2a} y\,dx = \int_0^{2a} \sqrt{\frac{x^3}{2a-x}}\,dx$

Put $x = 2a\sin^2\theta$

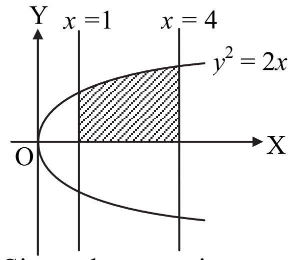
$\Rightarrow dx = 4a\sin\theta\cos\theta\,d\theta$

$\therefore$ required area $= \int_0^{\frac{\pi}{2}} \frac{\sqrt{8a^3\sin^6\theta}.4a\sin\theta\cos\theta}{\sqrt{2a - 2a\sin^2\theta}}\,d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{2a.\sin^3\theta.4a\sin\theta\cos\theta}{\cos\theta}\,d\theta$

$= 8a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\,d\theta = 8a^2.\frac{3.1}{4.2}.\frac{\pi}{2} = \frac{3\pi a^2}{2}$

12.

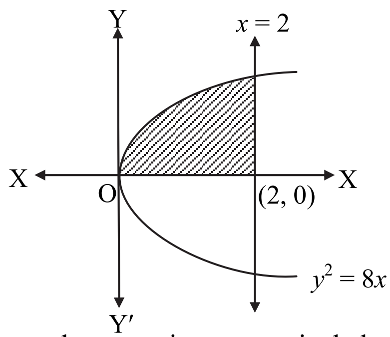


Since, the curve is symmetrical about X-axis.

$\therefore$ Required area $= 2\int_1^4 y\,dx$

$= 2\int_1^4 \sqrt{2}x^{\frac{1}{2}}\,dx = \frac{28\sqrt{2}}{3}$ sq. units

13.



Since, the curve is symmetrical about X-axis.

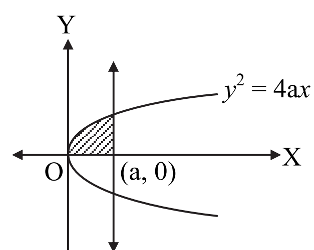
$\therefore$ Required area $= 2\int_0^2 y\,dx$

$= 2\int_0^2 \sqrt{8x}\,dx = 4\sqrt{2}\int_0^2 \sqrt{x}\,dx$

$= 4\sqrt{2}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^2 = \frac{8\sqrt{2}}{3}\left(2\sqrt{2}\right)$

$= \frac{32}{3}$ sq. units

14.

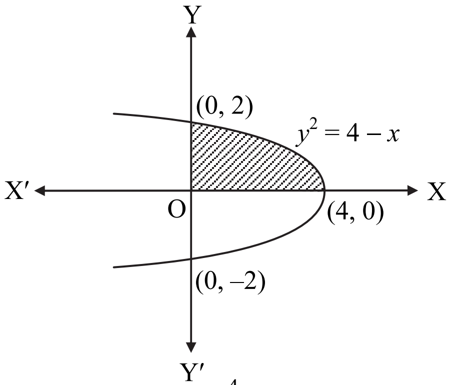


Required area $= 2\int_0^a y\,dx = 2\int_0^a \sqrt{4ax}\,dx$

$= 2 \times 2\sqrt{a} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^a$

$= \frac{8}{3}a^2$ sq. units

15.

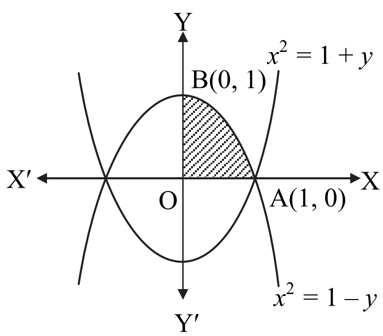


Required area $= 2\int_0^4 \sqrt{4-x}\,dx = 2\left[-\frac{2}{3}(4-x)^{3/2}\right]_0^4$

$= 2\left[0+\frac{2}{3}(4)^{\frac{3}{2}}\right] = \frac{4\times 8}{3}$

$= \frac{32}{3}$ sq. units

16.



Required area = 4(area of the region OABO)

$= 4\int_0^1 (1-x^2)\,dx$

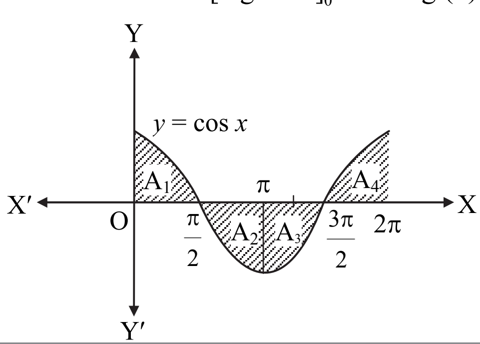
$= 4\left[x-\frac{x^3}{3}\right]_0^1$

$= \frac{8}{3}$

17. Required area $= 2\int_0^{\pi/3} \tan x\,dx$

$= 2\,[\log \sec x]_0^{\pi/3} = 2\log(2)$

18.



Required area = $A_1 + A_2 + A_3 + A_4$

$= 4\,A_1$

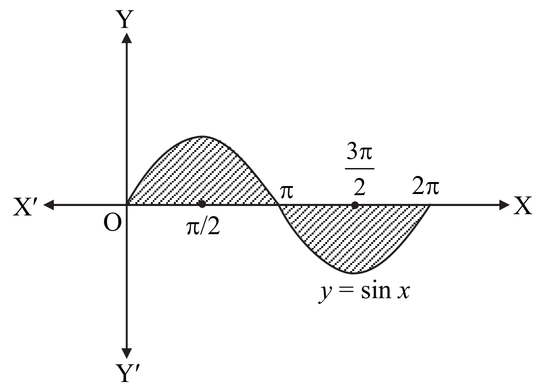
$= 4\int_0^{\pi/2} \cos x\,dx$

$= 4[\sin x]_0^{\pi/2} = 4\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= 4(1-0)$

$= 4$

19.



Required area $= 4\int_0^{\pi/2} y\,dx$

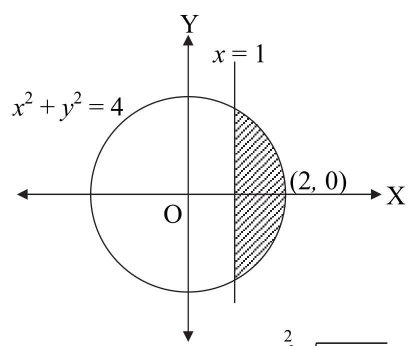
$= 4\int_0^{\pi/2} \sin x\,dx$

$= 4[-\cos x]_0^{\pi/2}$

$= -4\left(\cos\frac{\pi}{2} - \cos 0\right)$

$= 4$ sq. units

20.



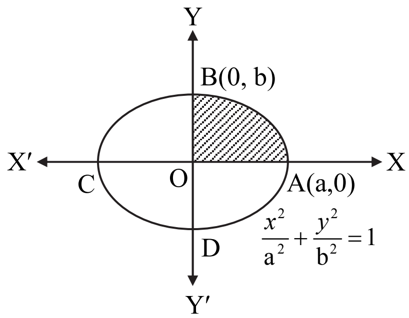
Area of smaller part $= 2\int_1^2 \sqrt{4-x^2}\,dx$

$= 2\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_1^2$

$= 2\left[2.\frac{\pi}{2} - \left(\frac{\sqrt{3}}{2} + 2.\frac{\pi}{6}\right)\right]$

$= \frac{4\pi}{3} - \sqrt{3}$

21.



Since, the curve is symmetrical about Y-axis as well as X-axis.

$\therefore$ the area of the given ellipse

$= 4(\text{area of OABO}) = 4\int_0^a y\,dx$

$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta . a\cos\theta\, d\theta$ ….[Put $x = a\sin\theta$]

$= 4ab\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$

$= 2ab\left[[\theta]_0^{\pi/2} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}\right] = \pi ab$ sq. units

22. $16x^2 + 9y^2 = 144 \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1$

Area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is π ab sq. units.

Here, a = 3, b = 4

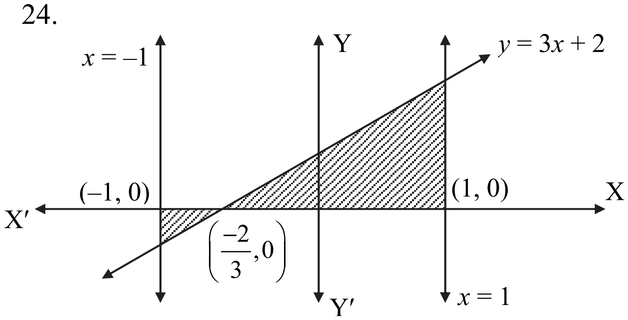
$\therefore$ Required area = $\pi ab = \pi(3)(4) = 12\pi$

23. Required area $= \int_{-1}^{1} x|x|\,dx$

$= \left|\int_{-1}^{0} -x^2\,dx\right| + \int_0^1 x^2\,dx$

$= \left|\frac{-1}{3}\right| + \frac{1}{3} = \frac{2}{3}$

24.



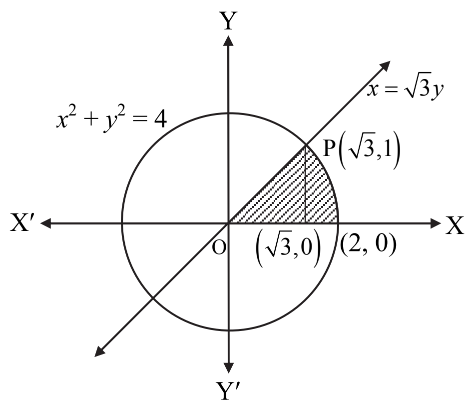
Required area $= \left|\int_{-1}^{\frac{-2}{3}} y\,dx\right| + \int_{\frac{-2}{3}}^{1} y\,dx$

$= \left|\int_{-1}^{-\frac{2}{3}} (3x+2)\,dx\right| + \int_{\frac{-2}{3}}^{1}(3x+2)\,dx$

$= \left|\left[\frac{3x^2}{2}+2x\right]_{-1}^{\frac{-2}{3}}\right| + \left[\frac{3x^2}{2}+2x\right]_{\frac{-2}{3}}^{1}$

$= \frac{1}{6}+\frac{25}{6} = \frac{13}{3}$ sq. units.

25.

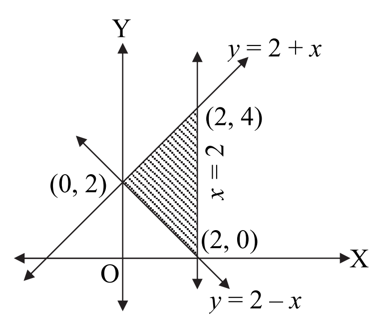


Required area $= \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\,dx + \int_{\sqrt{3}}^{2}\sqrt{4-x^2}\,dx$

$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$

$= \frac{\sqrt{3}}{2} + \left[\pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3}\right] = \frac{\pi}{3}$

26.

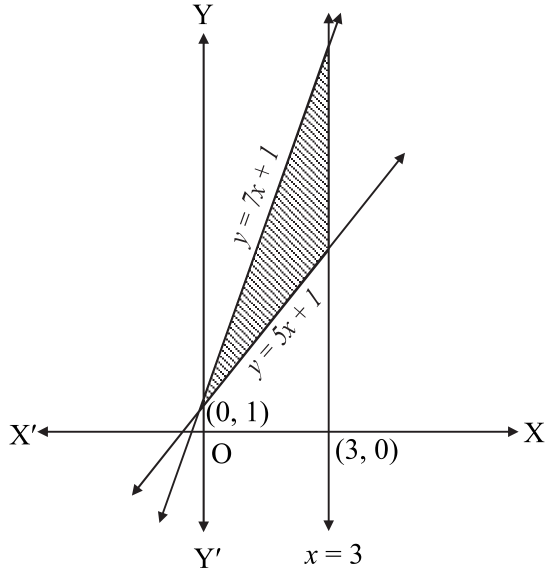


Required area $= \int_0^2 [(2+x)-(2-x)]\,dx$

$= \left[x^2\right]_0^2$

= 4 sq. units

27.



Required area $= \int_0^3 \left[(7x+1)-(5x+1)\right] dx$

$= \int_0^3 2x \, dx$

$= 2\left[\frac{x^2}{2}\right]_0^3 = 9$ sq. units

28. The curves $y = x$ and $y = x + \sin x$ intersect at $(0, 0)$ and (π, π).

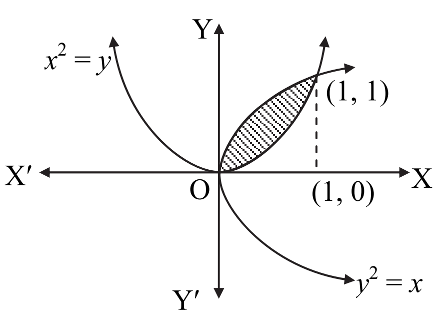
$\therefore$ required area

$= \int_0^\pi (x+\sin x)\, dx - \int_0^\pi x \, dx = \int_0^\pi \sin x \, dx$

$= [-\cos x]_0^\pi = -\cos \pi + \cos 0$

$= -(-1) + 1 = 2$

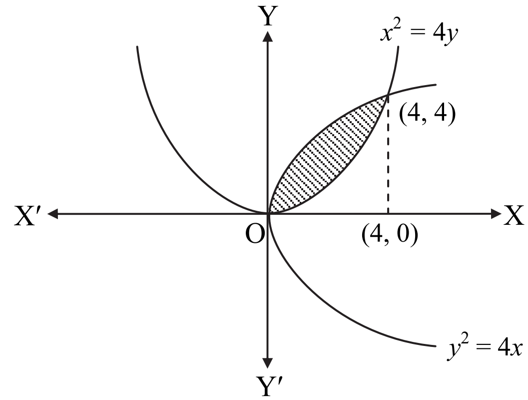
29. The two curves intersect at (0, 0) and (1, 1).



Required area $= \int_0^1 (\sqrt{x} - x^2)\, dx$

$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

30. The two parabolas intersect at (0, 0) and (4, 4).

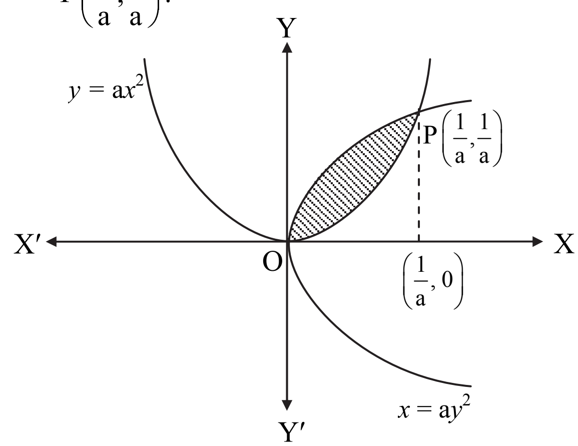


Required area $= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$

$= 2\left[\frac{2x^{3/2}}{3}\right]_0^4 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$

$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$ sq. units

31. The two curves intersect at O(0, 0) and $P\left(\frac{1}{a}, \frac{1}{a}\right)$.



According to the given condition,

$\int_0^{\frac{1}{a}} \left(\sqrt{\frac{x}{a}} - ax^2\right) dx = 1$

$\Rightarrow \left[\frac{2}{3\sqrt{a}} x^{3/2} - \frac{ax^3}{3}\right]_0^{1/a} = 1$

$\Rightarrow \frac{2}{3\sqrt{a}} \times \frac{1}{a^{3/2}} - \frac{a}{3} \times \frac{1}{a^3} = 1$

$\Rightarrow \frac{2}{3a^2} - \frac{1}{3a^2} = 1 \Rightarrow \frac{1}{3a^2} = 1$

$\Rightarrow a = \frac{1}{\sqrt{3}}$ ....[$\because a > 0$]

32. The area of the region bounded by $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16ab}{3}$ sq. units.

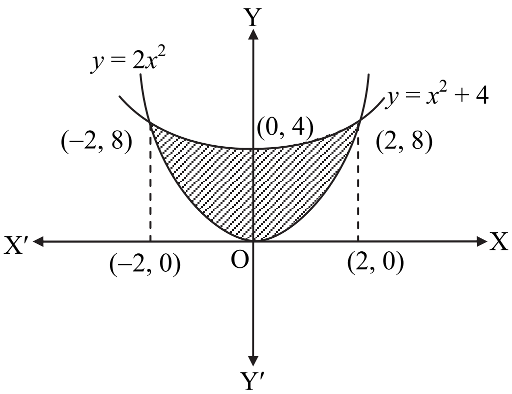
Given parabolas are $y^2 = \frac{9}{4}x$ and $x^2 = \frac{16}{3}y$

Here, $a = \frac{9}{16}$, $b = \frac{4}{3}$

$\therefore$ Required area $= \frac{16}{3} \times \frac{9}{16} \times \frac{4}{3}$

$= 4$ sq. units

33.

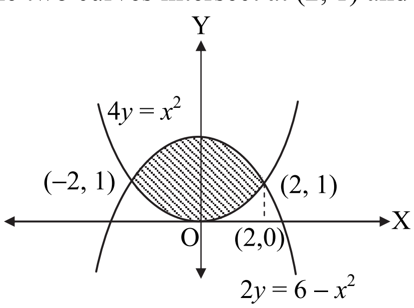


Required area $= \int_{-2}^{2} (x^2 + 4 - 2x^2)\, dx$

$= \int_{-2}^{2} (4 - x^2)\, dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$

$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}$

34. The two curves intersect at (2, 1) and (–2, 1).



Required area $= 2\int_{0}^{2} \left(\frac{6 - x^2}{2} - \frac{x^2}{4}\right) dx$

$= 2\int_{0}^{2} \left(3 - \frac{3x^2}{4}\right) dx = 6\left[x - \frac{x^3}{12}\right]_0^2$

$= 6\left(2 - \frac{8}{12}\right) = 6 \times \frac{16}{12}$

$= 8$ sq. units

35. The area of the region bounded by the parabola $y^2 = 4ax$ and the line $y = mx$ is $\frac{8a^2}{3m^3}$ sq.units.

36. The area bounded by $x^2 = 4ay$ and the line $y = mx$ is $\frac{8a^2m^3}{3}$.

Given, $x^2 = 2y \Rightarrow x^2 = 4\left(\frac{1}{2}\right)y$ and $y = 3x$

Here, $a = \frac{1}{2}$ and $m = 3$

$\therefore$ Required area $= \frac{8}{3} \times \frac{1}{4} \times 3 \times 3 \times 3 = 18$ sq. units

37. Given curves are $y = x^2$ and $y = x$.
On solving, we get $x = 0$, $x = 1$

$\therefore$ Required area $= \int_0^1 (x - x^2)\, dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

38. $y^2 = x$ and $2y = x$

$\therefore$ $\left(\frac{x}{2}\right)^2 = x \Rightarrow x^2 = 4x \Rightarrow x = 0, 4$

$\therefore$ Required area $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx$

$= \left[\frac{2x^{3/2}}{3} - \frac{x^2}{4}\right]_0^4$

$= \frac{4}{3}$ sq. units

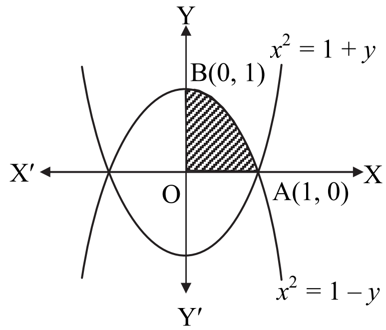
39. Given curves are $y = x^3$ and $y = \sqrt{x}$.
On solving, we get $x = 0$, $x = 1$

$\therefore$ Required area $= \int_0^1 \left(\sqrt{x} - x^3\right) dx$

$= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^4}{4}\right]_0^1$

$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$

40.



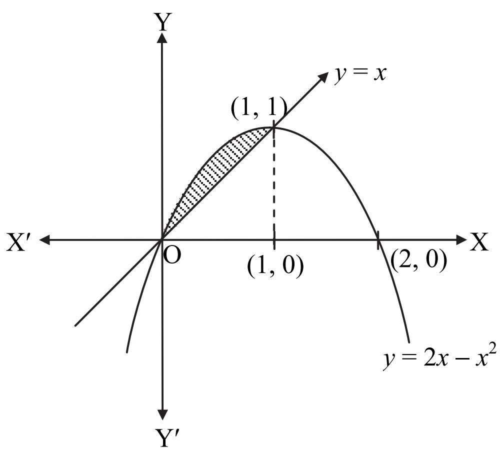
Required area = 4(area of the region OABO)

$$= 4\int_0^1 (1 - x^2)\,dx$$

$$= 4\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{8}{3}$$

41.

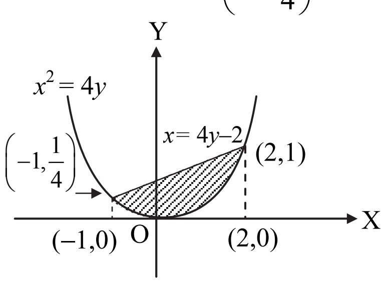


Required area = $\int_0^1 \left[(2x - x^2) - x\right]dx$

$$= \int_0^1 (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

42. The points of intersection of $x^2 = 4y$ and $x = 4y - 2$ are (2, 1) and $\left(-1, \frac{1}{4}\right)$.



Required area

$$= \int_{-1}^{2} \frac{1}{4}(x + 2)\,dx - \int_{-1}^{2} \frac{1}{4}x^2\,dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{8} \text{ sq. units}$$

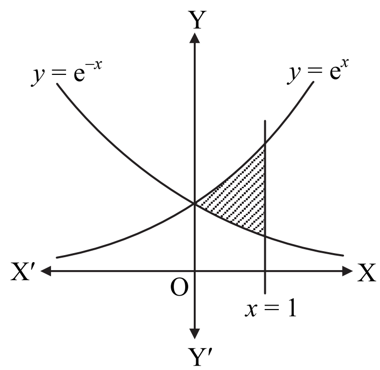
43. The two curves $y^2 = 4ax$ and $y = mx$ intersect at (0, 0) and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.

According to the given condition,

$$\int_0^{\frac{4a}{m^2}} (\sqrt{4a\,x} - mx)\,dx = \frac{a^2}{3}$$

$$\Rightarrow \frac{8}{3} \cdot \frac{a^2}{m^3} = \frac{a^2}{3} \Rightarrow m^3 = 8 \Rightarrow m = 2$$

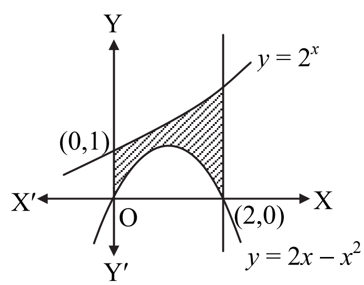
44.



Required area = $\int_0^1 (e^x - e^{-x})\,dx$

$$= \left[e^x + e^{-x}\right]_0^1 = e + \frac{1}{e} - 2$$

45.



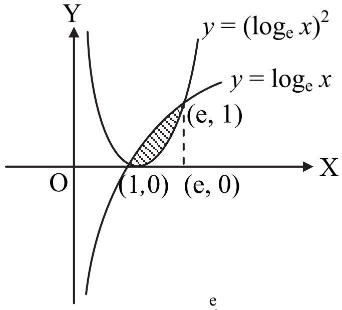
Required area $= \int_0^2 [2^x - (2x - x^2)]dx$

$$= \left[\frac{2^x}{\log 2} - x^2 + \frac{x^3}{3}\right]_0^2$$

$$= \frac{4}{\log 2} - 4 + \frac{8}{3} - \frac{1}{\log 2}$$

$$= \frac{3}{\log 2} - \frac{4}{3}$$

46.



Required area = $\int_1^e [\log x - (\log x)^2]\,dx$

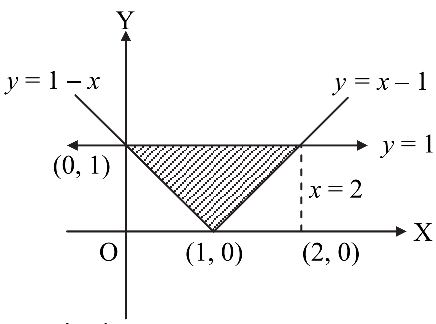
$= \int_1^e \log x\,dx - \int_1^e (\log x)^2\,dx$

$= [x\log x - x]_1^e - [x(\log x)^2 - 2x\log x + 2x]_1^e$

$= [e - e - (-1)] - [e(1)^2 - 2e + 2e - (2)]$

$= 1 - (e - 2) = 3 - e$

47.



Required area

$= \int_0^2 \left(1 - |x-1|\right)dx$

$= \int_0^2 dx - \left[\int_0^1 (1-x)dx + \int_1^2 (x-1)dx\right]$

$= [x]_0^2 - \left[\left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2\right]$

$= 2 - \left[1 - \frac{1}{2} + (2-2) - \left(\frac{1}{2} - 1\right)\right]$

$= 2 - \left(\frac{1}{2} + \frac{1}{2}\right) = 1$

48. The area between $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is $\frac{1}{4}\pi ab - \frac{1}{2}ab$ sq.units.

Here, a = 3, b = 2

$\therefore$ Required area = $\frac{1}{4}\pi(3)(2) - \frac{1}{2}(3)(2)$

$= \frac{3}{2}(\pi - 2)$ sq.units

Competitive Thinking

1. Required area = $\int_3^5 (3x-5)\,dx$

$= \frac{3}{2}\left[x^2\right]_3^5 - 5[x]_3^5 = \frac{3}{2}(25-9) - 5(5-3)$

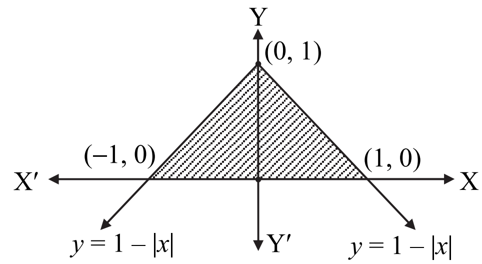
= 24 – 10 = 14 sq. units

2. According to the given condition,

$\int_1^2 mx\,dx = 6 \Rightarrow m\left[\frac{x^2}{2}\right]_1^2 = 6$

$\Rightarrow m\left(\frac{3}{2}\right) = 6 \Rightarrow m = 4$

3.



Required area $= 2\int_0^1 (1-x)\,dx = 2\left[x - \frac{x^2}{2}\right]_0^1$

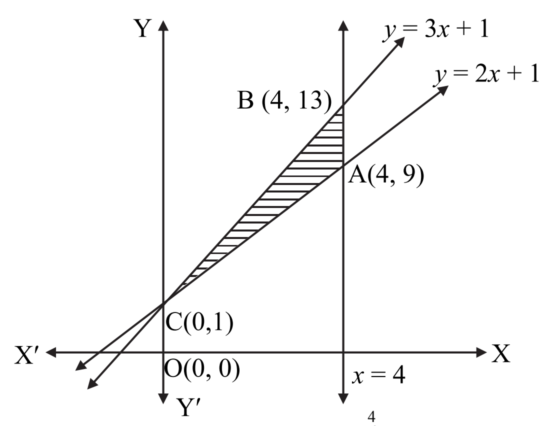
$= 2\left[1 - \frac{1}{2}\right] = 1$ sq. unit.

Alternate method:

Required area $= \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2} \times 2 \times 1 = 1$ sq. unit.

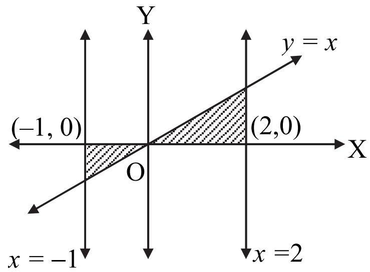
4.



Area bounded by the lines = $\int_0^4 (3x + 1 - 2x - 1)\,dx$

$= \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4$

= 8 sq. units

5. 

$$\text{Required area} = \left|\int_{-1}^{0} y\,dx\right| + \int_{0}^{2} y\,dx$$

$$= \left|\int_{-1}^{0} x\,dx\right| + \int_{0}^{2} x\,dx = \left|\frac{-1}{2}\right| + 2$$

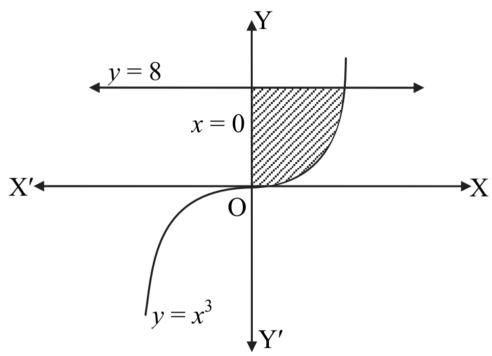
$$= 2 + \frac{1}{2} = \frac{5}{2} \text{ sq. units}$$

Alternate method:

$$\text{Required area} = \frac{1}{2} \times 1 \times 1 + \frac{1}{2} \times 2 \times 2$$

$$= \frac{1}{2} + 2$$

$$= \frac{5}{2} \text{ sq. units.}$$

6. 

$$\text{Required area} = \int_{0}^{8} x\,dy = \int_{0}^{8} y^{1/3}\,dy$$

$$= \frac{3}{4}\left[y^{4/3}\right]_0^8 = \frac{3}{4}(8^{4/3} - 0)$$

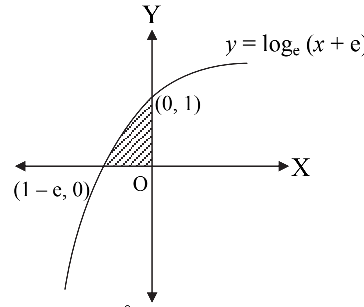
$$= 12$$

7. $$\text{Required area} = \int_{1}^{3} |x - 2|\,dx$$

$$= \int_{1}^{2} (2 - x)\,dx + \int_{2}^{3} (x - 2)\,dx$$

$$= \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^3$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

8. 

$$\text{Required area} = \int_{1-e}^{0} \log_e(x + e)\,dx$$

$$= \int_{1}^{e} \log t\,dt \quad \text{....[Put } x + e = t]$$

$$= \left[t \log t - t\right]_1^e = 1 \text{ sq. unit}$$

9. For X-axis, $y = 0$

$\therefore$ $2x - x^2 = 0$

$\Rightarrow x(2 - x) = 0 \Rightarrow x = 0, 2$

$$\text{Required area} = \int_{0}^{2} \left(2x - x^2\right) dx$$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3} \text{ sq.units}$$

10. For X-axis, $y = 0$

$\therefore$ $1 - x - 6x^2 = 0$

$\Rightarrow (2x + 1)(3x - 1) = 0$

$\Rightarrow x = \frac{-1}{2}$ or $x = \frac{1}{3}$

$\therefore$ $$\text{Required area} = \int_{\frac{-1}{2}}^{\frac{1}{3}} \left(1 - x - 6x^2\right) dx$$

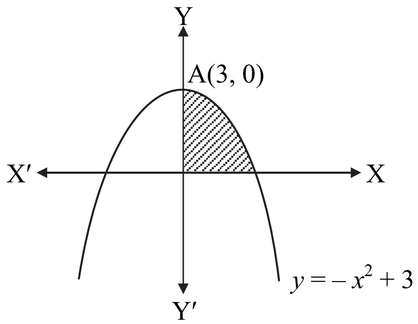
$$= \left[x - \frac{x^2}{2} - 2x^3\right]_{\frac{-1}{2}}^{\frac{1}{3}}$$

$$= \left[\frac{1}{3} - \frac{1}{18} - \frac{2}{27}\right] - \left[\frac{-1}{2} - \frac{1}{8} + \frac{1}{4}\right]$$

$$= \frac{11}{54} + \frac{3}{8}$$

$$= \frac{125}{216} \text{ sq. units.}$$

11.



Area of the required region = $2\int_0^{\sqrt{3}}\left(-x^2+3\right)dx$

$= 2\left[\frac{-x^3}{3} + 3x\right]_0^{\sqrt{3}}$

$= 4\sqrt{3}$ sq. Units

12. In both cases area will be same.

∴ A : B = 1 : 1

13. According to the given condition,

$\int_1^b f(x)\,dx = \sqrt{b^2+1} - \sqrt{2}$

$= \sqrt{b^2+1} - \sqrt{1+1} = [\sqrt{x^2+1}]_1^b$

∴ $f(x) = \frac{d}{dx}\left(\sqrt{x^2+1}\right) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$

14. According to the given condition,

$\int_{\frac{\pi}{4}}^{\beta} f(x)\,dx = \beta\sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta$

Differentiating w.r.t. β, we get

$f(\beta) = \sin\beta + \beta\cos\beta - \frac{\pi}{4}\sin\beta + \sqrt{2}$

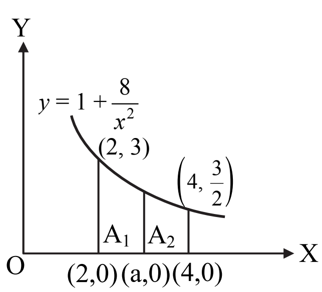
∴ $f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2} - \frac{\pi}{4}\sin\frac{\pi}{2} + \sqrt{2}$

$= 1 - \frac{\pi}{4} + \sqrt{2}$

15. $A_1 = \int_2^a\left(1+\frac{8}{x^2}\right)dx$

$= \left[x - \frac{8}{x}\right]_2^a$

$= a - \frac{8}{a} + 2$



$A_2 = \int_a^4\left(1+\frac{8}{x^2}\right)dx = \left[x - \frac{8}{x}\right]_a^4 = 2 - a + \frac{8}{a}$

According to the given condition,

$a - \frac{8}{a} + 2 = 2 - a + \frac{8}{a}$

$\Rightarrow 2a = \frac{16}{a} \Rightarrow a^2 = 8$

$\Rightarrow a = 2\sqrt{2}$ ....[∵ a > 0]

16. The given curve passes through (1, 2).

∴ 2 = a + b .....(i)

According to the given condition,

$\int_0^4 (a\sqrt{x} + bx)\,dx = 8$

$\Rightarrow \frac{2a}{3}[x^{3/2}]_0^4 + \frac{b}{2}[x^2]_0^4 = 8 \Rightarrow \frac{2a}{3}.8 + 8b = 8$

$\Rightarrow$ 2a + 3b = 3 ....(ii)

From (i) and (ii), we get

a = 3, b = –1

17. $R_1 = \int_{-1}^{2} x f(x)\,dx$

$= \int_{-1}^{2}(1-x)f(1-x)\,dx$

....$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$

$= \int_{-1}^{2}(1-x)f(x)\,dx$

....[∵ f(x) = f(1 – x) (given)]

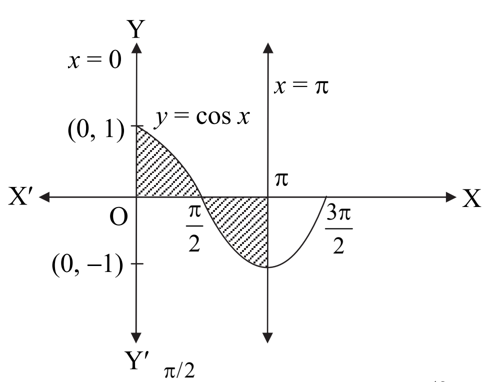
∴ $R_1 = \int_{-1}^{2} f(x)\,dx - R_1 \Rightarrow 2R_1 = \int_{-1}^{2} f(x)\,dx$

According to the given condition,

$R_2 = \int_{-1}^{2} f(x)\,dx$

∴ $R_2 = 2R_1$

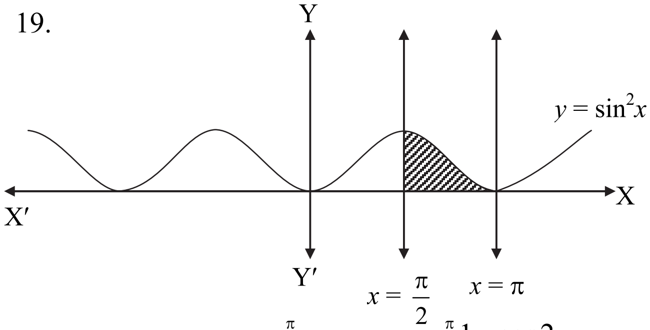
18.



Required area = $2\int_0^{\pi/2}\cos x\,dx = 2[\sin x]_0^{\pi/2}$

= 2 sq. units

19.

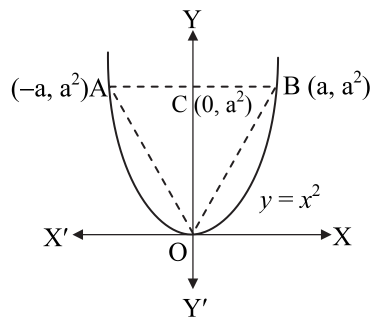


$$\text{Required area} = \int_{\frac{\pi}{2}}^{\pi} \sin^2 x \, dx = \int_{\frac{\pi}{2}}^{\pi} \frac{1-\cos 2x}{2} dx$$

$$= \left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \text{ sq. units}$$

20.



Area of the region AOB

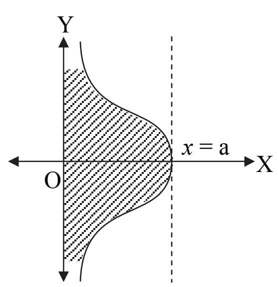
$$= 2\int_0^{a^2} x \, dy = 2\int_0^{a^2} \sqrt{y} \, dy = 2\left[\frac{y^{3/2}}{\frac{3}{2}}\right]_0^{a^2} = \frac{4}{3}a^3$$

Now, area of $\Delta AOB = \frac{1}{2} \times AB \times OC$

$$= \frac{1}{2} \times 2a \times a^2 = a^3$$

$\therefore \quad \frac{\text{Area of } \Delta AOB}{\text{Area of the region AOB}} = \frac{a^3}{\frac{4}{3}a^3} = \frac{3}{4}$

21.



Since the curve is symmetrical about X-axis.

$\therefore \quad$ Required area $A = 2\int_0^a a\sqrt{\frac{a-x}{x}} \, dx$

Put $x = a\sin^2\theta$

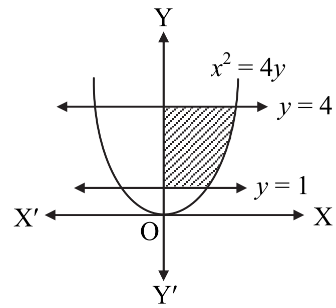
$\Rightarrow dx = 2a \sin\theta.\cos\theta \, d\theta$

$\therefore \quad A = 2\int_0^{\pi/2} a\sqrt{\frac{a\cos^2\theta}{a\sin^2\theta}} . 2a\sin\theta\cos\theta \, d\theta$

$$= 4a^2 \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta} \sin\theta\cos\theta \, d\theta$$

$$= 4a^2 \int_0^{\pi/2} \cos^2\theta \, d\theta = 4a^2 . \frac{1}{2} . \frac{\pi}{2} = \pi a^2$$

22.



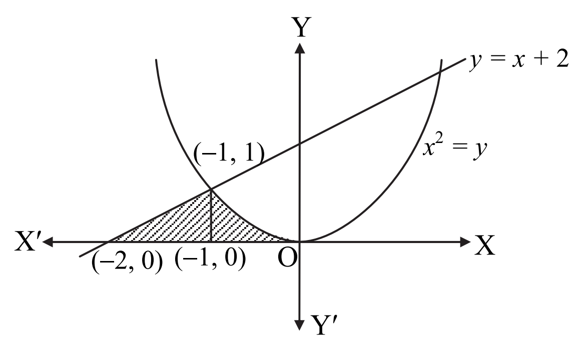
$$\text{Required area} = \int_1^4 x \, dy$$

$$= \int_1^4 2\sqrt{y} \, dy$$

$$= 2\left(\frac{2}{3}\right)\left[y^{3/2}\right]_1^4$$

$$= \frac{28}{3} \text{ sq. units.}$$

23.



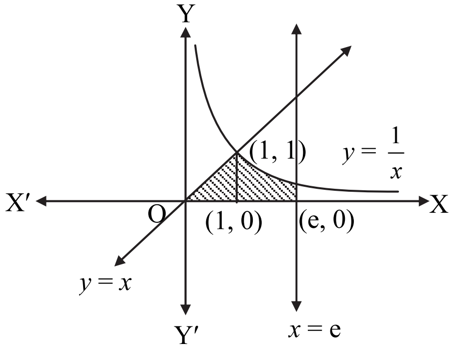
$$\text{Required area} = \int_{-2}^{-1} (x+2) \, dx + \int_{-1}^{0} x^2 \, dx$$

$$= \left[\frac{x^2}{2} + 2x\right]_{-2}^{-1} + \left[\frac{x^3}{3}\right]_{-1}^{0}$$

$$= \left[\left(\frac{1}{2} - 2\right) + 2(-1+2)\right] + \left(0 + \frac{1}{3}\right)$$

$$= \frac{5}{6}$$

24.



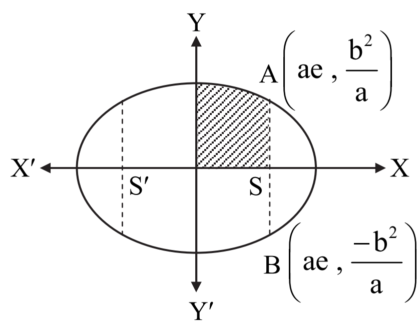
Required area $= \int_0^1 x\,dx + \int_1^e \frac{1}{x}dx$

$= \left[\frac{x^2}{2}\right]_0^1 + \left[\log x\right]_1^e$

$= \frac{1}{2} + 1$

$= \frac{3}{2}$ sq.units

25.



Since the curve is symmetrical about X-axis and Y-axis,

Area of region between the two latus-rectum

= 4 (Area of the shaded region)

$= 4\int_0^{ae} y\,dx$

$= 4\int_0^{ae} \frac{b}{a}\sqrt{a^2 - x^2}\ dx$

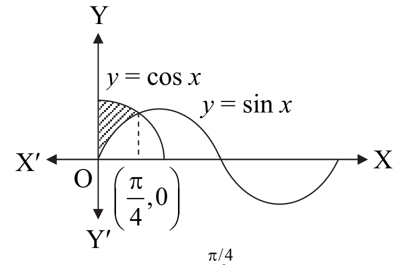
$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^{ae}$

$= \frac{4b}{a}\left[\frac{ae}{2}\sqrt{a^2\left(1-e^2\right)} + \frac{a^2}{2}\sin^{-1} e\right]$

$= \frac{4b}{a}\left[\frac{abe}{2} + \frac{a^2}{2}\sin^{-1} e\right] \ldots\left[\because b = a\sqrt{1-e^2}\right]$

$= 2b\,(be + a\sin^{-1} e)$

26.

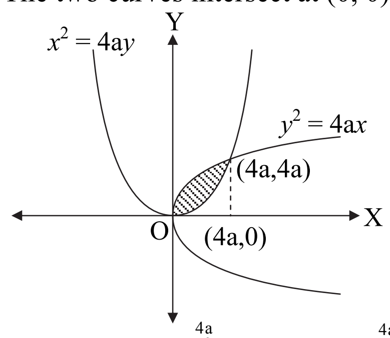


Required area $= \int_0^{\pi/4} (\cos x - \sin x)\ dx$

$= [\sin x + \cos x]_0^{\pi/4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$

27. The two curves intersect at (0, 0) and (4a, 4a).



Required area $= \int_0^{4a} 2a^{1/2}x^{1/2}\ dx - \int_0^{4a} \frac{x^2}{4a}\ dx$

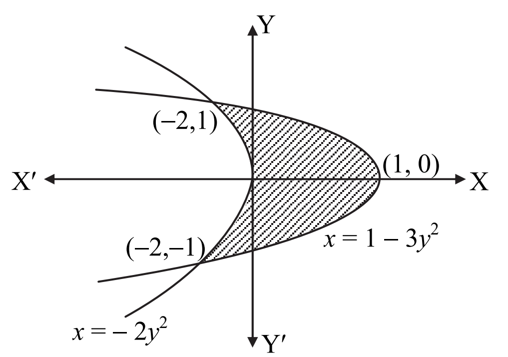
$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$ sq. units

28. The two parabolas intersect at (0, 0) and (1, 1).

$\therefore$ required area $= \int_0^1 \left(\sqrt{x} - x^2\right)dx$

$= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

29.



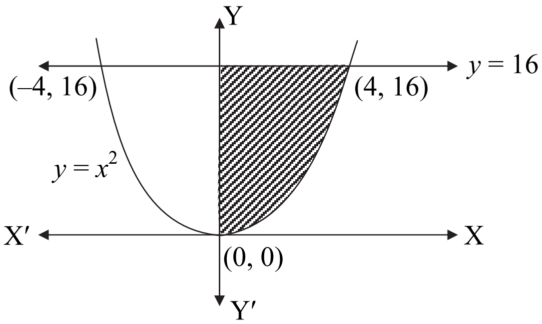
Area bounded by the parabolas

$= 2\int_0^1 \left(1 - 3y^2 + 2y^2\right)dy$

$= 2\int_0^1 \left(1 - y^2\right) dy = 2\left[y - \frac{y^3}{3}\right]_0^1$

$= 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$ sq. units

30.



