for any positive m, n. And again our conventions "think for us": the same formula is also true for negative m and n. For example,

$$(a^{-2})^3 = \left(\frac{1}{a^2}\right)^3 = \frac{1}{a^2} \cdot \frac{1}{a^2} \cdot \frac{1}{a^2} = \frac{1}{a^6} = a^{-6} = a^{(-2)\cdot 3}.$$

Problem 56. Check this formula for other combinations of signs (if $m > 0$, $n < 0$; if both m and n are negative; if one of them is equal to zero).

The last formula about powers:

$$\boxed{(ab)^n = a^n \cdot b^n}$$

Problem 57. Check this formula for positive and negative integers n.

Problem 58. What is $(-a)^{775}$? Is it a^{775} or $-a^{775}$?

Problem 59. Invent a formula for $\left(\frac{a}{b}\right)^n$.

Now a^n is defined for any integer n (positive or not) and for any a. But that is not the end of the game, because n may be a number that is not an integer.

Problem 60. Give some suggestions: What might $4^{1/2}$ be? And $27^{1/3}$? Motivate your suggestions as well as you can.

The definition of $a^{m/n}$ will be given later. (But that also is not the last possible step.)

22 Formula for short multiplication: The square of a sum

As we have seen already,

$$(a+b)(m+n) = am + an + bm + bn$$

(to multiply two sums you must multiply each term of the first sum by each term of the second sum and then add all the products). Now consider the case when the letters inside the parentheses are the same:

$$(a+b)(a+b) = aa + ab + ba + bb.$$

Remember that $ab = ba$ and that aa and bb are usually denoted as a^2 and b^2; we get

$$(a+b)(a+b) = a^2 + 2ab + b^2,$$

or

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

Problem 61. (a) Compute 101^2 without pencil and paper.

(b) Compute 1002^2 without pencil and paper.

Problem 62. Each of the two factors of a product becomes 10 percent bigger. How does the product change?

The rule in words: "The square of the sum of two terms is the sum of their squares plus two times the product of the terms".

Be careful here: "the square of the sum" and "the sum of the squares" sound very similar, but are different; the square of the sum is $(a+b)^2$ and the sum of the squares is $a^2 + b^2$.

Problem 63. Are the father of the son of NN and the son of the father of NN the same person?

23 How to explain the square of the sum formula to your younger brother or sister

A kind wizard liked to talk with children and to make them gifts. He was especially kind when many children came together; each of them got as many candies as the number of children. (So if you came alone, you got one, and if you came with a friend you got two and your friend got two.)

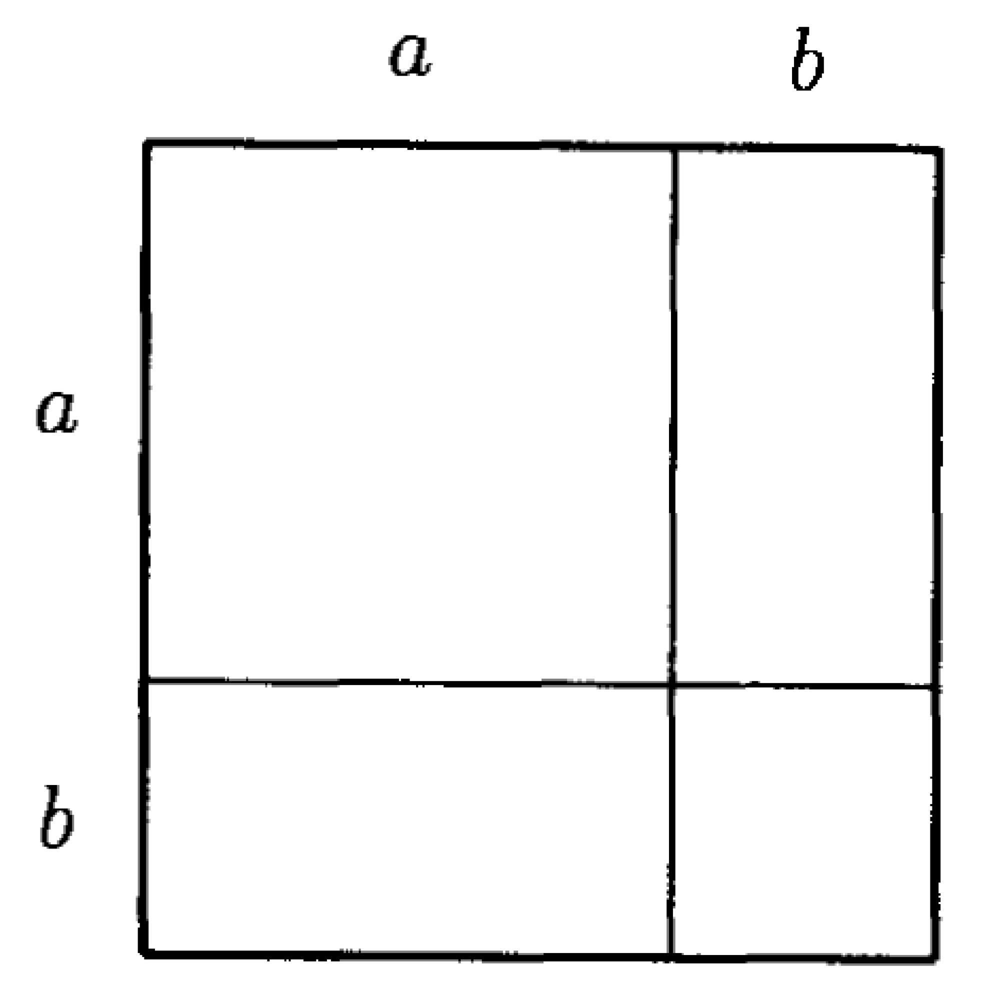
Once, a boys came together. Each of them got a candies – together they got a^2 candies. After they went away with the candies, b girls came and got b candies each – so the girls together got b^2 candies. So that day, the boys and girls got $a^2 + b^2$ candies together.

