A Square-Root Nyquist (M) Filter Design for Digital Communication Systems

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Abstract

Designing matched transmit and receive filters such that their combination satisfies the Nyquist condition is a classical problem in digital communication systems. In this paper, we propose a novel method for designing such filters. The proposed method is based on a cost function whose minimization leads to designs that can strike a balance between the stopband attenuation, the residual intersymbol interference (ISI), robust sensitivity to timing jitter and/or reduced peak-to-average power ratio (PAR). An iterative algorithm for finding the global minimum of the proposed cost function is suggested and its excellent performance is shown by presenting a variety of design examples.

Index Terms – Nyquist filters, Filter design.

I. Introduction

A classical problem in data communication is to design a pair of matched transmit and receive filters whose cascade is a Nyquist pulse-shape. Mathematically, this problem is phrased as follows. We wish to design a filter $H(z)$, with real-valued coefficients, such that $G(z) = H(z)H(z^{-1})$ satisfies the Nyquist criterion

$$\sum_{k=0}^{M-1} G\left(ze^{-j2\pi kf}\right) = M \tag{1}$$

where M is an integer called the over sampling factor. It indicates the number of filter coefficients per symbol interval. Equation (1) expresses the Nyquist criterion in the frequency domain. In the time domain, the Nyquist criterion finds the form

$$g(n) = \begin{cases} 1, & n = 0 \\ 0, & n = mM,\ m \neq 0 \\ \text{arbitrary}, & n \neq mM \end{cases} \tag{2}$$

where $g(n)$ is the inverse z-transform of $G(z)$. Also, for our further reference later, we note that $g(n) = h(n) \star h(-n)$, where $h(n)$ is the inverse z-transform of $H(z)$ and $\star$ denotes convolution.

A filter $G(z)$ that satisfies (1) is called Nyquist (M), [1], [2]. Moreover, when $|z| = 1$, $G(z) = H(z)H(z^{-1}) = |H(z)|^2$. Thus, $|H(z)| = \sqrt{G(z)}$, and we refer to $H(z)$ as a root-Nyquist (M) filter.

A design $H(z)$ that satisfies the Nyquist conditions (1) and (2) exactly is generally too restrictive and thus may not lead to a satisfactory filter. There are other aspects in a real-world design that one may wish to consider and a design that strikes a good balance between these aspects is often more desirable.

The various aspects that may be considered while designing $H(z)$ are:

1. The length of $H(z)$ should be kept as small as possible to minimize the implementation cost.
2. The Nyquist criterion of (1) and (2) should be satisfied as closely as possible to minimize interference among successive data symbols, when channel distortion is absent/negligible.
3. The transmission bandwidth and the stopband attenuation of $H(z)$ are system parameters. These are often dictated by a frequency mask in the relevant standards and, thus, the designed $H(z)$ must fit within the mask.
4. To provide immunity against timing jitter, the magnitude of the side-lobes of the impulse response $g(n) = h(n) \star h(-n)$ should be reduced.
5. To reduce the peak-to-average power ratio (PAR) of the modulated signal, one should design a pulse-shape $h(n)$ with a reduced tail size. Such modulated signals are useful in applications where power amplifiers with limited dynamic range are available.

Clearly, these are conflicting requirements, and one must give due consideration to the underlying tradeoffs during the design. This is what makes the design of Nyquist filters a challenging task, compared with the conventional filter design. Several techniques exist in the literature for the design of digital Nyquist and/or digital matched filters whose cascade is a Nyquist filter. Some of these works have proposed design approaches that consider items 1 to 4. However, there is very limited work that addresses the problem of PAR. A review of these works is presented in Section II.

The goal of this paper is to give a novel formulation of the design of root-Nyquist (M) filters that takes into account all the above issues and allow the designer to trade among the different aspects. By adopting a soft constraint approach and assigning a selectable weight to each constraint, the designer is given the freedom of tightening or loosening each constraint. As will be demonstrated through a number of design examples, the proposed approach provides the designer with a great deal of flexibility.

We acknowledge that the proposed cost function and the design algorithm developed in this paper follow a heuristic approach and thus may be naive in delivering solutions with proven global optimality. However, as we demonstrate through design examples, it is capable of delivering designs that are superior to those that have been reported in the past literature; see, for instance, the discussion in the next section on the possibility of limiting the designs to linear phase filters and a related design example that is presented in Fig. 1 in Section V-A.

II. Review of Related Works

A number of researchers have limited their study to design of Nyquist filters without any consideration of how such filters my be partitioned between the transmit and receive sides of the channel. Design of FIR Nyquist filters by minimizing the energy in the stopband was reported by Mueller [4], Halpern [5], and Panayirci and Tugbay [6]. Saramaki and Neuvo [7] proposed an iterative technique to design equiripple FIR Nyquist filters, and Vaidyanathan and Nguyen [8] proposed an "eigenfilter" approach for the design of approximately equiripple or nonequiripple Nyquist filters. Maximization of the energy in the middle of the impulse response for achieving robustness to timing jitter was used by Tugbay and Panayirci [9] and Tuncer [11]. Muravchik and Guisantes [10] improved on this approach by enforcing flat zero crossing in the designed pulse-shapes. The latest development along this line is the work of Boonyanant and Tantaratana [13] where by using an affine linear programming algorithm, [14], a class of Nyquist filters with very low sensitivity to timing jitter are reported. However, none of these designs considers separation of the designed filters between the transmitter and receiver.

Techniques for designing matched optimum FIR transmit and receive filters so that their cascade is a Nyquist filter were reported by a number of researchers. Chevillat and Ungerboeck [15] propose a design algorithm based on an iterative projected gradient method. The proposed algorithm finds an optimum root-Nyquist (M) filter with minimum stop-band energy; see equation (11) for the definition of the stopband energy. Salazar and Lawrence [16] and Samueli [17] design equiripple matched transmit and receive filters using the linear programming technique. Coleman and Lytle [18], [19] improve on these designs by introducing a desired frequency mask as a design constraint. Additional constraints that improve on the robustness of the receiver to timing jitters have been added and discussed by Coleman [20]. Further developments along this line can be found in the recent works of Davidson *et al.* [21], [22]. Other related works are [23], [24], [25], and [26]. With the exception of [15], the general approach that has been taken in these works is to design a (quasi-)Nyquist filter $G(z)$ first and then factorize it into a pair of matched transmit and receive filters $H(z)$ and $H(z^{-1})$. The filters $H(z)$ and $H(z^{-1})$ that are obtained in this way, are always non-linear phase.

The approach that is taken in this paper is different from the above works in the sense that it designs $H(z)$ directly. As we demonstrate throughout the paper, direct design of $H(z)$ (at least) has the following advantages:

- To reduce the complexity of realization, $H(z)$ can be constrained to be linear phase.
- Since there is a direct control over the transmit pulse shape, $H(z)$ can be designed for a reduced PAR.