Area bounded by $y = x^2$ and line $y = 16$ is $2\int_0^4 \left(x^2 - 16\right) dx$

$= 2\left[\frac{x^3}{3} - 16x\right]_0^4 = \frac{-256}{3}$

But area cannot be negative

∴ Required area $= \frac{256}{3}$ sq. units

31. The points of intersection of $y^2 = 4ax$ and $y = 2ax$ are given by

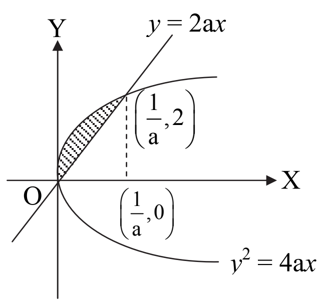
$(2ax)^2 = 4ax$

$\Rightarrow 4ax(ax - 1) = 0$

$\Rightarrow x = 0$ or $x = \frac{1}{a}$

When $x = 0$, $y = 0$ and when $x = \frac{1}{a}$, $y = 2$

∴ the points of intersection are (0, 0) and $\left(\frac{1}{a}, 2\right)$.



Required area $= \int_0^{1/a} \left[\sqrt{4ax} - 2ax\right] dx$

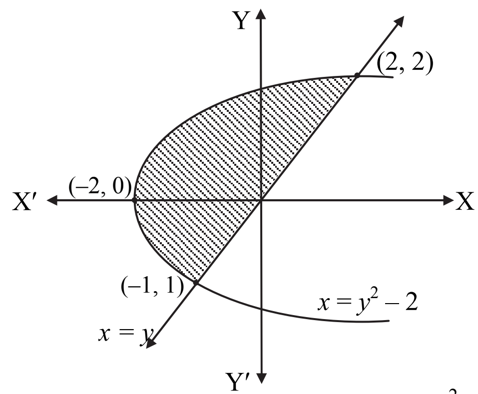
$= \frac{1}{3a}$ sq . units

32. The area of the region bounded by the parabola $y^2 = 4ax$ and the line $y = mx$ is $\frac{8a^2}{3m^3}$ sq. units.

Here, $a = \frac{1}{2}$ and $m = 1$

∴ Required area $= \frac{8\left(\frac{1}{2}\right)^2}{3(1)^3} = \frac{2}{3}$ sq. units.

33.



The points of intersection of $x = y^2 - 2$ and $x = y$ are (–1, –1) and (2, 2)

∴ Required area $= \int_{-1}^{2} \left(y^2 - 2 - y\right) dy$

$= \left[\frac{y^3}{3} - 2y - \frac{y^2}{2}\right]_{-1}^{2}$

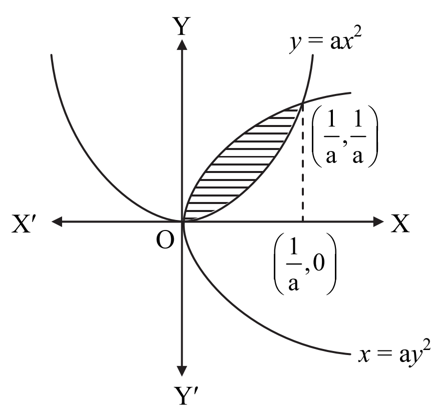
$= \left[\frac{8}{3} - 4 - \frac{4}{2}\right] - \left[-\frac{1}{3} + 2 - \frac{1}{2}\right]$

$= \frac{-9}{2}$

But area cannot be negative.

∴ Required area $= \frac{9}{2}$ sq. Units

34.



The two curves intersect at (0, 0) and $\left(\frac{1}{a}, \frac{1}{a}\right)$.

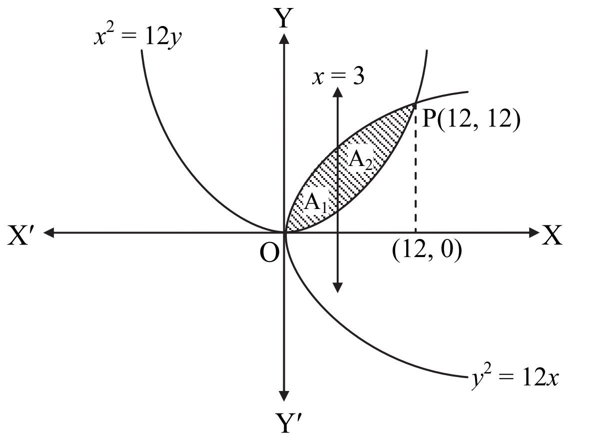
Required area $= \int_0^{\frac{1}{a}} \frac{\sqrt{x}}{\sqrt{a}} dx - \int_0^{\frac{1}{a}} ax^2 dx$

$\Rightarrow 1 = \frac{2}{3\sqrt{a}}\left[x^{\frac{3}{2}}\right]_0^{\frac{1}{a}} - \frac{a}{3}\left[x^3\right]_0^{\frac{1}{a}}$

$= \frac{2}{3\sqrt{a}}\left(\frac{1}{a}\right)^{\frac{3}{2}} - \frac{a}{3}\left(\frac{1}{a}\right)^3 = \frac{2}{3a^2} - \frac{1}{3a^2}$

$\Rightarrow \frac{1}{3a^2} = 1 \Rightarrow a = \frac{1}{\sqrt{3}}$

35.



The given curves intersect at O(0, 0) and P(12, 12).

$A_1 = \int_0^3 \sqrt{12x}\, dx - \int_0^3 \frac{x^2}{12} dx$

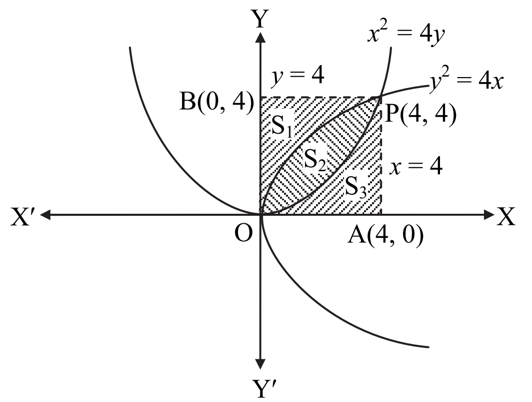
$= \sqrt{12}\left[\frac{2x^{3/2}}{3}\right]_0^3 - \left[\frac{x^3}{36}\right]_0^3 = \frac{45}{4}$

$A_2 = \int_3^{12} \sqrt{12x}\, dx - \int_3^{12} \frac{x^2}{12} dx$

$= \sqrt{12}\left[\frac{2x^{3/2}}{3}\right]_3^{12} - \left[\frac{x^3}{36}\right]_3^{12} = \frac{147}{4}$

$\therefore$ $A_1 : A_2 = 45 : 147 = 15 : 49$

36.



The two parabolas $y^2 = 4x$ and $x^2 = 4y$ intersect at O(0, 0) and P(4, 4).

$\therefore$ $S_3 = \int_0^4 \frac{x^2}{4} dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{16}{3}$

and $S_2 + S_3 = \int_0^4 \sqrt{4x}\, dx = 2\int_0^4 \sqrt{x}\, dx = \frac{4}{3}\left[x^{3/2}\right]_0^4$

$\therefore$ $S_2 + S_3 = \frac{32}{3}$ ….(i)

$\therefore$ $S_2 = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$

Since, $S_1 + S_2 + S_3$ = Area of square OAPB $= 4 \times 4$

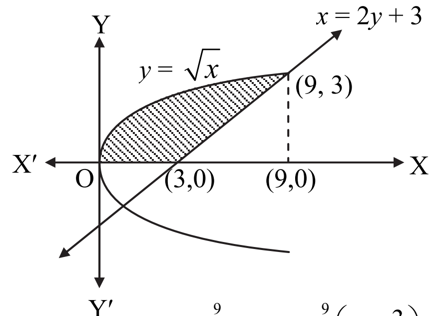
$\therefore$ $S_1 + S_2 + S_3 = 16$

$\Rightarrow S_1 + \frac{32}{3} = 16$ ….[From (i)]

$\Rightarrow S_1 = \frac{16}{3}$

$\therefore$ $S_1 : S_2 : S_3 = 1 : 1 : 1$

37.



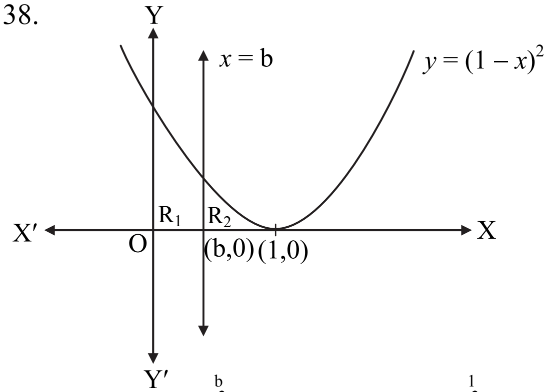
Required area $= \int_0^9 \sqrt{x}\, dx - \int_3^9 \left(\frac{x-3}{2}\right) dx$

$= \left[\frac{2x^{3/2}}{3}\right]_0^9 - \frac{1}{2}\left[\frac{x^2}{2} - 3x\right]_3^9$

$= \frac{2}{3}(27 - 0) - \frac{1}{2}(36 - 18)$

= 9 sq.units

38.



Here, $R_1 = \int_0^b (1-x)^2 dx$ and $R_2 = \int_b^1 (1-x)^2 dx$

$\therefore$ $R_1 = \left[\frac{(x-1)^3}{3}\right]_0^b$ and $R_2 = \left[\frac{(x-1)^3}{3}\right]_b^1$

$\Rightarrow R_1 = \frac{(b-1)^3}{3} + \frac{1}{3}$ and $R_2 = -\frac{(b-1)^3}{3}$

Since, $R_1 - R_2 = \frac{1}{4}$

$\therefore$ $\frac{(b-1)^3}{3} + \frac{1}{3} + \frac{(b-1)^3}{3} = \frac{1}{4}$

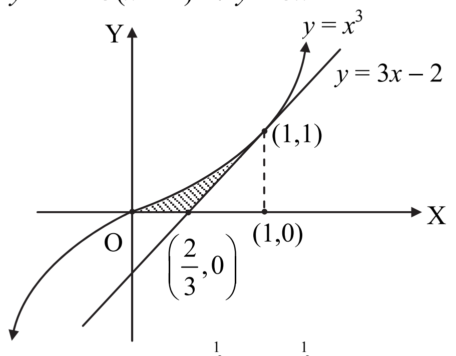
$\Rightarrow \frac{2}{3}(b-1)^3 = -\frac{1}{12} \Rightarrow (b-1)^3 = -\frac{1}{8}$

$\Rightarrow b - 1 = -\frac{1}{2} \Rightarrow b = \frac{1}{2}$

39. $y = x^3$

$\therefore$ $\frac{dy}{dx} = 3x^2 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = 3$

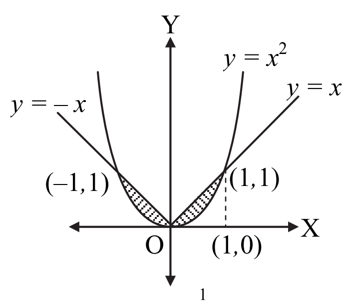
$\therefore$ equation of the tangent at (1, 1) is
$y - 1 = 3(x - 1) \Rightarrow y = 3x - 2$



Required area $= \int_0^1 x^3 dx - \int_{\frac{2}{3}}^1 (3x-2)\,dx$

$= \left[\frac{x^4}{4}\right]_0^1 - \left[\frac{3x^2}{2} - 2x\right]_{\frac{2}{3}}^1 = \frac{1}{12}$

40.

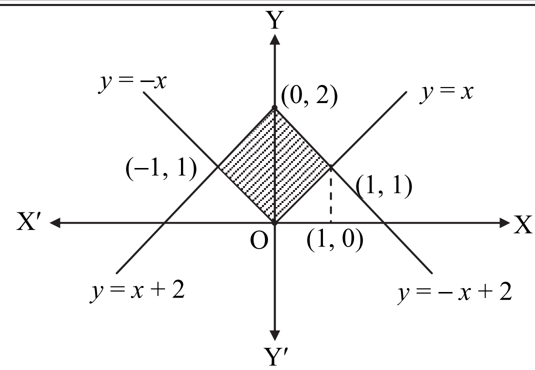


Required area $= 2\int_0^1 (x - x^2)\,dx$

$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$ sq. units

41.

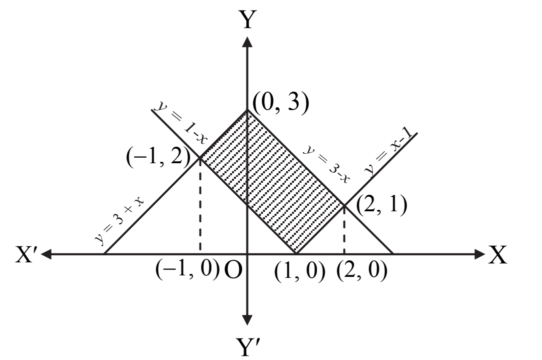


Required area $= 2\int_0^1 (-x + 2 - x)\,dx$

$= 2\int_0^1 (-2x + 2)\,dx$

$= 4\left[\frac{-x^2}{2} + x\right]_0^1 = 2$ sq. units

42.



$3 - |x| = |x - 1| \Rightarrow |x| + |x - 1| = 3 \Rightarrow x = -1, 2$

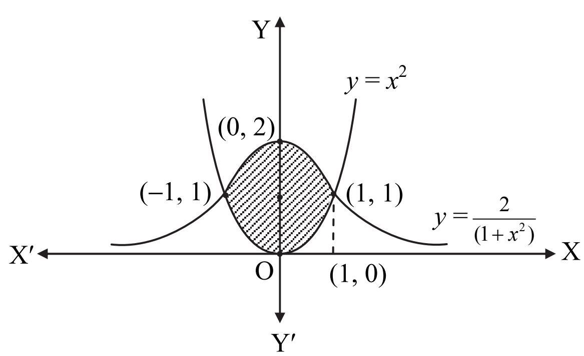
Required area

$= \int_{-1}^{2} (3 - |x| - |x-1|)\,dx$

$= \left[3x - \frac{x|x|}{2} - \frac{(x-1)|x-1|}{2}\right]_{-1}^{2}$

$= \left(6 - 2 - \frac{1}{2}\right) - \left(-3 + \frac{1}{2} + 2\right) = 4$ sq. units

43.



The given curves intersect at (–1, 1) and (1, 1).

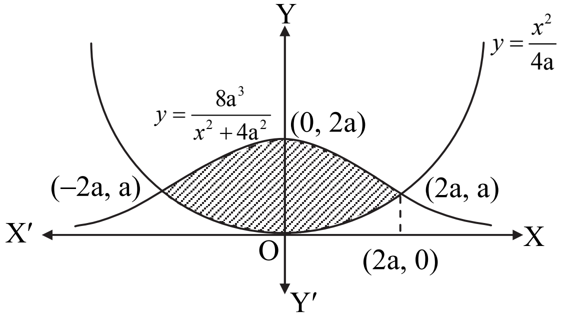
$\therefore$ Required area

$= 2\left[\int_0^1 \frac{2}{(1+x^2)}dx - \int_0^1 x^2\,dx\right]$

$= 2\left\{2\left[\tan^{-1}x\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right\}$

$= 2\left[2\left(\frac{\pi}{4}-0\right)-\left(\frac{1}{3}-0\right)\right] = \pi - \frac{2}{3}$

44.



The given curves intersect at (–2a,a) and (2a,a).

$\therefore$ Required area $= 2\left[\int_0^{2a} \frac{8a^3}{x^2+4a^2}dx - \int_0^{2a}\frac{x^2}{4a}dx\right]$

$= 2\left\{\frac{8a^3}{2a}\left[\tan^{-1}\left(\frac{x}{2a}\right)\right]_0^{2a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{2a}\right\}$

$= 2\left[4a^2\left(\tan^{-1}1-0\right)-\frac{1}{4a}\left(\frac{8a^3}{3}-0\right)\right]$

$= 2\left(4a^2.\frac{\pi}{4}-\frac{2a^2}{3}\right) = a^2\left(2\pi-\frac{4}{3}\right)$

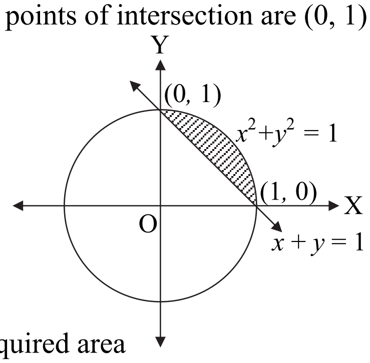
45. The points of intersection of $x^2 + y^2 = 1$ and $x + y = 1$ are given by

$x^2 + (1 - x)^2 = 1$

$\Rightarrow 2x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$

When $x = 0$, $y = 1$ and when $x = 1$, $y = 0$

$\therefore$ the points of intersection are (0, 1) and (1, 0).



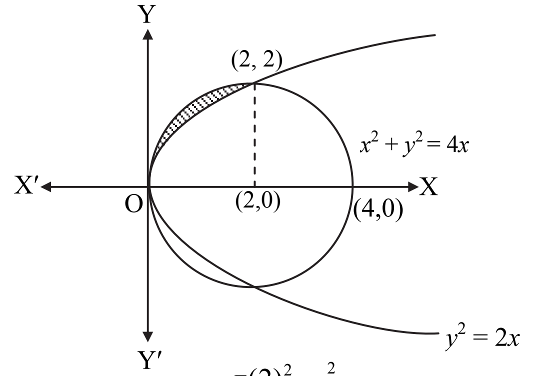
Required area

$= \int_0^1\left[\sqrt{1-x^2}-(1-x)\right]dx$

$= \left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x - x + \frac{x^2}{2}\right]_0^1$

$= \frac{1}{2}.\frac{\pi}{2} - 1 + \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$

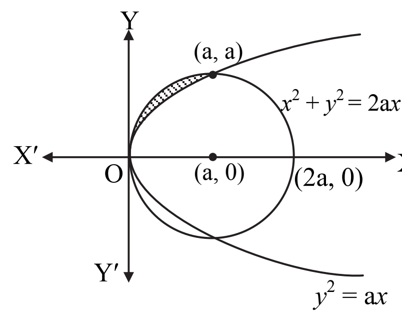
46.



Required area $= \frac{\pi(2)^2}{4} - \int_0^2 \sqrt{2x}\,dx$

$= \pi - \frac{2\sqrt{2}}{3}\left[x^{\frac{3}{2}}\right]_0^2 = \pi - \frac{8}{3}$

47.

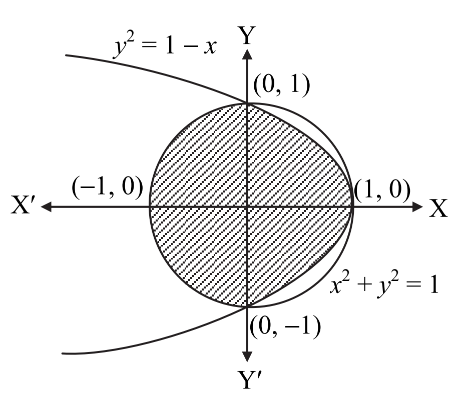


Required area $= \frac{\pi a^2}{4} - \int_0^a \sqrt{ax}\,dx$

$= \frac{\pi a^2}{4} - \frac{2\sqrt{a}}{3}\left[x^{\frac{3}{2}}\right]_0^a$

$= \frac{\pi a^2}{4} - \frac{2a^2}{3} = a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$ sq. units.

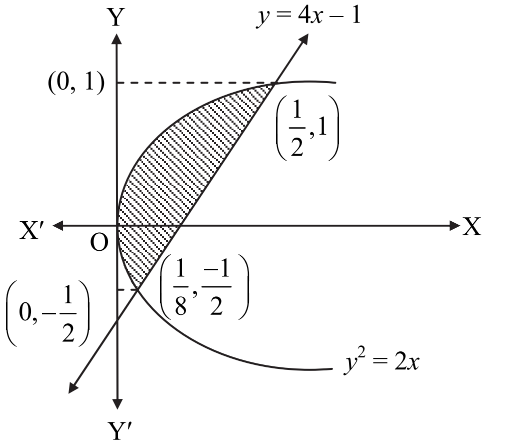
48.



Required area $= \frac{1}{2}\times\pi + 2\int_0^1 \sqrt{1-x}\,dx$

$= \frac{\pi}{2} + 2\left[\frac{(1-x)^{\frac{3}{2}}}{-\frac{3}{2}}\right]_0^1 = \frac{\pi}{2} - \frac{4}{3}(0-1) = \frac{\pi}{2} + \frac{4}{3}$

49.



Putting $x = \frac{y^2}{2}$ in $y = 4x - 1$, we get

$y = 4\left(\frac{y^2}{2}\right) - 1 \Rightarrow 2y^2 - y - 1 = 0$

$\Rightarrow (y - 1)(2y + 1) = 0 \Rightarrow y = 1, \frac{-1}{2}$

$\therefore$ required area $= \int_{-1/2}^{1}\left(\frac{y+1}{4}\right)dy - \int_{-1/2}^{1}\frac{y^2}{2}dy$

$= \frac{1}{4}\left[\frac{y^2}{2} + y\right]_{-1/2}^{1} - \frac{1}{2}\left[\frac{y^3}{3}\right]_{-1/2}^{1}$

$= \frac{1}{4}\left[\left(\frac{1}{2} - \frac{1}{8}\right) + \left(1 + \frac{1}{2}\right)\right] - \frac{1}{2}\left(\frac{1}{3} + \frac{1}{24}\right)$

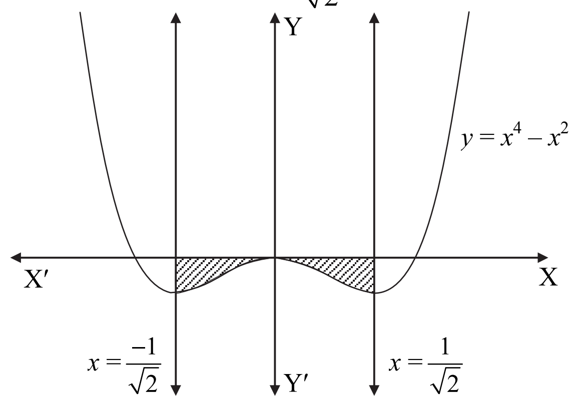
$= \frac{1}{4}\left(\frac{15}{8}\right) - \frac{1}{2}\left(\frac{9}{24}\right) = \frac{15}{32} - \frac{3}{16} = \frac{9}{32}$

50. $y = x^4 - x^2$

$\therefore$ $\frac{dy}{dx} = 4x^3 - 2x$

for minimum, $\frac{dy}{dx} = 0$

$\Rightarrow 4x^3 - 2x = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$



Required area $= 2\int_{0}^{\frac{1}{\sqrt{2}}}\left(x^4 - x^2\right)dx$

$= 2\left[\frac{x^5}{5} - \frac{x^3}{3}\right]_0^{\frac{1}{\sqrt{2}}}$

$= 2\left[\frac{1}{5\left(\sqrt{2}\right)^5} - \frac{1}{3\left(\sqrt{2}\right)^3}\right]$

$= \frac{2}{\left(\sqrt{2}\right)^3}\left[\frac{1}{5\left(\sqrt{2}\right)^2} - \frac{1}{3}\right]$

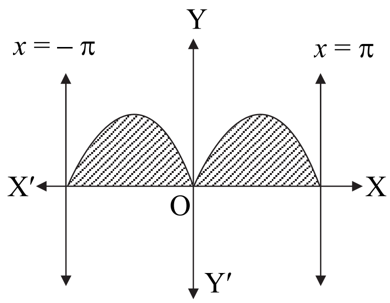
$= \frac{1}{\sqrt{2}}\left[\frac{1}{10} - \frac{1}{3}\right] = \frac{-7}{30\sqrt{2}}$

But, area cannot be negative.

$\therefore$ Required area $= \frac{7}{30\sqrt{2}}$

Evaluation Test

1.



Required area $= \int_{-\pi}^{\pi}|\sin x|\,dx$

$= \int_{-\pi}^{0}|\sin x|\,dx + \int_{0}^{\pi}|\sin x|\,dx$

$= \int_{-\pi}^{0}(-\sin x)\,dx + \int_{0}^{\pi}\sin x\,dx$

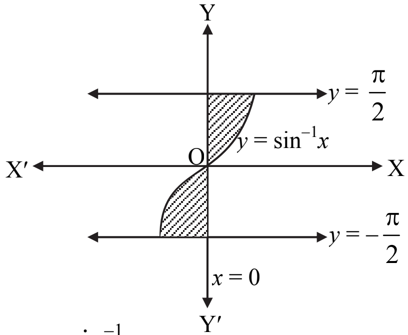
$= [\cos x]_{-\pi}^{0} - [\cos x]_{0}^{\pi}$

$= \cos 0 - \cos(-\pi) - (\cos\pi - \cos 0)$

$= 1 - (-1) - (-1 - 1)$

$= 1 + 1 - (-2)$

$= 4$ sq. units

2. 

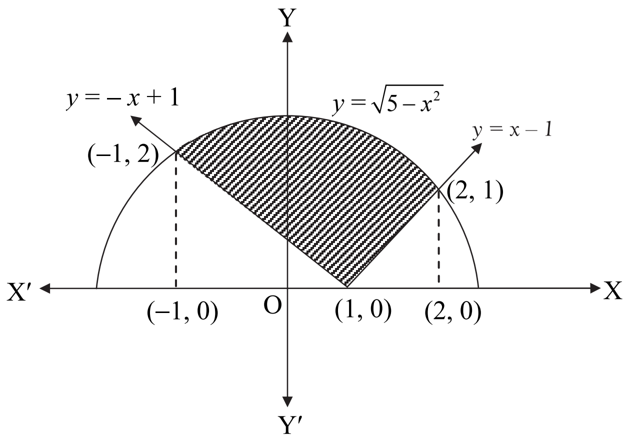
$y = \sin^{-1} x$

$\therefore \quad x = \sin y$

Required area $= 2\int_0^{\frac{\pi}{2}} x\, dy = 2\int_0^{\frac{\pi}{2}} \sin y\, dy$

$= -2\left[\cos y\right]_0^{\pi/2}$

$= -2(0 - 1) = 2$ sq.units

3. 

Required area $= \int_{-1}^{2} \sqrt{5 - x^2}\, dx - \int_{-1}^{1} (1 - x)\, dx$

$- \int_{1}^{2} (x - 1)\, dx$

$= \left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_{-1}^{2} - \left[x - \frac{x^2}{2}\right]_{-1}^{1}$

$- \left[\frac{x^2}{2} - x\right]_1^2$

$= 1 + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \left(-\frac{1}{2}\times 2 + \frac{5}{2}\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right)$

$- \left[1 - \frac{1}{2} - \left(-1 - \frac{1}{2}\right)\right] - \left(2 - 2 - \frac{1}{2} + 1\right)$

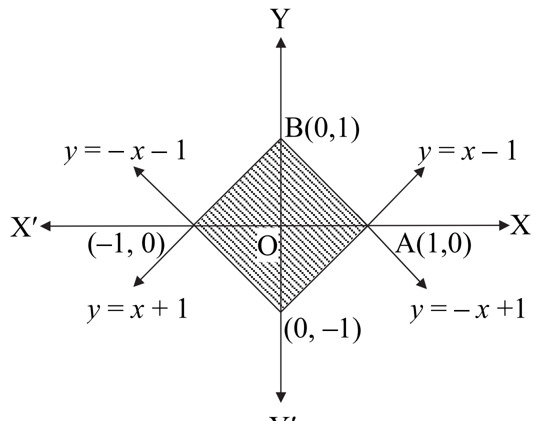
$= 1 + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) + 1 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - 2 - \frac{1}{2}$

$= \frac{5}{2}\left[\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right] - \frac{1}{2}$

$= \frac{5}{2}\left[\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) + \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right] - \frac{1}{2}$

....$\left[\because \sin^{-1} x = \cos^{-1}\sqrt{1 - x^2}\right]$

$= \frac{5}{2}\cdot\frac{\pi}{2} - \frac{1}{2} = \left(\frac{5\pi}{4} - \frac{1}{2}\right)$ sq.unit

4. 

Required area = 4 (area of Δ OAB)

$= 4\int_0^1 y\, dx$

$= 4\int_0^1 (-x + 1) dx$

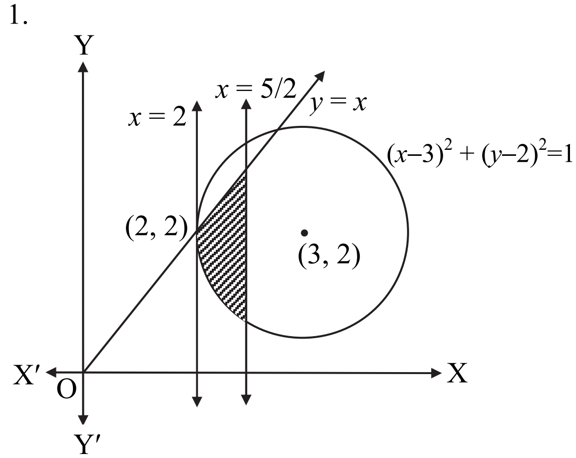
$= 4\left[\frac{-x^2}{2} + x\right]_0^1 = 2$ sq. units

5. The given equation can be written as

$x^2 - 6x + 9 + y^2 - 4y + 4 = 9 + 4 - 12$

$\therefore \quad (x - 3)^2 + (y - 2)^2 = 1$

This is a circle with centre at (3, 2) and radius 1.



$\therefore$ Required area

$= \int_2^{\frac{5}{2}} x\, dx - \int_2^{\frac{5}{2}} \left[2 - \sqrt{1 - (x - 3)^2}\, dx\right]$

$$= \left[\frac{x^2}{2}\right]_2^{5/2} - \left[2x\right]_2^{5/2}$$

$$+ \left[\left(\frac{x-3}{2}\right)\sqrt{1-(x-3)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-3}{1}\right)\right]_2^{5/2}$$

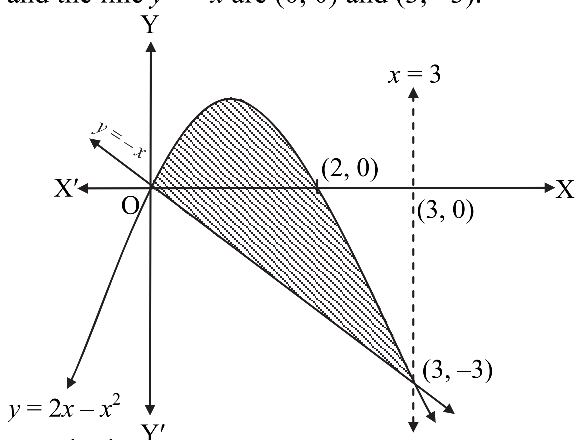
$$= \frac{1}{2}\left(\frac{25}{4}-4\right) - 2\left(\frac{5}{2}-2\right) + \left(\frac{-\frac{1}{2}}{2}\right)\sqrt{1-\frac{1}{4}}$$

$$+\frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right) - \left[0+\frac{1}{2}\sin^{-1}(-1)\right]$$

$$= \frac{9}{8} - 1 - \frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{\pi}{6}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right)$$

$$= \frac{1}{8} - \frac{\sqrt{3}}{8} + \frac{\pi}{4} - \frac{\pi}{12} = \left(\frac{\pi}{6} - \frac{\sqrt{3}-1}{8}\right) \text{sq.unit}$$

6. The point of intersection of the curve $y = 2x - x^2$ and the line $y = -x$ are (0, 0) and (3, –3).

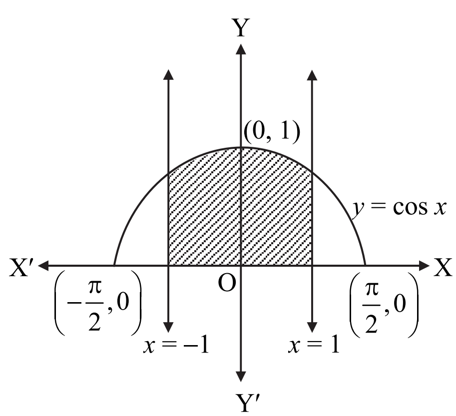


∴ Required area

$$= \int_0^3 [(2x - x^2) - (-x)]\mathrm{d}x$$

$$= \int_0^3 (3x - x^2)\,\mathrm{d}x = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{9}{2} \text{ sq. unit}$$

7.



$|x| = 1 \Rightarrow x = 1$ or $x = -1$

Required Area $= \int_{-1}^{1} \cos x\,\mathrm{d}x$

$$= 2\int_0^1 \cos x\,\mathrm{d}x$$

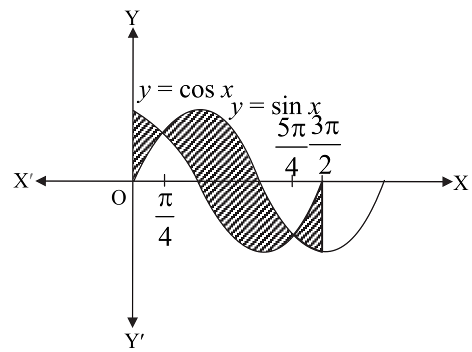
....[$\because$ cos x is an even f[n]]

$$= 2\left[\sin x\right]_0^1$$

$= 2(\sin 1 - 0)$

$= 2 \sin 1$

8.



Required area

$$= \int_0^{\frac{3\pi}{2}} |\cos x - \sin x|\,\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\mathrm{d}x$$

$$+ \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (\cos x - \sin x)\mathrm{d}x$$

$$= \left[\sin x + \cos x\right]_0^{\pi/4} - \left[\cos x + \sin x\right]_{\pi/4}^{5\pi/4}$$

$$+ \left[\sin x + \cos x\right]_{5\pi/4}^{3\pi/2}$$

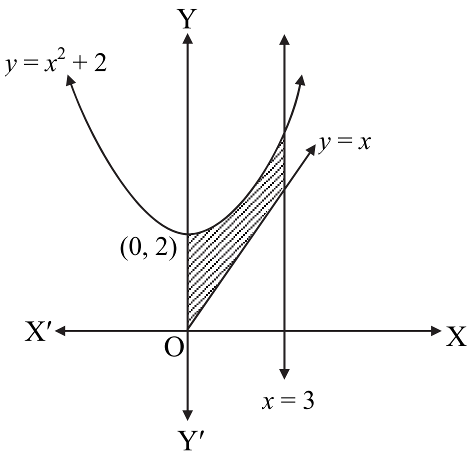
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0+1) - \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right]$$

$$+ (-1) + 0 - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2} - 1 + \sqrt{2} + \sqrt{2} - 1 + \sqrt{2}$$

$$= \left(4\sqrt{2} - 2\right) \text{ sq.unit}$$

9.

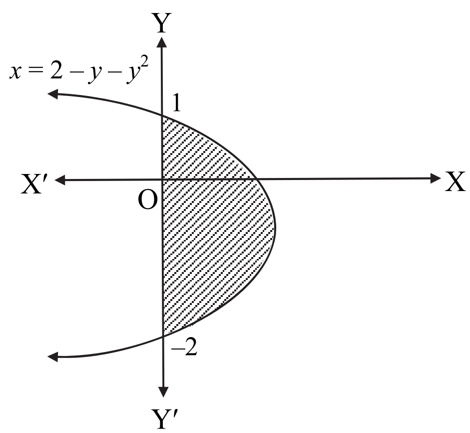


Required area $= \int_0^3 (x^2 + 2 - x)dx$

$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3$

$= 9 + 6 - \frac{9}{2} - 0 = \frac{21}{2}$ sq. unit

10.



Putting $x = 0$ in the given equation, we get $y = 1$ or $y = -2$

$\therefore$ Required Area $= \int_{-2}^{1} x\,dy = \int_{-2}^{1} (2 - y - y^2)dy$

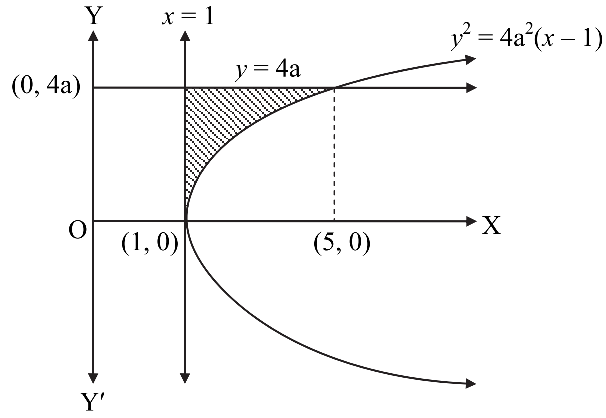
$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$

$= 2 - \frac{1}{2} - \frac{1}{3} - \left[-4 - 2 + \frac{8}{3}\right]$

$= 2 - \frac{1}{2} - \frac{1}{3} + 6 - \frac{8}{3}$

$= 8 - 3 - \frac{1}{2} = \frac{9}{2}$ sq.units

11.



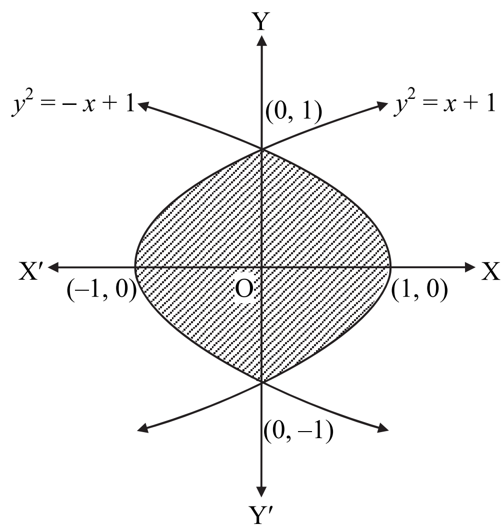
Required area $= \int_0^{4a} \left(\frac{y^2}{4a^2} + 1 - 1\right)dy$

$= \int_0^{4a} \frac{y^2}{4a^2}dy = \frac{1}{4a^2}\left[\frac{y^3}{3}\right]_0^{4a}$

$= \frac{1}{4a^2} \times \frac{1}{3} \times (64a^3 - 0)$

$= \frac{16a}{3}$ sq. units

12.



Required area

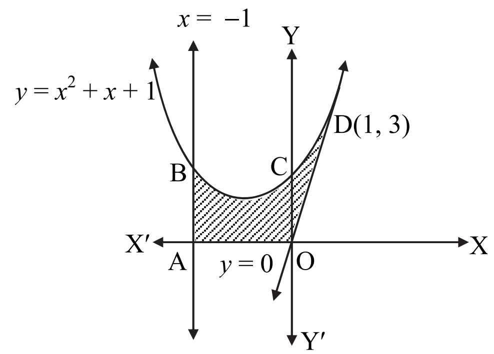
$= 2\int_{-1}^{0} \sqrt{x+1}\,dx + 2\int_0^1 \sqrt{-x+1}\,dx$

$= 2.\frac{2}{3}\left[(x+1)^{3/2}\right]_{-1}^{0} + 2.\frac{2}{3}\left[\frac{(-x+1)^{3/2}}{-1}\right]_0^1$

$= \frac{4}{3} + \frac{4}{3}$

$= \frac{8}{3}$ sq. units

13.



$y = x^2 + x + 1 \Rightarrow \frac{dy}{dx} = 2x + 1$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(1,3)} = 2(1) + 1 = 2 + 1 = 3$

$\therefore$ The equation of the tangent at the point (1, 3) is $y - 3 = 3(x - 1)$ i.e., $y = 3x$.

$\therefore$ It passes through origin.

$\therefore$ Required area

= area of the region OABCO + area of the region OCDO

$= \int_{-1}^{0} y\,dx + \int_{0}^{1} (y_1 - y_2)\,dx$

$= \int_{-1}^{0} (x^2 + x + 1)\,dx + \int_{0}^{1} (x^2 + x + 1 - 3x)\,dx$

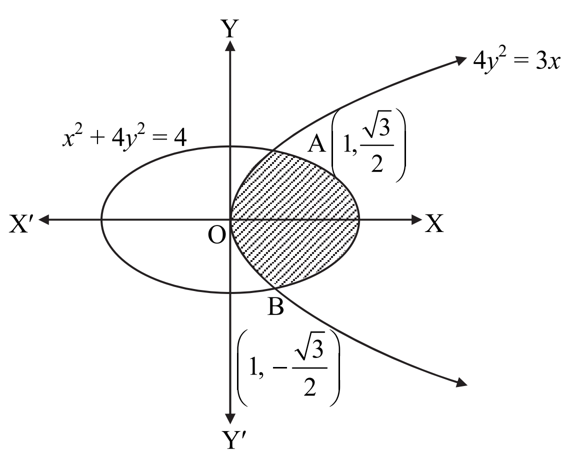
$= \int_{-1}^{0} (x^2 + x + 1)\,dx + \int_{0}^{1} (x^2 - 2x + 1)\,dx$

$= \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{0} + \left[\frac{x^3}{3} - x^2 + x\right]_{0}^{1}$

$= 0 - \left(-\frac{1}{3} + \frac{1}{2} - 1\right) + \frac{1}{3} - 1 + 1 - 0$

$= \frac{4}{3} - \frac{1}{2} + \frac{1}{3} = \frac{8 - 3 + 2}{6} = \frac{7}{6}$ sq. unit

14.



The equation $x^2 + 4y^2 = 4$ is of ellipse with centre at origin and the equation $4y^2 = 3x$ is of a parabola with vertex at origin.

Solving the equations, we get $x^2 + 3x - 4 = 0$

$\therefore$ $(x + 4)(x - 1) = 0$

But $x = -4$ is not possible, since both points of intersection lie on the right hand side of Y-axis.

$\therefore$ $x = 1$ and $y = \pm \frac{\sqrt{3}}{2}$

$\therefore$ The points of intersection are $A\left(1, \frac{\sqrt{3}}{2}\right)$ and $B\left(1, -\frac{\sqrt{3}}{2}\right)$.

$\therefore$ Required area

$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} (x_2 - x_1)\,dy$

$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left[\sqrt{4 - 4y^2} - \frac{4y^2}{3}\right] dy$

$= 2\int_{0}^{\frac{\sqrt{3}}{2}} \left[\sqrt{4 - 4y^2} - \frac{4y^2}{3}\right] dy$

….[$\because$ the function is even]

$= 4\int_{0}^{\frac{\sqrt{3}}{2}} \sqrt{1 - y^2}\,dy - \frac{8}{3}\int_{0}^{\frac{\sqrt{3}}{2}} y^2\,dy$

$= 4\left[\frac{y}{2}\sqrt{1 - y^2} + \frac{1}{2}\sin^{-1}(y)\right]_{0}^{\frac{\sqrt{3}}{2}} - \frac{8}{3}\left[\frac{y^3}{3}\right]_{0}^{\frac{\sqrt{3}}{2}}$

$= 2\left[\frac{\sqrt{3}}{2}\sqrt{\frac{1}{4}} + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - 0\right] - \frac{8}{3 \times 3} \times \left(\frac{3\sqrt{3}}{8} - 0\right)$

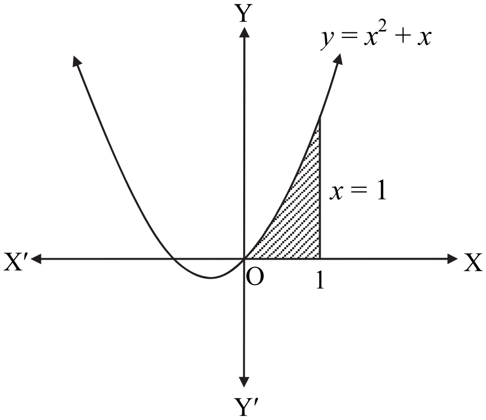
$= 2\left[\frac{\sqrt{3}}{4} + \frac{\pi}{3}\right] - \frac{8}{9} \times \frac{3\sqrt{3}}{8}$

$= \frac{2\pi}{3} + \frac{2\sqrt{3}}{4} - \frac{\sqrt{3}}{3}$

$= \frac{2\pi}{3} + \frac{2\sqrt{3}}{12}$

$= \left(\frac{2\pi}{3} + \frac{1}{2\sqrt{3}}\right)$ sq. unit

15.



Slope of tangent = $\frac{dy}{dx} = 2x + 1$

$\therefore$ $y = \int(2x+1)dx = x^2 + x + c$

The curve passes through the point (1, 2).

$\therefore$ $2 = 1^2 + 1 + c$ $\quad\therefore\quad c = 0$

$\therefore$ The equation of the curve is $y = x^2 + x$, which is a parabola as shown in the figure.

$\therefore$ Required area = $\int_0^1 (x^2 + x)dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$

$= \frac{1}{3} + \frac{1}{2} - 0$

$= \frac{5}{6}$ sq. unit

16. Draw AP $\perp$ to X-axis.

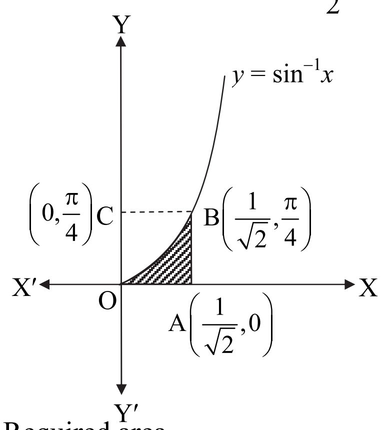
$A_1 = A(\Delta OAP) = \frac{1}{2} \times a \times a^2 = \frac{a^3}{2}$

A_2 = Area bounded the curve OA and the lines OP and AP

$= \int_0^a y\,dx = \int_0^a x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$

$\therefore$ Required area $= A_1 - A_2 = \frac{a^3}{2} - \frac{a^3}{3} = \frac{a^3}{6}$ sq. unit

17.