The next day, a boys and b girls decided to come together. Each of $a+b$ children got $a+b$ candies, so all the children together got $(a+b)^2$ candies. Did they get more or fewer candies than yesterday – and how big is the difference?

To answer this question we may use the following argument. The second time, each of the a boys got b more candies (because of the b girls), so all the boys together got ab more candies. Each girl got a more candies (because of the a boys), so all the girls got ba additional candies. So together, the boys and girls got $ab+ba=2ab$ candies more than on the previous day. So $(a+b)^2$ is $2ab$ more than a^2+b^2, that is, $(a+b)^2=a^2+b^2+2ab$.

Problem 64. Cut a square with edge $a+b$ into one square $a\times a$, one square $b\times b$ and two rectangles $a\times b$.

Solution.



The formula $(a+b)^2=a^2+b^2+2ab$ may be considered as a generic formula for infinitely many equalities like $(5+7)^2=5^2+2\cdot5\cdot7+7^2$ or $(13+\frac{1}{3})^2=13^2+2\cdot13\cdot\frac{1}{3}+\left(\frac{1}{3}\right)^2$; we get these equalities by replacing a and b by specific numbers. These number may, of course, be negative. For example, for $a=7$, $b=-5$ we get

$$(7+(-5))^2=7^2+2\cdot7\cdot(-5)+(-5)^2.$$

Plus times minus is minus, and minus times minus is plus, so we get

$$(7-5)^2=7^2-2\cdot7\cdot5+5^2.$$

The same thing could be done for any other numbers, so the general rule is that

$$\boxed{(a-b)^2=a^2-2ab+b^2}$$

Or in words: "The square of the difference is the sum of the squares minus two times the product of the terms".

Problem 65. Compute (**a**) 99^2; (**b**) 998^2 without pencil and paper.

Problem 66. What do the formulas $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$ give when (**a**) $a = b$; (**b**) $a = 2b$?

24 The difference of squares

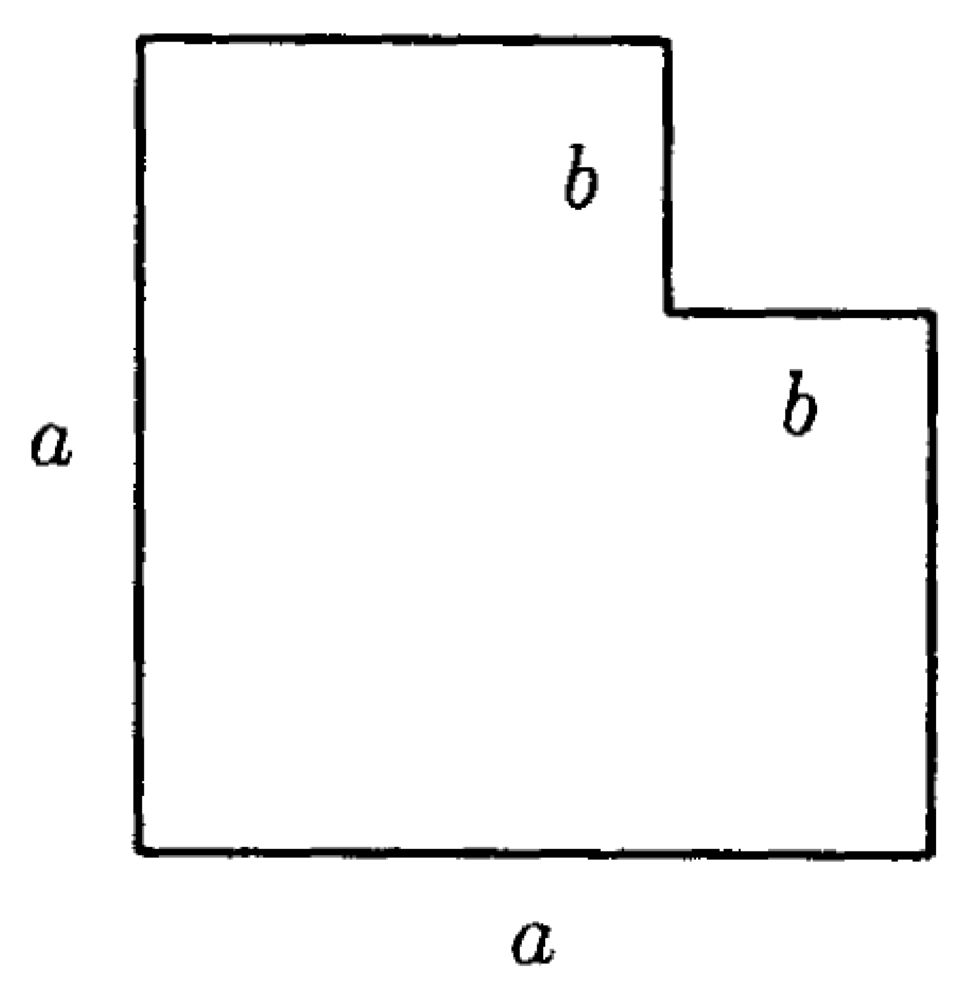
Problem 67. Multiply $a + b$ and $a - b$.

Solution. $(a+b)(a-b) = a(a-b) + b(a-b) = a^2 - ab + ba - b^2 = a^2 - b^2$ (here ab and ba compensate for each other). So we get the formula

$$\boxed{a^2 - b^2 = (a+b)(a-b)}$$

Problem 68. Multiply $101 \cdot 99$ without pencil and paper.

Problem 69. A piece of size $b \times b$ was cut from an $a \times a$ square.



Cut the remaining part into pieces and combine the pieces into a rectangle with sides $a - b$ and $a + b$.

These three formulas – the square of a sum, the square of a difference, and the difference of squares – are called "short multiplication formulas".

Problem 70. Two integers differ by 2. Multiply them and add 1 to the product. Prove that the result is a perfect square (the square of an integer). For example,

$$\begin{aligned} 3 \cdot 5 + 1 &= 16 = 4^2, \\ 13 \cdot 15 + 1 &= 196 = 14^2. \end{aligned}$$

Solution. (First version.) Let n denote the smaller number. Then the other number is $n+2$. Their product is $n(n+2) = n^2+2n$. Adding 1, we get $n^2+2n+1 = (n+1)^2$ (the formula for the square of the sum).

