1.1 Continuous and Discrete Signals and Systems

A continuous signal is a mathematical function of an independent variable $t \in \Re$, where $\Re$ represents a set of real numbers. It is required that signals are *uniquely* defined in t except for a finite number of points. For example, the function $f(t) = \sqrt{t}$ does not qualify for a signal even for $t > 0$ since the square root of t has two values for any non negative t. A continuous signal is represented in Figure 1.1. Very often, especially in the study of dynamic systems, the independent variable t represents time. In such cases $f(t)$ is a time function.

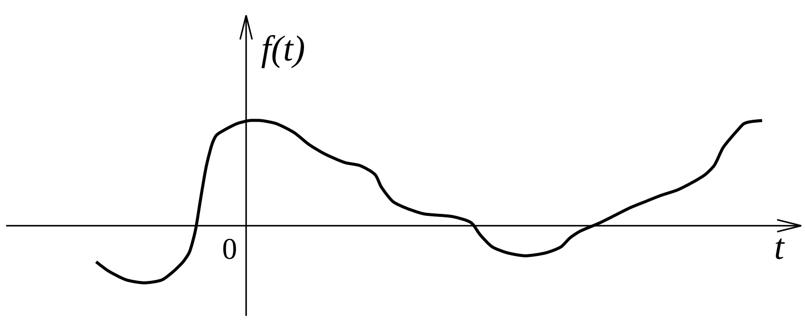


Figure 1.1: A continuous signal

Note that signals are real mathematical functions, but some transforms applied on signals can produce complex signals that have both real and imaginary parts. For example, in analysis of alternating current electrical circuits we use phasors, rotating vectors in the complex plane, $I(j\omega) = |I(j\omega)|\measuredangle I(j\omega)$, where ω represents the angular frequency of rotation, $\measuredangle I(j\omega)$ denotes phase, and $|I(j\omega)|$ is the amplitude of the alternating current. The complex plane representation is useful to simplify circuit analysis, however, the above defined complex signal represents in fact a real sinusoidal signal, oscillating with the corresponding amplitude, frequency, and phase, represented by $|I(j\omega)|\sin(\omega t + \measuredangle I(j\omega))$. Complex signal representation of real signals will be encountered in this textbook in many application examples. In addition, in several chapters on signal transforms (Fourier, Laplace, $\mathcal{Z}$-transform) we will present complex domain equivalents of real signals.

A discrete signal is a uniquely defined mathematical function (single-valued function) of an independent variable $k \in Z$, where Z denotes a set of integers. Such a signal is represented in Figure 1.2. In order to clearly distinguish between continuous and discrete signals, we will use in this book parentheses for arguments of continuous signals and square brackets for arguments of discrete signals, as demonstrated in Figures 1.1 and 1.2. If k represents discrete time (counted in the number of seconds, minutes, hours, days, ...) then $g[k]$ defines a discrete-time signal.

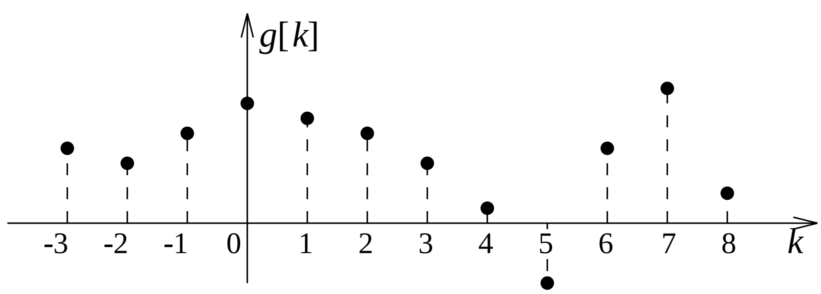


Figure 1.2: A discrete signal

Sampling

Continuous and discrete signals can be related through the sampling operation in the sense that a discrete signal can be obtained by performing sampling on a continuous-time signal with the uniform sampling period T as presented in Figure 1.3. Since T is a given quantity, we will use $f(kT) \triangleq f[k]$ in order to simplify notation.

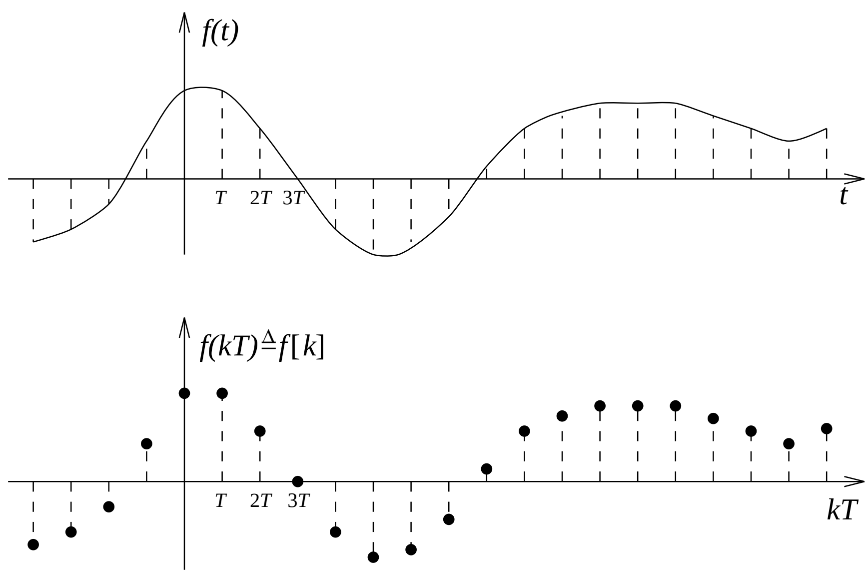


Figure 1.3: Sampling of a continuous signal

More about sampling will be said in Chapter 9.

