1.3 EVALUATING LIMITS ANALYTICALLY

Properties of Limits

For many *well – behaved* functions, evaluating the limit can be done by direct substitution. That is,

$$\lim_{x \to c} f(x) = f(c)$$

Such *well – behaved* functions are **continuous at *c***. We will study continuity of a function in §1.4.
The following theorems describe limits that can be evaluated by direct substitution.

Theorem 1.1 Some Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

Theorem 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

4. $\lim_{x \to c} \left[b \cdot f(x) \right] = bL$
5. $\lim_{x \to c} \left[f(x) \pm g(x) \right] = L \pm K$
6. $\lim_{x \to c} \left[f(x) \cdot g(x) \right] = LK$
7. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$;provided $K \neq 0$
8. $\lim_{x \to c} \left[f(x) \right]^n = L^n$

Theorem 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = \dfrac{p(x)}{q(x)}$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}$$

Theorem 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f\left(g(x)\right) = f(L)$$

Theorem 1.6 Limits of Trigonometric Functions

1. $\lim_{x \to c} \sin x = \sin c$	3. $\lim_{x \to c} \tan x = \tan c$	5. $\lim_{x \to c} \sec x = \sec c$
2. $\lim_{x \to c} \cos x = \cos c$	4. $\lim_{x \to c} \cot x = \cot c$	6. $\lim_{x \to c} \csc x = \csc c$

Example: Find each limit

(a) $\lim_{x \to 1} \left(-x^2 + 1\right)$

(b) $\lim_{x \to 4} \sqrt[3]{x+4}$

(c) $\lim_{x \to 3} \frac{\sqrt{x+1}}{x-4}$

(d) $\lim_{x \to 7} \sec\left(\frac{\pi x}{6}\right)$

Example: Use the given information to evaluate the limits: $\lim_{x \to c} f(x) = 2$ and $\lim_{x \to c} g(x) = 3$

(a) $\lim_{x \to c} \left[5g(x)\right]$

(b) $\lim_{x \to c} \left[f(x) + g(x)\right]$

(c) $\lim_{x \to c} \left[f(x)g(x)\right]$

(d) $\lim_{x \to c} \frac{f(x)}{g(x)}$

A Strategy for Finding Limits

If a limit cannot be found using direct substitution, then we will use the following theorem and some other techniques to evaluate the limit.

Theorem 1.7 Functions That Agree at All But One Point

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ <u>in an open interval containing c</u>. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

♬: Keep in mind that some functions do not have limits.

If direct substitution yields the meaningless result $\frac{0}{0}$, then you cannot determine the limit in this form. The expression that yields this result is called an **indeterminate form**. When you encounter this form, you must rewrite the fraction so that the new <u>denominator</u> does not have 0 as its limit. One way to do this is to *cancel like factors*, and a second way is to *rationalize the <u>numerator</u>*.

Example: Find the limit: $\lim_{x \to -1} \frac{2x^2 - x - 3}{x + 1}$

Example: Find the limit (if it exists): $\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3}$

Example: Find the limit (if it exists): $\lim_{x \to 0} \frac{\left[1/(x+4)\right]-(1/4)}{x}$

The Squeeze Theorem

Our immediate motivation for the squeeze theorem is to so that we can evaluate the following limits, which are necessary in determining the derivatives of sin and cosine:

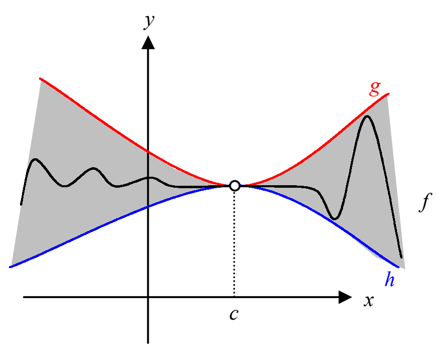
$$\lim_{x \to 0} \frac{\sin x}{x} \text{ and } \lim_{x \to 0} \frac{1-\cos x}{x}$$

Theorem 1.8 The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$, then

$$\lim_{x \to c} f(x) = L.$$

$h(x) \le f(x) \le g(x)$



Theorem 1.9 Two Special Trigonometric Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

2. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Proof:

Example: $\lim_{x \to 0} \frac{\tan x}{x}$

Example: $\lim_{x \to 0} \frac{\sin 4x}{x}$

The position function $s(t) = -16t^2 + h_0$ gives the height (in feet) of an object that has fallen for t seconds from a height of h_0 feet. The velocity at time $t = a$ seconds is given by

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

Example: Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme rocket pack strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Seconds later, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust. How long after Wile E. Coyote jumped, did he hit the road? How fast was he traveling at that moment?