(Second version.) Let n denote the bigger number. Then the smaller one is $n-2$. The product is $n(n-2) = n^2-2n$. Adding 1 we get $n^2-2n+1 = (n-1)^2$ (the square of the difference formula).

(Third version.) If we want to be fair and not choose between the bigger and the smaller number, let us denote by n the number halfway between the numbers. Then the smaller number is $n-1$, the bigger one is $n+1$, and the product is $(n+1)(n-1) = n^2-1$ (the difference of squares formula), that is, it is a perfect square minus one.

Problem 71. Write the sequence of squares of $1, 2, 3, \ldots$

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ \ldots$$

and under any two consecutive numbers of this sequence write their difference:

$$\begin{array}{ccccccccccccccc} 1 & & 4 & & 9 & & 16 & & 25 & & 36 & & 49 & \ldots \\ & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & \ldots \end{array}$$

In the second sequence any two consecutive numbers differ by 2. Can you explain why?

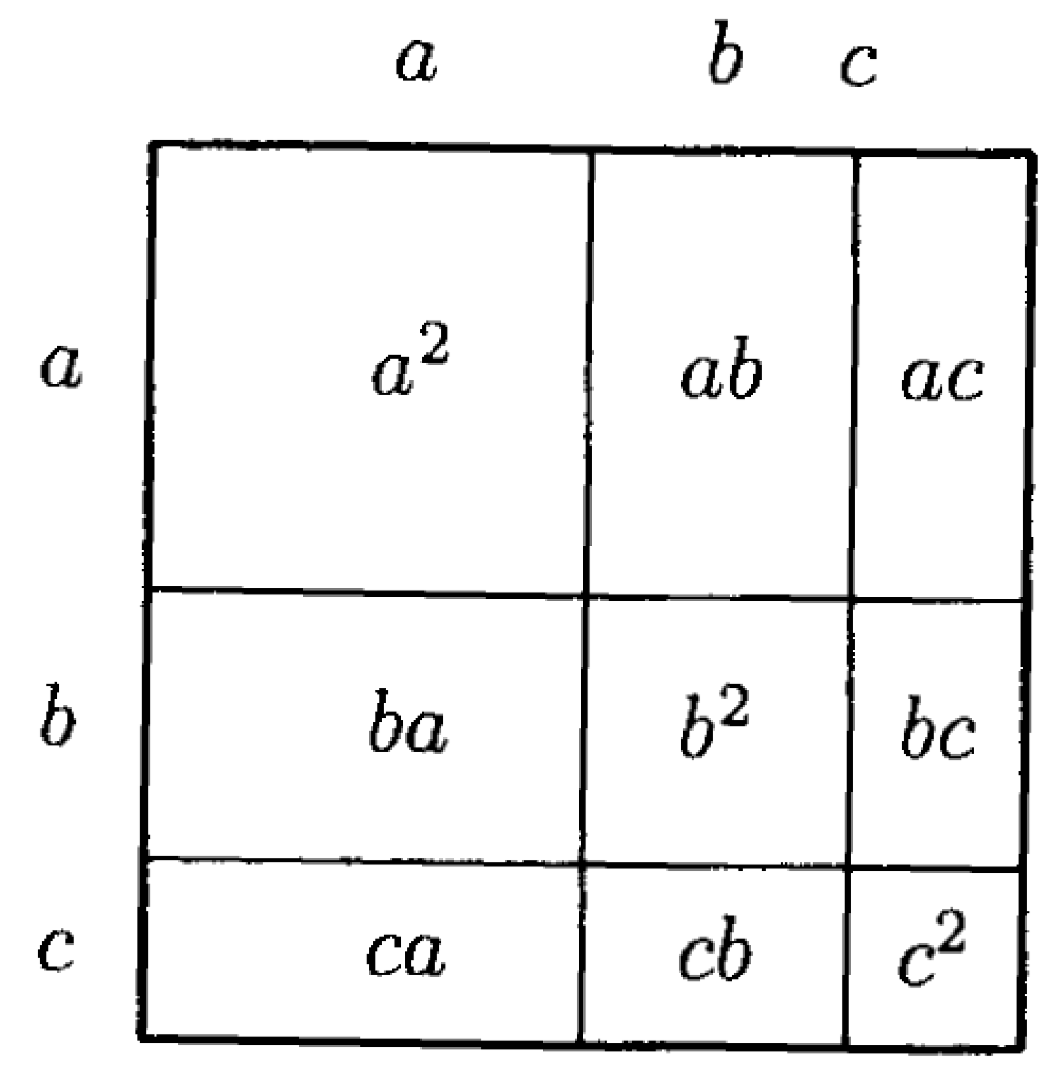
Solution. The consecutive numbers n and $n+1$ have squares n^2 and $(n+1)^2 = n^2+2n+1$. The difference between these squares is $2n+1$, and it becomes greater by 2 if we add 1 to n.

Remark. A sequence where each term is greater than the preceding one by a fixed constant (as in $3, 5, 7, 9, \ldots$) is called an arithmetic (pronounced "arithmEtic", not "arIthmetic") progression. We shall meet progressions again later.

Problem 72. There is a rule that allows us to square any number with the last digit 5, namely, "Drop this last digit out and get some n; multiply n by $n+1$ and add the digits 2 and 5 to the end". For example, for 35^2, we delete 5 and get 3, multiplying 3 and 4 we get 12, adding "2" and "5" we get the answer: 1225. Explain why this rule works.

Problem 73. Compute $(a+b+c)^2$.

Solution. $(a+b+c)^2 = (a+b+c)(a+b+c) = a^2+ab+ac+ba+b^2+bc+ca+cb+c^2 = a^2+b^2+c^2+2ab+2ac+2bc$.



Problem 74. Compute $(a+b-c)^2$.

