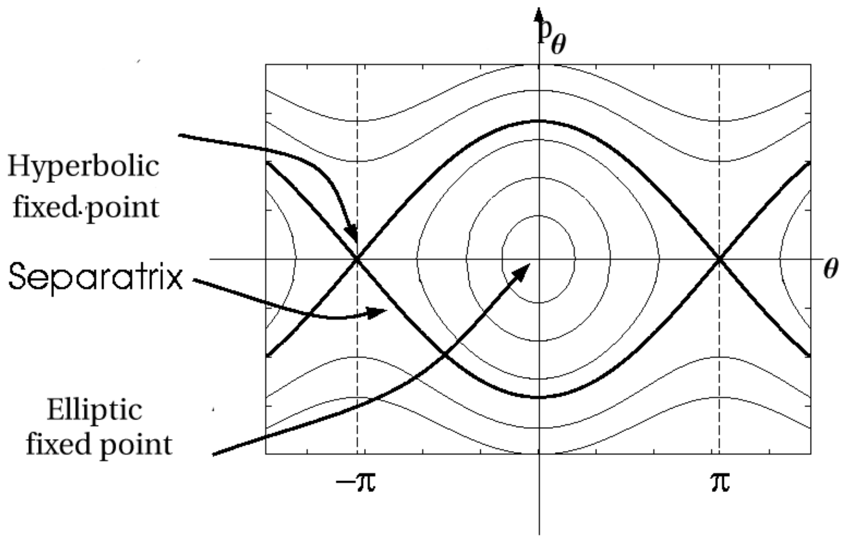
CLASSICAL DYNAMICS
Introduction

Chennai Mathematical Institute

January - April 2005



Chennai, April 12, 2006

Preface

This set of lecture notes is an attempt to convey the excitement of classical dynamics from a contemporary point of view. The contents of this course were developed for undergraduate students in their second semester at Chennai Mathematical Institute.

With recent interest and developments in non-linear dynamics and chaos it was thought appropriate to develop a mathematical framework for describing dynamical systems in general, linear or non-linear. Beginning with the description of differential equations governing the time evolution of a dynamical system, these notes discuss how the qualitative features of a dynamical system may be gleaned by simply studying its local behaviour and stability around the fixed points. The time evolution of any dynamical system is described by the flow of th system in phase space. Systems described by Hamiltonians are but a special case of more general dynamical systems. For completeness Lagrangian mechanics is also discussed and connections with Hamiltonian formalism established. In the end we discuss semi-quantitatively some simple systems which show chaotic behaviour.

The contents of this lecture notes evolved from an earlier set of lectures given by Professor Matthias Brack to students at the University of Regensburg in Germany. The students of B.Sc.(Physics), A.B. Belliappa, Argya Mondal and Shouvik Sur have contributed enormously to enrich the contents of these notes. Apart from going through the material they have written and edited parts of the material.

The lecture notes is our tribute to the memory of late Deep Roy, a student of the same batch, who passed away in July 2005. Deep was very enthusiastic about documenting the course of lectures and had volunteered many corrections and comments on the preliminary version. We would like to think that he would have appreciated what we have together done.

I would like to thank all the students of B.Sc (Physics), batch of 2004, for participating in this project. Encouragement from Deep Roy's father Amitava Roy, Madhavan Mukund and P.P. Divakaran to undertake this project is gratefully acknowledged.

M.V.N. Murthy
The Institute of Mathematical Sciences
Chennai
April 2006.

Deep Roy (1985-2005): A personal tribute

Deep Roy was born on 1st October, 1985, in Bangalore, to parents Ms. Nivedita Roy and Amitava Roy. He studied at the National Public School, Bangalore and passed +2 in the year 2003. He spent a year at National Institute of Technology, Trichy before joining B.Sc (Physics) program at Chennai Mathematical Institute in the year 2004. He was a fellow of the KVPY program and earned a Gold medal in the science exam conducted by the University of New South Wales. His CGPA at CMI was 9.4 where he was the topper in his batch. We consider ourselves lucky that he was amongst us.

He was with us for a very short while, just less than an year in fact. But, it was not short enough to prevent him from being an admirable influence on us. One of the most remarkable qualities that crosses our mind in retrospect, was his extreme sensitivity to joys and sorrows in others. All of us, particularly Shouvik and Arghya who came close to him during the last year of his life, have experienced that wondrous warmth of his affection. To us, he was a brother, a great friend, and more often than not, a mentor. Besides being a caring friend he was strict, sometimes unreasonably rigid, when it came to principles. Now after months of separation, we realize how much he cared for us, and this nature of his earned him the mandate to act a bit rude, a bit nutty and at times strict.

Deep was an avid music lover, an excellent guitarist and a very good singer. He had a collection of music which ranged from classical Indian and Western classical music to modern day heavy metal. Some of us picked up our love for rock and metal from him. Some others got initiated in to the language of pure notes going beyond the lyrics. Apart from his musical talent, he was academically brilliant. He spent a lot of time trying to teach us little nuances as well as the beautiful concepts that he had come across. Just by being with him, few of us did acquire the capacity to think big.

Most importantly, he did not believe that the sky was the limit and guided us to think big, think beyond the horizon. More importantly he taught us to believe in our abilities. He often used to quote Feynman-"What one ass can do, another too can.". This, we believe, by and large expresses aptly the emotion in us all.

Deep had a spiritual side to his personality. He was closely linked to the Ramakrishna Math and Mission. He was deeply influenced by the philosophy and ideals of Swami Vivekananda. His school life was probably not very smooth in spite of his excellent academic records; he sought refuge in the kind hearted monks in the order and the healing words of Maa Sarada Devi and the inspiring speeches of Swami Vivekananda. He owed almost all his confidence, determination and perseverance to Swamiji's speeches and letters. Given a spiritual mould such as this and a strong character, as we observed, the last act of his life remains puzzling. Whatever the cause was and however deeply disturbing it might have been, we hope he is at peace now.

A.B. Belliappa, Argya Mondal and Shouvik Sur
B.Sc. Physics, 2004.

Contents

Chapter 1

Introduction

This is an introductory course in classical dynamics from a contemporary view point. Classical mechanics occupies a different position in recent times as compared to what it was about three or four decades back. The syllabus for this course **Classical Mechanics II** actually starts with generalised coordinates, and goes on to Lagrangian and Hamiltonian mechanics. Essentially the syllabus is about what we call "Analytical Mechanics".

We are going to take liberties with the rough outline of the course and formulate the course in a way that coincides with more modern point of view while at the same time satisfying the syllabus "requirements" to the extent possible. We shall start with a very general definition of "Dynamical Systems" and discuss many concrete examples encompassing many areas, including but not restricted to Newtonian Systems. Indeed Hamiltonian dynamics will be discussed as a special case of the general dynamical systems.

While the basic concepts and general features of dynamical systems is of relatively recent origin, the concept of motion in phase-space and its geometrical depiction is simple. The application of these techniques is vast, from simple to more complex systems of mechanical or biological origin. Through this one can understand various types of dynamical evolution or "motion" before specialising to systems that are governed by Lagrangian (Hamiltonian) dynamics - mainly conservative systems. Thus Hamiltonian flows in here are treated as a special case of more general phase flows.

The prehistory of dynamics is indeed the development of the laws of mechanics. Following in the footsteps of Galileo and Newton we know that underlying the development of mechanics is the central tenet that Nature obeys unchanging laws that mathematics can describe. From seventeenth to the late nineteenth century the developments in classical mechanics, mainly due to the contributions from Newton, Euler, Lagrange, Hamilton, Jacobi..., was one of the main driving forces in the development of physics as a whole. The successes of classical mechanics during this period was very impressive. In fact in *Philosophical Essay on Probabilities (1812)* Laplace wrote:

> We ought then to consider the present state of the universe as the effect of its previous state and the cause of that which is to follow.... nothing

> would be uncertain, and the future like the past, would be open to its eyes.

While this this is indeed a forceful depiction of the determinism which is a very basic property of dynamical systems there are problems to be explored. The following quote from Poincare, about hundred years later, makes it amply clear:

> If we could know exactly the laws of nature and the situation of the universe at the initial instant, we should be able to predict exactly the situation of this same universe at a subsequent instant. But even when the natural laws should have no further secret for us, we could know the initial situation only approximately. If that permits us to foresee the subsequent situation with the same degree of approximation, this is all we require, we say the phenomenon has been predicted, that is it ruled by laws. But this is not always the case; it may happen that slight differences in the initial conditions produce very great differences in the final phenomena; a slight error in the former would make an enormous error in the latter. Prediction becomes impossible and we have the fortuitous phenomenon.

It is this insight that underlies the study of modern dynamics. The study of long term behaviour of systems, with sensitivity to initial conditions, cannot always be done with a prior knowledge of exact solutions, because such solutions may not be available. Thus geometric and probabilistic methods play an important role in understanding the qualitative features of such systems.

The approach to modern dynamics thus derives its foundations from the works of Poincare and Liapounov. Poincare (1899) studied nonlinear dynamics in the context of the n-body problem in celestial mechanics. Besides developing and illustrating the use of perturbation methods, Poincare presented a geometrically inspired qualitative point of view of dynamical systems by introducing the notion of a flow in a phase space.

In recent times the subject of dynamics is changing with emphasis on non-linear dynamics in general and the novel phenomenon of chaos in particular. Systems that can be modelled by nonlinear algebraic and/or nonlinear differential equations are called nonlinear systems. Examples of such systems occur in many disciplines. These new and substantive developments were made possible by the use of qualitative but powerful geometrical methods (with less emphasis on obtaining exact solutions) and of course the arrival of powerful computers making visualisation possible.

In the following we first develop a mathematical framework for describing dynamical systems in general, linear or non-linear. Beginning with the description of differential equations governing the time evolution of a dynamical system, we discuss how the qualitative features of a dynamical system may be gleaned by simply studying the behaviour of a dynamical system and its stability around the fixed points (described later). The flow of a system in phase space can be described by attributing a (velocity) vector to every point in the phase space, i.e. in terms of a vector field. Systems described by Hamiltonians are but a special case of dynamical systems since the velocity fields are obtained from a single function, namely the Hamiltonian

H. Such systems are discussed in detail. In the end we discuss semi-quantitatively some simple systems which show chaotic behaviour.

References:

1. I. Percival and D. Richards, *Introduction to Dynamics*, Cambridge University Press, 1991 (main text)

2. Boris Hasselblatt and Anatol Katok, *A first course in Dynamics*, Cambridge University Press, 2002 (slightly more mathematical than Percival's book).

3. J.V. Jose and E.J. Saletan, *Classical Dynamics: A contemporary approach*, Cambridge University Press, 1998 (advanced graduate level book- for the bold amongst you).

4. K.N. Srinivasa Rao, *Classical Mechanics*, Universities Press, 2003 (for Rigid body dynamics, small oscillations, analytical mechanics).

5. H. Goldstein, C.P. Poole and J.L. Safko, *Classical Mechanics* (III Edition), Addison Wesley Pub. (A much used text for classical mechanics moulded in the old school of thought).

Chapter 2

Dynamical Systems: Mathematical Preliminaries

We start with a very broad definition of a dynamical system and introduce the key theoretical concepts of phase space and fixed points (or limit cycles). While the definition may not cover all dynamical systems, it does cover a whole variety systems of interest to us. We give examples of systems with very simple dynamics to begin with and prepare ground for the treatment of more complicated systems which may not be analytically solvable.

2.1 Dynamical systems of order n

A **dynamical system** of order n is defined as follows:

1. The state of the system at any time t is represented by n-real variables

$$\{x_1, x_2, x_3, \ldots, x_n\} \Rightarrow \vec{r}$$

as coordinates of a vector $\vec{r}$ in an abstract $n-$ dimensional space. We refer to this space as the **state space** of simply **phase space**[1] in keeping with its usage in Hamiltonian dynamics. Thus the state of the system at any given time is a point in this phase space.

2. The time evolution of the system, **motion** is represented by a set of first order equations- the so called "equations of motion":

$$\begin{aligned}
\frac{dx_1}{dt} &= v_1(x_1, x_2, \ldots, x_n, t) \\
\frac{dx_2}{dt} &= v_2(x_1, x_2, \ldots, x_n, t) \\
&\cdots \\
\frac{dx_n}{dt} &= v_n(x_1, x_2, \ldots, x_n, t).
\end{aligned} \tag{2.1}$$

[1] Phase space description was first introduced by Henri Poincare and is widely used in statistical mechanics after it was adopted by J.Willard Gibbs

or simply

$$\frac{d\vec{r}}{dt} = \vec{v}(\vec{r}, t).$$

where

$$\vec{v} \Rightarrow \{v_1, v_2, \ldots, v_n\}.$$

is called the **velocity** function. While we use the notation x and v in analogy with mechanics, they do not always have the usual meaning of position and velocity.

The statements 1 and 2 together define a **dynamical system of order n**. If the velocity function does not depend on time explicitly, then the system is time-independent or **autonomous**. The set of all possible motions in called **phase flow**. The name phase flow may be understood by imagining a fluid flowing in the phase space with velocity $\vec{v}$. The direction and magnitude of the flow at any point in phase space is determined by the velocity vector.

We also require that the solution to be unique which requires $\vec{v}$ to obey certain conditions. Without going to mathematical details, it suffices to say that the solution of the differential equations in (2.1) are unique if the velocity vector $\vec{v}$ is a continuous function of its arguments and at least once differentiable. With the time evolution, the initial state of the system (denoted by a point in the phase space) evolves and follows a continuous trajectory which we shall call a **phase curve** which may be closed or open. Distinct phase curves are obtained when the initial state of the system is specified by a point which is not one of the points on the other trajectory. This leads to an important fact that two distinct trajectories can not intersect in a finite time period. The no intersection of phase space trajectories has to do with the fact that the evolution is **deterministic**. This is an important concept to which we shall return later.

2.2 First order systems

This is the simplest case of a dynamical system. The equation of motion[2] is given by

$$\frac{dx}{dt} = v(x, t)$$

where v is the velocity function. For any given $v(x, t)$, $x(t)$ is completely determined given $x(t)$ at some $t = t_0$. If, in particular, the system is autonomous, or v is not explicitly dependent on time then the solution can be written as,

$$t - t_0 = \int_{x(t_0)}^{x(t)} \frac{dx'}{v(x')}$$

Thus the solution $x(t)$ depends only on the difference $(t - t_0)$. Thus the time evolution of the system depends entirely on the time elapsed no matter where the origin of time is fixed.

[2] we call this "equation of motion" by habit. A more appropriate name is to call this *time evolution* equation of the system.

Example 1 - Radio-activity : A classic example of a dynamical system of first order is the Radio-active decay of a nucleus modelled by the equation

$$\frac{dN}{dt} = -\sigma N$$

where N is number of nuclei present at some time t. The solution of course is well known,

$$N(t) = N_0 \exp\{-\sigma(t - t_0)\},$$

where N_0 is the number of unstable nuclei present at t_0.

Example 2- Spread of epidemics :

Unlike radio-decay, here the growth is usually exponential at least in the initial period. If σ is an explicit function of time, as in the case of diseases one may obtain a power-law growth instead of exponential growth. A case which has been studied in detail is the threat of AIDS which has devastated many parts of Africa and is threatening many other countries like India.

In an effort to make quantitative assessment of the threat, efforts have been made to look at the reliable data compiled by Centres for Disease Control in the USA as a function of time. If I is the number of infected persons in a population of size N, then the rate of change of I may be given by

$$\frac{dI}{dt} \approx \alpha I$$

which gives rise to exponential growth in the initial phases which is usually true in an epidemic. However, it has been observed that in the case of AIDS that the growth shows a cubic dependence on time and not exponential. How is this achieved? Suppose the relative growth rate α is not a constant in time but a decreasing function of time, say,

$$\alpha = m/t$$

where m is a constant. Then the equation may be written as

$$\frac{dI}{dt} \approx mI/t$$

which has a power law solution, namely,

$$I = I_1 t^m + I_0$$

It turns out that for AIDS $m = 3$.

Example 3- Population growth :

A similar exponential growth also occurs in the population growth of various species. When 24 wild rabbits from Europe, not indigenous to Australia, were introduced in Australia they had a disaster on hand. With abundant food with no natural enemies they were in millions within a few years. The impact was so deep and widespread that it was called a national tragedy. Since the birth rate is proportional to the size of the population, one gets an equation of motion

$$\frac{dN}{dt} = \sigma N$$

which is similar to radioactive decay equation, but with a positive sign indicating an exponential growth (σ is positive).

The population problem, however, gets more complicated when considerations such as food and predators are introduced. We will return to this later. In all these cases there is only one function which varies with time.

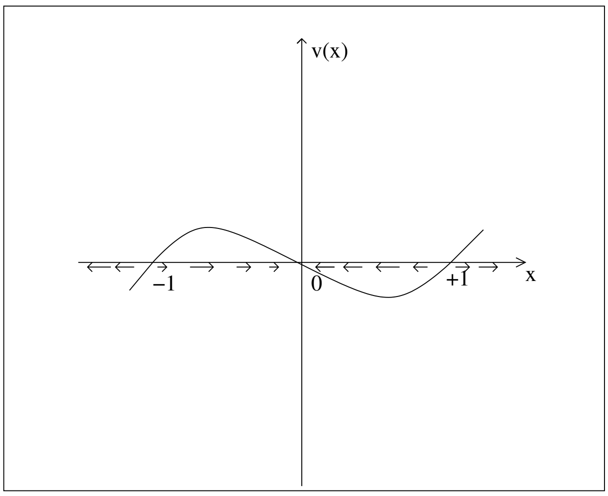
Example 4- a non-linear equation :

The equations of motion can, except in simple situations, get extremely complicated and the solutions are often not easy to obtain. However many qualitative features of a dynamical system may be obtained without actually solving the equations of motion. We will illustrate this with an example below- Consider the equation

$$\frac{dx}{dt} = v(x) = -x(1 - x^2)$$

which is similar to the radioactive decay problem with a nonlinear term added. Let us look at the properties of the velocity function:

- The system is autonomous- no explicit time dependence.
- The velocity function has zeros at $x_k = 0, \pm 1$- x_k are the roots of $v(x)$. If the system is at x_k at any time, it will continue to remain there for all times. x_k are therefore called **fixed points**. The system is said to be in **equilibrium** when it is at a fixed point.
- The phase space is one dimensional. The phase flows may be indicated by a set of arrows, for example, pointing left(right) if the sign of $v(x)$ is positive(negative) and whose length is proportional to the magnitude of $v(x)$ as shown in figure below. For reference we have also shown $v(x)$ as a function of x. The x-axis is the one dimensional phase space of the system.



A slightly simpler equation but the one that is used more often is the so called **Logistic Equation**. The simplest model of population growth is of course the exponential model which is not valid for all times since other impeding factors will come into play at some time. Obviously unchecked growth is not a realistic model. The simplest model that invokes factors such as competition for resources is given by the following equation:

$$\frac{dx}{dt} = v(x) = ax(1 - \alpha x)$$

where a and α are positive constants. The second term is a linear correction to the growth rate and α represents the control parameter to reduce the growth rate. We can simplify this equation further by making a change of variable $y = \alpha x$-we get

$$\frac{dy}{dt} = v(y) = ay(1 - y).$$

The system has two fixed points corresponding to

$$v(x) = 0 \Rightarrow x_1 = 0, \quad x_2 = 1.$$

Again one can go through the stability analysis and phase portrait to obtain qualitatively the properties of the system. However, in this case we may actually solve the system to check if our qualitative analysis is reasonable.

The solution is easily obtained by separating the variables:

$$a = \frac{dy/dt}{y(1-y)} = \frac{dy/dt}{y} + \frac{dy/dt}{1-y}$$

Integrating over t

$$\int adt = \int dt \left[\frac{dy/dt}{y} + \frac{dy/dt}{1-y}\right]$$

we have

$$at + C = \log|y| - \log|1 - y|$$

At time $t = 0$, we have

$$C = \log|y_0| - \log|1 - y_0|$$

Exponentiating,

$$|\frac{y-1}{y}| = \exp(-at - C) = \exp(-at)|\frac{y_0 - 1}{y_0}|$$

Since the quantities in absolute signs always agree in sign, we can write the solution as,

$$y(t) = \frac{y_0}{y_0 + (1 - y_0)\exp(-at)}$$

We may now look at the asymptotic behaviour of the system:

- The two fixed points at $y = 0$ and $y = 1$ obviously yield constant solutions $y(t) = 0, 1$, that is if y_0 is equal to any of these values then the dynamical evolution is trivial.

- For any other value of y_0, the system goes to $y = 1$ as $t \to \infty$. We have considered here only positive times and positive values of y since that is the physical situation in population growth problem. However the solution can be analysed in more general terms without these restrictions.

- Putting back $x = y/\alpha$, where x denotes the population, it is clear that the fixed point at $y = 1$ actually denotes the saturation in the population with $1/\alpha$ as the limiting population.

Properties of Fixed Points :
Generalising from the above examples, we may state:

- The set of zeros of the function $v(x)$ are called the fixed points of the system. The fixed points divide the phase space into several regions.

- x_k is a **stable fixed point or sink** if the flow is directed towards the fixed point, otherwise x_k is an **unstable fixed point or repellor**. In the above example obviously $x_k = 0$ is a stable fixed point where as $x_k = \pm 1$ are unstable. It is easy to see that the system evolves towards stable fixed point and away from an unstable fixed point. That is, if x_0 is a fixed point then define

$$\lambda = \frac{dv}{dx}|_{x_0}.$$

  If $\lambda < 0$ the fixed point is stable otherwise it is unstable. λ is called the characteristic value or some times called Lyapunov exponent.

- A fixed point may be both stable and unstable, ie., the neighbouring states approach the fixed point on one side but leave from the other side. We call this a **saddle point**. This property leads to the so called structural instability, that is even a small perturbation some time can change the nature of fixed point. For example analyse the system with $v(x) = x^2$.

- The system can not cross a fixed point, by definition. The motion is therefore bounded by the fixed point or fixed points. A system which starts out in the open interval between fixed points remains there for arbitrarily long periods of time. These intervals are therefore called **invariant sets**. In the example-4 above the invariant sets are $(-\infty, -1)$; $(-1, 0)$; $(0, 1)$; $(1, \infty)$

Often the motion may be **terminating**. The terminating motion happens when at some time t the solution of the differential equation is undefined.

Maps and discrete dynamics :
It often happens that the evolution of a dynamical system may not be continuous, but happens in discrete steps. This is especially true when important changes occur at definite intervals in time instead of changing continuously as envisaged by Eq.(2.1). Often discrete dynamics may be used as a convenience to solve the differential equation numerically.

An example of discrete dynamics :

For example the population of a given species may change only at a particular time of the year (mating or breeding season). Leonardo of Pisa in 1202 considered the following problem where a small number of rabbits grow in discrete steps. It is immaterial what species it is- it could be butterflies for example. He asked the question- How many rabbits can be bred from one pair in one year, if each pair of rabbits breed one pair every month except in the first month of their birth ?

The solution is straightforward and you will immediately see that the population of the rabbits grows. In the first twelve months the population for each month is the sequence of numbers - **1,1,2,3,5,8,13,21,34,..**. This is easily recognised to be the Fibonacci sequence which is generated by the equation:

$$p_{n+1} = p_n + p_{n-1}$$

with $p_0, p_1 = 1$,

How does this compare with the exponential growth in the population? It is easy to check that the discrete version of exponential growth may be written as

$$N_{n+1} = aN_n$$

where N_n is the measured population at some discrete time step n. Substituting this as a possible solution in the equation that generates Fibonacci sequence

$$N_{n+1} = N_n + N_{n-1} \;\Rightarrow\; a^2 N_{n-1} = aN_{n-1} + N_{n-1}$$

immediately fixes the value of the coefficient of growth a through

$$a^2 = a + 1,$$

where $a = [1 + \sqrt{5}]/2$ is the golden mean.

2.3 Second order systems

The phase space is two dimensional and each point in the phase space is characterised by two real numbers (x, y).

$$\vec{r}(t) \Rightarrow (x(t), y(t)).$$

Dynamical evolution of the system is governed by the system of equations,

$$\frac{dx}{dt} = v_x(x, y, t)$$

$$\frac{dy}{dt} = v_y(x, y, t)$$

or simply

$$\frac{d\vec{r}}{dt} = \vec{v}(x, y, t).$$

The solution of the equations defined by the velocity vector $\vec{v}$ has a unique solution for all time with the initial condition $\vec{r}(t_0) = (x_0, y_0)$. If the system is autonomous then of course there is no explicit dependence on time in the velocity function.

The solution $\vec{r}(t)$ obtained with a particular initial condition defines a continuous curve called the **phase curve**. The set of all phase curves tracing the actual motion is called **phase flow**. Note the phase curves exist only $d \geq 2$. In $d = 1$ there are only phase flows. The equation of the phase curve for an autonomous system of order 2 is given by

$$\frac{dy}{dx} = \frac{v_y(x,y)}{v_x(x,y)}$$

Example: Falling body in a gravitational field :

Let x denote the height at some time t. The force equation may be written as two first order equations:

$$\frac{dx}{dt} = v_x(x,y) = y$$

$$\frac{dy}{dt} = v_y(x,y) = -g$$

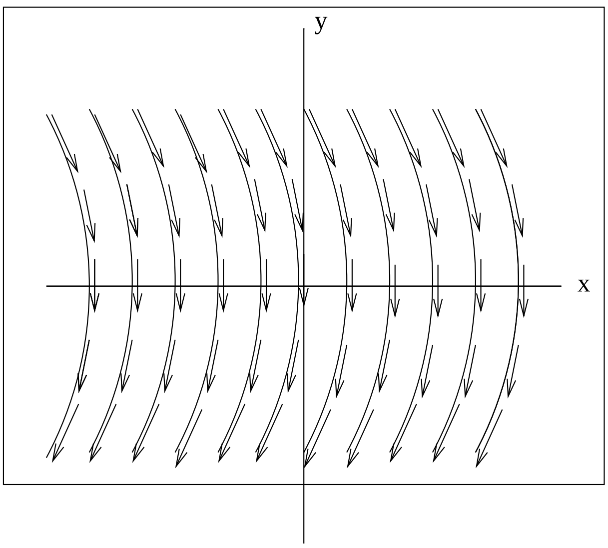
where g is the acceleration due to gravity. Thus the velocity field is given by,

$$\vec{v} = (y, -g)$$

Since g is never zero there are no fixed points in this system. The equation of the phase curve and its solution is

$$\frac{dy}{dx} = -\frac{g}{y} \Rightarrow x = x_0 - y^2/2g$$

where x_0 is the height at $t = 0$. The phase curves are therefore parabolas as shown in the figure below.



Different phase curves correspond to different initial conditions. The phase flows are characterised by the arrows of length and orientation given by

$$|\vec{v}| = \sqrt{y^2 + g^2}, \quad \tan(\theta) = -g/y$$

.

2.3.1 Linear Stability Analysis

We shall confine our analysis to 2nd order autonomous systems, for example, motion of a particle in one dimension or systems with one degree of freedom (dof). The motivation, as we shall see below, for a general stability analysis is to derive the local and where possible global properties of a dynamical system qualitatively, that is, without actually solving the evolution equations.

For an autonomous system of order 2 we have

$$\frac{dx}{dt} = v_x(x, y)$$

$$\frac{dy}{dt} = v_y(x, y)$$

where (x, y) define the state of the system in the phase space. We shall assume that $\vec{v}$ is some function of x, y which may be *non-linear* and therefore have many roots. The fixed point is defined through

$$v_x(x_k, y_k) = 0 = v_y(x_k, y_k)$$

and there may be many solutions. The nature of fixed points and the phase flows in the neighbourhood will be determined by the derivatives evaluated at the fixed point.

