Topic 1 Notes
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1 Complex algebra and the complex plane

We will start with a review of the basic algebra and geometry of complex numbers. Most likely you have encountered this previously in 18.03 or elsewhere.

1.1 Motivation

The equation $x^2 = -1$ has no real solutions, yet we know that this equation arises naturally and we want to use its roots. So we make up a new symbol for the roots and call it a complex number.

Definition. The symbols $\pm i$ will stand for the solutions to the equation $x^2 = -1$. We will call these new numbers complex numbers. We will also write

$$\sqrt{-1} = \pm i$$

Note: Engineers typically use j while mathematicians and physicists use i. We'll follow the mathematical custom in 18.04.

The number i is called an **imaginary number**. This is a historical term. These are perfectly valid numbers that don't happen to lie on the real number line.[1] We're going to look at the algebra, geometry and, most important for us, the exponentiation of complex numbers.

Before starting a systematic exposition of complex numbers, we'll work a simple example.

Example 1.1. Solve the equation $z^2 + z + 1 = 0$.

Solution: We can apply the quadratic formula to get

$$z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\sqrt{-1}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2}.$$

Think: Do you know how to solve quadratic equations by completing the square? This is how the quadratic formula is derived and is well worth knowing!

1.2 Fundamental theorem of algebra

One of the reasons for using complex numbers is because allowing complex roots means every polynomial has exactly the expected number of roots. This is called the fundamental theorem of algebra.

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[1] Our motivation for using complex numbers is *not* the same as the historical motivation. Historically, mathematicians were willing to say $x^2 = -1$ had no solutions. The issue that pushed them to accept complex numbers had to do with the formula for the roots of cubics. Cubics always have at least one real root, and when square roots of negative numbers appeared in this formula, even for the real roots, mathematicians were forced to take a closer look at these (seemingly) exotic objects.

Fundamental theorem of algebra. A polynomial of degree n has exactly n complex roots, where repeated roots are counted with multiplicity.

In a few weeks, we will be able to prove this theorem as a remarkably simple consequence of one of our main theorems.

1.3 Terminology and basic arithmetic

Definitions

- **Complex numbers** are defined as the set of all numbers

$$z = x + yi,$$

  where x and y are real numbers.

- We denote the set of all complex numbers by **C**. (On the blackboard we will usually write $\mathbb{C}$ –this font is called *blackboard bold*.)

- We call x the **real part** of z. This is denoted by $x = \text{Re}(z)$.

- We call y the **imaginary part** of z. This is denoted by $y = \text{Im}(z)$.

Important: The imaginary part of z is a **real number**. It **does not** include the i.

The basic arithmetic operations follow the standard rules. All you have to remember is that $i^2 = -1$. We will go through these quickly using some simple examples. It almost goes without saying that in 18.04 it is essential that you become fluent with these manipulations.

- **Addition:** $(3 + 4i) + (7 + 11i) = 10 + 15i$

- **Subtraction:** $(3 + 4i) - (7 + 11i) = -4 - 7i$

- **Multiplication:**

$$(3 + 4i)(7 + 11i) = 21 + 28i + 33i + 44i^2 = -23 + 61i.$$

  Here we have used the fact that $44i^2 = -44$.

Before talking about division and absolute value we introduce a new operation called conjugation. It will prove useful to have a name and symbol for this, since we will use it frequently.

Complex conjugation is denoted with a bar and defined by

$$\overline{x + iy} = x - iy.$$

If $z = x + iy$ then its conjugate is $\bar{z} = x - iy$ and we read this as "z-bar $= x - iy$".

Example 1.2. $\overline{3 + 5i} = 3 - 5i$.

The following is a very **useful property of conjugation**: If $z = x + iy$ then

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2.$$

Note that $z\overline{z}$ is real. We will use this property in the next example to help with division.

Example 1.3. (Division.)   Write $\dfrac{3+4i}{1+2i}$ in the standard form $x+iy$.