Hint. Use the answer of the preceding problem.

Problem 75. Compute $(a+b+c)(a+b-c)$.

Hint. Use the difference-of-squares formula.

Problem 76. Compute $(a+b+c)(a-b-c)$.

Hint. The difference-of-squares formula is useful here also.

Problem 77. Compute $(a+b-c)(a-b+c)$.

Hint. Even here the difference-of-squares formula can be used!

Problem 78. Compute $(a^2-2ab+b^2)(a^2+2ab+b^2)$.

Solution. This is equal to

$$(a-b)^2(a+b)^2 = ((a-b)(a+b))^2 = (a^2-b^2)^2 = a^4-2a^2b^2+b^4.$$

Another solution:

$$\begin{aligned}&(a^2-2ab+b^2)(a^2+2ab+b^2) =\\ &= ((a^2+b^2)+2ab)((a^2+b^2)-2ab) = (a^2+b^2)^2-(2ab)^2 =\\ &= a^4+2a^2b^2+b^4-4a^2b^2 = a^4+b^4-2a^2b^2.\end{aligned}$$

25 The cube of the sum formula

Let us derive the formula for $(a+b)^3$. By definition,

$$(a+b)^3 = (a+b)(a+b)(a+b),$$

and we may start here. But part of the job is done already:

$$(a+b)^3 = (a+b)^2(a+b) = (a^2+2ab+b^2)(a+b).$$

Now we have to multiply each term of the first sum by each term of the second one and take the sum of all products:

$$\begin{aligned}(a^2+2ab+b^2)(a+b) &= \\ = \ a^2\cdot a + 2ab\cdot a + b^2\cdot a &+ \\ + \ a^2\cdot b + 2ab\cdot b + b^2\cdot b&.\end{aligned}$$

Remembering how to multiply powers with a common base (that is, that $a^m \cdot a^n = a^{m+n}$) and putting a-factors first, we get

$$\begin{aligned}a^3 + 2a^2b + \ ab^2 &+ \\ + \ a^2b + 2ab^2 + b^3&.\end{aligned}$$

Here some terms are similar (only the numerical factors are different); they are written one under another. Collecting them, we get

$$\boxed{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$$

Problem 79. Compute 11^3 without pencil and paper.

Hint. $11 = 10 + 1$.

Problem 80. Compute 101^3 without pencil and paper.

Problem 81. Compute $(a-b)^3$.

Solution. We may compute it in the same way as before, writing $(a-b)^3 = (a-b)^2(a-b) = (a^2-2ab+b^2)(a-b)$ etc. But an easier way is to substitute $(-b)$ for b in the formula for $(a+b)^3$:

$$(a+(-b))^3 = a^3 + 3a^2\cdot(-b) + 3a(-b)^2 + (-b)^3$$

or

$$\boxed{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$$

(recall that minus times minus is plus and plus times minus is minus).

26 The formula for $(a+b)^4$

Before computing $(a+b)^4$ let us try to guess the answer. To do so, look at the formulas we already have:

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3.\end{aligned}$$

To get more "experimental data" we can add the formula

$$(a+b)^1 = a+b\,.$$

So we have:

$$\begin{aligned}(a+b)^1 &= a+b\\ (a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ (a+b)^4 &= ???\end{aligned}$$

How many additive terms do you expect in $(a+b)^4$? Five, of course. What is the first term? Definitely, a^4. The next term is a more difficult puzzle. (To tell you the truth, it will be $4a^3b$.) To explain how it can be guessed let us divide our question into two parts:

(1) What powers of a and b will appear?

(2) What numeric coefficients will appear?

Part (1) is simpler. If the formula for

$(a+b)^1$ uses a and b,
$(a+b)^2$ uses a^2, ab and b^2,
$(a+b)^3$ uses a^3, a^2b, ab^2 and b^3,

we may expect that

$(a+b)^4$ uses a^4, a^3b, a^2b^2, ab^3, and b^4.

Now look at the coefficients (we write the factor "1" to make our formulas more uniform):

$$\begin{aligned}(a+b)^1 &= 1a+1b\\ (a+b)^2 &= 1a^2+2ab+1b^2\\ (a+b)^3 &= 1a^3+3a^2b+3ab^2+1b^3\end{aligned}$$

or, without terms (only the coefficients):

$$\begin{array}{ccccc} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ ? & ? & ? & ? & ? \end{array}$$

(we have already said that we expect five terms in the $(a+b)^4$ formula). The first coefficient is, of course, 1. It seems that the second is 4 (because in the second column we have 1, 2 and 3). So we get

$$\begin{array}{ccccc} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & ? & ? & ? \end{array}$$

Two more coefficients can be guessed. In $(a+b)^4$, the letters a and b have equal rights, so b^4 must have the same coefficient as a^4, and ab^3 must have the same coefficient as a^3b – to avoid discrimination:

$$\begin{array}{ccccc} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & ? & 4 & 1\,. \end{array}$$

Now only a^2b^2 remains, and if we cannot guess it, we must compute it by brute force:

$$(a+b)^4 = (a+b)^3(a+b) = (a^3+3a^2b+3ab^2+b^3)(a+b) =$$

$$\begin{array}{lllllll} = & a^3\cdot a & +\ 3a^2b\cdot a & +\ 3ab^2\cdot a & +\ b^3\cdot a & + & \\ & & +\ a^3\cdot b & +\ 3a^2b\cdot b & +\ 3ab^2\cdot b & +\ b^3\cdot b = & \\ = & a^4 & +\ 3a^3b & +\ 3a^2b^2 & +\ ab^3 & + & \\ & & +\ a^3b & +\ 3a^2b^2 & +\ 3ab^3 & +\ b^4 & \end{array}$$

(again the similar terms are written one under another). Collecting them, we get

$$\boxed{(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4}$$

All our guesses turn out to be true and we find the remaining coefficient of a^2b^2, which turns out to be 6.