Required area

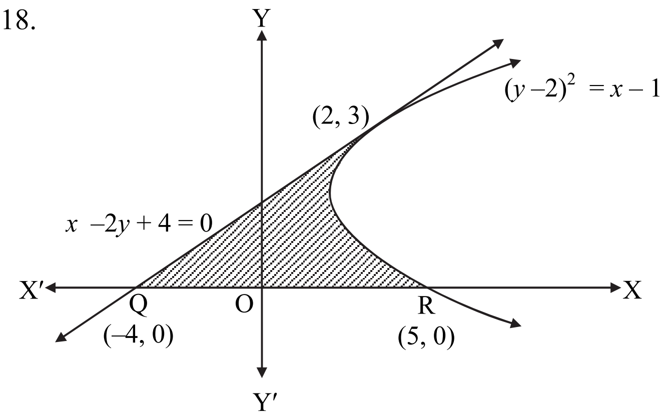
= area of the rectangle OABC

− area of the region OBCO

$= \frac{\pi}{4} \times \frac{1}{\sqrt{2}} - \int_0^{\frac{\pi}{4}} \sin y\,dy = \frac{\pi}{4\sqrt{2}} - [-\cos y]_0^{\pi/4}$

$= \left[\frac{\pi}{4\sqrt{2}} + \left(\frac{1}{\sqrt{2}} - 1\right)\right]$ sq. units

18.



The equation of the parabola is

$(y - 2)^2 = x - 1$

Diff. w.r.t. x, we get

$2(y-2)\frac{dy}{dx} = 1$

$\therefore$ $\frac{dy}{dx} = \frac{1}{2(y-2)}$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(2,3)} = \frac{1}{2(3-2)} = \frac{1}{2}$

$\therefore$ Equation of tangent is $y - 3 = \frac{1}{2}(x - 2)$

$\therefore$ $2y - 6 = x - 2$

$\therefore$ $x - 2y + 4 = 0$

It cuts the X-axis at the point Q (–4, 0) and the parabola cuts the X-axis at the point R(5, 0).

$\therefore$ required area = $\int_0^3 (x_1 - x_2)dy$

$= \int_0^3 \left[(y-2)^2 + 1 - (2y-4)\right]dy$

$= \int_0^3 \left(y^2 - 6y + 9\right)dy$

$= \left[\frac{y^3}{3} - 3y^2 + 9y\right]_0^3$

$= 9 - 27 + 27 - 0$

= 9 sq. units

19. Required area = $\int_0^{\frac{\pi}{4}}\left(\sqrt{\frac{1+\sin x}{\cos x}}-\sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$\ldots\left[\because \frac{1+\sin x}{\cos x}>\frac{1-\sin x}{\cos x}>0\right]$

$$=\int_0^{\frac{\pi}{4}}\left(\sqrt{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}}-\sqrt{\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}}\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\left(\sqrt{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}}-\sqrt{\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}}\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}}-\sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\frac{1+\tan\frac{x}{2}-1+\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx = \int_0^{\frac{\pi}{4}}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$\therefore$ required area = $\int_0^{\tan\frac{\pi}{8}}\frac{4t}{\left(1+t^2\right)\sqrt{1-t^2}}dt$

$$=\int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$$

$\ldots\left[\because \tan\frac{\pi}{8}=\sqrt{2}-1\right]$

Textbook Chapter No.

07 Differential Equations

Hints

Classical Thinking

1. Here, the highest order derivative is $\frac{d^2s}{dt^2}$ with power 2.

$\therefore$ order = 2 and degree = 2

2. Here, the highest order derivative is $\frac{d^2y}{dx^2}$ with power 3.

$\therefore$ order = 2 and degree = 3

3. Here, the highest order derivative is $\frac{d^2y}{dx^2}$ with power 3.

$\therefore$ order = 2 and degree = 3

4. Here, the highest order derivative is $\frac{d^3y}{dx^3}$ with power 1.

$\therefore$ order = 3 and degree = 1

5. In option (B), y'' is the highest order derivative, of order 2.

$\therefore$ option (B) is the correct answer.

6. Here, the highest order derivative is $\frac{d^4y}{dx^4}$ with power 1.

$\therefore$ order = 4 and degree = 1

7. $y = 4 \sin 3x$ ....(i)

$\Rightarrow \frac{dy}{dx} = 12 \cos 3x$

$\Rightarrow \frac{d^2y}{dx^2} = -36 \sin 3x = -9 \times 4 \sin 3x = -9y$ ....[From (i)]

$\Rightarrow \frac{d^2y}{dx^2} + 9y = 0$

8. $y = A \sin x + B \cos x$ ….(i)

$\Rightarrow \frac{dy}{dx} = A \cos x - B \sin x$

$\Rightarrow \frac{d^2y}{dx^2} = -A \sin x - B \cos x$

$= -(A \sin x + B \cos x)$

$= -y$ ….[From (i)]

$\therefore \frac{d^2y}{dx^2} + y = 0$

9. $y = a \cos (x + b)$ .....(i)

$\Rightarrow \frac{dy}{dx} = -a \sin(x + b)$

$\Rightarrow \frac{d^2y}{dx^2} = -a \cos(x + b) = -y$ ....[From (i)]

$\Rightarrow \frac{d^2y}{dx^2} + y = 0$

10. $y = ce^{\sin^{-1} x}$ ....(i)

$\Rightarrow \frac{dy}{dx} = ce^{\sin^{-1} x} . \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}$ ....[From (i)]

11. $y = (x + k)e^{-x}$ ....(i)

$\Rightarrow \frac{dy}{dx} = -(x + k) e^{-x} + e^{-x}$

$\Rightarrow \frac{dy}{dx} = -y + e^{-x}$ ....[From (i)]

$\Rightarrow \frac{dy}{dx} + y = e^{-x}$

12. $x^2y = a$

Differentiating w.r.t. x, we get

$x^2 \frac{dy}{dx} + 2xy = 0$

$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = 0$

13. $x^2 + y^2 = a^2$

Differentiating w.r.t. x, we get

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow x + y\frac{dy}{dx} = 0$

14. $x^2 \frac{dy}{dx} = 2$

Integrating on both sides, we get

$\int dy = \int \frac{2}{x^2} dx + c$

$\Rightarrow y = -\frac{2}{x} + c$

15. $\frac{dy}{dx} = x^2 + \sin 3x$

Integrating on both sides, we get

$\int dy = \int\left(x^2 + \sin 3x\right)dx + c$

$\Rightarrow y = \frac{x^3}{3} - \frac{\cos 3x}{3} + c$

16. $\frac{dy}{dx} = (ae^{bx} + c\cos mx)$

Integrating on both sides, we get

$\int dy = \int\left(ae^{bx} + c\cos mx\right)dx + k$

$\Rightarrow y = \frac{ae^{bx}}{b} + \frac{c\sin(mx)}{m} + k$

17. $\frac{dy}{dx} = \sec x(\sec x + \tan x)$

Integrating on both sides, we get

$\int dy = \int\left(\sec^2 x + \sec x \tan x\right)dx + c$

$\Rightarrow y = \tan x + \sec x + c$

18. $\frac{dy}{dx} = e^x(\sin x + \cos x)$

Integrating on both sides, we get

$\int dy = \int e^x\left(\sin x + \cos x\right)dx + c$

$\Rightarrow y = e^x \sin x + c$

19. $\frac{dy}{dx} = e^x + \cos x + x + \tan x$

Integrating on both sides, we get

$\int dy = \int\left(e^x + \cos x + x + \tan x\right)dx + c$

$\Rightarrow y = e^x + \sin x + \frac{x^2}{2} + \log \sec x + c$

20. $(1 + x^2)\frac{dy}{dx} = 1$

Integrating on both sides, we get

$\int dy = \int \frac{1}{1+x^2}dx + c$

$\Rightarrow y = \tan^{-1} x + c$

21. $\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} = 0$

Integrating on both sides, we get

$\int dy + \int \frac{1}{\sqrt{1-x^2}}dx = c$

$\Rightarrow y + \sin^{-1} x = c$

22. $\frac{dy}{dx} + \sin^2 y = 0$

$\Rightarrow \frac{dy}{dx} = -\sin^2 y$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{cosec}^2 y}$

Integrating on both sides, we get

$\int dx = -\int \operatorname{cosec}^2 y\, dy + c$

$\Rightarrow x = \cot y + c$

23. $\frac{dy}{dx} + \frac{1+x^2}{x} = 0$

Integrating on both sides, we get

$\int dy + \int\left(\frac{1}{x} + x\right)dx = c$

$\Rightarrow y + \log x + \frac{x^2}{2} = c$

24. $(1 + x^2)\frac{dy}{dx} = x \Rightarrow dy = \frac{x}{1+x^2}dx$

Integrating on both sides, we get

$\int dy = \int \frac{x}{1+x^2}dx + c$

$\Rightarrow y = \frac{1}{2}\log_e(1 + x^2) + c$

25. $\frac{dy}{dx} = \left(\frac{y}{x}\right)^{1/3}$

$\Rightarrow \frac{dy}{y^{1/3}} = \frac{dx}{x^{1/3}}$

Integrating on both sides, we get

$\int \frac{dy}{y^{1/3}} - \int \frac{dx}{x^{1/3}} = c_1$

$\Rightarrow \frac{3}{2}y^{2/3} - \frac{3}{2}x^{2/3} = c_1$

$\Rightarrow y^{2/3} - x^{2/3} = c$, where $c = \frac{2c_1}{3}$

26. $\frac{dy}{dx} = (1 + x)(1 + y^2)$

Integrating on both sides, we get

$\int \frac{dy}{1+y^2} = \int (1+x)dx + c$

$\Rightarrow \tan^{-1} y = \frac{x^2}{2} + x + c$

$\Rightarrow y = \tan\left(\frac{x^2}{2} + x + c\right)$

27. $\frac{dy}{dx} = x \log x \Rightarrow dy = x \log x dx$

Integrating on both sides, we get

$\int dy = \int x \log x dx + c$

$\Rightarrow y = \frac{x^2}{2} \log x - \frac{x^2}{4} + c$

28. $\frac{dy}{dx} = (x+y)^2$ ....(i)

Put $x + y = v$ ....(ii)

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$\frac{dv}{dx} - 1 = v^2$

$\Rightarrow \frac{dv}{dx} = v^2 + 1$

$\Rightarrow \frac{dv}{v^2+1} = dx$

Integrating on both sides, we get

$\tan^{-1} v = x + c \Rightarrow v = \tan(x + c)$

$\Rightarrow x + y = \tan(x + c)$

29. $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$

This is the linear differential equation of the form

$\frac{dy}{dx} + P.y = Q$, where $P = \frac{1}{x}$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

30. The given equation is of the form

$\frac{dy}{dx} + Py = Q.$

Here, $P = \frac{1}{3}$ and $Q = 1$

$\therefore$ I.F. $= e^{\int \frac{1}{3} dx} = e^{\frac{x}{3}}$

$\therefore$ solution of the given equation is

$y(\text{I.F.}) = \int Q(\text{I.F.}) dx + c$

$\Rightarrow y.\ e^{\frac{x}{3}} = \int 1.e^{\frac{x}{3}} dx + c$

$\Rightarrow y.\ e^{\frac{x}{3}} = 3e^{\frac{x}{3}} + c$

$\Rightarrow y = 3 + c.\,e^{-\frac{x}{3}}$

31. $\log\left(\frac{dy}{dx}\right) = x + y$

$\Rightarrow \frac{dy}{dx} = e^{x+y}$

$\Rightarrow \frac{dy}{dx} = e^x.e^y$

Integrating on both sides, we get

$\int e^x dx - \int e^{-y} dy = c$

$\Rightarrow e^x + e^{-y} = c$

32. Here, $P = \frac{1}{x}$ and $Q = x^2$

$\therefore$ I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\therefore$ solution of the given equation is

$y.x = \int x^2.x\, dx + c_1$

$\Rightarrow xy = \frac{x^4}{4} + c_1 \Rightarrow 4xy = x^4 + c$, where $c = 4c_1$

33. $x\frac{dy}{dx} + 3y = x \Rightarrow \frac{dy}{dx} + \frac{3y}{x} = 1$

$\therefore$ I.F. $= e^{3\int \frac{1}{x} dx} = e^{3\log x} = x^3$

$\therefore$ solution of the given equation is

$yx^3 = \int x^3.1 dx + c \Rightarrow yx^3 = \frac{x^4}{4} + c$

34. $\frac{dy}{dx} + \frac{y}{x} = \sin x$

$\therefore$ I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\therefore$ solution of the given equation is

$y.x = \int x \sin x\, dx + c$

$\Rightarrow yx = -x \cos x + \sin x + c$

$\Rightarrow x(y + \cos x) = \sin x + c$

35. $\frac{dy}{dx} + y = \cos x$

Here, $P = 1$ and $Q = \cos x$

$\therefore$ I.F. $= e^{\int 1 dx} = e^x$

$\therefore$ solution of the given equation is

$y.e^x = \int \cos x.e^x dx + c$

$\Rightarrow y.e^x = \frac{e^x(\cos x + \sin x)}{2} + c$

$\Rightarrow y = \frac{1}{2}(\cos x + \sin x) + c.e^{-x}$

36. $\cos x\dfrac{dy}{dx}+y\sin x=1$

$\Rightarrow \dfrac{dy}{dx}+y\tan x=\sec x$

$\therefore$ I.F. $=e^{\int \tan x dx}=e^{\log\sec x}=\sec x$

$\therefore$ solution of the given equation is

$y\sec x=\int \sec^2 x+c=\tan x+c$

37. I.F. $=e^{\int Pdx}=e^{\int \cot x dx}=e^{\log\sin x}=\sin x$

$\therefore$ solution of the given equation is

$y.\sin x=\int 2\cos x\sin x\,dx+c_1$

$\Rightarrow y\sin x=\int \sin 2x\,dx+c_1$

$\Rightarrow y\sin x=-\dfrac{1}{2}\cos 2x+c_1$

$\Rightarrow 2y\sin x+\cos 2x=c$, where $c=2c_1$

Critical Thinking

1. $\left(\dfrac{d^2y}{dx^2}\right)^3=\left(1+\dfrac{dy}{dx}\right)^{1/2}\Rightarrow\left(\dfrac{d^2y}{dx^2}\right)^6=1+\dfrac{dy}{dx}$

Here, the highest order derivative is $\dfrac{d^2y}{dx^2}$ with power 6.

$\therefore$ degree = 6

2. $\dfrac{d^2y}{dx^2}+\sqrt{1+\left(\dfrac{dy}{dx}\right)^3}=0$

$\Rightarrow\left(\dfrac{d^2y}{dx^2}\right)^2=\left[-\sqrt{1+\left(\dfrac{dy}{dx}\right)^3}\right]^2$

$\Rightarrow\left(\dfrac{d^2y}{dx^2}\right)^2=1+\left(\dfrac{dy}{dx}\right)^3$

Here, the highest order derivative is $\dfrac{d^2y}{dx^2}$ with power 2.

$\therefore$ degree = 2

3. $3\dfrac{d^2y}{dx^2}=\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$

$\Rightarrow 9\left(\dfrac{d^2y}{dx^2}\right)^2=\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^3$

Here, the highest order derivative is $\dfrac{d^2y}{dx^2}$ with power 2.

$\therefore$ degree = 2

4. $\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^{\frac{1}{3}}+x^{\frac{1}{4}}=0$

$\Rightarrow\left(\dfrac{d^2y}{dx^2}+x^{\frac{1}{4}}\right)^3=\left[-\left(\dfrac{dy}{dx}\right)^{\frac{1}{3}}\right]^3$

$\Rightarrow\left(\dfrac{d^2y}{dx^2}+x^{\frac{1}{4}}\right)^3=-\dfrac{dy}{dx}$

Here, the highest order derivative is $\dfrac{d^2y}{dx^2}$ with power 3.

$\therefore$ order = 2 and degree = 3

5. $\dfrac{d^2y}{dx^2}=\left\{y+\left(\dfrac{dy}{dx}\right)^2\right\}^{1/4}$

$\Rightarrow\left(\dfrac{d^2y}{dx^2}\right)^4=\left\{\left[y+\left(\dfrac{dy}{dx}\right)^2\right]^{1/4}\right\}^4$

$\Rightarrow\left(\dfrac{d^2y}{dx^2}\right)^4=y+\left(\dfrac{dy}{dx}\right)^2$

Here, the highest order derivative is $\dfrac{d^2y}{dx^2}$ with power 4.

$\therefore$ order = 2 and degree = 4

6. Since, the given differential equation cannot be expressed as a polynomial in differential coefficients, the degree is not defined.

7. $\sqrt{\dfrac{dy}{dx}}-4\dfrac{dy}{dx}-7x=0$

$\Rightarrow\left(\sqrt{\dfrac{dy}{dx}}\right)^2=\left(4\dfrac{dy}{dx}+7x\right)^2$

$\Rightarrow\dfrac{dy}{dx}=16\left(\dfrac{dy}{dx}\right)^2+56x\dfrac{dy}{dx}+49x^2$

This is a differential equation of order 1 and degree 2.

8. $\dfrac{d^2y}{dx^2}-\sqrt{\dfrac{dy}{dx}-3}=x$

$\Rightarrow\left(\dfrac{d^2y}{dx^2}-x\right)^2=\left(\sqrt{\dfrac{dy}{dx}-3}\right)^2$

$\Rightarrow\left(\dfrac{d^2y}{dx^2}\right)^2-2x.\dfrac{d^2y}{dx^2}+x^2=\dfrac{dy}{dx}-3$

Here, the highest order derivative is $\dfrac{d^2y}{dx^2}$ with power 2.

$\therefore$ degree = 2

9. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/4} = \left(\frac{d^2y}{dx^2}\right)^{1/3}$

$\Rightarrow \left\{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/4}\right\}^4 = \left(\frac{d^2y}{dx^2}\right)^{4/3}$

$\Rightarrow \left\{\left[1+\left(\frac{dy}{dx}\right)^2\right]^3\right\}^3 = \left\{\left[\frac{d^2y}{dx^2}\right]^{4/3}\right\}^3$

$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^9 = \left(\frac{d^2y}{dx^2}\right)^4$

Here, the highest order derivative is $\frac{d^2y}{dx^2}$ with power 4.

$\therefore$ degree = 4

10. $y = x\frac{dy}{dx} + \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$

$\Rightarrow y - x\frac{dy}{dx} = \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$

Squaring on both sides, we get

$y^2 - 2xy\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2 = a^2\left(\frac{dy}{dx}\right)^2 + b^2$

This is a differential equation of order 1 and degree 2.

11. $\left[1+\left(\frac{d^2y}{dx^2}\right)^3\right]^{4/5} = \left(\frac{m}{m+1}\right)\frac{d^3y}{dx^3}$

$\Rightarrow \left\{\left[1+\left(\frac{d^2y}{dx^2}\right)^3\right]^{4/5}\right\}^5 = \left(\frac{m}{m+1}\right)^5\left(\frac{d^3y}{dx^3}\right)^5$

$\Rightarrow \left[1+\left(\frac{d^2y}{dx^2}\right)^3\right]^4 = \left(\frac{m}{m+1}\right)^5\left(\frac{d^3y}{dx^3}\right)^5$

Here, the highest order derivative is $\frac{d^3y}{dx^3}$ with power 5.

$\therefore$ order = 3 and degree = 5

12. $\left(\frac{d^2y}{dx^2}\right)^5 + 4\frac{\left(\frac{d^2y}{dx^2}\right)^3}{\left(\frac{d^3y}{dx^3}\right)} + \frac{d^3y}{dx^3} = x^2 - 1$

$\therefore \left(\frac{d^2y}{dx^2}\right)^5 . \left(\frac{d^3y}{dx^3}\right) + 4\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{d^3y}{dx^3}\right)^2 = \left(x^2-1\right).\frac{d^3y}{dx^3}$

Here, the highest order derivative is $\frac{d^3y}{dx^3}$ with power 2.

$\therefore$ order = 3 and degree = 2

$\therefore$ m = 3 and n = 2

13. Option (A) has order = 4, degree = 1
Option (B) has order = 3, degree = 4
Consider option (C),

$\left[1+\left(\frac{dy}{dx}\right)^3\right]^{2/3} = 4\frac{d^3y}{dx^3}$

Cubing on both sides, we get

$\left[1+\left(\frac{dy}{dx}\right)^3\right]^2 = 4^3\left(\frac{d^3y}{dx^3}\right)^3$

Here, order = 3 and degree = 3

$\therefore$ option (C) is the correct answer.

14. Since, the given equation has 3 arbitrary constants i.e., g, f and c, therefore order of the given differential equation is 3.

15. Since, the given equation has 3 arbitrary constants i.e., a, b and c, therefore order of the given differential equation is 3.

16. The equation of a family of circles of radius r passing through the origin and having centre on Y-axis is $(x - 0)^2 + (y - r)^2 = r^2$ or $x^2 + y^2 - 2ry = 0$.
Since this equation has one arbitrary constant, its order is 1.

17. The equation of the family of circles which touch both the axes is $(x - a)^2 + (y - a)^2 = a^2$, where a is a parameter.
Since this equation has one arbitrary constant, its order is 1.

18. $y = ae^{mx} + be^{-mx}$ ....(i)

$\Rightarrow \frac{dy}{dx} = mae^{mx} - mbe^{-mx}$

$\Rightarrow \frac{d^2y}{dx^2} = m^2ae^{mx} + m^2be^{-mx}$

$\Rightarrow \frac{d^2y}{dx^2} = m^2(ae^{mx} + be^{-mx}) = m^2y$ ....[From (i)]

$\Rightarrow \frac{d^2y}{dx^2} - m^2y = 0$

19. $y = cx + c - c^3$ ....(i)

$\Rightarrow \frac{dy}{dx} = c$ ....(ii)

Substituting (ii) in (i), we get

$y = \frac{dy}{dx}.x + \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3$

20. $y = A \cos \omega t + B \sin \omega t$ ....(i)

$\Rightarrow y' = -A\, \omega \sin \omega\, t + B\, \omega \cos \omega\, t$

$\Rightarrow y'' = -A\, \omega^2 \cos \omega\, t - B\, \omega^2 \sin \omega\, t$

$\Rightarrow y'' = -\omega^2 (A \cos \omega\, t + B \sin \omega\, t)$

$\Rightarrow y'' = -\omega^2 y$ ....[From (i)]

21. $y = ax^{n+1} + bx^{-n}$ ....(i)

$\Rightarrow \frac{dy}{dx} = a.(n+1)x^n - bnx^{-n-1}$

$\Rightarrow \frac{d^2y}{dx^2} = a(n+1)nx^{n-1} + n(n+1)bx^{-n-2}$

$\Rightarrow x^2\frac{d^2y}{dx^2} = a(n+1)nx^{n+1} + bn(n+1).x^{-n}$

$= n(n+1)(ax^{n+1} + bx^{-n})$

$\therefore \quad x^2\frac{d^2y}{dx^2} = n(n+1)y$ ....[From (i)]

22. $y = c_1 \cos ax + c_2 \sin ax$ ....(i)

$\Rightarrow \frac{dy}{dx} = -c_1 a \sin ax + c_2 a \cos ax$

$\Rightarrow \frac{d^2y}{dx^2} = -c_1a^2 \cos ax - c_2a^2 \sin ax$

$\Rightarrow \frac{d^2y}{dx^2} = -a^2(c_1 \cos ax + c_2 \sin ax)$

$\Rightarrow \frac{d^2y}{dx^2} = -a^2y$ ....[From (i)]

$\Rightarrow \frac{d^2y}{dx^2} + a^2y = 0$

23. $\sin^{-1} x + \sin^{-1} y = c$

Differentiating w.r.t. x, we get

$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\;dy + \sqrt{1-y^2}\;dx = 0$

24. $y = (\sin^{-1}x)^2 + A\cos^{-1}x + B$

$\Rightarrow y = (\sin^{-1}x)^2 + A\left(\frac{\pi}{2} - \sin^{-1}x\right) + B$

....$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow y = (\sin^{-1}x)^2 - A\sin^{-1}x + \frac{\pi A}{2} + B$ ....(i)

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} - \frac{A}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = (2\sin^{-1}x - A)^2$

$= 4(\sin^{-1}x)^2 - 4A\sin^{-1}x + A^2$

$= 4[(\sin^{-1}x)^2 - A\sin^{-1}x] + A^2$

$= 4\left(y - \frac{\pi A}{2} - B\right) + A^2$

....[From (i)]

$\therefore \quad (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y - 2\pi A - 4B + A^2$

Differentiating w.r.t. x, we get

$(1-x^2).2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx}$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$

25. The equation of all the straight lines passing through the origin is

$y = mx$ ....(i)

$\therefore \quad \frac{dy}{dx} = m$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$ ....[From (i)]

26. $v = \frac{A}{r} + B$

Differentiating w.r.t. r, we get

$\frac{dv}{dr} = -\frac{A}{r^2}$ ....(i)

$\therefore \quad \frac{d^2v}{dr^2} = 2A.r^{-3}$

$= -2r^2\frac{dv}{dr}.r^{-3}$ ....[From (i)]

$\therefore \quad \frac{d^2v}{dr^2} + \frac{2}{r}\cdot\frac{dv}{dr} = 0$

27. The equation of the family of lines passing through (1, −1) is

$y + 1 = m(x - 1)$

$\Rightarrow y = m(x - 1) - 1$ ....(i)

$\Rightarrow \dfrac{dy}{dx} = m$

Substituting the value of m in (i), we get

$y = \dfrac{dy}{dx}(x - 1) - 1$

28. $y = x.e^{cx}$

Taking logarithm on both sides, we get

$\log y = \log x + cx$ ....(i)

$\Rightarrow c = \dfrac{1}{x}\log\dfrac{y}{x}$ ....(ii)

Differentiating (i) w.r.t. x, we get

$\dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{1}{x} + c$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{1}{x}\log\left(\dfrac{y}{x}\right)$ ....[From (ii)]

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x}\left[1+\log\left(\dfrac{y}{x}\right)\right]$

29. The system of circles which passes through origin and whose centre lies on Y − axis is

$x^2 + y^2 - 2ay = 0$ ....(i)

Differentiating w.r.t. x, we get

$2x + 2y\dfrac{dy}{dx} - 2a\dfrac{dy}{dx} = 0$

$\Rightarrow 2a = 2y + 2x\dfrac{dx}{dy}$ ....(ii)

Substituting (ii) in (i), we get

$x^2 + y^2 - 2y^2 - 2xy\dfrac{dx}{dy} = 0$

$\Rightarrow (x^2 - y^2)\dfrac{dy}{dx} - 2xy = 0$

30. $\dfrac{dy}{dx} = e^{x-y} + x^2e^{-y}$

$\Rightarrow \dfrac{dy}{dx} = e^{-y}(e^x + x^2)$

Integrating on both sides, we get

$\int e^y dy = \int (e^x + x^2)dx + c$

$\Rightarrow e^y = e^x + \dfrac{x^3}{3} + c$

31. $\log\left(\dfrac{dy}{dx}\right) = x + y$

$\Rightarrow \dfrac{dy}{dx} = e^{x+y} \Rightarrow \dfrac{dy}{dx} = e^x.e^y$

Integrating on both sides, we get

$\int e^x dx - \int e^{-y} dy = c$

$\Rightarrow e^x + e^{-y} = c$

32. $x \cos y dy = (xe^x \log x + e^x)dx$

$\Rightarrow \cos y dy = e^x\left(\log x + \dfrac{1}{x}\right)dx$

Integrating on both sides, we get

$\sin y = e^x \log x + c$

33. $\dfrac{dy}{dx} = 2^{y-x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2^y}{2^x}$

Integrating on both sides, we get

$\int 2^{-y} dy - \int 2^{-x} dx = c_1$

$\Rightarrow \dfrac{-2^{-y}}{\log 2} + \dfrac{2^{-x}}{\log 2} = c_1$

$\Rightarrow \dfrac{1}{2^x} - \dfrac{1}{2^y} = c_1 \log 2$

$\Rightarrow \dfrac{1}{2^x} - \dfrac{1}{2^y} = c$, where $c = c_1 \log 2$

34. $\dfrac{dy}{dx} + 2xy = y \Rightarrow \dfrac{dy}{dx} = y(1 - 2x)$

Integrating on both sides, we get

$\int\dfrac{dy}{y} = \int(1-2x)dx + c_1$

$\Rightarrow \log y = x - x^2 + c_1$

$\Rightarrow y = e^{x-x^2}.e^{c_1}$

$\Rightarrow y = c.e^{x-x^2}$, where $c = e^{c_1}$

35. $\dfrac{dy}{dx} = \dfrac{1-\cos x}{1+\cos x} = \tan^2\dfrac{x}{2}$

$\Rightarrow dy = \left(\sec^2\dfrac{x}{2} - 1\right)dx$

Integrating on both sides, we get

$y = 2\tan\dfrac{x}{2} - x + c$

36. $\dfrac{dy}{dx}+\dfrac{1+\cos 2y}{1-\cos 2x}=0$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{2\cos^2 y}{2\sin^2 x}=0$

$\Rightarrow \dfrac{dy}{\cos^2 y}+\dfrac{dx}{\sin^2 x}=0$

Integrating on both sides, we get

$\int \sec^2 y\,dy+\int \text{cosec}^2 x\,dx=c$

$\Rightarrow \tan y - \cot x = c$

37. $x(e^{2y}-1)dy+(x^2-1)e^y dx=0$

$\Rightarrow x\,(e^{2y}-1)\,dy=(1-x^2)\,e^y\,dx$

Integrating on both sides, we get

$\int \dfrac{e^{2y}-1}{e^y}dy=\int \dfrac{1-x^2}{x}dx+c$

$\Rightarrow \int e^y dy-\int e^{-y}\,dy = \int \dfrac{1}{x}dx - \int x\,dx+c$

$\Rightarrow e^y + e^{-y} = \log x - \dfrac{x^2}{2} + c$

38. $x^2dy=-2xydx$

$\Rightarrow \dfrac{dy}{y} = -\dfrac{2x}{x^2}dx$

Integrating on both sides, we get

$\int \dfrac{2x}{x^2}dx+\int \dfrac{dy}{y} = \log c$

$\Rightarrow \log x^2 + \log y = \log c$

$\Rightarrow \log x^2y = \log c \qquad \Rightarrow x^2y = c$

39. $\cot y\,dx = x\,dy$

Integrating on both sides, we get

$\int \dfrac{dx}{x} = \int \tan y\,dy+\log c$

$\Rightarrow \log x = \log(\sec y) + \log c$

$\Rightarrow \log x = \log(c.\sec y) \qquad \Rightarrow x = c\sec y$

40. $\dfrac{dy}{dx} = \cot x\cot y$

$\Rightarrow \cot x\,dx - \tan y\,dy = 0$

Integrating on both sides, we get

$\log(\sin x) - \log(\sec y) = \log c$

$\Rightarrow \log\left(\dfrac{\sin x}{\sec y}\right) = \log c \qquad \Rightarrow \sin x = c\sec y$

41. $x\sec y\dfrac{dy}{dx} = 1 \Rightarrow \sec y dy = \dfrac{dx}{x}$

Integrating on both sides, we get

$\log(\sec y + \tan y) = \log x + \log c$

$\Rightarrow \log(\sec y + \tan y) = \log(cx)$

$\Rightarrow \sec y + \tan y = cx$

42. $(e^y + 1)\cos x\,dx + e^y\sin x\,dy = 0$

$\Rightarrow \dfrac{e^y}{e^y+1}dy + \dfrac{\cos x}{\sin x}dx = 0$

Integrating on both sides, we get

$\int \dfrac{e^y}{e^y+1}dy + \int \dfrac{\cos x}{\sin x}dx = \log c$

$\Rightarrow \log(e^y + 1) + \log(\sin x) = \log c$

$\Rightarrow (e^y + 1)\sin x = c$

43. $y\,dx + (1 + x^2)\tan^{-1}x\,dy = 0$

Integrating on both sides, we get

$\int \dfrac{dx}{(1+x^2)\tan^{-1}x} + \int \dfrac{dy}{y} = \log c$

$\Rightarrow \log(\tan^{-1}x) + \log y = \log c$

$\Rightarrow \log(\tan^{-1}x.y) = \log c$

$\Rightarrow y\tan^{-1}x = c$

44. $3e^x\tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$

$\Rightarrow \dfrac{\sec^2 y}{\tan y}dy = -3\dfrac{e^x}{1-e^x}dx$

Integrating on both sides, we get

$\int \dfrac{\sec^2 y}{\tan y}dy = -3\int \dfrac{e^x}{1-e^x}dx + \log c$

$\Rightarrow \log(\tan y) = 3\log(1 - e^x) + \log c$

$\Rightarrow \log(\tan y) = \log[(1 - e^x)^3 c]$

$\Rightarrow \tan y = c(1 - e^x)^3$

45. $(\sin x + \cos x)dy + (\cos x - \sin x)\,dx = 0$

$\Rightarrow dy = -\left(\dfrac{\cos x - \sin x}{\sin x + \cos x}\right)dx$

Integrating on both sides, we get

$y = -\log(\sin x + \cos x) + \log c$

$\Rightarrow y = \log\left(\dfrac{c}{\sin x + \cos x}\right)$

$\Rightarrow e^y(\sin x + \cos x) = c$

46. $\dfrac{dy}{dx} = \dfrac{xy + y}{xy + x}$

$\Rightarrow \left(\dfrac{1+y}{y}\right)dy=\left(\dfrac{1+x}{x}\right)dx$

Integrating on both sides, we get

$\log y + y = \log x + x + \log A$

$\Rightarrow \log\left(\dfrac{y}{Ax}\right) = x - y \Rightarrow y = Axe^{x-y}$

47. $x\dfrac{dy}{dx} + y = y^2 \Rightarrow x\dfrac{dy}{dx} = y^2 - y$

$\Rightarrow \dfrac{dy}{y^2 - y} = \dfrac{dx}{x}$

$\Rightarrow \left[\dfrac{1}{y-1} - \dfrac{1}{y}\right]dy = \dfrac{dx}{x}$

Integrating on both sides, we get

$\log(y - 1) - \log y = \log x + \log c$

$\Rightarrow \log\left(\dfrac{y-1}{y}\right) = \log(x\,c)$

$\Rightarrow \dfrac{y-1}{y} = xc \Rightarrow y = 1 + cxy$

48. $(2y - 1)\,dx - (2x + 3)\,dy = 0$

Integrating on both sides, we get

$\int\dfrac{dx}{2x+3} - \int\dfrac{dy}{2y-1} = \log c_1$

$\Rightarrow \dfrac{1}{2}\log(2x + 3) - \dfrac{1}{2}\log(2y - 1) = \log c_1$

$\Rightarrow \log(2x + 3) - \log(2y - 1) = 2\log c_1$

$\Rightarrow \log\left(\dfrac{2x+3}{2y-1}\right) = \log c_1^2$

$\Rightarrow \dfrac{2x+3}{2y-1} = c$, where $c = c_1^2$

49. $(x - y^2x)\,dx = (y - x^2y)\,dy$

$\Rightarrow x(1 - y^2)\,dx = y(1 - x^2)dy$

Integrating on both sides, we get

$\int\dfrac{x}{1-x^2}dx - \int\dfrac{y}{1-y^2}dy = \log c$

$\Rightarrow -\dfrac{1}{2}\log(1 - x^2) + \dfrac{1}{2}\log(1 - y^2) = \log c$

$\Rightarrow \log(1 - y^2) - \log(1 - x^2) = 2\log c$

$\Rightarrow \dfrac{1-y^2}{1-x^2} = c^2$

$\Rightarrow 1 - y^2 = c^2(1 - x^2)$

50. $(1 - x^2)dy + xydx = xy^2dx$

$\Rightarrow (1 - x^2)dy = x(y^2 - y)\,dx$

Integrating on both sides, we get

$\int\dfrac{dy}{y(y-1)} = \int\dfrac{x}{1-x^2}dx + \log c$

$\Rightarrow \int\left(\dfrac{1}{y-1} - \dfrac{1}{y}\right)dy = \dfrac{-1}{2}\int\dfrac{-2x}{1-x^2}dx + \log c$

$\Rightarrow \log(y - 1) - \log y = \dfrac{-1}{2}\log(1-x^2) + \log c$

$\Rightarrow 2\log(y - 1) + \log(1 - x^2) = 2\log c + 2\log y$

$\Rightarrow \log[(y - 1)^2(1 - x^2)] = \log c^2y^2$

$\Rightarrow (y - 1)^2(1 - x^2) = c^2y^2$

51. $(x^2 - yx^2)\dfrac{dy}{dx} + y^2 + xy^2 = 0$

$\Rightarrow x^2(1 - y)\dfrac{dy}{dx} + y^2(1 + x) = 0$

$\Rightarrow \dfrac{(1-y)}{y^2}dy + \dfrac{(1+x)}{x^2}dx = 0$

Integrating on both sides, we get

$\int\left(\dfrac{1}{y^2} - \dfrac{1}{y}\right)dy + \int\left(\dfrac{1}{x^2} + \dfrac{1}{x}\right)dx = c$

$\Rightarrow -\dfrac{1}{y} - \log y - \dfrac{1}{x} + \log x = c$

$\Rightarrow \log\left(\dfrac{x}{y}\right) = \dfrac{1}{x} + \dfrac{1}{y} + c$

52. $\dfrac{dy}{dx}\tan y = \sin(x + y) + \sin(x - y)$

$\Rightarrow \dfrac{dy}{dx}\cdot\dfrac{\sin y}{\cos y} = 2\sin x\cos y$

$\Rightarrow \dfrac{\sin y}{\cos^2 y}dy = 2\sin x\,dx$

Integrating on both sides, we get

$\int\dfrac{\sin y}{\cos^2 y}dy - 2\int\sin x\,dx = c$

$\Rightarrow \dfrac{1}{\cos y} + 2\cos x = c$

$\Rightarrow \sec y + 2\cos x = c$

53. $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$

$\Rightarrow \dfrac{dy}{1+y^2} - \dfrac{dx}{1+x^2} = 0$

Integrating on both sides, we get

$\int\dfrac{dy}{1+y^2} - \int\dfrac{dx}{1+x^2} = \tan^{-1}c$

$\Rightarrow \tan^{-1}y - \tan^{-1}x = \tan^{-1}c$

$\Rightarrow \tan^{-1}\left(\dfrac{y-x}{1+xy}\right) = \tan^{-1}c$

$\Rightarrow y - x = c(1 + xy)$

54. $xy\dfrac{dy}{dx} = \dfrac{(1+y^2)(1+x+x^2)}{(1+x^2)}$

Integrating on both sides, we get

$$\int \frac{y}{1+y^2}dy = \int \frac{1+x^2+x}{x(1+x^2)}dx + c$$

$$\Rightarrow \frac{1}{2}\int \frac{2y}{1+y^2}dy = \int \frac{1}{x}dx + \int \frac{dx}{1+x^2} + c$$

$$\Rightarrow \frac{1}{2}\log(1+y^2) = \log x + \tan^{-1} x + c$$

55. $(\text{cosec } x \log y)dy + (x^2y)dx = 0$

$$\Rightarrow \frac{1}{y}\log y dy + x^2 \sin x dx = 0$$

Integrating on both sides, we get

$$\frac{(\log y)^2}{2} + [x^2(-\cos x) + \int 2x\cos x dx] = c$$

$$\Rightarrow \frac{(\log y)^2}{2} - x^2\cos x + 2(x\sin x + \cos x) = c$$

$$\Rightarrow \frac{(\log y)^2}{2} + (2 - x^2)\cos x + 2x\sin x = c$$

56. $\dfrac{dy}{dx} = \dfrac{x\log x^2 + x}{\sin y + y\cos y}$

Integrating on both sides, we get

$$\int(\sin y + y\cos y)dy = \int(x\log x^2 + x)dx + c$$

$$\Rightarrow -\cos y + y\sin y + \cos y$$

$$= \frac{x^2}{2}\log x^2 - \int\frac{x^2}{2}.\frac{1}{x^2}.2x\,dx + \int x\,dx + c$$

$$\Rightarrow y\sin y = \frac{x^2}{2}\times 2\log x - \int x\,dx + \int x\,dx + c$$

$$\Rightarrow y\sin y = x^2\log x + c$$

57. $\cos y \log(\sec x + \tan x)dx$

$= \cos x \log(\sec y + \tan y)\, dy$

Integrating on both sides, we get

$$\int \sec x \log(\sec x + \tan x)dx$$

$$= \int \sec y \log(\sec y + \tan y)dy + c$$

Put $\log(\sec x + \tan x) = t \Rightarrow \sec x\, dx = dt$

and $\log(\sec y + \tan y) = z \Rightarrow \sec y\, dy = dz$

$\therefore$ $\int t\,dt = \int z\,dz + c$

$$\Rightarrow \frac{t^2}{2} = \frac{z^2}{2} + c$$

$$\Rightarrow \frac{[\log(\sec x + \tan x)]^2}{2} = \frac{[\log(\sec y + \tan y)]^2}{2} + c$$

58. $\sqrt{a+x}\dfrac{dy}{dx} + x = 0$

Integrating on both sides, we get

$$\int dy + \int \frac{x}{\sqrt{a+x}}dx = c$$

$$\Rightarrow y + \int \frac{x+a-a}{\sqrt{a+x}}dx = c$$

$$\Rightarrow y + \int\left(\sqrt{a+x} - \frac{a}{\sqrt{a+x}}\right)dx = c$$

$$\Rightarrow y + \frac{2}{3}(a+x)^{\frac{3}{2}} - 2a\sqrt{a+x} = c$$

$$\Rightarrow 3y + 2(a+x)^{\frac{3}{2}} - 6a\sqrt{a+x} = 3c$$

$$\Rightarrow 3y + 2\sqrt{a+x}\,(a + x - 3a) = 3c$$

$$\Rightarrow 3y + 2\sqrt{a+x}\,(x - 2a) = 3c$$

59. $ydx + xdy + xy^2\, dx - x^2ydy = 0$

$$\Rightarrow \frac{ydx + xdy}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

$$\Rightarrow \frac{d(xy)}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating on both sides, we get

$$-\frac{1}{xy} + \log x - \log y = k$$

$$\Rightarrow \log\frac{x}{y} = \frac{1}{xy} + k$$

60. $y\,e^{-x/y}dx - (xe^{-x/y} + y^3)dy = 0$

$$\Rightarrow e^{-x/y}(ydx - xdy) = y^3dy$$

$$\Rightarrow e^{-x/y}\frac{(ydx - xdy)}{y^2} = ydy$$

$$\Rightarrow e^{-x/y}d\left(\frac{x}{y}\right) = ydy$$

Integrating on both sides, we get

$$-e^{-x/y} = \frac{y^2}{2} + c \Rightarrow \frac{y^2}{2} + e^{-x/y} = k,$$

where $k = -c$

61. $y' = 1 + x + y^2 + xy^2$

$$\Rightarrow \frac{dy}{dx} = (1+x)(1+y^2)$$

Integrating on both sides, we get

$$\int\frac{dy}{1+y^2} = \int(1+x)dx + c$$

$$\Rightarrow \tan^{-1}y = x + \frac{x^2}{2} + c \quad \text{….(i)}$$

Since, $y(0) = 0$ i.e., $y = 0$, when $x = 0$

$\therefore$ $\tan^{-1}(0) = 0 + c \Rightarrow c = 0$

$\therefore$ $\tan^{-1}y = x + \frac{x^2}{2}$ ….[From (i)]

$\Rightarrow y = \tan\left(x + \frac{x^2}{2}\right)$

62. $y' - y = 1$

$\Rightarrow \frac{dy}{dx} - y = 1$

$\Rightarrow \frac{dy}{dx} = 1 + y$

Integrating on both sides, we get

$\int \frac{dy}{1+y} = \int dx + c$

$\Rightarrow \log(1 + y) = x + c$

Since, $y(0) = 1$ i.e., $y = 1$, when $x = 0$

$\therefore$ $\log(1 + 1) = 0 + c \Rightarrow c = \log 2$

$\therefore$ $\log(1 + y) = x + \log 2$

$\Rightarrow \log\left(\frac{1+y}{2}\right) = x$

$\Rightarrow \frac{1+y}{2} = e^x$

$\Rightarrow y = 2e^x - 1$

$\Rightarrow y(x) = 2\exp(x) - 1$

63. $e^{dy/dx} = (x + 1)$

$\Rightarrow \frac{dy}{dx} = \log(x + 1)$

Integrating on both sides, we get

$\int dy = \int \log(x+1)\,dx + c$

$\Rightarrow y = x\log(x + 1) - \int \frac{x}{x+1}dx + c$

$= x\log(x + 1) - \int \frac{x+1-1}{x+1}dx + c$

$= x\log(x + 1) - \int \left(1 - \frac{1}{x+1}\right)dx + c$

$\therefore$ $y = x\log(x + 1) - x + \log(x + 1) + c$ ….(i)

Since, $y(0) = 3$ i.e., $y = 3$, when $x = 0$

$\therefore$ $3 = 0 + c \Rightarrow c = 3$

$\therefore$ $y = x\log(x + 1) + \log(x + 1) - x + 3$ ….[From (i)]

$\therefore$ $y = (x + 1)\log(x + 1) - x + 3$

64. $\frac{dy}{dx} = e^{x+y} - 1$ ….(i)

Put $x + y = v$ ….(ii)