Many of the works that were mentioned above, and some by others, [27], [28], [29], include the impact of the analog filters at the transmitter and/or receiver in their designs. The design formulation that is presented in this paper does not take the impact of analog filters into account. This, we believe, is a proper approach, since the current trend in the industry is to implement most of the transmit and receive processing in the digital domain. Analog filters are only applied at an intermediate frequency (IF) or even at the radio frequency (RF) band where the baseband signals have already gone through digital interpolation with a relatively large interpolation factor. In such systems analog filters have no significant impact on the equivalent baseband channel model and thus may be ignored while designing the pulse-shaping/matched filters. However, there might be some distortion introduced by the digital filters that are used for interpolations/decimations. For instance, to reduce the complexity of both transmitter and receiver, cascaded integrator-comb (CIC) filters [3], that may introduce significant baseband distortion, are commonly used in practice. Although it is possible to modify the design procedure that is presented in this paper to include and mitigate the distortion arising from other blocks in the transmission path (when they are known *a priori*), we have chosen to limit our presentation to the case where interpolation and decimation filters have an ideal response over the band of interest.

III. Problem Formulation

The problem of designing a root-Nyquist (M) filter

$$H(z) = \sum_{n=0}^{N} h(n) z^{-n} \tag{3}$$

may be formulated as follows. Let $\mathbf{h} = \begin{bmatrix} h(0) & h(1) & \cdots & h(N) \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{e}(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-N} \end{bmatrix}^{\mathrm{T}}$, where the superscript T denotes transposition. Note that (3) may be written as

$$H(z) = \mathbf{h}^{\mathrm{T}} \mathbf{e}(z). \tag{4}$$

Using (4) and recalling that $G(z) = H(z)H(z^{-1})$, we get

$$\begin{aligned} G(z) &= \left(\mathbf{h}^{\mathrm{T}}\mathbf{e}(z)\right)\left(\mathbf{h}^{\mathrm{T}}\mathbf{e}(z^{-1})\right)^{\mathrm{T}} \\ &= \mathbf{h}^{\mathrm{T}}\mathbf{e}(z)\mathbf{e}^{\mathrm{T}}(z^{-1})\mathbf{h} \\ &= \mathbf{h}^{\mathrm{T}}\mathbf{R}(z)\mathbf{h}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \mathbf{R}(z) &= \mathbf{e}(z)\mathbf{e}^{\mathrm{T}}(z^{-1}) \\ &= \begin{pmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-N} \end{pmatrix} \begin{pmatrix} 1 & z & \cdots & z^{N} \end{pmatrix} \\ &= \sum_{n=-N}^{N} z^{-n}\mathbf{S}_n \end{aligned} \tag{6}$$

and $\mathbf{S}_n$ are constant matrices whose elements are given by

$$[\mathbf{S}_n]_{k,l} = \begin{cases} 1, & k-l=n \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Substituting (6) in (5), we obtain

$$G(z) = \sum_{n=-N}^{N} \left(\mathbf{h}^{\mathrm{T}}\mathbf{S}_n\mathbf{h}\right) z^{-n}. \tag{8}$$

For $G(z)$ to be a Nyquist (M) filter, $\mathbf{h}$ has to be chosen such that

$$\mathbf{h}^{T}\mathbf{S}_n\mathbf{h} = \begin{cases} 1, & n=0 \\ 0, & n=mM,\ m\neq 0 \\ \text{arbitrary}, & n\neq mM. \end{cases} \tag{9}$$

These are a set of constraints that must be imposed while optimizing the coefficients of $H(z)$.

On the other hand, we note that $H(z)$ is a lowpass filter and as part of the design goal the magnitude response of $H(z)$ over its stopband has to be minimized. Following the notation of the (square-root) raised-cosine filters [30], assuming a rolloff factor α, and recalling that $H(z)$ is to be designed for a sampling rate M times faster than the symbol rate, we find that the stopband of $H(z)$ starts at the normalized frequency

$$f_o = \frac{1+\alpha}{2M} \tag{10}$$

and ends at $1-f_o$. Noting this, we define the cost function

$$\xi_s = \int_{f_o}^{1-f_o} |H(e^{j2\pi f})|^2 df \tag{11}$$

and as part of the design we seek an $H(z)$ that results in a small ξ_s. Moreover, recalling that according to the Parseval's relation $\mathbf{h}^{\mathrm{T}}\mathbf{h} = \int_0^1 |H(e^{j2\pi f})|^2 df$, we rarrange (11) as

$$\xi_s = \mathbf{h}^{\mathrm{T}}\mathbf{h} - \int_{-f_o}^{f_o} |H(e^{j2\pi f})|^2 df. \tag{12}$$

We also note that

$$\begin{aligned} \int_{-f_o}^{f_o} |H(e^{j2\pi f})|^2 df &= \int_{-f_o}^{f_o} \mathbf{h}^{\mathrm{T}}\mathbf{e}(e^{j2\pi f})\mathbf{e}^{\mathrm{T}}(e^{-j2\pi f})\mathbf{h} df \\ &= \mathbf{h}^{\mathrm{T}} \left(\int_{-f_o}^{f_o} \mathbf{e}(e^{j2\pi f})\mathbf{e}^{\mathrm{T}}(e^{-j2\pi f}) df \right) \mathbf{h}. \end{aligned} \tag{13}$$

Substituting (13) in (11), we obtain

$$\xi_s = \mathbf{h}^{\mathrm{T}}\mathbf{\Phi}\mathbf{h} \tag{14}$$

where

$$\mathbf{\Phi} = \mathbf{I} - \int_{-f_o}^{f_o} \mathbf{e}(e^{j2\pi f})\mathbf{e}^{\mathrm{T}}(e^{-j2\pi f}) df. \tag{15}$$

Performing the relevant integrals, the elements of $\mathbf{\Phi}$ are obtained as

$$\phi_{kl} = \begin{cases} 1 - 2f_o, & k = l \\ -2f_o \mathrm{sinc}(2f_o(k-l)), & k \neq l. \end{cases} \tag{16}$$

To summarize, the design of a root-Nyquist (M) filter is performed by minimizing the cost function ξ_s of (14), subject to the constraints (9). Chevillat and Ungerboeck [15] have also stated a slightly different formulation of the same problem and proposed an iterative projected gradient procedure for solving it. The problem formulation in [15] is different from the above in the sense that it is based on the maximization of the spectral energy of $H(e^{j2\pi f})$ over the passband range $-f_o < f < f_o$, as opposed to our formulation that seeks the minimization of the spectral energy of $H(e^{j2\pi f})$ over the stopband range $f_o < f < 1 - f_o$. When the aim is to satisfy constraints (9) perfectly, both solutions lead to the same design. However, as discussed below, since we set a design goal that strikes a balance between the accuracy of the constraints (9) and the stopband gain of $H(e^{j2\pi f})$ (and other design goals that will be introduced as we proceed), the cost function ξ_s, as will be found, is the natural choice.

IV. Design Procedure

In most applications, FIR filters are designed to have a linear phase response. Among different reasons, a major motivation to design FIR filters with constrained linear phase has been to reduce the cost of realization of digital filters [31]. This is because linear phase translates to a symmetry in filter coefficients and this in turn allows one to cut the number of multipliers to one half when the filter is programmed on a digital signal processor or implemented on a custom chip. Noting this, we limit our study and give all the derivations for the cases where $H(z)$ is a linear phase FIR filter. A numerical example that compares the computational complexity of an existing (optimal) design and a design obtained using the method of this paper is presented in the next section.

