1. Algebra and Functions

1.1.1 Equations and Inequalities

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1.1.1 Equations and Inequalities

- **Polynomials: linear, quadratic, higher order**

  $$ax + b = 0, ax^2 + bx + c = 0$$

- **Exponential and logarithms**

  $$y = a^x, y = \log_a(x)$$

- **Absolute Value**

  $$y = |a|$$

- **Equations and inequalities are often solved with algebraic manipulations.**
- **Equations typically have *discrete points* as solutions; inequalities typically have *regions* as solutions.**
- **Certain rules may be useful for solving particular equations, like the quadratic formula.**

1.1.2 The Quadratic Formula

Quadratic Formula

A formulaic approach to solving quadratic equations is the *quadratic formula:*

$$0 = ax^2 + bx + c \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In particular, quadratic equations have two distinct roots, unless $b^2 - 4ac = 0$ **.**

Solve $x^2 - 7x + 9 = 0$

Solve $2x^2 + x - 3 = 0$

1.1.3 Exponents and Logarithms

Properties of Exponents

Basic Rules:

- $a^{x+y} = a^x a^y$ **(same base, different exponents)**
- $a^x b^x = (ab)^x$ **(different base, same exponent)**
- $(a^x)^y = a^{xy}$ **(iterated exponents)**
- $x^0 = 1$ **for any value of x (convention)**

Properties of Logarithms

Logarithms enjoy certain algebraic properties, related to the exponential properties we have already studied.

- $\log_a(xy) = \log_a(x) + \log_a(y)$

  (logarithm of a product)

- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

  (logarithm of a quotient)

- $\log_a(x^y) = y\log_a(x)$

  (logarithm of an exponential)

- $\log_a(1) = 0$ **(logarithm of 1 equals 0)**

  $\log_a(a) = 0$

Solve $3^{x^2-1} = 1$

Solve $2^{2x-3} = \dfrac{1}{4}$

Solve $\log_7(-x+1) = 2$

Solve $\ln(2x + 2) = 0$

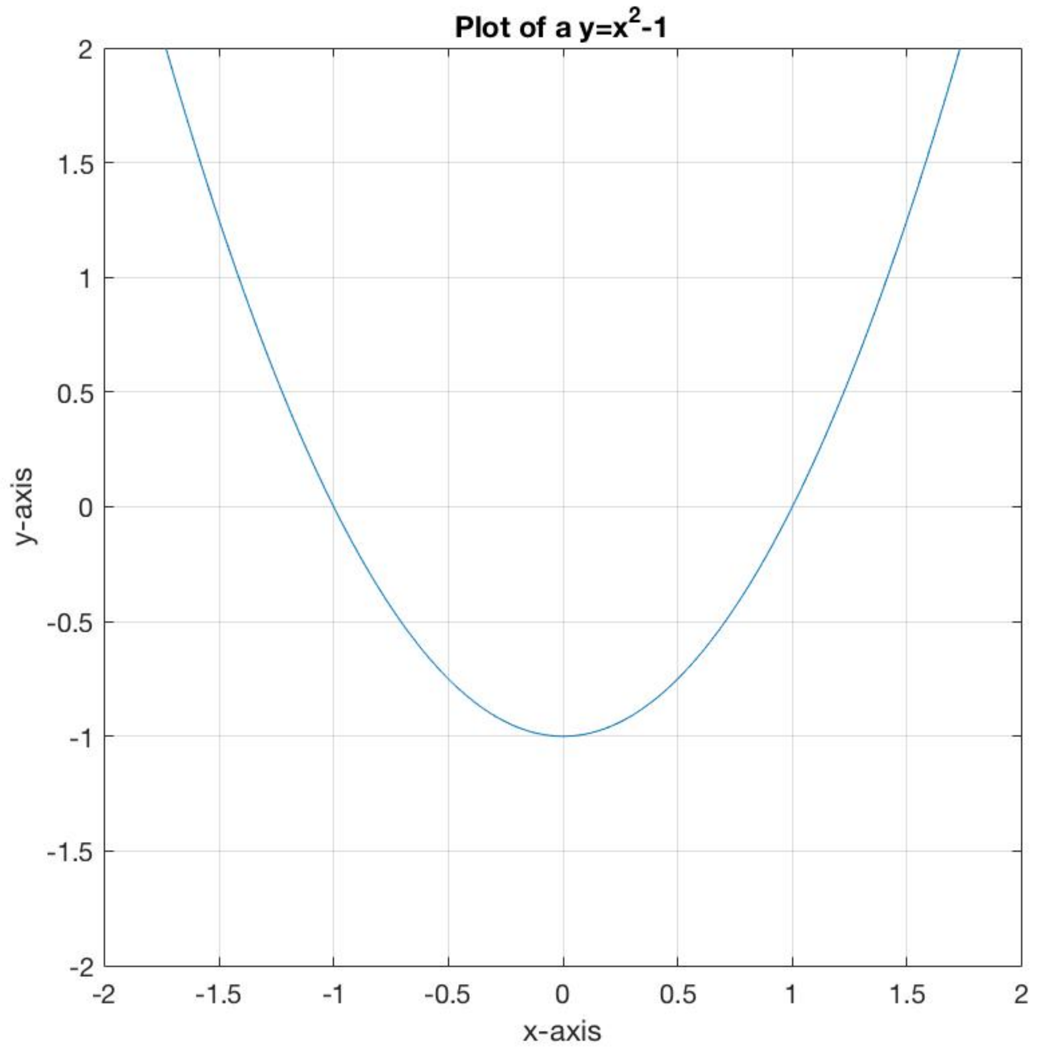
1.2 Introduction to Functions

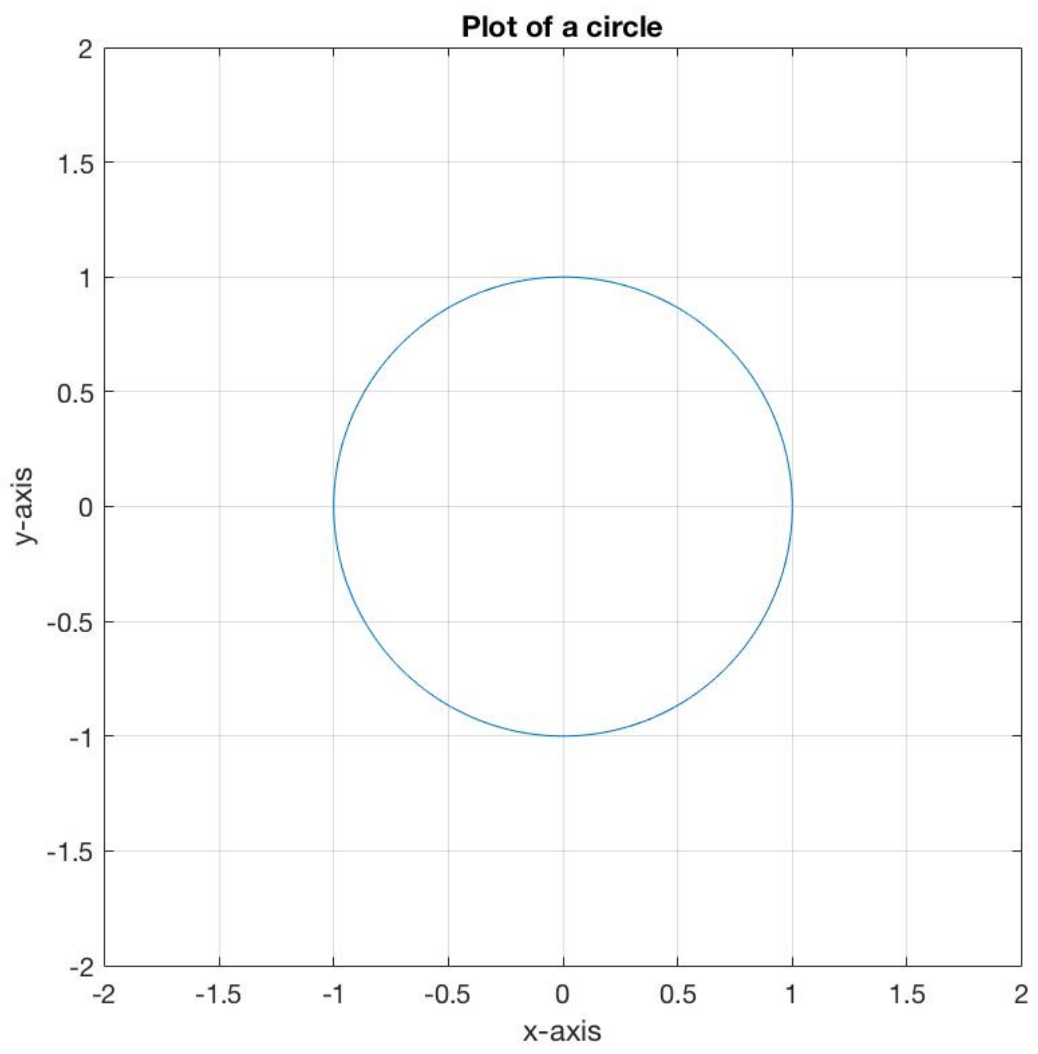
What is a Function?

- Functions are mathematical objects that send an input to a unique output.
- They are often, but not always, numerical.
- The classic notation is that $f(x)$ denotes the output of a function f at input value x.
- Functions are abstractions, but are very convenient for drawing mathematical relationships, and for analyzing these relationships.