Continuous- and discrete-time, *linear, time invariant, dynamic systems* are described, respectively, by *linear* differential and difference equations with *constant coefficients*. Mathematical models of such systems that have one input and one output are defined by

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

and

$$y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_1 y[k+1] + a_0 y[k] = f[k]$$

where n is the order of the system, $y(t)$ is the *system output* and $f(t)$ is the external forcing function representing the *system input*. In this textbook, we study only *time invariant* continuous and discrete linear systems for which the *coefficients* $a_i, i = 0, 1, ..., n-1$, *are constants*.

Linear time varying systems, whose coefficients vary in time are difficult for analysis, and they are studied in a graduate course on linear systems.

Initial Conditions

In addition to the *external forcing function,* the system is also driven by its *internal forces* coming from the *system initial conditions* (accumulated system energy at the given initial time). It is well known from elementary differential equations that in order to be able to find the solution of a differential equation of order n, the set of n initial conditions must be specified as

$$y(t_0),\ \frac{dy(t_0)}{dt},\ \ldots,\ \frac{d^{n-1}y(t_0)}{dt^{n-1}}$$

where t_0 denotes the initial time.

In the discrete-time domain, for a difference equation of order n, the set of n initial conditions must be specified. For the difference equation, the initial conditions are given by

$$y[k_0],\ y[k_0+1],\ \ldots,\ y[k_0+n-1]$$

It is interesting to point out that *in the discrete-time domain the initial conditions carry information about the evolution of the system output* in time, from some initial time k_0 to k_0+n-1. Those values are the system output past values, and they have to be used to determine the system output current value, that is, $y[k_0+n]$. In contrast, *for continuous-time systems all initial conditions are defined at the initial time* t_0.

System Response

The main goal in the analysis of dynamic systems is to find the system response (system output) due to external (system inputs) and internal (system initial conditions) forces. It is known from elementary theory of differential equations that the solution of a linear differential equation has two additive components: the *homogenous and particular solutions*. The homogenous solution is contributed by the initial conditions and the particular solution comes from the forcing function. In engineering, the homogenous solution is also called the *system natural response*, and the particular solution is called the *system forced response*. Hence, we have

$$y(t) = y_h(t) + y_p(t)$$

$$y[k] = y_h[k] + y_p[k]$$

Homogeneous and particular solutions of differential equations correspond, respectively (not identically, in general, see Example 1.1), to the so-called zero-input and zero-state responses of dynamic systems.

Definition 1.1: The continuous-time (discrete-time) linear system response solely contributed by the system initial conditions is called the *system zero-input (forcing function is set to zero) response*. It is denoted by y_{zi}.

Definition 1.2: The continuous-time (discrete-time) linear system response solely contributed by the system forcing function is called the *system zero-state response (system initial conditions are set to zero)*. It is denoted by y_{zs}.

In view of Definitions 1.1 and 1.2, it also follows that the linear system response has two components: one component contributed by the system initial conditions, y_{zi}, and another component contributed by the system forcing function (input), y_{zs}, that is, the following holds for continuous-time linear systems

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

and for discrete-time linear systems, we have

$$y[k] = y_{zi}[k] + y_{zs}[k]$$

Sometimes, in the linear system literature, the zero-state response is superficially called the system *steady state response*, and the zero-input response is called the system *transient response*. More precisely, the *transient response represents the system response in the time interval immediately after the initial time, say from*

$t_0 = 0$ *to* t_1, *contributed by both the system input and the system initial conditions.* The *system steady state response stands for the system response in the long run after some* $t > t_1$. This distinction between the transient and steady state responses is demonstrated in Figure 1.4. The component of the system transient response, contributed by the system initial conditions, in most cases decays quickly to zero. Hence, after a certain time interval, the system response is most likely determined by the forcing function only. Note that the steady state is not necessarily constant in time, as demonstrated in Example 1.2.

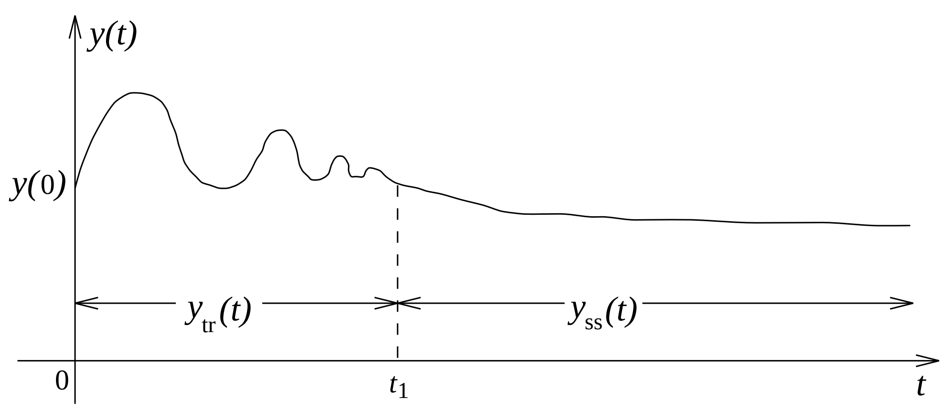


Figure 1.4: Transient and steady state responses

Example 1.2: Consider the same system as in Example 1.1 with the same initial conditions, but take the forcing function as $f(t) = \sin(t)$, that is

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \sin(t), \quad t \geq 0, \quad y(0) = 1, \quad \frac{dy(0)}{dt} = 1$$

The solution is derived in the textbook as

$$y(t) = \frac{9}{4}e^{-t} - \frac{21}{20}e^{-3t} + \frac{1}{10}\sin(t) - \frac{2}{10}\cos(t), \quad t \geq 0$$

It is easy to see that the system response exponential functions decay to zero pretty rapidly so that the system steady state response is determined by

$$y_{ss}(t) \approx \frac{1}{10}\sin(t) - \frac{2}{10}\cos(t), \quad t \geq t_1$$

The plots of $y(t)$ and $y_{ss}(t)$ are given in Figure 1.5.