27 Formulas for $(a+b)^5$, $(a+b)^6$, ... and Pascal's triangle

In $(a+b)^5$ we expect terms

$$a^5 \quad a^4b \quad a^3b^2 \quad a^2b^3 \quad ab^4 \quad b^5$$

with coefficients

$$1 \quad 5 \quad ? \quad ? \quad 5 \quad 1$$

To find the two remaining coefficients (they are expected to be equal, of course) let us proceed as usual:

$$(a+b)^5 = (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)(a+b) =$$

$$\begin{aligned} = a^4 \cdot a &+ 4a^3b \cdot a + 6a^2b^2 \cdot a + 4ab^3 \cdot a + b^4 \cdot a + \\ &+ a^4 \cdot b + 4a^3 \cdot b + 6a^2b^2 \cdot b + 4ab^3 \cdot b + b^4 \cdot b = \\ = a^5 &+ 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5. \end{aligned}$$

So our table of coefficients has one more row:

1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

Probably you have already figured out the rule: Each coefficient is equal to the sum of the coefficient above it and the one to the left of it: $1+4=5$, $4+6=10$, $6+4=10$, $4+1=5$.

The reason this is so becomes clear if we look at our computation ignoring everything except coefficients:

$$\begin{aligned} 1\ldots &+ 4\ldots + 6\ldots + 4\ldots + 1\ldots + \\ &+ 1\ldots + 4\ldots + 6\ldots + 4\ldots + 1\ldots = \\ 1\ldots &+ 5\ldots + 10\ldots + 10\ldots + 5\ldots + 1\ldots \end{aligned}$$

They are added exactly as the rule says.

For aesthetic reasons, we may write the table in a more symmetric way and add "1" on the top (because $(a+b)^0 = 1$). We get a triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

which can be continued using the rule that each number is the sum of the two numbers immediately above it (except for the first and the last numbers, which are equal to 1). For example, the next row will be

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

and it corresponds to the formula

$$\boxed{(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6}$$

This triangle is called *Pascal's triangle* (Blaise Pascal [1623–1662] was a French mathematician and philosopher.)

Problem 82. Compute 11^3, 11^4, 11^5 and 11^6.

Problem 83. Write a formula for $(a+b)^7$.

Problem 84. Find formulas for $(a-b)^4$, $(a-b)^5$ and $(a-b)^6$.

Problem 85. Compute the sums of all the numbers in the first, second, third, etc., rows of Pascal's triangle. Can you see the rule? Can you explain the rule?

Problem 86. What do the formulas for $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, etc., give when $a = b$?

Problem 87. Do you see the connection between the two preceding problems?

Problem 88. What do the formulas for $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, etc., give when $a = -b$?

28 Polynomials

By a *polynomial* we mean an expression containing letters (called *variables*), numbers, addition, subtraction and multiplication. Here are some examples:

$$a^4 + a^3b + ab^3 + b^4$$

$$(5 - 7x)(x - 1)(x - 3) + 11$$

$$(a + b)(a^3 + b^3)$$

$$(a + b)(a + 2b) + ab$$

$$(x + y)(x - y) + (y - x)(y + x)$$

$$0$$

$$(x + y)^{100}$$

These examples contain not only addition, subtraction and multiplication, but also positive integer constants as powers. These are legal because they can be considered as shortcuts (for example, a^4 may be considered as short notation for $a \cdot a \cdot a \cdot a$, which is perfectly legal). But a^{-7} or x^y are *not* polynomials.

A *monomial* is a polynomial that does not use addition or subtraction, that is, a product of letters and numbers. Here are some examples of monomials:

$$5 \cdot a \cdot 7 \cdot b \cdot a$$

$$127a^{15}$$

$$(-2)a^2b$$

(in the last example the minus sign is not subtraction but a part of the number "-2").

Usually numbers and identical letters are collected: for example, $5 \cdot a \cdot 7 \cdot b \cdot a$ is written as $35a^2b$.

Please keep in mind that a monomial is a polynomial, so sometimes for a mathematician one ("mono") is many ("poly").

Each polynomial can be converted into the sum of monomials if we remove parentheses. For example,

$$(a + b)(a^3 + b^3) = aa^3 + ab^3 + ba^3 + bb^3 = a^4 + ab^3 + ba^3 + b^4,$$

$$(a+b)(a+2b) = a^2 + 2ab + ba + 2b^2.$$

When doing so we can get similar monomials (having the same letters with the same powers and differing only in the coefficients). For example, in the second polynomial above, the terms $2ab$ and ba are similar. They can be collected into $3ab$ and we get

$$(a+b)(a+2b) = a^2 + 2ab + ba + 2b^2 = a^2 + 3ab + 2b^2.$$

Problem 89. Convert $(1+x-y)(12-zx-y)$ into a sum of monomials and collect the similar terms.

Solution.

$$\begin{aligned}
&(1+x-y)(12-zx-y) = \\
&\quad = \quad 12 - zx \underline{-y} + 12x - xzx - xy \underline{-12y} + yzx + y^2 = \\
&\quad = \quad 12 - zx - 13y + 12x - zx^2 - yx + yzx + y^2.
\end{aligned}$$

(The similar terms are underlined.)

Strictly speaking, this is not enough, because we need a sum of monomials and now we have subtraction. Therefore we need to do one more step to get

$$12 + (-1)zx + (-13)y + 12x + (-1)zx^2 + (-1)yx + 1yzx + 1y^2$$

(to make the terms more uniform we added the factor "1" before xyz and before y^2).

A *standard form* of a polynomial is a sum of monomials, where each monomial is a product of a number (called a *coefficient*) and of powers of different letters, and where all similar monomials are collected.