Function or not?

One of the key properties of a function is that it **assigns a unique output to an input.**

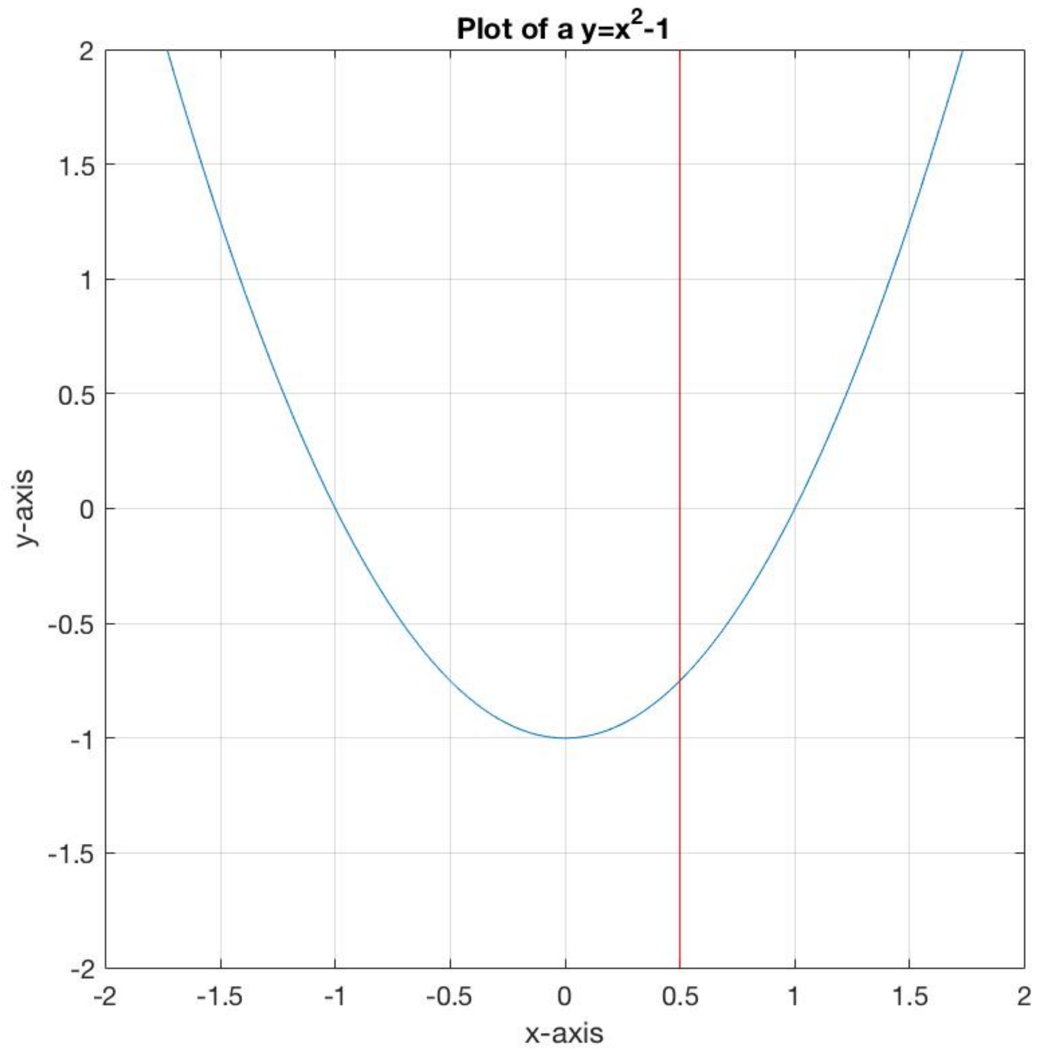


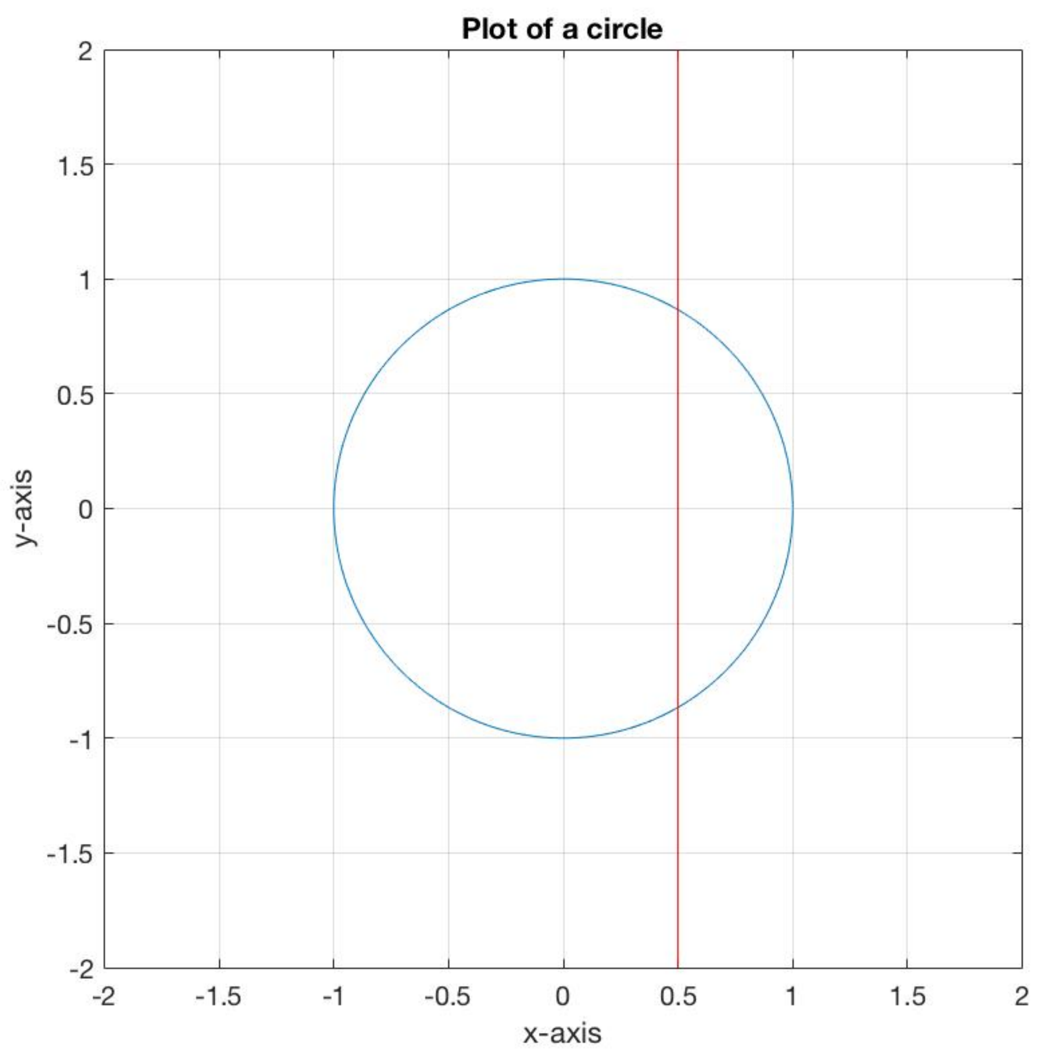


Vertical Line Test

A trick for checking if a mathematical relationship plotted in the Cartesian plane is a function is the *vertical line test.*

VLT: A plot is a function if and only if every vertical line intersects the plot in at most one place.





Plot $y = x^2 + 4$

Plot $x = y^2 + 4$

Function or not?

Representing with Functions

Functions are convenient for describing numerical relationships.