Consider one such fixed point (x_0, y_0). The stability around this fixed point may be obtained by giving a small displacement around the fixed point:

$$x(t) = x_0 + \delta x(t).$$

$$y(t) = y_0 + \delta y(t).$$

Now Taylor expand the velocity function $\vec{v}$ around the fixed point.

$$v_x(x, y) = v_x(x_0, y_0) + \frac{\partial v_x}{\partial x}|_{x_0,y_0}\delta x + \frac{\partial v_x}{\partial y}|_{x_0,y_0}\delta y + \ldots$$

$$v_y(x, y) = v_y(x_0, y_0) + \frac{\partial v_y}{\partial x}|_{x_0,y_0}\delta x + \frac{\partial v_y}{\partial y}|_{x_0,y_0}\delta y + \ldots$$

By definition the first term is zero since (x_0, y_0) is a fixed point. For an infinitesimal variation in $(\delta x, \delta y)$ we may linearise the equations of motion-

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} d\delta x/dt \\ d\delta y/dt \end{pmatrix} = \begin{pmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

or in a short form, and shifting the origin to (x_0, y_0),

$$\frac{d\delta\vec{r}}{dt} = A\delta\vec{r}$$

where A is a 2×2 matrix given by,

$$A = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{pmatrix}_{x_0, y_0}$$

The local stability analysis is best done in the eigenbasis or in some other convenient basis which we shall call the *Standard Basis*.

If the system is already linear the above analysis is globally, not just locally, valid since

$$\frac{dx}{dt} = v_x(x, y) = ax + by$$
$$\frac{dy}{dt} = v_y(x, y) = cx + dy$$

and the matrix A is given by,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

However, we need not restrict the analysis only to linear systems.

Consider the change of basis

$$M \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \delta X \\ \delta Y \end{pmatrix}$$

such that

$$B = MAM^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where λ_i are the eigenvalues of the stability matrix given by

$$\lambda_1 = \frac{1}{2}(\tau - \sqrt{\tau^2 - 4\delta})$$
$$\lambda_2 = \frac{1}{2}(\tau + \sqrt{\tau^2 - 4\delta})$$

where

$$\tau = V_{xx} + V_{yy}$$
$$\delta = V_{xx}V_{yy} - V_{xy}V_{yx}$$

are respectively the trace and the determinant of the stability matrix. The eigenvalues are real or complex depending on whether $\tau^2 \geq 4\delta$ or $\tau^2 < 4\delta$. We shall consider these cases separately.

In the transformed coordinates the linearised equations of motion and their solutions in the neighbourhood of the fixed point are given by

$$\delta\dot{X} = \lambda_1 \delta X \Rightarrow \delta X(t) = C_1 \exp(\lambda_1 t)$$
$$\delta\dot{Y} = \lambda_2 \delta Y \Rightarrow \delta Y(t) = C_2 \exp(\lambda_2 t)$$

The equation for the phase curve is

$$(\delta X/C_1)^{\lambda_2} = (\delta Y/C_2)^{\lambda_1}.$$

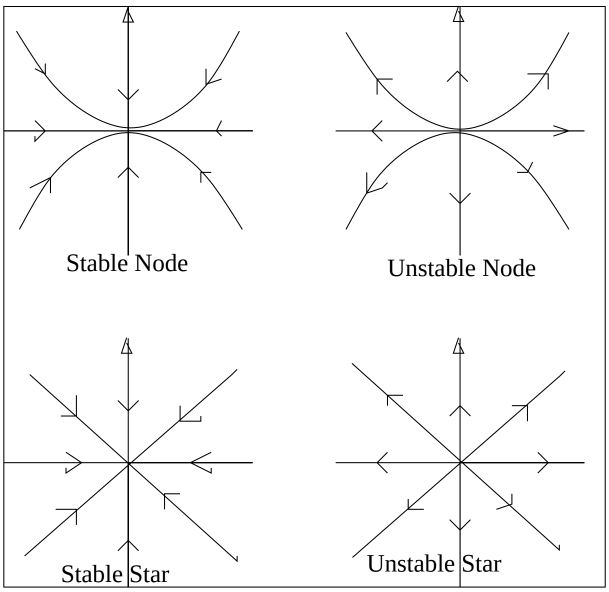
2.3.2 Classification of the fixed points

We use the properties of the eigenvalues to classify the fixed points. While in the first order systems motion either moves towards or away from the fixed point, the second order systems are richer in the sense there is much more variety in the nature of fixed points.

1. **Stable Node**, $\lambda_1, \lambda_2 < 0$.
$$\delta X, \delta Y \to 0$$
as $t \to \infty$.

   If in particular $\lambda_1 = \lambda_2 < 0$ and the equation is separable, $A = \lambda I$, it is called a stable star. (See Figure). If not a change of basis may be induced such that the matrix
$$B = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}$$
or equivalently $c = 0$ and $b \neq 0$. In this case
$$\delta\dot{X} = \lambda\delta X \Rightarrow \delta X(t) = C_1 \exp(\lambda t)$$
$$\delta\dot{Y} = c\delta X + \lambda\delta Y \Rightarrow \delta Y(t) = (C_2 + C_1 ct) \exp(\lambda t)$$
For $\lambda < 0$ this is called an improper node.

2. **Unstable Node**, $\lambda_1, \lambda_2 > 0$.
$$\delta X, \delta Y \to \infty$$
as $t \to \infty$. If in particular $\lambda_1 = \lambda_2 > 0$ and $A = \lambda I$ it is called a unstable star. (See Figure).



3. **Hyperbolic fixed point**, $\lambda_1 > 0, \lambda_2 < 0$.

$$|\delta X| \to \infty, |\delta Y| \to 0$$

as $t \to \infty$. This fixed point is also called a saddle point. These are rather special. If in particular the system has only one fixed point then the saddle point divides the phase space into four quadrants, each of which is an invariant subspace. That is a trajectory starting in one of the quadrants remains confined to the same quadrant for all times. These regions are separated by trajectories heading towards or away from the fixed point. The set of points along these trajectories are called invariant manifolds.

Next consider the case when the eigenvalues are complex.

4. **Stable spiral fixed point**, $\lambda_1 = -\alpha + i\beta, \lambda_2 = -\alpha - i\beta$ where $\alpha, \beta > 0$. Correspondingly we have,

$$\delta X = C_1 e^{-\alpha t + i\beta t}, \quad \delta Y = C_2 e^{-\alpha t - i\beta t}.$$

By a change of basis the solutions may be written as

$$\delta X' = e^{-\alpha t}(C_1 \cos\theta t + C_2 \sin\theta t)$$

$$\delta Y' = e^{-\alpha t}(-C_1 \sin\theta t + C_2 \cos\theta t)$$

The fixed point is called the spiral fixed point by looking at the behaviour of the real and imaginary parts as shown in the figure or the solutions given above explicitly in terms of the rotation angle θ. It is stable since for large times the system tends towards the fixed point.

5. **Unstable spiral fixed point**, $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$ where $\alpha, \beta > 0$. Correspondingly we have,

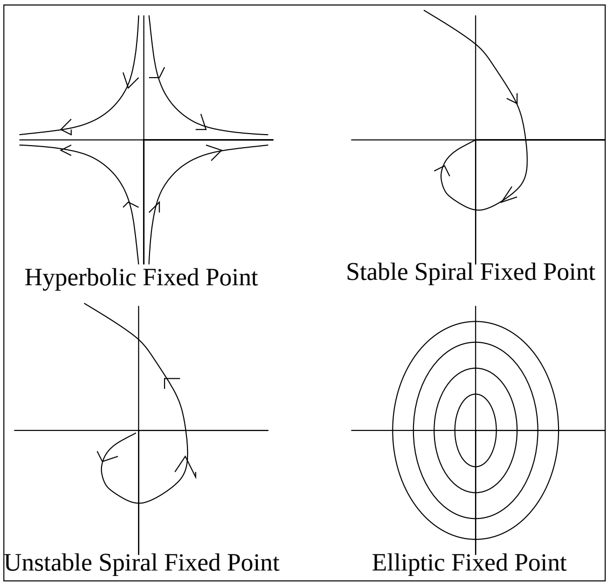
$$\delta X = C_1 e^{\alpha t + i\beta t}, \quad \delta Y = C_2 e^{\alpha t - i\beta t}.$$

The fixed point is unstable since for large times the system moves away from the fixed point.

6. **Elliptic fixed point**, $\lambda_1 = i\omega = -\lambda_2$. Correspondingly we have,

$$\delta X = C_1 e^{i\omega t}, \quad \delta Y = C_2 e^{-i\omega t}.$$

The system is confined to ellipses around the fixed point- each ellipse corresponds to a given initial condition.



In first-order systems all motions tend to fixed points or to infinity. In second order systems, in addition to fixed points, the system may also exhibit a **limit cycle** which is a closed trajectory encircling the fixed point. The trajectories within and outside tend towards the limiting trajectory as time $t \to \infty$. Such a behaviour occurs typically in systems that are oscillatory. A rigorous analysis of limit cycles, their occurrence and stability is beyond the scope of these lectures. However, it may be noted that they occur in systems where the long-term motion of the system is limited to some finite region in the state space. Because of the no intersection theorem, it is entirely reasonable to assume that in this bounded region the trajectory either approaches a fixed point or a closed trajectory (cycle) as $t \to \infty$. Indeed this is the content of the famous **Poincare-Bendixson Theorem**.

Example 1 : An example of this type of motion is the so called **Ricatti Equations**. This is a system of coupled equations of the form

$$\dot{x} = v_x = x + y - x(x^2 + y^2)$$

$$\dot{y} = v_y = -x + y - y(x^2 + y^2)$$

Obviously the origin (0,0) is a fixed point. The eigenvalues are

$$\lambda_{1,2} = 1 \pm i$$

and therefore the origin is an unstable spiral fixed point. Even though the equations are coupled and nonlinear, the equation system can be solved explicitly. Let

$$R(t) = x^2 + y^2$$

$$\tan\theta = y/x.$$

The equation for $R(t)$ and $\theta(t)$ may be written as

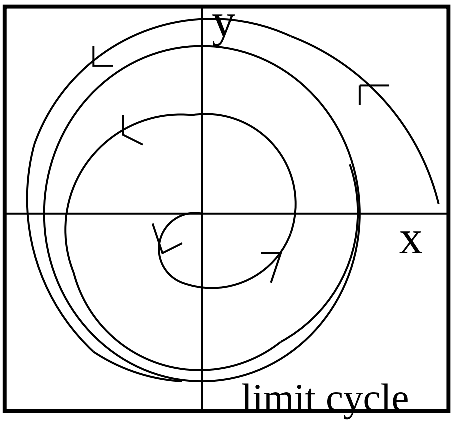
$$dR/dt = 2R - 2R^2$$

$$d\theta/dt = \frac{x\dot{y} - y\dot{x}}{R} = -1$$

whose solutions are

$$R(t) = \frac{C_1 e^{2t}}{C_1 e^{2t} + C_2} \qquad \theta(t) = -t + C_3$$

Therefore as $t \to \infty$ the system tends to $R(\infty) = 1$, which the unit circle in the original phase space. This is true for arbitrary values of the constants.



The result may be seen even more easily- think of the equation for R(t) as a first order system with a fixed point at $R = 0, 1$. It is easy to see that the fixed point at $R = 0$ is unstable and $R = 1$ is stable - in two dimensional phase space this is the limit cycle.

The result may be generalised by considering systems which are separable in polar coordinates. For example

$$\dot{x} = v_x = y + xf(r)$$

$$\dot{y} = v_y = -x + yf(r)$$

where $r^2 = x^2 + y^2$ and $f(r)$ is any function of r. The existence and stability of limit cycles then depends on the zeros of the function $f(r)$.

Example 2 :

The Lotka-Volterra equations, also known as the predator-prey equations, are a pair of first order, non-linear, differential equations. The are frequently used to describe the dynamics of biological systems (proposed independently by Alfred J. Lotka in 1925 and Vito Volterra in 1926). The Lotka-Volterra Model is an example of a second order autonomous system from population dynamics in biological systems. The model involves two populations, the prey denoted by x and predator denoted by y. The equations governing their populations are obviously coupled.

$$\frac{dx}{dt} = v_x(x, y) = Ax - Bxy$$

$$\frac{dy}{dt} = v_y(x, y) = Dxy - Cy$$

where $A, B, C, D \geq 0$. It is easy to see how the equations come about- The prey population x, will grow on its own but is diminished by the predator population y. On the other hand, the predator would starve if left alone and grows by feeding on its prey. We take x, y to be positive always. We also assume that there is no time gap between the causes and effects.

Note when there is no predator, prey's population increases indefinitely as it should assuming there is plenty of food. When there is no prey, of course the predator becomes extinct. There are two fixed points in the system

$$(x, y) = (0, 0)$$

$$(x, y) = (C/D, A/B)$$

Linearising the equations near the origin, it is easy to see that it is a hyperbolic fixed point. The solutions close to the origin are given by

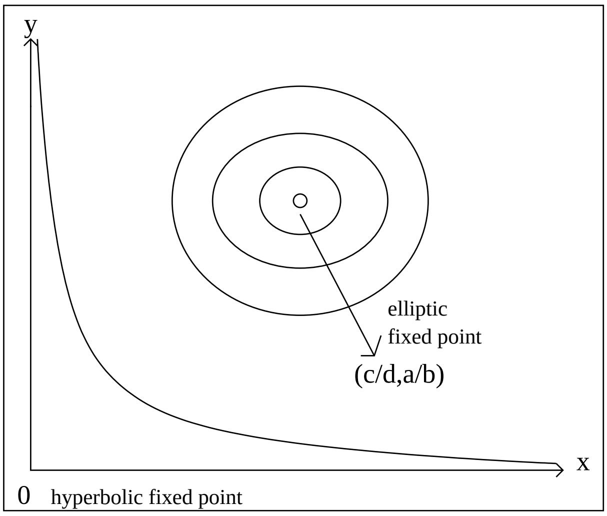
$$x = x_0 e^{At}, \quad y = y_0 e^{-Ct}$$

- whose long time behaviour is obvious.

The second fixed point is more interesting since the eigenvalues of the stability matrix around this fixed point are given by,

$$\lambda_{1,2} = \pm i(AC)^{1/2}$$

which is an elliptic fixed point. For any given initial values of (x,y) the populations vary cyclically around the elliptic fixed point in its neighbourhood. This is a more stable situation and is a good model for natural phenomena involving two species.



It is also easy to verify that there is a constant of motion in the system, namely

$$E = x^C e^{-Dx} y^A e^{-By}.$$

Obviously the phase curves are given by constant E curves.

2.4 Problems

Given below and in the following chapters are a set of sample problems given as assignments during the course.[3]

1. Sketch the phase diagram for systems with the following velocity functions, where a and b are constants with $b > a > 0$:

$$v_x(x,y) = a,\ v_y(x,y) = b; \quad v_x(x,y) = a,\ v_y(x,y) = x$$

$$v_x(x,y) = x^2,\ v_y(x,y) = y; \quad v_x(x,y) = 2xy,\ v_y(x,y) = y^2 - x^2$$

   Find the fixed points in each of these systems and classify them.

2. Classify the fixed points of the following linear systems and state whether they are stable or unstable:

$$\dot{x} = 3x + 4y, \quad \dot{y} = 2x + y$$

$$\dot{x} = 3x, \quad \dot{y} = 2x + y$$

$$\dot{x} = x + 2y, \quad \dot{y} = -2x + 5y$$

$$\dot{x} = x + 2y, \quad \dot{y} = -5x + 4y$$

3. Classify the fixed points of the equation system,

$$\frac{dx}{dt} = -y + xr^2 \sin(\pi/r)$$

$$\frac{dy}{dt} = x + yr^2 \sin(\pi/r)$$

   where

$$r^2 = x^2 + y^2.$$

   Draw the phase curves for $r < 1$, $r = 1$ and $r > 1$. Is there anything special if $r = 1/n$, where n is an integer ?

4. A system of order 2 is described by the velocity fields,

$$v_x = x - 3y - (x^3 + y^3 + xy^2 + yx^2),$$

$$v_y = y + x + (3x^3 - y^3 + 3xy^2 - yx^2).$$

   Draw the phase portrait of the system and show that it has a limit cycle.

[3]The problems are almost entirely taken from the book Dynamics by Percival and Richards. There is a rich array of problems in this book.

5. A second order system is described by the velocity fields,

$$v_x = x^3 - y; \qquad v_y = y^3 - x.$$

Show that if you begin with a small patch in the phase space of this system then the patch will grow in size with the passage of time. Hence show that the system cannot have any stable fixed point. Classify all the fixed points of the system and show that indeed, none of the fixed points are stable. Draw the phase portrait of the system.

6. A second order system is described by the velocity fields,

$$v_x = -y^3 - x^2 y - 3y,$$

$$v_y = x + 3x^3 + 3xy^2.$$

Show that the circle $x^2 + y^2 = 1$ is a phase curve of this system. Is it a limit cycle?

7. A first order dynamical system is described by the velocity function

$$v(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$$

Find the fixed points of the system. Draw the phase portrait and discuss the stability of the fixed points.

8. A falling body in air experiences resistance proportional to the square of its velocity V(t). Determine the equation for the rate of change of the velocity of the falling body. Describe the changes of V(t) for arbitrary initial conditions $V(0) > 0$. Draw the phase portrait of the system. What is the limiting behaviour as $t \to \infty$.

9. The angle $\psi (0 \leq \psi \leq 2\pi)$ has a time variation given by the equation

$$\frac{d\psi}{dt} = a + b\sin(\psi); \quad (a > 0, b > 0)$$

Find the fixed points and invariant sets of motion when $a < b; a = b; a > b$. Which one of these regimes would result in a motion corresponding to rotation? Find the period of rotation?

10. Show that a growth sequence given by the recurrence relation

$$a_{n+1} = (a_n + a_{n-1})/2$$

can never be exponential. Further writing the recurrence relation as a two component matrix evolution equation, find a general solution for a_n given $a_0 = x, a_1 = y$.

11. Classify all the fixed points and limit cycles (if any) of the two-dimensional flow defined by the equations,

$$\frac{dx}{dt} = x(a\sin(br^2)/r^2) - cy$$

$$\frac{dy}{dt} = y(a\sin(br^2)/r^2) + cx$$

where a, b, c are positive constants and $r^2 = x^2 + y^2$.

Answer: The problem is similar to the Ricatti equations. Use the same technique. Multiply the first equation by x and the second by y:

$$x\frac{dx}{dt} = x^2(a\sin(br^2)/r^2) - cxy$$

$$y\frac{dy}{dt} = y^2(a\sin(br^2)/r^2) + cyx$$

Adding the two equations we get

$$\frac{1}{2}\frac{dr^2}{dt} = r^2(a\sin(br^2)/r^2)$$

$$\frac{1}{2}\frac{dr^2}{dt} = a\sin(br^2)$$

Defining $x = r\cos\theta, y = r\sin\theta$, we have

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\ \dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\ \dot{\theta}$$

Using the equation for r^2, we get

$$\dot{\theta} = c \Rightarrow \theta(t) = ct + c_0.$$

However the equation for r, is now a one dimensional equation , each value of r specifies a circle in the two dimensional phase space. The fixed points of this equation are given by,

$$\frac{\sin(br^2)}{r} = 0 \Rightarrow r = \sqrt{n\pi/b}; n \geq 0, b > 0$$

Thus the fixed points in the values of r correspond to limit cycles. There are infinitely many starting from $r = \sqrt{\pi/b}$ with radius increasing as $\sqrt{n}$.

The stability is obtained from the sign of the velocity function. Since a, r, b are all positive, it is just the sign of the sine function which alternates. Thus the phase curve diverge from every odd limit cycle and converge asymptotically to the even limit cycle.

12. The Dutch engineer van der Pol modelled a vacuum tube circuit by the differential equation

$$\frac{d^2x}{dt^2} + \epsilon(x^2 - 1)\frac{dx}{dt} + x = 0$$

(a) Rewrite the above equation, as a set of first order dynamical equations and obtain the fixed points of the system.

(b) Describe the behaviour of the system around the fixed points for $\epsilon = 0$ and $\epsilon = 1$.

(c) Comment on the deformation of the phase curves as ϵ varies from 0 to 1 (qualitative behaviour is sufficient).

13. A population T of tigers feeds on a population D of deers. The total birth rate of tigers is proportional to its population and to the amount of food available. Their death rate is proportional to their population. The birth rate of the deers is also proportional the population and they die through encounters with tigers. With these assumptions, obtain the set of first order differential equations for the rate of change of populations. Find the fixed points of the system and describe the dynamical evolutions of the populations locally around the fixed points.

Chapter 3

Newtonian Systems

We will first introduce some basic notions about Newtonian systems as further examples of second order systems. In fact the oldest of the second order systems to be analysed is the Newtonian system. Newton's profound discovery that the mechanical systems are described by laws which may be written in the form of differential equations laid the foundation of modern theoretical physics- method of describing and predicting and indeed the deterministic evolution of any physical system.

For now, we shall assume that the dynamical system that we are interested consists of a single point mass. Going back to the standard notation in mechanics, let us denote the state of the system in the two dimensional phase space by two real variables (q, p), where q is the position (or more generally a configuration space coordinate) and p is the momentum. The state of the system at time t then determines the motion for all times.

The central law of mechanics is the Newton's law that an external force causes a proportional change in the velocity. That is,

$$F = ma = m\frac{d^2q}{dt^2}$$

where a is the acceleration and m denotes the mass of the particle.

For a particle moving in a "line" (more generally one degree of freedom) under the action of a force $F(q, t)$ the first order equations of motion are

$$dq/dt = p/m,$$

which defines the momentum, and

$$dp/dt = F(q, t).$$

Since the particle has only one degree of freedom, the force can always be derived from a **potential** $V(q, t)$ such that

$$F(q, t) = -\frac{\partial V}{\partial q}(q, t)$$

where

$$V(q,t) \equiv -\int_0^q dq' F(q',t)$$

and

$$\frac{p}{m} = \frac{d(p^2/m)}{dp}$$

Thus we can define a **Hamiltonian function**[1] for a mechanical system obeying Newton's laws as

$$H(q,p,t) = \frac{p^2}{2m} + V(q,t) \tag{3.1}$$

and the equations of motion may be written in terms of the partial derivatives of the Hamiltonian function.

$$dq/dt = \frac{\partial H}{\partial p} \tag{3.2}$$

$$dp/dt = -\frac{\partial V}{\partial q} = -\frac{\partial H}{\partial q} \tag{3.3}$$

The **phase velocity** is therefore given by

$$\vec{v} = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q})$$

The phase velocity is orthogonal to the gradient of H:

$$\vec{\nabla} H = (\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}); \quad \vec{v}.\vec{\nabla} H = 0.$$

Usually we refer to q as the generalised coordinate and p the generalised momentum, though they need not always correspond to the physical position and momentum in one dimensional motion.

The rate of change of the Hamiltonian for a given motion is given by,

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial q}\frac{dp}{dt} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

by virtue of the equations of motion. When the system is **conservative** the Hamiltonian does not explicitly depend on time and therefore it is conserved. For a Newtonian system this is a statement of **energy conservation**. It may seem that this is a highly idealised system since most naturally occurring systems are not isolated and are mostly dissipative. Often the time scale of dissipation is so large that we may, during the short intervals that the observations are made, treat them as conservative. The most important of such systems is the solar system itself.

We note some properties of the phase space (q,p):

- The velocity function or the phase velocity, $\vec{v}$, is normal to the gradient of H and of the same magnitude.

[1] A more general analysis of the Hamiltonian systems is given subsequently

- Since the Hamiltonian is conserved, the motion of the system is along phase curves corresponding to constant H.
- The contours corresponding a given H are invariant sets of the system since the motion is confined to a given contour for all times. The fixed points, or points of equilibrium, of the system are defined as usual by the vanishing of the velocity function.

3.1 Phase Portraits

Let us illustrate the properties of the phase space with a few examples, mainly conservative Hamiltonian systems.

Example 1 : Consider the example of one dimensional harmonic oscillator. The potential is given by,

$$V(q) = \frac{m}{2}\omega^2 q^2 \tag{3.4}$$

and the equations of motion are

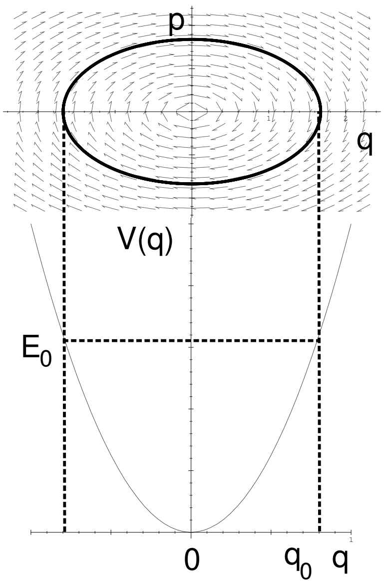
$$dq/dt = v_q(q,p) = p/m$$
$$dp/dt = v_p(q,p) = -m\omega^2 q.$$

Origin (0,0) is the fixed point and the eigenvalues of the stability matrix around the origin are imaginary $\pm i\omega$. Infant, the linear stability analysis around the fixed point provides the complete solution to the system. The phase curves are ellipses corresponding to constant energy as shown in the figure below.

$$E = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2$$

where

$$q(t) = A\cos(\omega t + C), \quad p(t) = -m\omega A \sin(\omega t + C)$$



The period is of oscillation is of course given by $T = 2\pi/\omega$ and is independent of energy.

Example 2 : Consider the example of one dimensional system with a linear repulsive force (inverted oscillator),

$$V(q) = -\frac{m}{2}\omega^2 q^2 \tag{3.5}$$

and the equations of motion are

$$dq/dt = v_q(q,p) = p/m$$

$$dp/dt = v_p(q,p) = m\omega^2 q.$$

Origin (0,0) is the fixed point and the eigenvalues of the stability matrix around the origin are equal in magnitude and opposite in sign $\pm\omega$. Therefore the origin is a hyperbolic fixed point.

The equation of the phase curves is obtained from the energy expression itself since it is a conservative system (E is conserved): The phase curves are hyperbolas in general corresponding to constant energy E¿0.

$$2mE = p^2 - m^2\omega^2 q^2 = (p - m\omega q)(p + m\omega q).$$

When $E = 0$ the phase curves are straight lines corresponding to $p = \pm m\omega q$. These obviously pass through the fixed point at the origin. Such phase curves which pass through a hyperbolic fixed fixed points are called **Separatrixes**. They mark boundaries between phase curves which are distinct and can not be deformed continuously into one another.

Example 3 : We can consider many variations on oscillator theme, for example the quartic oscillator.

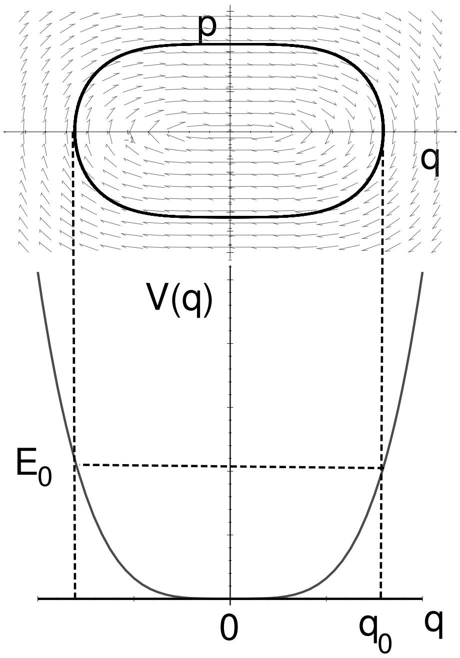
$$V(q) = q^4/4 \tag{3.6}$$

and the equations of motion are

$$dq/dt = v_q(q,p) = p/m$$

$$dp/dt = v_p(q,p) = -q^3.$$

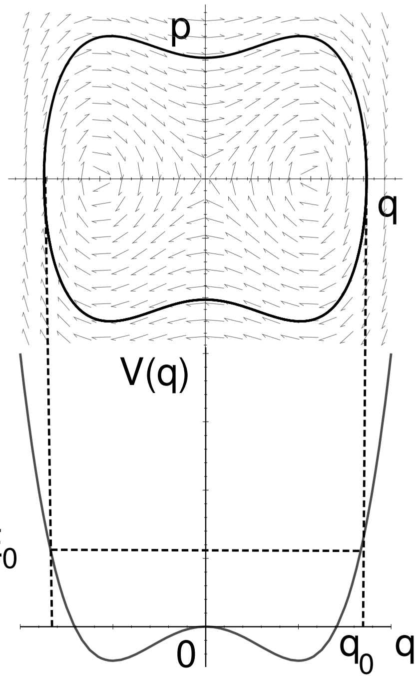
Again the origin is the only fixed point of the system. The phase curves correspond to the constant energy contours. The phase portrait is given below.