To add two polynomials in standard form we must add the coefficients of similar terms. If we get a zero coefficient, the corresponding term vanishes:

$$(1x + (-1)y) + (1y + (-2)x + 1z) =$$

$$(1 + (-2))x + ((-1) + 1)y + 1z = (-1)x + 0y + 1z = (-1)x + 1z.$$

To multiply two polynomials in standard form we need to multiply each term of the first polynomial by each term of the second polynomial. When multiplying monomials, we add powers of each variable:

$$(a^5b^7c)\cdot(a^3bd^4) = a^{5+3}b^{7+1}cd^4 = a^8b^8cd^4.$$

After this is done, we have to collect similar terms. For example,

$$(x-y)(x^2+xy+y^2) = x^3 + \underline{x^2y} + \underline{\underline{xy^2}} - \underline{yx^2} - \underline{\underline{xy^2}} - y^3 = x^3 - y^3.$$

(The pedantic reader may find that we have violated the rules adopted for the standard form of a polynomial, because the coefficients -1 and 1 are omitted.)

Problem 90.

(a) Multiply $(1+x)(1+x^2)$.

(b) Multiply $(1+x)(1+x^2)(1+x^4)(1+x^8)$.

(c) Compute $(1+x+x^2+x^3)^2$.

(d) Compute $(1+x+x^2+x^3+\cdots+x^9+x^{10})^2$.

(e) Find the coefficients of x^{30} and x^{29} in $(1+x+x^2+x^3+\cdots+x^9+x^{10})^3$.

(f) Multiply $(1-x)(1+x+x^2+x^3+\cdots+x^9+x^{10})$.

(g) Multiply $(a+b)(a^2-ab+b^2)$.

(h) Multiply $(1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10})$ by $(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10})$.

29 A digression: When are polynomials equal?

The word "equal" for polynomials may be understood in many different ways. The first possibility: Polynomials are equal if they can be transformed into one another by using algebraic rules (removing parentheses, collecting similar terms, finding common factors, and so on). Another possibility: Two polynomials are considered to be equal if after substituting any numbers for the variables they have the same numeric value. It turns out that these two definitions are equivalent; if two polynomials are equal in the sense of one of these definitions they are also equal in the sense of the other one. Indeed, if one polynomial can be converted into another using algebraic transformations, these transformations are still valid when variables are replaced by numbers.

So these polynomials have the same numeric value after replacement. It is not easy to prove the reverse statement: *If two polynomials are equal for any values of variables, they can be converted into each other by algebraic transformations.* So we shall use it – sorry! – without proof.

If we want to convince somebody that two given polynomials are equal, the first version of the definition is preferable; it is enough to show the sequence of algebraic transformations needed to get the second polynomial from the first one. On the other hand, if we want to convince somebody that two polynomials are not equal, the second definition is better; it is enough to find numbers that lead to the different values of the polynomials.

Problem 91. Prove that

$$(x-1)(x-2)(x-3)(x-4) \neq (x+1)(x+2)(x+3)(x+4)$$

without computations.

Solution. When $x = 1$ the left-hand side is equal to zero and the right-hand side is not, therefore these polynomials are not equal according to the second definition.

Problem 92. In the (true) equality

$$(x^2-1)(x+\cdots) = (x-1)(x+3)(x+\cdots)$$

some numbers are replaced by dots. What are these numbers?

Hint. Substitute -1 and -3 for x.

Now assume that somebody gives us two polynomials, not saying whether they are different or equal. How can we check this? We can try to substitute different numbers for the variables. If at least once these polynomials have different numeric values we can be sure that they are different. Otherwise we may suspect that these polynomials are in fact equal.

Problem 93. George tries to check whether the polynomials $(x+1)^2 - (x-1)^2$ and $x^2 + 4x - 1$ are equal or not by substituting 1 and -1 for x. Is it a good idea?

Solution. No. These polynomials have equal values for $x = -1$ (both values are -4) and for $x = 1$ (both give 4). However, they are not equal; for example, they have different values for $x = 0$.

To check whether two polynomials are equal or not in a more regular way, we may convert them to a standard form. If after this they differ only in the order of the monomials (or in the order of the factors inside the monomials), then the polynomials are equal. If not, it is possible to prove that the polynomials are different.

Sometimes equal polynomials are called "identically equal", meaning that they are equal for all values of variables. So, for example, $a^2 - b^2$ is identically equal to $(a-b)(a+b)$.

Remark. Later we shall see that sometimes a finite number of tests is enough to decide whether two polynomials are equal or not.

30 How many monomials do we get?

Problem 94. Each of two polynomials contains four monomials. What is the maximal possible number of monomials in their product?

Remark. Of course, any polynomial can be extended by monomials with zero coefficients like this:

$$x^3 + 4 = x^3 + 0x^2 + 0x + 4$$

Such monomials are ignored.

Solution. Multiply $(a+b+c+d)$ by $(x+y+z+u)$:

$$\begin{aligned}(a+b+c+d)(x+y+z+u) &=\\ &= ax+ay+az+au+\\ &\quad bx+by+bz+bu+\\ &\quad cx+cy+cz+cu+\\ &\quad dx+dy+dz+du\,.\end{aligned}$$

We get 16 terms. It is clear that 16 is the maximum possible number (because each of 4 monomials of the first polynomial is multiplied by each of 4 monomials of the second one).

Problem 95. Each of two polynomials contains four monomials. Is it possible that their product contains fewer than 16 monomials?

Solution. Yes, if there are similar monomials among the products. For example,

$$(1+x+x^2+x^3)(1+x+x^2+x^3) = 1+2x+3x^2+4x^3+3x^4+2x^5+x^6,$$

that is, after collecting similar terms we get 7 monomials instead of 16.

Problem 96. Is it possible when multiplying two polynomials that, after collecting similar terms, all terms vanish (have zero coefficients)?

Answer. No.

Remark. Probably this problem seems silly; it is clear that it cannot happen. If you think so, please reconsider the problem several years from now.

Problem 97. Is it possible when multiplying two polynomials that after the collection of similar terms all terms vanish (have zero coefficients) except one? (Do not count the case when each of the polynomials has only one monomial.)

Problem 98. Is it possible that the product of two polynomials contains fewer monomials than each of the factors?