Solution: We use the useful property of conjugation to clear the denominator:

$$\frac{3+4i}{1+2i} = \frac{3+4i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{11-2i}{5} = \frac{11}{5} - \frac{2}{5}i.$$

In the next section we will discuss the geometry of complex numbers, which give some insight into the meaning of the magnitude of a complex number. For now we just give the definition.

Definition. The **magnitude** of the complex number $x+iy$ is defined as

$$|z| = \sqrt{x^2 + y^2}.$$

The magnitude is also called the **absolute value**, **norm** or **modulus**.

Example 1.4. The norm of $3+5i = \sqrt{9+25} = \sqrt{34}$.

Important. The norm is the sum of x^2 and y^2. **It does not include the i and is therefore always positive**.

## 1.4   The complex plane

1.4.1   The geometry of complex numbers

Because it takes two numbers x and y to describe the complex number $z = x + iy$ we can visualize complex numbers as points in the xy-plane. When we do this we call it the complex plane. Since x is the real part of z we call the x-axis the real axis. Likewise, the y-axis is the imaginary axis.
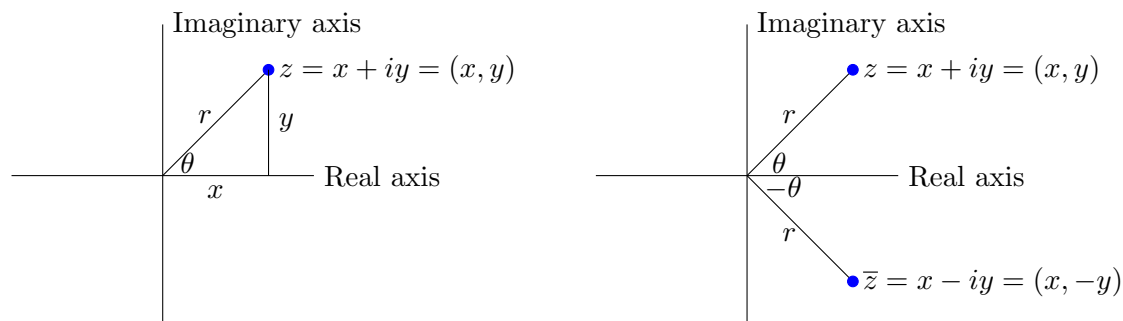
1.4.2   The triangle inequality

The triangle inequality says that for a triangle the sum of the lengths of any two legs is greater than the length of the third leg.
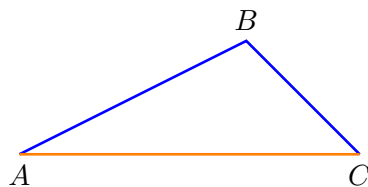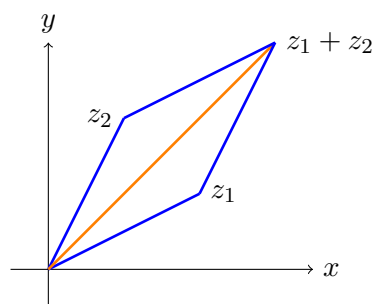
Triangle inequality: $|AB| + |BC| > |AC|$

For complex numbers the triangle inequality translates to a statement about complex magnitudes. Precisely: for complex numbers z_1, z_2

$$|z_1| + |z_2| \geq |z_1 + z_2|$$

with equality only if one of them is 0 or if $\arg(z_1) = \arg(z_2)$. This is illustrated in the following figure.

Triangle inequality: $|z_1| + |z_2| \geq |z_1 + z_2|$

We get equality only if z_1 and z_2 are on the same ray from the origin, i.e. they have the same argument.

1.5 Polar coordinates

In the figures above we have marked the length r and polar angle θ of the vector from the origin to the point $z = x + iy$. These are the same polar coordinates you saw in 18.02 and 18.03. There are a number of synonyms for both r and θ

$$r = |z| = \text{magnitude} = \text{length} = \text{norm} = \text{absolute value} = \text{modulus}$$
$$\theta = \arg(z) = \text{ argument of } z = \text{polar angle of } z$$

As in 18.02 you should be able to visualize polar coordinates by thinking about the distance r from the origin and the angle θ with the x-axis.