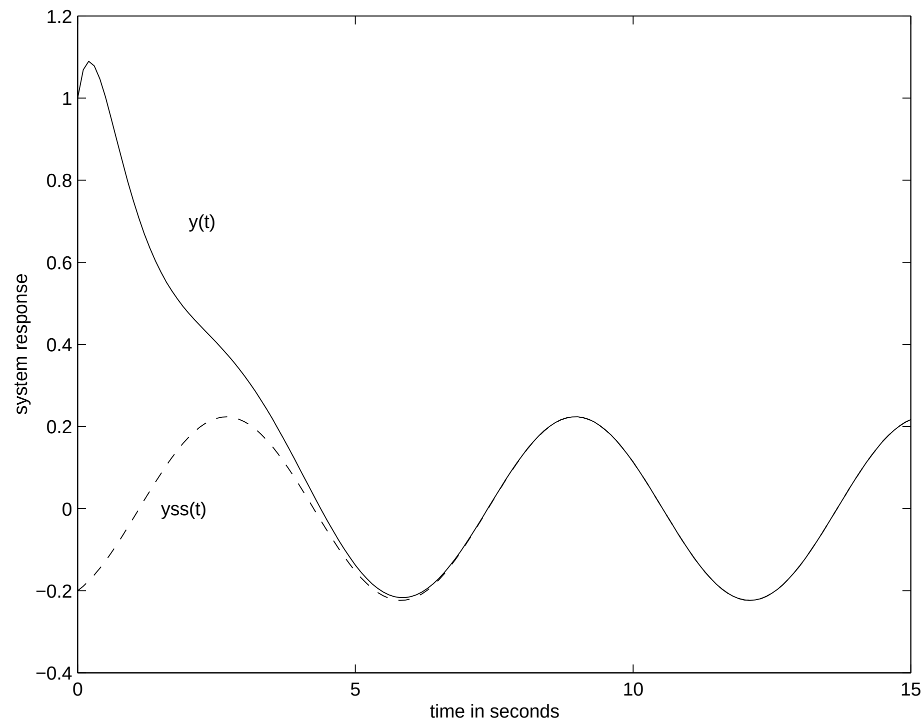


Figure 1.5: System complete response (solid line) and its steady state response (dashed line) for Example 1.2

It can be seen from the above figure that the transient ends roughly at $t_1 = 6$ s, hence after that time the system is in its steady state.

Linear dynamic systems process input signals in order to produce output signals. The processing rule is given in the form of differential/difference equations. Sometimes, linear dynamic systems are called linear signal processors. A block diagram representation of a linear system, processing one input and producing one output, is given in Figure 1.6.

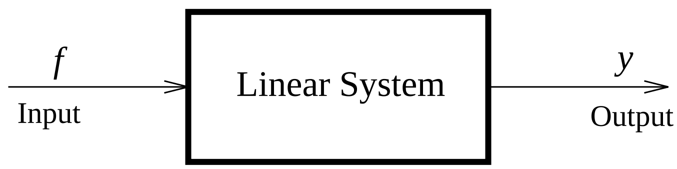


Figure 1.6: Input–output block diagram of a system

In general, the *system input signal can be differentiated by the system* so that the more general description of time invariant linear continuous-time systems is

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m\frac{d^m f(t)}{dt^m} + b_{m-1}\frac{d^{m-1}f(t)}{dt^{m-1}} + \cdots + b_1\frac{df(t)}{dt} + b_0 f(t)$$

This system differentiation of input signals leads to some interesting system properties (to be discussed in Chapters 3 and 4). The corresponding general form of time invariant linear discrete-time systems is

$$y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_1 y[k+1] + a_0 y[k]$$
$$= b_m f[k+m] + b_{m-1}f[k+m-1] + \cdots + b_1 f[k+1] + b_0 f[k]$$

The coefficients $a_i, i = 0, 1, 2, ..., n-1,$ and $b_j, j = 0, 1, ..., m,$ are constants. Note that for real physical systems $n \geq m$.

Due to the presence of the derivatives of the input signal on the right-hand side of the general system equation, impulses that instantly change system initial conditions are generated at the initial time. These impulses are called the impulse delta functions (signals). The impulse delta signal and its role in the derivative operation will be studied in detail in Chapter 2. We will learn in this textbook a method for solving the considered differential equations based on the Laplace transform. The Laplace transform will be presented in Chapter 4. Another method for solving the general system differential equation requires using $y(t) = y_h(t) + y_p(t)$ with *the particular solution being obtained through the convolution operation*. The convolution operation will be introduced in Chapter 2 and used in Chapters 3 and 4 for analysis of linear time invariant systems. The convolution operation will be studied in detail in Chapter 6 and its use in linear system theory will be fully demonstrated in Sections 7.1 and 8.2.