Solution. Yes:

$$\begin{aligned}
&(x^2+2xy+2y^2)(x^2-2xy+2y^2) = \\
&= ((x^2+2y^2)+2xy)((x^2+2y^2)-2xy) = \\
&= (x^2+2y^2)^2-(2xy)^2 = \\
&= x^4+4x^2y^2+4y^4-4x^2y^2 = \\
&= x^4+4y^4.
\end{aligned}$$

31 Coefficients and values

Recall Pascal's triangle and the formulas for $(a+b)^n$ for different n:

$$\begin{array}{ccccc}
1 & \quad & (a+b)^0 & = & 1 \\
1\ 1 & & (a+b)^1 & = & 1a+1b \\
1\ 2\ 1 & & (a+b)^2 & = & 1a^2+2ab+1b^2 \\
1\ 3\ 3\ 1 & & (a+b)^3 & = & 1a^3+3a^2b+3ab^2+1b^3 \\
1\ 4\ 6\ 4\ 1 & & (a+b)^4 & = & 1a^4+4a^3b+6a^2b^2+4ab^3+1b^4
\end{array}$$

etc. Each of these formulas is an equality between two polynomials.

Problem 99. What do we get for $a = 1$, $b = 1$?

Solution.

$$\begin{aligned}(1+1)^0 &= 1\\ (1+1)^1 &= 1+1\\ (1+1)^2 &= 1+2+1\\ (1+1)^3 &= 1+3+3+1\\ (1+1)^4 &= 1+4+6+4+1\end{aligned}$$

etc. Recall that $1 + 1 = 2$; so we proved that the sum of any row of Pascal's triangle is a power of 2. For example, the sum of the 25th row of Pascal's triangle is equal to 2^{24}.

Problem 100. Add the numbers of some row of Pascal's triangle with alternating signs. You get 0:

$$\begin{aligned}&1-1=0\\ &1-2+1=0\\ &1-3+3-1=0\\ &1-4+6-4+1=0\end{aligned}$$

etc. Why does this happen?

Hint. Try $a = 1$, $b = -1$.

Problem 101. Imagine that the polynomial $(1 + 2x)^{200}$ is converted to the standard form (the sum of powers of x with numerical coefficients). What is the sum of all the coefficients?

Hint. Try $x = 1$.

Problem 102. The same question for the polynomial $(1 - 2x)^{200}$ instead of $(1 + 2x)^{200}$.

Problem 103. Imagine that the polynomial $(1 + x - y)^3$ is converted to the standard form. What is the sum of its coefficients?

Problem 104. (*continued*) What is the sum of the coefficients of the terms not containing y?

Problem 105. (*continued*) What is the sum of the coefficients of the terms containing x?

32 Factoring

To multiply polynomials you may need a lot of patience, but you do not need to think; just follow the rules carefully. But to reconstruct factors if you know only their product you sometimes need a lot of ingenuity. And some polynomials cannot be decomposed into a product of nontrivial (nonconstant) factors at all. The decomposition process is called *factoring*, and there are many tricks that may help. We'll show some tricks now.

Problem 106. Factor the polynomial $ac + ad + bc + bd$.

Solution. $ac + ad + bc + bd = a(c + d) + b(c + d) = (a + b)(c + d)$.

Problem 107. Factor the following polynomials:

(a) $ac + bc - ad - bd$;

(b) $1 + a + a^2 + a^3$;

(c) $1 + a + a^2 + a^3 + \cdots + a^{13} + a^{14}$;

(d) $x^4 - x^3 + 2x - 2$.

Sometimes we first need to cut one term into two pieces before it is possible to proceed.

Problem 108. Factor $a^2 + 3ab + 2b^2$.

Solution. $a^2 + 3ab + 2b^2 = a^2 + ab + 2ab + 2b^2 = a(a + b) + 2b(a + b)$ $= (a + 2b)(a + b)$.

Remark. When multiplying two polynomials we collect the similar terms into one term. So it is natural to expect that when going in the other direction we may have to split a term into a sum of several terms.

Problem 109. Factor:

(a) $a^2 - 3ab + 2b^2$;

(b) $a^2 + 3a + 2$.

The formula for the square of the sum can be read "from right to left" as an example of factoring: the polynomial $a^2 + 2ab + b^2$ is factored into $(a + b)(a + b)$. The same factorization can also be obtained as follows:

$$a^2 + 2ab + b^2 = a^2 + ab + ab + b^2 = a(a + b) + b(a + b) = (a + b)(a + b).$$

Problem 110. Factor:

(a) $a^2 + 4ab + 4b^2$;

(b) $a^4 + 2a^2b^2 + b^4$;

(c) $a^2 - 2a + 1$.

Sometimes it is necessary to add and subtract some monomial (reconstructing the annihilated terms). We show this trick factoring $a^2 - b^2$ (though we know the factorization in advance: it is the difference-of-squares formula):

$$a^2 - b^2 = a^2 - ab + ab - b^2 = a(a - b) + b(a - b) = (a + b)(a - b).$$

Problem 111. Factor $x^5 + x + 1$.

Solution. $x^5 + x + 1 = x^5 + x^4 + x^3 - x^4 - x^3 - x^2 + x^2 + x + 1 = x^3(x^2 + x + 1) - x^2(x^2 + x + 1) + (x^2 + x + 1) = (x^3 - x^2 + 1)(x^2 + x + 1)$.

Probably you are discouraged by this solution because it seems impossible to invent it. The authors share your feeling.

Let us look at the factorization $a^2 - b^2 = (a + b)(a - b)$ once more from another viewpoint. If $a = b$, then the right-hand side is equal to zero (one of the factors is zero). Therefore the left-hand side must be zero, too. Indeed, $a^2 = b^2$ when $a = b$. Similarly, if $a + b = 0$ then $a^2 = b^2$ (in this case $a = -b$ and $a^2 = b^2$ because in changing the sign we do not change the square).

Problem 112. Prove that if $a^2 = b^2$ then $a = b$ or $a = -b$.

The moral of this story: When trying to factor a polynomial it is wise to see when it has a zero value. This may give you an idea what the factors might be.

Problem 113. Factor $a^3 - b^3$.