Example 1.5. Let's make a table of z, r and θ for some complex numbers. Notice that θ is not uniquely defined since we can always add a multiple of 2π to θ and still be at the same point in the plane.

$z = a + bi$	r	θ	
1	1	$0, 2\pi, 4\pi, \ldots$	Argument $= 0$, means z is along the x-axis
i	1	$\pi/2, \pi/2 + 2\pi \ldots$	Argument $= \pi/2$, means z is along the y-axis
$1 + i$	$\sqrt{2}$	$\pi/4, \pi/4 + 2\pi \ldots$	Argument $= \pi/4$, means z is along the ray at $45°$ to the x-axis
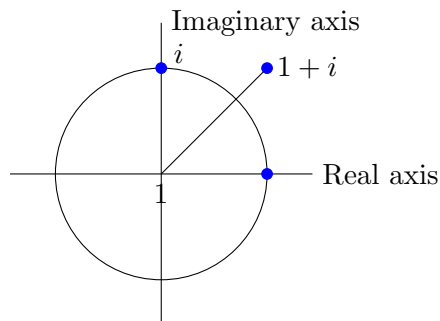
When we want to be clear which value of θ is meant, we will specify a **branch** of arg. For example, $0 \le \theta < 2\pi$ or $-\pi < \theta \le \pi$. This will be discussed in much more detail in the coming weeks. Keeping careful track of the branches of arg will turn out to be one of the key requirements of complex analysis.

1.6   Euler's Formula

Euler's (pronounced 'oilers') formula connects complex exponentials, polar coordinates, and sines and cosines. It turns messy trig identities into tidy rules for exponentials. We will use it a lot. The formula is the following:

$$\mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{1}$$

There are many ways to approach Euler's formula. Our approach is to simply take Equation 1 as the definition of complex exponentials. This is legal, but does not show that it's a good definition. To do that we need to show the $\mathrm{e}^{i\theta}$ obeys all the rules we expect of an exponential. To do that we go systematically through the properties of exponentials and check that they hold for complex exponentials.

1.6.1 $\mathrm{e}^{i\theta}$ behaves like a true exponential

P1. e^{it} differentiates as expected:
$$\frac{d\mathrm{e}^{it}}{dt} = i\mathrm{e}^{it}.$$

Proof. This follows directly from the definition:

$$\frac{d\mathrm{e}^{it}}{dt} = \frac{d}{dt}(\cos(t) + i\sin(t)) = -\sin(t) + i\cos(t) = i(\cos(t) + i\sin(t)) = i\mathrm{e}^{it}.$$

P2. $\mathrm{e}^{i \cdot 0} = 1$.

Proof. $\mathrm{e}^{i \cdot 0} = \cos(0) + i\sin(0) = 1$.

P3. The usual rules of exponents hold:

$$\mathrm{e}^{ia}\mathrm{e}^{ib} = \mathrm{e}^{i(a+b)}.$$

Proof. This relies on the cosine and sine addition formulas.

$$
\begin{aligned}
e^{ia} \cdot e^{ib} &= (\cos(a) + i\sin(a)) \cdot (\cos(b) + i\sin(b)) \\
&= \cos(a)\cos(b) - \sin(a)\sin(b) + i\,(\cos(a)\sin(b) + \sin(a)\cos(b)) \\
&= \cos(a+b) + i\sin(a+b) = e^{i(a+b)}.
\end{aligned}
$$

P4. The definition of $e^{i\theta}$ is consistent with the power series for e^x.