$$\text{Input} \xrightarrow{f} \text{Output}$$

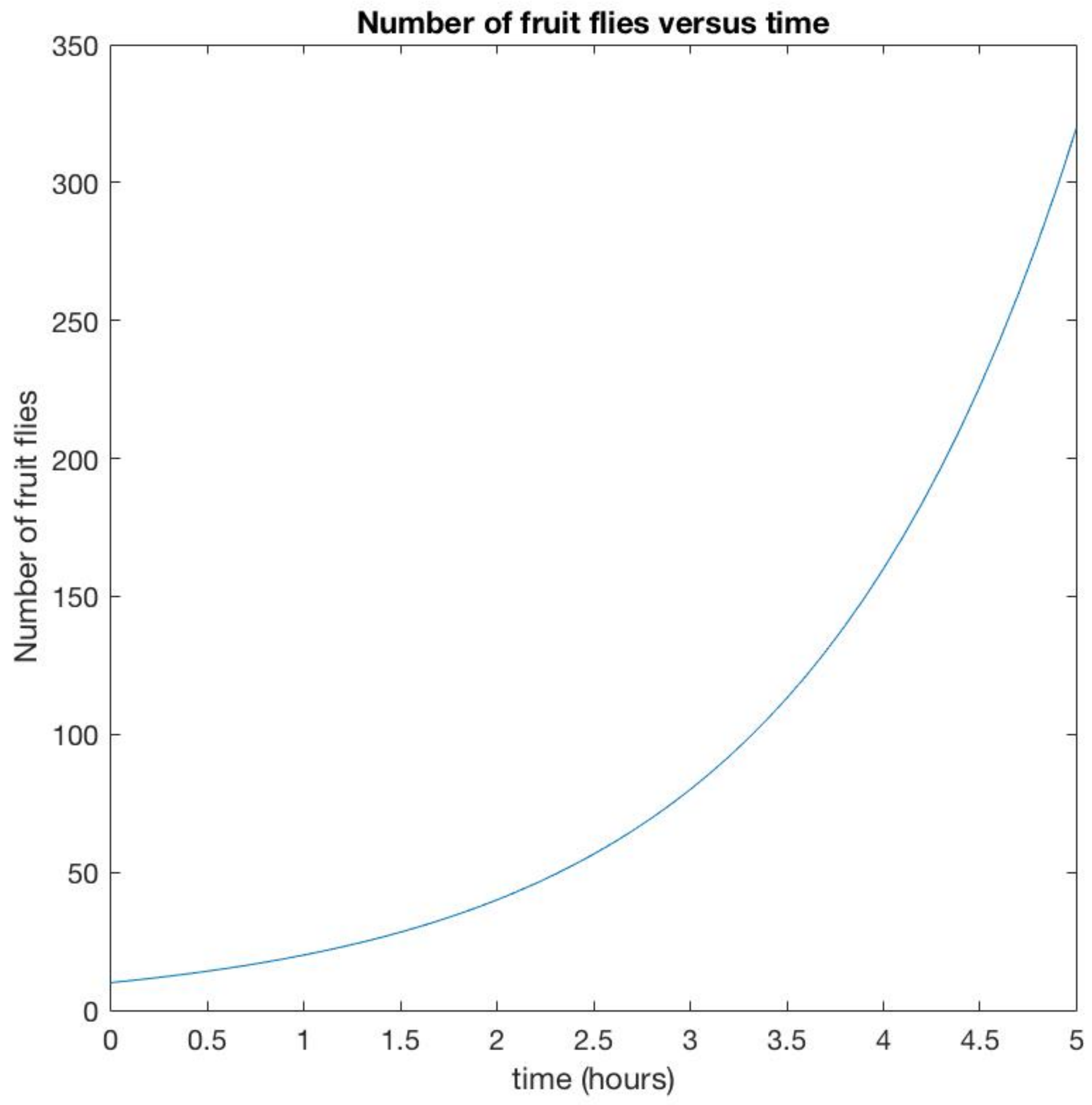
To model a relationship with functions, you simply need to understand how your input depends on your output.

Exponential Modeling

Exponential functions are more complicated than linear functions, but are very useful for things that, for example double in magnitude at a certain rate.

For example, suppose a colony of fruit flies starts with 10, and doubles every hour. Then the population of fruit flies at time t in hours is given as

$$P(t) = 10 \cdot 2^t$$



1.3 Domain and Range

Domain and Range of a Function

Let $f(x)$ be a function.

- The *domain* of $f(x)$ is the set of allowable inputs.
- The *range* of $f(x)$ is the set of possible outputs for the function.
- These can depend on the relationship the functions are modeling, or be intrinsic to the mathematical function itself.
- They can also be inferred from the plot of $f(x)$, if it is available.

Intrinsic Domain Limitations

Some mathematical objects have intrinsic limitations on their domains and ranges. Classic examples include:

- $f(x) = x^2$ has domain $(-\infty, \infty)$, range $[0, \infty)$.
- $f(x) = \sqrt{x}$ has domain $[0, \infty)$, range $[0, \infty)$.
- $f(x) = \log(x)$ has domain $(0, \infty)$, range $(-\infty, \infty)$.
- $f(x) = a^x$ has domain $(-\infty, \infty)$, range $(0, \infty)$.
- $f(x) = \frac{1}{x}$ has domain and range $(\infty, 0) \cup (0, \infty)$.

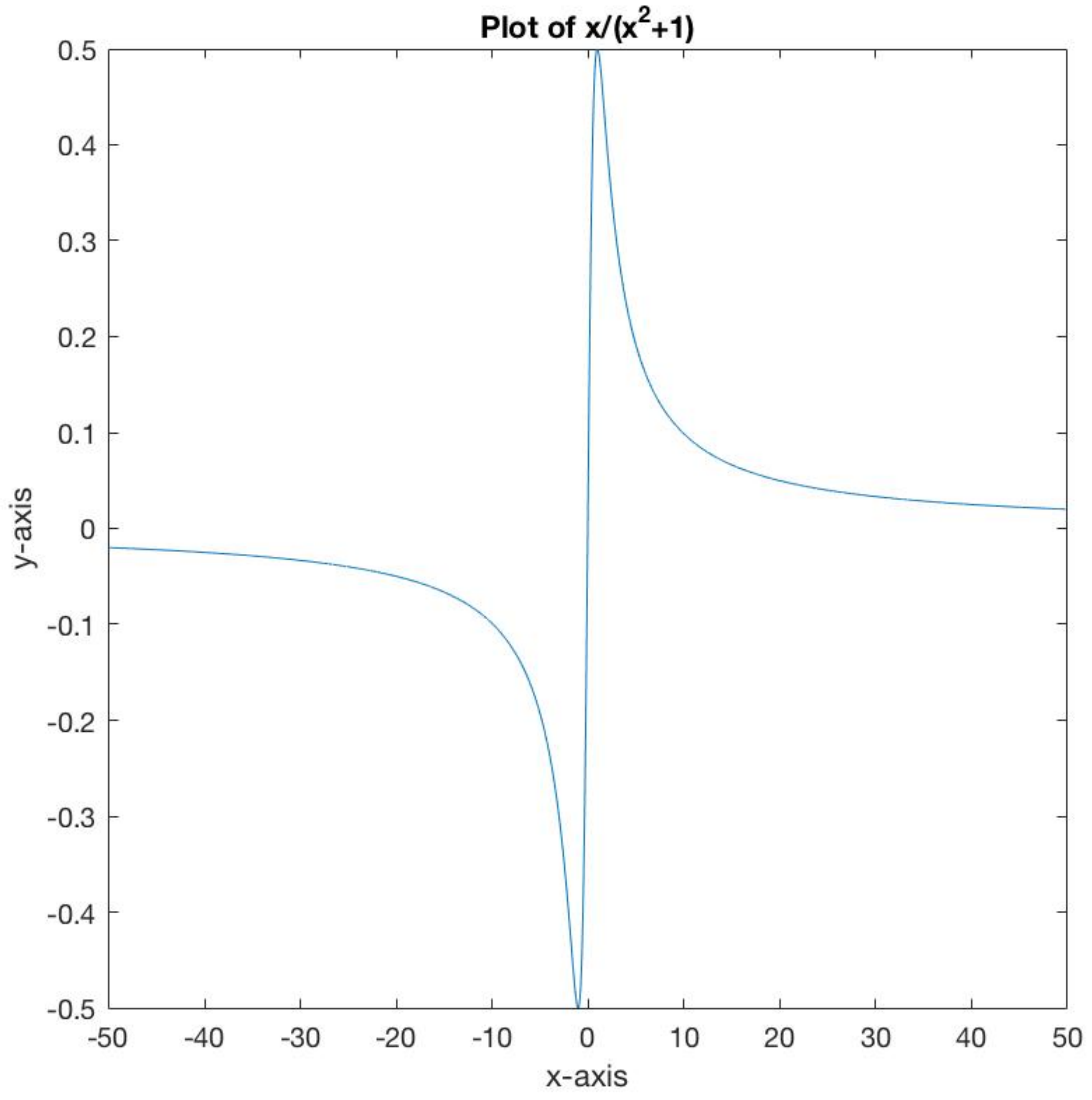
Visualizing Domain and Range

Given a plot of $f(x)$, one can observe the domain and range by considering what x and y values are achieved.

The function

$$f(x) = \frac{x}{x^2+1}$$

is hard to analyze, but its plot helps us guess its domain and range.