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$\frac{dv}{dx} = e^v$

Integrating on both sides, we get

$\int e^{-v}dv = \int dx + c$

$\Rightarrow -e^{-v} = x + c$

$\Rightarrow x + e^{-v} + c = 0$

$\Rightarrow x + e^{-(x+y)} + c = 0$

65. $\frac{dy}{dx} = \sin(x + y)$ ….(i)

Put $x + y = v$ ….(ii)

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$\frac{dv}{dx} - 1 = \sin v$

$\Rightarrow \frac{dv}{1+\sin v} = dx \Rightarrow \frac{1-\sin v}{\cos^2 v}dv = dx$

Integrating on both sides, we get

$\int \sec^2 v\,dv - \int \sec v \tan v\,dv = \int dx + c$

$\Rightarrow \tan v - \sec v = x + c$

$\Rightarrow \tan(x + y) - \sec(x + y) = x + c$

66. $\frac{dy}{dx} = \sin(x + y)\tan(x + y) - 1$ ….(i)

Put $x + y = v$ ….(ii)

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$\frac{dv}{dx} = \sin v \tan v \Rightarrow \frac{dv}{dx} = \frac{\sin^2 v}{\cos v}$

Integrating on both sides, we get

$\int dx - \int \frac{\cos v}{\sin^2 v}dv = c \Rightarrow x - \left(-\frac{1}{\sin v}\right) = c$

….[Put $\sin v = t \Rightarrow \cos v\,dv = dt$]

$\Rightarrow x + \operatorname{cosec} v = c$

$\Rightarrow x + \operatorname{cosec}(x + y) = c$

67. $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$ ....(i)

Put $x + y = v$ ....(ii)

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$\frac{dv}{dx} - 1 = \cos v + \sin v$

$\Rightarrow \frac{dv}{dx} = 1 + \cos v + \sin v$

Integrating on both sides, we get

$\int \frac{dv}{1+\cos v+\sin v} = \int dx + c$

$\Rightarrow \int \frac{dv}{1+\frac{1-\tan^2 v/2}{1+\tan^2 v/2}+\frac{2\tan v/2}{1+\tan^2 v/2}} = x + c$

$\Rightarrow \int \frac{\sec^2(v/2)}{2(1+\tan v/2)} dv = x + c$

$\Rightarrow \log|1+\tan v/2| = x + c$

$\Rightarrow \log\left|1+\tan\left(\frac{x+y}{2}\right)\right| = x + c$

68. $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$ ....(i)

Put $y = vx$ ....(ii)

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$v + x\frac{dv}{dx} = \frac{x^2+vx^2+v^2x^2}{x^2} = 1 + v + v^2$

$\Rightarrow x\frac{dv}{dx} = 1 + v^2 \Rightarrow \frac{dv}{1+v^2} = \frac{dx}{x}$

Integrating on both sides, we get

$\tan^{-1} v = \log x + c$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \log x + c$

69. $\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x}+1\right)$ ....(i)

Put $y = vx$ ....(ii)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$v + x\frac{dv}{dx} = v(\log v + 1) \Rightarrow x\frac{dv}{dx} = v\log v$

Integrating on both sides, we get

$\int \frac{dv}{v\log v} = \int \frac{dx}{x} + \log c$

$\Rightarrow \log(\log v) = \log x + \log c$

....$\left[\text{Put} \log v = t \Rightarrow \frac{1}{v}dv = dt\right]$

$\Rightarrow \log v = xc \Rightarrow \log\left(\frac{y}{x}\right) = cx$

70. $(x + y)\,dx + x\,dy = 0$

$\Rightarrow \frac{dy}{dx} = -\left(\frac{x+y}{x}\right)$ ....(i)

Put $y = vx$ ....(ii)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$v + x\frac{dv}{dx} = \frac{-x-vx}{x} = -1 - v$

$\Rightarrow x\frac{dv}{dx} = -1 - 2v$

Integrating on both sides, we get

$\int \frac{dv}{1+2v} = -\int \frac{dx}{x} + \log c_1$

$\Rightarrow \frac{1}{2}\log(1 + 2v) = -\log x + \log c_1$

$\Rightarrow \log\left(1+2\frac{y}{x}\right) = 2\log\frac{c_1}{x}$

$\Rightarrow \frac{x+2y}{x} = \left(\frac{c_1}{x}\right)^2$

$\Rightarrow x^2 + 2xy = c_1^2$

$\Rightarrow x^2 + 2xy = c$, where $c = c_1^2$

71. $x + y\frac{dy}{dx} = 2y \Rightarrow \frac{x}{y} + \frac{dy}{dx} = 2$ ....(i)

Put $y = vx$ ....(ii)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$\frac{1}{v} + v + x\frac{dv}{dx} = 2$

$\Rightarrow v + x.\frac{dv}{dx} = \frac{2v-1}{v} \Rightarrow \frac{v}{(v-1)^2}dv = -\frac{dx}{x}$

$\Rightarrow \frac{v-1+1}{(v-1)^2}dv = -\frac{dx}{x}$

$\Rightarrow \left[\frac{1}{(v-1)} + \frac{1}{(v-1)^2}\right]dv = -\frac{dx}{x}$

Integrating on both sides, we get

$$\int \frac{dv}{v-1} + \int \frac{dv}{(v-1)^2} = -\int \frac{dx}{x} + c$$

$$\Rightarrow \log(v-1) - \frac{1}{v-1} = -\log x + c$$

$$\Rightarrow \log(y-x) = \frac{x}{y-x} + c$$

72. $y^2dx + (x^2 - xy + y^2)\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{x^2 - xy + y^2} \quad \text{....(i)}$$

Put $y = vx$ ....(ii)

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{....(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2 - x.vx + v^2x^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^2}{1-v+v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v-v^3}{1-v+v^2}$$

Integrating on both sides, we get

$$\int \frac{v^2-v+1}{v(v^2+1)}dv = -\int \frac{dx}{x} + c$$

$$\Rightarrow \int\left(\frac{1}{v} - \frac{1}{v^2+1}\right)dv = -\int \frac{dx}{x} + c$$

$$\Rightarrow \log v - \tan^{-1} v = -\log x + c$$

$$\Rightarrow \log\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{x}\right) = -\log x + c$$

$$\Rightarrow \log y = \tan^{-1}\left(\frac{y}{x}\right) + c$$

73. $2xy\,\frac{dy}{dx} = x^2 + 3y^2$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+3y^2}{2xy} \quad \text{....(i)}$$

Put $y = vx$ ....(ii)

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{....(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$v + x\frac{dv}{dx} = \frac{x^2+3v^2x^2}{2vx^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{x^2(1+3v^2)}{2x^2v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

Integrating on both sides, we get

$$\int \frac{2v}{1+v^2}dv = \int \frac{dx}{x} + \log p$$

$$\Rightarrow \log(1+v^2) = \log x + \log p$$

$$\Rightarrow \log\left(\frac{1+v^2}{x}\right) = \log p$$

$$\Rightarrow \frac{1+v^2}{x} = p \qquad \Rightarrow \frac{x^2+y^2}{x^3} = p$$

$$\Rightarrow x^2 + y^2 = p.x^3$$

74. $\frac{dy}{dx} = \frac{x}{2y-x}$ ....(i)

Put $y = vx$ ....(ii)

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{....(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$v + x\frac{dv}{dx} = \frac{x}{2vx-x} = \frac{1}{2v-1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{2v-1} - v = \frac{1-2v^2+v}{2v-1}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{(v-1)(2v+1)}{2v-1}$$

$$\Rightarrow \frac{(2v-1)}{(2v+1)(v-1)}dv = \frac{-dx}{x}$$

$$\Rightarrow \frac{1}{3(v-1)} + \frac{4}{3(2v+1)} = \frac{-dx}{x}$$

Integrating on both sides, we get

$$\frac{1}{3}\log(v-1) + \frac{4}{3}\cdot\frac{1}{2}\log(2v+1) = -\log x + \log c_1$$

$$\Rightarrow \log(v-1)^{1/3} + \log(2v+1)^{2/3} = \log\frac{c_1}{x}$$

$$\Rightarrow (v-1)^{1/3}(2v+1)^{2/3} = \frac{c_1}{x}$$

$$\Rightarrow \left(\frac{y-x}{x}\right)\left(\frac{2y+x}{x}\right)^2 = \frac{c_1^3}{x^3}$$

$$\Rightarrow (x-y)(x+2y)^2 = c, \text{ where } c = -c_1^3$$

75. $\frac{dy}{dx} = \frac{x-y}{x+y}$ ....(i)

Put $y = vx$ ....(ii)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$v + x\frac{dv}{dx} = \frac{x - vx}{x + vx}$

$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v}$

Integrating on both sides, we get

$\int \frac{1+v}{2-(1+v)^2}dv = \int \frac{dx}{x} + \log c_1$

$\Rightarrow -\frac{1}{2}\log[2-(1+v)^2] = \log x + \log c_1$

$\Rightarrow -\frac{1}{2}\log(1-2v-v^2) = \log(xc_1)$

$\Rightarrow \log\left(\frac{1}{\sqrt{1-2v-v^2}}\right) = \log(xc_1)$

$\Rightarrow \frac{1}{\sqrt{1-2v-v^2}} = xc_1$

$\Rightarrow \frac{1}{1-2v-v^2} = x^2c_1^2$

$\Rightarrow x^2c_1^2(1-2v-v^2) = 1$

$\Rightarrow x^2c_1^2\left(1 - \frac{2y}{x} - \frac{y^2}{x^2}\right) = 1$

$\Rightarrow c_1^2(x^2 - 2xy - y^2) = 1$

$\Rightarrow y^2 + 2xy - x^2 = c$, where $c = \frac{-1}{c_1^2}$

76. A differential equation in which the dependent variable (y) and its differential coefficient occur only in the first degree and are not multiplied together is called a linear differential equation.

Hence, $y\frac{dy}{dx} + 4x = 0$ is a non-linear differential equation.

77. $x^2\frac{dy}{dx} + y = e^x$ can be written as

$\frac{dy}{dx} + \frac{y}{x^2} = \frac{e^x}{x^2}$, which is a linear equation.

78. $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{x^2-1}{x^2+1}$

$\therefore$ I.F. $= e^{\int\frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

79. $(x\log x)\frac{dy}{dx} + y = 2\log x$

$\Rightarrow \frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x}$

$\therefore$ I.F. $= e^{\int\frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

80. $(1-x^2)\frac{dy}{dx} - xy = 1$

$\Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2}.y = \frac{1}{1-x^2}$

$\therefore$ I.F. $= e^{\int\frac{-x}{1-x^2}dx} = e^{\frac{1}{2}\log(1-x^2)} = e^{\log\left(\sqrt{1-x^2}\right)} = \sqrt{1-x^2}$

81. $x\frac{dy}{dx} + y\log x = x.e^x x^{-\frac{1}{2}\log x}$

$\Rightarrow \frac{dy}{dx} + \frac{\log x}{x}\cdot y = e^x x^{-\frac{1}{2}\log x}$

$\therefore$ I.F. $= e^{\int\frac{\log x}{x}dx} = e^{\frac{1}{2}(\log x)^2}$

$= \left(e^{\frac{1}{2}\log x}\right)^{\log x}$ ....[$\because (a^m)^n = a^{mn}$]

$= \left(\sqrt{x}\right)^{\log x}$

82. $(1+y^2)\,dx - (\tan^{-1}y - x)\,dy = 0$

$\Rightarrow (1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$

$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$

This is the linear differential equation of the form $\frac{dx}{dy} + P.x = Q$, where $P = \frac{1}{1+y^2}$

$\therefore$ I.F. $= e^{\int P dy} = e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

83. $\frac{dy}{dx} + 2\cot x.y = 3x^2\text{cosec}^2 x$

$\therefore$ I.F. $= e^{2\int\cot x dx} = e^{2\log\sin x} = \sin^2 x$

$\therefore$ solution of the given equation is

$y.\sin^2 x = \int 3x^2\text{cosec}^2 x.\sin^2 x dx + c$

$\Rightarrow y\sin^2 x = \int 3x^2 dx + c \Rightarrow y\sin^2 x = x^3 + c$

84. $\dfrac{dy}{dx} + 2y\tan x = \sin x$

Here, P = 2 tan x and Q = sin x

$\therefore$ I.F. = $e^{\int 2\tan x\,dx}$

$= e^{2\log(\sec x)} = e^{\log \sec^2 x} = \sec^2 x$

$\therefore$ solution of the given equation is

$y(\sec^2 x) = \int \sin x \sec^2 x\,dx + c$

$\Rightarrow y\sec^2 x = \int \sec x \tan x\,dx + c$

$\Rightarrow y\sec^2 x = \sec x + c$

85. $\dfrac{dy}{dx} = \dfrac{1}{x+y+1} \Rightarrow \dfrac{dx}{dy} = x + y + 1$

$\Rightarrow \dfrac{dx}{dy} - x = y + 1$

$\therefore$ I.F. = $e^{\int -1dy} = e^{-y}$

$\therefore$ solution of the given equation is

$x.e^{-y} = \int (y+1)e^{-y}\,dy + c$

$\Rightarrow xe^{-y} = e^{-y}(-y-2) + c$

$\Rightarrow x = ce^{y} - y - 2$

86. $x\dfrac{dy}{dx} + y = x^2 + 3x + 2 \Rightarrow \dfrac{dy}{dx} + \dfrac{y}{x} = x + 3 + \dfrac{2}{x}$

Here, P = $\dfrac{1}{x}$, Q = $x + 3 + \dfrac{2}{x}$

$\therefore$ I.F. = $e^{\int \frac{1}{x}dx} = e^{\log x} = x$

$\therefore$ solution of the given equation is

$y.\,x = \int \left(x + 3 + \dfrac{2}{x}\right) x\,dx + c$

$\Rightarrow xy = \int x^2 dx + \int 3x\,dx + \int 2\,dx + c$

$\Rightarrow xy = \dfrac{x^3}{3} + \dfrac{3x^2}{2} + 2x + c$

87. $x\log x\dfrac{dy}{dx} + y = 2\log x$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x\log x}y = \dfrac{2}{x}$

$\therefore$ I.F. = $e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

$\therefore$ solution of the given equation is

$y\log x = \int \dfrac{2}{x}.\log x\,dx + c$

$\Rightarrow y\log x = (\log x)^2 + c$

88. $\dfrac{dy}{dx} + \dfrac{3x^2}{1+x^3}y = \dfrac{\sin^2 x}{1+x^3}$

Here, P = $\dfrac{3x^2}{1+x^3}$ and Q = $\dfrac{\sin^2 x}{1+x^3}$

$\therefore$ I.F. = $e^{\int \frac{3x^2}{1+x^3}dx} = e^{\log(1+x^3)} = 1 + x^3$

$\therefore$ solution of the given equation is

$y.(1+x^3) = \int \dfrac{\sin^2 x}{1+x^3}(1+x^3)dx$

$\Rightarrow y(1+x^3) = \int \dfrac{1-\cos 2x}{2}dx$

$\Rightarrow y(1+x^3) = \dfrac{1}{2}x - \dfrac{\sin 2x}{4} + c$

89. $\dfrac{dy}{dx} + y\sec^2 x = \tan x \sec^2 x$

Here, P = $\sec^2 x$, Q = $\tan x \sec^2 x$

$\therefore$ I.F. = $e^{\int \sec^2 x\,dx} = e^{\tan x}$

$\therefore$ solution of the given equation is

$y.e^{\tan x} = \int \tan x.\sec^2 x\,e^{\tan x}\,dx + c$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\therefore$ $y\,e^{\tan x} = \int t\,e^t\,dt + c$

$\Rightarrow y\,e^{\tan x} = t\,e^t - e^t + c$

$\Rightarrow y\,e^{\tan x} = e^{\tan x}(\tan x - 1) + c$

$\Rightarrow y = \tan x - 1 + c\cdot e^{-\tan x}$

90. $(x + 2y^3)\dfrac{dy}{dx} - y = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x+2y^3}$

$\Rightarrow \dfrac{dx}{dy} = \dfrac{x+2y^3}{y} \Rightarrow \dfrac{dx}{dy} - \dfrac{x}{y} = 2y^2$

$\therefore$ I.F. = $e^{\int -\frac{1}{y}dy} = e^{-\log y} = \dfrac{1}{y}$

$\therefore$ solution of the given equation is

$x(\text{I.F.}) = \int Q(\text{I.F.})\,dy + c$

$\Rightarrow x.\dfrac{1}{y} = \int 2y^2.\dfrac{1}{y}dy + c$

$\Rightarrow \dfrac{x}{y} = y^2 + c$

$\Rightarrow x = y^3 + c.y$

$\Rightarrow y^3 - x = -cy$

$\Rightarrow y^3 - x = Ay$, where A = – c

91. $x\mathrm{d}y + y\mathrm{d}x + \log y\mathrm{d}y = 0$

$\Rightarrow x\mathrm{d}y + y\mathrm{d}x = -\log y\mathrm{d}y$

$\Rightarrow y\dfrac{\mathrm{d}x}{\mathrm{d}y} + x = -\log y \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}y} + \dfrac{x}{y} = -\dfrac{\log y}{y}$

$\therefore$ I.F. $= e^{\int \frac{1}{y}\mathrm{d}y} = e^{\log y} = y$

$\therefore$ solution of the given equation is

$x.y = -\int y.\dfrac{\log y\mathrm{d}y}{y} + c$

$\Rightarrow xy = -(y\log y - y) + c$

$\Rightarrow xy + (y\log y - y) = c$

92. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y\tan x - y^2\sec x$

$\Rightarrow \dfrac{1}{y^2}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x} - \dfrac{1}{y}\tan x = -\sec x$ ….(i)

Put $v = y^{-1}$

$\Rightarrow \dfrac{\mathrm{d}v}{\mathrm{d}x} = -\dfrac{1}{y^2}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x}$

$\therefore$ $-\dfrac{\mathrm{d}v}{\mathrm{d}x} - v\tan x = -\sec x$ ….[From (i)]

$\Rightarrow \dfrac{\mathrm{d}v}{\mathrm{d}x} + v\tan x = \sec x$

This is the standard form of the linear differential equation.

$\therefore$ I.F. $= e^{\int \tan x\,\mathrm{d}x} = e^{\log\sec x} = \sec x$

93. $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y + x^3e^x \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} - \dfrac{2}{x}y = x^2e^x$

$\therefore$ I.F. $= e^{-\int\frac{2}{x}\mathrm{d}x} = e^{-2\log x} = e^{\log x^{-2}} = \dfrac{1}{x^2}$

$\therefore$ solution of the given equation is

$y\cdot\dfrac{1}{x^2} = \int x^2e^x\cdot\dfrac{1}{x^2}\mathrm{d}x + c$

$\Rightarrow \dfrac{y}{x^2} = e^x + c$ ….(i)

Since, $y = 0$, when $x = 1$

$\therefore$ $0 = e^1 + c \Rightarrow c = -e$

$\therefore$ $\dfrac{y}{x^2} = e^x - e$ ….[From (i)]

$\Rightarrow y = x^2(e^x - e)$

94. $x\mathrm{d}y = y(\mathrm{d}x + y\mathrm{d}y)$

$\Rightarrow y\mathrm{d}x = (x - y^2)\mathrm{d}y \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}y} + \left(-\dfrac{1}{y}\right)x = -y$

$\therefore$ I.F. $= e^{\int -\frac{1}{y}\mathrm{d}y} = e^{-\log y} = \dfrac{1}{y}$

$\therefore$ solution of the given equation is

$x\cdot\dfrac{1}{y} = \int -y\cdot\dfrac{1}{y}\mathrm{d}y + c$

$\Rightarrow \dfrac{x}{y} = -y + c$ ....(i)

Since, $y(1) = 1$ i.e., $y = 1$, when $x = 1$

$\therefore$ $1 = -1 + c \Rightarrow c = 2$

$\therefore$ $\dfrac{x}{y} = -y + 2$ ....[From (i)]

Putting $x = -3$, we get

$-\dfrac{3}{y} = -y + 2$

$\Rightarrow y^2 - 2y - 3 = 0$

$\Rightarrow (y - 3)(y + 1) = 0$

Since $y(x) > 0$, $y = 3$

95. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + 2x$

$\therefore$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = 2x$

$\therefore$ I.F. $= e^{\int -1.\mathrm{d}x} = e^{-x}$

$\therefore$ solution of the given equation is

$ye^{-x} = \int 2x\,e^{-x}\,\mathrm{d}x + c$

$= -2xe^{-x} - 2e^{-x} + c$

$\therefore$ $ye^{-x} = -2(x + 1)e^{-x} + c$

$\therefore$ $y = -2(x + 1) + ce^x$ ….(i)

Since, the curve passes through origin (0, 0).

$\therefore$ $0 = -2(0 + 1) + ce^0$

$\Rightarrow c = 2$

$\therefore$ $y = -2(x + 1) + 2e^x$ ….[From (i)]

$\therefore$ $y + 2(x + 1) = 2e^x$

96. Let P be the population at time t years.

Then, $\dfrac{\mathrm{dP}}{\mathrm{dt}} = \mathrm{kP}$

$\Rightarrow \dfrac{\mathrm{dP}}{\mathrm{P}} = \mathrm{kdt}$

Integrating on both sides, we get

$\log P = kt + c$

When $t = 0$, $P = 40000$

$\therefore$ $\log 40000 = 0 + c \Rightarrow c = \log 40000$

$\therefore$ $\log P = kt + \log 40000$

$\Rightarrow \log\left(\dfrac{P}{40000}\right) = kt$ ....(i)

When $t = 40$ yrs, $P = 60000$

$\therefore$ $\log\left(\dfrac{60000}{40000}\right) = 40\,k \Rightarrow k = \dfrac{1}{40}\log\left(\dfrac{3}{2}\right)$

$\therefore \quad \log\left(\dfrac{P}{40000}\right) = \dfrac{t}{40}\log\left(\dfrac{3}{2}\right)$ ....[From (i)]

When t = 60 yrs, we have

$\log\left(\dfrac{P}{40000}\right) = \dfrac{60}{40}\log\left(\dfrac{3}{2}\right)$

$\Rightarrow \dfrac{P}{40000} = \left(\dfrac{3}{2}\right)^{\frac{3}{2}} \Rightarrow \dfrac{P}{40000} = \dfrac{3}{2}\left(\dfrac{3}{2}\right)^{\frac{1}{2}}$

$\Rightarrow P = 40000 \times \dfrac{3}{2} \times 1.2247 = 73482$

97. Let P_0 be the initial population and let the population after t years be P. Then,

$\dfrac{dP}{dt} = kP \Rightarrow \dfrac{dP}{P} = k\,dt$

Integrating on both sides, we get

$\log P = kt + c$

When t = 0, $P = P_0$

$\therefore \quad \log P_0 = 0 + c \Rightarrow c = \log P_0$

$\therefore \quad \log P = kt + \log P_0$

$\Rightarrow \log\dfrac{P}{P_0} = kt$ ....(i)

When t = 5 hrs, $P = 2P_0$

$\therefore \quad \log\dfrac{2P_0}{P_0} = 5k$

$\Rightarrow k = \dfrac{\log 2}{5}$

$\therefore \quad \log\dfrac{P}{P_0} = \dfrac{\log 2}{5}t$ ....[From (i)]

When t = 25 hrs, we have

$\log\dfrac{P}{P_0} = \dfrac{\log 2}{5} \times 25 = 5\log 2 = \log 32$

$\therefore \quad P = 32P_0$

98. Let 'θ' be the temperature of the body at any time 't'.

$\therefore \quad \dfrac{d\theta}{dt} \propto (\theta - 20)$

$\therefore \quad \dfrac{d\theta}{dt} = k(\theta - 20)$

Integrating on both sides, we get

$\log(\theta - 20) = kt + c$

When t = 0, θ = 100° C

$\therefore \quad \log(100 - 20) = k(0) + c \quad \Rightarrow c = \log 80$

$\therefore \quad \log(\theta - 20) = kt + \log 80$ ….(i)

When t = 20, θ = 60° C

$\therefore \quad \log(60 - 20) = k(20) + \log 80$

$\Rightarrow k = \dfrac{1}{20}\log\left(\dfrac{1}{2}\right)$

$\therefore \quad \log(\theta - 20) = \dfrac{t}{20}\log\left(\dfrac{1}{2}\right) + \log 80$ ….[From (i)]

When θ = 30° C, we have

$\log(30 - 20) = t\left(\dfrac{1}{20}\right)\log\left(\dfrac{1}{2}\right) + \log 80$

$\Rightarrow \log 10 - \log 80 = \dfrac{t}{20}\log\left(\dfrac{1}{2}\right)$

$\Rightarrow \log\left(\dfrac{1}{8}\right) = \dfrac{t}{20}\log\left(\dfrac{1}{2}\right)$

$\Rightarrow 3\log\left(\dfrac{1}{2}\right) = \dfrac{t}{20}\log\left(\dfrac{1}{2}\right)$

$\Rightarrow \dfrac{t}{20} = 3$

$\Rightarrow t = 60$ minutes

99. Let 'x' be the number of bacteria present at time 't'.

$\therefore \quad \dfrac{dx}{dt} \propto x$

$\therefore \quad \dfrac{dx}{dt} = kx$

Integrating on both sides, we get

$\log x = kt + c$

When t = 0, $x = 1000$

$\therefore \quad \log(1000) = k(0) + c$

$\Rightarrow c = \log(1000)$

$\therefore \quad \log x = kt + \log(1000)$ ....(i)

When t = 1, $x = 2000$

$\therefore \quad \log(2000) = k(1) + \log(1000)$

$\Rightarrow k = \log\left(\dfrac{2000}{1000}\right) = \log 2$

$\therefore \quad \log x = t\log 2 + \log(1000)$ ….[From (i)]

When $t = 2\dfrac{1}{2} = \dfrac{5}{2}$, we have

$\log x = \left(\dfrac{5}{2}\right)\log 2 + \log(1000)$

$= \log\left(2^{\frac{5}{2}}\right) + \log(1000)$

$= \log\left(4\sqrt{2}\right) + \log(1000)$

$= \log(4000\sqrt{2})$

$= \log(4000 \times 1.414)$

$\therefore \quad \log x = \log(5656)$

$\Rightarrow x = 5656$

Competitive Thinking

1. Here, the highest order derivative is $\frac{d^3y}{dx^3}$.

$\therefore$ order = 3

2. Here, the order of the differential equation is 1.

3. Here, the highest order derivative is $\frac{d^3y}{dx^3}$.

$\therefore$ order = 3

6. Here, the highest order derivative is $\frac{d^3y}{dx^3}$ with power 2.

$\therefore$ order = 3, degree = 2

7. $y = x\frac{dy}{dx} + \frac{2}{\frac{dy}{dx}}$

$\Rightarrow y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + 2$

$\therefore$ order = 1, degree = 2

8. $y = x\left(\frac{dy}{dx}\right)^2 + \left(\frac{dx}{dy}\right)^2$

$\Rightarrow y\left(\frac{dy}{dx}\right)^2 = x\left(\frac{dy}{dx}\right)^4 + 1$

$\therefore$ order = 1, degree = 4

9. $\left(1+3\frac{dy}{dx}\right)^{\frac{2}{3}} = 4\ \frac{d^3y}{dx^3}$

$\Rightarrow \left(1+3\frac{dy}{dx}\right)^2 = 4^3\left(\frac{d^3y}{dx^3}\right)^3$

Here, the highest order derivative is $\frac{d^3y}{dx^3}$ with power 3.

$\therefore$ order = 3 and degree = 3

10. $\frac{d^2y}{dx^2} = \sqrt[3]{1+\left(\frac{dy}{dx}\right)^2}$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^3 = 1 + \left(\frac{dy}{dx}\right)^2$

Here, the highest order derivatives is $\frac{d^2y}{dx^2}$ with power 3

$\therefore$ Order = 2 and degree = 3

11. $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{7}{3}} = 7\left(\frac{d^2y}{dx^2}\right)$

$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^3\right]^7 = 7^3\left(\frac{d^2y}{dx^2}\right)^3$

Here, the highest order derivative is $\frac{d^2y}{dx^2}$ with power 3.

$\therefore$ order = 2 and degree = 3

12. $\frac{d^3y}{dx^3} = \sqrt[5]{1-\left(\frac{dy}{dx}\right)^7}$

$\Rightarrow \left(\frac{d^3y}{dx^3}\right)^5 = 1-\left(\frac{dy}{dx}\right)^7$

$\therefore$ order = 3, degree = 5

13. $p.\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$

$\Rightarrow \left(p.\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$

$\therefore$ order = 2 and degree = 2

14. $\left(\frac{d^2y}{dx^2}\right)^{1/3} + \left(x+\frac{dy}{dx}\right)^{1/2} = 0$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^{1/3} = -\left(x+\frac{dy}{dx}\right)^{1/2}$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left(x+\frac{dy}{dx}\right)^3$

$\therefore$ order = 2 and degree = 2

15. $\left(1+y_1^2\right)^{2/3} = y_2$

$\Rightarrow \left(1+y_1^2\right)^2 = \left(y_2\right)^3$

$\therefore$ order(n) = 2, degree(m) = 3

$\therefore$ $\frac{m+n}{m-n} = \frac{3+2}{3-2} = 5$

16. $y = px + \sqrt[3]{a^2p^2+b^2}$

$\Rightarrow (y - px)^3 = a^2p^2 + b^2$

$\Rightarrow y^3 - 3y^2px + 3p^2x^2y - p^3x^3 = a^2p^2 + b^2$

Here, $p = \frac{dy}{dx}$

$\therefore$ order = 1, degree = 3

17. $\left[1+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)\right]^{\frac{3}{4}}=\frac{d^2y}{dx^2}$

$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)\right]^3=\left(\frac{d^2y}{dx^2}\right)^4$

Here, the highest order derivative is $\frac{d^2y}{dx^2}$.

$\therefore$ order = 2

Since, the given differential equation cannot be expressed as polynomial in differential coefficients, the degree is not defined.

18. $y_2^{3/2}-y_1^{1/2}-4=0$

$\Rightarrow y_2^{3/2}=y_1^{1/2}+4$

Squaring on both sides, we get

$y_2^3=\left(y_1^{1/2}+4\right)^2=y_1+16+8y_1^{1/2}$

$\Rightarrow y_2^3-y_1-16=8y_1^{1/2}$

Squaring on both sides, we get

$\left(y_2^3-y_1-16\right)^2=64y_1$

Here, the highest order derivative is y_2 with power 6.

$\therefore$ degree = 6

19. $\sqrt{\sin x}\,(dx+dy) = \sqrt{\cos x}\,(dx-dy)$

$\Rightarrow \frac{\sqrt{\sin x}}{\sqrt{\cos x}}(dx+dy)=dx-dy$

$\Rightarrow \sqrt{\tan x}\left(1+\frac{dy}{dx}\right)=1-\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{1-\sqrt{\tan x}}{1+\sqrt{\tan x}}$

This is a differential equation of order 1 and degree 1.

20. $y(x)=1+\frac{dy}{dx}+\frac{1}{1.2}\left(\frac{dy}{dx}\right)^2+\frac{1}{1.2.3}\left(\frac{dy}{dx}\right)^3+\ldots.$

$\Rightarrow y(x)=e^{\frac{dy}{dx}}$ ....$\left[\because e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots.\right]$

$\Rightarrow \frac{dy}{dx}=\log y$

This is a differential equation of degree 1.

21. $y=C_1e^{2x+C_2}+C_3e^x+C_4\sin(x+C_5)$

$=C_1.e^{C_2}e^{2x}+C_3e^x+C_4(\sin x\cos C_5+\cos x\sin C_5)$

$=Ae^{2x}+C_3e^x+B\sin x+D\cos x$,

where $A=C_1e^{C_2}$, $B=C_4\cos C_5$, $D=C_4\sin C_5$

Since, this equation consists of four arbitrary constants.

$\therefore$ order of differential equation = 4

22. Consider option (C),

$y=2x-4$

$\therefore \frac{dy}{dx}=2$

$\therefore \left(\frac{dy}{dx}\right)^2-\frac{xdy}{dx}+y=2^2-2x+2x-4=0$

23. $y=a+\frac{b}{x}$

$\Rightarrow \frac{dy}{dx}=-\frac{b}{x^2}$ ….(i)

$\Rightarrow \frac{d^2y}{dx^2}=\frac{2b}{x^3} \Rightarrow x\frac{d^2y}{dx^2}=\frac{2b}{x^2}$

$\Rightarrow x\frac{d^2y}{dx^2}-\frac{2b}{x^2}=0$

$\Rightarrow x\frac{d^2y}{dx^2}+\frac{2dy}{dx}=0$ ….[From (i)]

24. $y=e^{-x}\cos 2x$

$\Rightarrow \frac{dy}{dx}=-2e^{-x}\sin 2x-e^{-x}\cos 2x$

$\Rightarrow \frac{d^2y}{dx^2}=4e^{-x}\sin 2x-3e^{-x}\cos 2x$

$\therefore \frac{d^2y}{dx^2}+\frac{2dy}{dx}+5y=0$

25. $y=mx+\frac{4}{m}$ ….(i)

$\Rightarrow \frac{dy}{dx}=m$

Putting $m=\frac{dy}{dx}$ in (i), we get

$y\left(\frac{dy}{dx}\right)=x\left(\frac{dy}{dx}\right)^2+4$

26. $y=ae^{bx}$ ….(i)

$\Rightarrow \frac{dy}{dx}=abe^{bx}$

$\Rightarrow \frac{dy}{dx}=by$ ….(ii) [From (i)]

$\Rightarrow \frac{d^2y}{dx^2}=b\frac{dy}{dx} \Rightarrow y\frac{d^2y}{dx^2}=by\frac{dy}{dx}$

$\Rightarrow y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=0$ ….[From (ii)]

27. $y = \dfrac{A}{x} + Bx^2$

$\Rightarrow xy = A + Bx^3$

Differentiating w.r.t. x, we get

$x\dfrac{dy}{dx} + y = 3Bx^2$ ….(i)

Again, differentiating w.r.t. x, we get

$x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 6Bx$

$\Rightarrow x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 6Bx^2$

$\Rightarrow x^2\dfrac{d^2y}{dx^2} + 2(3Bx^2 - y) = 6Bx^2$ ….[From (i)]

$\Rightarrow x^2\dfrac{d^2y}{dx^2} = 2y$

28. $y = e^{mx}$

$\Rightarrow \log y = mx$ …(i)

Differentiating w.r.t. x, we get

$\dfrac{1}{y}\cdot\dfrac{dy}{dx} = m$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{\log y}{x}$ ….[From (i)]

$\Rightarrow \dfrac{dy}{dx} = \left(\dfrac{y}{x}\right)\log y$

29. $y = e^x(A\cos x + B\sin x)$

$\Rightarrow y' = e^x(A\cos x + B\sin x) + e^x(B\cos x - A\sin x)$

$\Rightarrow y' = y + e^x(B\cos x - A\sin x)$ …(i)

$\therefore$ $y'' = y' + e^x(B\cos x - A\sin x) - e^x(A\cos x + B\sin x)$

$\Rightarrow y'' = y' + (y' - y) - y$ …[From (i)]

$\Rightarrow y'' - 2y' + 2y = 0$

30. $y = a\sin(\log x) + b\cos(\log x)$ ….(i)

$\Rightarrow \dfrac{dy}{dx} = \dfrac{a\cos(\log x)}{x} - \dfrac{b\sin(\log x)}{x}$

$\Rightarrow x\dfrac{dy}{dx} = a\cos(\log x) - b\sin(\log x)$

Differentiating w.r.t. x, we get

$x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = \dfrac{-a\sin(\log x)}{x} - \dfrac{b\cos(\log x)}{x}$

$\Rightarrow x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = -\dfrac{1}{x}[a\sin(\log x) + b\cos(\log x)]$

$\Rightarrow x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = -y$ ….[From (i)]

$\Rightarrow x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

31. Differentiating the given equation, we get $\dfrac{dy}{dx} = A$

$\therefore$ $y = x\left(\dfrac{dy}{dx}\right) + \left(\dfrac{dy}{dx}\right)^3$, which is of degree 3

32. $x = A\cos(nt + \alpha)$

Diferentiating w.r.t t, we get

$\dfrac{dx}{dt} = -A\sin(nt + \alpha)\,.\,n$

$\Rightarrow \dfrac{dx}{dt} = -An\sin(nt + \alpha)$

Differentiating w.r.t t, we get

$\dfrac{d^2x}{dt^2} = -An^2\cos(nt + \alpha)$

$\Rightarrow \dfrac{d^2x}{dt^2} = -n^2x$

$\Rightarrow \dfrac{d^2x}{dt^2} + n^2x = 0$

33. $y = e^{a\sin x}$

$\Rightarrow \log y = a\sin x$ ….(i)

Differentiating w.r.t. x, we get

$\dfrac{1}{y}.\dfrac{dy}{dx} = a\cos x$

$\Rightarrow a = \dfrac{1}{y\cos x}.\dfrac{dy}{dx}$

Putting the value of a in (i), we get

$y\log y = \tan x\dfrac{dy}{dx}$

34. $y^2 = 2d\left(x + \sqrt{d}\right)$ ....(i)

Differentiating w.r.t. x, we get

$2y\dfrac{dy}{dx} = 2d$ ….(ii)

Substituting (ii) in (i), we get

$y^2 = 2y\dfrac{dy}{dx}\left(x + \sqrt{y\dfrac{dy}{dx}}\right)$

$\Rightarrow y = 2x\dfrac{dy}{dx} + 2\dfrac{dy}{dx}.\sqrt{y\dfrac{dy}{dx}}$

$\Rightarrow \left(y - 2x\dfrac{dy}{dx}\right)^2 = 4y\left(\dfrac{dy}{dx}\right)^3$

This is a differential equation of order 1 and degree 3.

35. Required equation of parabola is

$(y - k)^2 = 4a(x - h)$

Since, this equation has two arbitary constants, it 's order is 2.

36. Equation of family of parabolas whose axis is X-axis is $y^2 = 4a(x - h)$
Differentiating w.r.t. x, we get
$$2y\frac{dy}{dx} = 4a$$
$$\Rightarrow y\frac{dy}{dx} = 2a$$
Again, differentiating w.r.t. x, we get
$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$
$\therefore$ order = 2 and degree = 1

37. $y = ax\cos\left(\frac{1}{x} + b\right)$ ....(i)
$$\Rightarrow \frac{dy}{dx} = -ax\sin\left(\frac{1}{x}+b\right).\left(-\frac{1}{x^2}\right) + a\cos\left(\frac{1}{x}+b\right)$$
$$\Rightarrow \frac{dy}{dx} = \frac{a}{x}.\sin\left(\frac{1}{x}+b\right) + a\cos\left(\frac{1}{x}+b\right)$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{a}{x}\cos\left(\frac{1}{x}+b\right).\left(-\frac{1}{x^2}\right)$$
$$-\frac{a}{x^2}\sin\left(\frac{1}{x}+b\right) - a\sin\left(\frac{1}{x}+b\right).\left(-\frac{1}{x^2}\right)$$
$$= -\frac{a}{x^3}\cos\left(\frac{1}{x}+b\right) = -\frac{ax}{x^4}\cos\left(\frac{1}{x}+b\right)$$
$\therefore$ $\frac{d^2y}{dx^2} = -\frac{y}{x^4}$ ....[From (i)]
$$\Rightarrow x^4\frac{d^2y}{dx^2} + y = 0 \Rightarrow x^4y_2 + y = 0$$

38. The differential equation representing the family of parabolas having vertex at origin is
$y^2 = 4ax$ ....(i)
Differentiating w.r.t. x, we get
$$2y\frac{dy}{dx} = 4a$$
$$\Rightarrow 2y\frac{dy}{dx} = \frac{y^2}{x} \quad \text{....[From (i)]}$$
$$\Rightarrow 2yx\frac{dy}{dx} = y^2$$
$$\Rightarrow y^2 - 2xy\frac{dy}{dx} = 0$$

39. Equation of family of parabolas with focus at (0, 0) and X-axis as axis is $y^2 = 4a(x + a)$ ....(i)
Differentiating (i) w.r.t. x, we get
$$2y\frac{dy}{dx} = 4a \quad \text{....(ii)}$$
Substituting (ii) in (i), we get
$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$$
$$\Rightarrow y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$$
$$\Rightarrow -y\left(\frac{dy}{dx}\right)^2 = 2x\frac{dy}{dx} - y$$

40. Equation of family of parabolas with focus at (0, 0) and X-axis as axis is $y^2 = 4a(x + a)$ ....(i)
Differentiating (i) w.r.t. x, we get
$$2y\frac{dy}{dx} = 4a \quad \text{....(ii)}$$
Substituting (ii) in (i), we get
$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$$
$$\Rightarrow y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$$
$\therefore$ order = m = 1, degree = n = 2
Now, mn – m + n = 1(2) – 1 + 2 = 3

41. Axis of parabola = X-axis
vertex = (m, 0)
Equation of all parabolas is
$(y - 0)^2 = 4a(x - m)$
$\Rightarrow y^2 = 4ax - 4am$
$$\therefore \quad 2y\frac{dy}{dx} = 4a$$
$$\Rightarrow y\frac{dy}{dx} = 2a$$
$$\therefore \quad y\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{dy}{dx}\right) = 0$$
$$\Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

42. Axis of parabola = Y-axis
Vertex = (0, m)
$\therefore$ Equation of parabola is
$(x - 0)^2 = 4a(y - m)$
$\Rightarrow x^2 = 4ay - 4am$
Differentiating w.r.t. x, we get
$$2x = 4a\frac{dy}{dx}$$
$$\Rightarrow \frac{1}{x}\frac{dy}{dx} = \frac{1}{2a} \Rightarrow \frac{1}{x}\frac{d^2y}{dx^2} - \frac{dy}{dx}\frac{1}{x^2} = 0$$
$$\Rightarrow x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$$

43. The differential equation of the family of circles touching Y- axis at the origin is
$x^2 + y^2 - 2ax = 0$ ….(i)
Differentiating w.r.t.x, we get
$$2x + 2y\frac{dy}{dx} - 2a = 0$$
$$\Rightarrow 2a = 2x + 2y\frac{dy}{dx} \quad ….(ii)$$
Substituting (ii) in (i), we get
$$x^2 + y^2 - 2x^2 - 2xy\frac{dy}{dx} = 0$$
$$\Rightarrow x^2 - y^2 + 2xy\frac{dy}{dx} = 0$$

44. The system of circles which passes through origin and whose centre lies on X-axis is $x^2 + y^2 - 2bx = 0$ ...(i)
Differentating w.r.t x, we get
$$2x + 2y\frac{dy}{dx} = 2b \quad ...(ii)$$
Substituting (ii) in (i), we get
$$x^2 + y^2 - 2x^2 - 2xy\frac{dy}{dx} = 0$$
$$y^2 - x^2 - 2xy\frac{dy}{dx} = 0$$

45. The equation of the family of lines which are at a unit distance from the origin is $x\cos\alpha + y\sin\alpha = 1$ ….(i)
Differentiating w.r.t. x, we get
$$\cos\alpha + \sin\alpha\frac{dy}{dx} = 0 \quad ….(ii)$$
By (i) – x × (ii), we get
$$\sin\alpha\left(y - x\frac{dy}{dx}\right) = 1$$
$$\Rightarrow y - x\frac{dy}{dx} = \operatorname{cosec}\alpha \quad ….(iii)$$
From (ii), $\left(\frac{dy}{dx}\right)^2 = \cot^2\alpha = \operatorname{cosec}^2\alpha - 1$
$$\therefore \left(\frac{dy}{dx}\right)^2 = \left(y - x\frac{dy}{dx}\right)^2 - 1 \quad ….[\text{From (iii)}]$$
$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = \left(y - x\frac{dy}{dx}\right)^2$$

46. $\sec x\,dy - \operatorname{cosec} y\,dx = 0$
$\Rightarrow \cos x\,dx - \sin y\,dy = 0$
Integrating on both sides, we get
$\sin x + \cos y = c$

47. $$\frac{dy}{dx} = 1 + x + y + xy$$
$$\Rightarrow \frac{dy}{dx} = (1 + x)(1 + y) \Rightarrow \frac{dy}{1+y} = (1 + x)dx$$
Integrating on both sides, we get
$$\log(1 + y) = x + \frac{x^2}{2} + c$$

48. $$9y\frac{dy}{dx} + 4x = 0$$
Integrating on both sides, we get
$$9\int y\,dy + 4\int x\,dx = c_1$$
$$\Rightarrow 9.\frac{y^2}{2} + 4.\frac{x^2}{2} = c_1$$
$$\Rightarrow \frac{y^2}{4} + \frac{x^2}{9} = \frac{c_1}{18}$$
$$\Rightarrow \frac{y^2}{4} + \frac{x^2}{9} = c, \text{ where } c = \frac{c_1}{18}$$

49. $(1 + y^2)\tan^{-1}x\,dx + (1 + x^2)\,2y\,dy = 0$
$$\Rightarrow \frac{\tan^{-1}x\,dx}{1+x^2} + \frac{2y}{1+y^2}dy = 0$$
Integrating on both sides, we get
$$\frac{\left(\tan^{-1}x\right)^2}{2} + \log|1 + y^2| = c_1$$
$\Rightarrow (\tan^{-1}x)^2 + 2\log|1 + y^2| = c$, where $c = 2c_1$

50. $$x\left(\frac{dy}{dx}\right)^2 + 2\sqrt{xy}\frac{dy}{dx} + y = 0$$
$$\Rightarrow \left(\sqrt{x}\frac{dy}{dx} + \sqrt{y}\right)^2 = 0$$
$$\Rightarrow \sqrt{x}\frac{dy}{dx} + \sqrt{y} = 0$$
Integrating on both sides, we get
$$\int\frac{dy}{\sqrt{y}} + \int\frac{dx}{\sqrt{x}} = c$$
$$\Rightarrow 2\sqrt{y} + 2\sqrt{x} = c$$
$$\Rightarrow \sqrt{x} + \sqrt{y} = \sqrt{a}, \text{ where } \sqrt{a} = \frac{c}{2}$$

51. $$\frac{dx}{x} + \frac{dy}{y} = 0$$
Integrating on both sides, we get
$\log x + \log y = \log c$
$\Rightarrow \log(xy) = \log c \Rightarrow xy = c$

52. $x\dfrac{dy}{dx} - y = 3$

$\Rightarrow x\dfrac{dy}{dx} = 3 + y$

$\Rightarrow \int \dfrac{1}{3+y}dy = \int \dfrac{1}{x}dx$

$\Rightarrow \log|y + 3| = \log|x| + \log c$

$\Rightarrow y + 3 = xc$

$\Rightarrow y = xc - 3$

This is the equation of family of straight line.