To include the symmetry of the filter coefficients in the design formulation, we define the column vector $\mathbf{h}'$ as

$$\mathbf{h}' = \left[\begin{array}{cccc} h(0) & h(1) & \cdots & h((N-1)/2) \end{array} \right]^T$$

when N is odd, and

$$\mathbf{h}' = \left[\begin{array}{cccc} h(0) & h(1) & \cdots & h(N/2) \end{array} \right]^T$$

when N is even. The vector $\mathbf{h}$ thus may be written in terms of $\mathbf{h}'$ as

$$\mathbf{h} = \mathbf{E}\mathbf{h}' \tag{17}$$

where $\mathbf{E} = \left[\begin{array}{c} \mathbf{I} \\ \mathbf{J} \end{array} \right]$, $\mathbf{I}$ is the identity matrix and $\mathbf{J}$, for N odd, is the antidiagonal matrix with the antidiagonal elements of 1 and, for N even, is obtained by removing the first row of the latter antidiagonal matrix. Clearly, the sizes of $\mathbf{I}$ and $\mathbf{J}$ should be chosen such that they will be compatible with $\mathbf{h}'$. Using (17), (9) and (14) are, respectively, rearranged as

$$\mathbf{h}'^T \mathbf{S}'_n \mathbf{h}' = \begin{cases} 1, & n = 0 \\ 0, & n = mM, \; m \neq 0 \\ \text{arbitrary}, & n \neq mM. \end{cases} \tag{18}$$

and

$$\xi_s = \mathbf{h}'^{\mathrm{T}} \mathbf{\Phi}' \mathbf{h}', \tag{19}$$

where

$$\mathbf{S}'_n = \mathbf{E}^{\mathrm{T}} \mathbf{S}_n \mathbf{E} \tag{20}$$

and

$$\mathbf{\Phi}' = \mathbf{E}^{\mathrm{T}} \mathbf{\Phi} \mathbf{E}. \tag{21}$$

The equalities defined by (18) suggest a set of hard constraints which may be unnecessary in an actual design. By relaxing these constraints, one will gain by reducing ξ_s to a lower value, i.e., in improving the stopband attenuation. We also note that to improve on the received signal's robustness to timing jitter, one may extend (18) to include other elements of $g(n)$, e.g., the samples in the tails of $g(n)$. For this purpose, we replace (18) by the set of *soft* equalities

$$\mathbf{h}'^{T}\mathbf{S}'_n\mathbf{h}' \approx d_n, \quad n = 0, 1, \cdots, N \tag{22}$$

where d_n are a set of desired/target values.

Next, to combine (19) and (22), we first note that the set of equations (22) may be combined and written in the compact form

$$\mathbf{B}\mathbf{h}' \approx \mathbf{d} \tag{23}$$

where

$$\mathbf{B} = (\mathbf{I} \otimes \mathbf{h}'^{\mathrm{T}})\mathbf{S}', \tag{24}$$

$\otimes$ denotes the Kronecher product, $\mathbf{I}$ is the identity matrix of size $N+1$,

$$\mathbf{S}' = \left[\mathbf{S}_0'^{\mathrm{T}}\ \mathbf{S}_1'^{\mathrm{T}}\ \cdots\ \mathbf{S}_N'^{\mathrm{T}}\right]^{\mathrm{T}} \tag{25}$$

and

$$\mathbf{d} = [d_0\ d_1\ \cdots\ d_N]^{\mathrm{T}}. \tag{26}$$

We also apply the Cholesky factorization to expand $\mathbf{\Phi}'$ as $\mathbf{\Phi}' = \mathbf{C}^{\mathrm{T}}\mathbf{C}$, where $\mathbf{C}$ is an upper triangular matrix and use this to rearrange (19) as

$$\xi_s = (\mathbf{C}\mathbf{h}')^{\mathrm{T}}\mathbf{C}\mathbf{h}' = \|\mathbf{C}\mathbf{h}'\|^2. \tag{27}$$

Here, $\|\cdot\|^2$ denotes the norm of a vector. From this, we argue that to minimize ξ_s, one may choose to minimize the length of the vector $\mathbf{C}\mathbf{h}'$. Accordingly, we may also say that as part of our design goal, we wish to find a vector $\mathbf{h}'$ which also satisfies the soft equation

$$\mathbf{C}\mathbf{h}' \approx \mathbf{0} \tag{28}$$

where $\mathbf{0}$ is a column vector with zero elements.

Combining (23) and (28), we get

$$\mathbf{D}\mathbf{h}' \approx \mathbf{u} \tag{29}$$

where $\mathbf{D} = \begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix}$. The approximation (29) is an over-determined system of soft equations for which we seek a solution for the unknown vector $\mathbf{h}'$. We also note that since some of the rows of $\mathbf{D}$ contain linear combination of the elements of $\mathbf{h}'$, (29) is quadratic in $\mathbf{h}'$.

To solve (29), we define the error vector

$$\mathbf{v} = \mathbf{\Gamma}(\mathbf{D}\mathbf{h}' - \mathbf{u}) \tag{30}$$

where $\mathbf{\Gamma}$ is a diagonal matrix whose diagonal elements are a set of weights given to the elements of the difference $\mathbf{D}\mathbf{h}' - \mathbf{u}$. Larger weights are assigned to those elements whose minimization should be emphasized. Zero weight is assigned to those elements that should be treated as *don't care.* The optimum value of $\mathbf{h}'$ is obtained by minimizing the norm of the vector $\mathbf{v}$, i.e., the cost function

$$\xi = \|\mathbf{v}\|^2. \tag{31}$$

We note that since $\mathbf{D}\mathbf{h}'$ is quadratic in $\mathbf{h}'$, ξ is is a fourth order function of $\mathbf{h}'$. Hence, (31), in general, is a multi-modal function and its global minimum can only be found iteratively, if a proper initial choice of $\mathbf{h}'$ (within the vicinity of its global minimum) could be made. Through numerical experiments, we found that the root-raised-cosine pulse is a good initial choice for $\mathbf{h}$. However, we acknowledge that this initial choice cannot guarantee convergence of $\mathbf{h}$ to the global minimum of ξ. Nevertheless, many examples that we tried using this initial choice led to designs that we argue are at least near optimum because of their very good responses, both in the time- and frequency-domain.

Table I summarizes the above results and lists an iterative algorithm for finding the vector $\mathbf{h}$ that minimizes (31). In this algorithm, the steps listed under iterations are executed multiple times until $\mathbf{h}'$ converges. It is also worth noting that the algorithm presented in Table I is similar to those that have been developed in [32] and [33] and successfully used in designing filter banks.

The MATLAB function 'rNyquistM.m', presented in Appendix A, is used to generate all the numerical results in the next section. This program has the following inputs:

- `N:` The filter order.
- `M:` The oversampling factor.
- `alpha:` The rolloff factor, α
- `gmaZ:` The weight factor for the center coefficient and all the zero-crossing points in $g(n)$. It is defined below as γ.

TABLE I
SQUARE-ROOT NYQUIST (M) FILTER DESIGN ALGORITHM.