The problem of finding the system response for the given input signal $f(t)$ *or* $f[k]$ *is the central problem in the analysis of linear systems.* It is basically the problem of solving the corresponding linear differential or difference equation. This problem can be solved either by using knowledge from the mathematical theory of linear differential and/or difference equations or the engineering frequency domain approach—based on the concept of the system transfer function, which leads to the conclusion that linear systems can be studied either in the time domain (to be generalized in Chapter 8 to the state space approach) or in the frequency domain (transfer function approach). Chapters 3–5 of this textbook will be dedicated to the frequency domain techniques and Chapters 6–8 will study time domain techniques for the analysis of continuous- and discrete-time linear time-invariant systems.

The system considered so far and symbolically represented in Figure 1.6, has only one input f and one output y. Such systems are known as *single-input single-output systems*. In general, systems may have several inputs and several outputs, say r inputs $f_1, f_2, ..., f_r$, and p outputs, $y_1, y_2, ..., y_p$. These systems are known as *multi-input multi-output systems*. They are also called *multivariable systems*. A block diagram for a multi-input multi-output system is represented in Figure 1.7.

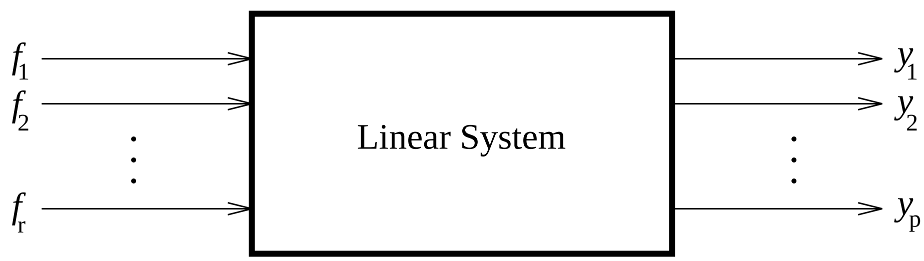


Figure 1.7: Block diagram of a multi-input multi-output system

The problem of obtaining differential (difference) equations that describe dynamics of real physical systems is known as *mathematical modeling*. In Section 1.3 mathematical models for several real physical systems will be derived.

1.2 System Linearity and Time Invariance

In Section 1.1, the concept of system linearity is tacitly introduced by stating that linear dynamic systems are described by linear differential/difference equations. We have also stated that the concept of time invariance is related to differential/difference equations with constant coefficients. In this section we discuss the concepts of system linearity and time invariance in more details.

1.2.1 System Linearity

The concept of system linearity is presented for continuous-time systems. Similar derivations and explanations are valid for presentation of the linearity concept of discrete-time linear dynamic systems. Before we derive and state the linearity property of continuous-time linear dynamic systems, we need the following definition.

Definition 1.3: The *system at rest* is the system that has no initial internal energy, that is, all its initial conditions are equal to zero.

It follows from Definition 1.3 that for a system at rest, the initial conditions are set to zero, that is

$$y(t_0) = 0,\ \frac{dy(t_0)}{dt} = 0,\ \ldots,\ \frac{d^{n-1}y(t_0)}{dt^{n-1}} = 0$$

Systems at rest are also called systems with zero initial conditions.

The linearity property of continuous-time linear dynamic systems is the consequence of the linearity property of mathematical derivatives, that is

$$\frac{d^i}{dt^i}(y_1(t)) + \frac{d^i}{dt^i}(y_2(t)) = \frac{d^i}{dt^i}(y_1(t) + y_2(t)), \quad i = 1, 2, \ldots$$

Consider the general nth order continuous-time linear differential equation that describes the behavior of an nth order linear dynamic system. Assume that the *system is at rest*, and that it is driven either by $f_1(t)$ or $f_2(t)$, which respectively produce the system outputs $y_1(t)$ and $y_2(t)$, that is

$$\frac{d^n y_1(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y_1(t)}{dt^{n-1}} + \cdots + a_1\frac{dy_1(t)}{dt} + a_0 y_1(t)$$
$$= b_m\frac{d^m f_1(t)}{dt^m} + b_{m-1}\frac{d^{m-1} f_1(t)}{dt^{m-1}} + \cdots + b_1\frac{df_1(t)}{dt} + b_0 f_1(t)$$

and

$$\frac{d^n y_2(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y_2(t)}{dt^{n-1}} + \cdots + a_1\frac{dy_2(t)}{dt} + a_0 y_2(t)$$
$$= b_m\frac{d^m f_2(t)}{dt^m} + b_{m-1}\frac{d^{m-1} f_2(t)}{dt^{m-1}} + \cdots + b_1\frac{df_2(t)}{dt} + b_0 f_2(t)$$

Assume now that the *same system at rest* is driven by a linear combination $\alpha f_1(t) + \beta f_2(t)$ where α and β are known constants. Multiplying the first equation by α and multiplying the second equation by β and adding the two differential equations, we obtain the following differential equation

$$\frac{d^n(\alpha y_1 + \beta y_2)}{dt^n} + a_{n-1}\frac{d^{n-1}(\alpha y_1 + \beta y_2)}{dt^{n-1}} + \cdots + a_0(\alpha y_1 + \beta y_2)$$
$$= b_m \frac{d^m(\alpha f_1 + \beta f_2)}{dt^m} + b_{m-1}\frac{d^{m-1}(\alpha f_1 + \beta f_2)}{dt^{m-1}}$$
$$+ \cdots + b_0(\alpha f_1 + \beta f_2)$$

It is easy to conclude that the output of the system at rest (the solution of the corresponding differential equation) due to a linear combination of system inputs $\alpha f_1(t) + \beta f_2(t)$ is equal to the corresponding linear combination of the system of outputs, that is $\alpha y_1(t) + \beta y_2(t)$. This is basically the *linearity principle*. Note that the linearity principle is valid under the assumption that the system initial conditions are zero (system at rest). The linearity principle is, in fact, the *superposition principle*, the well known principle of elementary circuit theory.