Solution. The expression $a^3 - b^3$ has a zero value when $a = b$. So it is reasonable to expect a factor $a - b$. Let us try: $a^3 - b^3 = a^3 - a^2b + a^2b - ab^2 + ab^2 - b^3 = a^2(a - b) + ab(a - b) + b^2(a - b) = (a^2 + ab + b^2)(a - b)$.

Problem 114. Factor $a^3 + b^3$.

Solution. $a^3 + b^3 = a^3 + a^2b - a^2b - ab^2 + ab^2 + b^3 = a^2(a+b) - ab(a+b) + b^2(a+b) = (a^2 - ab + b^2)(a+b)$.

The same factorization can be obtained from the solution of the preceding problem by substituting $(-b)$ for b.

Problem 115. Factor $a^4 - b^4$.

Solution. $a^4 - b^4 = a^4 - a^3b + a^3b - a^2b^2 + a^2b^2 - ab^3 + ab^3 - b^4 = a^3(a-b) + a^2b(a-b) + ab^2(a-b) + b^3(a-b) = (a-b)(a^3 + a^2b + ab^2 + b^3)$.

Problem 116. Factor:

(a) $a^5 - b^5$;

(b) $a^{10} - b^{10}$;

(c) $a^7 - 1$.

Another factorization of $a^4 - b^4$:

$$a^4 - b^4 = (a^2 - b^2)(a^2 + b^2).$$

These two factorizations are in fact related; both can be obtained from

$$(a^4 - b^4) = (a-b)(a+b)(a^2 + b^2)$$

by a suitable grouping of factors.

Problem 117. Factor $a^2 - 4b^2$.

Solution. Using that $4 = 2^2$ we write:

$$a^2 - 4b^2 = a^2 - 2^2b^2 = a^2 - (2b)^2 = (a - 2b)(a + 2b)$$

Let us try to apply the same trick to $a^2 - 2b^2$. Here we need a number called "the square root of two" and denoted by $\sqrt{2}$. It is approximately equal to $1.4142\ldots$; its main property is that its square is equal to 2: $(\sqrt{2})^2 = 2$. (Generally speaking, a square root of a nonnegative number a is defined as a nonnegative number whose square is equal to a. It is denoted by $\sqrt{a}$. Such a number always exists and is defined uniquely; see below.)

Using the square root of two we may write:

$$a^2 - 2b^2 = a^2 - (\sqrt{2}b)^2 = (a - \sqrt{2}b)(a + \sqrt{2}b).$$

So we are able to factor $a^2 - 2b^2$, though we are forced to use $\sqrt{2}$ as a coefficient.

Remark. Look at the equality

$$a - b = (\sqrt{a})^2 - (\sqrt{b})^2 = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}).$$

So we have factored $a-b$, haven't we? No, we haven't, because $\sqrt{a}-\sqrt{b}$ is not a polynomial; taking the square root is not a legal operation for polynomials – only addition, subtraction and multiplication are allowed. But how about $a-\sqrt{2}b$? Why do we consider it as a polynomial? Because our definition of a polynomial allows it to be constructed from letters and numbers using addition, subtraction, and multiplication. And $\sqrt{2}$ is a perfectly legal number (though it is defined as a square root of another number). So in this case everything is O.K.

Problem 118. Factor: **(a)** a^2-2; **(b)** a^2-3b^2; **(c)** $a^2+2ab+b^2-c^2$; **(d)** $a^2+4ab+3b^2$.

Problem 119. Factor a^4+b^4. (The known factorization of a^4-b^4 seems useless because substituting $(-b)$ for b we get nothing new.)

Solution. A trick: add and subtract $2a^2b^2$. It helps:

$$\begin{aligned} a^4+b^4 &= a^4+2a^2b^2+b^4-2a^2b^2 = \\ &= (a^2+b^2)^2-(\sqrt{2}ab)^2 = (a^2+b^2+\sqrt{2}ab)(a^2+b^2-\sqrt{2}ab). \end{aligned}$$

Let us see what we now know. We can factor a^n-b^n for any positive integer n (one of the factors is $a-b$). If n is odd, the substitution of $-b$ for b gives a factorization of a^n+b^n (one of the factors is $a+b$). But what about a^2+b^2, a^4+b^4, a^6+b^6, etc.? We have just factored the second one.

Problem 120. Can you factor any other polynomial of the form $a^{2n}+b^{2n}$?

Hint. $a^6+b^6 = (a^2)^3+(b^2)^3$. The same trick may be used if n has an odd divisor greater than 1.

But the simplest case, a^2+b^2, remains unsolved. It would be possible to write

$$a^2+b^2 = a^2-(\sqrt{-1}\cdot b)^2 = (a-\sqrt{-1}\cdot b)(a+\sqrt{-1}\cdot b)$$

if a square root of -1 exists. But – alas – it is not the case (the square of any nonzero number is positive and therefore not equal to -1). But mathematicians are tricky; if such a number does not exist, it must be invented. So they invented it, and got new numbers called *complex numbers*. But this is another story.

Problem 121. What would you suggest as the product of two complex numbers $(2+3\sqrt{-1})$ and $(2-3\sqrt{-1})$?

Let us finish this section with more difficult problems.

Problem 122. Factor:

(a) x^4+1;

(b) $x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)$;

(c) $a^{10}+a^5+1$;

(d) $a^3+b^3+c^3-3abc$;

(e) $(a+b+c)^3-a^3-b^3-c^3$;

(f) $(a-b)^3+(b-c)^3+(c-a)^3$.

Problem 123. Prove that if $a,b>1$ then $a+b<1+ab$.

Hint. Factor $(1+ab)-(a+b)$.

Problem 124. Prove that if $a^2+ab+b^2=0$ then $a=0$ and $b=0$.

Hint. Recall the factorization of a^3-b^3. (Another solution will be discussed later when speaking about quadratic equations.)

Problem 125. Prove that if $a+b+c=0$ then $a^3+b^3+c^3=3abc$.

Problem 126. Prove that if

$$\frac{1}{a+b+c}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

then there are two opposite numbers among a,b,c (i.e. $a=-b$, $a=-c$ or $b=-c$).