Inputs
N: filter order M: oversampling factor α: rolloff factor $\boldsymbol{\Gamma}$: diagonal matrix of weight factors.
Initialization
○ Construct $\mathbf{S}'$ using (7), (20) and (25). ○ Construct $\boldsymbol{\Phi}'$ using (16) and (21). ○ Apply Cholesky factorization to obtain $\mathbf{C}$ from $\boldsymbol{\Phi}' = \mathbf{C}^{\mathrm{T}}\mathbf{C}$. ○ Choose a desired/target vector $\mathbf{d}$ and form the vector $\mathbf{u}$, accordingly. ○ Construct the initial vector $\mathbf{h}'_0$ from the samples of a square-root raised-cosine pulse-shape with the rolloff factor α. ○ Let $i = 0$.
Iterations
○ $\mathbf{B} = (\mathbf{I} \otimes \mathbf{h}_i'^{\mathrm{T}})\mathbf{S}'$ ○ $\mathbf{D} = \begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix}$ ○ $\mathbf{h}' = \left(\mathbf{D}^{\mathrm{T}}\boldsymbol{\Gamma}^2\mathbf{D}\right)^{-1}\mathbf{D}^{\mathrm{T}}\boldsymbol{\Gamma}^2\mathbf{u}$ ○ $\mathbf{h}'_{i+1} = (\mathbf{h}'_i + \mathbf{h}')/2$ ○ Increment i
Final step
○ $\mathbf{h}' = \mathbf{h}'_i$ ○ Construct $\mathbf{h}$ from $\mathbf{h}'$

• `gmaT`: The weight factor for the tails of $g(n)$. It is defined below as γ' and is used to improve the robustness of the receiver to timing phase error/jitter.

• `eta`: The weight factor for the tails of $h(n)$. It is defined below as η and is used to improve on the peak-to-average power ratio (PAR) of the transmit signal.

Using this function, an interested reader can replicate all the results that are presented in the following section.

It is also worth noting that while the diagonal elements of $\boldsymbol{\Gamma}$ can, in general, be selected independently, in the MATLAB function 'rNyquistM.m', we have chosen to limit these choices to the three independent variables γ, γ', and η, in order to reduce the design parameters. Our experiments, as demonstrated in the next section, through a variety of numerical examples, show that these parameters provide sufficient flexibility to design filters with very good responses.

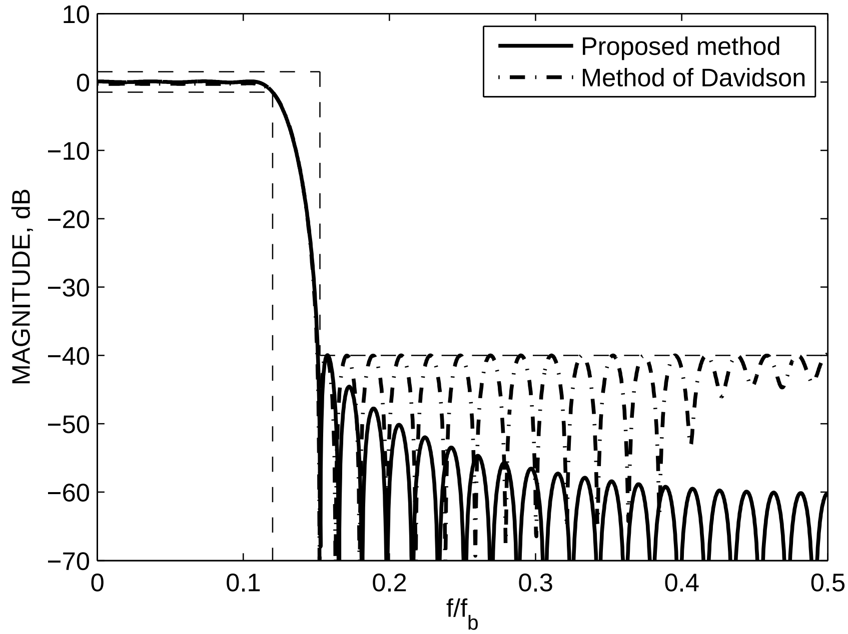

Fig. 1. Magnitude responses of two root-Nyquist (M) filters, designed by the method of this paper and the method of Davidson [22]. Dashed lines show the IS95 frequency mask.

V. Numerical Examples

A. *Comparison with existing designs*

To give an idea of how our designs compare with those in the past literature, we pick a filter that was designed by Davidson and presented in Fig. 5(b) of [22] and compare it with a filter that we design using our proposed algorithm. Both filters are designed to satisfy the requirement imposed by the IS95 standard. For our design, we run 'rNyquistM.m' with the parameters $N = 52$, $M = 4$, $\alpha = 0.19$, $\gamma = 0.4$, $\gamma' = 0$ and $\eta = 0$. The magnitude responses of the two designs are presented in Fig. 1. The Davidson's design is a non-linear phase filter with 48 coefficients. Our design is a linear phase filter with 53 coefficients. Hence, while the Davidson's filter requires 48 multiplications and 47 additions, to calculate each output sample of the filter, our design requires only 27 multiplications and 52 additions. Moreover, our design has a larger stopband attenuation than the Davidson's design. Furthermore, to make sure that a fair comparison is presented, we compare the two designs by exploring their peak ISI. Peak ISI, as defined in [22], is given by the summation $\sum_{n \neq 0} |g(n)|$. Direct numerical evaluation of the two designs show that the peak ISI in the Davidson filter is 0.0490 and in the filter designed here it is 0.0325. Hence, our design is superior to that of Davidson's in all three aspects that were explored here; namely, complexity, stopband attenuation, and peak ISI.

In the following, we proceed with the presentation of a number of designs whose goal is to demonstrate the capabilities of the proposed algorithm as well as to familiarize the reader with the MATLAB function 'rNyquistM.m'. Also, to evaluate our designs, they are compared with the truncated root-raised-cosine filters of the same length. We have chosen the root-raised-cosine filter for these comparisons (and not the more elegant designs that have been reported in other works, e.g., [18]-[22]), because this is the only known linear phase design and the most widely used pulse-shaping filter in practice. Moreover, the example given above shows that the linear phase property of our design allows us to increase the filter length and remain competitive with the existing (optimal) design methods that, in general, are non-linear phase.

B. Reducing the residual ISI and/or improving on the stopband attenuation

Tables II and III present the results of a series of Nyquist (M) filters that are designed using the MATLAB function 'rNyquistM.m'. The results compare the designed filters with the truncated root-raised-cosine filters of the same length.

Table II compares the stopband attenuation of the two designs according to the formula

$$\rho_{\rm SB} = 10 \log \frac{\int_{f_0}^{1-f_0} |H_{\rm rrc}(e^{j2\pi f})|^2 df}{\int_{f_0}^{1-f_0} |H_{\rm rNyq}(e^{j2\pi f})|^2 df} \tag{32}$$

where $H_{\rm rNyq}(e^{j2\pi f})$ and $H_{\rm rrc}(e^{j2\pi f})$ are the frequency responses of the root-Nyquist (M) and root-raised-cosine filters, respectively.

Table III presents the relative ISI level of the two designs when both are sampled optimally at the middle of the corresponding pulse-shapes. The relative ISI level is defined as

$$\rho_{\rm ISI} = 10 \log \frac{\sum\limits_{n=mM, m\neq 0} g_{\rm rrc}^2(n)}{\sum\limits_{n=mM, m\neq 0} g_{\rm rNyq}^2(n)} \tag{33}$$

where $g_{\rm rNyq}(n)$ and $g_{\rm rrc}(n)$ are the pulse-shapes resulting from the root-Nyquist (M) and root-raised-cosine designs, respectively.