The linearity principle can be put in a formal mathematical framework as follows. If we introduce the symbolic notation, the solutions of the considered equations can be recorded as

$$y_1(t) = \mathbf{L}\{f_1(t)\}, \quad y_2(t) = \mathbf{L}\{f_2(t)\}$$

where $\mathbf{L}$ stands for a linear integral type operator. In order to get a solution of an nth order differential equation, the corresponding differential equation has to be integrated n-times. That is why, linear dynamic systems can be modelled as integrators. Note that the considered differential equations can be multiplied by some constants, say α and β producing

$$\alpha y_1(t) = \mathbf{L}\{\alpha f_1(t)\} = \alpha \mathbf{L}\{f_1(t)\}$$

$$\beta y_2(t) = \mathbf{L}\{\beta f_2(t)\} = \beta \mathbf{L}\{f_2(t)\}$$

Adding these equations leads to the conclusion that

$$\alpha y_1(t) + \beta y_2(t) = \mathbf{L}(\alpha f_1(t)) + \mathbf{L}(\beta f_2(t))$$

It follows that the linearity principle can be mathematically stated as follows

$$\mathbf{L}\{\alpha f_1(t) + \beta f_2(t)\} = \alpha \mathbf{L}\{f_1(t)\} + \beta \mathbf{L}\{f_2(t)\}$$

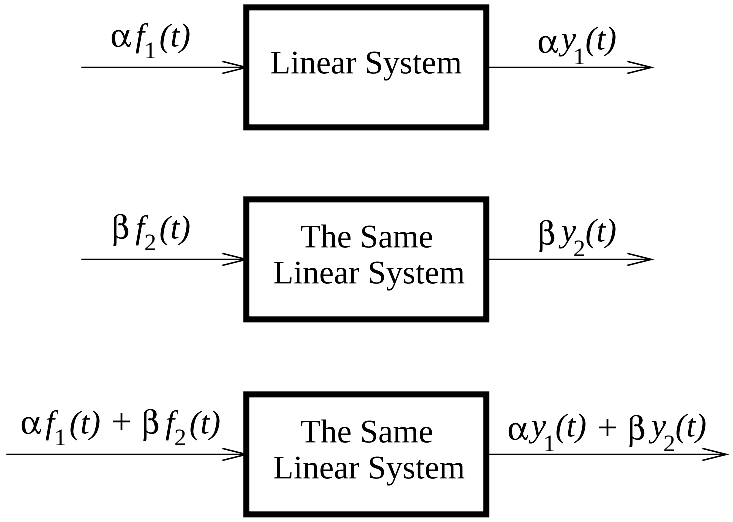
Using a similar reasoning, we can state the linearity principle for an arbitrary number of inputs, that is

$$\mathbf{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t) + \cdots + \alpha_N f_N(t)\}$$

$$= \alpha_1 \mathbf{L}\{f_1(t)\} + \alpha_2 \mathbf{L}\{f_2(t)\} + \cdots + \alpha_N \mathbf{L}\{f_N(t)\}$$

where $\alpha_i, i = 1, 2, ..., N,$ are constants.

The linearity principle is demonstrated graphically in Figure 1.8.



1.8: Graphical representation of the linearity principle for a system at rest

It is straightforward to verify, by using similar arguments that the linear difference equation also obeys to the linearity principle, that is

$$\mathbf{L}\{\alpha_1 f_1[k] + \alpha_2 f_2[k] + \cdots + \alpha_N f_N[k]\}$$

$$= \alpha_1 \mathbf{L}\{f_1[k]\} + \alpha_2 \mathbf{L}\{f_2[k]\} + \cdots + \alpha_N \mathbf{L}\{f_N[k]\}$$

where $\alpha_i, i = 1, 2, ..., N,$ are constants.

1.2.2 Linear System Time Invariance

For a general nth order linear dynamic system, represented by

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y(t)}{dt^{n-2}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m\frac{d^m f(t)}{dt^m} + b_{m-1}\frac{d^{m-1}f(t)}{dt^{m-1}} + \cdots + b_1\frac{df(t)}{dt} + b_0 f(t)$$

the coefficients $a_i, i = 0, 1, 2, ..., n-1$, and $b_i, i = 0, 1, 2, ..., m$, are assumed to be constant, which indicates that the given *system is time invariant*.

Here, we give an additional clarification of the system time invariance. Consider *a system at rest*. The system time invariance is manifested by the constant shape in time (waveform) of the system output response due to the given input. The output response of a system at rest is invariant regardless of the initial time of the input. If the system input is shifted in time, the system output response due to the same input will be shifted in time by the same amount and, in addition, it will preserve the same waveform. The corresponding graphical interpretation of the time invariance principle is shown in Figure 1.9.