Find the domain and range of the following functions

$$f(x) = \sqrt{x + 1}$$

$$g(x) = -\log_{10}(3x + 2)$$

$$g(x) = e^{x+2}$$

$$f(x) = x^2 + 2$$

$$f(x) = -x - 7$$

$$h(x) = -|x| + 1$$

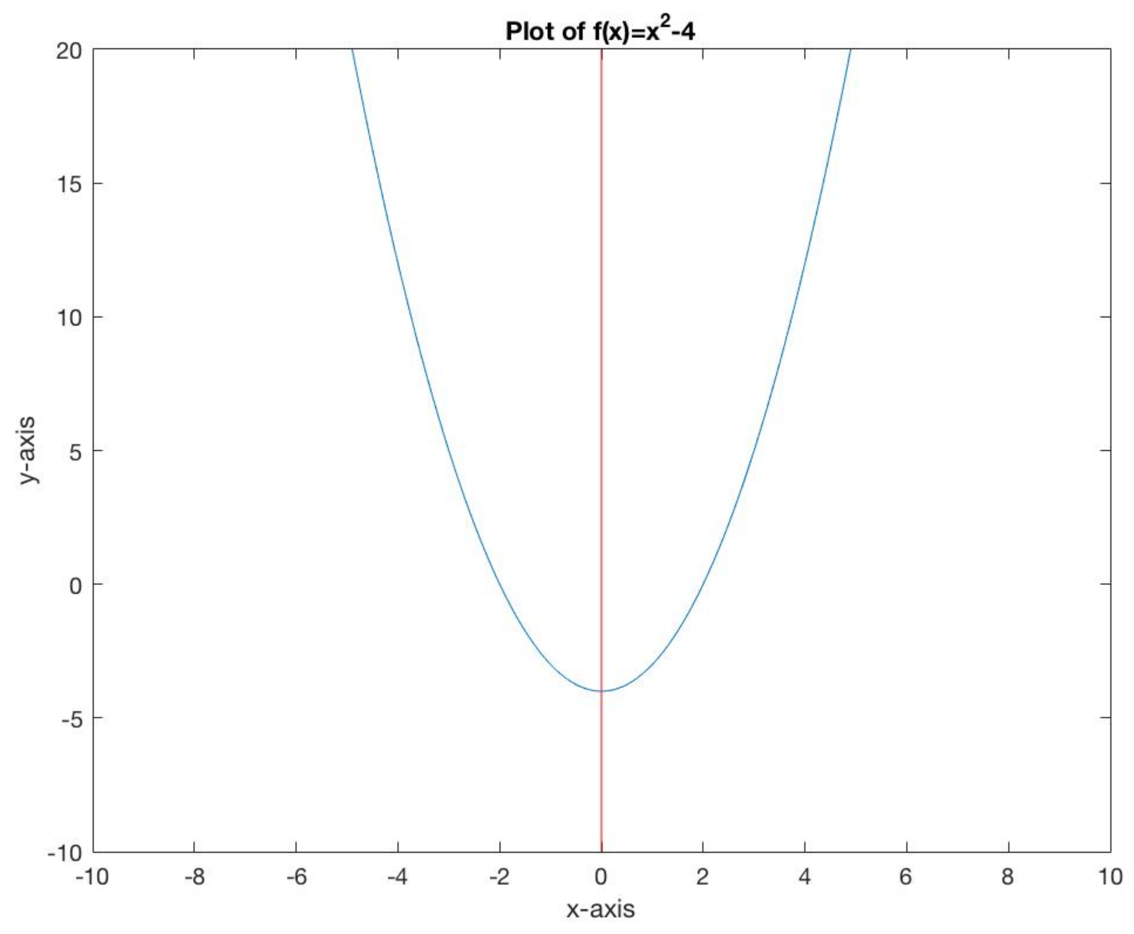
1.4.1 Graphing Functions

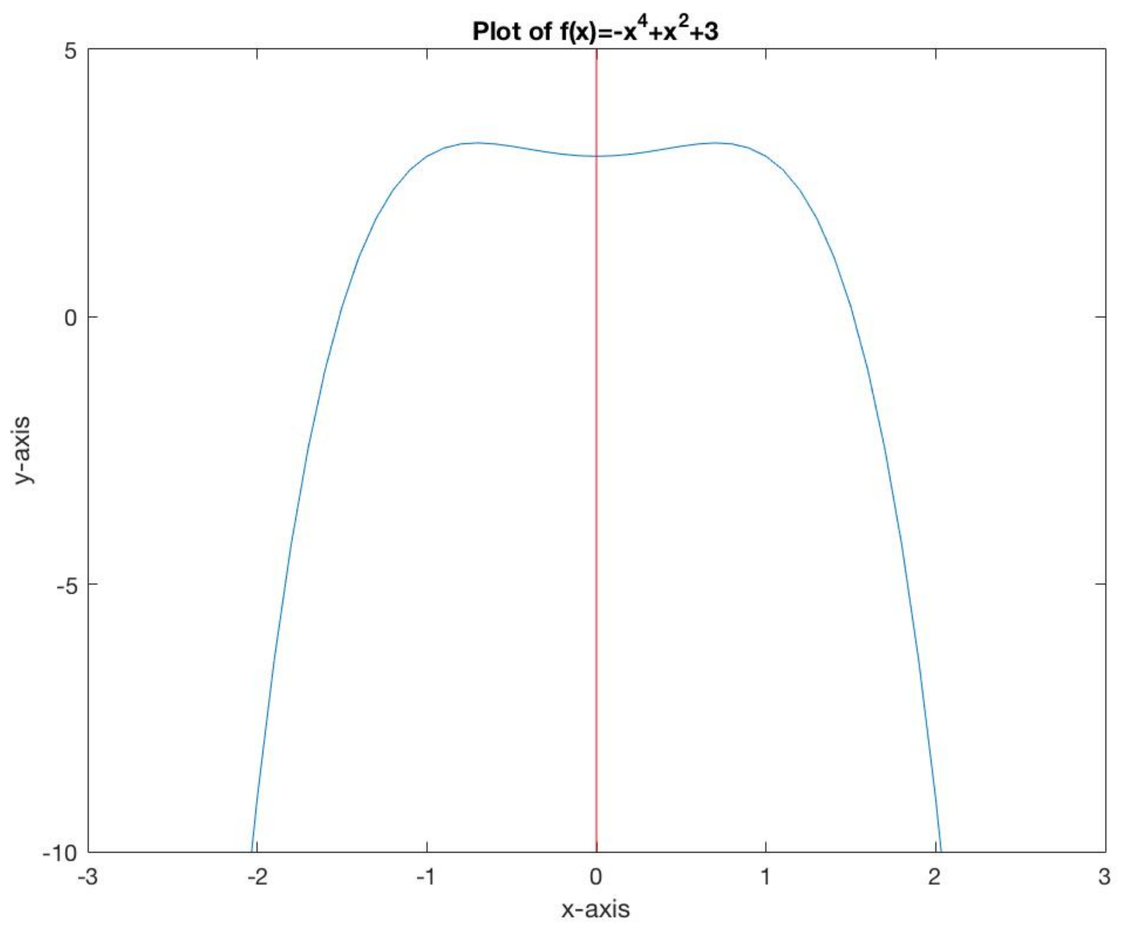
Plotting Functions

- Drawing a function in the Cartesian plane is extremely useful in understand the relationship it defines.
- One can always attempt to plot a function by computing many pairs $(x, f(x))$, and plotting these on the Cartesian plane.
- However, simpler qualitative observations may be more efficient. We will discuss of a few of these notions before moving on to some standard function plots to know.

Symmetry of Functions

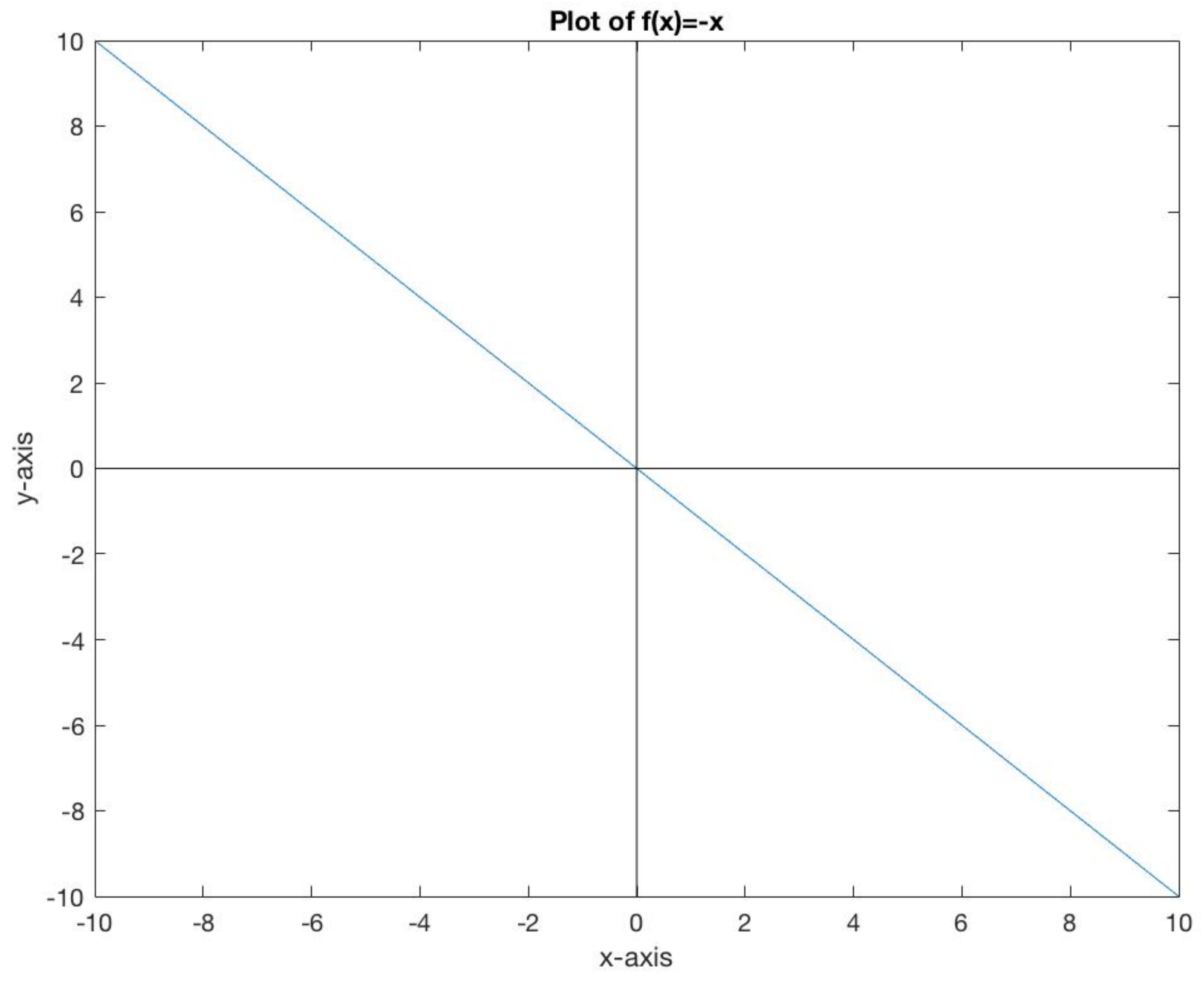
- **A function $f(x)$ is said to be *even/is symmetric about the y-axis* if for all values of $x, f(x) = f(-x)$.**
- **Functions that are even are mirror images of themselves across the y -axis.**