53. $\dfrac{dy}{dx} = \dfrac{(1+x)y}{(y-1)x}$

$\Rightarrow \dfrac{y-1}{y}\,dy = \dfrac{(1+x)}{x}dx$

$\Rightarrow \left(1-\dfrac{1}{y}\right)dy = \left(1+\dfrac{1}{x}\right)dx$

Integrating on both sides, we get

$x + \log x = y - \log y + c$

$\Rightarrow x - y + \log xy = c$

54. $y(1 + \log x)\,\dfrac{dx}{dy} - x\log x = 0$

$\Rightarrow y(1 + \log x)\,\dfrac{dx}{dy} = x\log x$

$\Rightarrow \left(\dfrac{1+\log x}{x\log x}\right)dx = \dfrac{dy}{y}$

Integrating on both sides, we get

$\log(\log x) + \log x = \log y + \log c$

$\Rightarrow \log(x\log x) = \log(yc)$

$\Rightarrow x\log x = cy$

55. $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$

$\Rightarrow y - ay^2 = a\dfrac{dy}{dx} + x\dfrac{dy}{dx}$

$\Rightarrow y(1 - ay) = (a + x)\dfrac{dy}{dx}$

Integrating on both sides, we get

$\int \dfrac{dx}{a+x} = \int \dfrac{dy}{y(1-ay)} + \log c$

$\Rightarrow \int \dfrac{dx}{a+x} = \int\left[\dfrac{1}{y} + \dfrac{a}{1-ay}\right]dy + \log c$

$\Rightarrow \log(a + x) = \log y - \log(1 - ay) + \log c$

$\Rightarrow \log[(a + x)(1 - ay)] = \log cy$

$\Rightarrow (x + a)(1 - ay) = cy$

56. $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$

Integrating on both sides, we get

$\int \dfrac{dy}{\sqrt{1-y^2}} + \int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1}c$

$\Rightarrow \sin^{-1}y + \sin^{-1}x = \sin^{-1}c$

$\Rightarrow \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) = \sin^{-1}c$

$\Rightarrow x\sqrt{1-y^2} + y\sqrt{1-x^2} = c$

57. $\dfrac{dy}{dx} + \sin\left(\dfrac{x+y}{2}\right) = \sin\left(\dfrac{x-y}{2}\right)$

$\Rightarrow \dfrac{dy}{dx} = \sin\left(\dfrac{x-y}{2}\right) - \sin\left(\dfrac{x+y}{2}\right)$

$\Rightarrow \dfrac{dy}{dx} = -2\sin\left(\dfrac{y}{2}\right).\cos\left(\dfrac{x}{2}\right)$

Integrating on both sides, we get

$\int \operatorname{cosec}\left(\dfrac{y}{2}\right)dy = -\int 2\cos\left(\dfrac{x}{2}\right)dx + c_1$

$\Rightarrow \dfrac{\log\tan\left(\dfrac{y}{4}\right)}{\dfrac{1}{2}} = -\dfrac{2\sin\left(\dfrac{x}{2}\right)}{\dfrac{1}{2}} + c_1$

$\Rightarrow \log\tan\left(\dfrac{y}{4}\right) = c - 2\sin\left(\dfrac{x}{2}\right)$, where $c = \dfrac{1}{2}c_1$

58. $\dfrac{dy}{dx} = 1 + y + y^2 + x + xy + xy^2$

$\Rightarrow \dfrac{dy}{dx} = (1 + y + y^2)(x + 1)$

Integrating on both sides, we get

$\int \dfrac{dy}{1+y+y^2} = \int (x+1)dx + c_1$

$\Rightarrow \int \dfrac{dy}{\left(y+\dfrac{1}{2}\right)^2 + \dfrac{3}{4}} = \dfrac{x^2}{2} + x + c_1$

$\Rightarrow \dfrac{1}{\sqrt{3}/2}.\tan^{-1}\left(\dfrac{y+\dfrac{1}{2}}{\sqrt{3}/2}\right) = \dfrac{x^2}{2} + x + c_1$

$\Rightarrow 4\tan^{-1}\left(\dfrac{2y+1}{\sqrt{3}}\right) = \sqrt{3}\left(x^2 + 2x\right) + c$,

where $c = 2\sqrt{3}\,c_1$

59. $\frac{dy}{dx} = 3^{x+y}$

$\Rightarrow 3^x dx - 3^{-y} dy = 0$

Integrating on both sides, we get

$3^x + 3^{-y} = c$

When $x = 0 = y$,

$3^0 + 3^0 = c \Rightarrow c = 2$

$\therefore$ $3^x + 3^{-y} = 2$

$\Rightarrow 3^x + 3^{-y} - 2 = 0$

60. $x dy + 2y\,dx = 0$

$\Rightarrow \frac{dy}{y} + 2\frac{dx}{x} = 0$

$\Rightarrow \int \frac{dy}{y} + 2\int \frac{dx}{x} = 0$

$\Rightarrow \log y + 2\log x = \log c$

$\Rightarrow \log y + \log x^2 = \log c$

$\Rightarrow \log x^2 y = \log c$

$\Rightarrow x^2 y = c$

Given that $x = 2$ and $y = 1$

$\therefore$ $(2)^2\,1 = c$

$\Rightarrow c = 4$

$\therefore$ $x^2 y = 4$ is the particular solution.

61. $2x\frac{dy}{dx} - y = 0$

$\Rightarrow 2x dy = y dx$

$\Rightarrow 2\int \frac{1}{y} dy = \int \frac{1}{x} dx$

$\Rightarrow 2\log y = \log x + \log c$

$\Rightarrow y^2 = xc$

Since, $y(1) = 2$ i.e., $y = 2$ when $x = 1$

$\therefore$ $2^2 = 1 \times c \Rightarrow c = 4$

$\therefore$ $y^2 = 4x$

This represents the equation of parabola.

62. $e^{\frac{dy}{dx}} = x$

$\Rightarrow \frac{dy}{dx} = \log x$

$\Rightarrow \int dy = \int \log x\,dx$

$\Rightarrow y = \log x.(x) - \int dx + c$

$\Rightarrow y = x\log x - x + c$

Since, $y(1) = 3$ i.e., $y = 3$ when $x = 1$

$\therefore$ $3 = \log 1 - 1 + c$

$\Rightarrow c = 4$

$\therefore$ $y = x\log x - x + 4$

63. $\log \frac{dy}{dx} = x$

$\Rightarrow \frac{dy}{dx} = e^x \Rightarrow dy = e^x\,dx$

Integrating on both sides, we get

$y = e^x + c$

At $x = 0$ and $y = 1$,

$1 = e^0 + c$

$\Rightarrow c = 0$

$\therefore$ $y = e^x$

64. $\log\left(\frac{dy}{dx}\right) = 3x + 4y$

$\Rightarrow \frac{dy}{dx} = e^{3x+4y}$

$\Rightarrow e^{-4y}\,dy = e^{3x}dx$

Integrating on both sides, we get

$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + c$

When $y = 0 = x$,

$-\frac{1}{4} = \frac{1}{3} + c \Rightarrow c = -\frac{7}{12}$

$\therefore$ $\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12}$

$\Rightarrow 4e^{3x} + 3e^{-4y} - 7 = 0$

65. $\frac{dy}{dx} = y + 3$

$\Rightarrow \frac{dy}{y+3} = dx$

Integrating on both sides, we get

$\int \frac{dy}{y+3} = \int dx + c$

$\Rightarrow \log(y + 3) = x + c$ ....(i)

Since, $y(0) = 2$ i.e., $y = 2$, when $x = 0$

$\therefore$ $\log(2 + 3) = 0 + c \Rightarrow c = \log 5$

$\therefore$ $\log(y + 3) = x + \log 5$ ....[From (i)]

$\Rightarrow y + 3 = 5e^x$

$\Rightarrow y = 5e^x - 3$

$\therefore$ $y(\log 2) = 5e^{\log 2} - 3 = 10 - 3$

$= 7$

66. $\frac{dy}{dx} = 1 - y$

$\int \frac{dy}{1-y} = \int dx + c$

$\Rightarrow -\log(1 - y) = x + \log c$

$-\log(1 - y) - \log c = x$

$\log(1 - y)c = -x$

$(1 - y)c = e^{-x}$ ...(i)

Since, $y(0) = 3$ i.e., $y = 3$, when $x = 0$

$\therefore$ $-2c = e^0 \Rightarrow c = \frac{-1}{2}$

$\therefore$ $(1 - y)\left(\frac{-1}{2}\right) = e^{-x}$ ...[From (i)]

$\Rightarrow y - 1 = 2e^{-x}$

$\Rightarrow y = 2e^{-x} + 1$

$\Rightarrow y = 2e^{-\log_e 8} + 1$

$\Rightarrow y = 2 \times \frac{1}{8} + 1 = \frac{5}{4}$

67. $y(1 + \log x)\frac{dx}{dy} - x\log x = 0$

$\Rightarrow \frac{1+\log x}{x\log x}dx = \frac{dy}{y}$

Integrating on both sides, we get

$\log(x \log x) = \log y + \log c$

$\Rightarrow \log(x \log x) = \log(y\, c)$

$\Rightarrow x \log x = y\, c$ .....(i)

When $x = e$, $y = e^2$

$\therefore$ $e \log e = e^2 c \Rightarrow c = \frac{1}{e}$

$\therefore$ $x \log x = \frac{y}{e}$ ….[From (i)]

$\Rightarrow y = e\, x \log x$

68. $\sin\left(\frac{dy}{dx}\right) = a$

$\Rightarrow \frac{dy}{dx} = \sin^{-1}a \Rightarrow dy = \sin^{-1} a\, dx$

Integrating on both sides, we get

$y = (\sin^{-1}a)x + c$ ….(i)

Since, $y(0) = 1$ i.e., $y = 1$, when $x = 0$

$\therefore$ $1 = 0 + c \Rightarrow c = 1$

$\therefore$ $y = x\sin^{-1}a + 1$ ….[From (i)]

$\Rightarrow \frac{y-1}{x} = \sin^{-1}a$

$\Rightarrow \sin\left(\frac{y-1}{x}\right) = a$

69. $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$

$\Rightarrow \frac{dy}{1+y} = \left(\frac{-\cos x}{2+\sin x}\right)dx$

Integrating on both sides, we get

$\int\frac{dy}{1+y} + \int\frac{\cos x}{2+\sin x}dx = \log c$

$\Rightarrow \log(1 + y) + \log(2 + \sin x) = \log c$

$\Rightarrow (y + 1)(2 + \sin x) = c$ ....(i)

Since, $y(0) = 1$ i.e., $y = 1$, when $x = 0$

$\therefore$ $(1 + 1)(2 + \sin 0) = c \Rightarrow c = 4$

$\therefore$ $(y + 1)(2 + \sin x) = 4$ ....[From (i)]

$\Rightarrow y = \frac{4}{2+\sin x} - 1$

$\therefore$ $y\left(\frac{\pi}{2}\right) = \frac{4}{2+\sin\frac{\pi}{2}} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$

70. $y(1 + xy)\,dx = x\,dy$

$\Rightarrow \frac{y\,dx - x\,dy}{y^2} = -x\,dx$

$\Rightarrow d\left(\frac{x}{y}\right) = -x\,dx$

Integrating on both sides, we get

$\frac{x}{y} = \frac{-x^2}{2} + c$ ….(i)

Since, the curve passes through $(1, -1)$.

$\therefore$ $-1 = \frac{-1}{2} + c \Rightarrow c = \frac{-1}{2}$

$\therefore$ $\frac{x}{y} = \frac{-x^2}{2} - \frac{1}{2}$ ….[From (i)]

$\Rightarrow y = \frac{-2x}{x^2+1}$

i.e., $f(x) = \frac{-2x}{x^2+1}$

$\therefore$ $f\left(-\frac{1}{2}\right) = \frac{4}{5}$

71. $f(x) = \frac{-2x}{1+x^2}$

$\therefore$ $f\left(\frac{1}{2}\right) = \frac{-4}{5}$

72. $\frac{dy}{dx} = x + xy$

$\Rightarrow \frac{1}{1+y}dy = x\,dx$

Integrating on both sides, we get

$\int\frac{1}{1+y}dy = \int x\,dx + c$

$\log(1 + y) = \frac{x^2}{2} + c$ …(i)

Since, the required curve passes through (0, 1).

$\therefore$ $c = \log 2$

$\therefore \quad \log(1+y) = \dfrac{x^2}{2} + \log 2 \quad$ ….[From (i)]

$\Rightarrow \log\left(\dfrac{1+y}{2}\right) = \dfrac{x^2}{2}$

$\Rightarrow y = 2e^{\frac{x^2}{2}} - 1$

73. $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+y-1}$ ....(i)

Put $x + y = v$ ....(ii)

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$\dfrac{dv}{dx} - 1 = \dfrac{v+1}{v-1}$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{2v}{v-1} \Rightarrow \dfrac{v-1}{2v} dv = dx$

Integrating on both sides, we get

$\dfrac{v}{2} - \dfrac{1}{2}\log v = x + c_1$

$\Rightarrow v - \log v = 2x + 2c_1$

$\Rightarrow x + y - \log(x+y) = 2x + 2c_1$

$\Rightarrow y = x + \log(x+y) + c$, where $c = 2c_1$

74. $(x+y)^2 \dfrac{dy}{dx} = a^2$

Put $x + y = v \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \quad v^2\left(\dfrac{dv}{dx} - 1\right) = a^2$

$\Rightarrow v^2 \dfrac{dv}{dx} - v^2 = a^2$

$\Rightarrow \dfrac{v^2}{v^2 + a^2} dv = dx$

$\Rightarrow \displaystyle\int \dfrac{v^2}{v^2+a^2} dv = \int dx$

$\Rightarrow \displaystyle\int \dfrac{v^2 + a^2 - a^2}{v^2 + a^2} dv = \int dx$

$\Rightarrow v - a \tan^{-1}\dfrac{v}{a} + c = x$

$\Rightarrow x + y - a\tan^{-1}\left(\dfrac{x+y}{a}\right) + c = x$

$\Rightarrow \tan^{-1}\left(\dfrac{x+y}{a}\right) = \dfrac{y+c}{a}$

$\Rightarrow \dfrac{x+y}{a} = \tan\left(\dfrac{y+c}{a}\right)$

75. $(x+y-1)dx + (2x+2y-3)dy = 0$

$\Rightarrow \dfrac{dy}{dx} = -\left(\dfrac{x+y-1}{2x+2y-3}\right)$ ….(i)

Put $x + y = v$ ….(ii)

$\Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$\dfrac{dv}{dx} - 1 = -\left(\dfrac{v-1}{2v-3}\right)$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{1-v}{2v-3} + 1 \Rightarrow \dfrac{dv}{dx} = \dfrac{v-2}{2v-3}$

Integrating on both sides, we get

$\displaystyle\int \dfrac{2v-3}{v-2} dv = \int dx + c$

$\Rightarrow \displaystyle\int \dfrac{2(v-2)+1}{v-2} dv = x + c$

$\Rightarrow 2v + \log(v-2) = x + c$

$\Rightarrow 2(x+y) + \log(x+y-2) = x + c$

$\Rightarrow 2y + x + \log(x+y-2) = c$

76. $\dfrac{dy}{dx} = \dfrac{x-2y+1}{2(x-2y)}$ ….(i)

Put $x - 2y = v$ ….(ii)

$\Rightarrow 1 - 2\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}\left(1 - \dfrac{dv}{dx}\right)$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$\dfrac{1}{2}\left(1 - \dfrac{dv}{dx}\right) = \dfrac{v+1}{2v}$

$\Rightarrow \dfrac{dv}{dx} = -\dfrac{1}{v}$

Integrating on both sides, we get

$\displaystyle\int v\,dv = -\int dx + c_1$

$\Rightarrow \dfrac{v^2}{2} = -x + c_1$

$\Rightarrow (x-2y)^2 = -2x + 2c_1$

$\Rightarrow (x-2y)^2 + 2x = c$, where $c = 2c_1$

77. $\dfrac{dy}{dx} = -\dfrac{x-2y+1}{2(x-2y)+3}$ ....(i)

Put $x - 2y = v$ ....(ii)

$\Rightarrow 1 - \dfrac{2dy}{dx} = \dfrac{dv}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}\left(1 - \dfrac{dv}{dx}\right)$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$\dfrac{1}{2}\left(1 - \dfrac{dv}{dx}\right) = -\dfrac{v+1}{2v+3}$

$\Rightarrow \dfrac{2v+3}{4v+5}dv = dx$

Integrating on both sides, we get

$\int\left[\dfrac{\frac{1}{2}(4v+5)+\frac{1}{2}}{4v+5}\right]dv = \int dx + c_1$

$\Rightarrow \dfrac{1}{2}v + \dfrac{1}{2}.\dfrac{1}{4}\log(4v+5) = x + c_1$

$\Rightarrow \dfrac{1}{2}(x-2y) + \dfrac{1}{8}\log\left[4(x-2y)+5\right] = x + c_1$

$\Rightarrow \log[4(x-2y)+5] = 8x - 4(x-2y) + 8c_1$

$\Rightarrow \log[4(x-2y)+5] = 4(x+2y) + c$,

where $c = 8c_1$

78. $\dfrac{dy}{dx} = \dfrac{1}{\cos(x+y)}$ ....(i)

Put $x + y = v$ ....(ii)

$\Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$\dfrac{dv}{dx} - 1 = \dfrac{1}{\cos v}$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{1}{\cos v} + 1 \Rightarrow \dfrac{\cos v}{1+\cos v}dv = dx$

$\Rightarrow \left[\dfrac{2\cos^2(v/2)-1}{2\cos^2(v/2)}\right]dv = dx$

$\Rightarrow \left[1 - \dfrac{1}{2}\sec^2(v/2)\right]dv = dx$

Integrating on both sides, we get

$\int dv - \dfrac{1}{2}\int \sec^2(v/2)\,dv + c = \int dx$

$\Rightarrow v - \tan\left(\dfrac{v}{2}\right) + c = x$

$\Rightarrow x + y - \tan\left(\dfrac{x+y}{2}\right) + c = x$

$\Rightarrow y + c = \tan\left(\dfrac{x+y}{2}\right)$

79. $\dfrac{dy}{dx} = \tan\left(\dfrac{y}{x}\right) + \dfrac{y}{x}$

Put $\dfrac{y}{x} = v \Rightarrow y = vx$

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore \quad v + x\dfrac{dv}{dx} = \tan v + v$

$\Rightarrow x\dfrac{dv}{dx} = \tan v \Rightarrow \int\dfrac{dv}{\tan v} = \int\dfrac{dx}{x}$

$\Rightarrow \log|\sin v| = \log x + \log c$

$\Rightarrow \log\sin\left(\dfrac{y}{x}\right) = \log xc$

$\Rightarrow \sin\left(\dfrac{y}{x}\right) = xc$

80. $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ ...(i)

Put $y = vx$ ...(ii)

$\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ ...(iii)

Substituting (ii) and (iii) in (i), we get

$v + x\dfrac{dv}{dx} = \dfrac{1+v}{1-v}$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{1+v^2}{1-v} \Rightarrow \dfrac{1-v}{1+v^2}dv = \dfrac{dx}{x}$

Integrating both sides, we get

$\int\dfrac{1-v}{1+v^2}dv = \int\dfrac{dx}{x}$

$\Rightarrow \int\dfrac{1}{1+v^2}dv - \dfrac{1}{2}\int\dfrac{2v}{1+v^2} = \log x + c$

$\Rightarrow \tan^{-1}v - \dfrac{1}{2}\log(1+v^2) = \log x + c$

$\Rightarrow \tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{1}{2}\log\dfrac{x^2+y^2}{x^2} + \log x + c$

$\Rightarrow \tan^{-1}\left(\dfrac{y}{x}\right) = \log\dfrac{\sqrt{x^2+y^2}}{x} + \log x + c$

$\Rightarrow \tan^{-1}\left(\dfrac{y}{x}\right) = \log\sqrt{x^2+y^2} + c$

81. $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ ….(i)

Put $y = vx$ ….(ii)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$v + x\frac{dv}{dx} = \frac{v^2x^2}{x.vx - x^2} = \frac{v^2}{v-1}$

$\Rightarrow x\frac{dv}{dx} = \frac{v^2}{v-1} - v \Rightarrow x\frac{dv}{dx} = \frac{v}{v-1}$

Integrating on both sides, we get

$\int\left(\frac{v-1}{v}\right)dv = \int\frac{dx}{x} + \log k$

$\Rightarrow v - \log v = \log x + \log k$

$\Rightarrow v = \log(x.v.k)$

$\Rightarrow e^v = xvk \Rightarrow e^{y/x} = ky$

82. $(x^2 + xy)\, y' = y^2$

$\therefore \frac{dy}{dx} = \frac{y^2}{x^2 + xy}$ ...(i)

Put $y = vx$ ...(ii)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ...(iii)

Substituting (ii) and (iii) in (i), we get

$v + x\frac{dv}{dx} = \frac{v^2x^2}{x^2 + vx^2}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2}{1+v}$

$\Rightarrow x\frac{dv}{dx} = \frac{v^2}{1+v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{-v}{1+v}$

$\Rightarrow \int -\frac{1+v}{v}dv = \int\frac{1}{x}dx$

$\Rightarrow -(\log v + v) = \log x + \log c$

$\Rightarrow -v = \log xvc$

$\Rightarrow -\frac{y}{x} = \log y\, c \quad \Rightarrow e^{-\frac{y}{x}} = cy$

83. $(x^2 + y^2)dx = 2xydy$

$\Rightarrow \frac{dy}{dx} = \frac{1+\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$ ….(i)

Put $y = vx$ ….(ii)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{2v}$

Integrating on both sides, we get

$\int\frac{dx}{x} - \int\frac{2v}{1-v^2}dv = c_1$

$\Rightarrow \log x + \log(1 - v^2) = c_1$

$\Rightarrow (1 - v^2).x = e^{c_1}$

$\Rightarrow \left(1 - \frac{y^2}{x^2}\right).x = e^{c_1}$

$\Rightarrow x^2 - y^2 = e^{c_1}.x$

$\Rightarrow c.(x^2 - y^2) = x$, where $c = \frac{1}{e^{c_1}}$

84. $\frac{dy}{dx} = \frac{y-x}{y+x}$ ….(i)

Put $y = vx$ ….(ii)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$v + x\frac{dv}{dx} = \frac{vx - x}{vx + x} = \frac{v-1}{v+1}$

$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v \quad \Rightarrow x\frac{dv}{dx} = \frac{-(v^2+1)}{v+1}$

Integrating on both sides, we get

$\int\frac{v+1}{v^2+1}dv = -\int\frac{dx}{x} + c_1$

$\Rightarrow \frac{1}{2}\int\frac{2v}{v^2+1}dv + \int\frac{1}{v^2+1}dv = -\int\frac{dx}{x} + c_1$

$\Rightarrow \frac{1}{2}\log(v^2+1) + \tan^{-1}v = -\log x + c_1$

$\Rightarrow \log\left(\frac{y^2+x^2}{x^2}\right) + 2\tan^{-1}\left(\frac{y}{x}\right) = -2\log x + 2c_1$

$\Rightarrow \log(x^2 + y^2) - 2\log x + 2\tan^{-1}\left(\frac{y}{x}\right)$

$= -2\log x + 2c_1$

$\Rightarrow \log(x^2 + y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = c$, where $c = 2c_1$

85. $x\mathrm{d}y - y\mathrm{d}x = \sqrt{x^2+y^2}\,\mathrm{d}x$

$\Rightarrow x\mathrm{d}y = \sqrt{x^2+y^2}\,\mathrm{d}x + y\mathrm{d}x$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{x^2+y^2}+y}{x}$ ….(i)

Put $y = \mathrm{v}x$ ….(ii)

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{v} + x\frac{\mathrm{dv}}{\mathrm{d}x}$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$\mathrm{v} + x\frac{\mathrm{dv}}{\mathrm{d}x} = \frac{\sqrt{x^2+\mathrm{v}^2x^2}+\mathrm{v}x}{x}$

$\Rightarrow \mathrm{v} + x\frac{\mathrm{dv}}{\mathrm{d}x} = \sqrt{1+\mathrm{v}^2}+\mathrm{v}$

$\Rightarrow x\frac{\mathrm{dv}}{\mathrm{d}x} = \sqrt{1+\mathrm{v}^2}$

Integrating on both sides, we get

$\int\frac{\mathrm{dv}}{\sqrt{1+\mathrm{v}^2}} = \int\frac{\mathrm{d}x}{x} + \log \mathrm{c}$

$\Rightarrow \log\left(\mathrm{v}+\sqrt{1+\mathrm{v}^2}\right) = \log x + \log \mathrm{c}$

$\Rightarrow \mathrm{v} + \sqrt{1+\mathrm{v}^2} = x\mathrm{c}$

$\Rightarrow \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = x\mathrm{c}$

$\Rightarrow y + \sqrt{x^2+y^2} = x^2\mathrm{c}$

86. Putting $y = \mathrm{v}x$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{v} + x\frac{\mathrm{dv}}{\mathrm{d}x}$ in the given equation, we get

$\mathrm{v}x\left(\mathrm{v}+x\frac{\mathrm{dv}}{\mathrm{d}x}\right) = x\left[\mathrm{v}^2+\frac{\phi(\mathrm{v}^2)}{\phi'(\mathrm{v}^2)}\right]$

Integrating on both sides, we get

$\int\frac{\mathrm{v}\phi'(\mathrm{v}^2)}{\phi(\mathrm{v}^2)}\mathrm{dv} = \int\frac{\mathrm{d}x}{x} + \log \mathrm{c}_1$

$\Rightarrow \frac{1}{2}\log[\phi(\mathrm{v}^2)] = \log x + \log \mathrm{c}_1$

$\Rightarrow \log\left[\phi\left(\frac{y^2}{x^2}\right)\right] = 2\log x + 2\log \mathrm{c}_1$

$\Rightarrow \log\left[\phi\left(\frac{y^2}{x^2}\right)\right] = \log x^2\mathrm{c}_1^2$

$\Rightarrow \phi\left(\frac{y^2}{x^2}\right) = \mathrm{c}x^2$, where $\mathrm{c} = \mathrm{c}_1^2$

87. $\left(1+\mathrm{e}^{\frac{x}{y}}\right)\mathrm{d}x + \mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\mathrm{d}y = 0$

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{-\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+\mathrm{e}^{\frac{x}{y}}}$ ....(i)

Put $\frac{x}{y} = \mathrm{u}$ ....(ii)

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{u} + y\frac{\mathrm{du}}{\mathrm{d}y}$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$\mathrm{u} + y\frac{\mathrm{du}}{\mathrm{d}y} = \frac{-\mathrm{e}^{\mathrm{u}}(1-\mathrm{u})}{1+\mathrm{e}^{\mathrm{u}}}$

$\therefore \quad y\frac{\mathrm{du}}{\mathrm{d}y} = \frac{-\mathrm{e}^{\mathrm{u}}-\mathrm{u}}{1+\mathrm{e}^{\mathrm{u}}}$

$\therefore \quad \frac{1+\mathrm{e}^{\mathrm{u}}}{\mathrm{u}+\mathrm{e}^{\mathrm{u}}}\mathrm{du} = \frac{-\mathrm{d}y}{y}$

Integrating on both sides, we get

$\int\frac{1+\mathrm{e}^{\mathrm{u}}}{\mathrm{u}+\mathrm{e}^{\mathrm{u}}}\mathrm{du} = -\int\frac{\mathrm{d}y}{y}$

$\therefore \quad \log|\mathrm{u}+\mathrm{e}^{\mathrm{u}}| = -\log|y| + \log|\mathrm{c}|$

$\therefore \quad \log\left|\frac{x}{y}+\mathrm{e}^{\frac{x}{y}}\right| = \log\left|\frac{\mathrm{c}}{y}\right|$

$\therefore \quad \frac{x}{y} + \mathrm{e}^{\frac{x}{y}} = \frac{\mathrm{c}}{y}$

$\therefore \quad y\mathrm{e}^{\frac{x}{y}} + x = \mathrm{c}$

88. $y\cos\frac{y}{x}(x\,\mathrm{d}y - y\,\mathrm{d}x) + x\sin\frac{y}{x}(x\,\mathrm{d}y + y\,\mathrm{d}x) = 0$

$\Rightarrow \frac{y}{x}\cos\frac{y}{x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{y}{x}\right) + \sin\frac{y}{x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{x}\right) = 0$ ….(i)

Put $y = \mathrm{v}x$ ….(ii)

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{v} + x\frac{\mathrm{dv}}{\mathrm{d}x}$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$\mathrm{v}\cos\mathrm{v}\left(\mathrm{v}+x\frac{\mathrm{dv}}{\mathrm{d}x}-\mathrm{v}\right) + \sin\mathrm{v}\left(\mathrm{v}+x\frac{\mathrm{dv}}{\mathrm{d}x}+\mathrm{v}\right) = 0$

$\Rightarrow (\mathrm{v}\cos\mathrm{v})x\frac{\mathrm{dv}}{\mathrm{d}x} + \sin\mathrm{v}\left(2\mathrm{v}+x\frac{\mathrm{dv}}{\mathrm{d}x}\right) = 0$

$\Rightarrow x(\mathrm{v}\cos\mathrm{v}+\sin\mathrm{v})\frac{\mathrm{dv}}{\mathrm{d}x} + 2\mathrm{v}\sin\mathrm{v} = 0$

Integrating on both sides, we get

$\int\frac{\mathrm{v}\cos\mathrm{v}+\sin\mathrm{v}}{\mathrm{v}\sin\mathrm{v}}\mathrm{dv} = -2\int\frac{\mathrm{d}x}{x} + \log\mathrm{c}$

$\Rightarrow \log (\text{v} \sin \text{v}) = -2 \log x + \log \text{c}$

$\Rightarrow \log (\text{v} \sin \text{v}) = \log \dfrac{\text{c}}{x^2}$

$\Rightarrow \text{v} \sin \text{v} = \dfrac{\text{c}}{x^2}$

$\Rightarrow \dfrac{y}{x} \sin \dfrac{y}{x} = \dfrac{\text{c}}{x^2} \quad \Rightarrow y \sin \dfrac{y}{x} = \dfrac{\text{c}}{x}$

Since, $y(1) = \dfrac{\pi}{2}$, i.e., $y = \dfrac{\pi}{2}$, when $x = 1$

$\therefore \quad \dfrac{\pi}{2} \sin \dfrac{\pi}{2} = \text{c} \Rightarrow \text{c} = \dfrac{\pi}{2}$

$\therefore \quad y \sin \dfrac{y}{x} = \dfrac{\pi}{2x}$

89. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ ….(i)

Put $y = \text{v}x$ ….(ii)

$\Rightarrow \dfrac{dy}{dx} = \text{v} + x\dfrac{d\text{v}}{dx}$ ….(iii)

Substituting (ii) and (iii) in (i), we get

$\text{v} + x\dfrac{d\text{v}}{dx} = \dfrac{x^2 + \text{v}^2x^2}{2x\text{v}x} = \dfrac{1+\text{v}^2}{2\text{v}}$

$\Rightarrow x\dfrac{d\text{v}}{dx} = \dfrac{1+\text{v}^2}{2\text{v}} - \text{v} \quad \Rightarrow x\dfrac{d\text{v}}{dx} = \dfrac{1-\text{v}^2}{2\text{v}}$

Integrating on both sides, we get

$\int \dfrac{dx}{x} - \int \dfrac{2\text{v}}{1-\text{v}^2} d\text{v} = \log \text{c}$

$\Rightarrow \log x + \log (1 - \text{v}^2) = \log \text{c}$

$\Rightarrow \log x + \log\left(1 - \dfrac{y^2}{x^2}\right) = \log \text{c}$

$\Rightarrow x\left(1 - \dfrac{y^2}{x^2}\right) = \text{c}$

$\Rightarrow x^2 - y^2 = \text{c}x$ ….(iv)

Since, the required curve passes through (2, 1).

$\therefore \quad 4 - 1 = 2\text{c} \Rightarrow \text{c} = \dfrac{3}{2}$

$\therefore \quad x^2 - y^2 = \dfrac{3}{2}x$ ….[From (iv)]

$\Rightarrow 2(x^2 - y^2) = 3x$

90. $\dfrac{dy}{dx} - \dfrac{y}{x} = x^3 - 3$

$\therefore$ I. F. $= e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \dfrac{1}{x}$

91. $x\dfrac{dy}{dx} + 2y = x^2 \Rightarrow \dfrac{dy}{dx} + \dfrac{2y}{x} = x$

$\therefore$ I.F. $= e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2} = x^2$

92. $\dfrac{dy}{dx} + \dfrac{y}{1+x^2} = \dfrac{e^{\tan^{-1} x}}{1+x^2}$

$\therefore$ I.F. $= e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1} x}$

93. $\cos x\dfrac{dy}{dx} + y\sin x = 1$

$\Rightarrow \dfrac{dy}{dx} + y\tan x = \sec x$

$\therefore$ I.F. $= e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

94. $\dfrac{dy}{dx} + y = \dfrac{1+y}{x}$

$\Rightarrow \dfrac{dy}{dx} + y = \dfrac{1}{x} + \dfrac{y}{x}$

$\Rightarrow \dfrac{dy}{dx} + \left(1 - \dfrac{1}{x}\right)y = \dfrac{1}{x}$

$\therefore$ I.F. $= e^{\int\left(1-\frac{1}{x}\right)dx} = e^{x - \log x} = \dfrac{e^x}{x}$

95. $\dfrac{dy}{dx} = \dfrac{1}{x+y+2}$

$\Rightarrow \dfrac{dx}{dy} = x + y + 2 \qquad \Rightarrow \dfrac{dx}{dy} - x = y + 2$

$\therefore$ I.F. $= e^{\int -dy} = e^{-y}$

96. I. F. $= \sin x$

$\therefore \quad e^{\int P\,dx} = \sin x$

$\Rightarrow \int P\,d x = \log(\sin x) \quad \Rightarrow P = \dfrac{d}{dx}[\log(\sin x)]$

$\Rightarrow P = \dfrac{1}{\sin x} \times \cos x = \cot x$

97. $\dfrac{dy}{dx} + P(x).y = 0$

Here, Q = 0

$\therefore$ I.F. $= e^{\int P dx}$

$\therefore$ solution of the given equation is

$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$

$\Rightarrow y.e^{\int P dx} = 0 + c \Rightarrow y = c.e^{-\int P dx}$

98. I.F. $= e^{\int a\,dx} = e^{ax}$

$\therefore$ solution of the given equation is

$y.e^{ax} = \int e^{mx}.e^{ax}\,dx + c = \dfrac{e^{(a+m)x}}{a+m} + c$

$\Rightarrow y = \dfrac{e^{mx}}{a+m} + ce^{-ax}$

$\Rightarrow y(a + m) = e^{mx} + c(a + m)\,e^{-ax}$

99. $\dfrac{dy}{dx} + y = 1$

I.F. $= e^{\int dx} = e^{x}$

$\therefore$ Solution of the differential equation is

$ye^{x} = \int e^{x}\,dx + c_1$

$\Rightarrow ye^{x} = e^{x} + c_1 \qquad \Rightarrow e^{x}(1 - y) = -c_1$

$\Rightarrow \log e^{x} + \log(1 - y) = -\log c_1$

$\Rightarrow x + c = -\log(1 - y)$

$\Rightarrow \log\left|\dfrac{1}{1-y}\right| = x + c$

100. $\dfrac{dy}{dx} = \dfrac{y}{x} + x$

$\Rightarrow \dfrac{dy}{dx} - \dfrac{y}{x} = x$

$\therefore$ I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\log x} = \dfrac{1}{x}$

$\therefore$ solution of the given equation is

$\dfrac{y}{x} = \int x.\dfrac{1}{x}dx + a$

$\Rightarrow \dfrac{y}{x} = x + a \quad \Rightarrow y = x^2 + ax$

101. $x\dfrac{dy}{dx} + 2y = x^2$

$\Rightarrow \dfrac{dy}{dx} + \left(\dfrac{2}{x}\right)y = x$

$\therefore$ I.F. $= e^{2\int \frac{1}{x}dx} = e^{2\log x} = x^2$

$\therefore$ solution of the given equation is

$yx^2 = \int x^3\,dx + c_1$

$\Rightarrow yx^2 = \dfrac{x^4}{4} + c_1 = \dfrac{x^4 + C}{4}$, where $C = 4c_1$

$\Rightarrow y = \dfrac{x^4 + C}{4x^2}$

102. $y + x^2 = \dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} - y = x^2$

$\therefore$ I.F. $= e^{\int -dx} = e^{-x}$

$\therefore$ solution of the given equation is

$ye^{-x} = \int e^{-x}\cdot x^2 dx + c$

$\Rightarrow y.e^{-x} = -x^2.e^{-x} - 2xe^{-x} - 2e^{-x} + c$

$\Rightarrow y + x^2 + 2x + 2 = c.e^{x}$

103. $(1 + y^2) + \left(x - e^{\tan^{-1}y}\right)\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{e^{\tan^{-1}y}}{1+y^2}$

$\therefore$ I.F. $= e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

$\therefore$ solution of the given equation is

$x.e^{\tan^{-1}y} = \int \dfrac{e^{\tan^{-1}y}}{1+y^2}.e^{\tan^{-1}y}dy + c_1$

$\Rightarrow xe^{\tan^{-1}y} = \int \dfrac{e^{2\tan^{-1}y}}{1+y^2}dy + c_1$

$\Rightarrow xe^{\tan^{-1}y} = \dfrac{e^{2\tan^{-1}y}}{2} + c_1$

$\Rightarrow 2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + c$, where $c = 2c_1$

104. $(y - 3x^2)\,dx + x\,dy = 0$

$\Rightarrow (y - 3x^2) = -x\dfrac{dy}{dx} \qquad \Rightarrow \dfrac{dy}{dx} = \dfrac{-y}{x} + 3x$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x}(y) = 3x$

$\therefore$ I.F. $= e^{\int \frac{1}{x}dx} = e^{\log x} = x$

$\therefore$ Solution of the given differential equation is

$xy = \int 3x^2 + c$

$\Rightarrow xy = \dfrac{3x^3}{3} + c$

$\Rightarrow y = x^2 + \dfrac{c}{x}$

105. $(x - 4y^3)\dfrac{dy}{dx} - y = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x - 4y^3}$

$\Rightarrow \dfrac{dx}{dy} = \dfrac{x - 4y^3}{y} \qquad \Rightarrow \dfrac{dx}{dy} - \dfrac{x}{y} = -4y^2$

$\therefore$ I.F. $= e^{\int -\frac{1}{y}dy} = e^{-\log y} = \dfrac{1}{y}$

$\therefore$ Solution of the given equation is

$$x \cdot \frac{1}{y} = \int -4y^2 \cdot \frac{1}{y}\,dy + c$$

$$\Rightarrow \frac{x}{y} = -2y^2 + c \Rightarrow x + 2y^3 = cy$$

106. $y\,dx + (x + x^2y)\,dy = 0$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = -x^2$$

$$\Rightarrow -\frac{1}{x^2}\cdot\frac{dx}{dy} + \left(\frac{-1}{y}\right)\cdot\frac{1}{x} = 1 \quad \text{....(i)}$$

Put $v = \frac{1}{x}$

$$\Rightarrow \frac{dv}{dy} = -\frac{1}{x^2}\cdot\frac{dx}{dy}$$

$$\therefore \frac{dv}{dy} + \left(\frac{-1}{y}\right)\cdot v = 1 \quad \text{....[From (i)]}$$

$$\therefore \text{I.F.} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

$\therefore$ solution of the given equation is

$$v \cdot \frac{1}{y} = \int \frac{1}{y}dy + c_1$$

$$\Rightarrow \frac{1}{xy} = \log y + c_1$$

$$\Rightarrow -\frac{1}{xy} + \log y = -c_1$$

$$\Rightarrow -\frac{1}{xy} + \log y = c, \text{ where } c = -c_1$$

107. $\frac{dx}{dy} + \frac{x}{y} = x^2$

$$\Rightarrow \frac{1}{x^2}\cdot\frac{dx}{dy} + \frac{1}{y}\left(\frac{1}{x}\right) = 1 \quad \text{....(i)}$$

Put $v = \frac{1}{x}$

$$\Rightarrow \frac{dv}{dy} = -\frac{1}{x^2}\cdot\frac{dx}{dy}$$

$$\therefore -\frac{dv}{dy} + \frac{1}{y}\cdot v = 1 \quad \text{....[From (i)]}$$

$$\Rightarrow \frac{dv}{dy} + \left(-\frac{1}{y}\right)\cdot v = -1$$

$$\therefore \text{I.F.} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

$\therefore$ solution of the given equation is

$$v\cdot\left(\frac{1}{y}\right) = \int (-1)\cdot\frac{1}{y}dy + c$$

$$\Rightarrow \frac{1}{x}\cdot\frac{1}{y} = -\log y + c \Rightarrow \frac{1}{x} = cy - y\log y$$

108. $\cos x\,dy = y(\sin x - y)\,dx$

$$\Rightarrow \frac{dy}{dx} = y\tan x - y^2\sec x$$

$$\Rightarrow \frac{1}{y^2}\cdot\frac{dy}{dx} + \tan x\left(-\frac{1}{y}\right) = -\sec x \quad \text{....(i)}$$

Put $v = -\frac{1}{y} \quad \Rightarrow \frac{dv}{dx} = \frac{1}{y^2}\cdot\frac{dy}{dx}$

$$\therefore \frac{dv}{dx} + (\tan x)v = -\sec x \quad \text{....[From (i)]}$$

$$\therefore \text{I.F.} = e^{\int \tan x\,dx} = e^{\log(\sec x)} = \sec x$$

$\therefore$ solution of the given equation is

$$v.\sec x = \int -\sec x.\sec x\,dx + c_1$$

$$\Rightarrow v\sec x = -\tan x + c_1$$

$$\Rightarrow -\frac{1}{y}\sec x = -\tan x + c_1$$

$$\Rightarrow \sec x = y(\tan x + c), \text{ where } c = -c_1$$

109. $(xy^4 + y)\,dx - x\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{xy^4 + y}{x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = y^4$$

$$\Rightarrow y^{-4}\cdot\frac{dy}{dx} - \frac{y^{-3}}{x} = 1 \quad \text{....(i)}$$

Put $v = y^{-3}$

$$\Rightarrow \frac{dv}{dx} = -3y^{-4}\cdot\frac{dy}{dx}$$

$$\therefore -\frac{1}{3}\cdot\frac{dv}{dx} - \frac{v}{x} = 1 \quad \text{....[From (i)]}$$

$$\Rightarrow \frac{dv}{dx} + \frac{3}{x}\cdot v = -3$$

$$\therefore \text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\log x} = x^3$$

$\therefore$ solution of the given equation is

$$v.x^3 = \int -3.x^3\,dx + c_1$$

$$\Rightarrow v.x^3 = \frac{-3x^4}{4} + c_1 \Rightarrow \frac{x^3}{y^3} = -\frac{3}{4}x^4 + c_1$$

$$\Rightarrow 4x^3 + 3x^4y^3 = 4c_1.y^3$$

$$\Rightarrow 4x^3 + 3x^4y^3 = c.y^3, \text{ where } c = 4c_1$$

110. $\frac{dy}{dx} + y \tan x = \sec x$

I.F. $= e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$

$\therefore$ Solution of the given differential equation is

$y \sec x = \int \sec^2 x \, + c$

$\Rightarrow y \sec x = \tan x + c$

$y(0) = 0 \Rightarrow c = 0$

$\therefore$ $y \sec x = \tan x$

111. I.F. $= e^{\int \frac{dx}{x \log_e x}} = e^{\log(\log_e x)} = \log_e x$

$\therefore$ solution of the given equation is

$y.\log_e x = \int \frac{\log_e x}{x} dx + c$

$\Rightarrow y \log_e x = \frac{(\log_e x)^2}{2} + c$

When $x = e$, $y = 1$

$\therefore$ $1 = \frac{1}{2} + c \Rightarrow c = \frac{1}{2}$

$\therefore$ $y \log_e x = \frac{(\log_e x)^2}{2} + \frac{1}{2}$

$\Rightarrow 2y = \log_e x + \frac{1}{\log_e x}$

112. Given, $(x \log x) \frac{dy}{dx} + y = 2x \log x$

When $x = 1$, $y = 0$

$(x \log x) \frac{dy}{dx} + y = 2x \log x$

$\therefore$ $\frac{dy}{dx} + \frac{y}{x \log x} = 2$

$\therefore$ I.F. $= e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x$

$\therefore$ solution of the given equation is

$y.\log x = \int 2 \log x \, dx + c$

$\therefore$ $y \log x = 2(x \log x - x) + c$

When $x = 1$, $y = 0$

$\therefore$ $0 = -2 + c \Rightarrow c = 2$

$\therefore$ $y \log x = 2(x \log x - x) + 2$

$\therefore$ $y(e) = 2(e - e) + 2 = 2$

113. $y^2 dx + \left(x - \frac{1}{y}\right) dy = 0$

$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2} x = \frac{1}{y^3}$

$\therefore$ I.F. $= e^{\int \frac{1}{y^2} dy} = e^{-\frac{1}{y}}$

$\therefore$ solution of the given equation is

$x.\, e^{-\frac{1}{y}} = \int \frac{1}{y^3}.e^{-\frac{1}{y}} dy + c$

$\Rightarrow x\, e^{-\frac{1}{y}} = e^{-\frac{1}{y}}\left(\frac{1}{y} + 1\right) + c$ ....(i)