Example 4 : The Quartic Oscillator with a saddle point has a potential given by

$$V(q) = \frac{1}{4}q^4 - \frac{1}{2}q^2 \tag{3.7}$$

Note that the system has three fixed points (q,p) phase space, two stable elliptic fixed points at, $(q,p) = (\pm 1, 0)$, and one hyperbolic fixed point at the origin, $(q,p) = (0,0)$, as shown in the figure. The nature of motion $E < E^*, E = E^* and E > E^*$ (E^* = energy at the local maximum $q = 0$) are very different.

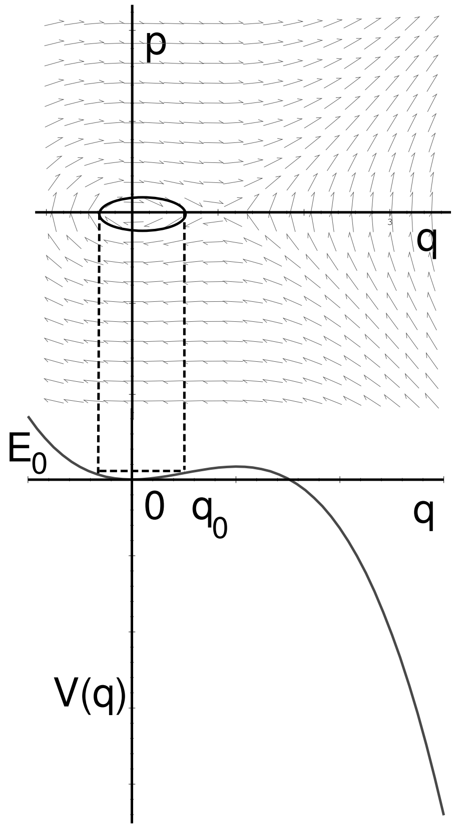


Note that the above examples correspond to cases where the phase curves are compact. Now consider some in which it is mixed:

Example 5 : The Cubic Oscillator

$$V(q) = \frac{1}{2}q^2 - \frac{1}{3}q^3 \tag{3.8}$$

The fixed points of the system are given by $(q,p) = (0,0),\ (1,0)$. While the first one is an elliptic fixed point, the second one is a hyperbolic fixed point. The phase space is not compact. For energies $E > E^*$ (E^* = Energy at the local maxima) the particle goes of to infinity at large times.



Example 6 :

We now consider an example of a non-conservative system- namely the damped harmonic oscillator.

$$\ddot{q} + c\dot{q} + \omega^2 q = 0\,. \tag{3.9}$$

or equivalently,

$$\dot{q} = p, \quad \dot{p} = -cp - \omega^2 q. \tag{3.10}$$

The fixed point is of course the origin. The nature of motion depends on the value of the damping coefficient c. The characteristic equation is of course

$$\lambda^2 + \lambda c + \omega^2 = 0$$

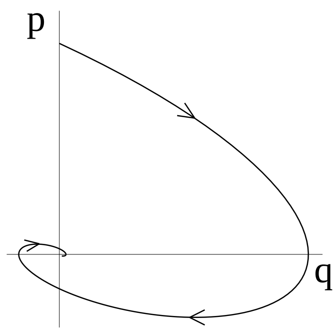
whose eigenvalues are

$$\lambda_{1,2} = -\frac{1}{2}[c \pm \sqrt{c^2 - 4\omega^2}].$$

We shall first consider the case when c is positive. The solution for $\omega > c/2$ is given by

$$q(t) = q_0 e^{-ct/2} \sin(\nu\, t + \alpha) \tag{3.11}$$

where $\nu = \sqrt{\omega^2 - c^2/4}$. It is a case of weak damping. The fixed point is a stable spiral fixed point for this case.



If $\omega < c/2$, then strong damping occurs since both the eigenvalues are real and positive with the solution

$$q(t) = A \exp(-\lambda_1 t) + B \exp(-\lambda_2 t)$$

where A and B are determined by the initial conditions. The phase curves do not show any oscillatory behaviour, tend to the origin as t increases. The fixed point is therefore a stable node.

If c is negative, and $\omega > c/2$, we have the opposite situation with unstable spiral and unstable node in the two cases that we considered above. It is an interesting exercise to show that such damped (or dissipative) as the one considered here can never be Hamiltonian!

Example 7 : Consider the free rotations about an axis. The variable is the angle $\theta(t)$. For free rotations,

$$\dot{\theta} = p_\theta$$

which is a constant and therefore the angular momentum

$$\dot{p}_\theta = 0.$$

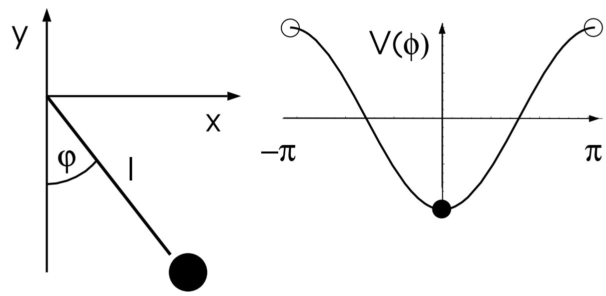
The Hamiltonian is

$$H = \frac{p_\theta^2}{2I}$$

where I is the momentum of inertia of the freely rotating body. The phase space trajectories are straight lines parallel to the θ axis.

3.2 The Pendulum

The mathematical pendulum is the prototype of one-dimensional nonlinear systems. The actual solutions are complicated involving the elliptic integrals and Jacobian elliptic functions.



Since the length of the pendulum $l = const.$ the system has only one degree of freedom, namely the angle ϕ.

$$p_\phi = ml^2\dot\phi\,, \tag{3.12}$$

$$H(p_\phi,\phi) = \frac{p_\phi^2}{2ml^2} - mgl\cos\phi\,. \tag{3.13}$$

The phase space is defined by variables (p_ϕ, ϕ). Note that $I = ml^2$ is the moment of inertia of the system and further the angular momentum $L = I\omega = I\dot\phi = p_\phi$ both defined with respect an axis of rotation. In this case it is the line passing through the support, perpendicular to the plane of vibration (rotation) or the pendulum.

While the Newtonian equation of motion is of second order,

$$l\ddot\phi + g\sin\phi = 0 \tag{3.14}$$

and the first order evolution equations in the phase space are given by

$$\begin{aligned}\dot\phi &= \frac{p_\phi}{ml^2}\,,\\ \dot p_\phi &= -mgl\sin\phi\,.\end{aligned} \tag{3.15}$$

The fixed points of the pendulum are given by

$$\vec v(q,p) = 0 \Rightarrow p = 0, q = \phi = n\pi, \quad n = 0, \pm1, \pm2, \ldots.$$

It is easy to check that the fixed points are alternatively elliptic (n-even) and hyperbolic (n-odd). Since the motion is periodic, we can limit the angle $\phi \in [-\pi, \pi]$ and $-\infty < p_\phi < \infty$. In fact both π and $-\pi$ correspond to the same point - if one folds the edges, the phase space corresponds to a cylinder of infinite extent in the y-direction while the configuration space is compact and reduces to S^1 with the above mapping.

The energy of the system may be obtained as the first integral from the equations of motion.

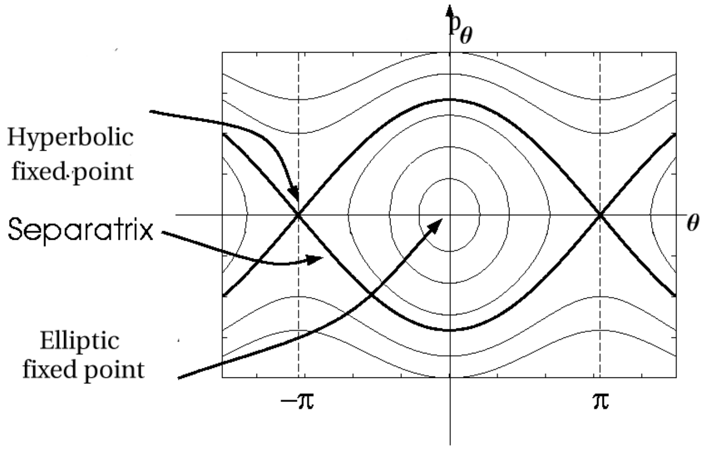
$$E = \frac{m}{2}l^2\dot\phi^2 - mgl\cos\phi\,, \tag{3.16}$$

The special case when $E = mgl$ corresponds to the energy along the special trajectory which meets the hyperbolic fixed point- called the **separatrix**.

3.2.1 *Phase portrait*

As mentioned before, when the motion is periodic, we can limit the angle $\phi \in [-\pi, \pi]$ and $-\infty < p_\phi < \infty$. The phase space corresponds to a cylinder of infinite extent in the y-direction.

The phase portrait of a pendulum is given below:



1. *Elliptic fixed point*: For small ϕ - the pendulum just oscillates about the fixed point at the origin. Close to the origin linear stability analysis shows that that this is an elliptic fixed point which corresponds to $E = -mgl$. The motion in the region $-mgl < E < mgl$ is usually referred to as **Libration** which is characterised by an average momentum: $\langle p_\phi \rangle = 0$. The phase curves are approximate ellipses around the fixed point.

2. *Unstable hyperbolic fixed point*: The hyperbolic point corresponds to $\phi = \pm\pi$- the point where the pendulum is held vertically upwards corresponding to $(E = mgl)$; This is an unstable situation since a small displacement forces the mass to move away from the fixed point unlike the stable case. The $E = mgl$ curve in the phase space passing through the hyperbolic fixed point is called the **separatrix** whose equation is given by
$$\frac{1}{2}mgl\cos^2(\phi/2) = \frac{p_\phi^2}{2m}$$

3. The motion in the region $E > mgl$ is **rotation**. Here p_ϕ does not change sign and $\langle p_\phi \rangle \neq 0$.

4. The separatrix divides the phase space into two disconnected regions: inside the separatrix the phase curves are closed and may be continuously deformed into one another by changing energy. Outside the separatrix the motion is characterised by rotations with p_ϕ having a definite sign through out the phase curve and hence open. Thus there are two distinct **homotopy classes**[2] separated by the separatrix.

[2] Let f_1 and f_2 be two functions such that $f_1\ , f_2\colon \mathbf{I} \to \mathbf{R}$, where $\mathbf{I} \subseteq \mathbf{R}$. Let $F\colon [0,1] \times \mathbf{I} \to \mathbf{R}$. If $F(0,\mathbf{I}) = f_1$ and $F(1,\mathbf{I}) = f_2$, then f_1 is said to be *homotopic* to f_2.

3.2.2 Elliptic functions

The time taken to cover an angle ϕ is

$$t(\phi) = l\sqrt{\frac{m}{2}} \int_0^{\phi} \frac{d\phi'}{\sqrt{E + mgl\cos\phi'}}. \tag{3.17}$$

which is an elliptic integral and $\phi(t = 0) = 0$. Let us now consider different kinds of motion possible for a pendulum depending on the energy.

Case $E < mgl$:

If the pendulum oscillates but does not rotate (does not have enough energy to cross the separatrix) then we can write,

$$E = -mgl\cos\phi_0\,. \tag{3.18}$$

Obviously the angle ϕ_0 corresponds to the maximum deviation from the mean position when the velocity is zero. The integral in (3.17) can then be converted into the standard form as follows:

$$t(\phi) = \sqrt{\frac{l}{2g}} \int_0^{\phi} \frac{d\phi'}{\sqrt{\cos\phi' - \cos\phi_0}}. \tag{3.19}$$

Let

$$\cos\phi' - \cos\phi_0 = 2\left[\cos^2(\phi'/2) - \cos^2(\phi_0/2)\right] = 2\left[\sin^2(\phi_0/2) - \sin^2(\phi'/2)\right]$$

and further

$$\sin\xi(\phi') = \frac{\sin(\phi'/2)}{\sin(\phi_0/2)} \tag{3.20}$$

Substituting we get

$$\begin{aligned} t(\phi) &= \sqrt{\frac{l}{g}} \frac{1}{2\sin(\phi_0/2)} \int_0^{\phi} \frac{d\phi'}{\sqrt{1 - \sin^2\xi(\phi')}} \\ &= \sqrt{\frac{l}{g}} \int_0^{\gamma} \frac{d\xi}{\sqrt{1 - k^2\sin^2\xi}} \quad = \quad \sqrt{\frac{l}{g}}\, F(\gamma, k)\,, \end{aligned} \tag{3.21}$$

where we used the following definitions

$$\gamma = \arcsin\left[\frac{\sin(\phi/2)}{\sin(\phi_0/2)}\right], \tag{3.22}$$

$$k^2 = \sin^2(\phi_0/2)\,. \qquad (k^2 < 1) \tag{3.23}$$

The integral in (3.21) is usually known as the elliptic integral of the first kind.

$$F(\gamma, k) := \int_0^{\gamma} \frac{d\xi}{\sqrt{1 - k^2\sin^2\xi}}\,. \tag{3.24}$$

The period of the pendulum T corresponds to a change in angle such that the pendulum returns to its original position. that is, $\phi = \phi_0$, corresponding to $\sin\xi = 1 \Rightarrow \xi = \pi/2$. In this limit we have

$$K(k) := F(\frac{\pi}{2}, k)\,. \tag{3.25}$$

where $K(k)$ is the Jacobian (or complete) elliptic function. The period of the pendulum is given by

$$T = 4\sqrt{\frac{l}{g}}K(k)\,. \tag{3.26}$$

The factor 4 comes from the fact that a single period T consists of four traversals $0 \le \phi \le \phi_0$.

Following remarks are useful:

1. When the energy $E \to mgl$ the parameter $k \to 1$. In this limit $K(k)$ diverges and the period $T \to \infty$.
2. In the limit of small energies,$E \to -mgl$ we have $\phi_0 \to 0$ $k \to 0$ and $K(0) = \frac{\pi}{2}$, the period of the pendulum is that given usually by the period of a simple pendulum- small amplitude vibration: $T = 2\pi\sqrt{\frac{l}{g}}$

Case $E > mgl$ Here we define a constant μ such that

$$\mu = mgl/E < 1\,. \tag{3.27}$$

The integral (3.17) can be written as

$$\int_0^\phi \frac{d\xi}{\sqrt{1+\mu\cos\xi}} = F\left(\frac{\phi}{2}, \sqrt{\frac{2\mu}{1+\mu}}\right). \qquad (\mu < 1) \tag{3.28}$$

Define

$$q^2 = \frac{2\mu}{1+\mu} = \frac{2mlg}{E+mlg} < 1 \tag{3.29}$$

we get

$$t(\phi) = \sqrt{\frac{2ml^2}{E+mlg}}\,F\left(\frac{\phi}{2}, q\right). \tag{3.30}$$

The period is given by

$$T = t(\pi) - t(-\pi) = 2\sqrt{\frac{2ml^2}{E+mlg}}\,K(q) \tag{3.31}$$

which is divergent when $E \to mgl$, $q \to 1$. For $E \to \infty$ we have $q \to 0$ and the only motion is regular rotation.

Few remarks are in order: The motion can be discussed in the following phase domains:

- *Approximately harmonic motion:*
  $-mgl \leq E \ll mgl$; $\Rightarrow \phi \ll 1$, $\sin\phi \simeq \phi$, and the differential equation becomes linear: $\ddot{\phi} + \frac{g}{l}\phi = 0$. The solution is $\phi(t) = a\sin(\omega t + \alpha)$ where $\omega = \sqrt{g/l}$; The period is $T = 2\pi/\omega = 2\pi\sqrt{l/g}$.

- *Rotations:*
  $E \gg mgl$; $T \gg V \Rightarrow V(\phi)$ can be neglected.
  $\Rightarrow E = T$ and $\dot{\phi} := \omega = \sqrt{2E/ml^2}$
  $\Rightarrow \phi(t) = \omega t + \phi_0; \quad T = 2\pi/\omega = 2\pi\sqrt{ml^2/2E}$.

3.3 Problems

1. Classify the fixed point/s of a dynamical system described by the second order equation

$$\frac{d^2x}{dt^2} + x - x^2 = 0$$

and draw the phase curves of the system. (This is the one-dimensional analogue of the so called Henon-Heiles system in two dimensions which is chaotic.).

2. A boat is rowed across a river of width a. It is rowed so that it always points towards a fixed point on the opposite bank. The boat moves at a constant speed u relative to the water which flows at a constant speed v.

   (a) Write down the equations of motion of the system.

   (b) Obtain the equation of the phase curves and sketch the phase diagram.

3. Draw the phase portrait of a damped pendulum described by the equation,

$$\frac{d^2\theta}{dt^2} + \alpha\frac{d\theta}{dt} + \frac{g}{L}\sin(\theta) = 0$$

and classify the fixed points. (θ is the angle made by the pendulum with respect to downward vertical).

4. Sketch the phase curve of a particle of mass moving in the potential

$$V(q) = -A\ q^4 \quad (A > 0)$$

Give the equation of the separatrix and find the motion, $q_s(t)$ on the separatrix. Demonstrate that this motion terminates.

The particle starts at time $t = 0$ at the point $q_0 > 0$ with momentum $p_0 > 0$. Obtain an integral expression for the time t at which it reaches the point $q > q_0$, and hence show that the motion terminates.

5. " An autonomous Hamiltonian system cannot have a limit cycle". If you believe in this statement, prove it. If you do not, give a counter example.

6. (a) Given a set of first order equations for a second order autonomous system:

$$\frac{dq}{dt} = v_q(q, p)$$

$$\frac{dp}{dt} = v_p(q, p)$$

derive the condition for this system to be a Hamiltonian system assuming p to be momentum conjugate to q.

(b) Choosing $p = f(q)\frac{dq}{dt}$, show that the equation

$$\frac{d^2q}{dt^2} + (\frac{dq}{dt})^2 + q = 0$$

may be put in a Hamiltonian form, if at all.

7. A damped vertical pendulum has the equation of motion

$$\ddot{\psi} + \alpha\dot{\psi} + \omega^2 \sin\psi$$

where α is chosen to be positive and ψ is the angle between the pendulum and the downward vertical. Determine the position and the nature of the fixed points. Sketch the phase curves in a sufficiently large region of phase space to show all qualitative features of the motion. Describe what happens to the motion and fixed points as $\alpha \to 0$.

8. A particle is constrained to slide under gravity on a smooth wire in the shape of a vertical circle of radius R, as the wire rotates about a vertical diameter with constant angular velocity ω. If ψ is the angular displacement of the particle from the downward vertical,

   (a) write down the Hamiltonian of the system.

   (b) What is the order of the system?

   (c) Find the fixed points of the system and classify them.

9. Show that the equation of motion

$$\frac{d^2q}{dt^2} + G(q)[\frac{dq}{dt}]^2 - F(q) = 0$$

may be put into Hamiltonian form by choosing the conjugate momentum

$$p = f(q)\frac{dq}{dt},$$

where,

$$f(q) = exp[q \int^q dq' G(q')].$$

With this choice, show that the Hamiltonian is,

$$H(q,p) = \frac{p^2}{2f(q)} + V(q),$$

where,

$$V(q) = -\int^q dq' F(q') f(q').$$

10. Consider a bead of mass m sliding smoothly on a wire of shape $y = f(x)$ in the vertical (y, x) plane. The potential energy is given by $V(y) = mgy$. (a) Show that the Lagrangian of the system can be written as,

$$L = \frac{1}{2}(\frac{dx}{dt})^2[1 + (\frac{df}{dx})^2] - mgf(x).$$

Obtain the corresponding Hamiltonian.

(b) If $f(x) = -\cos(\pi x/2a)$; $\quad -a \le x \le a$ discuss the motion in the neighbourhood of $x = 0$.

Chapter 4

Variational Principle and Lagrange equations

Until now we have not introduced the Lagrangian for a mechanical system even though the traditional route to Hamiltonian mechanics is through Lagrangian mechanics. In the preceding chapter we introduced the Hamiltonian directly from Newtonian mechanics and showed that Hamiltonian dynamics is a special case of more general dynamics. The first order dynamical equations were written directly from Newton's equations of motion by defining the momentum through $p = m\ dq/dt$. However, it is not always possible to define momentum this way especially in the presence of constraints. The relationship between velocities and momenta is not one of simple scaling.

It is easier to formulate equations of motion in terms of configuration space variable $q(t)$ and the corresponding velocitydq/dt, while the dynamical evolution of the system is more easily described in the phase space. As we shall see later while the phase space evolution is area preserving, the $q, \dot{q}$ is not always area preserving.

4.1 Constraints and generalised coordinates

Consider a system of n-particles in a d dimensional space. If there are no constraints, the number of degrees of freedom $f = dn$. That is, the state of the system in the **configuration space** is described by dn real numbers. However, if the system is constrained the position vectors at any time of the n- particles may not all be independent. The geometric properties of the system decide the nature of **generalised coordinates**.For example a bead constrained move along a string or a material point confined to move on a sphere in three-dimensional space.

A constraint is analytically expressed by an equation (or equations) of the form

$$\phi(\vec{r_i}, t) \leq 0.$$

Such a constraint is called a *unilateral* constraint. However, a constraint of the form

$$\phi(\vec{r_i}, t) = 0$$

is a *bilateral* constraint[1] Such constraints are called *holonomic* constraints. Constraints which are not of the form of holonomic constraints are referred to as *non-holonomic* constraints. Constraints are non-holonomic if they :

- Contain velocities or, differentials only of the radius vector(s).
  Example: A wheel rolling on a surface, has constraint given by : $\vec{v} = \dot{\vec{\theta}} \wedge \vec{r}$. This constraint relates the angular velocity $\dot{\vec{\theta}}$ to the linear velocity $\vec{v}$. This motion cannot be integrated, without knowing the motion of the wheel.

- Appear only in *inequalities*.
  Example: A particle confined in a closed box, such that its configuration space is limited by the dimensions of the box.

If holonomic constraints are independent of time, then they would be called *scleronomic* constraints. If they explicitly depend on time then they are called *rheonomic*. Here the analysis is confined to only holonomic constraints. In general there may be k- such independent equations of constraint[2]:

$$\phi_j(\vec{r_i}, t) = 0, \quad \text{where} \;\; j \in \{1, \ldots, k\}$$

so that the state of the system is described by $N = dn - k$ generalised coordinates $q_i(t)$ such that,

$$\begin{aligned} q_1 &= q_1(x_1, x_2, ..., x_N) \\ q_2 &= q_2(x_1, x_2, ..., x_N) \\ q_3 &= q_3(x_1, x_2, ..., x_N) \\ &\;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ q_N &= q_1(x_1, x_2, ..., x_N) \end{aligned}$$

where the generalised coordinates q_i are functions of x_i obtained after imposing constraints, ie., $x_i, i = N+1, ..., dn$ are treated as coordinates dependent on $x_i, i = 1, ..., N$. The transformation from natural coordinates to generalised coordinates must be non-singular, that is, the Jacobian of the transformation

$$\det\left(\frac{\partial(q_1, \ldots, q_N)}{\partial(x_1, \ldots, x_N)}\right) \neq 0 \qquad .$$

4.2 Calculus of variations

We begin this section with an old but classic problem, which marked the beginning of *variational calculus*. Known as the *brachistochrone problem*, meaning the problem

[1]The word bilateral constraint comes from the fact that $\phi = 0$ may be looked upon as a combination of two unilateral constraints $\phi > 0$ and $\phi < 0$.

[2]Mathematically speaking, the constraints are given by the kernel of the map(vector field) $\vec{\Phi}\colon \mathbf{R}^{dn} \to \mathbf{R}^k \;\;,\;\; \textit{where} \;\;\; \vec{\Phi} := \vec{\Phi}\{\phi_1, \ldots, \phi_k\}$

of least time, was one of the earliest problems to apply the calculus of variations. It was originally posed by Johann Bernoulli in 1696, and solved by Newton (within an evening! It was an open problem at that time...). The problem may be simply stated as:

> Let there be two nails fixed to a wall, at random positions, such that one is above the other. A string is attached between the nails, and a bead is introduced such that it can fall freely along the string due to gravity. What would be the shape of the string such that the bead takes the least time to go from the upper nail (say nail A) to its lower counterpart (say nail B) ?

Some ground rules should be kept in mind: The problem assumes no friction or any other form of dissipation. The size of the bead is also of no consequence. It is an ideal setting which exists only our minds. Before analysing the problem, note that the shortest distance between any two points of course is a straight line. While this is the shortest path, is it the quickest? To realise that the shortest path need not be the quickest, all one has to do is to put the nails at the same height and start reducing the height of one of the nails.

To proceed further, let us first calculate the time taken to travel from a point A to B. We have,

$$vdt = ds,$$

where ds is the length of the arc covered in time dt when the velocity is v, actually speed. Equivalently,

$$T_{AB} = \int_A^B \frac{ds}{v},$$

and

$$v = \sqrt{2gy},$$

where g is acceleration due to gravity and y is the height at some point along the actual path. Since the motion is in a plane, $x-y$, where x denotes horizontal distance and y vertical distance, we may parametrise the path by $y = y(x)$. Therefore,

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (\frac{dy}{dx})^2 dx}$$

. The time taken is the given as an integral,

$$T_{AB} = \int_A^B dx \sqrt{\frac{1 + y'^2}{2gy}}.$$

The solution to the Brachistochrone problem is obtained by determining $y(x)$ such that the time taken between two points is a minimum.

In solving the above problem, Bernoulli developed new methods which were later refined by Leonhard Euler into what we now call the calculus of variations. The general problem may be stated as follows: Given a function W,

$$W = \int_A^B dxF(y, y', x).$$

the curve that gives the extremal or stationary value to W, that is

$$\delta W = \delta \int_A^B dxF(y, y', x) = 0$$

must satisfy the differential equation

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0. \tag{4.1}$$

This equation was obtained by Euler using the calculus of variations.

Going back to the Brachistochrone problem, we have

$$F = \sqrt{\frac{1 + y'^2}{2gy}}$$

and using the Euler equation we obtain the equation of the curve along which the time taken is the least. Thus

$$F - y'\frac{\partial F}{\partial y'} = C$$

where C is a constant.

The solution of the above equation is a cycloid which is given by the following parametric equations

$$x = \frac{1}{2}k^2(\theta - \sin\theta), \quad y = \frac{1}{2}k^2(1 - \cos\theta)$$

where $k^2 = 1/2gC^2$. Note that the brachistochrone does not depend on the mass or the strength of gravitational constant.

4.3 Variational principle and the Lagrangian

Now we return to dealing with the description of the motion of a system of **n** particles. To describe the motion of the system, we also require N generalised velocities, $\dot{q}_i = \frac{d}{dt}q_i(t)$. We often use the shorthand notation:

$$\mathbf{q}(t) = \{q_i(t)\}, \quad \dot{\mathbf{q}}(t) = \{\dot{q}_i(t)\}. \qquad (i = 1, 2, 3, \ldots N) \tag{4.2}$$

Suppose we want to describe the motion of this many particle system at a point A at time t_1 and at point B at a later time t_2 with the end points fixed. We assume that there exists a function, L called the **Lagrangian**, which has all the necessary information about the dynamical evolution of the system.

In order to derive the Lagrange's equations of motion, we first define **Hamilton's Principle Function** (also called the *action integral*) as:

$$R = \int_{t_1}^{t_2} L[\mathbf{q}(t), \dot{\mathbf{q}}(t), t]\, \mathrm{d}t\,. \tag{4.3}$$

Note that R is a *functional*[3] of $q(t)$, it depends on not just one value t, but on the function q and all of t in a given interval t_1 to t_2. The Hamilton's **variational principle** states that the **physical trajectory** is one for which the action R is a minimum (extremum)- also some times referred to as the principle of least action. This principle is the most general formulation of the law governing the time evolution of mechanical systems. Minimising the action leads to the Lagrange equations of motion.