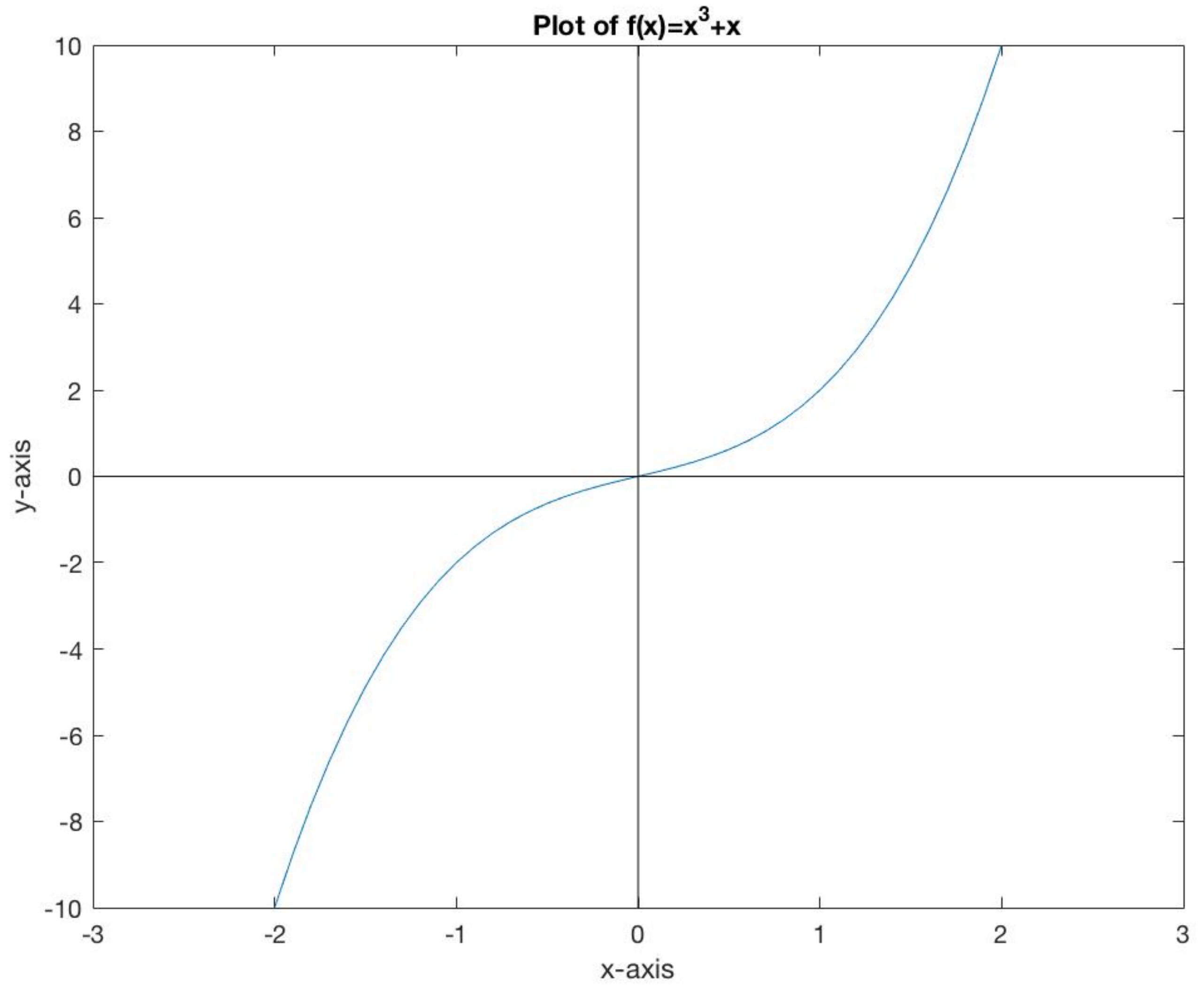




Symmetry of Functions

- A function $f(x)$ is said to be *odd/has symmetry about the origin* if for all values of x , $f(-x) = -f(x)$.
- Functions that are odd can be reflected over the x-axis, then the y -axis.





Are the following functions even, odd, or neither?

1.4.2 Transformation of Functions

Transformations of Functions

It is also convenient to consider some standard transformations for functions, and how they manifest visually:

- $f(x) \mapsto f(x+a)$ shifts the function to the left by a if a is positive, and to the right by a if a is negative.
- $f(x) \mapsto f(x)+b$ shifts the function up by b if b is positive, and down by b if b is negative.
- $f(x) \mapsto f(-x)$ reflects the function over the y-axis.
- $f(x) \mapsto -f(x)$ reflects the function over the x-axis.

Plot the following functions:

$$f(x) = 3x - 1$$

$$g(x) = -(x-1)^2$$

$$g(x) = e^{-x+1}$$

1.4.3 Inverse Functions

Inverse Functions

Let $f(x)$ be a function. The *inverse function f* is the function that "undoes" $f(x)$; it is denoted $f^{-1}(x)$.

More precisely, for all x in the domain of $f(x)$,

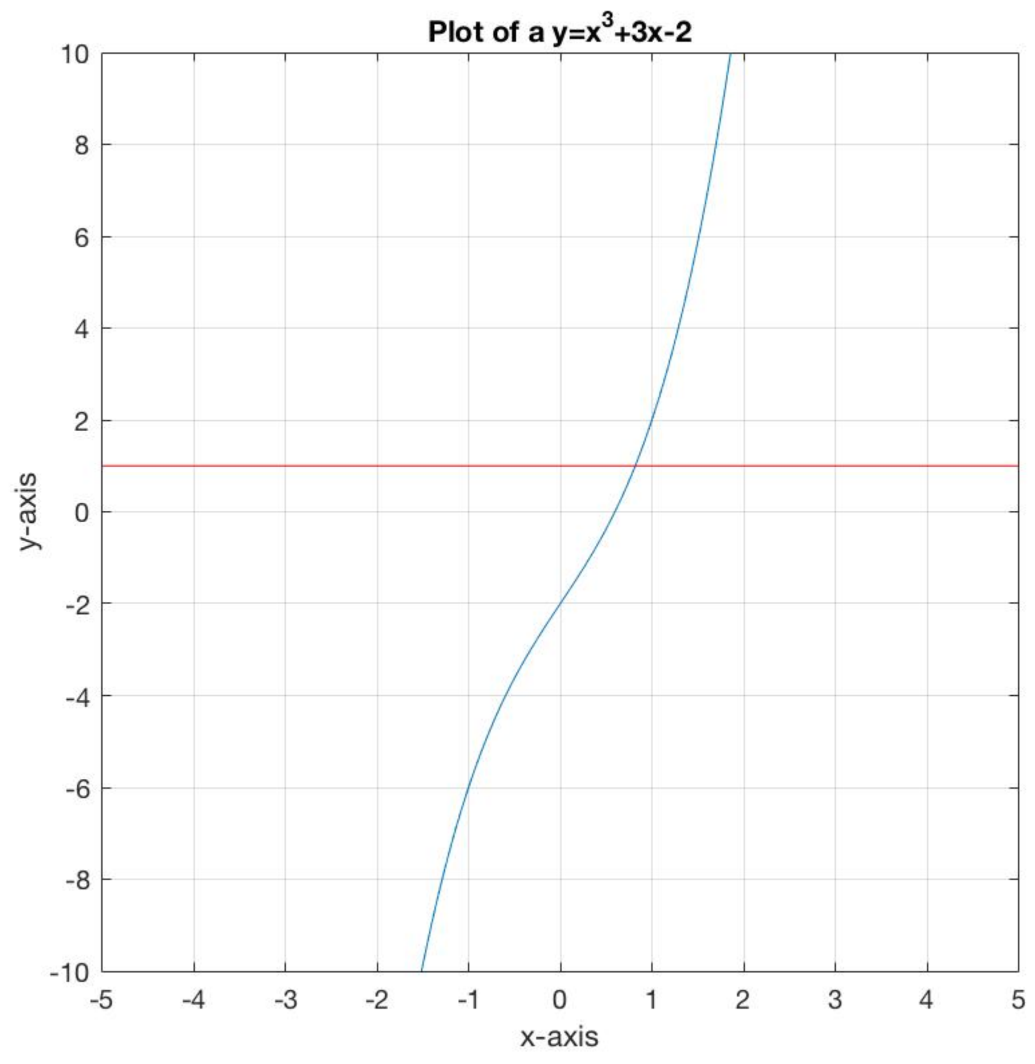
$$(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$$