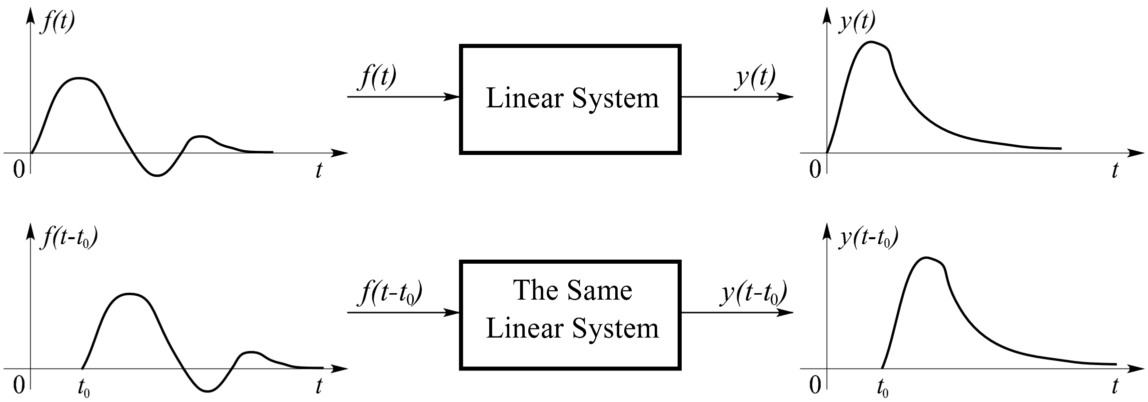


Figure 1.9: Graphical representation of system time invariance

The same arguments presented for the time invariance of continuous-time linear dynamic systems, described by differential equations, hold for the time invariance of discrete-time linear dynamic systems described by difference equations.

The system linearity and time invariance principles will be used in the follow up chapters to simplify the solution to the main linear system theory problem, the problem of finding the system response due to arbitrary input signals.

1.3 Mathematical Modeling of Systems

An Electrical Circuit

Consider a simple RLC electrical circuit presented in Figure 1.10.

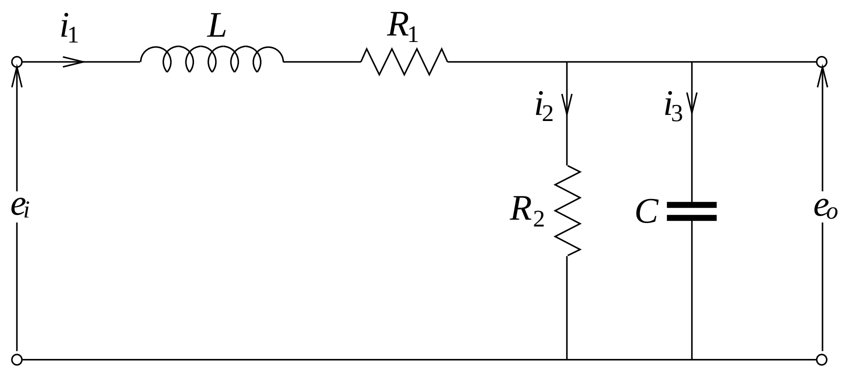


Figure 1.10: An RLC network

Applying the basic circuit laws for voltages and currents, we obtain

$$e_i(t) = L\frac{di_1(t)}{dt} + R_1 i_1(t) + e_0(t)$$

and

$$e_0(t) = R_2 i_2(t) = \frac{1}{C}\int_0^t i_3(\tau)d\tau + v_C(0) \quad \Rightarrow \quad i_3 = C\frac{de_0(t)}{dt}$$

$$i_1(t) = i_2(t) + i_3(t)$$

It follows from the above equations that

$$i_1(t) = \frac{1}{R_2}e_0(t) + C\frac{de_0(t)}{dt}$$

From these equations we obtain the desired second-order differential equation, which relates the input and output of the system, and represents a mathematical model of the circuit given in Figure 1.10

$$\frac{d^2e_0(t)}{dt^2} + \left(\frac{L + R_1R_2C}{R_2LC}\right)\frac{de_0(t)}{dt} + \left(\frac{R_1 + R_2}{R_2LC}\right)e_0(t) = \frac{1}{LC}e_i(t)$$

In order to be able to solve this differential equation for $e_0(t)$, the initial conditions $e_0(0)$ and $de_0(0)/dt$ must be known (determined). For electrical circuits, the initial conditions are usually specified in terms of capacitor voltages and inductor currents. Hence, in this example, $e_0(0)$ and $de_0(0)/dt$ should be expressed in terms of $v_C(0)$ and $i_1(0)$. Note that in this mathematical model $e_i(t)$ represents the system input and $e_0(t)$ is the system output. However, any of the currents and any of the voltages can play the roles of either input or output variables.

A Mechanical System

A translational mechanical system is represented in Figure 1.11.

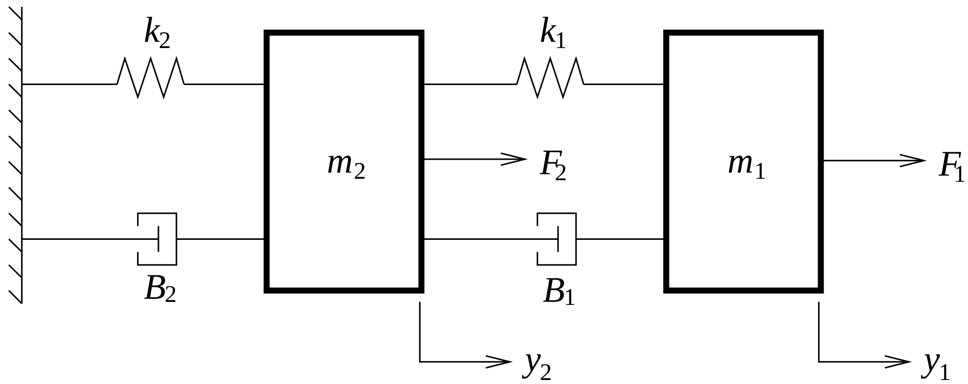


Figure 1.11: A translational mechanical system

Using the basic laws of dynamics, we obtain

$$F_1 = m_1 \frac{d^2 y_1(t)}{dt^2} + B_1 \left(\frac{dy_1(t)}{dt} - \frac{dy_2(t)}{dt} \right) + k_1 (y_1(t) - y_2(t))$$

and

$$F_2 = m_2 \frac{d^2 y_2(t)}{dt^2} + B_2 \frac{dy_2(t)}{dt} + k_2 y_2(t) - B_1 \left(\frac{dy_1(t)}{dt} - \frac{dy_2(t)}{dt} \right)$$
$$-k_1(y_1(t) - y_2(t))$$