To obtain the results of Tables II and III the diagonal elements of the weight factor matrix $\mathbf{\Gamma}$ are selected as follows. Unit weights are assigned to the elements of $\mathbf{Ch}'$. These elements are related to and control the stopband attenuation of the filter. A weight factor γ is assigned to the elements of $\mathbf{Bh}'$ that correspond to the constraints (9), i.e., the zeroth, Mth, $2M$th, $\cdots$ elements of $\mathbf{Bh}'$. Zero weights are

TABLE II

THE RELATIVE STOPBAND ATTENUATION, (32), OF THE ROOT-NYQUIST (M) AND THE TRUNCATED ROOT-RAISED-COSINE FILTERS.

		γ			
	N	0.5000	1.0000	2.0000	10.0000
$\alpha = 0.5$	20	1.6178	0.4063	0.0740	−0.1704
	30	9.3794	9.0566	8.9711	8.0121
	40	19.1303	19.0376	18.9644	12.5494
	50	30.1334	29.9167	28.0668	14.9912
	60	41.0508	36.3266	29.9738	24.0538
$\alpha = 0.25$	20	3.5810	−1.0794	−2.7703	−3.3811
	30	2.5102	−0.4043	−1.3489	-1.7130
	40	2.2428	0.4603	−0.0709	-0.3763
	50	3.9911	2.9134	2.6099	2.2819
	60	8.9934	8.3452	8.1682	7.6964

TABLE III

THE RELATIVE ISI LEVEL, (33), IN THE ROOT-NYQUIST (M) AND THE TRUNCATED ROOT-RAISED-COSINE FILTERS.

		γ			
	N	0.5000	1.0000	2.0000	10.0000
$\alpha = 0.5$	20	−18.7242	−8.2342	0.2200	3.4130
	30	12.6339	20.4501	22.3180	23.6261
	40	2.9576	4.0373	4.2225	6.8532
	50	23.0144	23.2353	23.9055	33.4174
	60	9.5795	11.1440	15.1168	25.9084
$\alpha = 0.25$	20	−4.0740	3.3571	13.7849	30.0271
	30	−5.2105	3.7271	14.4272	25.5533
	40	−16.2490	−6.3214	3.5310	9.5329
	50	−0.1070	10.2161	17.8382	20.7652
	60	9.3846	19.1388	23.5555	24.9116

assigned to the rest of the elements of $\mathbf{Bh}'$. Accordingly, by increasing γ, one can make the constraints (9) tighter, i.e., improve on residual ISI. This will be at the cost of some loss in the stopband attenuation of the designed filters. The results presented in Tables II and III show how the stopband attenuation and residual ISI can be traded against each other by varying the weight factor γ. We also remind the reader that the weight factor γ is called `gmaZ` in the function 'rNyquistM.m', and to obtain the results of Tables II and III we have set `gmaT` and `eta` equal to zero.

Next, to further explore the results of the Nyquist (M) designs, we pick one of the designs from Tables II and III and study the corresponding time-domain and frequency-domin responses. Let us consider the case where $\alpha = 0.5$, $N = 30$, and $\gamma = 2$. In this case, there is a moderate 8.97 dB improvement in the stopband attenuation and a significant 22.3 dB improvement in the residual ISI level. The magnitude responses of the two designs are presented in Fig. 2. As predicted above, the root-Nyquist (M) filter has a better stopband attenuation. Also, to show the impact of the reduced ISI, in Figs. 3(a) and (b), we have presented the received signal constellations when a 64-QAM sequence is passed through a pair of matched filters obtained from the two designs. That is, we have assumed an ideal (i.e., distortionless) channel and no additive noise. As observed, the residual ISI arising from the root-raised-cosine design results in spread of constellation points. On the other hand, the root-Nyquist (M) design results in constellation points with no noticeable spreading.

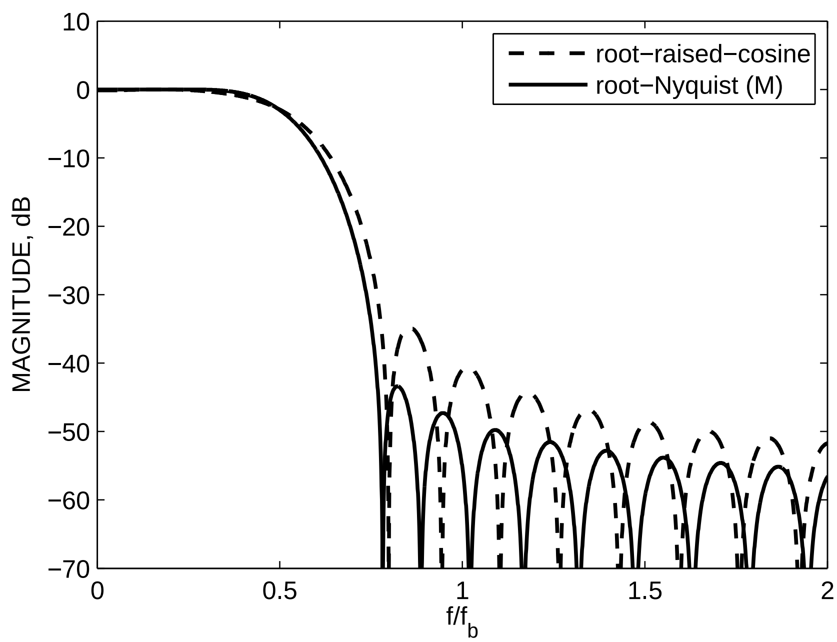

Fig. 2. Magnitude responses of a root-Nyquist (M) filter and a root-raised-cosine filter. The filter parameters in both cases are $M = 5$, $N = 30$ and $\alpha = 0.5$.

It appears that by adopting the proposed design strategy, one can gain both in the time and frequency domain. Is this true? We answer this question by exploring the impulse response of the system, assuming an ideal channel. Such impulse responses which are obtained by combining a pair of transmit and receive filters from each design are presented in Figs. 4. From these plots, we make the following observation. The tails of the system impulse response are larger in the case of root-Nyquist (M). Larger tail in the impulse response results in a higher sensitivity to timing jitter [9], [11], [25]. This is the price paid for higher attenuation in the stopband and lower residual ISI.