Proof. To see this we have to recall the power series for e^x, $\cos(x)$ and $\sin(x)$. They are

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots
\end{aligned}
$$

Now we can write the power series for $e^{i\theta}$ and then split it into the power series for sine and cosine:

$$
\begin{aligned}
e^{i\theta} &= \sum_0^\infty \frac{(i\theta)^n}{n!} \\
&= \sum_0^\infty (-1)^k \frac{\theta^{2k}}{(2k)!} + i \sum_0^\infty (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \\
&= \cos(\theta) + i\sin(\theta).
\end{aligned}
$$

So the Euler formula definition is consistent with the usual power series for e^x.

Properties **P1-P4** should convince you that $e^{i\theta}$ behaves like an exponential.

1.6.2   Complex exponentials and polar form

Now let's turn to the relation between polar coordinates and complex exponentials.

Suppose $z = x + iy$ has polar coordinates r and θ. That is, we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Thus, we get the important relationship

$$
z = x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}.
$$

This is so important you shouldn't proceed without understanding. We also record it without the intermediate equation.

$$
z = x + iy = re^{i\theta}. \tag{2}
$$

Because r and θ are the polar coordinates of (x, y) we call $z = re^{i\theta}$ the polar form of z.

Let's now verify that magnitude, argument, conjugate, multiplication and division are all easy to compute from the polar form of z.

Magnitude. $|e^{i\theta}| = 1$.

Proof.

$$|e^{i\theta}| = |\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

In words, this says that $e^{i\theta}$ is always on the unit circle – this is useful to remember!

Likewise, if $z = re^{i\theta}$ then $|z| = r$. You can calculate this, but it should be clear from the definitions: $|z|$ is the distance from z to the origin, which is exactly the same definition as for r.

Argument. If $z = re^{i\theta}$ then $\arg(z) = \theta$.

Proof. This is again the definition: the argument is the polar angle θ.

Conjugate. $\overline{(re^{i\theta})} = re^{-i\theta}$.

Proof.

$$\overline{(re^{i\theta})} = \overline{r(\cos(\theta) + i\sin(\theta))} = r(\cos(\theta) - i\sin(\theta)) = r(\cos(-\theta) + i\sin(-\theta)) = re^{-i\theta}.$$

In words: complex conjugation changes the sign of the argument.

**Multiplication.** If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

This is what mathematicians call trivial to see, just write the multiplication down. In words, the formula says the for $z_1 z_2$ the magnitudes multiply and the arguments add.

Division. Again it's trivial that

$$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

Example 1.6. (Multiplication by $2i$) Here's a simple but important example. By looking at the graph we see that the number $2i$ has magnitude 2 and argument $\pi/2$. So in polar coordinates it equals $2e^{i\pi/2}$. This means that multiplication by $2i$ multiplies lengths by 2 and adds $\pi/2$ to arguments, i.e. rotates by $90°$. The effect is shown in the figures below
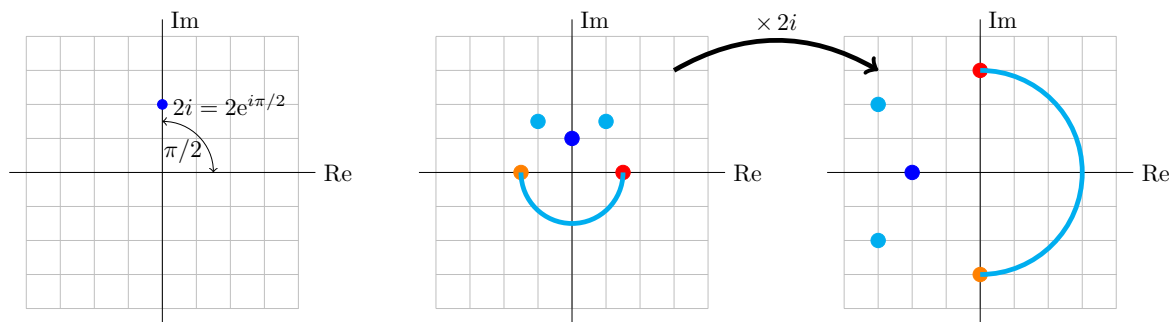
$|2i| = 2, \quad \arg(2i) = \pi/2$ Multiplication by $2i$ rotates by $\pi/2$ and scales by 2