The equations of motion are derived by infinitesimal variation of the path defined by the set $\mathbf{q}(t)$ with the initial and end points fixed, that is

$$\mathbf{q}(t) \to \mathbf{q}(t) + \delta\mathbf{q}(t)\,, \qquad \delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = 0\,, \tag{4.4}$$

such that the action integral is a minimum, or equivalently it is stationary under the variation δq_i:

$$\frac{\delta R}{\delta q_i(t)} = 0 \quad + \ \mathcal{O}[(\delta q_i)^2]\,. \tag{4.5}$$

Immediately it is possible to derive the **Lagrange equations of motion:**

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0\,. \qquad (i = 1, 2, 3 \ldots N) \tag{4.6}$$

Note that the variational principle is a global statement which leads to the Lagrange equations which are local differential equations.

Remarks :

- The Lagrange equations for a system with N degrees of freedom are a set of N differential equations of second order. They are therefore solved by specifying $2N$ initial conditions such as $q_i(0)$, $\dot{q}_i(0)$ at some time $t = 0$, say. This will completely specify the time evolution of the system. On the other hand one may also specify N coordinates at two different times: $q_i(t_1)$, $q_i(t_2)$.

- $\mathbf{q}(t)$ denotes the **Trajectory** or *path* followed by the system.

- Any function $f[\mathbf{q}(t), \dot{\mathbf{q}}(t)]$, which remains constant along the physical trajectory and does not explicitly dependent on time is obviously conserved, that is

$$f[\mathbf{q}(t), \dot{\mathbf{q}}(t)] = \mathit{const.} \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} f = 0\,. \tag{4.7}$$

[3]If Λ is a functional, then $\Lambda\colon \mathbf{F} \to \mathbf{R}$,or $\mathbf{C}$, where $\mathbf{F}$ is a space of real or complex functions, $\mathbf{R}$ and $\mathbf{C}$ are the spaces of real and complex nos. respectively.

- The equations motion derived from the variational principle do not correspond to a unique Lagrangian. In particular suppose we effect a change in the Lagrangian by a total derivative, then

$$L'[\mathbf{q}(t), \dot{\mathbf{q}}(t), t] = L[\mathbf{q}(t), \dot{\mathbf{q}}(t), t] + \dot{f}(q). \tag{4.8}$$

  The action

$$R' = \int_{t_1}^{t_2} L'[\mathbf{q}(t), \dot{\mathbf{q}}(t), t] \, \mathrm{d}t \,. = \int_{t_1}^{t_2} L[\mathbf{q}(t), \dot{\mathbf{q}}(t), t] \, \mathrm{d}t \,. + f(q_2, t_2) - f(q_1, t_1). \tag{4.9}$$

  differs by the change induced by the function f at the end points. This change however drops out since the end points are held fixed against any variation - that is the equations of motion are the same for any two Lagrangians which differ by a function which can be written as a total time derivative.

The real power of Lagrangian mechanics lies in the description of the mechanical system in a coordinate independent way and can be broadened to accommodate a large class of other types of systems. For example in Newtonian mechanics, for a conservative system in 1 dimension, the equation of motion is of course given by,

$$m\ddot{x} = F(x) = -V'(x),$$

where the 'prime' denotes the derivative with respect to x. A coordinate transformation of the form $x = f(y)$ will transform the equation of motion in y as

$$m(f'(y)\ddot{y} + f''(y)(\dot{y}^2)) = F(f(y))$$

whose structure is different from the original equation of motion.

Lagrangian formulation of mechanics, however, provides a method for describing the mechanical systems such that coordinate transformations leave the structure of the equations of motion invariant even in the presence of constraints.

4.3.1 Examples

Determining the form of a Lagrangian, modulo total time derivative, is in general non-trivial. We discuss the case of free particle first.

System of free particles: In this case we can use some basic principles like homogeneity and isotropy of space to determine the form of the Lagrangian. In order to describe the dynamics of a mechanical system we first choose a reference frame. This is straight forward for a free particle.

The properties of a free particle should be the same no matter where it is, at different instances of time, and what its orientation is. Thus we require the space to be homogeneous and isotropic. In particular we call this frame "inertial" since space and time are homogeneous.

Experiments show that for a free particle the velocity

$$\vec{v} = \dot{\vec{q}} = \text{constant}.$$

This is the law of inertia.

Invoking the property of homogeneity, the Lagrangian can not depend on position and time explicitly. From the equation of motion it is clear that the Lagrangian has the form:

$$L = L(v^2); \quad v^2 = \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + \cdots + \dot{q}_n^2.$$

It is independent of position and orientation. The free particle equation of motion is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0\,.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^2}\frac{\partial v^2}{\partial \dot{q}_i}\right) = 0\,. \qquad \Rightarrow \quad \frac{\partial L}{\partial v^2}\ddot{q}_i = 0\,.$$

which is indeed the law of motion.

Thus we have the necessary condition that the Lagrangian has to be a function of the modulus of the velocity. We may thus write

$$L = Cv^{2\alpha}.$$

where α can not be determined from the properties of space and time alone. The law of inertial is governed for arbitrary values of α.

We now invoke the **principle of relativity** to obtain the precise form of the Lagrangian. Consider two reference frames I_1 and I_2 where I_2 is a co-moving frame with a constant relative velocity of $\vec{V}$. According to the principle of relativity the laws of motion must be the same in two frames.

We assume that the coordinates in the two frames are related through the Galilean transformation:

$$\vec{r}' = \vec{r} + \vec{V}t,$$

$$t' = t.$$

Since we already know that $L = L(v^2)$ we have

$$I_1: \quad L = L(v^2)$$

$$I_2: \quad L = L(u^2)$$

where u is the modulus of velocity in I_2. Since $\vec{u} = \vec{v} + \vec{V}$, we have

$$u^2 = v^2 + 2\vec{v}.\vec{V} + V^2 = v^2 + \frac{d}{dt}(2\vec{r}.\vec{V} + tV^2)$$

Thus

$$L' = Cu^{2\alpha}. = C[v^2 + \frac{d}{dt}(2\vec{r}.\vec{V} + tV^2)]^{\alpha}$$

Thus the Lagrangians in two different frames may be considered equivalent (describing the same system) only when $\alpha = 1$ since then they would differ by a total derivative. Thus

$$L(v^2) = L' = Cu^2. = C[v^2 + \frac{d}{dt}(2\vec{r}.\vec{V} + tV^2)]$$

The constant C however remains arbitrary but the action has a minimum only when $C > 0$ since v^2 is positive definite. In particular we may chose,

$$L = \frac{1}{2}mv^2$$

where m is the mass of the particle and the free particle Lagrangian is simply its kinetic energy.

Newtonian systems :

For systems obeying Newtonian mechanics, the Lagrange's equations are equivalent to Newton's second law. For particle motion in one dimension we have,

$$F = -\frac{dV}{dx} = m\ddot{x}.$$

Assuming m to be a constant we can write the above equation as,

$$\frac{-dV}{dx} - \frac{d}{dt}(m\dot{x})$$

$$\frac{-dV}{dx} - \frac{d}{dt}(\frac{d}{d\dot{x}}\frac{m\dot{x}^2}{2}).$$

Adding terms whose partial derivatives are equal to zero, we have

$$\frac{\partial}{\partial x}[\frac{m\dot{x}^2}{2} - V(x)] - \frac{d}{dt}(\frac{d}{d\dot{x}}\frac{m\dot{x}^2}{2}).$$

We can add the potential term in the second term without altering the nature of the equation,

$$\frac{\partial}{\partial x}[\frac{m\dot{x}^2}{2} - V(x)] - \frac{d}{dt}(\frac{d}{d\dot{x}}\frac{m\dot{x}^2}{2} - V(x)).$$

In general therefore for systems with kinetic energy T and potential energy$V$, we define

$$L = T - V$$

and we recover the Lagrangian equation of motion for the Newtonian particle,

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{4.10}$$

For example the Lagrangian of a plane pendulum is given by

$$L(\psi, \dot{\psi}) = \frac{1}{2}ml^2\dot{\psi}^2 + mglcos\psi.$$

4.4 Symmetry and its Applications

Noether's Theorem :

If a Lagrangian is invariant under a **family** of transformations, its dynamical system possesses a **constant of motion** and it can be found from a knowledge of the Lagrangian and the transformation(s). The invariance properties are valid only in those areas where, the homogeneity and/or isotropy of space and time is preserved.

4.4.1 Co-ordinate independence of the Lagrangian

The Lagrangian, is independent of the choice of a specific set of generalised coordinates. What this means is that, if a set $\{q_i\}$ is used to write the Lagrangian, then another set $\{\tilde{q}_i\}$ also fits the prescription, given that the map $\tilde{q}_i := \tilde{q}_i\{q_1, \ldots, q_N\}$ is invertible. The physical implication of the invertibility condition is that, the particle trajectory described by $\{q_i\}$ is the same as that described by $\{\tilde{q}_i\}$. Hence we can write :

$$\tilde{L}(\tilde{q}, \dot{\tilde{q}}, t) \equiv L(q(\tilde{q}), \dot{q}(\tilde{q}, \dot{\tilde{q}}, t), t) \equiv L(q, \dot{q}, t)$$

This property implies that the Lagrangian is a **Scalar Function**.

4.4.2 Symmetry and conservation laws

Previously, we have argued that the Lagrangian for a free particle is proportional to v^2, using properties of space-time and the principle of relativity. For a system of particles we generalise this to

$$L = \frac{1}{2} m \sum_i^N \dot{q}_i^2$$

consistent with the equations of motion. Beyond this we can not determine the Lagrangian for systems which are in general not free- may be confined and interacting. Invoking *experience* and known *experimental facts* we deduce the correct form of L, modulo the total derivative. For dynamical systems which have kinetic energy T and potential energy V, the form of the Lagrangian is given by

$$L[\mathbf{q}(t), \dot{\mathbf{q}}(t), t] = T - V \,. \tag{4.11}$$

Here T denotes the kinetic energy, which is in general bilinear in velocities

$$T = \sum_{i,j=1}^N \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j \,, \tag{4.12}$$

and V is the Potential

$$V = V(\mathbf{q}, t) \,. \tag{4.13}$$

The mass tensor in general can be quite complicated but often can be written as $m_{ij} = m_i \delta_{ij}$.

The equations of motion are:

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i} = F_i;$$

Note that the potential is also defined only up to an additive constant.

An important insight that can be gained through the Lagrangian formulation of mechanics is the relationship between the symmetries of the Lagrangian and conservation laws.

Consider a *closed system* (the meaning of which will become clear soon) of degrees of freedom N, described by generalised coordinates, $q_1(t), q_2(t), \cdots, q_N(t)$ and associated velocities $\dot{q}_1, \cdots, \dot{q}_N$.

> There may exist in such systems quantities which do not change during motion but are specified once the initial conditions are given. These are called conserved quantities or integrals of motion– Nether Theorem.

We will illustrate this below.

Conservation of energy :

The Lagrangian of a system of particles $L = L(q_i, \dot{q}_i, t)$ may depend on time explicitly. The total time derivative of the Lagrangian is

$$\frac{dL}{dt} = \sum_i [\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial q_i}\dot{q}_i]\frac{\partial L}{\partial t}$$

Using the equations of motion we have

$$\frac{dL}{dt} = \sum_i [\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i] + \frac{\partial L}{\partial t} = \frac{d}{dt}\sum_i [\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i] + \frac{\partial L}{\partial t}$$

Suppose now that the Lagrangian does not explicitly depend on time, we have

$$\frac{d}{dt}[\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L] = 0.$$

As a consequence we have

$$E := [\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L]$$

to be a constant of time and we call this the energy of the system. For a mechanical system[4] since $L = T - V$

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2T$$

. Therefore $E = T + V$ and we are justified in calling E the total energy of the system.

[4]For the equation that we write in the next step, it is important that the term $\mathbf{T}$ is homogeneous in $\{\dot{q}\}$. This is possible only when the original coordinates are functions of the q_i's only, without any explicit dependence on time, i.e. $\frac{\partial x_j}{\partial t} = 0$.

Conservation of momentum :

Consider a system that is bodily displaced by an infinitesimal amount ϵ. Let us assume that this will imply changing each of the position vectors $\vec{r}_i$ by an amount, $\vec{r}_i \rightarrow \vec{r}_i + \vec{\epsilon}$. Invariance of the Lagrangian under this translation implies that the variation in L given by

$$\delta L = \sum_i \frac{\partial L}{\partial \vec{r}_i} \delta \vec{r}_i = \vec{\epsilon} \sum_i \frac{\partial L}{\partial \vec{r}_i} = 0.$$

Since ϵ is arbitrary we have

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = \frac{d}{dt}[\sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i}] = 0.$$

Thus translation invariance leads to a conserved quantity,

$$\sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} = \sum_i \vec{p}_i = \vec{P}$$

which we call as the total momentum of the system. Furthermore, we define the momentum conjugate to the vector $\vec{r}_i$ as

$$\vec{p}_i := \frac{\partial L}{\partial \dot{\vec{r}}_i}$$

The cyclic co-ordinates: We have already seen the general form of the Lagrangian. In particular, let's consider the Lagrangian of a free particle that we have already dealt with :

$$L = \frac{1}{2} m \dot{q}^2$$

where the $\dot{q}$ denotes the generalised velocity which is trivial here. Here the corresponding q i.e. the generalised co-ordinate is missing here. This then is the cyclic co-ordinate of this Lagrangian. So, let us formulate a generalised explanation of a cyclic co-ordinate. If our Lagrangian $L := L(q_i, \dot{q}_i)$, where $i = 1, 2, 3, ..., n$. But, say q_k is missing from the equation such that L is independent of q_k. Then this Lagrangian L is said to be cyclic in q_k. Now watch the power of the cyclic co-ordinate: We know, $\dot{p}_k = \frac{\partial L}{\partial q_k}$. But here, since L is independent of q_k, we have $\dot{p} = 0$. Thus the conjugate generalised momentum is conserved . Physically, $\frac{\partial L}{\partial q_k} = 0$ implies the homogeneity of space in the q_k "direction" as discussed before.

Gauge transformations :

Noether's theorem can be generalised. We have used the equations of motion which remain the same, for example, when the Lagrangian is changed by a total derivative of some function $f = f(q)$

$$L' = L + df/dt$$

While the equations of motions are unchanged under such a change of the Lagrangian, the conjugate momentum does change. For example

$$p = \frac{\partial L}{\partial \dot{q}}$$

and

$$p' = \frac{\partial L'}{\partial \dot{q}} = p + \frac{\partial f}{\partial q}$$

since $df/dt = (\partial f/\partial q)\dot{q}$. Such a transformation is called **gauge transformation**. The energy however is given by

$$E' = p'\dot{q} - L' = (p + df/dq)\dot{q} - (L + df/dq\ \dot{q}) = E.$$

If f depends on time explicitly then E' differs from E by the partial derivative of f with respect to time. For "closed systems" however the energy is unchanged during gauge transformations.

4.5 Applications of Lagrange equations

We now consider a few specific problems to illustrate the application of Lagrangian formulation. The object is to provide a consistent way of formulating the equations of motion without going into too much of details. Often the main difficulty in the solution of problems mechanics is the choice of co-ordinate system which will make the solution of the equations of motion simple.

A feature that is often useful is the appearance of *ignorable or cyclic* co-ordinates which allow a simple analysis of the system. If a generalised coordinate does not explicitly appear in the Lagrangian of the system, the corresponding conjugate momentum is conserved, that is it is a constant of motion.

Suppose $L = L(q_1, q_2, \cdots, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_i, \cdots, t)$ so that the generalised coordinate q_i does not explicitly appear in the Lagrangian. The corresponding Euler-Lagrange equation of motion is

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0\,. \qquad (4.14)$$

Since q_i does not appear in L, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad p_i = \frac{\partial L}{\partial \dot{q}_i} = \text{constant} \qquad (4.15)$$

The co-ordinate q_i is called a cyclic co-ordinate. There may be more than one such coordinate in the system. This profound result can be an important tool in simplifying the equations of motion.

The two-body problem :

Consider a two particle system with position vectors $\vec{r_1}, \vec{r_2}$ in three dimensions with mass m_1 and m_2 respectively. The Lagrangian of the system may be written as

$$L = T - V = \frac{1}{2}m_1\dot{\vec{r_1}}^2 + \frac{1}{2}m_1\dot{\vec{r_2}}^2 - V(\vec{r_1}, \vec{r_2}).$$

Quite often it is convenient to use the so called *centre-of-mass coordinate system*, where

$$\vec{r} = \vec{r_1} - \vec{r_2}$$

which is the relative coordinate between the two particles and

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}$$

which is the position vector of the centre of mass of the system.

In terms of these new coordinates the Lagrangian may be written as

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{R}, \vec{r}),$$

where

$$M = m_1 + m_2$$

and

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

which is called the reduced mass of the system. The importance of this separation lies in the fact that, most often the potential energy term of the Lagrangian splits into two parts,

$$V(\vec{R}, \vec{r}) = V_R(\vec{R}) + V_r(\vec{r}).$$

This then allows us to separate the Lagrangian in a separable form as

$$L = L_R + L_r$$

where

$$L_R = \frac{1}{2}M\dot{\vec{R}}^2 - V_R(\vec{R})$$

and

$$L_r = \frac{1}{2}m\dot{\vec{r}}^2 - V_r(\vec{r}).$$

Such a decomposition often simplifies the solution of the problem. Typical examples are Earth moving around the Sun, the electron orbiting around the nucleus. Typical of such problems is also the fact the two-body potential depends only on the relative distance between the two particles, that is

$$V(\vec{R}, \vec{r}) = V_r(r)$$

which is not only independent of the center-of-mass coordinate, but also independent of the relative orientation of the two particles. The first property leads to *translation invariance* while the second property implies conservation of angular momentum, often referred to as the central force problem.

Here after we shall assume that the system is translationally invariant. Therefore it is of little interest to consider the motion of the centre of mass. Properties of the system will not be affected the actual position of the centre of mass. We shall also assume that the force field is central and concentrate on the relative Lagrangian L_r. The problem is analysed more easily in spherical polar coordinates:

$$\begin{aligned} x &= r\ \sin\theta\ \cos\phi \\ y &= r\ \sin\theta\ \sin\phi \\ z &= r\ \cos\theta, \end{aligned} \tag{4.16}$$

where r, θ, ϕ are the new generalised co-ordinates. The Lagrangian in spherical polar coordinates is then given by

$$L_r = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2] - V(r) \tag{4.17}$$

It is easily seen that ϕ is an ignorable co-ordinate since it does not explicitly appear in the Lagrangian. As a consequence we have the corresponding conjugate momentum to be a constant of motion, that is,

$$p_\phi = mr^2\sin^2\theta\dot{\phi} = \text{constant.}$$

This is easily seen to be the angular momentum projection about the polar axis and is a constant for a particular choice of the axis. Since r, θ are in general functions of time, they will assume non-zero values. It follows therefore that $\dot{\phi}$ must vanish in order that p_ϕ continues to be a constant. Thus the motion takes place on a plane defined by a unique value of ϕ. Without loss of generality we can choose this to be the plane $\phi = 0$. This in fact the first of Kepler's results.

The reduced Lagrangian in the plane is therefore given by,

$$L_r = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] - V_r(r) \tag{4.18}$$

where θ is now an ignorable coordinate leading to yet another constant of motion

$$p_\theta = l = mr^2\dot{\theta} = \text{constant.}$$

which is a statement of the constancy of angular momentum about an axis through the origin and perpendicular to the coordinate plane. This is essentially the statement of Kepler's second law of motion for planetary motion.

The effective equation of motion for the system may therefore be written as

$$m\ddot{r} = mr\dot{\theta}^2 - V_r'(r) = -V_{eff}'(r) \tag{4.19}$$

where the effective potential $V_{eff}(r)$ is given by

$$V_{eff}(r) = V_r(r) + \frac{l^2}{2mr^2}$$

with a centrifugal term which is l dependent. Thus the central force problem is essentially reduced to describing one-dimensional radial motion in an effective potential that depends on the value of the angular momentum. In the actual Kepler problem we have $V_r(r) = -GMm/r$.

Coriolis force :

One of the main advantages of using Lagrangian formalism is that the equations of motion retain the same form no matter what reference frame one uses. Often the introduction of rotating coordinate system complicates a problem- for example a mechanical system in motion on Earth. To see how the problem may be simplified, consider the Lagrangian of a freely moving object in 3-dimensions.

$$L = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2 + \dot{z}^2].$$

Now look at the motion in a frame rotating about the z-axis, that is,

$$x = x' \cos \omega t - y' \sin \omega t$$

$$y = x' \sin \omega t - y' \cos \omega t$$

$$z = z'$$

The Lagrangian in the new frame may be written as

$$L = \frac{1}{2}m[\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2] + m\omega(x'\dot{y'} - y'\dot{x'}) + \frac{1}{2}m\omega^2(x'^2 + y'^2).$$

Using the Lagrange equations of motion we have

$$m\ddot{x'} = 2m\omega\dot{y'} + m\omega^2 x'$$

$$m\ddot{y'} = -2m\omega\dot{x'} + m\omega^2 y'$$

The two terms on the LHS of the above equations correspond to the Coriolis force and the centrifugal force respectively.

Pendulum revisited :

The period of a pendulum is independent of the amplitude only for very small oscillations. However, the amplitude dependence of the period for large amplitudes may be eliminated by rapping the string of the pendulum around a limiting curve.

Consider a particle of mass m constrained to move in a vertical plane, as in the case of a simple pendulum, but along a smooth cycloid under the influence of gravity. The cycloid is given by the parametric equations,

$$x = A(\psi + \sin \psi),$$

$$z = A(1 - \cos\psi),$$

where $-\pi < \psi < \pi$, and A is a positive constant with $z-$axis pointing vertically upwards. (Cycloid is the curve described by a point on the rim of a wheel as it rolls along a flat surface).

The kinetic energy of motion is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = 2mA^2\cos^2(\psi/2)\dot{\psi}^2$$

and therefore the Lagrangian is given by

$$L = T - V = 2mA^2\cos^2(\psi/2)\dot{\psi}^2 - mgA(1 - \cos\psi)$$

It is more convenient to express this in terms of the arc length s along the cycloid. Note that

$$ds^2 = dx^2 + dy^2 = 4A^2\cos^2(\psi/2)d\psi^2 \Rightarrow ds = \pm 2A\cos(\psi/2) - 1.$$

Let us make ds be positive by choosing $+$ sign without loss of generality. Then the arc length measured from $x = -\pi A, z = 2A$ is

$$s = \int_{-\pi}^{\psi} 2A\cos(\psi/2)d\psi = 4A[\sin\psi/2 + 1].$$

If the arc length measured from $x = 0$, that is from the bottom of the curve then

$$s = \int_{0}^{\psi} 2A\cos(\psi/2)d\psi = 4A\sin(\psi/2).$$

Either way,

$$\dot{s} = 2A\cos(\psi/2)\dot{\psi}.$$

Thus the kinetic energy is

$$T = \frac{1}{2}m\dot{s}^2 = 2mA^2\cos^2(\psi/2)\dot{\psi}^2$$

as before. The potential energy term is given by

$$V = mgA(1 - \cos\psi) = \frac{mgs^2}{8A}.$$

Thus the Lagrangian using the arc length as the dynamical parameter is given by,

$$L = \frac{1}{2}m\dot{s}^2 - \frac{mgs^2}{8A}.$$

To find the period use the equation of motion. We have

$$m\ddot{s} + \frac{mgs}{4A} = 0$$

or equivalently

$$\ddot{s} + \frac{g}{4A} s = 0.$$

Thus the frequency of oscillation is

$$\omega = \sqrt{g/4A}, \quad T = \frac{2\pi}{\omega} = 4\pi\sqrt{A/g}$$

which is independent of the amplitude of oscillation with the period identical to that of a *simple pendulum*.

Small oscillations :

As a further application of the Lagrangian formulation, consider the example of mechanical vibration of an atomic lattice, such as in a crystal. The mean positions of the atoms are given according to the crystal structure. However, atoms may undergo small displacements about the mean position if the crystal is not infinitely rigid. The mechanical vibrations of the atoms are in general complicated, but as we show below, may be resolved into the so called **normal modes of vibration** which consists of finite number of simple harmonic motions.

Consider a system of mutually interacting and vibrating particles. The stable equilibrium of the system is given by minimising the potential energy for an arrangement of the atoms. The only motion possible for individual particles is a small displacement about this equilibrium position.

If q_i are the generalised coordinates of the particles and q_{0i} are their equilibrium values, the potential energy may be expressed as a Taylor's series expansion:

$$V(q_i) = V(q_{0i}) + \sum_i \left(\frac{\partial V}{\partial q_i}\right)_0 \delta q_i + \frac{1}{2}\sum_i \sum_j \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \delta q_i \delta q_j + \cdots$$

where we have kept terms only up to second order consistent with our assumption that the displacements are small. Since the equilibrium values q_{0i} are fixed, we may simplify the notation as follows:

$$\delta q_i = \eta_i, \quad \frac{\partial}{\partial q_i} = \frac{\partial}{\partial \eta_i}$$

Thus η_i now play the role of generalised coordinates. The Taylor expansion above may be rewritten as

$$V(\eta_i) = V(0) + \sum_i \left(\frac{\partial V}{\partial \eta_i}\right)_0 \eta_i + \frac{1}{2}\sum_i \sum_j \left(\frac{\partial^2 V}{\partial \eta_i \partial \eta_j}\right)_0 \eta_i \eta_j + \cdots$$

At equilibrium, by definition,

$$\left(\frac{\partial V}{\partial \eta_i}\right)_0 = 0$$

and $V(0)$ is an arbitrary constant which may be set equal to zero without loss of generality. Thus we may write

$$V(\eta_i) = \frac{1}{2}\sum_i \sum_j V_{ij}\eta_i\eta_j + \cdots ,$$

where

$$V_{ij} = \left(\frac{\partial^2 V}{\partial\eta_i\partial\eta_j}\right)_0$$

and is independent of the new generalised coordinates η_i. Note that V_{ij} is a symmetric matrix.

The kinetic energy of the particles in cartesian coordinate system is given by

$$T = \frac{1}{2}\sum_i m_i\dot{x}_i^2$$

The (time-independent) transformation to generalised coordinates q_i is effected by the relations

$$q_i = q_i(x_j), \quad x_i = x_i(q_j).$$

Since $\dot{q}_i = \dot{\eta}_i$ we have

$$\dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j = \sum_j \frac{\partial x_i}{\partial \eta_j}\dot{\eta}_j$$

Substituting this in the kinetic energy expression

$$T = \frac{1}{2}\sum_i m_i \left(\sum_j \sum_k \left(\frac{\partial x_i}{\partial\eta_j}\right)_0 \left(\frac{\partial x_i}{\partial\eta_k}\right)_0\right)\dot{\eta}_j\dot{\eta}_k$$

where we have assumed

$$\frac{\partial x_i}{\partial\eta_j} = \left(\frac{\partial x_i}{\partial\eta_j}\right)_0$$

for small amplitude motion. Once again we may write the kinetic energy in the matrix form

$$T = \frac{1}{2}\sum_j \sum_k T_{jk}\dot{\eta}_j\dot{\eta}_k$$

where the symmetric matrix T_{jk} is

$$T_{jk} = \sum_i m_i \left(\frac{\partial x_i}{\partial\eta_j}\right)_0 \left(\frac{\partial x_i}{\partial\eta_k}\right)_0$$

and is independent of coordinates.

The conservative Lagrangian to this order is

$$L = T - V = \frac{1}{2}\sum_j \sum_k [T_{jk}\dot{\eta}_j\dot{\eta}_k - V_{jk}\eta_j\eta_k].$$

The equations of motion are given by

$$\frac{d}{dt}\sum_k T_{jk}\dot{\eta}_k = \sum_k T_{jk}\ddot{\eta}_k = -\sum_k V_{jk}\eta_k$$

We will now make the assumption that the motion is periodic, the solution may therefore be written in the form

$$\eta_k = \eta_k^0 e^{i\omega t}.$$

Substituting this in equation of motion

$$\sum_k [\omega^2 T_{jk} - V_{jk}]\eta_k^0 = 0$$

The solution exists provided the determinant

$$|\omega^2 T_{jk} - V_{jk}| = 0$$

The solution of this determinantal equation provides the values of the eigenfrequencies ω^2 which are interpreted as the frequencies of the *normal modes of vibration* of the system.