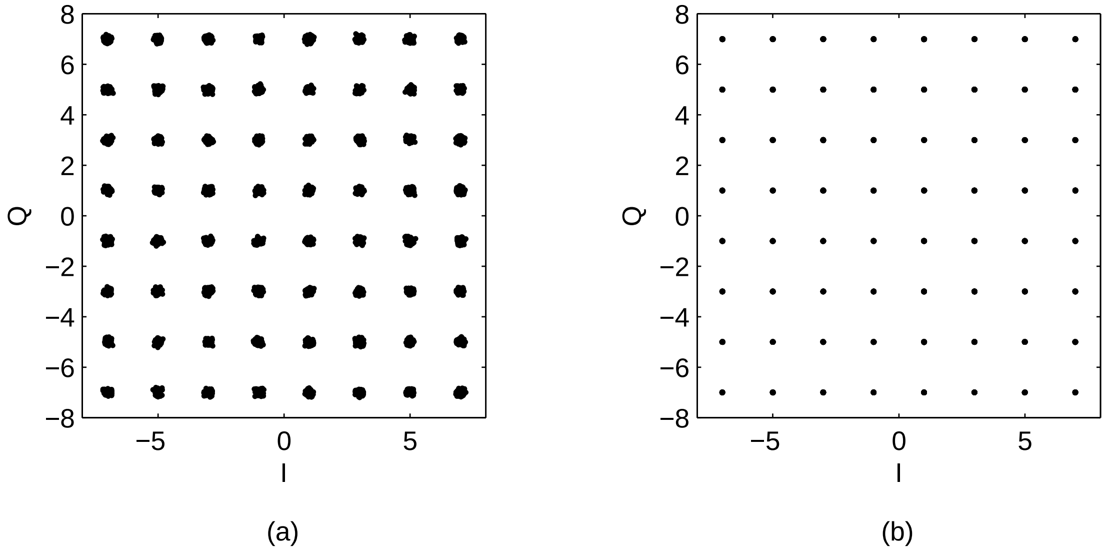

Fig. 3. Symbol constellation/eye pattern of a 64-QAM symbol sequence after passing through a pair of matched filters. (a) root-raised-cosine filter. (b) root-Nyquist (M) filter. Filter parameters are the same as those in Fig. 2.

C. Designs with robust behavior against timing jitter

A conclusion that may be derived from the above discussion is that to reduce the sensitivity of a communication system to timing jitter, one may choose to design a pulse-shape $h(n)$ that leads to a combined impulse response $g(n) = h(n) \star h(-n)$ with reduced tail sizes. This can easily be done within the design frame work that was developed in this paper. In order to reduce the tail sizes, we simply assign some non-zero weights to the elements of $\mathbf{Bh}'$ that correspond to the tails of $g(n)$. In the designs that were presented in Tables II and III and also in Figs. 2 to 4, we only emphasized on the response samples at the zero crossings, and at the middle of the pulse where the desired amplitude has to be unity. Here, we extend the constraints and include all the samples in the tails of the response. Fig. 5 presents the magnitude response of a root-Nyquist (M) filter that was designed in this way. Also shown, for comparison, is the magnitude response of a root-raised-cosine filter of the same length. For these designs we have chosen $M = 5$, $N = 40$ and $\alpha = 0.5$. Fig. 6 presents signal constellations at the output of the matched filter at the receiver, for the timing phases of 0, 5, 10 and 15% of a symbol period. As expected, the proposed Nyquist (M) design is less sensitive to timing offset. For instance, at the timing offset of 15%, in the raised-cosine design the received symbols begin to overlap and errors can occur, even in the absence of channel noise. On the other hand, for the same timing offset, in the Nyquist (M) design, the constellation clusters remain separated. The cost for this robust behavior, as observed in Fig. 5, is a 5

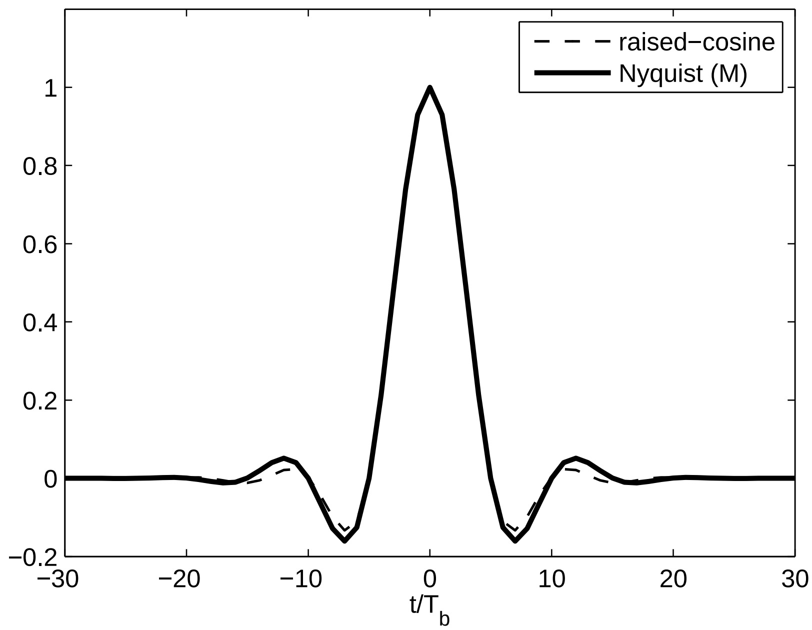

Fig. 4. Combined response of a pair of matched root-raised-cosine and root-Nyquist (M) filters. Filter parameters are the same as those in Fig. 2.

dB loss in the stopband.

The above design was arrived at through a few trials. In order to strike a balance between the reduced sensitivity to the timing offset and the stopband response of the design, we tried different choices of the weight factors. The design whose results are presented in Figs. 5 and 6 is obtained by using the following weight factors. A weight factor $\gamma = 5$ is assigned to the elements of $\mathbf{Bh}'$ that correspond to the zero crossing points and middle coefficient of the filter. A weight factor $\gamma' = 0.5$ is assigned to the rest of the tail samples of the response, i.e., to the elements of $g(n)$, for $|n| > M$ and $n \neq mM$, where m is an integer.

In our research we found that because of the non-linearity of the problem, there is no straightforward way of developing an algorithm for designing filters of the type discussed above. However, we found that it is also not difficult to come up with a design that strikes a good balance between the stopband attenuation and a desired response in the time domain, through a trial and error approach. What one needs to do is to try a few choices of the parameters γ and γ' (`gmaZ` and `gmaT`, respectively, in the MATLAB function 'rNyquistM.m') and improve on the design by observing the trend of the results (the filter response, constellation diagrams, ...) as γ and γ' change.

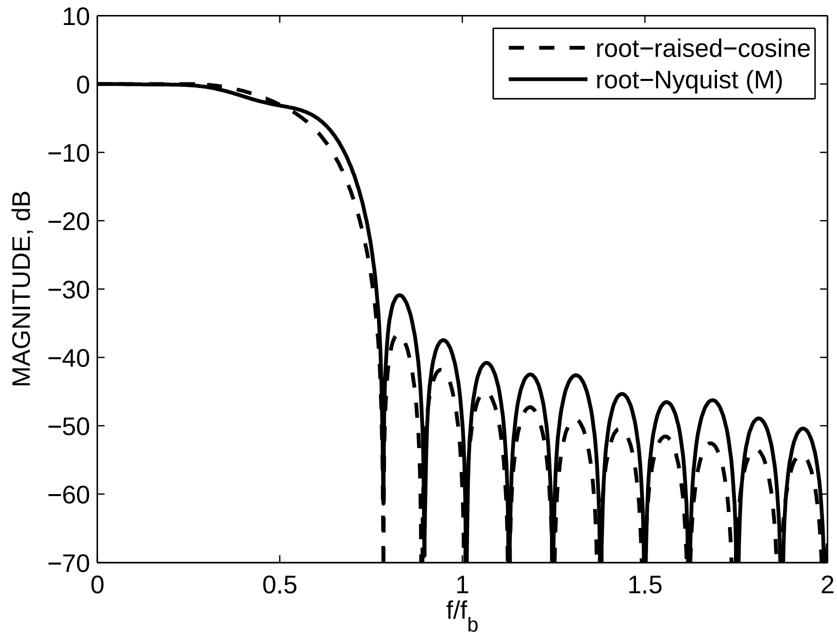

Fig. 5. Magnitude responses of a root-Nyquist (M) and a root-raised-cosine filter. The Nyquist filter is designed for robust behavior with respect to the timing phase error. The filters parameters are $M = 5$, $N = 40$ and $\alpha = 0.5$.