Since, $y(1) = 1$ i.e., $y = 1$, when $x = 1$

$\therefore$ $1.e^{-1} = e^{-1}(1 + 1) + c \Rightarrow c = -e^{-1}$

$\therefore$ $x\, e^{-\frac{1}{y}} = e^{-\frac{1}{y}}\left(\frac{1}{y} + 1\right) - e^{-1}$ ....[From (i)]

$\Rightarrow x = 1 + \frac{1}{y} - \frac{e^{1/y}}{e}$

114. $(1 + x^2) \frac{dy}{dx} + 2xy - 4x^2 = 0$

$\Rightarrow \frac{dy}{dx} + \left(\frac{2xy}{1 + x^2}\right) y = \frac{4x^2}{1 + x^2}$

$\therefore$ I.F. $= e^{\int \frac{2x}{1+x^2} dx} = e^{\log\left(1 + x^2\right)} = 1 + x^2$

$y.(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + c$

$\Rightarrow y(1 + x^2) = \int \frac{4x^2}{1 + x^2} \times (1 + x^2)\, dx + c$

$\Rightarrow y(1 + x^2) = \frac{4}{3}x^3 + c$

Since, $y(0) = -1$, i.e., when $x = 0$, $y = -1$

$\therefore$ $c = -1$

$\Rightarrow y(1 + x^2) = \frac{4}{3}x^3 - 1$

$\Rightarrow y = \frac{4x^3}{3\left(1 + x^2\right)} - \frac{1}{1 + x^2}$

$\therefore$ $y(1) = \frac{4}{6} - \frac{1}{2} = \frac{1}{6}$

115. $(1 + t)\frac{dy}{dt} - ty = 1 \Rightarrow \frac{dy}{dt} - \frac{t}{1 + t}.y = \frac{1}{1 + t}$

$\therefore$ I.F. $= e^{\int -\frac{t}{1+t} dt} = e^{-\int \frac{1+t-1}{1+t} dt}$

$= e^{-\int\left(1 - \frac{1}{1+t}\right) dt} = e^{-t + \log(1+t)} = (1 + t).e^{-t}$

$\therefore$ solution of the given equation is

$y.(1 + t).e^{-t} = \int (1 + t).e^{-t} \cdot \frac{1}{1 + t} dt + c$

$= \int e^{-t} dt + c$

$\therefore$ $y(1 + t)\, e^{-t} = -e^{-t} + c$

$\Rightarrow$ $y(1 + t) = -1 + c.e^{t}$ ….(i)

Since, $y(0) = -1$ i.e., $y = -1$, when $t = 0$

$\therefore$ $-1(1 + 0) = -1 + c.e^{0} \Rightarrow c = 0$

$\therefore \quad y(1 + t) = -1$ ....[From (i)]

$\Rightarrow y = \dfrac{-1}{1+t}$

$\therefore \quad y(1) = \dfrac{-1}{1+1} = \dfrac{-1}{2}$

116. $dy = \cos x(2 - y \operatorname{cosec} x)dx$

$\Rightarrow \dfrac{dy}{dx} = 2\cos x - y\cot x$

$\Rightarrow \dfrac{dy}{dx} + y\cot x = 2\cos x$

$\therefore \quad$ I.F. $= e^{\int \cot x dx} = e^{\log \sin x} = \sin x$

$\therefore \quad$ solution of the given equation is

$y.\sin x = \int 2\cos x.\sin x + c_1$

$\therefore \quad y.\sin x = \int \sin 2x + c_1 = -\dfrac{\cos 2x}{2} + c_1$

$= \dfrac{2\sin^2 x - 1}{2} + c_1 = \sin^2 x - \dfrac{1}{2} + c_1$

$\therefore \quad y\sin x = \sin^2 x + c$, where $c = c_1 - \dfrac{1}{2}$

When $x = \dfrac{\pi}{2}, y = 2$

$\therefore \quad 2\sin\dfrac{\pi}{2} = \sin^2\dfrac{\pi}{2} + c$

$\Rightarrow 2 = 1 + c \Rightarrow c = 1$

$\therefore \quad y\sin x = \sin^2 x + 1 \Rightarrow y = \sin x + \operatorname{cosec} x$

117. $\dfrac{dy(x)}{dx} + g'(x)y(x) = g(x)g'(x)$

$\therefore \quad$ I.F. $= e^{\int g'(x)dx} = e^{g(x)}$

$\therefore \quad$ solution of the given equation is

$y(x).e^{g(x)} = \int g(x)g'(x) . e^{g(x)}dx + c$

$\Rightarrow y(x).e^{g(x)} = e^{g(x)}[g(x) - 1] + c$ ....(i)

Putting $x = 0$ in (i), we get

$0 = e^0(0 - 1) + c$ ...[$\because y(0) = 0, g(0) = 0$ (given)]

$\Rightarrow c = 1$

$\therefore \quad y(x)e^{g(x)} = e^{g(x)}[g(x) - 1] + 1$ ....[From (i)]

Putting $x = 2$, we get

$y(2)e^0 = e^0(0 - 1) + 1$ ....[$\because$ g(2) = 0 (given)]

$\Rightarrow y(2) = 0$

118. $\sin x\dfrac{dy}{dx} + y\cos x = 4x$

$\Rightarrow \dfrac{dy}{dx} + y\cot x = \dfrac{4x}{\sin x}$

I. F. $= e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

General solution of the given equation is

$y. \sin x = \int \sin x \cdot \dfrac{4x}{\sin x} + c$

$\Rightarrow y\sin x = 2x^2 + c$ ...(i)

$y\left(\dfrac{\pi}{2}\right) = 0$ i.e. $y = 0$ when $x = \dfrac{\pi}{2}$

$0 = 2 . \dfrac{\pi^2}{4} + c \Rightarrow c = -\dfrac{\pi^2}{2}$

$y\sin x = 2x^2 - \dfrac{\pi^2}{2}$ ...[From (i)]

When $x = \dfrac{\pi}{6}$,

$y . \dfrac{1}{2} = 2 . \dfrac{\pi^2}{36} - \dfrac{\pi^2}{2}$

$\Rightarrow y\left(\dfrac{\pi}{6}\right) = \dfrac{-8}{9}\pi^2$

119. $y' - y\tan x = 2x\sec x$

$\Rightarrow \dfrac{dy}{dx} + (-\tan x)y = 2x\sec x$

$\therefore \quad$ I.F. $= e^{-\int \tan x dx} = e^{\log\cos x} = \cos x$

$\therefore \quad$ solution of the given equation is

$y.\cos x = \int 2x\sec x.\cos x\, dx + c$

$\Rightarrow y\cos x = x^2 + c$ ....(i)

Since, $y(0) = 0$ i.e., $y = 0$, when $x = 0$

$\therefore \quad 0 = 0 + c \Rightarrow c = 0$

$\therefore \quad y\cos x = x^2$ ....[From (i)]

$\Rightarrow y = x^2\sec x$ ....(ii)

$\Rightarrow y' = x^2\sec x\tan x + 2x\sec x$ ....(iii)

Putting $x = \dfrac{\pi}{4}$ in (ii) and (iii), we get

$y\left(\dfrac{\pi}{4}\right) = \dfrac{\pi^2}{8\sqrt{2}}$ and $y'\left(\dfrac{\pi}{4}\right) = \dfrac{\pi^2}{8\sqrt{2}} + \dfrac{\pi}{\sqrt{2}}$

Putting $x = \dfrac{\pi}{3}$ in (ii) and (iii), we get

$y\left(\dfrac{\pi}{3}\right) = \dfrac{2\pi^2}{9}$ and $y'\left(\dfrac{\pi}{3}\right) = \dfrac{2\pi^2}{3\sqrt{3}} + \dfrac{4\pi}{3}$

120. 'p' is the population at time 't'.

$\therefore \quad \dfrac{dp}{dt} = \dfrac{3}{100}p$

$\Rightarrow \dfrac{dp}{p} = \dfrac{3}{100}dt$

Integrating on both sides, we get

$\int \dfrac{dp}{p} = \dfrac{3}{100}\int dt$

$\Rightarrow \log p = \dfrac{3}{100}t + c_1$

$\Rightarrow p = e^{\frac{3}{100}t + c_1} \Rightarrow p = e^{\frac{3}{100}t}e^{c_1}$

$\Rightarrow p = c\,e^{\frac{3}{100}t}$ ...[where $e^{c_1} = c$]

121. $\dfrac{dV}{dt} = -k(T-t)$

$\Rightarrow dV = -k(T-t)dt$

Integrating on both sides, we get

$\int dV = -k\int (T-t)\,dt + c$

$\Rightarrow V(t) = \dfrac{k(T-t)^2}{2} + c$ ....(i)

Initially i.e., when t = 0, V(t) = I

$\therefore \quad I = \dfrac{kT^2}{2} + c \Rightarrow c = I - \dfrac{kT^2}{2}$

$\therefore \quad V(t) = \dfrac{k(T-t)^2}{2} + I - \dfrac{kT^2}{2}$ ....[From (i)]

When t = T,

$V(T) = I - \dfrac{kT^2}{2}$

122. $\dfrac{d(p(t))}{dt} = \dfrac{1}{2}p(t) - 200$

Integrating on both sides, we get

$\int \dfrac{d(p(t))}{\frac{1}{2}p(t) - 200} = \int dt + c_1$

$\Rightarrow 2\log\left(\dfrac{p(t)}{2} - 200\right) = t + c_1$

$\Rightarrow \dfrac{p(t)}{2} - 200 = e^{\frac{t}{2}}.c, \left(\text{where } c = e^{\frac{c_1}{2}}\right)$ ....(i)

Putting t = 0, we get

$\dfrac{p(0)}{2} - 200 = e^0.c$

$\Rightarrow \dfrac{100}{2} - 200 = c \Rightarrow c = -150$

$\therefore \quad \dfrac{p(t)}{2} - 200 = e^{\frac{t}{2}}(-150)$ ....[From (i)]

$\Rightarrow p(t) = 400 - 300e^{\frac{t}{2}}$

Evaluation Test

1. The given equation is

$\sqrt{1-x^4} + \sqrt{1-y^4} = a(x^2 - y^2)$

Put $x^2 = \sin\alpha$, $y^2 = \sin\beta$

The equation becomes

$\cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$

$2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) = 2a\sin\left(\dfrac{\alpha-\beta}{2}\right)\cos\left(\dfrac{\alpha+\beta}{2}\right)$

$\therefore \quad \cot\left(\dfrac{\alpha-\beta}{2}\right) = a$

$\therefore \quad \alpha - \beta = 2\cot^{-1}a$

$\therefore \quad \sin^{-1}x^2 - \sin^{-1}y^2 = 2\cot^{-1}a$

Differentiating w.r.t. x, we get

$\dfrac{1}{\sqrt{1-x^4}}\cdot 2x - \dfrac{1}{\sqrt{1-y^4}}\cdot 2y\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{x}{y}\sqrt{\dfrac{1-y^4}{1-x^4}}$

$\therefore$ Degree and order are both 1.

2. Since, the given differential equation cannot be expressed as a polynomial in differential coefficients, so its degree is not defined.

3. The equation of tangent at any point P(x, y) is

$Y - y = \dfrac{dy}{dx}(X - x)$

This meets the X-axis at $A\left(x - y\dfrac{dx}{dy}, 0\right)$.

Similarly, it meets the Y-axis at $B\left(0, y - x\dfrac{dy}{dx}\right)$.

According to the given condition,

P is the mid-point of AB.

$\therefore$ $2x = x - y\dfrac{dx}{dy}$ and $2y = y - x\dfrac{dy}{dx}$

$\therefore$ $x + y\dfrac{dx}{dy} = 0$ and $y + x\dfrac{dy}{dx} = 0$

Both of these equations reduce to

$\dfrac{1}{x}dx + \dfrac{1}{y}dy = 0$

Integrating both sides, we get

$\log x + \log y = \log c$

$\therefore$ $\log(xy) = \log c$

$\therefore$ $xy = c$, which is the equation of rectangular hyperbola.

4. $\sqrt{1+x^2} + \sqrt{1+y^2} = A\left(x\sqrt{1+y^2} - y\sqrt{1+x^2}\right)$

Put $x = \tan\alpha$, $y = \tan\beta$

The equation becomes

$\sec\alpha + \sec\beta = A(\tan\alpha\sec\beta - \tan\beta\sec\alpha)$

$\therefore$ $\dfrac{1}{\cos\alpha} + \dfrac{1}{\cos\beta} = A\left(\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{1}{\cos\beta} - \dfrac{\sin\beta}{\cos\beta}\cdot\dfrac{1}{\cos\alpha}\right)$

$\therefore$ $\dfrac{\cos\alpha + \cos\beta}{\cos\alpha\cos\beta} = A\left(\dfrac{\sin\alpha - \sin\beta}{\cos\alpha\cos\beta}\right)$

$\therefore$ $\cos\alpha + \cos\beta = A(\sin\alpha - \sin\beta)$

$\therefore$ $2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

$= A.2\sin\left(\dfrac{\alpha-\beta}{2}\right)\cos\left(\dfrac{\alpha+\beta}{2}\right)$

$\therefore$ $\cot\left(\dfrac{\alpha-\beta}{2}\right) = A$

$\therefore$ $\alpha - \beta = 2\cot^{-1}A$

$\therefore$ $\tan^{-1}x - \tan^{-1}y = 2\cot^{-1}A$

Differentiating w.r.t. x, we get

$\dfrac{1}{1+x^2} - \dfrac{1}{1+y^2}\cdot\dfrac{dy}{dx} = 0$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$

$\therefore$ Degree and order of the differential equation are both 1.

5. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{f\left(\dfrac{y}{x}\right)}{f'\left(\dfrac{y}{x}\right)}$

Put $y = vx$

$\therefore$ $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ The given equation becomes,

$v + x\dfrac{dv}{dx} = v + \dfrac{f(v)}{f'(v)}$

$\therefore$ $\dfrac{1}{x}dx = \dfrac{f'(v)}{f(v)}dv$

Integrating on both sides, we get

$\log x = \log f(v) + \log K$

$\Rightarrow x = f(v)K$

$\Rightarrow x = Kf\left(\dfrac{y}{x}\right)$

$\therefore$ $f\left(\dfrac{y}{x}\right) = \dfrac{1}{K}\cdot x = cx$, where $c = \dfrac{1}{K}$

6. The given equation is $\dfrac{dy}{dx} + f'(x)y = f(x)f'(x)$

$\therefore$ I.F. $= e^{\int f'(x)\,dx} = e^{f(x)}$

$\therefore$ the required solution is

$y.e^{f(x)} = \int e^{f(x)}.f(x)f'(x)dx$

$= \int e^t.t\,dt$, where $f(x) = t$

$= t.e^t - \int e^t.dt$

$= te^t - e^t + c$

$\therefore$ $y.e^{f(x)} = f(x)\,e^{f(x)} - e^{f(x)} + c$

$\therefore$ $y = f(x) - 1 + ce^{-f(x)}$

7. The given equation is

$(x+1)\,f'(x) - 2(x^2+x)\,f(x) = \dfrac{e^{x^2}}{x+1}$

If $y = f(x)$, the equation is

$\dfrac{dy}{dx} - 2xy = \dfrac{e^{x^2}}{(x+1)^2}$, which is a linear equation

$\therefore$ I.F. $= e^{-\int 2x\,dx} = e^{-x^2}$

$\therefore$ the required solution is

$y.e^{-x^2} = \int\dfrac{1}{(x+1)^2}dx + c = -\dfrac{1}{x+1} + c$

When $x = 0$, $y = 5$

$\therefore$ $c = 6$

$\therefore$ $y.e^{-x^2} = -\dfrac{1}{x+1} + 6$

$= \dfrac{-1+6x+6}{x+1} = \dfrac{6x+5}{x+1}$

$\therefore$ $y = f(x) = \left(\dfrac{6x+5}{x+1}\right)e^{x^2}$

8. $\dfrac{dy}{dx} = \dfrac{x^2+y^2+1}{2xy}$

$\therefore$ $2xydy = (x^2+1)\,dx + y^2\,dx$

$\therefore$ $2xydy - y^2dx = (x^2+1)\,dx$

Dividing by x^2, we get

$\dfrac{2xydy - y^2dx}{x^2} = \left(\dfrac{x^2+1}{x^2}\right)dx$

$\therefore$ $d\left(\dfrac{y^2}{x}\right) = \left(1+\dfrac{1}{x^2}\right)dx$

Integrating both sides, we get

$\dfrac{y^2}{x} = x - \dfrac{1}{x} + c$

$\therefore$ $y^2 = x^2 - 1 + cx$

When $x = 1$, $y = 0$, $\therefore$ c = 0

$\therefore$ the required solution is $y^2 = x^2 - 1$

i.e., $x^2 - y^2 = 1$, which is the equation of a hyperbola.

9. $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2+y^2} = 0$

$\therefore$ $\dfrac{1}{2}(2xdx + 2ydy) + \dfrac{1}{1+\dfrac{y^2}{x^2}}\left(\dfrac{xdy - ydx}{x^2}\right) = 0$

$\therefore$ $\dfrac{1}{2}d(x^2+y^2) + d\left(\tan^{-1}\dfrac{y}{x}\right) = 0$

Integrating both sides, we get

$\dfrac{1}{2}(x^2+y^2) + \tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{c}{2}$

$\therefore$ $x^2 + y^2 + 2\tan^{-1}\left(\dfrac{y}{x}\right) = c$

$\therefore$ $2\tan^{-1}\left(\dfrac{y}{x}\right) = c - x^2 - y^2$

$\therefore$ $\tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{c - x^2 - y^2}{2}$

$\therefore$ $\dfrac{y}{x} = \tan\left(\dfrac{c - x^2 - y^2}{2}\right)$

$\therefore$ $y = x\tan\left(\dfrac{c - x^2 - y^2}{2}\right)$

10. $2x^2y\dfrac{dy}{dx} = \tan(x^2y^2) - 2xy^2$

$\therefore$ $x^2.2y\dfrac{dy}{dx} + y^2.2x = \tan(x^2y^2)$

$\therefore$ $\dfrac{d}{dx}(x^2y^2) = \tan(x^2y^2)$

$\therefore$ $\dfrac{dz}{dx} = \tan z$, where $z = x^2y^2$

$\therefore$ $dx = \cot z\,dz$

Integrating both sides, we get

$x = \log(\sin z) + c \Rightarrow x = \log(\sin x^2y^2) + c$

When $x = 1$, $y = \sqrt{\dfrac{\pi}{2}}$, $\therefore$ $z = \dfrac{\pi}{2}$

$\therefore$ $1 = \log 1 + c$, $\therefore$ c = 1

$\therefore$ the required solution is

$x = \log(\sin x^2y^2) + 1$

$\therefore$ $\log(\sin x^2y^2) = x - 1$

$\therefore$ $\sin(x^2y^2) = e^{x-1}$

11. $x^3\dfrac{dy}{dx} + 4x^2\tan y = e^x\sec y$

Dividing by $x^3\sec y$, we get

$\dfrac{1}{\sec y}\cdot\dfrac{dy}{dx} + \dfrac{4\tan y}{x\sec y} = \dfrac{e^x}{x^3}$

$\therefore$ $\cos y\dfrac{dy}{dx} + (4\sin y).\dfrac{1}{x} = \dfrac{e^x}{x^3}$

Put $\sin y = t$, $\therefore$ $\cos y\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\therefore$ the equation becomes,

$\dfrac{dt}{dx} + \left(\dfrac{4}{x}\right)t = \dfrac{e^x}{x^3}$, which is a linear equation

$\therefore$ I.F. $= e^{\int\frac{4}{x}dx} = e^{4\log x} = e^{\log x^4} = x^4$

$\therefore$ the required solution is

$tx^4 = \int e^x.x\,dx$

$\therefore$ $tx^4 = xe^x - e^x + c$

$\therefore$ $\sin y.x^4 = xe^x - e^x + c$

When $x = 1$, $y = 0$

$\therefore$ c = 0

$\therefore$ $\sin y.x^4 = xe^x - e^x$

$\therefore$ $\sin y = (x-1)\,e^x.x^{-4}$

12. We have, $\dfrac{dy}{dx} = y\tan x - y^2\sec x$

$\therefore$ $\dfrac{1}{y^2}\cdot\dfrac{dy}{dx} = \dfrac{1}{y}\tan x - \sec x$

$\therefore$ $\dfrac{1}{y^2}\cdot\dfrac{dy}{dx} - \dfrac{1}{y}\tan x = -\sec x$

Put $\dfrac{1}{y} = t$, $\therefore$ $-\dfrac{1}{y^2}\cdot\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\therefore$ The equation becomes,

$\dfrac{dt}{dx} + (\tan x)\,t = \sec x$, which is linear equation.

$\therefore$ I.F. $= e^{\int \tan x\, dx}$

$= e^{\log(\sec x)} = \sec x$

$\therefore$ the required solution is

$t \sec x = \int \sec^2 x\, dx + c$

$\therefore$ $\frac{1}{y}\sec x = \tan x + c$

$\therefore$ $\sec x = y\,(c + \tan x)$

13. $(xy - x^2)\frac{dy}{dx} = y^2$

$\therefore$ $y^2\frac{dx}{dy} = xy - x^2$

Dividing by x^2y^2, we get

$\frac{1}{x^2}\frac{dx}{dy} = \frac{1}{x}\cdot\frac{1}{y} - \frac{1}{y^2}$

$\therefore$ $-\frac{1}{x^2}\frac{dx}{dy} + \frac{1}{x}\cdot\frac{1}{y} = \frac{1}{y^2}$

Put $\frac{1}{x} = t$

$\therefore$ $-\frac{1}{x^2}\cdot\frac{dx}{dy} = \frac{dt}{dy}$

$\therefore$ The equation becomes

$\frac{dt}{dy} + \frac{1}{y}\cdot t = \frac{1}{y^2}$, which is a linear equation

$\therefore$ I.F. $= e^{\int \frac{1}{y} dy} = e^{\log y} = y$

$\therefore$ the required solution is

$ty = \int \frac{1}{y}\, dy + c$

$\therefore$ $ty = \log y + c$

$\therefore$ $\frac{y}{x} = \log y + c$

$\therefore$ $y = x\,(\log y + c)$

The curve passes through the point $(-1, 1)$.

$\therefore$ $1 = -1(0 + c)$, $\therefore$ $c = -1$

$\therefore$ the required solution is $y = x\,(\log y - 1)$.

14. $\frac{dy}{dx} = e^{x-y}\,(1 - e^y)$

$\therefore$ $e^y\frac{dy}{dx} = e^x\,(1 - e^y)$

$\therefore$ $e^y\frac{dy}{dx} = e^x - e^x.e^y$

$\therefore$ $e^y\frac{dy}{dx} + e^xe^y = e^x$

Put $e^y = t$, $\therefore$ $e^y\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ The given equation becomes

$\frac{dt}{dx} + e^x.t = e^x$, which is a linear equation.

$\therefore$ I.F. $= e^{\int e^x dx} = e^{e^x}$

$\therefore$ the required solution is

$t.e^{e^x} = \int e^{e^x}.e^x\, dx$

$= \int e^z.dz$, where $e^x = z$

$= e^z + c$

$\therefore$ $t.e^{e^x} = e^{e^x} + c$

$\therefore$ $e^y.e^{e^x} = e^{e^x} + c$

15. The equation of the tangent to the curve $y = f(x)$ at P (x, y) is

$Y - y = \frac{dy}{dx}(X - x)$

This meets the X-axis at $\left(x - y\frac{dx}{dy}, 0\right)$.

According to the given condition,

$x - \frac{y}{\frac{dy}{dx}} = y$

$\therefore$ $x - y = \frac{y}{\frac{dy}{dx}}$

$\therefore$ $\frac{dy}{dx} = \frac{y}{x - y}$, which is a homogeneous d.E.

Put $y = vx$

$\therefore$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ the equation becomes,

$v + x\frac{dv}{dx} = \frac{vx}{x - vx} = \frac{v}{1 - v}$

$\therefore$ $x\frac{dv}{dx} = \frac{v}{1 - v} - v$

$= \frac{v - v + v^2}{1 - v}$

$\therefore$ $\frac{1 - v}{v^2}dv = \frac{1}{x}dx$

Integrating both sides, we get

$\int\left(v^{-2} - \frac{1}{v}\right)dv = \int\frac{1}{x}dx + c$

$\therefore \quad -\frac{1}{v} - \log v = \log x + c$

$\therefore \quad -\frac{x}{y} - \log\left(\frac{y}{x}\right) = \log x + c$

$\therefore \quad -\frac{x}{y} - \log y + \log x = \log x + c$

$\therefore \quad -\frac{x}{y} - \log y = c$

This curve passes through (1, 1).

$\therefore \quad c = -1$

$\therefore \quad -\frac{x}{y} - \log y = -1$

$\therefore \quad \frac{x}{y} + \log y = 1$

$\therefore \quad \log y = 1 - \frac{x}{y}$

$\therefore \quad y = e^{1-\frac{x}{y}} = e.e^{-\frac{x}{y}}$

$\therefore \quad y e^{\frac{x}{y}} = e$

16. $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$

$\therefore \quad \int \frac{y}{\sqrt{1-y^2}} dy = \int 1 dx$

$\therefore \quad -\sqrt{1-y^2} = x + c$

$\therefore \quad (x + c)^2 = 1 - y^2$

$\therefore \quad (x + c)^2 + y^2 = 1$

$\therefore$ Radius is fixed, which is 1 and the centre is (–c, 0) which is a variable centre on the X-axis.

17. (A) $f(tx, ty) = \frac{tx - ty}{t^2x^2 + t^2y^2} = t^{-1}\left(\frac{x-y}{x^2+y^2}\right)$

$= t^{-1} f(x, y)$

$\therefore$ Homogeneous of degree –1.

(B) $f(tx, ty) = (tx)^{\frac{1}{3}}(ty)^{-\frac{2}{3}} \tan^{-1}\left(\frac{tx}{ty}\right)$

$= (t)^{-\frac{1}{3}} x^{\frac{1}{3}} y^{-\frac{2}{3}} \tan^{-1}\left(\frac{x}{y}\right) = t^{-\frac{1}{3}} f(x, y)$

$\therefore$ Homogeneous of degree $-\frac{1}{3}$.

(C) $f(tx, ty) = tx\left[\log\sqrt{t^2x^2 + t^2y^2} - \log t y\right] + ty e^{\frac{tx}{ty}}$

$= t\left[x \log\frac{\sqrt{t^2x^2+t^2y^2}}{ty}\right] + ty e^{\frac{x}{y}}$

$= t\left[x\left(\log\sqrt{x^2+y^2} - \log y\right) + y e^{\frac{x}{y}}\right]$

$= t f(x, y)$

$\therefore$ Homogeneous of degree 1.

(D) $f(tx, ty)$

$= tx\left\{\log\left[\frac{2t^2x^2 + t^2y^2}{tx}\right] - \log(tx + ty)\right\} + t^2y^2 \tan\left(\frac{tx + 2ty}{3tx - ty}\right)$

$= tx\left[\log\frac{2x^2 + y^2}{x(x+y)}\right] + t^2y^2 \tan\left(\frac{x+2y}{3x-y}\right)$

$\therefore$ Non-Homogeneous.

18. $\frac{dy}{dx} = \frac{\sin 2y}{x + \tan y}$

$\therefore \quad \frac{dx}{dy} = \frac{x + \tan y}{\sin 2y}$

$\therefore \quad \frac{dx}{dy} - \frac{x}{\sin 2y} = \frac{\tan y}{\sin 2y}$,

which is a linear equation

$\therefore \quad$ I.F. $= e^{-\int \operatorname{cosec} 2y\, dy} = e^{-\frac{1}{2}\log(\tan y)} = e^{\log(\tan y)^{-\frac{1}{2}}}$

$= e^{\log\sqrt{\cot y}}$

$= \sqrt{\cot y}$

$\therefore$ the required solution is

$x\sqrt{\cot y} = \int \frac{\tan y}{\sin 2y}\sqrt{\cot y}\, dy + c$

$= \int \frac{1}{\sqrt{\tan y}} \cdot \frac{\sin y}{\cos y} \cdot \frac{1}{2 \sin y \cos y} dy + c$

$= \int \frac{1}{2\sqrt{\tan y}} \sec^2 y\, dy + c$

$\therefore \quad x\sqrt{\cot y} = \sqrt{\tan y} + c$

The curve passes through $\left(1, \frac{\pi}{4}\right)$.

$\therefore \quad 1 = 1 + c, \quad \therefore c = 0$

$\therefore$ the equation of the curve is

$x = \tan y$

19. The equation of hyperbola is $xy = 2$

$\therefore \quad y = \dfrac{2}{x}$

$\therefore \quad m_1 = \dfrac{dy}{dx} = -\dfrac{2}{x^2}$ (slope of tangent to the hyperbola)

$m_2 = \dfrac{dy}{dx}$ = slope of tangent to the required family of curves.

The curves are intersecting orthogonally,

$m_1m_2 = -1$

$\therefore \quad \dfrac{dy}{dx} \times \left(-\dfrac{2}{x^2}\right) = -1$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{x^2}{2}$

Integrating both sides, we get $y = \dfrac{x^3}{6} + c$, which is the equation of required family of curves.

20. The given equation is

$\dfrac{dy}{dx} = -\dfrac{(y+y^3)}{1+x+xy^2} = \dfrac{-(y+y^3)}{1+x(1+y^2)}$

$\therefore \quad \dfrac{dx}{dy} = -\dfrac{1+x(1+y^2)}{y(1+y^2)}$

$= -\dfrac{1}{y(1+y^2)} - \dfrac{x}{y}$

$\therefore \quad \dfrac{dx}{dy} + \dfrac{1}{y} \cdot x = -\dfrac{1}{y(1+y^2)}$, which is a linear equation

$\therefore \quad$ I.F. $= e^{\int \frac{1}{y} dy} = e^{\log y} = y$

$\therefore \quad$ the required solution is

$xy = -\int \dfrac{1}{1+y^2} dy + c$

$\therefore \quad xy = -\tan^{-1} y + c$

The curve passes through (0, 1) $\quad \therefore c = \dfrac{\pi}{4}$

$\therefore \quad$ the required equation of the curve is

$xy + \tan^{-1} y = \dfrac{\pi}{4}$

21. Slope of tangent $= \dfrac{dy}{dx}$

$\therefore \quad$ slope of normal $= -\dfrac{1}{\frac{dy}{dx}}$

The equation of the normal is

$Y - y = -\dfrac{1}{\frac{dy}{dx}}(X - x)$

$\therefore \quad (Y - y)\dfrac{dy}{dx} + (X - x) = 0$

The normal passes through the point (3, 0).

$\therefore \quad (0 - y)\dfrac{dy}{dx} + (3 - x) = 0$

$\therefore \quad y\dfrac{dy}{dx} = 3 - x$

$\therefore \quad y\,dy = (3 - x)\,dx$

Integrating both sides, we get $\dfrac{y^2}{2} = 3x - \dfrac{x^2}{2} + c$

The curve passes through (3 , 4), $\quad \therefore c = \dfrac{7}{2}$

$\therefore \quad$ the equation of the curve is $\dfrac{y^2}{2} = 3x - \dfrac{x^2}{2} + \dfrac{7}{2}$

$\therefore \quad x^2 + y^2 - 6x - 7 = 0$

22. The given equation can be written as

$x\,dy - y\,dx = xy^3 (1 + \log x)\,dx$

$\therefore \quad -\left(\dfrac{y\,dx - x\,dy}{y^2}\right) = xy (1 + \log x)\,dx$

$\therefore \quad -d\left(\dfrac{x}{y}\right) = xy (1 + \log x)\,dx$

$\therefore \quad -\dfrac{x}{y} d\left(\dfrac{x}{y}\right) = x^2 (1 + \log x)\,dx$

Integrating both sides, we get

$$-\dfrac{\left(\frac{x}{y}\right)^2}{2} = (1+\log x)\int x^2 dx - \int\left\{\dfrac{d}{dx}(1+\log x)\int x^2 dx\right\}dx + \dfrac{c}{2}$$

$\therefore \quad -\dfrac{x^2}{2y^2} = (1 + \log x) \cdot \dfrac{x^3}{3} - \int \dfrac{1}{x} \cdot \dfrac{x^3}{3} dx + \dfrac{c}{2}$

$\therefore \quad -\dfrac{x^2}{2y^2} = (1 + \log x) \cdot \dfrac{x^3}{3} - \dfrac{x^3}{9} + \dfrac{c}{2}$

$\therefore \quad -\dfrac{x^2}{y^2} = \dfrac{2x^3}{3}(1+\log x) - \dfrac{2x^3}{9} + c$

$= \dfrac{2x^3}{3}\left(1 + \log x - \dfrac{1}{3}\right) + c$

$\therefore \quad -\dfrac{x^2}{y^2} = \dfrac{2x^3}{3}\left(\dfrac{2}{3} + \log x\right) + c$

23. The equation of the tangent at P(x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

This meets the Y-axis at $\left(0, y - x\frac{dy}{dx}\right)$.

According to the given condition,

$$y - x\frac{dy}{dx} = x^3$$

$$\Rightarrow \frac{dy}{dx} + \left(-\frac{1}{x}\right)y = -x^2$$

$\therefore$ I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

$\therefore$ solution of the given equation is

$$y \cdot \frac{1}{x} = \int -x^2 \cdot \frac{1}{x} + c$$

$$\Rightarrow \frac{y}{x} = -\frac{x^2}{2} + c \Rightarrow y = -\frac{x^3}{2} + cx$$

$\therefore$ $f(x) = -\frac{x^3}{2} + cx$ ....(i)

$\therefore$ $f(1) = -\frac{1}{2} + c$

$\Rightarrow 1 = -\frac{1}{2} + c \Rightarrow c = \frac{3}{2}$

$\therefore$ $f(x) = -\frac{x^3}{2} + \frac{3}{2}x$ ....[From (i)]

$\therefore$ $f(-3) = \frac{27}{2} - \frac{9}{2} = 9$

24. $\frac{dV}{dt} = -k(T - t)$

$\Rightarrow dV = -k(T - t)dt$

Integrating on both sides, we get

$$\int dV = -k\int (T - t)\,dt + c$$

$\Rightarrow V(t) = \frac{k(T - t)^2}{2} + c$ ....(i)

Initially i.e., when $t = 0$, $V(t) = I$

$\therefore$ $I = \frac{kT^2}{2} + c \Rightarrow c = I - \frac{kT^2}{2}$

$\therefore$ $V(t) = \frac{k(T - t)^2}{2} + I - \frac{kT^2}{2}$ ....[From (i)]

When $t = T$,

$$V(T) = I - \frac{kT^2}{2}$$

25. $\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$ ....(i)

Put $y = vx$ ....(ii)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$$v + x\frac{dv}{dx} = v - \cos^2 v \quad \Rightarrow \quad x\frac{dv}{dx} = -\cos^2 v$$

Integrating on both sides, we get

$$\int \sec^2 v\,dv = -\int \frac{dx}{x} + c$$

$\Rightarrow \tan v = -\log x + c$

$\Rightarrow \tan\frac{y}{x} = -\log x + c$ ....(iv)

Since, the required curve passes through $\left(1, \frac{\pi}{4}\right)$.

$\therefore$ $\tan\frac{\pi}{4} = -\log 1 + c \Rightarrow c = 1$

$\therefore$ $\tan\frac{y}{x} = -\log x + 1$ ....[From (iv)]

$\Rightarrow \tan\frac{y}{x} = -\log x + \log e$

$$\Rightarrow y = x\tan^{-1}\left[\log\left(\frac{e}{x}\right)\right]$$

Textbook Chapter No.

08 Probability Distribution

Hints

Classical Thinking

1. The sum of all the probabilites in a probability distribution is always unity.

$\therefore$ $0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$\Rightarrow 4k + 0.6 = 1 \Rightarrow 4k = 0.4 \Rightarrow k = 0.1$

2. Since, $\sum_{x=1}^{4} P(X = x) = 1$

$\therefore$ $\frac{1}{8} + \frac{1}{2} + \frac{1}{4} + k = 1$

$\Rightarrow k + \frac{1+4+2}{8} = 1$

$\Rightarrow k = 1 - \frac{7}{8} = \frac{1}{8}$

3. Since, $\sum_{x=0}^{4} P(X = x) = 1$

$\therefore$ $k + 2k + 3k + 2k + k = 1$

$\Rightarrow 9k = 1 \Rightarrow k = \frac{1}{9}$

4. The probability distribution of X is

X	0	1	2
P (X)	k	2k	3k

Since, $\sum_{x=0}^{2} P(X = x) = 1$

$\therefore$ $k + 2k + 3k = 1$

$\Rightarrow 6k = 1 \Rightarrow k = \frac{1}{6}$

5. $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$= \frac{{}^5C_0}{2^5} + \frac{{}^5C_1}{2^5} + \frac{{}^5C_2}{2^5}$

$= \frac{1+5+10}{2^5} = \frac{16}{32}$

6. $F(-1) = 0.2 + 0.3 = 0.5$

7. $F(x_1) = p_1 = 0.05$

$F(x_2) = p_1 + p_2 = 0.05 + 0.2 = 0.25$

$F(x_3) = p_1 + p_2 + p_3 = 0.25 + 0.15 = 0.4$

$F(x_4) = p_1 + p_2 + p_3 + p_4 = 0.4 + 0.25 = 0.65$

$F(x_5) = p_1 + p_2 + p_3 + p_4 + p_5 = 0.65 + 0.35 = 1$

8. $E(X) = \sum x_i . P(x_i)$

$= 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$

$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{3}{2}$

9. $E(X) = \sum x_i \cdot P(x_i)$

$= 0\ (0.2) + 1\ (0.5) + 3\ (0.2) + 5\ (0.1)$

$= 0 + 0.5 + 0.6 + 0.5$

$= 1.6$

Variance $= \sum x_i^2 \cdot P(x_i) - [E(X)]^2$

$= (0)^2\ (0.2) + (1)^2\ (0.5) + (3)^2\ (0.2) + (5)^2\ (0.1) - (1.6)^2$

$= 4.8 - 2.56 = 2.24$

10. Since, $Var(X) = E(X^2) - [E(X)]^2$

$\therefore$ $4 = 13 - [E(X)]^2$

$\therefore$ $[E(X)]^2 = 13 - 4 = 9$

$\therefore$ $E(X) = 3$

11. Since, $Var(X) = E(X^2) - [E(X)]^2$

$\therefore$ $6 = E(X^2) - (5)^2$

$\therefore$ $E(X^2) = 25 + 6 = 31$

12. Mean $= E(X) = \sum x_i \cdot P(x_i)$

$= \frac{1}{6}(1) + \frac{1}{3}(2) + \frac{1}{3}(3) + \frac{1}{6}(4)$

$= \frac{1}{6} + \frac{2}{3} + 1 + \frac{4}{6} = \frac{1+4+6+4}{6}$

$= \frac{15}{6} = \frac{5}{2}$

Variance $= \sum x_i^2 \cdot P(x_i) - [E(X)]^2$

$= \frac{1}{6}(1)^2 + \frac{1}{3}(2)^2 + \frac{1}{3}(3)^2 + \frac{1}{6}(4)^2 - \left(\frac{5}{2}\right)^2$

$= \frac{1}{6} + \frac{4}{3} + \frac{9}{3} + \frac{16}{6} - \frac{25}{4}$

$= \frac{2+16+36+32-75}{12}$

$= \frac{86-75}{12} = \frac{11}{12}$

13. $E(X) = 1\left(\frac{1}{7}\right) + 2\left(\frac{2}{7}\right) + 3\left(\frac{3}{7}\right) + 4\left(\frac{1}{7}\right) = \frac{18}{7}$

$E(X^2) = (1^2)\left(\frac{1}{7}\right) + (2^2)\left(\frac{2}{7}\right) + (3^2)\left(\frac{3}{7}\right) + (4^2)\left(\frac{1}{7}\right)$

$= \frac{1}{7} + \frac{8}{7} + \frac{27}{7} + \frac{16}{7} = \frac{52}{7}$

$\therefore$ $Var(X) = E(X^2) - [E(X)]^2$

$= \frac{52}{7} - \left(\frac{18}{7}\right)^2 = \frac{40}{49}$

14. $P(1 < X < 3) = \int_1^3 f(x)dx$

$= \int_1^3 \frac{1}{5}dx = \frac{1}{5}[x]_1^3 = \frac{2}{5}$

15. $P\left(\frac{1}{3} < X < \frac{1}{2}\right) = \int_{1/3}^{1/2} f(x)dx$

$= \int_{1/3}^{1/2} 2x\,dx = [x^2]_{1/3}^{1/2} = \frac{5}{36}$

16. $P(0.5 \le X \le 1.5) = \int_{0.5}^{1.5} f(x)dx = \int_{0.5}^{1.5} 0.5x\,dx$

$= 0.5\left[\frac{x^2}{2}\right]_{0.5}^{1.5} = \frac{1}{2}$

17. $P(X > 3) = \int_3^4 f(x)dx$

$= \int_3^4 \frac{x}{8}dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_3^4 = \frac{7}{16}$

18. Since, $f(x)$ is the p.d.f. of X.

$\therefore$ $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_0^2 kx^2dx = 1 \Rightarrow k\left[\frac{x^3}{3}\right]_0^2 = 1$

$\Rightarrow k\left[\frac{8}{3}\right] = 1 \Rightarrow k = \frac{3}{8}$

19. The c.d.f. of X is

$F(x) = \int_{-1}^{x} \frac{x^2}{3}dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{x}$

$= \frac{1}{3}\left(\frac{x^3}{3} + \frac{1}{3}\right) = \frac{x^3}{9} + \frac{1}{9}$

Critical Thinking

1. The sum of all the probabilities in a probability distribution is always unity.
In option (A), we have $0.3 + 0.2 + 0.4 + 0.1 = 1$

2. Since, $\sum_{x=1}^{4} P(X = x) = 1$

$\therefore$ $k + 2k + 3k + 4k = 1$

$\Rightarrow 10k = 1 \qquad \Rightarrow k = \frac{1}{10}$

Now, $P(X < 3) = P(X = 1) + P(X = 2)$
$= k + 2k$
$= 3k = 3\left(\frac{1}{10}\right) = 0.3$

3. Since, $\sum_{x=0}^{4} P(X = x) = 1$

$\therefore$ $k + 3k + 5k + 2k + k = 1$

$\Rightarrow 12k = 1 \qquad \Rightarrow k = \frac{1}{12}$

Now, $P(X \ge 2) = P(X=2) + P(X=3) + P(X=4)$
$= 5k + 2k + k$
$= 8k = 8\left(\frac{1}{12}\right) = \frac{2}{3}$

4. Since, $\sum_{x=0}^{6} P(X = x) = 1$

$\therefore$ $k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$\Rightarrow 49k = 1 \Rightarrow k = \frac{1}{49}$

$\therefore$ $P(0 < X < 4)$
$= P(X = 1) + P(X = 2) + P(X = 3)$
$= 3k + 5k + 7k = 15k = \frac{15}{49}$

5. P(X is odd)
$= P(X = -3) + P(X = -1) + P(X = 1) + P(X = 3)$
$= 0.05 + 0.15 + 0.25 + 0.10 = 0.55$

6.

$X = x$	0	1	2	3	4
$P(X = x)$	k	4k	6k	4k	k

Since, $\sum_{x=0}^{4} P(X = x) = 1$

$\therefore$ $k + 4k + 6k + 4k + k = 1$
$\Rightarrow 16k = 1$
$\Rightarrow k = \frac{1}{16}$

7. Since, $\sum_{x=0}^{7} P(X = x) = 1$

$\therefore$ $0 + P + 2P + 2P + 3P + P^2 + 2P^2 + 7P^2 + P = 1$

$\therefore$ $10P^2 + 9P - 1 = 0$

$\therefore$ $(P + 1)(10P - 1) = 0$

$\therefore$ $P = \frac{1}{10}$ ….[$\because$ P $\geq$ 0, $\therefore$ P + 1 $\neq$ 0]

8. Since, $\sum_{x=0}^{2} P(X = x) = 1$

$\therefore$ $3k^3 + 4k - 10k^2 + 5k - 1 = 1$

$\Rightarrow 3k^3 - 10k^2 + 9k - 2 = 0$

$\Rightarrow (k - 1)(k - 2)(3k - 1) = 0$

$\Rightarrow k = 1$ or $k = 2$ or $k = \frac{1}{3}$

For k = 1 or k = 2, P (X = 1) < 0, which is not possible.