We shall illustrate this by solving an example. Consider a linear triatomic molecule. The model of the linear triatomic molecule consists of a particle of mass M coupled elastically to two other particles of mass m, say. Let the particles be positioned equally spaced along a line. We shall consider the motion along the straight line. Denote the displacements along the line from equilibrium position by η_1, η_2, η_3.

The kinetic energy and potential energy of the particles are given by

$$T = \frac{1}{2}[m(\dot{\eta}_1^2 + \dot{\eta}_3^2) + M\dot{\eta}_2^2]$$

$$V = \frac{1}{2}k[(\eta_1 - \eta_2)^2 + (\eta_2 - \eta_3)^2]$$

where k is the spring constant. The Lagrangian is

$$L = \frac{1}{2}[m(\dot{\eta}_1^2 + \dot{\eta}_3^2) + M\dot{\eta}_2^2] - \frac{1}{2}k[(\eta_1 - \eta_2)^2 + (\eta_2 - \eta_3)^2]$$

The equations of motion are given by

$$m\ddot{\eta}_1 = k(\eta_2 - eta_1)$$

$$m\ddot{\eta}_2 = k(\eta_3 - eta_2) - k(\eta_2 - \eta_1)$$

$$m\ddot{\eta}_3 = -k(\eta_3 - eta_2)$$

Assuming

$$\eta_k = \eta_k^0 e^{i\omega t}$$

the determinantal equation is

$$\begin{vmatrix} m\omega^2 - k & k & 0 \\ k & M\omega^2 - 2k & k \\ 0 & k & m\omega^2 - k \end{vmatrix} = 0\,.$$

The three eigenvalues of the equation are given by

$$\omega_1 = 0, \quad \omega_2 = \pm\sqrt{k/m}, \quad \omega_3 = \pm\sqrt{k(2m+M)/mM}.$$

The negative eigenvalues or frequencies have no particular significance. One may combine the exponentials to give either cos or sin functions.

The three solutions correspond to three physically possible solutions: $\omega_1 = 0$ corresponds to the situation when all the three particles are moving in phase undergoing the same linear motion. The solution corresponding to ω_2 corresponds to the motion in which the particle in the middle is stationary while the two outer particles move with opposite phase. The third solution represents the two outer particles moving in phase and out of phase with the one in the middle.

4.6 Problems

1. A particle moving in the plane $X - Y$ is described by the lagrangian,

$$L = (1/2)m[(\dot{x}^2 + \dot{y}^2) - \omega^2(x^2 + y^2)] + \alpha \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}.$$

   Obtain the Lagrangian equations of motion of the system. Obtain expressions for the momenta p_x, p_y. Is there anything peculiar about the system? If so, explain.

2. If a particle of mass m is projected vertically upwards, far from the surface of the earth, the Lagrangian is

$$L = m\dot{z}^2/2 + GMm/(R + z).$$

   G being the gravitational constant, M and R are the mass and radius o the Earth and z is the height of the particle above the surface. Find the Hamiltonian and sketch the phase diagram clearly showing the region where the motion is bounded and the region where it is unbounded. Find the equation of the separatrix. For the bounded motion find the time taken to go from the surface of the earth to a maximum height $Z_m ax$ and back.

3. A particle is constrained to slide under gravity on a smooth wire in the shape of a vertical circle with radius R, when the wire rotates about a vertical diameter with constant angular velocity Ω. Write the Lagrangian of the system and show that the Hamiltonian is

$$H = \frac{p^2}{2mR^2} - m[R^2\Omega^2 \sin^2(\psi) + gR\cos(\psi)],$$

   where ψ is the angular displacement of the particle from the downward vertical, with the centre of the circle as origin. Determine fixed points of the system and their stability.

4. Suppose a fictitious system of one degree of freedom has the Lagrangian,

$$L = \frac{M}{4}\dot{q}^4 - \frac{m}{2}\dot{q}^2 - \lambda\omega^2 q^2$$

   Show that it is impossible to obtain a Hamiltonian that will encode as much information about this system as the Lagrangian does. Can you generalise this result— for what type of Lagrangians is it possible to construct a Hamiltonian that will serve as an equally good description of the system?

5. A particle of mass m is constrained to move in a vertical, $x - z$ plane under the influence of gravity on the smooth curve with parametric equation

$$z = A\cos(\phi), \quad x = B\sin(\phi) \quad (-\pi < \phi < \pi),$$

where A and B are positive constants and the z axis is vertically upwards.

(a) Write the Lagrangian of the system and derive the equation of motion.

(b) Obtain the solution of the equation of motion (a) when $|\phi| < \phi_0 << \pi/2$ and (b) when the kinetic energy much greater than the gravitational potential energy.

Chapter 5

The Hamiltonian Formulation

The transition from Lagrange to Hamilton formalism is done by first defining **momentum** p_i (which is canonical conjugate to the configuration coordinate q_i):

$$p_i := \frac{\partial L}{\partial \dot{q}_i}\,, \quad \mathbf{p}(t) = \{p_i(t)\}\,. \qquad (i = 1, 2, 3, \dots N) \tag{5.1}$$

Note that the simple relation between the momentum p_i and the velocity $\dot{q}_i$ for Newtonian systems in Cartesian coordinates is rather deceptive.

An important property of momenta is that they may be expressed as gradient of action. Consider the variation of the action S with only one end point fixed, say the initial point:

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})\right]\delta q dt$$

Using the equation of motion,

$$\delta S = p\delta q \Rightarrow p = \frac{\partial S}{\partial q}$$

where p is now the gradient of action. This is, in general, not true of the velocities.

The **Hamiltonian** is then obtained through the Legendre-Transformation:

$$H(\mathbf{p}, \mathbf{q}, t) := \sum_{i=1}^{N} p_i \dot{q}_i - L(\mathbf{q}, \dot{\mathbf{q}}, t)\,, \tag{5.2}$$

which is a function of variables p_i, q_i alone.

For a large class of systems in Newtonian mechanics the Hamiltonian may be written in the form

$$H(p_i, q_i) = T + V = \frac{1}{2m}\sum_i p_i^2 + V(q_i)$$

where $p_i = m\dot{q}_i$. But the simple relation between the momentum and velocity is not always taken for granted especially when the system moves in the presence of constraints.

Example: A simple example where the relation between momentum and velocity is not simple is in the case of a particle sliding on a wire shape $z = f(x)$, say, under gravity. Since

$$T = \frac{m}{2}(\dot{x}^2 + \dot{z}^2) = \frac{m}{2}\dot{x}^2(1 + (df/dx)^2)$$

and

$$V = mgz = mgf(x).$$

The generalised momentum $p = m\dot{x}(1 + f'(x)^2)$ and

$$H = \frac{p^2}{2m(1 + f'(x)^2)} + mgf(x).$$

Remarks:

- p_i und q_i form a set of conjugate variables whose product $p_i q_i$ has the dimension of work.

- The equations of motion may be derived from a variational principle. On the one hand we have:

$$\begin{aligned}
\mathrm{d}H(\mathbf{p}, \mathbf{q}, t) &= \mathrm{d}\left(\sum_i p_i \dot{q}_i - L(\mathbf{q}, \dot{\mathbf{q}}, t)\right) \\
&= \sum_i \left(\mathrm{d}p_i \dot{q}_i + p_i \mathrm{d}\dot{q}_i - \frac{\partial L}{\partial q_i}\,\mathrm{d}q_i - \frac{\partial L}{\partial \dot{q}_i}\,\mathrm{d}\dot{q}_i\right) - \frac{\partial L}{\partial t}\,\mathrm{d}t \\
&= \sum_i \left(\mathrm{d}p_i \dot{q}_i + p_i \mathrm{d}\dot{q}_i - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\,\mathrm{d}q_i - \frac{\partial L}{\partial \dot{q}_i}\,\mathrm{d}\dot{q}_i\right) - \frac{\partial L}{\partial t}\,\mathrm{d}t \\
&= \sum_i (\mathrm{d}p_i \dot{q}_i + p_i \mathrm{d}\dot{q}_i - \dot{p}_i \mathrm{d}q_i - p_i \mathrm{d}\dot{q}_i) - \frac{\partial L}{\partial t}\,\mathrm{d}t \\
&= \sum_i (\mathrm{d}p_i \dot{q}_i - \dot{p}_i \mathrm{d}q_i) - \frac{\partial L}{\partial t}\,\mathrm{d}t\,.
\end{aligned}$$

  On the other hand we have the total derivative to be

$$\mathrm{d}H(\mathbf{p}, \mathbf{q}, t) = \sum_i \left(\frac{\partial H}{\partial p_i}\mathrm{d}p_i + \frac{\partial H}{\partial q_i}\mathrm{d}q_i\right) + \frac{\partial H}{\partial t}\mathrm{d}t\,, \tag{5.3}$$

  which immediately leads to the Hamilton equations of motion:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i\,, \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i\,. \tag{5.4}$$

- The Hamilton equations are a set of $2N$ first order differential equations, ideally suited for phase space description of the dynamical system.

- If $\frac{\partial L}{\partial t} = 0$, then it follows $\frac{\partial H}{\partial t} = 0$, so that $H = E = \mathit{constant}$, the energy is conserved.

The variational principle may thus be written in a short-hand notation as

$$\delta \int_{①}^{②} \left(\sum_i p_i \mathrm{d}q_i - H \mathrm{d}t \right) = 0 \,. \tag{5.5}$$

where ① und ② are used as short form for q_1, t_1 and q_2, t_2.

5.1 PBs- How to identify integrals of motion?

Let $f(\mathbf{q}, \mathbf{p}, t)$ be a function, differentiable, defined on the phase space. The total time derivative of f is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_i \left[\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right] = \frac{\partial f}{\partial t} + [f, H] \,. \tag{5.6}$$

Here we have made use of the Hamilton equations (5.4). The symbol $[H, f]$ is referred to as the **Poisson Bracket** or simply PB[1]. In general for any two functions f, g defined on the phase space the PB is defined in general as

$$[f, g] := \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) . \tag{5.7}$$

If a function does not explicitly depend on time, but only through phase space variables, then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = [f, H] \,. \tag{5.8}$$

Evidently, if $[f, H] = 0$ then $f(p, q)$ is a constant motion.

Properties of Poisson Brackets :

$$\begin{aligned} [f, g] &= -[g, f] \\ [f + g, h] &= [f, h] + [g, h] \\ [fg, h] &= f[g, h] + [f, h]g \\ [f, [g, h]] &+ [g, [h, f]] + [h, [f, g]] = 0 \ , \end{aligned} \tag{5.9}$$

where the last property is known as the Jacobi identity for PB. Note that the above identities define a Lie algebra- that is the functions on the phase space form a Lie algebra under PBs.

[1]In finite dimensions any simplectic manifold is defined by an operation of PB on a pair of scalar fields ψ_1, ψ_2 to produce another scalar field ψ, ie, $\psi = [\psi_1, \psi_2]$

Theorem of Poisson: If f and g are constants of motion, so is $[g, f]$.

Furthermore we have the following important PB relations:

$$\begin{aligned} [q_i, q_j] &= [p_i, p_j] = 0\,, \\ [q_i, p_j] &= \delta_{ij}\,. \end{aligned} \tag{5.10}$$

For the components of angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, we have:

$$[L_i, L_j] = \epsilon_{ijk} L_k\,. \qquad (i, j, k = 1, 2, 3) \tag{5.11}$$

Poisson Brackets and Hamiltonian dynamics: Not every conceivable motion on the phase space may be derived from Hamilton equations. The properties of PBs may be used to test whether dynamical system is Hamiltonian or not. A system is Hamiltonian if and only if

$$\frac{\mathrm{d}[f, g]}{\mathrm{d}t} = [\dot{f}, g] + [f, \dot{g}], \tag{5.12}$$

where f and g are two functions on the phase space. The result follows by first noting that

$$\begin{aligned} \frac{\mathrm{d}[f, g]}{\mathrm{d}t} &= [[f, g], H] + \frac{\partial [f, g]}{\partial t} \\ &= [[f, H], g] + [f, [g, H]] + [\frac{\partial f}{\partial t}, g] + [f, \frac{\partial g}{\partial t}] \\ &= [\dot{f}, g] + [f, \dot{g}], \end{aligned} \tag{5.13}$$

using the equation of motion. and also the Jacobi identity.

Example : Consider the dynamics defined by a set of first order equations given by,

$$\dot{q} = qp, \quad \dot{p} = -pq \tag{5.14}$$

Using Liebnitz rule for the PB $[q, p]$ one can easily show that this is not a Hamiltonian system[2] though the system has a conserved quantity, $p(t) + q(t)$, and is exactly solvable:

$$q(t) = const. \frac{q(0)\exp(ct)}{p(0) + q(0)\exp(ct)} \quad p(t) = const. \frac{p(0)}{p(0) + q(0)\exp(ct)} \tag{5.15}$$

5.2 Canonical Transformations

Consider the dynamics of a system described by a Hamiltonian, $H(\mathbf{q}, \mathbf{p})$. We therefore have

1. (q_i, p_i) are canonically conjugate variables in the sense that they follow the PB-Relations given by (5.10).

[2]There is even simpler way of showing this in this case, namely $\frac{\partial H}{\partial p \partial q}$ obtained from the two equations above are not equal. Hence there is no Hamiltonian.

2. Their time evolution is given by (5.4),
3. as obtained from the variational principle (5.5).

Consider a coordinate transformation $(\mathbf{q}, \mathbf{p}) \longrightarrow (\mathbf{Q}, \mathbf{P})$. We call this a **Canonical Transformation** or **CT** when

$$\begin{aligned} q_i &\longrightarrow Q_i(\mathbf{p}, \mathbf{q}, t) \\ p_i &\longrightarrow P_i(\mathbf{p}, \mathbf{q}, t) \\ H(\mathbf{p}, \mathbf{q}, t) &\longrightarrow \widetilde{H}(\mathbf{P}, \mathbf{Q}, t) \end{aligned} \tag{5.16}$$

such that under this transformation the Hamiltonian, $\widetilde{H}$, which is now a function of the canonical variables (Q_i, P_i) and preserves of the form of the equations of motion (5.4) and the variational relation (5.5), that is

$$\frac{\partial \widetilde{H}}{\partial P_i} = \dot{Q}_i\,, \qquad \frac{\partial \widetilde{H}}{\partial Q_i} = -\dot{P}_i\,. \tag{5.17}$$

In general the new phase space coordinates generated through CTs may not have the same meaning as the original coordinates, say as position and momentum.

Optimal transformations : A major motivation for the canonical transformation is to find a particular transformation which reduces the Hamiltonian to a very simple form. One particular interesting and useful time independent CT is when the transformed Hamiltonian depends exclusively on the new momenta

$$H(\mathbf{p}, \mathbf{q}) \longrightarrow \widetilde{H}(\mathbf{P}) \tag{5.18}$$

Immediately it follows that

$$\frac{\partial \widetilde{H}}{\partial P_i} = f_i(\mathbf{P})\,, \qquad \frac{\partial \widetilde{H}}{\partial Q_i} = 0\,. \tag{5.19}$$

We can easily integrate the equations of motion

$$Q_i(t) = f_i t + Q_i(0)$$

While the set $P_i, Q_i(0)$ constitute a set of $2N$ constants of integration, in particular P_i are referred to as constants of motion.

Example: A simple and well known example that illustrates the use of CTs is the the one dimensional harmonic oscillator-

$$H(p, q) = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2\,. \tag{5.20}$$

The following transformation is clearly canonical:

$$q = \sqrt{2P/m\omega}\,\sin Q\,, \quad p = \sqrt{2m\omega P}\,\cos Q\,. \tag{5.21}$$

The Hamiltonian after the canonical transformation is given as a function of P only and Q is some times called angle variable for obvious reasons.

$$\widetilde{H}(P,Q) = \omega P \cos^2 Q + \omega P \sin^2 Q = \omega P = \widetilde{H}(P) = E\,. \tag{5.22}$$

This is an optimal transformation since the integration of the Hamilton equations are trivial:

$$\begin{aligned} \frac{\partial \widetilde{H}}{\partial Q} &= -\dot{P} = 0 \quad \Rightarrow \quad P = const. = \frac{E}{\omega}\,, \\ \frac{\partial \widetilde{H}}{\partial P} &= \dot{Q} = \omega = const. \quad \Rightarrow \quad Q(t) = \omega t + \alpha\,. \end{aligned} \tag{5.23}$$

The solutions may now be expressed in the original coordinates giving

$$q(t) = \sqrt{2E/m\omega}\,\sin(\omega t + \alpha)\,, \tag{5.24}$$

$$p(t) = \sqrt{2Em\omega}\,\cos(\omega t + \alpha)\,, \tag{5.25}$$

which are of course well known.

5.3 Liouville Volume Theorem

As discussed before, in the Lagrangian dynamics the fundamental quantities are the configuration coordinates $\mathbf{q} \in Q$, where Q denotes the configuration space. The Lagrange equations are second order and are solved to obtained the trajectories in the configuration space. Often the the configuration space is the full Eucledian space, all the position vectors as also the velocity vectors are also contained in the Eucledian space. This however is not always the case- for example for particles confined to move on a circle S^1 or sphere S^2. While the trajectories are contained in some Q contained in the Eucledian space, the velocity vector lies on the tangent plane. Thus one needs an extension of Q, denoted TQ, in order to describe the motion. The Lagrangian then a scalar function on this space, that is the Lagrangian provides a map between TQ and the space of real functions R.

In Hamiltonian dynamics, however, the dynamics is described on the **phase space** Γ where the position q_i and momentum p_i are treated on equal footing, indeed they can be arbitrarily mixed in a coordinate transformation. If we denote the position space by $\mathcal{R}^N$ and momentum space $\mathcal{P}^N$:

$$\Gamma = \mathcal{R}^N \otimes \mathcal{P}^N, \qquad \mathbf{q} \in \mathcal{R}^N \quad \mathbf{p} \in \mathcal{P}^N. \tag{5.26}$$

The elements of Γ are the **phase space vectors** $\boldsymbol{\xi}$,

$$\boldsymbol{\xi} = \{\xi_i\} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} q_1 \\ \vdots \\ q_N \\ p_1 \\ \vdots \\ p_N \end{pmatrix}, \qquad \boldsymbol{\xi} \in \Gamma\,. \tag{5.27}$$

Defining a $2N$-dimensional gradient or Nabla-Vector

$$\boldsymbol{\nabla} := \frac{\partial}{\partial \boldsymbol{\xi}} = \left(\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_N}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_N}\right), \tag{5.28}$$

the equations of motion take a simple form

$$\dot{\boldsymbol{\xi}} = \begin{pmatrix} \frac{\partial}{\partial \mathbf{p}} H \\ -\frac{\partial}{\partial \mathbf{q}} H \end{pmatrix} = \mathbf{J} \cdot \boldsymbol{\nabla} H(\boldsymbol{\xi}, t)\,. \tag{5.29}$$

The $2N \times 2N$ dimensional matrix $\mathbf{J}$ is referred to as the **Simplectic Matrix** and is given by,

$$\mathbf{J} := \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{5.30}$$

where $\mathbf{1}$ refers to the unit matrix and $\mathbf{0}$ is the null matrix of dimension N. Again we note that the $\boldsymbol{\nabla} H(\boldsymbol{\xi}, t)$ denotes **Hamiltonian flow (-vector)** and the $\boldsymbol{\xi}(t)$ form the **Phases pace Trajectories(-path)**.

The **Theorem of Liouville** may now be stated in the following equivalent formulations:

- The Hamiltonian flows are divergence free:

$$\boldsymbol{\nabla} \cdot \dot{\boldsymbol{\xi}} = \sum_i \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) = 0 \tag{5.31}$$

  That is the phase space volume is incompressible.

- The volume element $d\Gamma$ of the phase space is invariant under canonical transformations:

$$d\Gamma := \left(\prod_{i=1}^{N} \mathrm{d}q_i\right)\left(\prod_{i=1}^{N} \mathrm{d}p_i\right) \stackrel{!}{=} \left(\prod_{i=1}^{N} \mathrm{d}Q_i\right)\left(\prod_{i=1}^{N} \mathrm{d}P_i\right). \tag{5.32}$$

- The Jacobian determinant under canonical transformation is unity:

$$\begin{aligned} J &= \det\left(\frac{\partial(Q_1, \ldots, Q_N, P_1, \ldots, P_N)}{\partial(q_1, \ldots, q_N, p_1, \ldots, p_N)}\right) \\ &= \begin{vmatrix} \frac{\partial Q_1}{\partial q_1} & \cdots & \frac{\partial Q_1}{\partial q_N} & \frac{\partial Q_1}{\partial p_1} & \cdots & \frac{\partial Q_1}{\partial p_N} \\ \vdots & & & & & \vdots \\ \frac{\partial P_N}{\partial q_1} & \cdots & \frac{\partial P_N}{\partial q_N} & \frac{\partial P_N}{\partial p_1} & \cdots & \frac{\partial P_N}{\partial p_N} \end{vmatrix} = 1\,. \end{aligned} \tag{5.33}$$

  For simplicity consider a system of order-2. The Jacobian is given by,

$$J = \det\left(\frac{\partial(Q, P)}{\partial(q, p)}\right)$$

$$= \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = [Q, P]_{(q,p)} = 1 \,. \tag{5.34}$$

Since it is a canonical transformation, we have

$$\dot{Q} = \frac{\partial \widetilde{H}}{\partial P} = \left[\frac{\partial \widetilde{H}}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial \widetilde{H}}{\partial p}\frac{\partial p}{\partial P} \right].$$

Since H is a scalar $\widetilde{H} = H$ independent of the representation. Thus,

$$\dot{Q} = \left[-\dot{p}\frac{\partial q}{\partial P} + \dot{q}\frac{\partial p}{\partial P} \right] = [q, p]_{Q,P}\dot{Q}.$$

Therefore we have the result

$$[q, p]_{Q,P} = 1$$

and since the inverse exists,

$$[Q, P]_{q,p} = 1$$

The PB of (p,q) calculated in any representation formed out of canonical transformations is representation independent. As a result

$$J = \det\left(\frac{\partial(Q, P)}{\partial(q, p)}\right) = [Q, P] = 1$$

when $(p, q) \to (P, Q)$ is a canonical transformation.

This can be generalised to arbitrary PB structures since

$$[f, g]_{P,Q} = [f, g]_{p,q}[q, p]_{P,Q} = [f, g]_{p,q}$$

by virtue of the above result.

Remarks :

- The canonical transformations are used to identify the so called **ideal** transformations where the new Hamiltonian may be independent of new coordinates Q_i, such a system is called **integrable** which will be discussed in detail later.
- It is easily shown that the Poisson Brackets $[f, g]_{p,q}$ are invariant under Canonical Transformations-
$$[f, g]_{p,q} = [\tilde{f}, \tilde{g}]_{P,Q} \,. \tag{5.35}$$
- As a consequence we have the following important relations-

$$\begin{aligned} [P_i, P_j]_{p,q} &= [Q_i, Q_j]_{p,q} = 0 \,, \\ [Q_i, P_k]_{q,p} &= \delta_{ik} \,, \end{aligned} \tag{5.36}$$

Example :

Consider a transformation of the form

$$Q = \alpha q + \beta p$$

$$P = \gamma q + \delta p$$

The Jacobian of the transformation is

$$[Q, P]_{(q,p)} = \alpha\delta - \beta\gamma.$$

Therefore, the transformation is canonical if the determinant of the transformation matrix, namely,

$$\alpha\delta - \beta\gamma = 1.$$

We may identify the following three cases:

- $\alpha = \delta = 1, \ \gamma = \beta = 0$ which is a trivial CT.
- $\alpha = 0, \beta = -\gamma = 1$ or equivalently $Q = -p, P = q + \delta p$.
- $\alpha = \delta = 1, \ \gamma = 0$ or $Q = q + \beta p, P = p$ which is a shear transformation in the p direction.
- $\alpha = \cos\theta = \delta, \ \gamma = -\beta = \sin\theta$ which is a rotation since

  $$Q = q\cos\theta - p\sin\theta$$

  $$P = q\sin\theta - p\cos\theta.$$

  All rotation of the type given above for arbitrary θ are Canonical Transformations.

Example :

We may re-interpret Liouville theorem by restating that the phase volume is conserved under Hamiltonian flows. That is the infinitesimal transformations generated by the Hamiltonian is itself a canonical transformation.

Let q_0 and p_0 be coordinate and momentum at time t. Consider the evolution of this system to a later time $t + \delta t$. Then

$$q_1(t) = q(t + \delta t) = q_0 + \frac{dq}{dt}\delta t + O(\delta t^2),$$

$$p_1(t) = p(t + \delta t) = p_0 + \frac{dp}{dt}\delta t + O(\delta t^2).$$

Substituting Hamiltonian equations of motion, we have

$$q_1(t) = q_0 + \frac{\partial H}{\partial p}\delta t + O(\delta t^2),$$

$$p_1(t) = p_0 - \frac{\partial H}{\partial q}\delta t + O(\delta t^2).$$

We may now treat q_1, p_1 as obtained from a transformation on q_0, p_0, thus

$$J = \det\left(\frac{\partial(q_1, p_1)}{\partial(q_0, p_0)}\right) \qquad = \begin{vmatrix} \frac{\partial q_1}{\partial q_0} & \frac{\partial q_1}{\partial p_0} \\ \\ \frac{\partial p_1}{\partial q_0} & \frac{\partial p_1}{\partial p_0} \end{vmatrix}$$

$$= \begin{vmatrix} 1 + \frac{\partial^2 H}{\partial q_0 \partial p_0}\delta t & \frac{\partial^2 H}{\partial p_0^2}\delta t \\ \\ -\frac{\partial^2 H}{\partial q_0^2}\delta t & 1 - \frac{\partial^2 H}{\partial p_0 \partial q_0}\delta t \end{vmatrix} = 1 + O(\delta t^2). \tag{5.37}$$

Thus, to the order we are interested, infinitesimal transformation generated by the Hamiltonian is a canonical transformation- that is the Hamiltonian flows preserve phase volume.

5.4 Generating function of canonical transformations

A fundamental property of canonical transformations is that it preserves the element of area in the phase space. For example, in the one dimensional case we have,

$$\int_R dpdq = \int_{R'} [Q, P]_{q,p} dQdP \tag{5.38}$$

where the Poisson bracket is the same as the Jacobian given in the previous section and is unity.(The arguments of this sections may be easily extended to systems with more than one degree of freedom.)

In general a transformation from one set of phase space variables to another set requires specifying two functions. However, the constraint emerging from the area preserving property reduces this to just one function.

For a region R enclosed by a curve C, by Stokes theorem we have

$$\int_R dpdq = \int_C pdq, \quad p = p(q); \tag{5.39}$$

and

$$\int_{R'} dPdQ = \int_{C'} PdQ, \quad P = P(Q) \tag{5.40}$$

The distinction between C and C' is only in terms of the parametrisations, but in an abstract sense they are the same. Using the area preserving property, we have

$$\int_C p\,\mathrm{d}q = \int_{C'} P\,\mathrm{d}Q \tag{5.41}$$

That is out of the four variables (p, q, P, Q) we may choose any two of them to be independent for variation. This implies that, given a CT, in each neighbourhood in the phase space there exists a function$F$ such that

$$\int_C [pdq - PdQ] = \int_C \mathrm{d}F(q, Q) = 0 \tag{5.42}$$

Since there is a choice of four variable from which to choose, we can define four types of generating functions:

1. Type I: $F = F_1(q, Q)$. The other two variables are then given by,

$$\frac{\partial F_1}{\partial q} = p\,, \quad \frac{\partial F_1}{\partial Q} = -P\,, \quad \widetilde{H} = H\,. \tag{5.43}$$

2. Type II: $F = F_2(q, P)$ with

$$\frac{\partial F_2}{\partial q} = p\,, \quad \frac{\partial F_2}{\partial P} = Q\,, \quad \widetilde{H} = H\,. \tag{5.44}$$

Note that the generating functions F_1 and F_2 may be related by the Legendre-Transformation $F_2(q, P, t) := F_1(q, Q, t) + QP$.