D. Designs with reduced peak-to-average power ratio (PAR)

The PAR is defined as the ratio of the peak signal power over the average power at the transmitter output. Signal peaks often arise at a timing phase where the transmit signal suffers from significant ISI; as an example, see Fig. 8. Since the amount of this ISI is dominantly determined by the tails of the transmitter filter $h(n)$, to reduce PAR, one may choose to reduce the size of the tails of $h(n)$. This can easily be included in the design formulation of this paper by adding an additional term $\eta \sum_{n \in \mathcal{T}_h} h^2(n)$ to the cost function ξ, where η is a weight factor and $\mathcal{T}_h$ is the set of n indices that correspond to the tails of $h(n)$. This design parameter is included and called `eta` in the MATLAB function 'rNyquistM.m'.

Fig. 7 presents the magnitude response of a root-Nyquist (M) filter that has been designed with the goal of reducing PAR. This filter is designed for a rolloff factor 0.25 and has the length of $N = 40$ and the oversampling factor $M = 5$. The response of the root-raised-cosine of the same length is also shown for comparison. Fig. 8 presents the eye patterns of binary transmit baseband signals for both cases. As seen, the signal arising from the root-raised-cosine design has a wider scattering than its counterpart from the root-Nyquist (M) design. Moreover, a detailed evaluation of the transmit signals arising from the two designs reveals that the root-Nyquist (M) design has a PAR which is 1.25 dB lower than that of the root-raised-cosine design. This gain comes at the cost of some distortion in the passband; see Fig. 7.

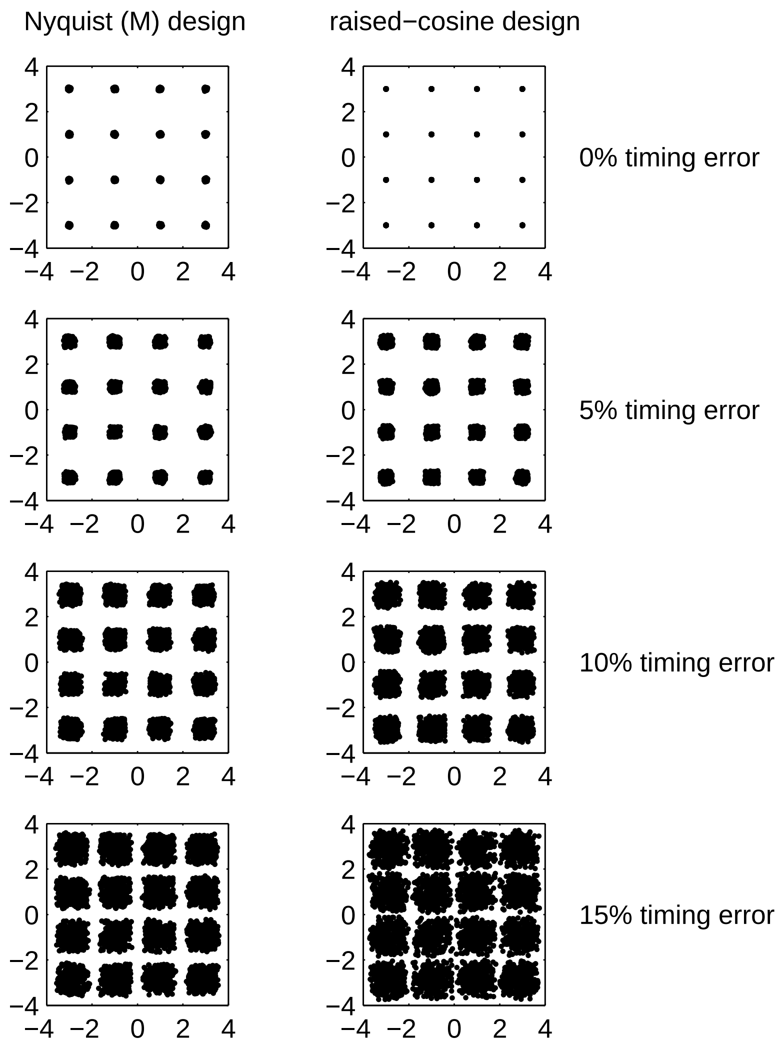

Fig. 6. Demonstration of the robust behavior of a Nyquist (M) filter to timing phase offset, through eye patterns. The eye patterns arising from a raised-cosine pulse are presented for comparison.

This distortion has some impact on the match filtered signal at the receiver. As a result, the latter may suffer from some residual ISI, even in the absence of channel distortion. However, fortunately, since the level of distortion is low, it may be tolerable in the cases of small constellations, or can be compensated easily by using a short length equalizer if required. A loss in the stopband attenuation may also be seen for other designs, not shown here. This is the price one should pay to obtain a lower PAR.

The weight factors that were used in the MATALB function 'rNyquistM.m' for the above design are: $\gamma = 0.1$, $\gamma' = 0$, and $\eta = 0.3$. These choices are obtained through a few trial an error attempts.

VI. Conclusion

We developed a generic cost function that could be used to design a wide range of transmit/receive filters in the application of digital communication systems. An iterative algorithm for minimization of the

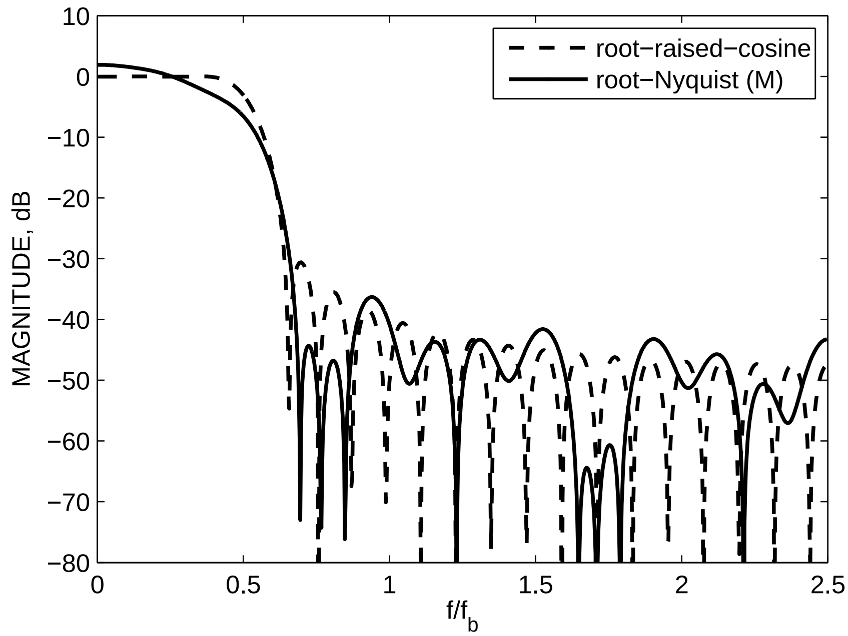

Fig. 7. Magnitude responses of a root-Nyquist (M) and a root-raised-cosine filter. The Nyquist filter is designed for a reduce PAR. The filters parameters are $M = 5$, $N = 40$ and $\alpha = 0.25$.