$\therefore$ $k = \frac{1}{3}$

9. Since, $\sum_{x=1}^{5} P(X = x) = 1$

$\therefore$ $\frac{1}{20} + \frac{3}{20} + a + b + \frac{1}{20} = 1$

$\Rightarrow a + b = 1 - \frac{5}{20}$

$\Rightarrow a + 2a = 1 - \frac{1}{4}$ ....[$\because$ b = 2a (given)]

$\Rightarrow 3a = \frac{3}{4}$

$\Rightarrow a = \frac{1}{4}$ and $b = 2\left(\frac{1}{4}\right) = \frac{1}{2}$

10. The probability distribution of X is

X	0	1	2	3	4
P(X)	0.1	k	2k	2k	k

Since, $\sum_{x=0}^{4} P(X = x) = 1$

$\Rightarrow 0.1 + k + 2k + 2k + k = 1$

$\Rightarrow 6k = 0.9$ $\Rightarrow k = 0.15$

11. Let $P(X = x_3) = k$. Then $P(X = x_1) = \frac{k}{2}$,

$P(X = x_2) = \frac{k}{3}$, $P(X = x_4) = \frac{k}{5}$

Since, $P(X = x_1) + P(X = x_2) + P(X = x_3) + P(X = x_4) = 1$

$\therefore$ $\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$ $\Rightarrow k = \frac{30}{61}$

$\therefore$ option (A) is the correct answer.

12. Let X denotes the number of heads.

Thus, the possible values of X are 0, 1, 2 and 3.

$P(X = 0)$ = P(getting no head)

= P(TTT) = $\frac{1}{8}$

$P(X = 1)$ = P(getting one head)

= P(HTT, THT, TTH) = $\frac{3}{8}$

$P(X = 2)$ = P(getting two heads)

= P(HHT, THH, HTH) = $\frac{3}{8}$

$P(X = 3)$ = P(getting three heads)

= P(HHH) = $\frac{1}{8}$

$\therefore$ Option (D) is the correct answer.

13. Let X denote the number of red balls drawn from the bag. There are 4 red balls and X can take values 0, 1, 2 and 3.

$P(X = 0)$ = Probability of getting no red ball

$= \frac{^6C_3}{^{10}C_3} = \frac{1}{6}$

$P(X = 1)$ = Probability of getting one red ball

$= \frac{^4C_1 \times {}^6C_2}{^{10}C_3} = \frac{1}{2}$

$P(X = 2)$ = Probability of getting two red balls

$= \frac{^4C_2 \times {}^6C_1}{^{10}C_3} = \frac{3}{10}$

$P(X = 3)$ = Probability of getting three red balls

$= \frac{^4C_3}{^{10}C_3} = \frac{1}{30}$

14. Let X denote the number of defective mangoes from the bag. X can take values 0, 1, 2, 3 and 4.

$P(X = 0)$ = Probability of getting no defective mango $= \frac{^{15}C_4}{^{20}C_4} = \frac{91}{323}$

$P(X = 1)$ = Probability of getting one defective mango $= \frac{^5C_1 \times {}^{15}C_3}{^{20}C_4} = \frac{455}{969}$

P(X = 2) = Probability of getting two defective mangoes $= \frac{{}^{5}C_2 \times {}^{15}C_2}{{}^{20}C_4} = \frac{70}{323}$

P(X = 3) = Probability of getting three defective mangoes $= \frac{{}^{5}C_3 \times {}^{15}C_1}{{}^{20}C_4} = \frac{10}{323}$

P(X = 4) = Probability of getting four defective mangoes $= \frac{{}^{5}C_4}{{}^{20}C_4} = \frac{1}{969}$

15. $P(x = 2) = F(2) - F(1) = 0.43 - 0.18 = 0.25$
$P(x = 3) = F(3) - F(2) = 0.54 - 0.43 = 0.11$
$P(x = 4) = F(4) - F(3) = 0.68 - 0.54 = 0.14$
$\therefore$ $P(1 < x < 5) = P(x = 2) + P(x = 3) + P(x = 4)$
$= 0.25 + 0.11 + 0.14 = 0.50$

16. $P(X = 1) = F(1) - F(0) = 0.65 - 0.5 = 0.15$
$P(X = 3) = F(3) - F(1) = 0.75 - 0.65 = 0.10$
$P(X = 5) = 0.85 - 0.75 = 0.10$
$P(X = 7) = 0.90 - 0.85 = 0.05$
$P(X = 9) = 1 - 0.90 = 0.10$
$\therefore$ $P(X \le 3 | X > 0)$
$$= \frac{P(X=1)+P(X=3)}{P(X=1)+P(X=3)+P(X=5)+P(X=7)+P(X=9)}$$
$$= \frac{0.15+0.1}{0.15+0.1+0.1+0.05+0.1}$$
$$= \frac{0.25}{0.50} = \frac{1}{2}$$

17. The sum of all the probabilities in a probability distribution is always unity.
$\therefore$ $0.1 + k + 0.2 + 2k + 0.3 + k = 1$
$\Rightarrow 0.6 + 4k = 1$
$\Rightarrow 4k = 0.4$
$\Rightarrow k = 0.1$
$\therefore$ $E(X) = (-2)(0.1) + (-1)(0.1) + 0(0.2) + 1(2 \times 0.1) + 2(0.3) + 3(0.1) = 0.8$

18. The sum of all the probabilities in a probability distribution is always unity.
$\therefore$ $0.2 + 0.1 + 0.3 + k = 1$
$\therefore$ $k = 1 - 0.6 = 0.4$
$E(X) = \sum x_i \cdot P(x_i)$
$= 1(0.2) + 2(0.1) + 3(0.3) + 4(0.4)$
$= 0.2 + 0.2 + 0.9 + 1.6 = 2.9$
$\text{Var}(X) = E(X^2) - [E(X)]^2$
$= (1)^2(0.2) + (2)^2(0.1) + (3)^2(0.3) + (4)^2(0.4) - (2.9)^2$
$= 0.2 + 0.4 + 2.7 + 6.4 - 8.41$
$= 9.7 - 8.41 = 1.29$

19. The sum of all the probabilities in a probability distribution is always unity.
$\therefore$ $k + 3k + 3k + k = 1$
$\Rightarrow 8k = 1$
$\Rightarrow k = \frac{1}{8}$
$$E(X) = 0\left(\frac{1}{8}\right)+1\left(\frac{3}{8}\right)+2\left(\frac{3}{8}\right)+3\left(\frac{1}{8}\right)$$
$$= \frac{3}{2}$$
$\text{Var}(X) = E(X^2) - [E(X)]^2$
$$= 0^2\left(\frac{1}{8}\right)+1^2\left(\frac{3}{8}\right)+2^2\left(\frac{3}{8}\right)+3^2\left(\frac{1}{8}\right)-\left(\frac{3}{2}\right)^2$$
$$= \frac{3}{4}$$

20. $\text{Mean} = E(X) = \sum x_i.P(x_i)$
$= 1(0.1) + 2(0.2) + 3(0.3) + 4(0.4) = 3$
$\text{Var}(X) = \sum x_i^2.P(x_i) - [E(X)]^2$
$= 1^2(0.1) + 2^2(0.2) + 3^2(0.3) + 4^2(0.4) - (3)^2$
$= 0.1 + 0.8 + 2.7 + 6.4 - 9 = 10 - 9 = 1$
$\therefore$ S.D. = 1

21. The p.m.f. of the r.v. X is as follows:

$X = x$	-1	0	1	2
$P(X = x)$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{10}$

$\therefore$ $E(X) = -1\left(\frac{2}{5}\right) + 0 + 1\left(\frac{1}{5}\right) + 2\left(\frac{1}{10}\right) = 0$

22.

$X = x$	1	2	3	4
$P(X = x)$	k	4k	9k	16k

Since, $P(1) + P(2) + P(3) + P(4) = 1$
$\therefore$ $k + 4k + 9k + 16k = 1$
$\Rightarrow 30k = 1$
$\Rightarrow k = \frac{1}{30}$
$\therefore$ $E(X) = 1.\frac{1}{30} + 2.\frac{4}{30} + 3.\frac{9}{30} + 4.\frac{16}{30}$
$$= \frac{100}{30}$$
$$= \frac{10}{3}$$

23. $P(1) = \frac{C}{1^3}$, $P(2) = \frac{C}{2^3}$, $P(3) = \frac{C}{3^3}$

Since, $P(1) + P(2) + P(3) = 1$

$\therefore \quad \frac{C}{1^3} + \frac{C}{2^3} + \frac{C}{3^3} = 1$

$\Rightarrow C\left(\frac{1}{1}+\frac{1}{8}+\frac{1}{27}\right) = 1$

$\Rightarrow C\left(\frac{216+27+8}{216}\right) = 1$

$\Rightarrow C = \frac{216}{251}$

$\therefore \quad E(X) = (1)\frac{C}{1^3} + (2)\frac{C}{2^3} + (3)\frac{C}{3^3}$

$= C\left(1+\frac{1}{4}+\frac{1}{9}\right) = C\left(\frac{36+9+4}{36}\right)$

$= \frac{216}{251} \times \frac{49}{36} = \frac{294}{251}$

24. $E(X) = \sum x_i \cdot P(x_i) = 1.6$

$Var(X) = \sum x_i^2 \cdot P(x_i) - [E(X)]^2$

$= 4.8 - 2.56$

$= 2.24$

Now, $4E(X^2) - Var(X)$

$= 4\sum x_i^2 \cdot P(x_i) - Var(X) = 4(4.8) - 2.24$

$= 19.2 - 2.24$

$= 16.96$

25. $E(X) = \sum x_i \cdot P(x_i)$

$= 0(q^2) + 1(2pq) + 2(p^2)$

$= 2pq + 2p^2$

$= 2p(q + p)$

$= 2p$ ....[$\because p + q = 1$]

$Var(X) = E(X^2) - [E(X)]^2$

$= 0(q^2) + 1^2(2pq) + 2^2(p^2) - (2p)^2$

$= 2pq + 4p^2 - 4p^2$

$= 2pq$

26. $E(X) = \sum x_i \cdot P(x_i)$

$= 0(q^3) + 1(3q^2p) + 2(3qp^2) + 3(p^3)$

$= 3pq(q + 2p) + 3p^3$

$= 3pq[(p + q) + p] + 3p^3$

$= 3pq(1 + p) + 3p^3$ ....[$\because p + q = 1$]

$= 3pq + 3p^2q + 3p^3$

$= 3pq + 3p^2(q + p)$

$= 3p(q + p)$ ....[$\because p + q = 1$]

$= 3p(1)$

$= 3p$

27. X can take values 0, 1, 2 and 3.

$P(X = 0)$ = Probability of getting no head

$= \frac{1}{8}$

$P(X = 1)$ = Probability of getting one head

$= \frac{3}{8}$

$P(X = 2)$ = Probability of getting two heads

$= \frac{3}{8}$

$P(X = 3)$ = Probability of getting three heads

$= \frac{1}{8}$

$\therefore \quad E(X) = (0)\left(\frac{1}{8}\right) + (1)\left(\frac{3}{8}\right) + (2)\left(\frac{3}{8}\right) + (3)\left(\frac{1}{8}\right)$

$= 0 + \frac{3}{8} + \frac{3}{4} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$

28. X can take values 0, 1 and 2.

$P(X = 0)$ = Probability of getting no tail

$= \frac{1}{4}$

$P(X = 1)$ = Probability of getting one tail

$= \frac{1}{2}$

$P(X = 2)$ = Probability of getting two tails

$= \frac{1}{4}$

$\therefore \quad E(X) = 0\left(\frac{1}{4}\right) + (1)\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right)$

$= 0 + \frac{1}{2} + \frac{1}{2} = 1$

$Var(X) = E(X^2) - [E(X)]^2$

$= 0^2\left(\frac{1}{4}\right) + 1^2\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{4}\right) - (1)^2$

$= \frac{1}{2}$

29. X can take values 0, 1 and 2.

$P(X = 0)$ = Probability of not getting six $= \frac{25}{36}$

$P(X = 1)$ = Probability of getting one six $= \frac{10}{36}$

$P(X = 2)$ = Probability of getting two sixes $= \frac{1}{36}$

$\therefore$ the probability distribution of X is given by

X	0	1	2
P (X)	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

$\therefore$ $E(X) = \sum x_i.P(x_i) = 0 \times \frac{25}{36} + 1 \times \frac{10}{36} + 2 \times \frac{1}{36}$

$= \frac{10}{36} + \frac{2}{36} = \frac{1}{3}$

30. In a single throw of a pair of dice, the sum of the numbers on them can be 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. So X can take values 2,3,4,…, 12. The probability distribution of X is

X:	2	3	4	5	6	7	8	9	10	11	12
P(X) :	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$\therefore$ $E(X) = \frac{1}{36} \times 2 + \frac{2}{36} \times 3 + \frac{3}{36} \times 4 + \frac{4}{36} \times 5$

$+ \frac{5}{36} \times 6 + \frac{6}{36} \times 7 + \frac{5}{36} \times 8 + \frac{4}{36} \times 9$

$+ \frac{3}{36} \times 10 + \frac{2}{36} \times 11 + \frac{1}{36} \times 12$

$\Rightarrow E(X) = \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40$

$+ 36 + 30 + 22 + 12)$

$\Rightarrow E(X) = \frac{252}{36} = 7$

31. $E(X) = \sum x_i.P(x_i)$

$= 1\left(\frac{1}{15}\right) + 2\left(\frac{1}{15}\right) + \ldots. + 14\left(\frac{1}{15}\right) + 15\left(\frac{1}{15}\right)$

$= \frac{1}{15}(1 + 2 + 3 + \ldots. + 14 + 15)$

$= \frac{1}{15}\left(\frac{15\times16}{2}\right)$ ....$\left[\because \sum_{r=1}^{n} r = \frac{n(n+1)}{2}\right]$

$= 8$

32.

X	1	2	3	….	n
P(X)	$\frac{2}{n(n+1)}$	$\frac{4}{n(n+1)}$	$\frac{6}{n(n+1)}$	….	$\frac{2n}{n(n+1)}$

$E(X) = \sum x_i \cdot P(x_i)$

$= 1.\frac{2}{n(n+1)} + 2.\frac{4}{n(n+1)} + 3.\frac{6}{n(n+1)}$

$+\ldots. + n.\frac{2n}{n(n+1)}$

$= \frac{2}{n(n+1)}(1 + 4 + 9 + \ldots + n^2)$

$= \frac{2}{n(n+1)}(1^2 + 2^2 + 3^2 + \ldots + n^2)$

$= \frac{2}{n(n+1)} \cdot \frac{n(n+1)(2n+1)}{6}$

$= \frac{2n+1}{3}$

33. $E(X) = \sum x_i \cdot P(x_i)$

$= 1\left(\frac{1}{n}\right) + 2\left(\frac{1}{n}\right) + 3\left(\frac{1}{n}\right) + \ldots. + n\left(\frac{1}{n}\right)$

$= \frac{1+2+3+\ldots.+n}{n}$

$= \frac{1}{n} \times \frac{n(n+1)}{2}$

$= \frac{n+1}{2}$

$E(X^2) = \sum x_i^2 \cdot P(x_i)$

$= 1^2\left(\frac{1}{n}\right) + 2^2\left(\frac{1}{n}\right) + 3^2\left(\frac{1}{n}\right) + \ldots. + n^2\left(\frac{1}{n}\right)$

$= \frac{1^2 + 2^2 + 3^2 + \ldots. + n^2}{n}$

$= \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6}$

$= \frac{(n+1)(2n+1)}{6}$

$\therefore$ $\text{Var}(X) = E(X^2) - [E(X)]^2$

$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$

$= \frac{n+1}{2}\left(\frac{2n+1}{3} - \frac{n+1}{2}\right)$

$= \frac{n^2-1}{12}$

$\therefore$ Standard deviation of X

$= \sqrt{\text{Var}(X)} = \sqrt{\frac{n^2-1}{12}}$

34. Let X = demand for each type of cake (according to the profit)

$P(X = 3) = 10\% = \frac{10}{100} = 0.1$

$P(X = 2.5) = 5\% = \frac{5}{100} = 0.05$

$P(X = 2) = 20\% = \frac{20}{100} = 0.2$

$P(X = 1.5) = 50\% = \frac{50}{100} = 0.5$

$P(X = 1) = 15\% = \frac{15}{100} = 0.15$

∴ The probability distribution table is as follows:

X	3	2.5	2	1.5	1
P(X)	0.1	0.05	0.2	0.5	0.15

$E(X) = \sum x_i \cdot P(x_i)$

$= 3(0.1) + 2.5(0.05) + 2(0.2) + 1.5(0.5) + 1(0.15)$

$= 0.3 + 0.125 + 0.4 + 0.75 + 0.15 = 1.725$

35. Since, f (x) is the p.d.f. of X

∴ $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$\Rightarrow \int_0^3 C(9 - x^2)dx = 1$

$\Rightarrow C\left[9x - \frac{x^3}{3}\right]_0^3 = 1$

$\Rightarrow C(27 - 9) = 1 \Rightarrow C = \frac{1}{18}$

36. Since, f(x) is the p.d.f. of X.

∴ $\int_{-\infty}^{\infty} f(x)dx = 1$

$\Rightarrow \int_{-\infty}^{0} f(x)dx + \int_0^1 f(x)dx + \int_1^{\infty} f(x)dx = 1$

$\Rightarrow 0 + \int_0^1 kx^2(1 - x)dx + 0 = 1$

$\Rightarrow k\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 1 \Rightarrow k = 12$

37. Since, f(x) is the p.d.f. of X.

∴ $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$\Rightarrow \int_0^3 \left(\frac{x}{6} + k\right) dx = 1$

$\Rightarrow \left[\frac{x^2}{12} + kx\right]_0^3 = 1 \Rightarrow \frac{3}{4} + 3k = 1$

$\Rightarrow 3k = \frac{1}{4} \Rightarrow k = \frac{1}{12}$

38. $P\left(\frac{1}{4} < X < \frac{1}{3}\right) = \int_{\frac{1}{4}}^{\frac{1}{3}} f(x)\, dx = \int_{\frac{1}{4}}^{\frac{1}{3}} 3(1 - 2x^2)\, dx$

$= [3x - 2x^3]_{1/4}^{1/3}$

$= \left(1 - \frac{2}{27}\right) - \left(\frac{3}{4} - \frac{1}{32}\right)$

$= \frac{1}{4} + \frac{1}{32} - \frac{2}{27} = \frac{179}{864}$

39. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

∴ $\int_0^4 \frac{K}{\sqrt{x}} dx = 1$

$\Rightarrow K\left[2\sqrt{x}\right]_0^4 = 1$

$\Rightarrow 2K\left[\sqrt{4} - \sqrt{0}\right] = 1$

$\Rightarrow 4K = 1$

$\Rightarrow K = \frac{1}{4}$

∴ $P(X \geq 1) = P(1 \leq X < 4)$

$= \int_1^4 f(x)\, dx = 2K\left[\sqrt{x}\right]_1^4$

$= 2 \times \frac{1}{4}(2 - 1) = \frac{1}{2} = 0.5$

40. $P(|X| < 1) = P(-1 < X < 1)$

$= \int_{-1}^{1}\left(\frac{x + 2}{18}\right)dx$

$= \frac{1}{18}\int_{-1}^{1}(x + 2)\, dx$

$= \frac{1}{18}\left[\frac{x^2}{2} + 2x\right]_{-1}^{1}$

$= \frac{1}{18}\left(\frac{5}{2} + \frac{3}{2}\right) = \frac{4}{18} = \frac{2}{9}$

41. $P(0.2 \leq X \leq 0.5) = \int_{0.2}^{0.5} \frac{x^2}{8}\, dx = \left[\frac{x^3}{24}\right]_{0.2}^{0.5}$

$= \frac{1}{24}\left[(0.5)^3 - (0.2)^3\right]$

$= \frac{0.125 - 0.008}{24} = \frac{0.117}{24}$

42. Since, f(x) is the p.d.f. of X.

$\therefore \quad \int_{-\infty}^{\infty} f(x)\, dx = 1$

$\therefore \quad \int_{0}^{\infty} K.e^{-\theta x}\, dx = 1$

$\therefore \quad K\left[\frac{e^{-\theta x}}{-\theta}\right]_0^{\infty} = 1$

$\Rightarrow -\frac{K}{\theta}\left[\frac{1}{e^{\theta x}}\right]_0^{\infty} = 1$

$\Rightarrow -\frac{K}{\theta}\left[\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right] = 1$

$\Rightarrow -\frac{K}{\theta}\left[\frac{1}{\infty} - \frac{1}{1}\right] = 1$

$\Rightarrow \frac{K}{\theta} = 1 \Rightarrow K = \theta$

43. P(0 < X < K) = 0.5

$\Rightarrow \int_0^K f(x)\, dx = \frac{1}{2}$

$\Rightarrow \int_0^K ae^{-ax}\, dx = \frac{1}{2}$

$\Rightarrow a\left[\frac{e^{-ax}}{-a}\right]_0^K = \frac{1}{2}$

$\Rightarrow -\left[e^{-ax}\right]_0^K = \frac{1}{2}$

$\Rightarrow -\left(e^{-aK} - e^{0}\right) = \frac{1}{2}$

$\Rightarrow -e^{-aK} + 1 = \frac{1}{2}$

$\Rightarrow e^{-aK} = \frac{1}{2}$

$\Rightarrow -aK = \log\left(\frac{1}{2}\right)$

$\Rightarrow aK = \log 2$

$\Rightarrow K = \frac{1}{a}\log 2$

44. Since, $f_X(x)$ is the p.d.f. of X.

$\therefore \quad \int_0^4 \frac{k}{\sqrt{x}}\, dx = 1$

$\Rightarrow k\left[2\sqrt{x}\right]_0^4 = 1$

$\Rightarrow k = \frac{1}{4}$ ….(i)

$F(x) = \int_0^x \frac{k}{\sqrt{x}}\, dx$

$= k\left[2\sqrt{x}\right]_0^x$

$= \frac{\sqrt{x}}{2}$ ....[From (i)]

45. $P(C_1 \cup C_2)$

$= P(C_1) + P(C_2)$

$= \int_1^2 f(x)dx + \int_4^5 f(x)dx$

$= \int_1^2 \frac{1}{x^2}dx + \int_4^5 \frac{1}{x^2}dx$

$= \left[\frac{-1}{x}\right]_1^2 + \left[\frac{-1}{x}\right]_4^5$

$= -\frac{1}{2} + 1 - \frac{1}{5} + \frac{1}{4}$

$= \frac{11}{20}$

46. Since, f(x) is the p.d.f. of X.

$\therefore \quad \int_0^2 f(x)dx = 1$

$\Rightarrow \int_0^2 \left(kx^2\right)dx = 1$

$\Rightarrow k\left[\frac{x^3}{3}\right]_0^2 = 1$

$\Rightarrow k = \frac{3}{8}$

$\therefore$ Required probability = $P(X \le 1) = \int_0^1 f(x)dx$

$= \int_0^1 \left(kx^2\right)dx$

$= \frac{3}{8}\int_0^1 x^2\, dx$

$= \frac{3}{8}\left[\frac{x^3}{3}\right]_0^1$

$= \frac{3}{8}\left(\frac{1}{3} - 0\right)$

$= \frac{1}{8}$

Competitive Thinking

1. Since, $\sum_{x=1}^{3} P(X = x) = 1$

∴ $0.3 + k + 2k + 2k = 1$

$\Rightarrow 5k = 0.7$

$\Rightarrow k = 0.14$

2. Since, $\sum_{x=1}^{6} P(X = x) = 1$

∴ $0.1 + 2k + k + 0.2 + 3k + 0.1 = 1$

∴ $6k = 1 - 0.4 = 0.6$

∴ $k = \frac{0.6}{6} = 0.1$

3. When we get 1, positive divisors = 1
When we get 2, positive divisors = 2
When we get 3, positive divisors = 2
When we get 4, positive divisors = 3
When we get 5, positive divisors = 2
When we get 6, positive divisors = 4

∴ range of random variable X = {1, 2, 3, 4}

4. When a coin is tossed 3 times possibilities are

	HHH	TTT	HHT	HTH
Absolute difference between Heads and Tails($X=x_i$)	$3 - 0 = 3$	$3 - 0 = 3$	$2 - 1 = 1$	$2 - 1 = 1$

	THH	HTT	TTH	THT
Absolute difference between Heads and Tails($X=x_i$)	$2 - 1 = 1$	$2 - 1 = 1$	$2 - 1 = 1$	$2 - 1 = 1$

∴ $P(X = 1) = \frac{6}{8} = \frac{3}{4}$

5.

X = k	0	1	2	3	4	5
P(X = k)	a	a	$\frac{3a}{4}$	$\frac{4a}{8}$	$\frac{5a}{16}$	$\frac{6a}{32}$

Since, $\sum_{k=0}^{5} P\ (X = k) = 1$

∴ $a + a + \frac{3a}{4} + \frac{4a}{8} + \frac{5a}{16} + \frac{6a}{32} = 1$

$\Rightarrow \frac{15}{4}a = 1 \Rightarrow a = \frac{4}{15}$

Now, P(X = prime value)

$= P(X = 2) + P(X = 3) + P(X = 5)$

$= \frac{3a}{4} + \frac{4a}{8} + \frac{6a}{32}$

$= \frac{23a}{16}$

$= \frac{23}{16} \times \frac{4}{15}$

$= \frac{23}{60}$

6. Mean $= (1)\left(\frac{1}{4}\right) + (2)\left(\frac{1}{8}\right) + (3)\left(\frac{5}{8}\right)$

$= \frac{19}{8}$

7. Mean $= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right) + 4\left(\frac{1}{6}\right)$

$= \frac{1}{6} + \frac{2}{3} + \frac{3}{3} + \frac{2}{3}$

$= \frac{1}{6} + \frac{7}{3}$

$= \frac{15}{6}$

$= \frac{5}{2}$

8. Since $\sum_{x=1}^{6} P(X = x) = 1$

∴ $a + a + a + b + b + 0.3 = 1$

$\Rightarrow 3a + 2b = 0.7$ ...(i)

Mean $= a + 2a + 3a + 4b + 5b + 6\ (0.3)$

$\Rightarrow 4.2 = 6a + 9b + 1.8$

$\Rightarrow 6a + 9b = 2.4$ ...(ii)

On solving (i) and (ii), we get

$a = 0.1,\ b = 0.2$

9. $E\ (X) = 3 \times \frac{1}{3} + 4 \times \frac{1}{4} + 12 \times \frac{5}{12}$

$= 7$

10. $y = 2x$

x	0	1	2	3
y	0	2	4	6
P(y)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

∴ Expected gain $= \sum y_i\, P(y_i)$

$= 0\left(\frac{1}{8}\right) + 2\left(\frac{3}{8}\right) + 4\left(\frac{3}{8}\right) + 6\left(\frac{1}{8}\right) = 3$

11. E (X) = 2(0.3) + 3(0.4) + 4 (0.3)
$= 0.6 + 1.2 + 1.2 = 3$

$\therefore$ $\text{Var}(X) = E(X^2) - [E(X)]^2$
$= 4(0.3) + 9(0.4) + 16(0.3) - (3)^2$
$= 9.6 - 9 = 0.6$

12. $E(X) = \sum x_i \cdot P(x_i)$
$= 0(0.1) + 1(0.4) + 2(0.3) + 3(0.2) + 4(0)$
$= 0 + 0.4 + 0.6 + 0.6 + 0$
$= 1.6$

Variance $= \sum x_i^2 \cdot P(x_i) - [E(x)]^2$
$= 0^2 (0.1) + 1^2 (0.4) + 2^2 (0.3) + 3^2 (0.2) + 4^2 (0) - 1.6^2$
$= 0 + 0.4 + 1.2 + 1.8 - 2.56$
$= 0.84$

13. $E(X) = \Sigma x_i . P(x_i)$
$= 0\left(\frac{25}{16}\right) + 1\left(\frac{5}{18}\right) + 2\left(\frac{1}{36}\right) = \frac{1}{3}$

$V(X) = \Sigma x_i^2 . P(x_i) - [E(X)]^2$
$= (0)^2\left(\frac{25}{36}\right) + (1)^2\left(\frac{5}{18}\right) + (2)^2\left(\frac{1}{36}\right) - \left(\frac{1}{3}\right)^2$
$= \frac{7}{18} - \frac{1}{9} = \frac{5}{18}$

S.D. $= \sqrt{\text{var}(X)} = \sqrt{\frac{5}{18}} = \frac{1}{3}\sqrt{\frac{5}{2}}$

14. $E(X) = \sum x_i \cdot P(x_i) = -\frac{1}{3} + 0 + \frac{1}{6} + \frac{2}{3} = \frac{1}{2}$

$\text{Var}(X) = E(X^2) - [E(X)]^2$
$= \frac{(-1)^2}{3} + 0 + \frac{1^2}{6} + \frac{2^2}{3} - \left(\frac{1}{2}\right)^2$
$= \frac{1}{3} + \frac{1}{6} + \frac{4}{3} - \frac{1}{4}$
$= \frac{11}{6} - \frac{1}{4} = \frac{19}{12}$

$\therefore$ $6E(X^2) - \text{Var}(X)$
$= 6\left(\frac{1}{3} + 0 + \frac{1}{6} + \frac{4}{3}\right) - \frac{19}{12}$
$= 11 - \frac{19}{12}$
$= \frac{113}{12}$

15. Let P(X = 3) = a, then

$P(X = 1) = \frac{a}{2}$, $P(X = 2) = \frac{a}{3}$ and $P(X = 4) = \frac{a}{5}$

Since, P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1

$\therefore$ $\frac{a}{2} + \frac{a}{3} + a + \frac{a}{5} = 1$

$\Rightarrow a = \frac{30}{61}$

Now,

$X = x$	1	2	3	4
$P(X = x)$	$\frac{a}{2}$	$\frac{a}{3}$	a	$\frac{a}{5}$

Now, μ = mean $= \frac{1}{2}a + \frac{2}{3}a + 3a + \frac{4}{5}a$
$= \frac{149}{30}a$

σ^2 = variance
$= \frac{1}{2}a + \frac{4}{3}a + 9a + \frac{16}{5}a - \left(\frac{149}{30}a\right)^2$
$= \frac{421}{30}a - \left(\frac{149}{30}a\right)^2$

Now, $\sigma^2 + \mu^2 = \frac{421}{30}a - \left(\frac{149}{30}a\right)^2 + \left(\frac{149}{30}a\right)^2$
$= \frac{421}{30} \times \frac{30}{61} = \frac{421}{61}$

16. $\text{Var}(X) = \sigma^2 = 5^2 = 25$
$\text{Var}(X) = E(X^2) - [E(X)]^2$
$\Rightarrow 25 = E(X^2) - 10^2$
$\Rightarrow E(X^2) = 125$

$E\left(\frac{X - 15}{5}\right)^2 = E\left(\frac{X^2 - 30X + 225}{25}\right)$
$= \frac{1}{25}\left[E(X^2) - 30E(X) + 225\right]$
$= \frac{1}{25}(125 - 300 + 225)$
$= 2$

17. Let x denote number of defective pens.
x can take the values 0, 1, 2.

$P(X = 0) = \frac{{}^4C_2}{{}^6C_2} = \frac{2}{5}$

$$P(X = 1) = \frac{{}^2C_1 \times {}^4C_1}{{}^6C_2} = \frac{8}{15}$$

$$P(X = 2) = \frac{{}^2C_2}{{}^6C_2} = \frac{1}{15}$$

X = x	0	1	2
P(x)	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$

$$E(X) = \sum x_i P(x_i)$$
$$= 0\left(\frac{2}{5}\right) + 1\left(\frac{8}{15}\right) + 2\left(\frac{1}{15}\right)$$
$$= \frac{10}{15}$$
$$= \frac{2}{3}$$

$$E(X^2) = \sum x_i^2 P(x_i)$$
$$= 0\left(\frac{2}{5}\right) + 1\left(\frac{8}{15}\right) + 4\left(\frac{1}{15}\right)$$
$$= \frac{12}{15} = \frac{4}{5}$$

Standard deviation (σ) $= \sqrt{E(X^2) - [E(X)]^2}$

$$= \sqrt{\frac{4}{5} - \frac{4}{9}}$$
$$= \frac{4}{3\sqrt{5}}$$

19. Required probability $= \int_0^4 f(x)\,dx$

$$= \int_0^4 \frac{1}{5}\,dx$$
$$= \frac{1}{5}[x]_0^4 = \frac{4}{5} = 0.8$$

20. $P(X = 4) = F(4) - F(3) = 0.62 - 0.48 = 0.14$

$P(X = 5) = F(5) - F(4) = 0.85 - 0.62 = 0.23$

$P(3 < X \le 5) = P(X = 4) + P(X = 5)$

$= 0.14 + 0.23 = 0.37$

Evaluation Test

1. Given,

$P(X= 3) = 2P(X= 1)$ and $P(X= 2) = 0.3$ ….(i)

Now, mean = 1.3

$\therefore$ $0 \times P(X = 0) + 1 \times P(X = 1) + 2 \times P(X = 2) + 3 \times P(X = 3) = 1.3$

$\Rightarrow 7P(X = 1) = 0.7$ ….[From (i)]

$\Rightarrow P(X = 1) = 0.1$

Also, $P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1$

$\Rightarrow P(X = 0) + 3P(X = 1) = 0.7$ ….[From (i)]

$\Rightarrow P(X = 0) + 0.3 = 0.7$

$\Rightarrow P(X = 0) = 0.4$

2. $\sum_{x=0}^{8} P(X = x) = 1$

$\Rightarrow a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$\Rightarrow 81a = 1$

$\Rightarrow a = \frac{1}{81}$

3. $P(E) = P(X = 2 \text{ or } X = 3 \text{ or } X = 5 \text{ or } X = 7)$

$= P(X = 2) + P(X = 3) + P(X = 5) + P(X= 7)$

$= 0.23 + 0.12 + 0.20 + 0.07 = 0.62$

$P(F) = P(X < 4)$

$= P(X = 1) + P(X = 2) + P(X = 3)$

$= 0.15 + 0.23 + 0.12 = 0.50$

$P(E \cap F) = P(X \text{ is a prime number less than } 4)$

$= P(X = 2) + P(X = 3)$

$= 0.23 + 0.12 = 0.35$

$\therefore$ $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= 0.62 + 0.50 - 0.35 = 0.77$

4. Here, $\frac{1+3p}{4}$, $\frac{1-p}{4}$, $\frac{1+2p}{4}$ and $\frac{1-4p}{4}$ are probabilities when X takes values -1, 0, 1 and 2 respectively. Therefore, each is greater than or equal to 0 and less than or equal to 1.

i.e., $0 \le \frac{1+3p}{4} \le 1$, $0 \le \frac{1-p}{4} \le 1$,

$0 \le \frac{1+2p}{4} \le 1$ and $0 \le \frac{1-4p}{4} \le 1$

$\Rightarrow -\frac{1}{3} \le p \le \frac{1}{4}$

$$\text{Mean}(X) = -1 \times \frac{1+3p}{4} + 0 \times \frac{1-p}{4} + 1 \times \frac{1+2p}{4} + 2 \times \frac{1-4p}{4}$$

$$= \frac{2-9p}{4}$$

Now, $-\frac{1}{3} \le p \le \frac{1}{4}$

$\Rightarrow 3 \ge -9p \ge -\frac{9}{4}$

$\Rightarrow -\frac{1}{4} \le 2 - 9p \le 5$

$\Rightarrow -\frac{1}{16} \le \frac{2-9p}{4} \le \frac{5}{4}$

5. $P(X > 1.5) = \int_{1.5}^{2} \frac{x}{2} dx = \left[\frac{x^2}{4}\right]_{1.5}^{2} = 0.4375$

and $P(X > 1) = \int_{1}^{2} \frac{x}{2} dx = \left[\frac{x^2}{4}\right]_{1}^{2} = 0.75$

$\therefore$ $P\left(\frac{X > 1.5}{X > 1}\right) = \frac{P(X > 1.5)}{P(X > 1)} = \frac{0.4375}{0.75} = \frac{7}{12}$

6. $P(X = x_i) = ki$, where $1 \le i \le 10$

$\therefore$ $\sum P(X = x_i) = 1$

$\Rightarrow (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10)k = 1$

$\Rightarrow k = \frac{1}{55}$

7. We have, $\sum_{x=0}^{\infty} P(X = x) = 1$

$\Rightarrow k \sum_{x=0}^{\infty} (x+1)\left(\frac{1}{5}\right)^x = 1$

$\Rightarrow k\left[1 + 2\left(\frac{1}{5}\right) + 3\left(\frac{1}{5}\right)^2 + 4\left(\frac{1}{5}\right)^3 + ...\right] = 1$

$\Rightarrow k\left[\frac{1}{1-\frac{1}{5}} + \frac{1 \times \frac{1}{5}}{\left(1-\frac{1}{5}\right)^2}\right] = 1$

$$....\left[\because a + (a+d)r + (a+2d)r^2 + ... = \frac{a}{1-r} + \frac{dr}{(1-r)^2}\right]$$

$\Rightarrow k\left(\frac{5}{4} + \frac{5}{16}\right) = 1$

$\Rightarrow \frac{25k}{16} = 1$

$\Rightarrow k = \frac{16}{25}$

Textbook Chapter No.

09 Binomial Distribution

Hints

Classical Thinking

2. $P(X = 1) = {}^{10}C_1 (0.2)(0.8)^9 = 0.2684$

3. Probability of getting head is $p = \frac{1}{2}$

$\therefore$ $q = 1 - \frac{1}{2} = \frac{1}{2}$

Also, n = 4

$\therefore$ Required probability = P (X = 3)

$$= {}^4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{4}$$

4. Here $p = \frac{1}{2}$, $q = \frac{1}{2}$, n = 10

$\therefore$ Required probability= P (X = 5)

$$= {}^{10}C_5\left(\frac{1}{2}\right)^5.\left(\frac{1}{2}\right)^5 = \frac{63}{256}$$

5. Probability of obtaining 5 is $p = \frac{1}{6}$

$\therefore$ $q = 1 - \frac{1}{6} = \frac{5}{6}$

Also, n = 7

$\therefore$ Required probability = P (X = 4)

$$= {}^7C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^3$$

6. Probability of getting on even number is

$p = \frac{3}{6} = \frac{1}{2}$

$\therefore$ $q = 1 - \frac{1}{2} = \frac{1}{2}$ and n = 5, r = 3

$\therefore$ Required probability

$$= {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5}{16}$$

7. Probability of getting an odd number, $p = \frac{3}{6} = \frac{1}{2}$

$\therefore$ $q = 1 - \frac{1}{2} = \frac{1}{2}$

Also, n = 2

$\therefore$ Required probability = P (X = 2)

$$= {}^2C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^0 = \frac{1}{4}$$

8. Here, $p = \frac{1}{2}$, $q = \frac{1}{2}$, n = 3

$\therefore$ Required probability = P (X ≥ 2)

$$= {}^3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) + {}^3C_3\left(\frac{1}{2}\right)^3$$

$$= \frac{4}{8} = \frac{1}{2}$$

9. Probability of getting an odd number,

$p = \frac{3}{6} = \frac{1}{2}$

$\therefore$ $q = 1 - \frac{1}{2} = \frac{1}{2}$ Also, n = 5

$\therefore$ Variance = npq = $5.\left(\frac{1}{2}\right).\left(\frac{1}{2}\right) = \frac{5}{4}$

Critical Thinking

1. Here, $q = \frac{1}{5}$

$\Rightarrow p = 1 - \frac{1}{5} = \frac{4}{5}$

Also, n = 5

$\therefore$ Required probability = ${}^5C_1\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^4$

{Here exactly one student is swimmer}

2. Probability of success is $p = \frac{3}{5}$

$\Rightarrow q = 1 - p = \frac{2}{5}$

Also, n = 5

$\therefore$ Required probability = P (X = 2)

$$= {}^5C_2\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3 = \frac{144}{625}$$

3. Probability that bulb will fuse, $p = 0.05$
$= \frac{1}{20}$

$\therefore$ Probability that bulb will not fuse,
$q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$
Also, $n = 5$

$\therefore$ Probability that out of 5 bulbs none will fuse
$= {}^5C_0\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^5 = \left(\frac{19}{20}\right)^5$

4. Probability of correct prediction,
$p = \frac{1}{3} \Rightarrow q = 1 - \frac{1}{3} = \frac{2}{3}$
Also, $n = 7$

$\therefore$ Required probability = P (X = 4)
$= {}^7C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^3 = \frac{280}{3^7}$

5. Here, $p = \frac{1}{2}, q = \frac{1}{2}, n = 3$

$\therefore$ Required probability
= Probability of getting exactly one head + probability of getting exactly two heads
$= {}^3C_1\left(\frac{1}{2}\right)^1 . \left(\frac{1}{2}\right)^2 + {}^3C_2\left(\frac{1}{2}\right)^2 . \left(\frac{1}{2}\right)^1$
$= \frac{3}{8} + \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$

6. Here, $p = \frac{1}{2}, q = \frac{1}{2}, n = 10$

$\therefore$ Required probability = P(X = 4)
$= {}^{10}C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6 = {}^{10}C_4\left(\frac{1}{2}\right)^{10}$
$= {}^{10}C_6\left(\frac{1}{2}\right)^{10}$ ....$\left[\because {}^nC_r = {}^nC_{n-r}\right]$

7. 9 P (X = 4) = P (X = 2)

$\therefore$ $9.{}^6C_4 p^4q^2 = {}^6C_2 p^2q^4 \Rightarrow 9p^2 = q^2$
Putting $q = 1 - p$, we get
$p = \frac{1}{4}$

8. Required probability
= P(exactly two success) + P(exactly three success)
$= {}^3C_2.\left(\frac{2}{6}\right)^2\left(\frac{4}{6}\right) + {}^3C_3\left(\frac{2}{6}\right)^3$
$= \frac{2}{9} + \frac{1}{27} = \frac{7}{27}$

9. We have, $p = \frac{3}{4}$
$\Rightarrow q = \frac{1}{4}$ and $n = 5$

$\therefore$ Required probability = $P(X \geq 3)$
$= {}^5C_3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + {}^5C_4\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + {}^5C_5\left(\frac{3}{4}\right)^5$
$= \frac{(10)(27)}{4^5} + \frac{(5)(81)}{4^5} + \frac{243}{4^5}$
$= \frac{270 + 405 + 243}{1024}$
$= \frac{459}{512}$

10. Required probability = $P(X \geq 1)$
$= {}^3C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + {}^3C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + {}^3C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0$
$= \frac{91}{216}$

11. Required probability = $P (X \geq 6)$
$= {}^8C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2 + {}^8C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right) + {}^8C_8\left(\frac{1}{2}\right)^8$
$= \frac{37}{256}$

12. Let the coin be tossed n times.
Then, P(7 heads) = ${}^nC_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^{n-7} = {}^nC_7\left(\frac{1}{2}\right)^n$
and P(9 heads) = ${}^nC_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^{n-9} = {}^nC_9\left(\frac{1}{2}\right)^n$
Now, P(7 heads) = P(9 heads)
$\Rightarrow {}^nC_7 = {}^nC_9$
$\Rightarrow n = 16$

$\therefore$ P (3 heads) = ${}^{16}C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{16-3}$
$= {}^{16}C_3\left(\frac{1}{2}\right)^{16} = \frac{35}{2^{12}}$

14. Here, $n = 3, p = \frac{1}{6}, q = \frac{5}{6}$
Mean $= np = 3 \times \frac{1}{6} = \frac{1}{2}$
Variance $= npq = 3 \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{12}$

15. Here, $p = \frac{2}{6} = \frac{1}{3} \Rightarrow q = 1 - \frac{1}{3} = \frac{2}{3}$
Also, n = 2
$\therefore$ Variance $= npq = 2 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}$

16. We have, mean = np = 2
and variance = npq = 1
$\Rightarrow q = \frac{1}{2}, p = \frac{1}{2}$ and n = 4
$\therefore$ $P(X \geq 1) = 1 - P(X = 0)$
$= 1 - {}^4C_0\left(\frac{1}{2}\right)^4$
$= \frac{15}{16}$

17. Here, np = 4 and npq = 3
$\Rightarrow p = \frac{1}{4}, q = \frac{3}{4}$
Also, n = 16
$\therefore$ $P(X = 6) = {}^{16}C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^{10}$

18. Probability of getting a red card is
$p = \frac{26}{52} = \frac{1}{2}$
$\therefore$ $q = 1 - \frac{1}{2} = \frac{1}{2}$ Also, n = 4
Mean $= np = 4\left(\frac{1}{2}\right) = 2$
Variance $= npq = 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1$

19. $\frac{P(X = k)}{P(X = k-1)} = \frac{{}^nC_k(p)^k(q)^{n-k}}{{}^nC_{k-1}(p)^{k-1}(q)^{n-k+1}}$
$= \frac{{}^nC_k}{{}^nC_{k-1}}.\frac{p}{q}$
$\therefore$ $\frac{P(X = k)}{P(X = k-1)} = \frac{n-k+1}{k}.\frac{p}{q}$

20. Let X denote the number of aces obtained in two draws. Then, X follows binomial distribution with n = 2, $p = \frac{4}{52} = \frac{1}{13}$ and
$q = \frac{12}{13}$
$\therefore$ Mean of number of aces $= np = \frac{2}{13}$

Competitive Thinking

1. Here, n = 5, $p = \frac{1}{3}$
and $q = 1 - \frac{1}{3} = \frac{2}{3}$
$\therefore$ $P(2 < X < 4) = P(X = 3)$
$= {}^5C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{40}{243}$