3. Type III: $F = F_3(p, Q)$. The other two variables are then given by,

$$\frac{\partial F_3}{\partial p} = -q\,, \quad \frac{\partial F_3}{\partial P} = -Q\,, \quad \widetilde{H} = H\,. \tag{5.45}$$

4. Type IV: $F = F_4(p, P)$ with

$$\frac{\partial F_4}{\partial p} = -q\,, \quad \frac{\partial F_4}{\partial P} = Q\,, \quad \widetilde{H} = H\,. \tag{5.46}$$

Generalisation to many degrees of freedom is rather straightforward.

Examples :

Below we give examples of some simple generating functions-

(A) Identity transformation:

$$F_2 = \sum_i q_i P_i \qquad \Rightarrow \qquad p_i = P_i\,, \quad Q_i = q_i\,. \tag{5.47}$$

(B) Point transformations:

$$F_2 = \sum_i P_i f_i(\mathbf{q}, t) \qquad \Rightarrow \qquad Q_i = f_i(\mathbf{q})\,, \quad p_i = \sum_j P_j \frac{\partial f_j}{\partial q_i}\,. \tag{5.48}$$

A special property of the point transformations is that the coordinates transform among themselves, as in $(x, y, z) \to (r, \theta, \phi)$. The new momenta P_i may however depend on both the old momenta and coordinates. Of course, by choosing f_i appropriately, it is possible to arrange momenta to be transformed among themselves.

(C) An interesting case is the following:

$$F_1 = \sum_i q_i Q_i \qquad \Rightarrow \qquad p_i = Q_i\,, \quad P_i = -q_i\,. \tag{5.49}$$

5.4.1 Time dependent transformation

Consider a time-dependent transformation in phase space given by

$$\begin{aligned} q_i &\longrightarrow Q_i(\mathbf{p},\mathbf{q},t) \\ p_i &\longrightarrow P_i(\mathbf{p},\mathbf{q},t) \\ H(\mathbf{p},\mathbf{q},t) &\longrightarrow \widetilde{H}(\mathbf{P},\mathbf{Q},t) \end{aligned} \tag{5.50}$$

The transformation is canonical if and only if the Jacobian

$$\det\left(\frac{\partial(Q_1,\ldots,Q_N,P_1,\ldots,P_N)}{\partial(q_1,\ldots,q_N,p_1,\ldots,p_N)}\right)_t = 1 \tag{5.51}$$

where t is held fixed for partial differentiation. Once this is satisfied, one may directly take over the theory of generating functions as outlined before: For example **1. Type I:** $F = F_1(\mathbf{q},\mathbf{Q},t)$.

$$\frac{\partial F_1}{\partial q_i} = p_i\,, \quad \frac{\partial F_1}{\partial Q_i} = -P_i\,, \quad \widetilde{H} = H + \frac{\partial F_1}{\partial t}\,. \tag{5.52}$$

Using Legendre-Transformation $F_2(\mathbf{q},\mathbf{P},t) := F_1(\mathbf{q},\mathbf{Q},t) + \sum_i Q_i P_i$ **2. Type II:** $F = F_2(\mathbf{q},\mathbf{P},t)$ mit

$$\frac{\partial F_2}{\partial q_i} = p_i\,, \quad \frac{\partial F_2}{\partial P_i} = Q_i\,, \quad \widetilde{H} = H + \frac{\partial F_2}{\partial t}\,. \tag{5.53}$$

Analogously we obtain 3. and 4. Type: $F_3(\mathbf{p},\mathbf{Q},t)$ and $F_4(\mathbf{p},\mathbf{P},t)$. In all these cases the Hamiltonian is given by,

$$\widetilde{H} = H + \partial F_n/\partial t$$

.

5.4.2 Group of Canonical Transformations

It is easy to see that the set of all canonical transformations form a group:

- Identity transformation is trivially canonical.
- The inverse of a canonical transformation is also a canonical transformation.
- Product of two canonical transformations is also a canonical transformation- it is associative.

The infinitesimal form of a canonical transformation may be obtained using, for example, the generating function $F_2(P,q)$. let

$$F_2(\mathbf{P},\mathbf{q}) = \mathbf{P}.\mathbf{q} + \epsilon W(\mathbf{P},\mathbf{q},\epsilon)$$

where ϵ is a small parameter. The first term is simply the identity. Using the equations of motion we have,

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i + \epsilon \frac{\partial W}{\partial q_i}$$
$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i + \epsilon \frac{\partial W}{\partial P_i}$$

In the limit $\epsilon \to 0$, we have to first order

$$P_i = p_i - \epsilon \frac{\partial W}{\partial q_i}$$

$$Q_i = q_i + \epsilon \frac{\partial W}{\partial P_i}.$$

Further to first order in ϵ,

$$\frac{\partial P_i}{\partial \epsilon} = -\frac{\partial W}{\partial Q_i}$$
$$\frac{\partial Q_i}{\partial \epsilon} = \frac{\partial W}{\partial P_i}.$$

Thus if we regard ϵ as the time parameter, then these are the Hamilton equations of motion with the generator W playing the role of the Hamiltonian. Thus the Hamiltonian itself is a generator of canonical transformations.

5.5 Action-Angle Variables

The simplest possible description of a *conservative system* is provided by the *action-angle* variables. The original variables (q, p) may not necessarily be the best suited variables to solve (integrate) the system even if the physics of the problem is best illustrated in terms of these variables. To be specific consider the simple case,

$$H(p, q) = \frac{p^2}{2m} + V(q),$$

where $p(q, E) = \pm\sqrt{2m(E - V)}$ is a multi valued function. We shall seek a new set of variables, (I, θ) such that

- each phase curve is labelled uniquely by I, called action, which is constant along the phase curve and
- each point is identified by a single variable θ, called angle.

The first requirement gives,

$$\frac{dI}{dt} = -\frac{\partial H}{\partial \theta} = 0 \Rightarrow H = H(I)$$
$$\frac{d\theta}{dt} = \frac{\partial H}{\partial I} = \omega,$$

where ω is now a constant since H is a function of I only.

For a one-dimensional system θ may be periodic (though this is not always the case), that is $\theta \to \theta + 2\pi$ after every period.

Consider a general Hamiltonian with one degree of freedom given above. The area enclosed by the phase curve at energy E, say, is

$$A(E) = \int dpdq = \int_C p(q,E)dq = 2\int_{q_1}^{q_2} dq\sqrt{2m(E - V(q))}$$

where q_i are the turning points. In the action-angle variables we have

$$A(E) = \int_0^{2\pi} d\theta I = 2\pi I.$$

Therefore we have

$$I = \frac{1}{\pi}\int_{q_1}^{q_2} dq\sqrt{2m(E - V(q))}$$

which is now only a function of the energy E.

5.5.1 Harmonic oscillator in N-dimension

We have already analysed the harmonic oscillator problem in one dimension in terms of action and angle variables. Consider a more general problem of oscillator in N-dimensions. For simplicity let us set $m = \omega = 1$ for all the oscillator. The Hamiltonian is given by

$$2H = \sum_{i=1}^{N}(p_i^2 + q_i^2).$$

In particular let us define

$$I_i = p_i^2 + q_i^2; \quad i = 1, \cdots, N.$$

Since

$$\dot{q}_i\delta_{ij} = \frac{\partial H}{\partial p_i} = \frac{\partial I_j}{\partial p_i}$$

and

$$\dot{p}_i\delta_{ij} = -\frac{\partial H}{\partial q_i} = -\frac{\partial I_j}{\partial q_i}$$

we have

$$[I_i, H] = 0$$

for $i = 1, ..., N$. Further it is easy to show that

$$[I_i, I_j] = 0$$

proving the fact that I_i are indeed constants of motion in **involution**, that is their PBs vanish. Thus the Hamiltonian in action variables may be written as

$$2\widetilde{H}(I_1, \cdots, I_N) = \sum_i I_i.$$

For example for a two dimensional oscillator the action variables are $I_1 = p_1^2 + q_1^2$ and $I_2 = p_2^2 + q_2^2$. The full phase space is four dimensional $(p_1, q_1; p_2, q_2)$. But the constants I_1 and I_2 define a two dimensional surface embedded in the 4-dimensional space. In particular I_1 and I_2 describe separately the equation of a circle. Thus the 2d surface on which the motion takes place is a torus.

In general for an n-dimensional oscillator system the motion takes place on an n-torus embedded in a $2n$- dimensional space.

5.6 Integrable systems

Mechanical systems that can be integrated (solved) completely and globally are rare and exceptional systems. In general the chances of finding a complete solution depends on the existence of **integrals of motion**.

Having gone through many examples, by now it seems that the existence of n-integrals of motion in involution (same as the number of degrees of the system) renders a 2n-canonical equations integrable. By this token, all conservative systems with one degree of freedom and a phase space dimension-2, are integrable. Similarly the central potential problem in 3-dimensions is also integrable since apart from energy, we have angular momentum in z-direction L_z and the square of the angular momentum vector L^2 as integrals of motion.

We shall make a general statement below, some times also called the Liouville integrability theorem, which clarifies the above point- without proof:

Let $I_1, \cdots, I_n$ be dynamical quantities defined on a 2n-dimensional phase space $T^*Q(q_i, p_i)$ of an autonomous Hamiltonian system, H. Let I_i be in involution,

$$[I_i, I_j] = 0; \quad i, j = 1, \cdots, n.$$

That is I_i's are independent in the sense that at each point on the n-dimensional surface $M = \{q_i, p_i\}$ the differentials $dI_1, dI_2, \cdots, dI_n$ may be effected in a linear independent way. Then

- M is a smooth surface that stays invariant under the flow (evolution) corresponding to H. If in addition M is compact and connected, then it can be mapped on to an $n-$ dimensional torus:

  $$T^n = S^1 \times S^1 \times S^1 \cdot \times S^1$$

  repeated $n-$ times and every S^1 is a circle.

- Every S^1 may be described by an angle variable $0 \leq \theta_i \leq 2\pi$. Most general motion on M is quasi periodic, which is given by the solution of the transformed equations of motion:

  $$\frac{d\theta_i}{dt} = \omega_i$$

- The canonical equations of motion are then solved by quadrature (integration).

When ω_i are rationally independent, that is

$$\sum_i r_i \omega_i = 0$$

if and only if $r_i = 0$ for all i, the motion is in general quasi periodic since any trajectory on T^n never closes.

5.6.1 Digression on the angle variable

For fixed I, the relation between θ and q is obtained by considering area between two neighbouring curves defined by, I and$I + \delta I$. The change in the area

$$\delta A = \int \int_S dp dq = \int dq[p(q, I + \delta I) - p(q, I)] = \delta I \int dq \frac{\partial p}{\partial I} + \cdots$$

Since in the I, θ plane,

$$\delta A = \delta I . \theta(q, I)$$

we have

$$\theta(q, I) = \frac{\partial}{\partial I} \int_0^q dq' p(q', I)$$

Typically p is positive as q is increasing and negative otherwise.

While we have assumed that θ is bounded and periodic, there are exceptional systems in one dimensions where θ is not bounded. For example the inverted oscillator in one-dimension is described by the Hamiltonian,

$$H(p, q) = \frac{p^2}{2m} - \frac{1}{2} m \omega^2 q^2$$

whose solutions are given by

$$q = (\frac{2I}{\omega})^{1/2} \sinh \theta$$

$$p = (2I\omega)^{1/2} \cosh \theta$$

and

$$\widetilde{H} = \omega I.$$

Even though we may write

$$\theta(t) = \omega t + \theta(0)$$

the "angle" is unbounded in this case and can not be interpreted as an angle variable as in the case of the harmonic oscillator. Similar situation occurs in the case of a free particle.

5.7 Problems

1. The Hamiltonian of a system is given by,

$$H = p^2 + (4q + 1)p + 4q^2 - 2q.$$

Show that under a canonical transformation the Hamiltonian reduces to the form

$$H = P^2/2m - mgQ,$$

resembling that of a particle falling under gravity. Here m and g are some fixed numbers.

Draw the phase portrait of the system in the $Q - P$ space and then transform it back to get the phase portrait in the original phase space.

2. Consider a Hamiltonian of the perturbed linear oscillator.

$$H(q, p) = \frac{p^2}{2m} + \frac{1/2}{m}\omega^2 q^2 + \frac{1}{4}\epsilon m q^4$$

Find the approximate change in the period of the oscillator. Only corrections to first order in ϵ are require.

3. The Hamiltonian of a system is

$$H = 8p^3 + (12q + 1)p^2 + 2(3q^2 - 1)p + q^3 - q,$$

Can you make a canonical transformation that will bring the Hamiltonian to the form,

$$H = P^2/2m + m\omega^2 Q^2/2$$

resembling that of a simple harmonic oscillator?

4. A transformation of the coordinate and momentum (q, p) of a certain system is given by,

$$T_1: \quad Q = 2q; \quad P = 2q + p/2.$$

Another transformation on the same system is given by,

$$T_2: \quad \tilde{Q} = \sqrt{3}Q + P; \quad \tilde{P} = Q/2 + \sqrt{3}P/2.$$

Show that these transformations are canonical. Find a generator of the product transformation T_1T_2. Can the same generator be used to generate the product transformation T_2T_1?

5. (a) Sketch the potential function

$$V(q) = -q^2/2 + q^4/4$$

and determine the stationary (equilibrium) points.

(b) Write down the Hamiltonian and the equations of motion assuming a particle of unit mass is moving under the influence of this potential. Find the fixed points.

(c) Define the separatrix and find the equation of the separatrix for the above Hamiltonian.

(d) Give a rough sketch of the phase curves.

6. A ball bounces vertically up and down, making elastic collisions with the floor. The Hamiltonian for the system is

$$H(q,p) = \frac{p^2}{2m} + mgq$$

where m is the mass of the ball, and g is the acceleration due to gravity.

(a) Show that when the energy of the system is E, the action is

$$I = \frac{2\sqrt{2m}}{3\pi mg} E^{3/2}.$$

(b) Find the Hamiltonian H(I) in the action angle variables. Determine the period T of the bouncing motion as a function of energy E.

7. The motion of a particle in two dimensions is described by a Lagrangian,

$$L = \frac{1}{2} m[(\frac{dr}{dt})^2 + r^2(\frac{d\phi}{dt})^2] - V(r),$$

where $V(r) = -e^2/r$.[Hydrogen atom in two-dimensions]. Obtain the corresponding Hamiltonian in terms of (p_r, p_ϕ, r, ϕ). Using the definition of action variables over one complete cycle, obtain expressions for I_r and I_ϕ and further show that the energy of the system may be written as,

$$E = -\frac{2\pi^2 me^2}{(I_r + |I_\phi|)^2}.$$

8. In the above problem, expressing the Hamiltonian in Cartesian coordinates, show that in addition to the constants of motion energy E and angular momentum —L—, there are following two additional classical constants of motion :

$$L_x = \frac{p_y}{m} L - \alpha \frac{x}{r}$$

$$L_y = -\frac{p_x}{m} L - \alpha \frac{y}{r}.$$

Further show that algebra of L_x, L_y and L closes.

9. Consider the motion of a particle of mass m in the potential

$$2V(r) = kr^2 + \alpha/r^2.$$

where k, α and r are all positive. Show that the energy and action are related by,

$$E = \sqrt{k\alpha} + 2I\sqrt{k/m}.$$

Calculate the frequency of motion. (You may assume the following integral,

$$\int (-Ax^2 + 2Bx - C)^{1/2}\frac{dx}{x} = \pi(B\sqrt{A} - \sqrt{C})$$

for positive A, B, C. The integrand vanishes at the limits.)

10. Consider the motion of a perfectly elastic ball of unit mass bouncing between two planes moving apart with a constant speed V. (you may assume one plane fixed and other moving). Show that the the action between successive collisions may be related by,

$$I_{n+1} = I_n(1 + \frac{2z_n^2}{1 - 3z_n}).$$

Find z_n and show that the character of motion changes after $v_0/2V$ collisions with the moving plane.

11. The Hamiltonian for a quartic oscillator is given by,

$$H = \frac{p^2}{2} + \frac{\omega_0^2 r^2}{2} + \frac{\epsilon r^4}{4}$$

Assume ϵ is a small parameter. Using first order canonical perturbation theory,

(a) Find the perturbed frequency ω.

(b) Find the generator of canonical transformation connecting (I, θ) of the perturbed system with (J, ϕ) (unperturbed) variables.

12. A particle is moving in a potential

$$V(q) = A(q/d)^{2n},$$

where A and d are positive constants and n is a positive integer. Show that the energy E and the action I are related by,

$$E = (n\pi I/dJ_n)^{2n/n+1}(1/2m)^{n/n+1}A^{1/n+1}$$

where

$$J_n = \int_0^1 dx(1-x)^{1/2}x^{-1+1/2n}$$

For large n, $J_n = 2n + O(1/n)$. Calculate the energy in this limit and interpret the result.

13. Use perturbation theory (first order) to obtain the first order correction to the motion of the free rotor Hamiltonian

$$H(p) = p^2/2G,$$

where G is the moment of inertial of the system, when perturbed by the potential

$$V(\psi) = \alpha \sin^3(\psi).$$

14. (a) Consider a Hamiltonian system with n degrees of freedom, that is, it is described by generalised coordinates $q_1, ..., q_n$ and momenta $p_1, ..., p_n$. In relation to this system, explain the terms "integrable" and "non-integrable".

    (b) For a Hamiltonian system with n-degrees of freedom, under what conditions is it possible, in principle to make a canonical transformation to angle-action variables? Assuming such a transformation is possible for a given system with 2 degree of freedom, describe the most general trajectory for the motion of the system in action-angle variables.

15. A light elastic string of length $4b$ is loaded with 3 beads each of mass m. Assuming the generalised coordinates q_1, q_2, q_3 to be lateral displacements,

    (a) write down the kinetic energy and the potential energy of the system.

    (b) In the limit of small oscillations find the frequency of the normal modes.

    (c) Sketch the oscillation profiles for the system of 3 beads in terms of the generalised coordinates corresponding to the three normal modes.

    You may choose to do this directly by going over to appropriate coordinate system or by following the standard method of small oscillations.

16. A system with configuration angle ψ has the Hamiltonian

$$H = (1/2)p^2 + p\sin(\psi), \quad (|\psi| \leq \pi)$$

Determine the position and the nature of the fixed points. Sketch the phase diagram, showing clearly where there is libration and where there is rotation. Write an expression for the period of librating motion at energy E (need not integrate).

17. A particle of mass m is constrained to move in a vertical plane under the influence of gravity on the smooth cycloid with parametric equation

$$z = A(1 - \cos(\phi)), \quad x = A(\phi + \sin(\phi)) \quad (-\pi < \phi < \pi),$$

where A is a positive constant and the z axis is vertically upwards. If s is the arc length along the cycloid, measured from the bottom of the curve, show that the Lagrangian of the system can be written in the form,

$$L = (1/2)m\dot{s}^2 - mgs^2/8A.$$

Find the period of motion.

18. The Hamiltonian for a SHO is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Introduce the complex coordinates

$$z = \sqrt{m\omega/2}(x + ip/m\omega)$$

$$z^* = \sqrt{m\omega/2}(x - ip/m\omega)$$

.

(1) Is this a Canonical Transformation? (2) Express H in terms of z, z^*. (3) Evaluate the Poisson brackets $[z, z^*], [z, H], [z, H^*]$. (4) Write down and solve the equations of motion for z, z^*, hence obtain the solutions for x, p.

19. The Hamiltonian of a freely falling body is

$$H(z, p) = p^2/2m + mgz,$$

z being the height of the particle above the ground and m its mass. By expressing z as a function of (p,P) and using the F_4 generating function find a canonical transformation so that the new Hamiltonian is

$$K(Q, P) = P$$

Solve for $(Q(t), P(t))$ and hence find the time dependence of $(z(t), p(t))$.

20. The motion of a projectile near the surface of the earth is governed by the Hamiltonian

$$H = \frac{1}{2m}[p_x^2 + p_z^2] + mgz$$

where x denotes the horizontal coordinate and z the vertical, while p_x, p_z are the conjugate momenta.

Set up the time independent of Hamiltonian-Jacobi equation for the problem and solve the same. Use the solution to obtain $x(t), z(t)$.

Chapter 6

Hamilton-Jacobi Theory

Consider a general Hamiltonian $H(\mathbf{p}, \mathbf{q})$, $\mathbf{p}$ and $\mathbf{q}$ may have the physical interpretation of momentum and position variables. Even in the case of an exactly solvable system, these may not necessarily be the best set of variables to solve the system in spite of the fact that the physics of the problem is best illustrated in terms of these variables. In fact the ideal situation would be to find a set of local coordinates on the phase space such that the Hamiltonian is a constant.The generating function of such a transformation is in general a solution of a non-linear partial differential equation, namely, the **Hamilton-Jacobi** equation or simply **HJ** equation.

The HJ equation are not easy to solve, but they lend themselves to perturbative calculations (for example the way it is used in celestial mechanics). Historically, they played an important role in the development of quantum mechanics.

6.1 The Hamilton-Jacobi equation

For a **conservative system** with N degrees of freedom, an optimal canonical transformation is of the form

$$H(\mathbf{p}, \mathbf{q}) \longrightarrow \widetilde{H}(\mathbf{P}, \mathbf{Q}) \stackrel{!}{=} \widetilde{H}(\mathbf{P}), \tag{6.1}$$

by which the new Hamiltonian function is a function of the new momenta P_i and is independent of the new coordinates Q_i. Such a transformation may be effected using the generator of Type 2 S given by,

$$S(\mathbf{q}, \mathbf{P}) := F_2(\mathbf{q}, \mathbf{P}). \tag{6.2}$$

For such a generator, we must have

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}, \quad \mathbf{Q} = \frac{\partial S}{\partial \mathbf{P}}, \quad \frac{\partial S}{\partial t} = 0. \tag{6.3}$$

If such a transformation satisfies the ansatz in (6.1), immediately we get the **time independent Hamilton-Jacobi equation:**

$$H\left(\frac{\partial S}{\partial \mathbf{q}}, \mathbf{q}\right) = E = H(\mathbf{P}, \tag{6.4}$$

where the RHS is a number for a given set of P_i which are constants of integration. Remarks:

- Thus (6.4) is a partial differential equation for the generating function $S(q_1, \ldots, q_N, P_1, \ldots$ in the N coordinates q_i, for a given set of P_i which are constants of integration. The equations are in general non-linear.

- Once it is shown that the (6.4) exists, the equations of motion are trivially solved:

$$\frac{\partial \widetilde{H}}{\partial Q_i} = -\dot{P}_i = 0 \quad \Rightarrow \quad P_i = const., \tag{6.5}$$

$$\frac{\partial \widetilde{H}}{\partial P_i} = \dot{Q}_i = \omega_i = const. \quad \Rightarrow \quad Q_i(t) = \omega_i t + \alpha_i \,. \tag{6.6}$$

- For a give set of P_i, S is often referred to as the action integral (also known as the Hamilton Characteristic Function) since it can be written as an integral over coordinates alone (apart from constants):

$$S = \sum_{i=1}^{N} \int_{q_i(0)}^{q_i(t)} p_i \mathrm{d}q_i = \int_{\mathbf{q}(0)}^{\mathbf{q}(t)} \mathbf{p} \cdot \mathrm{d}\mathbf{q} \,. \tag{6.7}$$

In the case of **time dependent transformations**, the Hamilton principal function (4.3) itself is the generator of the desired time independent transformation:

$$\begin{aligned} F_2(\mathbf{q}, \mathbf{P}, t) = R(t_0, t) &= \int_{t_0}^{t} L[\mathbf{q}(t'), \dot{\mathbf{q}}(t'), t']\, \mathrm{d}t' \\ &= \int_{t_0}^{t} \left(\sum_i p_i \dot{q}_i - H(\mathbf{q}, \dot{\mathbf{q}}, t') \right) \mathrm{d}t' \,, \end{aligned} \tag{6.8}$$

where P_i are again fixed to be constants of integration. Using (5.53) the time dependent Hamilton-Jacobi equation is given by

$$H\left(\frac{\partial R}{\partial \mathbf{q}}, \mathbf{q}, t \right) = -\frac{\partial R}{\partial t} \,. \tag{6.9}$$

where as in (6.4) the energy $H = E = const.$ with the ansatz

$$R = S - Et \,. \tag{6.10}$$

Examples

(A) *One dimensional systems:* Consider the motion of a particle in a potential $V(q)$ in one dimension. The transformed momentum can be chosen to be the energy since it is conserved, that is: $P = E = const$. The generating function is $S = S(q, E)$ which is a solution of the HJ equation

$$\widetilde{H}\left(\frac{\partial S}{\partial q}, q\right) = E \tag{6.11}$$

with

$$p = \frac{\partial S(q,E)}{\partial q}\,; \qquad Q = \frac{\partial S(q,E)}{\partial E}\,. \tag{6.12}$$

The solutions for the new variables is straight forward-

$$\begin{aligned} \frac{\partial \widetilde{H}}{\partial Q} &= -\dot{P} = 0\,, \\ \frac{\partial \widetilde{H}}{\partial P} &= \dot{Q} = \frac{\partial E}{\partial E} = 1 \quad \Rightarrow \quad Q(t) = t - t_0\,. \end{aligned} \tag{6.13}$$

The generating function is then given as a solution of the equation: (with $(P, Q) = (E, t - t_0)$)

$$H(p,q) = \frac{p^2}{2m} + V(q) = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) = E\,, \tag{6.14}$$

from which we get

$$\frac{\partial S}{\partial q} = \sqrt{2m[E - V(q)]} \quad \Rightarrow \quad S(q,E) = \int_{q_0}^{q} \sqrt{2m[E - V(q')]}\,\mathrm{d}q'. \tag{6.15}$$

For $Q = t - t_0$ we find

$$\begin{aligned} Q = t - t_0 &= \frac{\partial S}{\partial E} = \frac{\partial}{\partial E}\int_{q_0}^{q} \sqrt{2m[E - V(q')]}\,\mathrm{d}q' \\ &= \sqrt{\frac{m}{2}} \int_{q_0}^{q} \frac{\mathrm{d}q'}{\sqrt{E - V(q')}}\,. \end{aligned} \tag{6.16}$$

which upon inversion gives the solution for the variable $q(t)$.

(B) *Separable Systems:* If the Hamiltonian function has the form

$$H(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{N} h(p_i, q_i)\,, \tag{6.17}$$

then the system is separable. For example h may be of the form $h(p,q) = p^2/2m + V(q)$. Such a situation, as in (6.17), occurs in the mean field approach to many body problems (Hartree-Fock-Theory, Density Functional theory). When the Hamiltonian is of the form (6.17) the corresponding generating function $S(\mathbf{q}, \mathbf{P})$ also has a separable form:

$$S(\mathbf{q}, \mathbf{P}) = \sum_{i=1}^{N} s(q_i, P_i) \tag{6.18}$$

The problem then reduces to finding the solution for the generating function $s(q, P)$. The HJ-equation(6.4) may be broken into N equations given by

$$h\left(\frac{\partial s}{\partial q_i}, q_i\right) = E_i = const., \qquad (i = 1, \ldots, N) \tag{6.19}$$

where E_i is the energy associated with each degree of freedom. The total energy is of course

$$E = \sum_i E_i\,. \tag{6.20}$$

Chapter 7

Non-linear Maps and Chaos

7.1 Introduction

The Classical Mechanics that is often presented in class rooms is in most of the cases limited to specific kinds of systems which are exactly solvable. We have, however, extended our analytical tools to handle systems which are not exactly solvable with the following methods:

- Identifying the fixed points in the phase space.
- Do stability analysis to have local information even in the absence of global understanding.
- Use perturbation theory when exact solutions are not available.

While this enlarges the kind of problems that one can deal with, these methods do not go too far either. It is more a rule than an exception to have systems which are not amenable to analytical tools. Quite often we have to resort to numerical methods and big computers to solve complex systems.