proposed cost function and a MALTAB function for its implementation were presented. By choosing an example from the past literature and designing for the same specifications, using the proposed algorithm, it was demonstrated that one can design a filter with superior performance and lower computational complexity. This ability of the algorithm was attributed to the fact that it designs filters that are linear phase, and thus have symmetrical coefficients could be used to reduce the complexity of its realization. The previous methods in the literature design filters that are non-linear phase. In addition, we presented a number of design examples that demonstrated the capabilities as well as the versatility of the proposed method. Compared with a truncated root-raised-cosine filter, the presented designs were superior both with respect to stopband attenuation and residual intersymbol interference. Moreover, we showed that the proposed method allows designing filters that provide robustness to timing jitter and/or result in a reduced peak-to-average power ratio (PAR). Filter design for reduced PAR is a unique feature of our design technique that to the best of our knowledge have not been reported in any work in the past.

Acknowledgment

The author is grateful to Dr. Timothy Davidson, the associate editor in charge of this paper, for his numerous constructive comments that improved the quality of this paper significantly. Dr. Davidson also provided the coefficients of the filter that was used for comparisons in Section V-A.

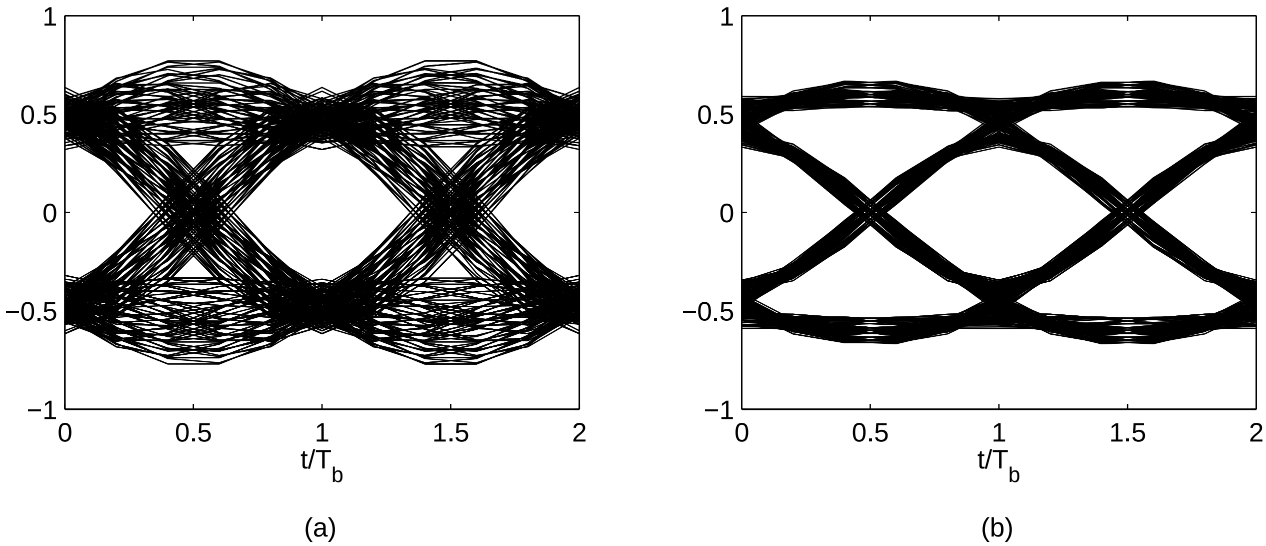

Fig. 8. Eye patterns of transmit signals in a system that uses a root-raised-cosine filter, (a); and in a system that uses a root-Nyquist (M) filter, (b).

Appendix A: MTALAB Functions for Root-Nyquist (M) Filter Design

```
function h=rNyquistM(N,M,alpha,gmaZ,gmaT,eta,itns);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% root-Nyquist (M) filter design                          %
% parameters:                                             %
% N: filter order (filter length = N+1)                   %
% M: number of samples per symbol period                  %
% alpha:  rolloff factor (range 0 to 1)                   %
% gmaZ: Weight factor for middle tap and zero crossings %
% gmaT: Weight factor for the tails of g=h*h (used for  %
%    designs with robust behavor against timing jitter) %
% eta:  Weight factor for tails of h (used for designs  %
%              with reduced PAR)                          %
% itns:  Number of iterations for the least-squares     %
%                  optimization                           %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Set up the weight matrix Gamma %%%
Gamma=zeros(1,1+N); Gamma(M+2:end)=gmaT;
Gamma(1:M:end)=gmaZ; Gamma=[Gamma ones(1,1+N/2)];
Gamma2=Gamma.^2; Gamma2=diag(Gamma2);
%%% Initial filter %%%
h=r_cos_p(N,M,alpha);
if rem(N+1,2)==0 h1=h(1:(N+1)/2);else h1=h(1:N/2+1); end
Lh1=length(h1);
%%% Set up constraint matrices, Sn %%%
S=zeros(N+1,N+1,N+1); temp=ones(N+1,1);
for n=1:N+1
    S(:,:,n)=spdiags(temp,-(n-1),N+1,N+1);
end
%%% Set up the matrix Phi %%%
Phi=zeros(N+1,N+1); fo=(1/2/M)*(1+alpha);
Phi=[1-2*fo -2*fo*sinc(2*fo*[1:N])]; Phi=toeplitz(Phi);
Phi=Phi+1e-10*eye(size(Phi)); %to stabilize Chol fac.
%%% Form the matrices S' and Phi' %%%
I=eye(Lh1); J=hankel([zeros(Lh1-1,1); 1]);
```

```
if rem(N+1,2)==1 J=J(2:end,:); end
E=[I; J]; Phi1=E'*Phi*E; S1=[];
for n=1:N+1  S1=[S1; E'*S(:,:,n)*E];  end
%%% Add tail constraint to reduce PAR %%%
X=zeros(Lh1,1); X(1:end-M)=eta; Phi1=Phi1+diag(X);
%%% Iterative lease-squares optimization %%%
C=chol(Phi1); % Choloskey factorization
for kk=1:itns
    B=kron(eye(N+1),h1')*S1; D=[B; C]; u=[1;zeros(N+Lh1,1)];
    h1=(h1+inv(D'*Gamma2*D)*(D'*Gamma2*u))/2;
end
h=E*h1;
```

The above program calls the function 'function h=r_cos_p(N,M,alpha)' which is used to generate a root-raised-cosine filter by sampling the root-raised-cosine pulse-shape

$$h_{\text{rrc}}(t) = \frac{\sin\left((1-\alpha)\frac{\pi t}{T}\right) + \frac{4\alpha t}{T}\cos\left((1+\alpha)\frac{\pi t}{T}\right)}{\frac{\pi t}{T}\left(1-\left(\frac{4\alpha t}{T}\right)^2\right)}. \tag{34}$$

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