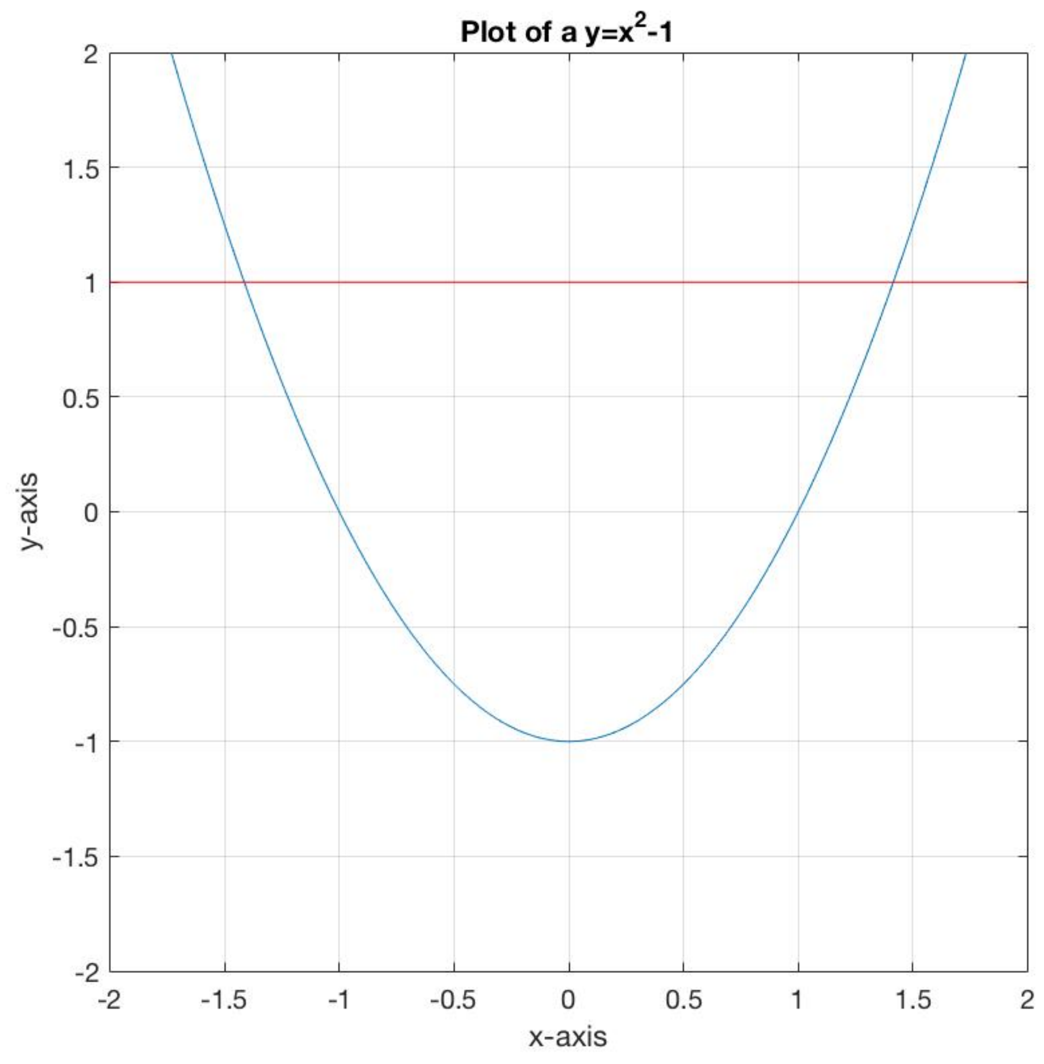
Remarks on Inverse Functions

- Not all functions have inverse functions; we will show how to check this shortly.
- Note that $f^{-1}(x) \neq (f(x))^{-1}$, that is, inverse functions are not the same as the reciprocal of a function. The notation is subtle.
- The domain of $f(x)$ is the range of $f^{-1}(x)$, and the range of $f(x)$ is the domain of $f^{-1}(x)$.
- Inverse functions can be plotted by taking the original function and reflecting across the line $y = x$

Horizontal Line Test

- Recall that one can check if a plot in the Cartesian plane is the plot of a function via the *vertical line test.*
- One can check whether a function $f(x)$ has an inverse function via the *horizontal line test*: the function has an inverse if every horizontal line intersects the plot of $f(x)$ at most once.





For each of the following functions, determine if it has an inverse function on its range. If so, find it.

$$f(x) = x^2 + 1$$

$$f(x) = \log(x - 2)$$

$$g(x) = -2x - 1$$

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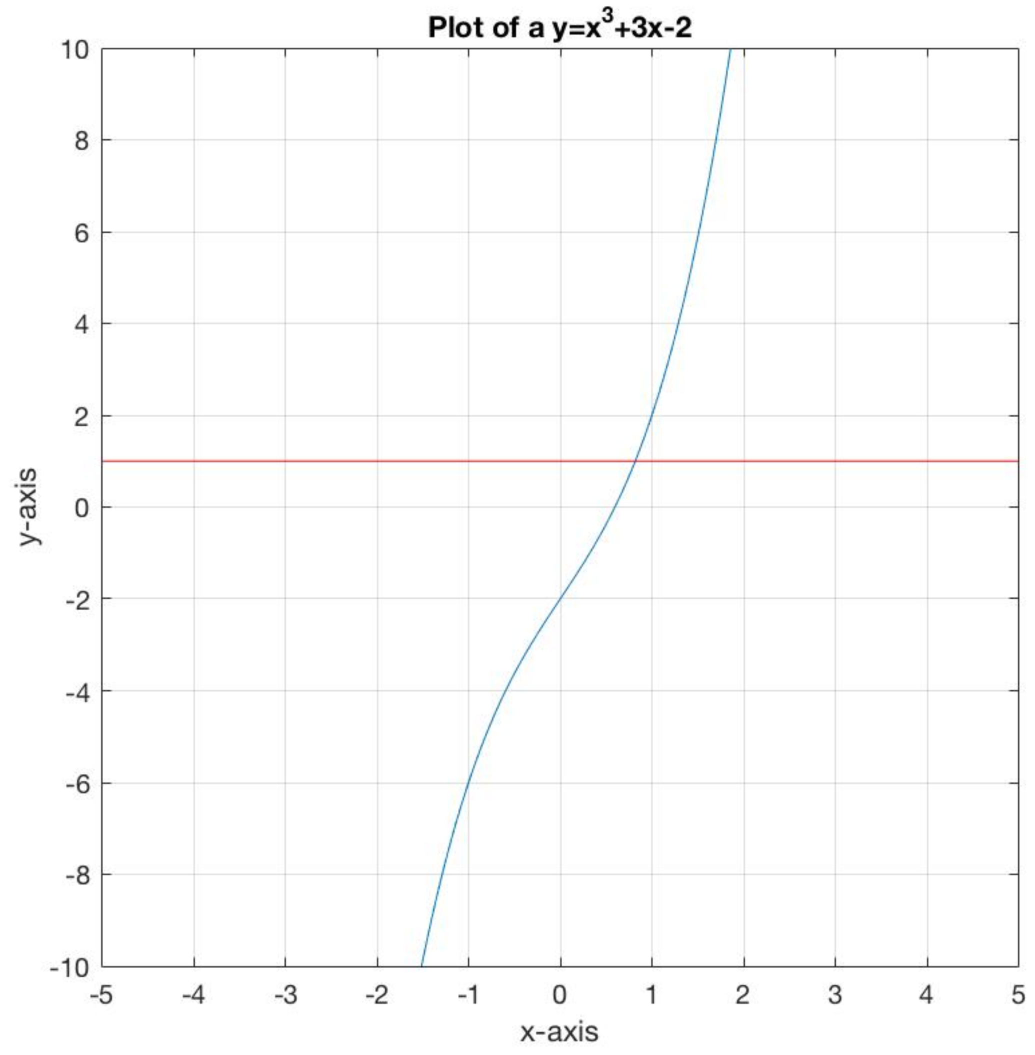