However, there are systems, more common than is normally expected, which are qualitatively different from any thing that we have known before. They may be **chaotic**. Even when the system does not look complicated, (e.g. the dynamical equations may look very simple), the system may exhibit a rich repertoire of dynamics and can go *beyond the scope of any theoretical prediction* within a very short time!!

Determinism vs. predictability :

The basic structure of Classical Mechanics tells us that once we know the differential equations governing the dynamics of some system (from Newton's second law, or from the knowledge of the Hamiltonian of the system) and the values of the quantities of interest at some point in time (initial conditions), we know *in principle* all about the past and the future of the system. Thus, we say Classical Mechanics is a *deterministic theory*.

For centuries, people including prolific mathematicians like Lagrange have thought that from a deterministic theory we can always *predict* everything about the system, given enough computing power. It is only recently that people have come to appreciate the important property about the chaotic systems where long term predictions are doomed no matter how large the computer is. In fact, we will see that the outcomes of such systems behave almost like random variates, which are by definition unpredictable[1].

Therefore, *determinism does not ensure predictability.*

What is Chaos ?

The essential property due to which systems become chaotic is called *sensitivity to initial conditions.* By this we mean, that any two phase space trajectories that start out very close to each other separate exponentially with time. In practice this means that after a small *time* the separation $\delta\vec{x}(t)$ attains the magnitude of L, the characteristic length scale of the system as a whole.

Physicist's definition of chaos :

A chaotic system must have exponential growth of error in computing a phase space trajectory and typically should also have an attractor with bounded phase space volume. To satisfy both these conditions, the phase trajectories should come back close together, a property called *mixing*). In general, chaotic systems have phase diagrams where a certain pattern seems to occur over and over infinitely many times, but never actually repeating.

We may therefore define chaos as:

> The system obeys deterministic laws of evolution, but the outcome is highly sensitive to small uncertainties in the specification of the initial state: Locally unstable and globally mixing.

Measure of sensitivity: Lyapunov exponents :

The property of *sensitivity to initial conditions* can be quantified as

$$|\delta\vec{x}(t)| \approx \exp \lambda t |\delta\vec{x}(0)|,$$

where λ, the mean rate of separation of the trajectories of the system, is called the **Lyapunov exponent**. For the system to be chaotic, the Lyapunov exponent has to be positive.

For any finite accuracy

$$|\delta\vec{x}(0)| = \delta x,$$

of the initial data, the dynamics is predictable only up to a finite **Lyapunov time**

$$T_L \approx -\frac{1}{\lambda} \log(\delta x/L).$$

Note: `Positive Lyapunov exponents` ***`does not`*** `guarantee chaos. This is because of the fact that Lyapunov exponents carry absolutely` ***`no information`*** `about the mixing property.`

[1] We should not confuse chaotic systems with *Stochastic systems* where the system is *intrinsically non-deterministic* due to the presence of external random disturbance or noise.

7.2 Modelling and Maps

To study the dynamics of a system, we first need to *model* them in a neat mathematical construct. This in most physical systems is a set of differential equations which describes how the quantities to be observed change with a parameter time. By discretising time, we can convert our differential equations to difference equations[2] or *maps*. For example, the differential equation

$$\frac{dx(t)}{dt} = g(t)$$

has the equivalent map

$$x_{n+1} = x_n + g_n$$

where the discretised time is labelled by n with some arbitrary constant time step Δt.

This eases out the computational task of solving or simulating the system to a great extent and allows us to concentrate on the qualitative nature of the dynamics without caring a great deal about the mathematical difficulties of the model[3].

More importantly, in practical applications, what we have is a set of observations, a sequence of states, at times differing by multiples of a period. Such a situation is more common in biological evolution, economics, agriculture or medical data. Maps turn out to be a natural choice to express their dynamics. The system may have many variables, and corresponding no. of dimensions in the phase space. We, henceforth, shall confine ourselves to maps involving one variable.

7.2.1 Linear and quadratic maps in 1-D

The motion of an one-dimensional system is represented by a sequence of real numbers labelled by integer time variable t and satisfying the relation—

$$\begin{aligned} f_r : \quad \mathrm{R} &\to \mathrm{R}\,, \\ x &\mapsto f_r(x)\,. \end{aligned} \tag{7.1}$$

that is,

$$x_{n+1} = f_r(x_n)\,. \qquad n \in N \tag{7.2}$$

Here the variable is x and r denotes the control parameter(s) which is system dependent.

To determine the variable after k time steps, the map can be iterated k times to obtain a new iterated map denoted by f^k which is given by:

$$f^k(x_n) := \underbrace{f(f(\dots f(x_n)\dots)}_{k\ \text{mal}} = x_{n+k}\,. \tag{7.3}$$

[2]also called Recurrence relations.

[3]Solving differential equations by numerical methods anyway gives us a discrete set of data points

Therefore, defining the extent of divergence for each iteration as $\Delta x_n = |x_{n+1} - x_n|$, the Lyapunov Exponent of an 1-D map can be formally defined as:–

$$\sigma(x_0) = \lim_{k\to\infty}\left[\frac{1}{k}\lim_{\epsilon\to 0}\ln\left|\frac{f^k(x_0+\epsilon) - f^k(x_0)}{\epsilon}\right|\right]. \tag{7.4}$$

A *Fixed point* of the map x_r^*, if it exists, is obtained by

$$x_r^* = f_r(x_r^*)\,. \tag{7.5}$$

which may be stable or unstable. Note that as before once the system is at a fixed point by definition, it stays there for all times– that is for all further iterations of the map.

The fixed point may be, in general, stable or unstable. It is stable if the system tends towards the fixed point, that is,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n(x_1) = x^* \quad \text{fr alle}\,. \tag{7.6}$$

Otherwise it is unstable.

Linear Map :

The general form of a linear map is

$$\begin{array}{rccc} f_r: & \mathrm{R} & \rightarrow & \mathrm{R} \\ & x & \mapsto & rx + c\,. \end{array} \tag{7.7}$$

For $r \neq 1$, the fixed point is $x^* = c$. Further iterations of the map gives,

$$f_r^k(x^*) = r^k x^* \tag{7.8}$$

Thus the system evolves according to two possible scenarios: When $|r| < 1$ and any initial value $x_0 \in \mathrm{R}$ the system converges to x^*. The fixed point is therefore stable.

For $|r| > 1$ the fixed point at the origin is unstable. The Lyapunov-Exponent is (from eq.7.4) –

$$\sigma(x_0) = \lim_{k\to\infty}\left[\frac{1}{k}\ln\left|\frac{\mathrm{d}f^k(x)}{\mathrm{d}x}\right|_{x=x_0}\right] \tag{7.9}$$

$$= \lim_{k\to\infty}\left[\frac{1}{k}\ln|r^k|\right] \tag{7.10}$$

$$= \ln|r| \tag{7.11}$$

and is independent of the initial starting pointx_0. The behaviour of the system may be summarised as follows:

$\lvert r\rvert < 1$	$\lim_{n\to\infty} x_n \to c$	$x^* = 0$ stable FP	$\sigma < 0$
$\lvert r\rvert > 1$	$\lim_{n\to\infty} x_n \to \infty$	$x^* = 0$ unstable FP	$\sigma > 0$

Quadratic maps : Quadratic 1-D maps can have the most general form—

$$\begin{aligned} f: \quad \mathrm{R} &\rightarrow \mathrm{R} \\ x &\mapsto gx^2 + hx + c\,. \end{aligned} \tag{7.12}$$

For simplicity, we shall consider one-parameter maps, i.e. maps where g and h can be parametrised by a single variable r as, $g = g(r)$ and $h = h(r)$. These maps are very important because:-

- Quadraticity is the minimum possible non-linearity. So, these are the simplest of all systems that exhibit chaos.
- They still manifest almost all the qualitatively different and interesting features of a typical chaotic system in their dynamics.
- As these have only one observable in phase space and only one variable in the parameter space, all of their asymptotic dynamics can be plotted on a single 2-D graph to give us a pictorial understanding of chaotic motion.

The prototype of 1-dimensional chaos is the famous **Logistic Map**. This produces so rich a dynamics with least possible analytic or computational clutter that we shall devote a whole section to the study of this map which will, in its course, reveals a host of interesting non-linear phenomena.

7.3 The Logistic Map

Logistic Equation was first introduced by P.F. Verhulst in 1845 to model the population dynamics of a single biological species constrained only by natural resources (food, say) available. The map was popularised in a seminal 1976 paper by the biologist Robert May. T

$$\dot{x} = rx(1-x).$$

This equation can be solved analytically and leads to stable population for some values of x. The implicit assumption here is that the birth and death rates are uniform over all time. This, however is not a reasonable assumption in many situations. Often species have their own *breeding seasons* and we are interested in their population data taken *yearly*, or *seasonally.* Here is where maps come handy since only periodic updates are needed without recourse to continuum dynamics.

As per the procedure of getting maps from differential equations mentioned earlier, we define **Logistic Map** as:

$$\begin{aligned} f_r: \quad [0,1] &\rightarrow [0,1] \\ x &\mapsto f_r(x) = r\,x(1-x) \\ \text{Therefore} \quad x_{n+1} &= r\,x_n(1-x_n). \end{aligned} \tag{7.13}$$

Note that the system is non-linear.

The fixed points of the map depend on the parameter r.

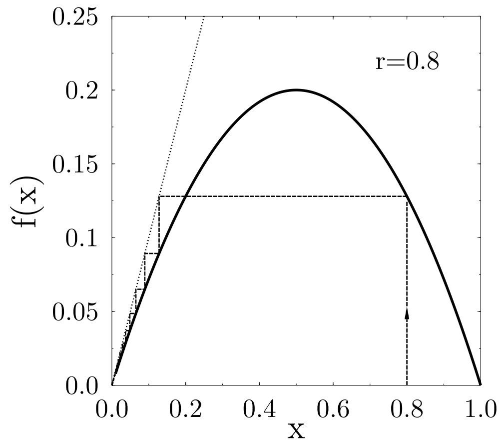


Figure 7.1: $r = 0.8$ (stable fixed point being $x^* = 0$)

- For $r < 1$, $x^* = 0$ is the only fixed point–$|f_r'(0)| = r < 1$ and is therefore stable.
- For $1 < r < 3$ there are two fixed points– namely $x_1^* = 0$ and $x_2^* = 1 - 1/r$. The first one now is unstable since ($|f_r'(0)| = r > 1$), and the second one is stable since ($|f_r'(x_2^*)| = |\,r - 2\,| < 1$).
- For $r > 3$ ist $x_2^* = 1 - 1/r$ is also not stable since $|f_r'(x_2^*)| = |\,r - 2\,| > 1$. Indeed both fixed points are unstable. Therefore, we do not get any fixed point final state of the system in practice.[4]

We shall use *cobweb graphs* here, as a nice tool to visualise the time evolution of logistic map at different r values. The method of drawing them is very simple and straightforward:

1. Draw the map and the line $x_n = x_{n+1}$ on the same graph.
2. Start from the x_0 on x-axis and move vertically upwards to hit the map.
3. Move horizontally to reach the $x_n = x_{n+1}$ line, you are at x_1 above x-axis now.
4. Move vertically again as in 2 and repeat.

The figures 7.1 and 7.2 illustrate the dynamics of the map for two parameter values in the fixed point regime. At $r = 3$ the stable fixed point at x_2^* has $|f'(x_2^*) = 1|$ and it becomes unstable for values of $r > 3$. Thus both fixed points of the first order quadratic map become unstable at this point. That is all the further iterates move away from these two fixed points unless we start *exactly* on them. The question then is where does the system go?

[4]We still get periodic points, and the system becomes 2,4,8... -periodic as we increase r further, until at $r = 3.56$ it becomes chaotic. We shall study this in sec. 2.1 in details.

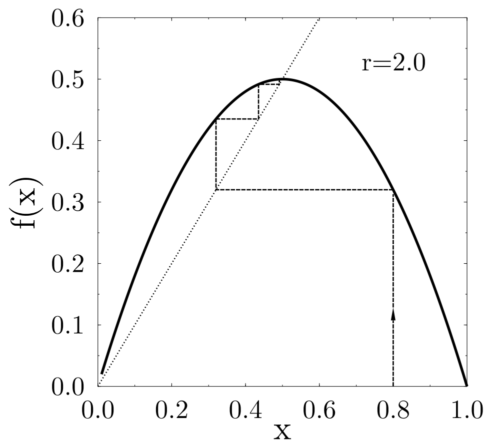


Figure 7.2: $r = 2.0$; stable fixed at $x_2^* = 0.5$, unstable fixed point at $x_1^* = 0$

The answer is hidden in the second order map, f^2, that is the map of second iterates: This map, which goes directly from x_n to x_{n+2}, has a mapping function given by,

$$g_r(x) := f_r(f_r(x)) = f_r^2(x) = r^2x(1-x) - r^3x^2(1-x)^2. \tag{7.14}$$

The fixed points of the map are determined by the quartic equation

$$x = r^2x(1-x) - r^3x^2(1-x)^2. \tag{7.15}$$

We immediately recognise that $x_1^* = 0$ and $x_2^* = 1-1/r$ must be roots of this equation since they are clearly fixed points of the first order map. With little algebra it is easy to see that it has two more roots:

$$x_{3,4}^* = \frac{r+1 \pm \sqrt{(r+1)(r-3)}}{2r}. \tag{7.16}$$

It turns out that both these fixed points are *attractive* for the map of the second iterates. Furthermore it is easy to show that

$$x_3^* = f(x_4^*), \quad x_4^* = f(x_3^*),$$

in terms of the first order map.

The values jump up and down after each map but rapidly approach the fixed points $x_{3,4}^*$ alternatively for large values of n. If T_0 is the time between successive iterations of the map, the map has a periodic motion with a period $2T_0$. We call this phenomenon a **bifurcation**. The asymptotic time evolution is now an oscillation between two states (see Figure 7.3).

The periodic doubling feature of the logistic map seen for $r > 3.0$ persists till r approaches the value 3.45.

The logistic map for $3.45 < r < 3.57$ becomes even more interesting. The two fixed points of the map remain stable or attractive fixed points until $r = 1 + \sqrt{6} =$

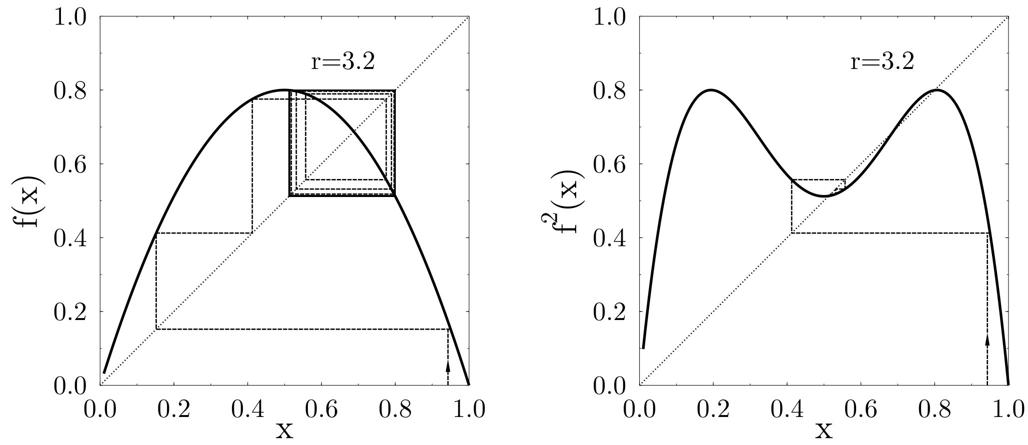


Figure 7.3: $r = 3.2$ Period 2 bifurcation of the logistic map

3.4495. As might be expected, the behaviour of the system undergoes a dramatic change after this value of the control parameter.

An understanding of what happens next is gained by looking at the fourth iterates:

$$x \to f_r^4(x) = f_r^2(f_r^2(x)).$$

Without submitting this map to a rigorous algebraic study, we may take resort to a computer experiment. All the four fixed points known until now become unstable, but four new attractive or stable fixed points appear in the place of old ones:

$$x_6^* = f(x_5^*), \quad x_7^* = f(x_6^*), x_8^* = f(x_7^*), \quad x_5^* = f(x_8^*),$$

The iterates of the original map, after the initial transient, approach these values and stay forever in the above cycle with period $4T_0$.

The point to note here is that *as we keep increasing the parameter r, the pattern of period doublings continues to occur.* Old fixed points become unstable, but reign of order is still found in higher order maps.[5]

Bifurcation diagram :

Equipped with the power of computers today, we are therefore in a position to ask the question: What happens to the system's long-term behaviour as we vary the parameter r over the whole continuous spectrum of allowed values? The set of instructions that we may give to the computer are:-

1. Set $r = 0.0$
2. Iterate the Logistic Map for 200 (say) times starting with a random initial condition.

[5]All fixed point of a map is also a fixed point of its higher order iterated maps, but converse is not true.

3. Plot the next 100 iteration values on a x_n vs. r graph.

4. If $r \geq 4.0$, terminate; *else* increase r by 0.001 and repeat from step 2.

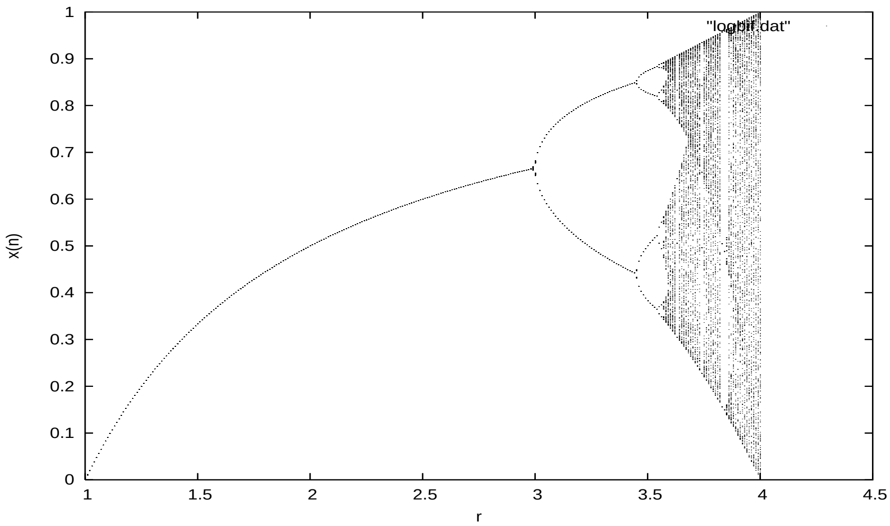


Figure 7.4: Bifurcation diagram for logistic map. r goes upto 4.5 to emphasise the fact that the points flies off to ∞ for $r > 4.0$

This produces figure 7.3. We observe exactly what is to be expected from the pattern above — and a lot more. We see the fixed point solution increasing and giving rise to periodic points, then 4-periods and so on. But what happens as r is increased further? At each branching the original fixed point becomes unstable and gives rise to two new stable fixed points. The values of r at which these bifurcations occur are called bifurcation points. As we have seen the first bifurcation occurs at $r = 3.0$, next at $r = 3.45$ and so on. If nth bifurcation occurs at a point r_n, it is clear from the figure 7.3 that $r_{n+1} - r_n$ decreases rapidly as n increases. By the time r reaches a value $r_\infty = 3.5699456\cdots$, an infinite number of bifurcations would have occurred with as many fixed points in the interval $0 < x < 1$.

We now consider the question of sensitivity to initial conditions. This is characterised by the Lyapunov exponent λ defined by the equation

$$\delta x(t) = \delta x(0) \exp \lambda t,$$

where $\delta x(0)$ is a tiny separation between two initial points and $\delta x(t)$ is the separation between the two trajectories after time t. If indeed the trajectories diverge chaotically, then λ will be found to be positive for almost all choices of the initial points. Computer calculations for $0 < r < r_\infty$ show that the Lyapunov exponent is either zero or negative throughout this range. This is to be expected since the motion after the initial transient is almost always periodic for $r < r_\infty$.

However, once $r > r_\infty$, the iterates of the logistic map jump chaotically in the interval $0 < x < 1$. This is confirmed by the calculations of the Lyapunov exponent which shows that λ is positive for almost all values of r in the range $r_\infty < r < 4.00$.

However, this is not the whole story. Within this regime of chaos, there exist small windows of regular behaviour where $\lambda < 0$. These, rather narrow, windows are characterised by odd cycles $3, 5, 6 \cdots$ instead of the even cycles characteristic of the regular domain $r < r_\infty$. What we find therefore is an extraordinary mixture of chaotic and regular behaviour.

The Feigenbaum constants α and δ :

No treatment of the logistic equation is complete without a mention of the seminal discoveries of M.J. Feigenbaum which led to the application of statistical methods in *chaos analysis*. We have already emphasised the cascade of bifurcations and the period-doublings as a function of the control parameter r.

It is clear from Fig.7.3 that each time a bifurcation takes place the new stable fixed points are distributed in the same way as before but appears reduced in size and resembles a pitchfork, hence the name pitch-fork bifurcation. Let us denote the width of a pitchfork by d_n at the nth bifurcation. Based on high precision numerical analysis, Feigenbaum found that the ratio of successive pitchfork widths approached a limiting value as n became large. This gave rise to the *universal constant* α defined as

$$\alpha = \frac{d_n}{d_{n+1}} = 2.5029078750957 \cdots . \tag{7.17}$$

Furthermore, the positions r_n at which bifurcations occur also show a *self similar* behaviour. Roughly speaking, if r_n and r_{n+1} are the values of successive points at which bifurcations occur, we define a second *universal constant* denoted as δ given by

$$\delta = \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669201609 \cdots . \tag{7.18}$$

as $n \to \infty$. The reason why we call them universal constants will be made clear soon. The existence of the constants α and δ in the case of logistic equation may not be wholly unexpected. What is remarkable is the discovery by Feigenbaum that a whole class of maps, called *quadratic maps*, display a bifurcation pattern where by the constants α, δ defined as above have exactly the same value as determined from logistic map above. It is in this sense we call these constants universal. Quadratic maps are defined as maps which have only a single maximum in the interval $0 < x < 1$. It is in this sense that we call all functions that give rise to bifurcations which scale according to same α and δ as belonging to the same *universality class*, a term borrowed from theory of critical phenomena in statistical mechanics.

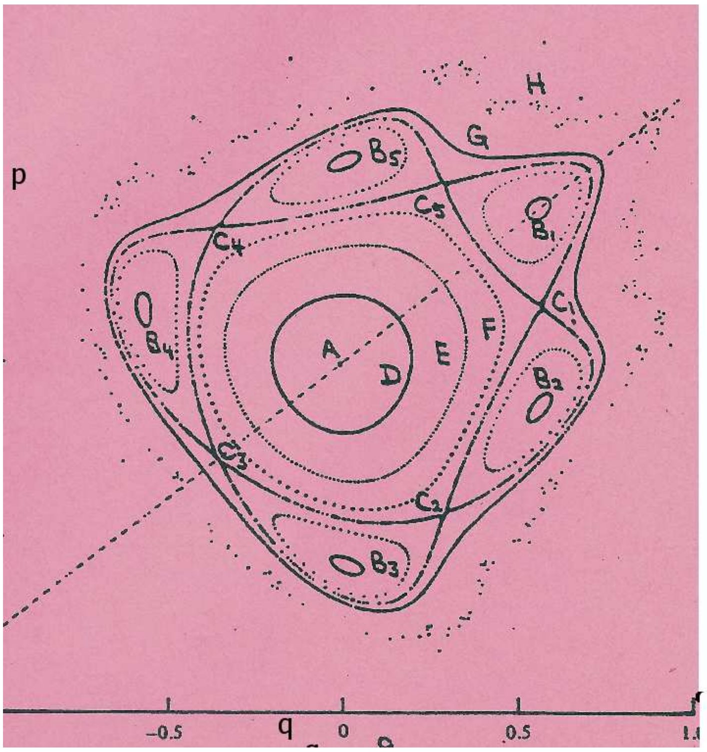
7.4 Problems

1. Consider a non-linear map,

$$q_{j+1} = \cos\alpha\ q_j - \sin\alpha\ (p_j - q_j^2)$$
$$p_{j+1} = \sin\alpha\ q_j + \cos\alpha\ (p_j - q_j^2)$$

(a) where q_j, p_j are coordinate and conjugate momentum. Show that this gives a canonical transformation from (q_j, p_j) to the new variables, (q_{j+1}, p_{j+1}).

(b) The points on the (q, p) phase-space plot in the figure were calculated by taking many iterations of the above map with $\cos\alpha = 0.24$ with several initial conditions. Discuss various features of this figure in as much a detail as possible using the labels in the figure. In particular your answer should include the following: elliptic and hyperbolic points, the presence or absence of a separatrix; regular or periodic motion.



2. The equation of motion for a first order linear system is given by,

$$\frac{dx}{dt} = a_1 x t \quad (0 \le t < T_1),$$

$$\frac{dx}{dt} = a_2 x t \quad (T_1 \le t \le T),$$

where T is the period of the system.

(a) Find the period propagator of the system. (b) Suppose $a_1 = -a_2 = 1$ and $T = 2$ and $T_1 = 1$, sketch the time evolution of the system. Qualitatively discuss the state of the system after long times when $a_1 = a_2$ are both positive.

3. The Ulam - von Neumann map is defined by

$$x_{n+1} = 1 - bx_n^2$$

in the interval $-1 \leq x_i \leq 1$.

(a) Determine the range of b necessary to keep the mappings on x in the interval.

(b) Determine the fixed points and their stability.

(c) Determine the range of b for 1-cycle, 2-cycle and 3-cycle.

(d) Assuming that the universality has already set in, determine the Feigenbaum constant δ and predict b_∞.

(e) Check the results numerically by drawing the bifurcation diagram for the fixed points of this map. Find the value of the other critical constant α.

4. The equation of motion of a system is $dx/dt = a(t)x$ where

$$a(t) = ct(0 \leq t < 1), \quad a(t) = a(t+1)$$

Obtain the period propagator K and determine the stability of motion for real non-zero values of c. Obtain also the period propagator $k(t+1, t)$ for all t and sketch the graph $x(t)$ between $t = -2 and t + 2$ if $x(-2) = 1$. How would the graph change if $x(-2) = 3$?

Appendix A

Tangent space and tangent bundle

Let us consider the Eucledian space[1] $\mathbf{E}^n$. Let $\mathbf{U}$ be a subspace of $\mathbf{E}^n$ i.e. $\dim(\mathbf{U}) \leq n$. Let $\vec{p} \in \mathbf{U}$. Let $\vec{v}$ and $\vec{w}$ be two vectors with their base at point $\vec{p}$ (this is similar to the effect on two radius vectors, when we shift the origin to the point $\vec{p}$). Let us denote the set $\mathbf{S}_p$ as the set of all such vectors with their base at point $\vec{p}$. Note that :

$$(\vec{v} + \vec{w}) \in \mathbf{S}_p$$

$$(\alpha\vec{v} + \beta\vec{w}) \in \mathbf{S}_p \quad \text{, where} \quad \alpha, \beta \in \mathbf{R}$$

If we have a set $\mathcal{B}$ of such vectors which are linearly independent, then we can say that $\mathbf{S}_p$ is a vector space with dimension equal to the cardinality of the set $\mathcal{B}$. Vector space $\mathbf{S}_p$ is called the *tangent space* at $\vec{p}$. Physically, the vectors of $\mathbf{S}_p$ are the tangent vectors to all possible curves, that pass through the point $\vec{p}$ and are constrained on the subspace $\mathbf{U}$. If $\dim(\mathbf{U}) < n$ then, the subspace $\mathbf{U}$ is called a *hyper surface* or, a *manifold*[2] $\mathcal{M}$, embedded in $\mathbf{E}^n$. Interestingly, $\dim(\mathbf{S}_p) = \dim(\mathbf{U})$. The union of the tangent spaces at every point on $\mathcal{M}$ is called the *tangent bundle*, i.e. tangent bundle $:= \bigcup_p \mathbf{S}_p \quad \forall \ \vec{p} \in \mathcal{M}$.

In the section below Q represents the configuration manifold, while TQ represents the tangent bundle associated with it.