Example 1.7. (Raising to a power) Let's compute $(1+i)^6$ and $\left(\frac{1+i\sqrt{3}}{2}\right)^3$

Solution: $1+i$ has magnitude $= \sqrt{2}$ and arg $= \pi/4$, so $1+i = \sqrt{2}e^{i\pi/4}$. Raising to a power is now easy:

$$(1+i)^6 = \left(\sqrt{2}e^{i\pi/4}\right)^6 = 8e^{6i\pi/4} = 8e^{3i\pi/2} = -8i.$$

Similarly, $\dfrac{1 + i\sqrt{3}}{2} = e^{i\pi/3}$, so $\left(\dfrac{1 + i\sqrt{3}}{2}\right)^3 = (1 \cdot e^{i\pi/3})^3 = e^{i\pi} = -1$

1.6.3   Complexification or complex replacement

In the next example we will illustrate the technique of complexification or complex replacement. This can be used to simplify a trigonometric integral. It will come in handy when we need to compute certain integrals.

Example 1.8. Use complex replacement to compute

$$I = \int e^x \cos(2x)\, dx.$$

Solution: We have Euler's formula

$$e^{2ix} = \cos(2x) + i\sin(2x),$$

so $\cos(2x) = \operatorname{Re}(e^{2ix})$. The complex replacement trick is to replace $\cos(2x)$ by e^{2ix}. We get (justification below)

$$I_c = \int e^x \cos 2x + i e^x \sin 2x\, dx, \qquad I = \operatorname{Re}(I_c).$$

Computing I_c is straightforward:

$$I_c = \int e^x e^{i2x}\, dx = \int e^{x(1+2i)}\, dx = \frac{e^{x(1+2i)}}{1 + 2i}.$$

Here we will do the computation first in rectangular coordinates. In applications, for example throughout 18.03, polar form is often preferred because it is easier and gives the answer in a more useable form.

$$
\begin{aligned}
I_c &= \frac{e^{x(1+2i)}}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} \\
&= \frac{e^x(\cos(2x) + i\,\sin(2x))(1 - 2i)}{5} \\
&= \frac{1}{5} e^x(\cos(2x) + 2\sin(2x) + i(-2\cos(2x) + \sin(2x)))
\end{aligned}
$$

So,

$$I = \operatorname{Re}(I_c) = \frac{1}{5} e^x(\cos(2x) + 2\sin(2x)).$$

Justification of complex replacement. The trick comes by cleverly adding a new integral to I as follows. Let $J = \int e^x \sin(2x)\, dx$. Then we let

$$I_c = I + iJ = \int e^x(\cos(2x) + i\sin(2x))\, dx = \int e^x e^{2ix}\, dx.$$

Clearly, by construction, $\operatorname{Re}(I_c) = I$ as claimed above.

Alternative using polar coordinates to simplify the expression for I_c:
In polar form, we have $1 + 2i = re^{i\phi}$, where $r = \sqrt{5}$ and $\phi = \arg(1 + 2i) = \tan^{-1}(2)$ in the first quadrant. Then:

$$I_c = \frac{e^{x(1+2i)}}{\sqrt{5}e^{i\phi}} = \frac{e^x}{\sqrt{5}}e^{i(2x-\phi)} = \frac{e^x}{\sqrt{5}}(\cos(2x - \phi) + i\,\sin(2x - \phi)).$$

Thus,

$$I = \text{Re}(I_c) = \frac{e^x}{\sqrt{5}}\cos(2x - \phi).$$

1.6.4 Nth roots

We are going to need to be able to find the nth roots of complex numbers, i.e., solve equations of the form

$$z^N = c,$$

where c is a given complex number. This can be done most conveniently by expressing c and z in polar form, $c = Re^{i\phi}$ and $z = re^{i\theta}$. Then, upon substituting, we have to solve

$$r^N e^{iN\theta} = Re^{i\phi}$$

For the complex numbers on the left and right to be equal, their magnitudes must be the same and their arguments can only differ by an integer multiple of 2π. This gives

$$r = R^{1/N} \qquad N\theta = \phi + 2\pi n, \text{ where } n = 0, \pm 1, \pm 2, \ldots$$

Solving for θ, we have

$$\theta = \frac{\phi}{N} + \frac{2\pi n}{N}.$$

Example 1.9. Find all 5 fifth roots of 2.