This system has *two inputs*, F_1 and F_2, and *two outputs*, $y_1(t)$ and $y_2(t)$. These equations can be rewritten as

$$m_1 \frac{d^2 y_1(t)}{dt^2} + B_1 \frac{dy_1(t)}{dt} + k_1 y_1(t) - B_1 \frac{dy_2(t)}{dt} - k_1 y_2(t) = F_1$$

and

$$-B_1 \frac{dy_1(t)}{dt} - k_1 y_1(t) + m_2 \frac{d^2 y_2(t)}{dt^2} + (B_1 + B_2) \frac{dy_2(t)}{dt}$$
$$+(k_1 + k_2) y_2(t) = F_2$$

Techniques for obtaining experimentally mathematical models of dynamic systems are studied within the scientific area called *system identification*.

Amortization Process Model

If we purchase a house, or a car, and take a loan of d dollars with a fixed interest rate of R percent per year ($r = R/12$ per month), then the loan is paid back through the process known in economics as amortization. Using simple logic, it is not hard to conclude that the outstanding principal, $y[k]$, at $k+1$ discrete time instant (month) is given by the following recursive formula (difference equation)

$$y[k+1] = y[k] + ry[k] - f[k+1] = (1+r)y[k] - f[k+1]$$

where $f[k+1]$ is the payment made in $(k+1)$st discrete-time instant (month).

Let us assume that the monthly loan payment is constant, say $f[k] = p$. The question that we wish to answer is: what is the monthly loan payment needed to pay back the entire loan of d dollars within N months?

The answer to this question can be easily obtain by iterating this difference equation and finding the corresponding solution formula. Since $y[0] = d$ and $f[1] = f[2] = \cdots = f[N] = p$ are known, we have for $k = 1$ and $k = 2$

$$y[1] = (1+r)y[0] - f[1] = (1+r)d - p$$

$$\begin{aligned} y[2] &= (1+r)y[1] - f[2] = (1+r)\{(1+r)d - p\} - p \\ &= (1+r)^2 d - (1+r)p - p \end{aligned}$$

Continuing this procedure for $k = 3, ..., N,$ we can recognize the pattern and get

$$y[3] = (1+r)^3 d - (1+r)^2 p - (1+r)p - p$$

$$\begin{aligned} y[N] &= (1+r)^N d - (1+r)^{N-1} p - \cdots - (1+r)^2 p - (1+r)p - p \\ &= (1+r)^N d - p \sum_{i=0}^{N-1} (1+r)^i \end{aligned}$$

The formula obtained represents the solution of the difference equation. The formula can be even further simplified using the known summation formula

$$\sum_{i=0}^{n} q^i = \frac{q^{n+1}-1}{q-1}, \qquad q \neq 1$$

Applying this formula, we obtain

$$y[N] = (1+r)^N d - p\frac{(1+r)^N - 1}{(1+r)-1} = (1+r)^N d - p\frac{(1+r)^N - 1}{r}$$

We conclude that the loan is paid back when $y[N] = 0$, which implies the formula for the required monthly payment as

$$0 = (1+r)^N d - p\frac{(1+r)^N - 1}{r} \quad \Rightarrow \quad p = \frac{r(1+r)^N}{(1+r)^N - 1}d$$

Heart Beat Dynamics

Dynamics of a heart beat (diastole is a relaxed state and systole is a contracted state of a heart) can be approximately described by the following set of linear differential equations

$$\dot{x}_1(t) = -\frac{2}{\epsilon}x_1(t) - \frac{1}{\epsilon}x_2(t) - \frac{1}{\epsilon}x_3(t)$$

$$\dot{x}_2(t) = -2x_1(t) - 2x_2(t)$$

$$\dot{x}_3(t) = -x_2(t)$$

where $x_1(t)$ is the length of muscle fibre, $x_2(t)$ represents the tension in the fiber caused by blood pressure, and $x_3(t)$ represents dynamics of an electrochemical process that governs the heart beat, and ϵ is a small positive parameter. The system is driven by the initial condition that characterizes the heart's diastolic state, whose normalized value, in this model, is equal to $(x_1(0), x_2(0), x_3(0)) = (1, -1, 0)$.

Eye Movement (Oculomotor Dynamics)

Dynamics of eye movement (muscles, eye, and orbit) can be modeled by the second-order system represented by

$$\frac{d^2y(t)}{dt^2} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\frac{dy(t)}{dt} + \frac{1}{\tau_1\tau_2}y(t) = \frac{1}{\tau_1\tau_2}f(t)$$

where $\tau_1 = 13\,\mathrm{ms}$ and $\tau_2 = 224\,\mathrm{ms}$ are respectively the minor and major eye time constants. $y(t)$ is the eye position in degrees and $f(t)$ is the eye stimulus force in degrees (reference eye position, target position). Several other mathematical models for eye movement exist in the biomedical engineering literature, including a more complex model of order six to be presented in Chapter 8, Problem 8.46.