2. Probability of getting head, $p = \frac{1}{2}$
$\therefore$ $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$
Also, n = 10
$\therefore$ Required probability = P (X = 6)
$= {}^{10}C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4$
$= \frac{10!}{6!4!}.\frac{1}{2^{10}} = \frac{105}{512}$

3. Probability of occurrence of '4' is $p = \frac{1}{6}$
$\therefore$ $q = 1 - \frac{1}{6} = \frac{5}{6}$
Also, n = 2,
$\therefore$ Required probability = P (X ≥ 1)
$= {}^2C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + {}^2C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^0$
$= \frac{11}{36}$

4. Probability that person will develop immunity (p) = 0.8
q = 1 – p = 0.2
$\therefore$ Required probability $= {}^8C_0\,(0.8)^8\,(0.2)^0$
$= (0.8)^8$

5. Probability of getting rotten egg is
$p = \frac{10}{100} = \frac{1}{10}$
$\therefore$ $q = 1 - \frac{1}{10} = \frac{9}{10}$
Also, n = 5
$\therefore$ The probability that no egg is rotten
$= {}^5C_0.\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^5$

6. Probability of disease to be fatal = p = 10%

$p = \frac{10}{100} = \frac{1}{10}, q = \frac{9}{10}$

Number of patients, n = 6

$\therefore$ Required probability $= {}^6C_3\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^3$

$= 1458 \times 10^{-5}$

7. Probability of getting a 'six' in one throw is

$p = \frac{1}{6}$

$\therefore$ $q = 1 - \frac{1}{6} = \frac{5}{6}$

Also, n = 4

$\therefore$ Required probability

$= P(x = 4) = {}^4C_4\left(\frac{1}{6}\right)^4 . \left(\frac{5}{6}\right)^0$

$= \frac{1}{1296}$

8. P(without defect) $= \frac{8}{10} = \frac{4}{5} = p$

P(defected) $= \frac{2}{10} = \frac{1}{5} = q$ and n = 2, r = 2

$\therefore$ Required probability $= {}^2C_2\left(\frac{4}{5}\right)^2 . \left(\frac{1}{5}\right)^0 = \frac{16}{25}$

9. 2P(2) = 3P(3)

$\Rightarrow 2\ {}^6C_2\ p^2 q^4 = 3\ {}^6C_3\, p^3 q^3$

Putting q = 1 – p, we get

$p = \frac{1}{3}$

10. 4P(X = 4) = P(X = 2)

$\Rightarrow 4.{}^6C_4\, p^4q^2 = {}^6C_2\, p^2q^4$

$\Rightarrow 4p^2 = q^2$

$\Rightarrow 4p^2 = (1 - p)^2$

$\Rightarrow 3p^2 + 2p - 1 = 0$

$\Rightarrow p = \frac{1}{3}$

11. Here, $p = q = \frac{1}{2}$

Probability that head occurs 6 times

$= {}^nC_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{n-6}$ and probability that head occurs 8 times $= {}^nC_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{n-8}$

$\therefore$ ${}^nC_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{n-6} = {}^nC_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{n-8}$

$\Rightarrow {}^nC_6 = {}^nC_8 \Rightarrow (n - 6)(n - 7) = 56 \Rightarrow n = 14$

12. We have, ${}^{100}C_{50}\, p^{50}(1 - p)^{50} = {}^{100}C_{51}\, p^{51}(1 - p)^{49}$

$\Rightarrow \frac{1-p}{p} = \frac{100!}{51!.\,49!} \times \frac{50!.\,50!}{100!}$

$= \frac{50}{51}$

$\Rightarrow 51 - 51p = 50p \Rightarrow p = \frac{51}{101}$

13. The required probability

= 1 – Probability of equal number of heads and tails

$= 1 - {}^{2n}C_n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{2n-n}$

$= 1 - \frac{(2n)!}{n!n!}\left(\frac{1}{4}\right)^n$

$= 1 - \frac{(2n)!}{(n!)^2} \times \frac{1}{4^n}$

14. Probability of failure, $q = \frac{1}{3}$

Probability for getting success, $p = 1 - \frac{1}{3} = \frac{2}{3}$

Also, n = 4

$\therefore$ Required probability = P (X ≥ 3)

$= {}^4C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^0 + {}^4C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$

$= \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$

$= \frac{16}{27}$

15. Probability for white ball, $p = \frac{2}{6} = \frac{1}{3}$

Probability for black ball, $q = \frac{4}{6} = \frac{2}{3}$

Also, n = 5

$\therefore$ Required probability = P (X ≥ 4)

$= {}^5C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0 + {}^5C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)$

$= \left(\frac{1}{3}\right)^4\left[\frac{1}{3} + (5)\frac{2}{3}\right]$

$= \frac{11}{3^5} = \frac{11}{243}$

16. Required probability = P (X < 2)

$= {}^8C_1\left(\frac{1}{20}\right)^1\left(\frac{19}{20}\right)^7 + {}^8C_0\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^8$

$= \frac{27}{20}\left(\frac{19}{20}\right)^7$

17. P(minimum face value not less than 2 and maximum face value is not greater than 5)
= P(2 or 3 or 4 or 5)

$= \frac{4}{6} = \frac{2}{3}$

$\therefore$ required probability $= {}^4C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^0 = \frac{16}{81}$

18. Here, p = probability of getting perfect square in any throw $= \frac{2}{6} = \frac{1}{3}$

$\therefore$ $q = \frac{2}{3}$ and n = 4

Now,
P(getting perfect square in at least one throw)
= 1 – P(not getting perfect square in any throw)
$\Rightarrow P(X \geq 1) = 1 - P(X = 0)$

$= 1 - {}^4C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4$

$= 1 - \left(\frac{2}{3}\right)^4 = \frac{65}{81}$

19. P(answer is correct) $= p = \frac{1}{2}$

$\therefore$ $q = 1 - \frac{1}{2} = \frac{1}{2}$

Also, n = 10

$\therefore$ P(at least 7 answers are correct) = $P(X \geq 7)$
= P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)

$= {}^{10}C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 + {}^{10}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + {}^{10}C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right) + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$

$= \left({}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right)\frac{1}{2^{10}}$

$= (120 + 45 + 10 + 1)\frac{1}{1024}$

$= \frac{176}{1024}$

$= \frac{11}{64}$

20. Probability of green ball (p) $= \frac{15}{25} = \frac{3}{5}$

Probability of yellow ball (q) $= \frac{10}{25} = \frac{2}{5}$

Also, n = 10

$\therefore$ Variance = npq

$= 10\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)$

$= \frac{12}{5}$

21. Probability of occurence of event A is p = 0.3

$\therefore$ q = 0.7

Also, n = 6

$\therefore$ Variance = npq

$= 6 \times 0.3 \times 0.7 = 1.26$

22. n = 10, p = 0.4

E(X) = np = 4

V(X) = npq = 10 × 0.4 × 0.6 = 2.4

$V(X) = E(X^2) - [E(X)]^2$

$\therefore$ $2.4 = E(X^2) - [4]^2$

$\Rightarrow E(X^2) = 18.4$

23. Given np = 6, npq = 4

$\therefore$ $\frac{npq}{np} = \frac{4}{6}$

$\Rightarrow q = \frac{2}{3}$ and $p = \frac{1}{3}$

$\therefore$ np = 6

$\Rightarrow n \times \frac{1}{3} = 6$

$\Rightarrow$ n = 18

24. Mean = np = 18

Variance = npq = 12

$\therefore$ $\frac{npq}{np} = \frac{12}{18} \Rightarrow q = \frac{2}{3}$

$p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$

Now, np = 18

$\Rightarrow n\left(\frac{1}{3}\right) = 18$

$\Rightarrow$ n = 54

$\therefore$ Values of x are 0, 1, 2, 3, …, 54 = 55 values

25. $\left.\begin{matrix} np = 4 \\ npq = 2 \end{matrix}\right\} \Rightarrow q = \frac{1}{2}, p = \frac{1}{2}, n = 8$

$\therefore$ $P(X = 1) = {}^8C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^7$

$= 8.\frac{1}{2^8} = \frac{1}{2^5} = \frac{1}{32}$

26. $\left.\begin{matrix} np = 8 \\ npq = 4 \end{matrix}\right\} \Rightarrow q = \frac{1}{2}, p = \frac{1}{2}, n = 16$

$\therefore$ $P(X = 1) = {}^{16}C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{15}$

$= 16 \times \frac{1}{2} \times \frac{1}{2^{15}}$

$= 2^4 \times \frac{1}{2} \times \frac{1}{2^{15}} = \frac{1}{2^{12}}$

27. $\left.\begin{matrix} np = 4 \\ npq = 2 \end{matrix}\right\} \Rightarrow q = \frac{1}{2}, p = \frac{1}{2}, n = 8$

$\therefore$ $P(X = 2) = {}^8C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^6$

$= 28.\frac{1}{2^8} = \frac{28}{256}$

28. $E(X) = 5$ and $Var(X) = 2.5$

$\Rightarrow np = 5$ and $npq = 2.5$

$\Rightarrow p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 10$

$\therefore$ $P(X < 1) = P(X = 0)$

$= {}^{10}C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10}$

29. $E(X) = 6$ and $V(X) = 2$

$\Rightarrow np = 6$ and $npq = 2$

$\Rightarrow q = \frac{1}{3}, p = \frac{2}{3}$ and $n = 9$

$P(5 \le X \le 7) = P(X = 5) + P(X = 6) + P(X = 7)$

$= {}^9C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^4 + {}^9C_6\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^3 + {}^9C_7\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^2$

$= \frac{2^5}{3^9}[{}^9C_5 + {}^9C_6 \times 2 + {}^9C_7 \times 4]$

$= \frac{2^5}{3^9}[126 + 168 + 144]$

$= \frac{2^5 \times 438}{3^9} = \frac{2^5 \times 146}{3^9} = \frac{4672}{6561}$

30. Probability of getting a success, $p = \frac{1}{4}$

Probability of not getting success, $q = \frac{3}{4}$

Standard deviation $= \sqrt{\text{Variance}}$

$\Rightarrow$ Variance $= 9$

$\Rightarrow npq = 9 \Rightarrow n.\frac{1}{4}.\frac{3}{4} = 9 \Rightarrow n = 48$

Mean $= np = \frac{1}{4} \times 48 = 12$

31. Let X = Number of heads appear in n tosses

$X \sim B\left(n, \frac{1}{2}\right)$

Now, $P(X \ge 1) = 1 - P(X = 0) = 1 - \frac{1}{2^n}$

Since, $P(X \ge 1) \ge 0.9$

$\therefore$ $1 - \frac{1}{2^n} \ge 0.9$

$\Rightarrow \frac{1}{2^n} \le \frac{1}{10} \Rightarrow 2^n \ge 10 \Rightarrow n \ge 4$

$\therefore$ minimum number of tosses = 4

Evaluation Test

1. We have, $p = \frac{10}{100} = \frac{1}{10}$

$\therefore$ $q = 1 - \frac{1}{10} = \frac{9}{10}$

According to the given condition,

$P(X \ge 1) \ge \frac{50}{100}$

$\Rightarrow 1 - P(X = 0) \ge \frac{1}{2}$

$\Rightarrow P(X = 0) \le \frac{1}{2}$

$\Rightarrow \left(\frac{9}{10}\right)^n \le \frac{1}{2}$,

which is possible if n is at least 7.

$\therefore$ $n = 7$

2. $P(X = 1) = 8 . P(X = 3)$, if $n = 5$

$\Rightarrow {}^5C_1 q^4 p^1 = 8.\ {}^5C_3 q^2 p^3$

$\Rightarrow \dfrac{5q^2}{p^2} = 8(10) \Rightarrow \dfrac{q^2}{p^2} = 16$

$\Rightarrow q = 4p$

$\Rightarrow 1 - p = 4p$

$\Rightarrow 5p = 1 \Rightarrow p = \dfrac{1}{5}$

3. $P(X = 0) = \dfrac{16}{81}$

$\Rightarrow {}^4C_0\, p^0\, q^4 = \dfrac{16}{81} \Rightarrow q^4 = \left(\dfrac{2}{3}\right)^4$

$\Rightarrow q = \dfrac{2}{3} \Rightarrow p = \dfrac{1}{3}$

$\therefore$ $P(X = 4) = {}^4C_4 p^4 q^0 = p^4 = \left(\dfrac{1}{3}\right)^4 = \dfrac{1}{81}$

4. Here $p = \dfrac{3}{6} = \dfrac{1}{2}$

and $q = 1 - p = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

$n = 100$

$\therefore$ variance $= npq = 100 \times \dfrac{1}{2} \times \dfrac{1}{2} = 25$

5. Here $n = 8$,

p = Probability of getting 1 or 3 $= \dfrac{2}{6} = \dfrac{1}{3}$

$\therefore$ $q = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$\therefore$ S.D. $= \sqrt{npq} = \sqrt{8 \times \dfrac{1}{3} \times \dfrac{2}{3}} = \sqrt{\dfrac{16}{9}} = \dfrac{4}{3}$

6. Let X denote the number of failures in 5 trials.
Then, $P(X = r) = {}^5C_r\, (1 - p)^r\, p^{5-r}$; $r = 0,1,2,...., 5$

$\therefore$ $P(X \geq 1) \geq \dfrac{31}{32}$

$\Rightarrow 1 - P(X = 0) \geq \dfrac{31}{32}$

$\Rightarrow 1 - p^5 \geq \dfrac{31}{32}$

$\Rightarrow \dfrac{1}{32} \geq p^5$

$\Rightarrow p \leq \dfrac{1}{2} \Rightarrow p \in \left[0, \dfrac{1}{2}\right]$

7. Required probability

$= {}^3C_0\left(\dfrac{1}{2}\right)^3 \times {}^3C_0\left(\dfrac{1}{2}\right)^3 + {}^3C_1\left(\dfrac{1}{2}\right)^3 \times {}^3C_1\left(\dfrac{1}{2}\right)^3$

$+ {}^3C_2\left(\dfrac{1}{2}\right)^3 \times {}^3C_2\left(\dfrac{1}{2}\right)^3 + {}^3C_3\left(\dfrac{1}{2}\right)^3 \times {}^3C_3\left(\dfrac{1}{2}\right)^3$

$= (1 + 9 + 9 + 1) \times \dfrac{1}{8} \times \dfrac{1}{8} = \dfrac{5}{16}$

8. P(at least one success) $\geq \dfrac{9}{10}$

$\Rightarrow P(X \geq 1) \geq \dfrac{9}{10}$

$\Rightarrow 1 - P(X = 0) \geq \dfrac{9}{10}$

$\Rightarrow P(X = 0) \leq \dfrac{1}{10}$

$\Rightarrow {}^nC_0\left(\dfrac{1}{4}\right)^0\left(\dfrac{3}{4}\right)^n \leq \dfrac{1}{10}$

$\Rightarrow \left(\dfrac{3}{4}\right)^n \leq \dfrac{1}{10}$

$\Rightarrow \left(\dfrac{4}{3}\right)^n \geq 10$

$\Rightarrow n\log_{10}\left(\dfrac{4}{3}\right) \geq \log_{10} 10$

$\Rightarrow n(\log_{10} 4 - \log_{10} 3) \geq 1$

$\Rightarrow n \geq \dfrac{1}{\log_{10} 4 - \log_{10} 3}$

9. According to the given condition,
$np + npq = 15$ and $(np)^2 + (npq)^2 = 117$

$\therefore$ $\dfrac{n^2p^2(1+q^2)}{(np+npq)^2} = \dfrac{117}{15^2}$

$\Rightarrow \dfrac{1+q^2}{(1+q)^2} = \dfrac{117}{225}$

$\Rightarrow 6q^2 - 13q + 6 = 0 \Rightarrow q = \dfrac{2}{3}$

$\therefore$ $p = 1 - \dfrac{2}{3} = \dfrac{1}{3}$

Since, $np + npq = 15$

$\Rightarrow n \times \dfrac{1}{3} + n \times \dfrac{2}{9} = 15$

$\Rightarrow n = 27$

$\therefore$ mean $= np = 27 \times \dfrac{1}{3} = 9$

10. P(getting head) = p = $\frac{1}{2}$

$\therefore$ $q = 1 - \frac{1}{2} = \frac{1}{2}$

Here, $P(X = r) = {}^nC_r p^r q^{n-r} = {}^nC_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{n-r}$

$= {}^nC_r \left(\frac{1}{2}\right)^n$

Since, P(X= 4), P(X= 5) and P(X= 6) are in A.P.

$\Rightarrow 2P(X = 5) = P(X = 4) + P(X = 6)$

$\Rightarrow 2\ {}^nC_5 \left(\frac{1}{2}\right)^n = {}^nC_4 \left(\frac{1}{2}\right)^n + {}^nC_6 \left(\frac{1}{2}\right)^n$

$\Rightarrow 2\ {}^nC_5 = {}^nC_4 + {}^nC_6$

$\Rightarrow 2 \frac{n!}{5!(n-5)!} = \frac{n!}{4!(n-4)!} + \frac{n!}{6!(n-6)!}$

$\Rightarrow \frac{2}{5(n-5)} = \frac{1}{(n-4)(n-5)} + \frac{1}{6 \times 5}$

$\Rightarrow n^2 - 21n + 98 = 0$

$\Rightarrow (n - 7)(n - 14) = 0$

$\Rightarrow n = 7$ or 14

11. Let the probability of success and failure be p and q respectively.

$\therefore$ $p = 2q$

Since, $p + q = 1$

$\therefore$ $3q = 1 \Rightarrow q = \frac{1}{3}$

$\therefore$ $p = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore$ required probability

$= {}^6C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 + {}^6C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)$

$+ {}^6C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^0$

$= \frac{240}{729} + \frac{192}{729} + \frac{64}{729} = \frac{496}{729}$

12. Mean = np and variance = npq

$\therefore$ np = 20 and npq = 16

$\therefore$ $20q = 16 \Rightarrow q = \frac{4}{5}$

$\therefore$ $p = 1 - \frac{4}{5} = \frac{1}{5}$

Since, np = 20

$\therefore$ $n \times \frac{1}{5} = 20 \Rightarrow n = 100$

MHT-CET 2019

6th May 2019 (Afternoon)

Hints

1. $\cos\theta + \sec\theta = 2$

$\Rightarrow \cos\theta + \dfrac{1}{\cos\theta} = 2$

$\Rightarrow \cos^2\theta + 1 = 2\cos\theta$

$\Rightarrow (\cos\theta - 1)^2 = 0$

$\Rightarrow \cos\theta = 1$

$\therefore$ $\sec\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta = 1 - 1 = 0$

$\therefore$ $\sec^2\theta - \sin^2\theta = 1 - 0 = 1$

2. $A - B = \dfrac{\pi}{4}$

$\therefore$ $\tan(A - B) = \tan\dfrac{\pi}{4}$

$\Rightarrow \dfrac{\tan A - \tan B}{1 + \tan A \tan B} = 1$

$\Rightarrow \tan A - \tan B - \tan A \tan B = 1$

$\Rightarrow 1 + \tan A - \tan B - \tan A \tan B = 2$

$\Rightarrow (1 + \tan A)(1 - \tan B) = 2$

3. Line AB and line BC are perpendicular.

$\therefore$ Slope of AB × slope of BC = –1

$\Rightarrow \dfrac{4-8}{-3-5} \times \dfrac{k-4}{7-(-3)} = -1$

$\Rightarrow \dfrac{1}{2} \times \dfrac{k-4}{10} = -1$

$\Rightarrow k = -16$

4. Equation of parabola is $y^2 = 16x$.

$\therefore$ $a = 4$

Given, $y = 8$

Substituting $y = 8$ in $y^2 = 16x$, we get

$(8)^2 = 16x$

$\Rightarrow x = \dfrac{64}{16} = 4$

Focal distance $= |x + a|$

$= 4 + 4 = 8$

5. Conjugate hyperbola of $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1$

is $\dfrac{y^2}{25} - \dfrac{x^2}{144} = 1$

Here, $b = 5$, $a = 12$

$e = \dfrac{\sqrt{a^2 + b^2}}{b} = \dfrac{\sqrt{144 + 25}}{5} = \dfrac{13}{5}$

$\therefore$ the foci are $(0, \pm be)$ i.e., (0 ± 13)

6. Consider option (A),

$f(x) = 3\cos x + 4$

$\therefore$ $f(-x) = 3\cos(-x) + 4$

$= 3\cos x + 4 = f(x)$

$\therefore$ $f(x)$ is an even function.

7. $\sin x \geq 0$

$\Rightarrow x \in [2n\pi, (2n + 1)\pi]$ ...(i)

$16 - x^2 \geq 0$

$\Rightarrow x^2 \leq 16$

$\Rightarrow -4 \leq x \leq 4$ ...(ii)

From (i) and (ii), we get

$x \in [-4, -\pi] \cup [0, \pi]$

8. $(x + y)\left(\dfrac{1}{x} + \dfrac{1}{y}\right) = (x + y)\left(\dfrac{x + y}{xy}\right)$

$= \dfrac{(x + y)^2}{xy}$

$= \dfrac{x^2 + y^2 + 2xy}{xy}$

$= \dfrac{x}{y} + \dfrac{y}{x} + 2$

Since, A. M. ≥ G. M.

$\therefore$ $\dfrac{\dfrac{x}{y} + \dfrac{y}{x}}{2} \geq \sqrt{\dfrac{x}{y} \cdot \dfrac{y}{x}}$

$\Rightarrow \dfrac{x}{y} + \dfrac{y}{x} \geq 2$

$\therefore$ $(x + y)\left(\dfrac{1}{x} + \dfrac{1}{y}\right) \geq 2 + 2 \geq 4$

$\therefore$ the minimum value is 4.

9. $\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} \frac{1}{(n+1)!}$

$= \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...$

$= \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\right) - \left(1 + \frac{1}{1!}\right)$

$= e - 2$

10. No. of defective bulbs = 6

$\therefore$ No. of non-defective bulbs = 4

3 bulbs can be selected out of 10 light bulbs in $^{10}C_3$ ways.

$\therefore$ $n(S) = {^{10}C_3}$

Let A be the event that room is lit.

$\therefore$ A' is the event that the room is not lit.

For A′ the bulbs should be selected from the 6 defective bulbs. This can be done in $^6C_3$ ways.

$\therefore$ $n(A') = {^6C_3}$

$\therefore$ $P(A') \equiv \frac{n(A')}{n(S)} = \frac{^6C_3}{^{10}C_3}$

$\therefore$ P(Room is lit) = 1 – P(Room is not lit)

$\therefore$ $P(A) = 1 - P(A')$

$= 1 - \frac{^6C_3}{^{10}C_3}$

$= 1 - \frac{6 \times 5 \times 4}{10 \times 9 \times 8}$

$= 1 - \frac{1}{6} = \frac{5}{6}$

12. $p \equiv F$

$p \leftrightarrow r \equiv T$ and $p \leftrightarrow q \equiv F$

$\Rightarrow p \equiv F, r \equiv F$ and $p \equiv F, q \equiv T$

$\therefore$ the truth values of q and r are T and F respectively.

14. $A (\text{Adj } A) = |A| . (I_n)$

$\therefore$ $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix}$

$\Rightarrow |A| = 10$

15. Since, inverse of matrix A does not exist.

$\therefore$ $|A| = 0$

$\Rightarrow \begin{vmatrix} 1 & 2 & x \\ 4 & -1 & 7 \\ 2 & 4 & -6 \end{vmatrix} = 0$

$\Rightarrow 1(6 - 28) - 2(-24 - 14) + x(16 + 2) = 0$

$\Rightarrow -22 + 76 + 18x = 0$

$\Rightarrow 18x = -54$

$\Rightarrow x = -3$

16. $3 \sin^2 x - 7 \sin x + 2 = 0$

$\Rightarrow 3\sin^2 x - 6 \sin x - \sin x + 2 = 0$

$\Rightarrow 3\sin x (\sin x - 2) - (\sin x - 2) = 0$

$\Rightarrow (3 \sin x - 1) (\sin x - 2) = 0$

$\Rightarrow \sin x = \frac{1}{3}$ or 2

$\Rightarrow \sin x = \frac{1}{3}$ ....[$\because \sin x \neq 2$]

Let $\sin^{-1} \frac{1}{3} = \alpha$, $0 < \alpha < \frac{\pi}{2}$ are the solutions in $[0, 5\pi]$. Then, α, $\pi - \alpha$, $2\pi + \alpha$, $3\pi - \alpha$, $4\pi + \alpha$, $5\pi - \alpha$ are the solutions in $[0, 5\pi]$.

$\therefore$ number of solutions = 6

17. Let $\sin^{-1} x = \theta$

$\Rightarrow x = \sin \theta$

$2x\sqrt{1 - x^2} = 2 \sin \theta \cos \theta$

$= \sin 2\theta$

$\therefore$ $\sin^{-1}\left(2x\sqrt{1 - x^2}\right) = \sin^{-1} (\sin 2\theta)$

$= \sin^{-1} [\sin (\pi - 2\theta)]$

$...\left[\begin{array}{l} \because \frac{1}{\sqrt{2}} \le x \le 1 \Rightarrow \frac{\pi}{4} \le \theta \le \frac{\pi}{2} \\ \Rightarrow \frac{\pi}{2} \le 2\theta \le \pi \\ \Rightarrow 0 \le \pi - 2\theta \le \frac{\pi}{2} \end{array}\right]$

$= \pi - 2\theta$

$= \pi - 2 \sin^{-1} x$

$\therefore$ $A = \pi$, $B = -2$

18. $\cot x = -\sqrt{3}$

$\Rightarrow \tan x = \frac{-1}{\sqrt{3}}$

$\Rightarrow \tan x = \tan \frac{5\pi}{6}$

$\Rightarrow x = n\pi + \frac{5\pi}{6}$, $n \in Z$

...[$\because \tan \theta = \tan \alpha \Rightarrow \theta = n\pi + \alpha$, $n \in Z$]

19. By sine rule,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin A}$

$\frac{b}{\sin B} = \frac{c}{\sin A}$

$\Rightarrow \dfrac{2c}{\sin 3C} = \dfrac{c}{\sin C}$ ...[$\because$ b = 2c, B = 3C]

$\Rightarrow \dfrac{\sin 3C}{\sin C} = 2$

$\Rightarrow \dfrac{3\sin C - 4\sin^3 C}{\sin C} = 2$

$\Rightarrow 3 - 4\sin^2 C = 2$

$\Rightarrow \sin C = \dfrac{1}{2} \Rightarrow C = 30°$

$B = 3 \times 30° = 90°$

$\therefore$ A = 60° ...[Remaining angle of ΔABC]

$\therefore$ $\sin A = \sin 60° = \dfrac{\sqrt{3}}{2}$

20. Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we get
$a = \sin^2\theta - 1 = -\cos^2\theta$, $b = \cos^2\theta$
Here, $a + b = -\cos^2\theta + \cos^2\theta = 0$
$\therefore$ the lines are perpendicular.
$\therefore$ $\theta = \dfrac{\pi}{2}$

21. Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we get
$a = 2$, $2h = -3$, $b = 1$

$m_1 + m_2 = \dfrac{-2h}{b} = 3$, $m_1.m_2 = \dfrac{a}{b} = 2$

$(m_1)^3 + (m_2)^3 = (m_1 + m_2)\left(m_1^2 - m_1m_2 + m_2^2\right)$
$= 3\,[(m_1 + m_2)^2 - 3m_1 m_2]$
$= 3\,[(3)^2 - 3(2)]$
$= 3(3) = 9$

22. Since, G is the centroid of ΔABC.
$\therefore$ $2 = \dfrac{3 + y + 2x}{3}$ and $1 = \dfrac{x - 2 + 2y}{3}$
$\Rightarrow 6 = 3 + y + 2x$ and $3 = x - 2 + 2y$
$\Rightarrow 2x + y = 3$ and $x + 2y = 5$
Solving these equations, we get
$x = \dfrac{1}{3}$ and $y = \dfrac{7}{3}$

23. $\bar{a}, \bar{b}, \bar{c}$ are coplanar vectors.
$\Rightarrow [\bar{a}\ \bar{b}\ \bar{c}] = 0$
Let $\bar{\alpha} = 2\bar{a} - \bar{b}$, $\bar{\beta} = 2\bar{b} - \bar{c}$ and $\gamma = 2\bar{c} - \bar{a}$.
Then,

$$[\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}] = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{vmatrix} [\bar{a}\ \bar{b}\ \bar{c}]$$

$\Rightarrow [\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}] = 7[\bar{a}\ \bar{b}\ \bar{c}] = 7(0) = 0$

24. Since, the given vectors are coplanar.

$\therefore$ $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$

$\Rightarrow -ac - a(b - c) + c^2 = 0$
$\Rightarrow -ac - ab + ac + c^2 = 0$
$\Rightarrow c^2 = ab$
$\Rightarrow$ c is the G. M. of a and b.

26. Putting $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in the given equation,
$x\hat{i} + y\hat{j} + z\hat{k} = \left(\hat{i} + \hat{j} + \hat{k}\right) + \lambda\left(\hat{j} + \hat{k}\right)$
$\therefore$ $x\hat{i} + y\hat{j} + z\hat{k} = \hat{i} + (1 + \lambda)\hat{j} + (1 + \lambda)\hat{k}$
$\therefore$ $x = 1, y = 1 + \lambda, z = 1 + \lambda$
$\therefore$ $x = 1, y = z$

27. Let the XZ plane divides the line segment joining the given points in the ratio k : 1 at the point P (x, y, z).

$\therefore$ $x = \dfrac{ka + 3}{k + 1}$, $y = \dfrac{-4k + 2}{k + 1}$

$z = \dfrac{3k + b}{k + 1}$

Since, P (x, y, z) lie on the XZ plane, its y co-ordinate will be zero.

$\therefore$ $0 = \dfrac{-4k + 2}{k + 1}$

$\Rightarrow -4k + 2 = 0$

$\Rightarrow k = \dfrac{1}{2}$

$\therefore$ k : 1 = 1 : 2

28. The vector equation of the plane passing through the point $A\left(\bar{a}\right)$ and parallel to the non-zero vectors $\bar{b}$ and $\bar{c}$ is
$\bar{r} \cdot \left(\bar{b} \times \bar{c}\right) = \bar{a} \cdot \left(\bar{b} \times \bar{c}\right)$

Here, $\bar{a} = -\hat{i} + 2\hat{j} - 5\hat{k}$, $\bar{b} = 4\hat{i} - \hat{j} + 3\hat{k}$,
$\bar{c} = \hat{i} + \hat{j} - \hat{k}$

$\therefore$ $\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ 1 & 1 & -1 \end{vmatrix}$

$= -2\hat{i} + 7\hat{j} + 5\hat{k}$

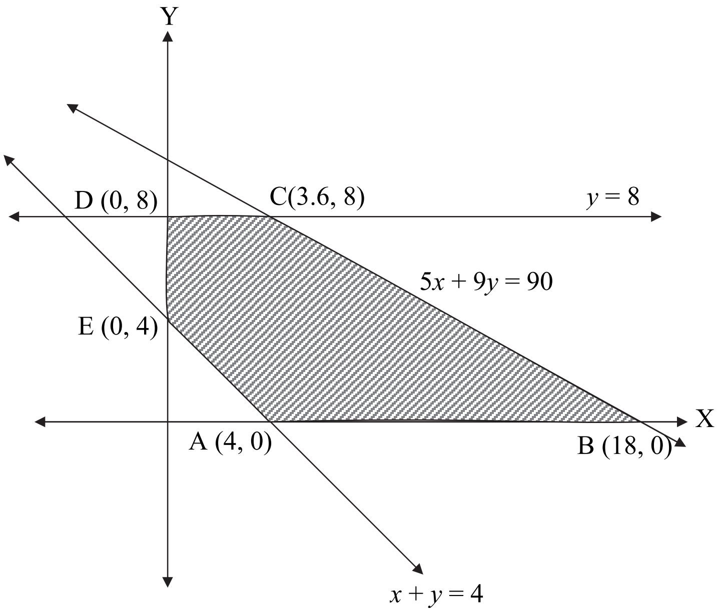
$\therefore$ $\bar{a}\cdot(\bar{b}\times\bar{c}) = (-\hat{i} + 2\hat{j} - 5\hat{k})\cdot(-2\hat{i} + 7\hat{j} + 5\hat{k})$

$= (-1)(-2) + 2(7) + (-5)(5)$

$= -9$

$\therefore$ The vector equation of the plane is $\bar{r}\cdot(-2\hat{i} + 7\hat{j} + 5\hat{k}) = -9$

29.



The corner points of the feasible region A(4, 0), B (18, 0), C (3.6, 8), D(0, 8) and E(0, 4).

At A (4, 0), z = 24

At B (18, 0), z = 108

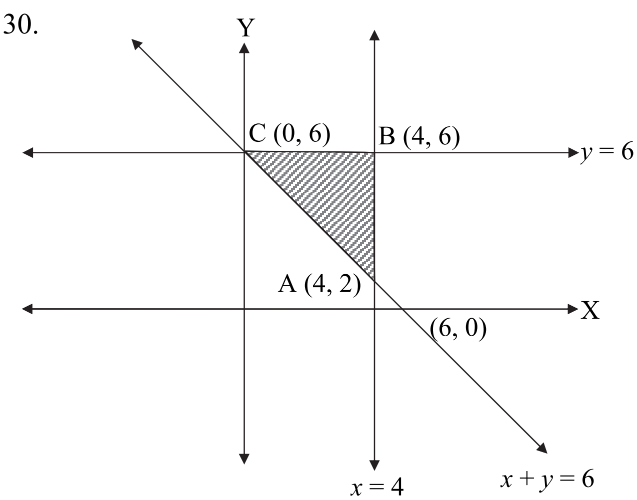
At C (3.6, 8), z = 37.6

At D (0, 8), z = 16

At E (0, 4), z = 8

$\therefore$ Minimum value of Z occurs at (0, 4).

30.



The corner points of the feasible region A(4, 2), B (4, 6) and C (0, 6).

At A (4, 2), z = 10

At B (4, 6), z = 14

At C (0, 6), z = 6

$\therefore$ Maximum value of Z is 14.

31. Since, f(x) is continuous at $x = 0$.

$\therefore$ $f(0) = \lim\limits_{x\to 0} f(x)$

$= \lim\limits_{x\to 0} (x+1)^{\cot x}$

$= \lim\limits_{x\to 0} \left[(1+x)^{\frac{1}{x}}\right]^{\frac{x}{\tan x}}$

$= e^1 = e$

32. Since, f(x) is continuous at $x = a$.

$\therefore$ $f(a) = \lim\limits_{x\to a} f(x)$

$$= \lim_{x\to a} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}}$$

$$= \lim_{x\to a} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x+a}.\sqrt{x-a}}$$

$$= \lim_{x\to a} \frac{1}{\sqrt{x+a}} \left(\frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x-a}}\right)$$

$$= \frac{1}{\sqrt{2a}} \lim_{x\to a} \left(\frac{\sqrt{x} - \sqrt{a}}{\sqrt{x-a}} + 1\right)$$

$$= \frac{1}{\sqrt{2a}} \lim_{x\to a} \left[\frac{(\sqrt{x})^2 - (\sqrt{a})^2}{\sqrt{x-a}(\sqrt{x}+\sqrt{a})} + 1\right]$$

$$= \frac{1}{\sqrt{2a}} \lim_{x\to a} \left(\frac{x-a}{\sqrt{x-a}(\sqrt{x}+\sqrt{a})} + 1\right)$$

$$= \frac{1}{\sqrt{2a}} \lim_{x\to a} \left[\frac{\sqrt{x-a}}{\sqrt{x}+\sqrt{a}} + 1\right]$$

$$= \frac{1}{\sqrt{2a}} (0+1) = \frac{1}{\sqrt{2a}}$$

33. $f'(2^-) = \lim\limits_{h\to 0} \dfrac{f(2+h) - f(2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{1 + 2 + h - 3}{h}$

$= \lim\limits_{h\to 0} \dfrac{h}{h} = 1$

$$f'(2^+) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$
$$= \lim_{h \to 0} \frac{5 - (2+h) - 3}{h}$$
$$= \lim_{h \to 0} \left(\frac{-h}{h}\right) = -1$$

$\therefore$ $f'(2^-) \neq f'(2^+)$

$\therefore$ $f'(2)$ does not exist.

34. $y = \operatorname{cosec}^{-1}\left(\frac{x^2+1}{x^2-1}\right) + \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

$$= \sin^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$
$$\ldots\left[\because \operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)\right]$$
$$= \frac{\pi}{2} \qquad \ldots\left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

$\therefore$ $\frac{dy}{dx} = 0$

35. $x = t \log t$ and $y = t^t$

$\therefore$ $x = \log t^t = \log y$

Differentiating both sides w.r.t. x, we get

$1 = \frac{1}{y} \cdot \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = y = t^t$

Since, $x = t \log t$

$\therefore$ $x = \log t^t$

$\Rightarrow e^x = t^t$

$\therefore$ $\frac{dy}{dx} = e^x$

36. $f(x) = x^3 - 3x$

$\therefore$ $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$

$f'(x) = 0$

$\Rightarrow 3x^2 - 3 = 0$

$\Rightarrow 3(x^2 - 1) = 0$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

For $x = 1$,

$f''(1) = 6 > 0$

For $x = -1$

$f''(-1) = -6 < 0$

$\therefore$ $f(x)$ attains minimum value at $x = 1$.

$\therefore$ $a = 1$

37. $f(x) = x^3 + bx^2 + ax - 6$

$f(1) = 1 + b + a - 6 = a + b - 5$

$f(3) = 27 + 9b + 3a - 6 = 3a + 9b + 21$

$f(1) = f(3)$

$\therefore$ $a + b - 5 = 3a + 9b + 21$

$\Rightarrow 2a + 8b = -26$

$\Rightarrow a + 4b = -13$

38. $y = e^x + e^{-x}$ ...(i)

$\therefore$ $\frac{dy}{dx} = e^x - e^{-x}$

The slope of the horizontal tangent is $\frac{dy}{dx} = 0$

$\therefore$ $0 = e^x - e^{-x}$

$\Rightarrow e^x = e^{-x}$

$\Rightarrow e^{2x} = 1$

$\Rightarrow x = 0$

Substituting $x = 0$ in (i), we get

$y = e^0 + e^0 = 2$

39. Let $I = \int \frac{1}{\sqrt{9 - 16x^2}} dx$

$$= \int \frac{1}{\sqrt{(3)^2 - (4x)^2}} dx$$
$$= \frac{1}{4} \sin^{-1}\left(\frac{4x}{3}\right) + c$$

Comparing with $a \sin^{-1}(bx) + c$, we get

$a = \frac{1}{4}$ and $b = \frac{4}{3}$

$\therefore$ $4a + 3b = 4\left(\frac{1}{4}\right) + 3\left(\frac{4}{3}\right) = 5$

40. Let $I = \int \frac{1}{3 + 2\cos^2 x} dx$

Dividing Nr and Dr by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x \, dx}{3\sec^2 x + 2}$$
$$= \int \frac{\sec^2 x \, dx}{3(1 + \tan^2 x) + 2}$$
$$= \int \frac{\sec^2 x}{3 + 3\tan^2 x + 2} dx$$
$$= \int \frac{\sec^2 x}{5 + 3\tan^2 x} dx$$

Put $\tan x = t$

$\therefore$ $\sec^2 x\, dx = dt$

$\therefore$ $I = \int \frac{1}{5+3t^2} dt$

$= \int \frac{1}{\left(\sqrt{5}\right)^2 + \left(\sqrt{3}\,t\right)^2} dt$

$= \frac{1}{\sqrt{3}\sqrt{5}} \tan^{-1}\left(\frac{\sqrt{3}\,t}{\sqrt{5}}\right) + c$

$\therefore$ $I = \frac{1}{\sqrt{15}} \tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{5}}\right) + c$

41. $\int x^2\ e^{3x}\, dx = x^2 \cdot \frac{e^{3x}}{3} - \int 2x \cdot \frac{e^{3x}}{3}\ dx$

$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left[x.\frac{e^{3x}}{3} - \int 1.\frac{e^{3x}}{3} dx\right]$

$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left(\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right) + c$

$= \frac{1}{3}\ x^2\ e^{3x} - \frac{2}{9}\ xe^{3x} + \frac{2}{27}\ e^{3x} + c$

$= \frac{e^{3x}}{27}\ (9x^2 - 6x + 2) + c$

$\therefore$ $f(x) = 9x^2 - 6x + 2$

42. Let $I = \int_0^1 \frac{1}{x + \sqrt{x}}\ dx$

$= \int_0^1 \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)} dx$

Put $t = \sqrt{x}\ + 1 \Rightarrow dt = \frac{1}{2\sqrt{x}}\ dx$

When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

$\therefore$ $I = 2\int_1^2 \frac{dt}{t}$

$= 2\left[\log t\right]_1^2$

$= 2\ (\log 2 - \log 1) = 2 \log 2 = \log 2^2 = \log 4$

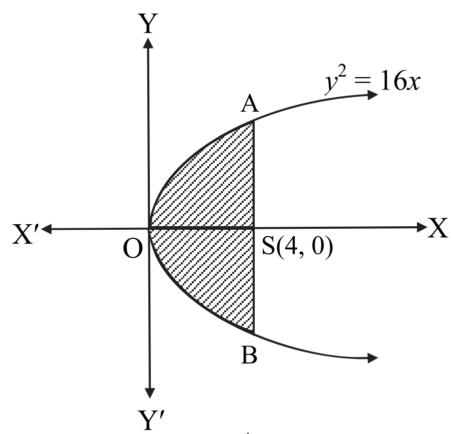
43. Let $f(x) = \log\left(\frac{2-\sin x}{2+\sin x}\right)$

$\therefore$ $f(-x) = \log\left(\frac{2-\sin(-x)}{2+\sin(-x)}\right)$

$= -\log\left(\frac{2-\sin x}{2+\sin x}\right) = -f(x)$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ $\int_{\frac{-\pi}{8}}^{\frac{\pi}{8}} \log\left(\frac{2-\sin x}{2+\sin x}\right) dx = 0$

44. 

Required area $= 2\int_0^4 4\sqrt{x}\ dx$

$= 8\int_0^4 \sqrt{x}\ dx$

$= 8\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$

$= \frac{16}{3}\left[4^{\frac{3}{2}} - 0\right]$

$= \frac{16}{3} \times 8$

$= \frac{128}{3}$ sq. units

45. $\frac{dy}{dx} = \frac{y+1}{x+1}$

$\therefore$ $\frac{dy}{y+1} = \frac{dx}{x+1}$

Integrating on both sides, we get

$\log (y + 1) = \log (x + 1) + \log c$

$\Rightarrow \log (y + 1) = \log (x + 1).c$

$\Rightarrow y + 1 = c(x + 1)$ ...(i)

Since, $y(1) = 2$ i.e., $y = 2$ when $x = 1$

$\therefore$ $3 = c(2) \Rightarrow c = \frac{3}{2}$

$\therefore$ $y + 1 = \frac{3}{2}\ (x + 1)$ ...[From (i)]

$\Rightarrow 3x - 2y + 1 = 0$

46. Let P be the number of bacteria present at time t.

$$\frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = k\,dt$$

Integrating on both sides, we get

$\log P = kt + c$

When $t = 0$, $P = P_0$

$\therefore$ $\log P_0 = 0 + c \Rightarrow c = \log P_0$

$\therefore$ $\log P = kt + \log P_0$

$$\Rightarrow \log \frac{P}{P_0} = kt \qquad ....(i)$$

When $t = 5$ hrs, $P = 3P_0$

$\therefore$ $\log \dfrac{3P_0}{P_0} = 5k$

$$\Rightarrow k = \frac{\log 3}{5}$$

$\therefore$ $\log \dfrac{P}{P_0} = \dfrac{\log 3}{5} t \qquad ....[\text{From (i)}]$

When $t = 10$ hrs, we have

$$\log \frac{P}{P_0} = \frac{\log 3}{5} \times 10 = 2 \log 3 = \log 9$$

$\therefore$ $P = 9P_0 = 9$ times the original

47. Equation of all rectangular hyperbolas is $xy = c^2$

Differentiating w.r.t.x, we get

$$x\frac{dy}{dx} + y = 0$$

Here, the order is 1.

48. Since, $f(x)$ is the p.d.f. of X.

$\therefore$ $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + \int_{0}^{2} kx\,dx + 0 = 1$$

$$\Rightarrow k\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$\Rightarrow k(2-0) = 1 \qquad \Rightarrow k = \frac{1}{2}$$

49. Since, $\displaystyle\sum_{x=1}^{5} P(X = x) = 1$

$\therefore$ $\dfrac{1}{20} + \dfrac{3}{20} + k + 2k + \dfrac{1}{20} = 1$

$$\Rightarrow 3k + \frac{5}{20} = 1$$

$$\Rightarrow 3k = 1 - \frac{5}{20}$$

$$\Rightarrow 3k = \frac{15}{20}$$

$$\Rightarrow k = \frac{1}{4}$$

50. P(success) = p, P(failure) = q

According to the given condition,

$$p = \frac{3}{4} q$$

$\therefore$ $1 - q = \dfrac{3}{4} q \qquad ...[\because p + q = 1]$

$$\Rightarrow q = \frac{4}{7}$$

P(at least one success) $= P(X \geq 1)$

$= 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - {}^5C_0\, p^0 q^5$

$= 1 - q^5$

$= 1 - \left(\dfrac{4}{7}\right)^5$