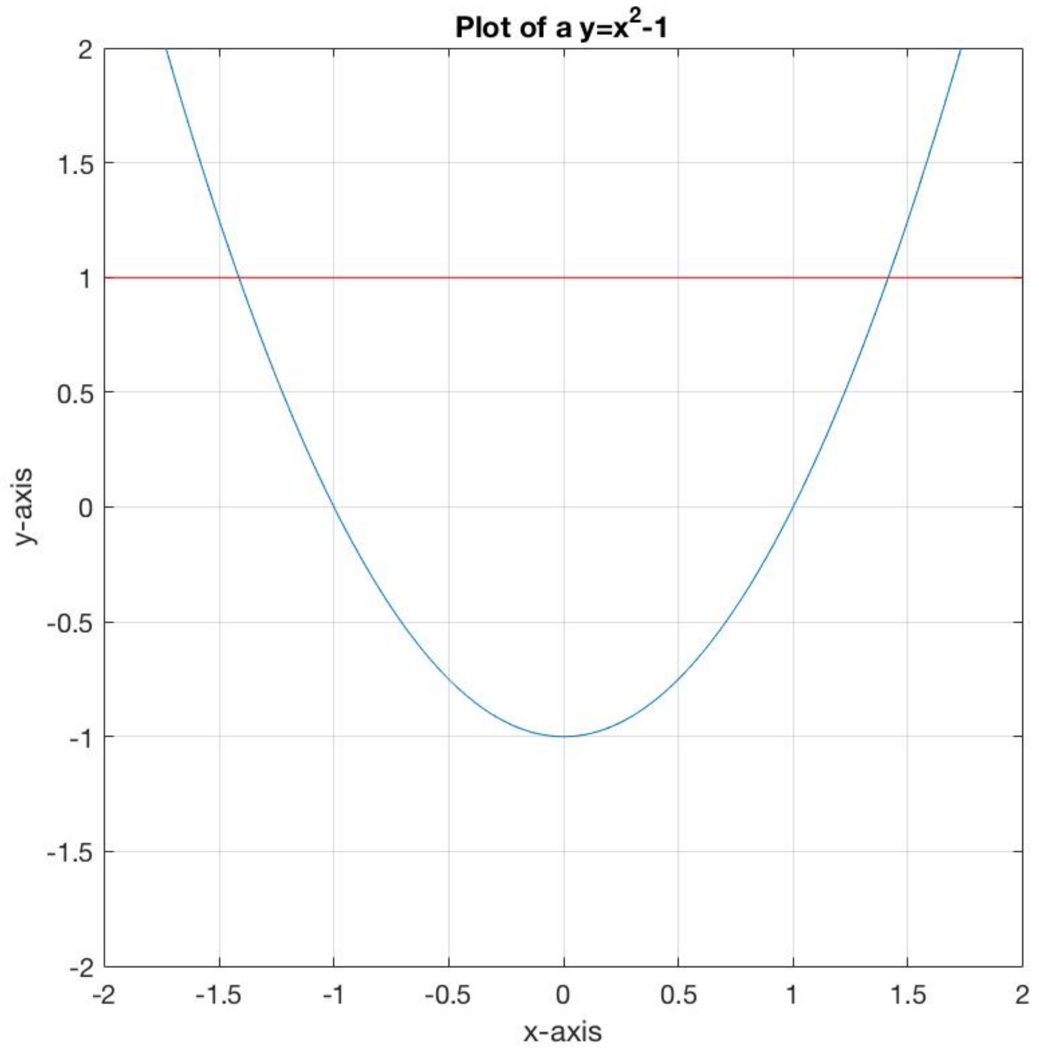

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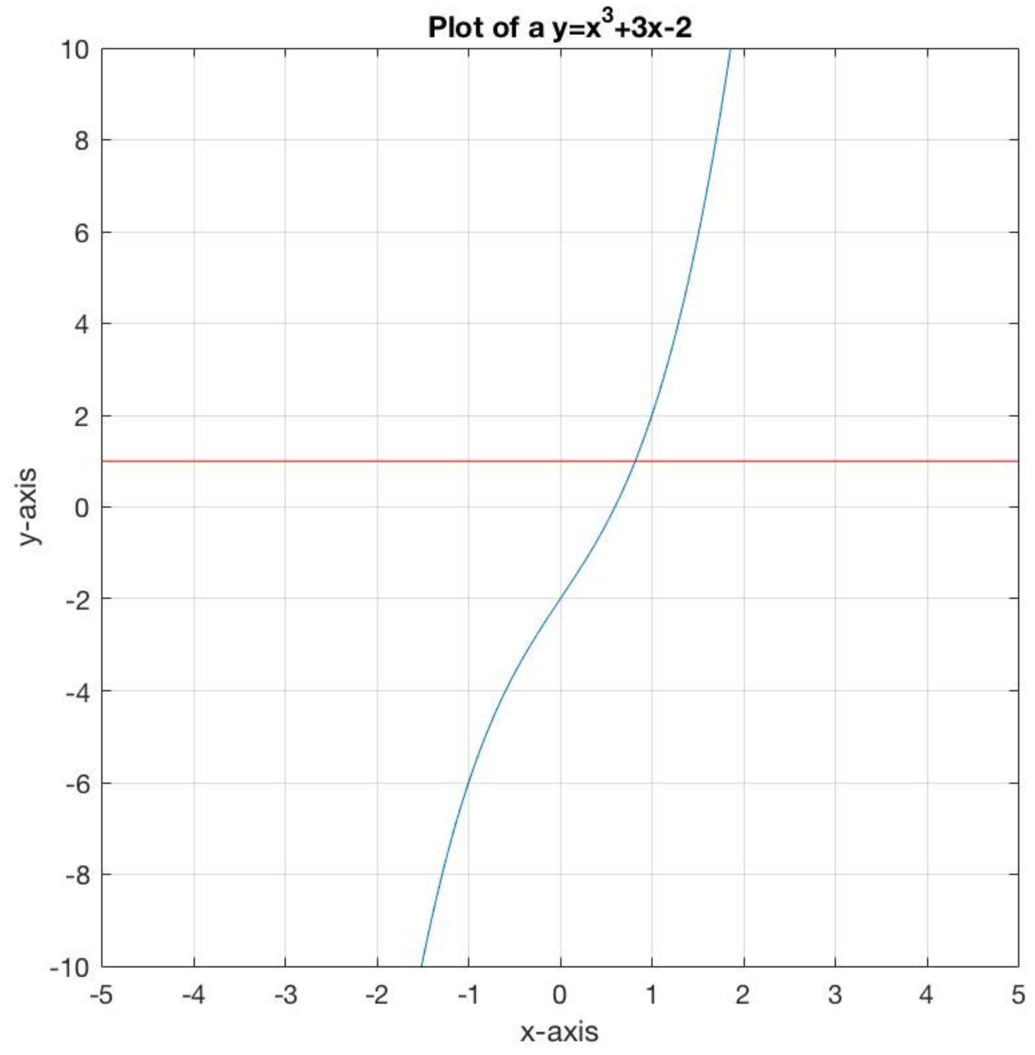

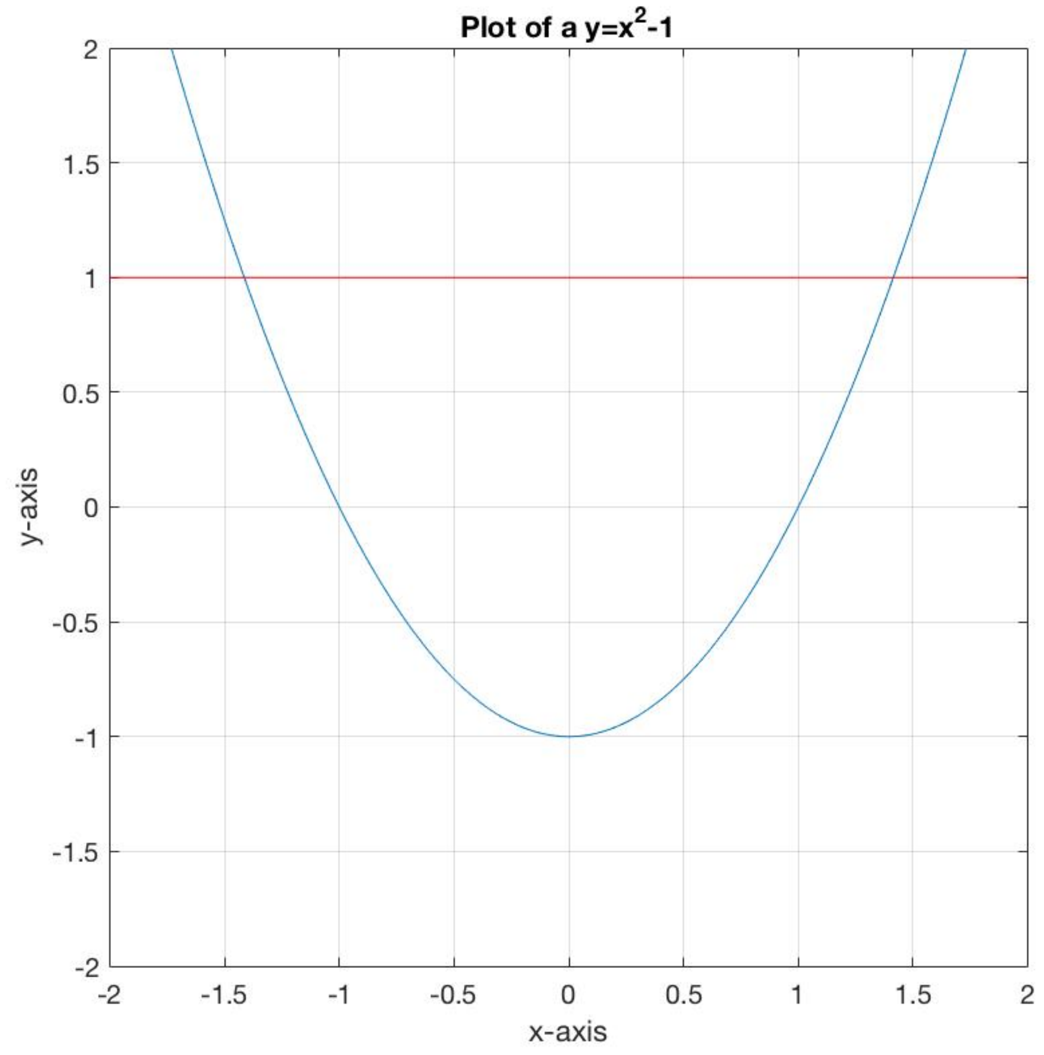

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1.5.1 Linear Growth

Linear and Exponential Growth

- One can use a variety of functions to *model* real world phenomenon in science, business, economics, public policy, and more.
- Models can be very complicated, but two relatively simple starting points are *linear* and *exponential* models.
- In linear models, rate of change is constant.
- In exponential models, rate of doubling is constant.