Notes on Q and TQ :

While discussing the Lagrangian equations we have concentrated on the configuration space Q. The equations of motion yield what we are really interested, namely $\vec{q}(t)$ on Q. In Eucledian space, all position vectors are contained in E^3, say. Similarly all velocity vectors $\vec{v}$ are also represented in the same space E^3 However, this may not always be the case since the configuration space Q is not always the Eucledian space E^3. For example consider a particle confined to move on a circle $Q = S^1$, or a sphere $Q = S^2$. While the configuration is embedded in E^3, the motion takes place in a subspace Q. The velocity vector $\vec{v}$ is tangent to $\vec{q}$ contained in Q at every point.

[1]There is a difference between an Euclidean space and a real vector space. A real vector space is not Euclidean by itself.

[2]A manifold has more structures than a mere subspace.

Thus we need to construct a new space TQ in which to describe motion, Although Euler-Lagrange equations are second order differential equation on Q, the Lagrangian $L(q, \dot{q}, t)$ depends on both positions and velocities, and may be time t. Thus the Lagrangian is function on a larger space, TQ which is also called the velocity phase space.

The **velocity phase space** is obtained from Q, by adjoining every point q on Q, all velocities allowed. That is all possible tangents to Q at that point denoted as $T_q\ Q$. The velocity phase space is a collection of all Q and T_qQ.

As an illustration, consider $Q = S^1$. To each point denoted by $q = \theta$ on S^1, adjoin all possible values of $\dot{\theta}$, which runs from $-\infty\ to\ +\infty$. This attaches an infinite line$T_\theta S^1$ on both sides of the point θ on the circle, thus generating an infinite cylinder, $TQ = TS^1$. A simple example where this happens is the Pendulum.

Note that this is in contrast to the Phase-space dynamics which we discussed earlier. The motion is entirely contained in the (q, p) phase space by definition. In order to distinguish the two spaces, the phase space is often denoted by T^*Q.

Appendix B

Legendre Transformation

Legendre transformation is the route that leads us from the **Lagrangian** to the **Hamiltonian** and vice-versa.

We had performed a Legendre transform when we wrote

$$p = \frac{\partial L}{\partial \dot{q}} \tag{B.1}$$

The method :

So, how does one construct a *Legendre Transform*?

Consider the function :

$$F = f(x).$$

Our aim would be to describe the same function *uniquely* using the tangent and the intercept of $f(x)$, in the *intercept-tangent* plane. The curve of the function lies in the $x - F$ plane. The tangent at any point of the curve is given by :

$$t(x) = \frac{dF}{dx} \tag{B.2}$$

The intercept of the curve at a given point x_0 is given by :

$$I(x_0) = f(x_0) - x_0 t(x_0)$$

Therefore at a general point x the intercept function is given by

$$I(x) = F - xt(x)$$

This gives the required curve in the $t - I$ plane. Next we have to construct a function $g(t)$ which is independent of x. This can be achieved by:

- Inverting[1] the function that we obtained in **116** (i.e. expressing x as a function of t).
- Replacing it in the expression for $I(x)$, thereby re-expressing it as $I(t)$.

[1]This invertibility is not obvious and may give rise to other complications.

- Define $G = g(t) = -I(t)^2$.

Now all that is left to prove, is a way to get back to the original curve $f(x)$ in our old plane. We do this in the following way :

- Draw straight lines with slope $t(x)$ and intercept $I(x)$, for each value of x in the $x - F$ plane.
- Sketch the envelope of this family.

This envelope gives us the required curve $f(x)$.

Mathematically, we do the same thing with G as what we did with F.

$$\begin{aligned}\frac{dG}{dt} &= \frac{d}{dt}\left(tx(t) - F\right) \\ &= x(t) + tx'(t) - \frac{df}{dx}\frac{dx}{dt} \\ &= x(t)\end{aligned}$$

We apply the same prescription as before and get back F. This can be easily generalised to higher dimensions with a few modifications.

Invertibility of the Legendre transforms and the Hessian condition: The invertibility of a function (map) requires it to be single valued (one-one). This rules out any possibility of the existence of an extrema in the range. The same is valid for the map **115**. For this map to be invertible the matrix formed by $\frac{\partial^2 L}{\partial q_i \partial q_j}$ should be non-singular. This is known as the *Hessian condition* and the matrix is called the *Hessian.* Let the matrix be denoted by $\mathcal{H}$. If $|\mathcal{H}| = 0$ then the transform is non-invertible and we land up with what is known as a *Singular Lagrangian.* Its evident, that such a Lagrangian arrests our progress to the corresponding (non-trivial) Hamiltonian and back, by the methods that we have dealt so far.

We get a another Hessian $\mathcal{H} = \frac{\partial^2 H}{\partial p_i \partial p_j}$ when we try to move from the Hamiltonian to the Lagrangian, requiring the same conditions for invertibility.

[2] This step was not necessary, but was done to draw an analogy with the Lagrangian and the Hamiltonian inter-conversions.

Appendix C

The Calculus of Variations

A long time ago, one of the Bernoulli brothers posed the following problem to the other:

> Let there be two nails fixed to a wall, at random positions, such that one is above the other. A string is attached between the nails, and a bead is introduced such that it can move freely on the string. What should be the shape of the string such that the bead takes the least time to go from the upper nail (say nail A) to its lower counterpart (say nail B) ?

We have to minimise

$$\begin{aligned} t &= \int dt \\ &= \int \frac{\mathrm{d}s}{v} \end{aligned}$$

Historically, the calculus of variations was motivated by problems like the one above, which is so famous that it has got its own name: it's called the Brachistochrone problem. In what follows, we will try to solve this problem.

Variational calculus is in many ways similar to the single- and multi-variable calculus and in some ways different. From a knowledge of calculus it's relatively easy to step into the calculus of variations. The single major difference can be illustrated using an example in two dimensions: in 'normal' calculus, say with two variables x and y, if we want to check whether some point is a local minimum of a function $g(x, y)$ then we only have to look at the function in the neighbourhood of that point, whereas for a problem in the calculus of variations such as the one above, we want to check whether the integral of some function $F(y, y\prime, x)$ along some *curve* $(x(t), y(t))$ between two predefined endpoints is a minimum and for that we have to look at all the curves in the 'neighbourhood' of that curve, that is all the curves infinitesimally close to the curve $(x(t), y(t))$. This integral is a function $A(y(x))$ from the space of all curves (in 2-space) to the real numbers [1]. Since it's actually simpler to take the general case (again, in 2 dimensions; the generalisation to any number of dimensions comes soon,

[1]Incidentally, a function from the space of functions to the reals is called a functional.

and is easy), let's try and find an extremum of A for the function $F(y, y\prime, x)$ along all curves $y(x)$ between endpoints (x_1) and (x_2). First of all, define

$$A(y) = \int_{x_1}^{x_2} F(y, y\prime, x)dx \tag{C.1}$$

To carry out the variation, define an arbitrary continuous function $\eta(x)$ and now define a function of *two* variables:

$$Y(x, \epsilon) = Y(x, 0) + \epsilon\eta(x)$$

where x and ϵ are independent and $y(x) = Y(x, 0)$. Since the endpoints were defined to be fixed, η is subject to the constraint

$$\eta(x_1) = \eta(x_2) = 0$$

Therefore

$$A(Y) = \int_{x_1}^{x_2} F(Y, Y', x)dx$$

Assume we already know $y(x)$ for which $A(y(x))$ is at an extremum. In that case,

$$\left.\frac{\delta A(Y(x, \epsilon))}{\delta\epsilon}\right|_{\epsilon=0} = 0$$

Differentiating,

$$\frac{\delta A}{\delta\epsilon} = \int_{x_1}^{x_2} \left(\frac{\delta F}{\delta Y}\frac{\delta Y}{\delta\epsilon} + \frac{\delta F}{\delta Y'}\frac{\delta Y'}{\delta\epsilon}\right)dx$$

Taking the second part of the LHS,

$$\begin{aligned}
\int_{x_1}^{x_2} \frac{\delta F}{\delta Y\prime}\frac{\delta Y\prime}{\delta\epsilon}dx &= \int_{x_1}^{x_2} \frac{\delta F}{\delta Y\prime}\frac{\delta^2 Y}{\delta x\delta\epsilon}dx \\
&= \left.\frac{\delta F}{\delta Y\prime}\frac{\delta Y}{\delta\epsilon}\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\delta F}{\delta Y\prime}\right)\frac{\delta Y}{\delta\epsilon}dx \\
&= \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\delta F}{\delta Y\prime}\right)\frac{\delta Y}{\delta\epsilon}dx
\end{aligned}$$

since $\frac{\delta Y}{\delta\epsilon} = \eta(x) = 0$ at x_1 and x_2. Putting it back in,

$$\begin{aligned}
\int_{x_1}^{x_2} \left(\frac{\delta F}{\delta Y}\frac{\delta Y}{\delta\epsilon} + \frac{d}{dx}\left(\frac{\delta F}{\delta Y\prime}\right)\frac{\delta Y}{\delta\epsilon}\right)dx &= 0 \\
\Rightarrow \int_{x_1}^{x_2} \left(\frac{\delta F}{\delta y} + \frac{d}{dx}\frac{\delta F}{\delta y\prime}\right)\frac{\delta Y}{\delta\epsilon}dx &= 0 \\
\Rightarrow \int_{x_1}^{x_2} \left(\frac{\delta F}{\delta y} + \frac{d}{dx}\frac{\delta F}{\delta y\prime}\right)\eta(x)dx &= 0
\end{aligned}$$

because $Y = y$, $Y' = y\prime$ at $\epsilon = 0$ and $\frac{\delta Y}{\delta e} = \eta(x)$. Now we've come to the key point of all we've been doing so far: the variation is zero for *any* arbitrary function $\eta(x)$. This can only happen if

$$\left(\frac{\delta F}{\delta y} + \frac{d}{dx}\frac{\delta F}{\delta y\prime}\right) = 0 \tag{C.2}$$

(Why?) This is sometimes called the 'fundamental lemma' of the calculus of variations. Note that we achieved the variation of the curve over *all* the neighbouring paths by first saying that for some η the variation over the paths was zero and then saying that the η chosen was arbitrary. The original problem (remember?) is a special case of this where

$$\begin{aligned} A &= t \\ F &= \frac{ds}{v} \\ &= \sqrt{\frac{1 + y\prime^2}{2gy}} \end{aligned}$$

which can be solved using equation (118).

The generalisation to n independent variables is straightforward. Let

$$A = \int_{t_1}^{t_2} L(q_1, q_2, \ldots q_n, q\prime_1, q\prime_2, \ldots q\prime_n, t)dx$$

Where $q_i = q_i(t)$. Let

$$\begin{aligned} Q_i(t, 0) &= q_i(t) \\ Q_i(t, \epsilon) &= Q_i(t, 0) + \epsilon\eta_i(t) \end{aligned}$$

and so on, in analogy with the two dimensional case above; Differentiating,

$$\begin{aligned} \frac{\delta A}{\delta \epsilon} &= \int_{t_1}^{t_2} \sum_{1 \le i \le n} \left(\frac{\delta F}{\delta Q_i}\frac{\delta Q_i}{\delta \epsilon} + \frac{\delta F}{\delta Q_i'}\frac{\delta Q_i'}{\delta \epsilon}\right)dt \\ \Rightarrow 0 &= \int_{t_1}^{t_2} \sum_{1 \le i \le n} \left(\frac{\delta F}{\delta Q_i} + \frac{d}{dt}\frac{\delta F}{\delta Q_i\prime}\right)\frac{\delta Q_i}{\delta \epsilon}\bigg|_0 dt \\ \Rightarrow 0 &= \int_{t_1}^{t_2} \sum_{1 \le i \le n} \left(\frac{\delta F}{\delta q_i} + \frac{d}{dt}\frac{\delta F}{\delta q_i\prime}\right)\eta_i(t)dt \end{aligned}$$

If each

$$\eta_i(t)$$

is arbitrary, we can set each

$$\left(\frac{\delta F}{\delta q_i} + \frac{d}{dt}\frac{\delta F}{\delta q_i\prime}\right)$$

to zero. The η_i's are arbitrary if and only if the variables q_i are independent. For if they are not, i.e. are related by one or more equations of constraint, then the variations will also be related to each other similarly if they are to be consistent with the equations of constraint [2].

[2] This is sometimes referred to as D'Alembert's Principle.

Example : Find the geodesic [3] on a sphere. (*Hint*: Transform to independent coordinates.) We could try to minimise

$$L = \sqrt{\dot{x} + \dot{y} + \dot{z}}\, dt$$

but we wouldn't be able to use the above algorithm for that because the coordinates x, y, z are related by the constraint

$$x^2 + y^2 + z^2 = R^2$$

All we have to do is transform to R, θ, ϕ coordinates, since θ and ϕ would be independent and R not required. In that case we can perform the variation over $R\, d\theta$ and $R \sin\theta\, d\phi$. Now if θ and ϕ are varied independently, the variation would be consistent with the equation of constraint; it would be *tangential* to the surface determined by the constraints. One could abstract out this notion and say that if one has n variables related by k independent equations of constraint, then the variation has to take place in the space of dimension $n - k$ tangential to the kernel of the constraint equations, which is a space of dimension $n - k$.

[3] The geodesic between two points is the path of shortest distance between the points.

Appendix D

Pendulum with a vibrating base

We consider perturbations of a Hamiltonian perturbed by rapid oscillations. Later we apply this to the case of a pendulum with a vibrating base. The basic idea is to see if the unstable fixed point of the *mathematical pendulum* can be made stable by perturbing with rapid oscillations of the base (pivot)[1]. We first consider the general theory and then the application[2].

D.1 Fast perturbations

To set the setting consider the motion of a free particle in a rapidly oscillating field:

$$H(p,q) = H_0(p,q) + qF\sin\omega, t$$

where F is a constant and $H_0 = p^2/2m$ is the unperturbed Hamiltonian. It is easy to see that the equation of motion is given by

$$m\ddot{q} = -F\sin\omega t.$$

Integrating we have

$$q(t) = q_0(t) + \xi(t) = q_0(t) - \frac{F\sin\omega t}{m\omega^2}$$

and

$$p(t) = p_0(t) + \eta(t) = q_0(t) + \frac{F\cos\omega t}{\omega}$$

Thus the average motion always follows the unperturbed motion given by $q_0(t) = vt$ which is linear in time. Without further proof we may in general assume that

$$\begin{aligned} q(t) &= q_0(t) + O(1/\omega^2) \\ p(t) &= p_0(t) + O(1/\omega). \end{aligned} \tag{D.1}$$

[1]This is a statement made by Feynman in his Lectures on Physics.
[2]See Percival and Richards, Introduction to Dynamics for more details

in the presence of fast perturbations, that is when ω is large. Thus in the order of perturbations we assume that $\eta^2 and \xi$ are of the same order. We will use this fact presently.

Consider now a general Hamiltonian with a perturbing field that is rapidly oscillating. We want to find the Hamiltonian of mean motion. Let

$$H(p,q,t) = H_0(p,q) + V(q)\sin\omega t, \tag{D.2}$$

where $V(q)\sin\omega t$ is an external perturbation. In the general the solution may be written as a combination of the smooth and oscillating part:

$$q(t) = q_0(t) + \xi(t), \quad p(t) = p_0(t) + \eta(t),$$

where we assume that the oscillating has a mean which is zero.

The Hamiltonian equations of motion are

$$\dot{q}(t) = \frac{\partial H}{\partial p} = \dot{q}_0(t) + \dot{\xi}(t) = \frac{\partial H_0}{\partial p},$$

$$\dot{p}(t) = -\frac{\partial H}{\partial p} = \dot{p}_0(t) + \dot{\eta}(t).$$

Now consider making a Taylor expansion of the full Hamiltonian given in Eq.D.2 in two variables around the variables of the mean motion (p_0, q_0):

$$H_0(p,q) = H_0(p_0,q_0) + \frac{\partial H}{\partial p_0}\eta + \frac{\partial^2 H_0}{\partial p_0^2}\frac{\eta^2}{2} + \frac{\partial H_0}{\partial q_0}\xi + \cdots$$

and

$$V(q) = V(q_0) + \frac{\partial V}{\partial q_0}\xi + \cdots,$$

where we have used the fact that ξ and η^2 are of the same order.

The equations of motion to first order in ξ are given by

$$\dot{q} = \dot{q}_0 + \dot{\xi} = \frac{\partial H_0}{\partial p_0} + \frac{\partial^2 H}{\partial p_0^2}\eta + \frac{\partial^3 H_0}{\partial p_0^3}\frac{\eta^2}{2} + \frac{\partial^2 H_0}{\partial q_0 \partial p_0}\xi + \cdots$$

$$\dot{p} = \dot{p}_0 + \dot{\eta} = -[\frac{\partial H_0}{\partial q_0} + \frac{\partial^2 H}{\partial p_0 \partial q_0}\eta + \frac{\partial^3 H_0}{\partial p_0^2 \partial q_0}\frac{\eta^2}{2} + \frac{\partial^2 H_0}{\partial q_0^2}\xi] - [\frac{\partial V(q_0)}{\partial q_0} + \frac{\partial^2 V}{\partial q_0^2}\xi]\sin\omega t] + \cdots$$

In order to get the mean motion we average over the period of rapid oscillations with $\langle\xi\rangle = 0 = \langle\eta\rangle$. We have

$$\langle\dot{q}\rangle = \frac{\partial H_0}{\partial p_0} + \frac{\partial^3 H_0}{\partial p_0^3}\langle\frac{\eta^2}{2}\rangle + \cdots$$

$$\langle\dot{p}\rangle = -\frac{\partial H_0}{\partial q_0} - \frac{\partial^3 H_0}{\partial p_0^2 \partial q_0}\langle\frac{\eta^2}{2}\rangle - \frac{\partial^2 V}{\partial q_0^2}\langle\xi\sin\omega t\rangle + \cdots$$

The equation of motion of the oscillatory terms keeping terms up to the leading order only is given by

$$\dot{\xi} = \frac{\partial^2 H}{\partial p_0^2}\eta + \cdots$$

and

$$\dot{\eta} = -\frac{\partial V(q_0)}{\partial q_0}\sin\omega t + \cdots ,$$

Assuming q_0, p_0 vary very little over a period, the approximate solutions for the oscillating parts, $\xi(t), \eta(t)$ from the above equations are therefore given by

$$\eta(t) = \frac{\cos\omega t}{\omega}\frac{\partial V(q_0)}{\partial q_0}$$

and

$$\xi(t) = \frac{\sin\omega t}{\omega^2}\frac{\partial V(q_0)}{\partial q_0}\frac{\partial^2 H}{\partial p_0^2}\eta$$

Further we also have, for the averages,

$$\langle \xi(t)\sin\omega t\rangle = \frac{1}{2\omega^2}\frac{\partial V(q_0)}{\partial q_0}\frac{\partial^2 H}{\partial p_0^2}\eta$$

and

$$\langle \eta^2(t)\rangle = \frac{1}{2\omega^2}(\frac{\partial V(q_0)}{\partial q_0})^2$$

We now substitute these results in the equations of mean motion:

$$\langle \dot{q}\rangle = \frac{\partial H_0}{\partial p_0} + \frac{1}{4\omega^2}(\frac{\partial V(q_0)}{\partial q_0})^2\frac{\partial^3 H_0}{\partial p_0^3}$$

and

$$\langle \dot{p}\rangle = -\frac{\partial H_0}{\partial q_0} - \frac{1}{4\omega^2}(\frac{\partial V(q_0)}{\partial q_0})^2\frac{\partial^3 H_0}{\partial p_0^2 \partial q_0} - \frac{1}{2\omega^2}\frac{\partial V(q_0)}{\partial q_0}\frac{\partial^2 H}{\partial p_0^2}\frac{\partial^2 V}{\partial q_0^2}$$

Both these equations may be combined and written in the form of Hamiltonian equations albeit with a new Hamiltonian K,

$$\langle \dot{q}\rangle = \frac{\partial K(q_0, p_0)}{\partial p_0}$$

and

$$\langle \dot{p}\rangle = -\frac{\partial K(q_0, p_0)}{\partial q_0}$$

where

$$K(q_0, p_0) = H_0 + \frac{1}{4\omega^2}(\frac{\partial V(q_0)}{\partial q_0})^2\frac{\partial^2 H_0}{\partial p_0^2} \tag{D.3}$$

which is the desired effective Hamiltonian of mean motion. Note the correction is of the order $1/\omega^2$.

D.2 Pendulum with a vibrating pivot

Consider a pendulum consisting of a mass m attached to a light stiff rod which is free to move vertically about the pivot. We choose $z-$ axis along the vertical and $x-$ axis to be horizontal. If the length of the rod is l, then the coordinates of the bob are given by

$$x = l\sin\psi, \quad y = -l\cos\psi - F(t)$$

where $F(t)$ is the time dependent vertical displacement of the rod about the pivot and ψ is the generalised coordinate which is the angle between the rod and the down vertical. We will choose a specific form for this later. The potential energy is then given by

$$V(z,t) = mgz = -mg[l\cos\psi + F(t)]$$

The Lagrangian of the system is given by

$$L = \frac{1}{2}m[\dot{x}^2 + \dot{z}^2] - V(z,t).$$

Substituting for the velocities,

$$L = \frac{1}{2}m[l^2\dot{\psi}^2 - 2l\dot{\psi}\dot{F}\sin\psi] + mgl\cos\psi + \frac{1}{2}m\dot{F}^2 + mgF$$

The last two terms are independent of ψ and its derivative and are functions of time only. They may be ignored as far as dynamics of the system is concerned.

We can further manipulate the Lagrangian by using the following identity:

$$\frac{d}{dt}[\dot{F}\cos\psi] = -\dot{F}\dot{\psi}\sin\psi + \ddot{F}\cos\psi.$$

Substituting the above in the Lagrangian above

$$L = \frac{1}{2}ml^2\dot{\psi}^2 + m(g - \ddot{F})l\cos\psi,$$

where we have ignored the total time derivative.

The conjugate momentum is given by

$$p = ml^2\dot{\psi}$$

and the Hamiltonian is

$$H = \frac{p^2}{2ml^2} - ml(g - \ddot{F})\cos\psi$$

This is indeed a nice form since the forced vertical movement can at best alter the acceleration, hence shift g.

Now consider the case when the F is given by an oscillating form:

$$F(t) = A\sin\omega t.$$

This is the same form as the rapidly oscillating perturbation that we considered in the general theory in the previous section. The Hamiltonian may be written in the form

$$H = \frac{p^2}{2ml^2} - mlg(1 + \frac{A\omega^2}{g}\sin\omega t)\cos\psi$$

solving this explicitly we may consider the Hamiltonian of the mean motion K given in Eq.D.3. The additional potential due to vibrations is given by

$$V(q)\sin\omega t = -ml\omega^2 A\sin\omega t.$$

Substituting this in Eq.D.3 we have

$$K(p_0,\psi) = \frac{p_0^2}{2m} - mgl[\cos\psi - k\sin^2\psi] = \frac{p_0^2}{2m} + V_{eff}(\psi) \tag{D.4}$$

where

$$k = \frac{A^2\omega^2}{4gl}.$$

The fixed points of the system are given by $p_0 = 0$ and $\psi = 0, \pi, \text{and}, \cos\psi = -1/2k$ where we have the first two are the old fixed points of the pendulum and the third one is due to the effective potential.

- At $\psi = 0$ is always a stable fixed point since

  $$\frac{d^2V_{eff}}{d\psi^2} = mgl(2k+1) > 0.$$

  This is of course expected.

- At $\psi = \pi$ we have

  $$\frac{d^2V_{eff}}{d\psi^2} = mgl(2k-1)$$

  Thus the fixed point is always stable if $2k > 1$ or equivalently

  $$A^2\omega^2\rangle 2gl,$$

  that is for fast enough oscillations the originally unstable fixed point can be made stable.

- At $\cos\psi = -1/2k$ we have

  $$\frac{d^2V_{eff}}{d\psi^2} = mgl(\frac{1}{2k} - 2k)$$

  which is always unstable since for any real ψ we have $|2k| \leq 1$.

Thus indeed the unstable fixed point of the unperturbed pendulum may be stabilised provided the oscillations are fast enough.

Appendix E

General problems

1. An electrically charged particle of mass m is constrained to lie on the straight line between two fixed charges C_1 and C_2 at a distance $2l$ apart. The total force on the particle is the sum of two forces of magnitude μ/r_j^2, where r_j is the distance from the particle to C_j, and each fixed charge repels the particle. Obtain the Hamiltonian and sketch the phase diagram.

   If the particle starts from rest at a distance kl from the mid point of C_1 and C_2, where $0 < k < 1$, write an expression for the period of oscillation.

2. Consider a tunnel across Earth. The distance between the center of the Earth and mid point along the tunnel is l. A particle of mass m moves freely in the tunnel without friction.

   (a) Show that the motion of the particle corresponds to that of an oscillator.

   (b) Determine the frequency of oscillation and the time taken for one full traversal along the tunnel.

3. A body of mass m falls freely from a height of 200 meters at the equator. Calculate the magnitude and direction of the deflection. What would the deflection be if it were to fall freely in Chennai?

4. Consider the continuous family of coordinate and time transformations

$$Q^\alpha = q^\alpha + \epsilon f^\alpha(q, t),$$

$$T = t + \epsilon\tau(q, t),$$

   where ϵ is a small parameter. Show that if this transformation preserves the action,

$$S = \int_{t_1}^{t_2} L(q, dq/dt, t)dt = \int_{T_1}^{T_2} L(Q, dQ/dT, T)dT,$$

   then

$$\frac{\partial L}{\partial \dot{q}^\alpha}(\dot{q}^\alpha \tau - f^\alpha) - L\tau$$

   is a constant of motion. Assume that q^α satisfy the Lagrange's equations and the invariance of action.

5. Consider the following transformation on a three particle system (equal masses) in one dimension:

$$
\begin{aligned}
x_1 &= Q/3 + \frac{q_1}{\sqrt{2}} + \frac{q_2}{\sqrt{6}} \\
x_2 &= Q/3 - \frac{q_1}{\sqrt{2}} + \frac{q_2}{\sqrt{6}} \\
x_3 &= Q/3 - 2\frac{q_2}{\sqrt{6}} \\
y_1 &= P + \frac{p_1}{\sqrt{2}} + \frac{p_2}{\sqrt{6}} \\
y_2 &= P - \frac{p_1}{\sqrt{2}} + \frac{p_2}{\sqrt{6}} \\
y_3 &= P - 2\frac{p_2}{\sqrt{6}}
\end{aligned}
$$

where (x_i, y_i) and $(Q, q_1, q_2, P, p_1, p_2)$ are the coordinates and momenta in the initial and the transformed systems. Is this a canonical transformation?

The Hamiltonian of a coupled oscillator system is given by,

$$H = \frac{1}{2m}(y_1^2 + y_2^2 + y_3^2) + \frac{1}{2}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2].$$

Express the Hamiltonian in the new coordinates using the transformation given above.

What is the physical interpretation of Q and P? What is a possible generalisation of these coordinates to a system of N particles?

6. Have fun doing this problem of Ant-Chase: Three ants move on a flat table top. Initially they are positioned at the vertices A, B, C of an equilateral triangle where each side measures a length $1cm$. Ant A is attracted to Ant B which is attracted to Ant C which in turn is attracted to Ant A. They head towards each other at a constant speed $1cm/sec$. At every moment therefore the ants are heading towards the instantaneous position of the target ant.

   Do the ants ever meet? If so how long does it take? What is the path followed by the ants. Generalise this to 4 ants.

7. (a) Show that a linear second-order recursion

$$a_{n+1} = pa_{n-1} + qa_n$$

   can be written as a first order matrix equation

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = M \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

   where M is a 2×2 matrix.

(b) If the eigenvalues of this matrix λ, μ are distinct, then show that every solution of the recursion is of the form,

$$a_n = x\lambda^n + y\mu^n,$$

where x, y are determined from the initial conditions a_0, a_1.

(c) Consider the recursion

$$a_{n+1} = a_{n-1}/2 + a_n/2.$$

Show that the solution is given by,

$$a_n = x + (-1/2)^n y$$

for some x and y.