BOEING Aircraft

The linearized equations governing the motion of a BOEING's aircraft are

$$\frac{d\alpha(t)}{dt} = -0.313\alpha(t) + 56.7q(t) + 0.232f_e(t)$$

$$\frac{dq(t)}{dt} = -0.0139\alpha(t) - 0.426q(t) + 0.0203f_e(t)$$

$$\frac{d\theta(t)}{dt} = 56.7q(t)$$

where $\alpha(t)$ in the aircraft angle of attack, $q(t)$ is the pitch rate, and $\theta(t)$ represents the aircraft pitch angle. The driving force $f_e(t)$ stands for the elevator deflection angle. Differentiating the above system of three first-order linear differential equations, it can be replaced by one third-order linear differential equation that gives direct dependence of $\theta(t)$ on $f_e(t)$, that is

$$\frac{d^3\theta(t)}{dt^3} + 0.739\frac{d^2\theta(t)}{dt^2} + 0.921\frac{d\theta(t)}{dt} = 1.151\frac{df_e(t)}{dt} + 0.1774f_e(t)$$

1.4 System Classification

Real-world systems are either ***static*** or ***dynamic***. Static systems are represented by algebraic equations, for example algebraic equations describing electrical circuits with resistors and constant voltage sources, or algebraic equations in statics indicating that at the equilibrium the sums of all forces are equal to zero.

Dynamic systems are, in general, described either by *differential/difference equations* (also known as ***systems with concentrated*** or ***lumped parameters***) or by *partial differential equations* (known as ***systems with distributed parameters***). For example, electric power transmission lines, wave propagation, behavior of antennas, propagation of light through optical fiber, and heat conduction represent dynamic systems described by partial differential equations. For example, one dimensional electromagnetic wave propagation is described by the partial differential equation

$$\frac{\partial^2 E(t,x)}{\partial t^2} + c^2 \frac{\partial^2 E(t,x)}{\partial x^2} = 0$$

$E(t,x)$ is electric filed, t represents time, x is the spacial coordinate and c is the constant that characterizes the medium. Systems with distributed parameters are also known as *infinite dimensional systems,* in contrast to systems with concentrated parameters that are known as *finite dimensional systems* (they are represented by differential/difference equations of finite orders, $n < \infty$).

Dynamic systems with lumped parameters can be either *linear* or *nonlinear*. ***Linear dynamic systems*** are described by linear differential/difference equations and they obey to the linearity principle. ***Nonlinear dynamic systems*** are described by nonlinear differential/difference equations. For example, a simple pendulum is described by the nonlinear differential equation

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{m}\sin\left(\theta(t)\right) = 0$$

$\theta(t)$ is the pendulum angle, $g = 9.8\,\mathrm{m/s^2}$ is the gravitational constant, and m is the pendulum mass, and $\sin\left(\theta(t)\right)$ is a nonlinear function.

We can also distinguish between *time invariant systems* (systems with constant coefficients) obeying to the time invariance principle and *time-varying systems* whose parameters change in time. For example, the linear time varying model of the Erbium-doped optical fiber amplifier is given by

$$\frac{dN(t)}{dt} + \frac{1}{\tau_l(t)}N(t) = b_p(t)P_p + \sum_{i=1}^{n} b_i(t)P_{si}$$

$N(t)$ represents deviation from the nominal value of the average level of the normalized number of Erbium atoms in the upper excited state, $\tau_l(t)$ is the time varying time constant, P_p and P_{si} are respectively laser pump and optical signal power deviations from their nominal values, and $b_p(t)$ and $b_i(t)$ are coefficients.

Some system parameters and variables can change according to random laws. For example, the generated power of a solar cell, house humidity and temperature. Sometimes system inputs are random signals. For example, aircraft under wind disturbances, electric current under electron thermal noise. Systems that have random parameters and/or process random signals are called ***stochastic systems***. Stochastic systems can be either linear or nonlinear, time invariant or time-varying, continuous or discrete. In contrast to stochastic systems, we have ***deterministic systems*** whose parameters and input signals are deterministic quantities.

Real world physical systems are known as ***nonanticipatory systems*** or ***causal systems***. Let the input $f(t_1)$ be applied to a system at time t_1. The real physical system can only to produce the system output at time equal to t_1, that is $y(t_1)$.

The real physical system cannot, at time t_1, produce information about $y(t)$ for $t > t_1$. That is, the system is unable to predict the future input values and produce the future system response $y(t)$, based on the information that the system has at time t_1. The system causality can also be defined with the statement that the system input $f(t_2)$ has no impact on the system output $y(t_1)$ for $t_2 > t_1$. In contrast to nonanticipatory (causal) systems, we have ***anticipatory*** or ***noncausal systems***. Anticipatory systems are encountered in digital signal processing—they are artificial systems.

Dynamic systems are also *systems with memory*. Namely, the system output at time t_1 depends not only on the system input at time t_1, but also on all previous values of the system input. Let $y(t) = \phi(f(t))$ be the solution of the corresponding differential equation representing a dynamic system.

The fact that the dynamic system possesses memory can be formally recorded as $y(t_1) = \phi(f(t)),\ t_0 \leq t \leq t_1$. In contrast, **static systems have no memory**. If the relationship $y(t) = \phi(f(t))$ came from a static system, then we would have $y(t_1) = \phi(f(t_1))$. That is, for static systems the output at time t_1 depends only on the input at the given time instant t_1. Static systems are known as ***memoryless systems*** or ***instantaneous systems***. For example, an electric resistor is a static system since its voltage (system output) is an instantaneous function of its current (system input) so that $y(t) = v(t) = Ri(t) = Rf(t)$ for any t.

Analog systems deal with continuous-time signals that can take a continuum of values with respect to the signal magnitude. ***Digital systems*** process digital signals whose magnitudes can take only a finite number of values. In digital systems, signals are discretized with respect to both time and magnitude (*signal sampling and quantization*).