Linear Equations

- Equations of the form $y = ax + b$; a is the slope or rate of growth.
- We want to compute values of x given y and vice versa.
- Sometimes we need to perform some algebraic rearrangements first

Suppose it costs \$20 to manufacture a widget, with a start-up cost of \$1,000. Write the cost function in terms of number of widgets manufactured.

1.5.2 Exponential Growth

Exponential Models

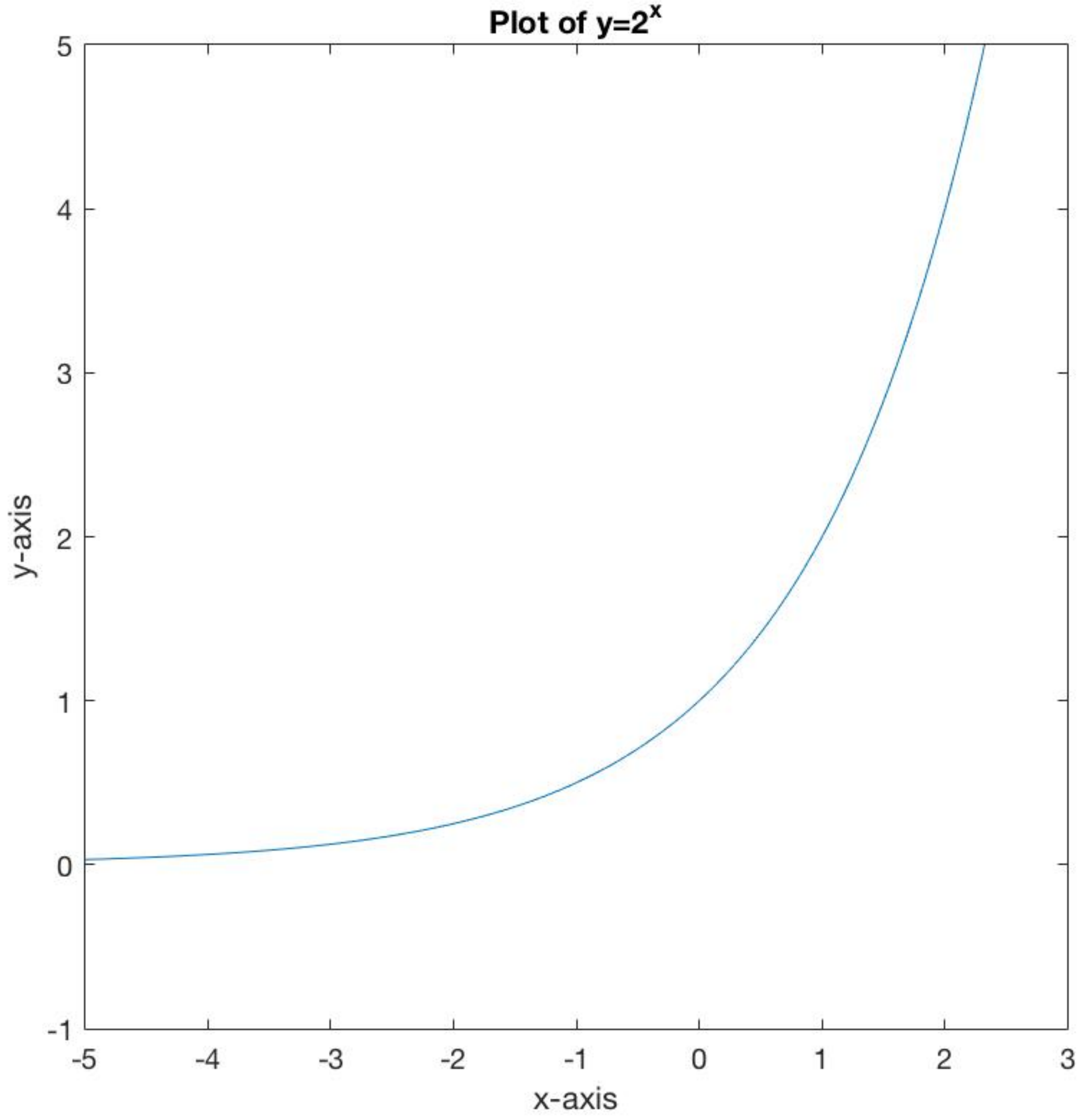
$$f(x) = a^x$$

- These may look daunting! However, we can use our exponential and logarithmic properties (tricks) to make our lives easier.
- Recall that $$y = a^x \Leftrightarrow \log_a(y) = x$$
- From this, we can approach many equations that look intimidating.

Properties of Exponents

Basic Rules:

- $a^{x+y} = a^x a^y$ **(same base, different exponents)**
- $a^x b^x = (ab)^x$ **(different base, same exponent)**
- $(a^x)^y = a^{xy}$ **(iterated exponents)**
- $x^0 = 1$ **for any value of x** **(convention)**


Plot of y=2^x
y-axis
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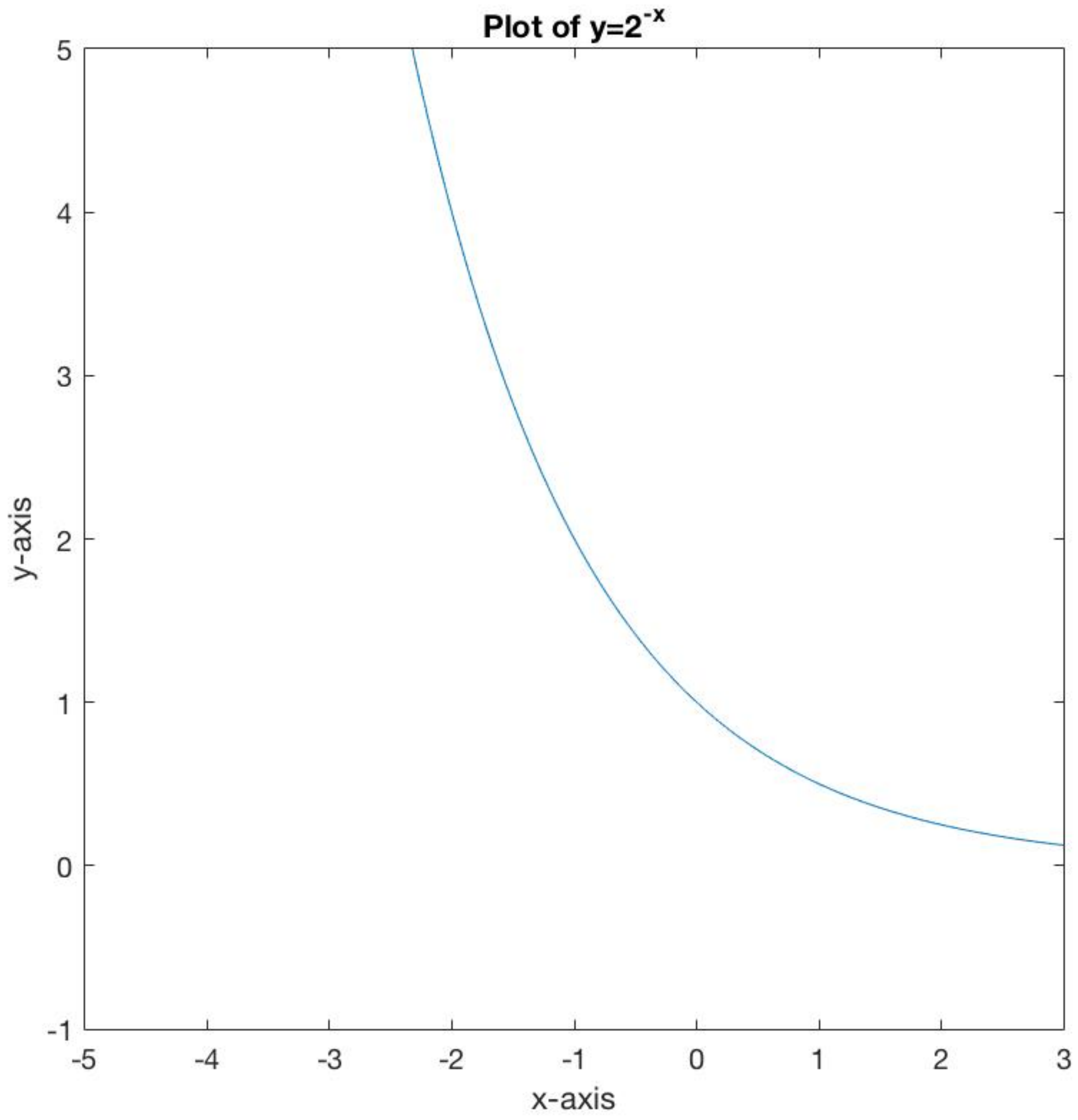
Plot of y=2^-x
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-5
-4
-3
-2
-1
0
1
2
3
x-axis
y-axis

Suppose a population of javelinas doubles in size every 16 months. If there are 100 to start,

A. **Write the population function in terms of time in months.**
B. **What is the population of javelinas after 6 years?**
C. **How long until there are 1000 javelinas?**