Solution: For $c = 2$, we have $R = 2$ and $\phi = 0$, so the fifth roots of 2 are

$$z_n = 2^{1/5}e^{2n\pi i/5}, \text{ where } n = 0, \pm 1, \pm 2, \ldots$$

Looking at the right hand side we see that for $n = 5$ we have $2^{1/5}e^{2\pi i}$ which is exactly the same as the root when $n = 0$, i.e. $2^{1/5}e^{0i}$. Likewise $n = 6$ gives exactly the same root as $n = 1$, and so on. This means, we have 5 different roots corresponding to $n = 0, 1, 2, 3, 4$.

$$z_n = 2^{1/5},\ 2^{1/5}e^{2\pi i/5},\ 2^{1/5}e^{4\pi i/5},\ 2^{1/5}e^{6\pi i/5},\ 2^{1/5}e^{8\pi i/5}$$

Similarly we can say that in general $c = Re^{i\phi}$ has N distinct Nth roots:

$$z_n = r^{1/N}e^{i\phi/N + i\,2\pi(n/N)} \text{ for } n = 0, 1, 2, \ldots N - 1.$$

Example 1.10. Find the 4 fourth roots of 1.

Solution: We need to solve $z^4 = 1$, so $\phi = 0$. So the 4 distinct fourth roots are in polar form

$$z_n = 1,\ e^{i\pi/2},\ e^{i\pi},\ e^{i3\pi/2}$$

and in Cartesian representation

$$z_n = 1, \; i, \; -1, \; -i.$$

Example 1.11. Find the 3 cube roots of -1.

Solution: $z^2 = -1 = e^{i\pi + i\,2\pi n}$. So, $z_n = e^{i\,\pi/3 + i\,2\pi(n/3)}$ and the 3 cube roots are $e^{i\pi/3}$, $e^{i\pi}$, $e^{i5\pi/3}$. Since $\pi/3$ radians is $60°$ we can simpify:

$$e^{i\pi/3} = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + i\,\frac{\sqrt{3}}{2} \quad \Rightarrow \quad z_n = -1, \; \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Example 1.12. Find the 5 fifth roots of $1 + i$.   *Solution:* $z^5 = 1 + i = \sqrt{2}e^{i(\pi/4 + 2n\pi)}$, for $n = 0, 1, 2, \ldots$. So, the 5 fifth roots are

$$2^{1/10}e^{i\pi/20}, \; 2^{1/10}e^{i9\pi/20}, \; 2^{1/10}e^{i17\pi/20}, \; 2^{1/10}e^{i25\pi/20}, \; 2^{1/10}e^{i33\pi/20}.$$

Using a calculator we could write these numerically as $a + bi$, but there is no easy simplification.

Example 1.13. We should check that our technique works as expected for a simple problem. Find the 2 square roots of 4.

Solution: $z^2 = 4e^{i\,2\pi n}$. So, $z_n = 2e^{i\,\pi n}$, with $n = 0, 1$. So the two roots are $2e^0 = 2$ and $2e^{i\pi} = -2$ as expected!

1.6.5   The geometry of Nth roots

Looking at the examples above we see that roots are always spaced evenly around a circle centered at the origin. For example, the fifth roots of $1 + i$ are spaced at increments of $2\pi/5$ radians around the circle of radius $2^{1/5}$.

Note also that the roots of real numbers always come in conjugate pairs.

Cube roots of -1   Fifth roots of $1 + i$
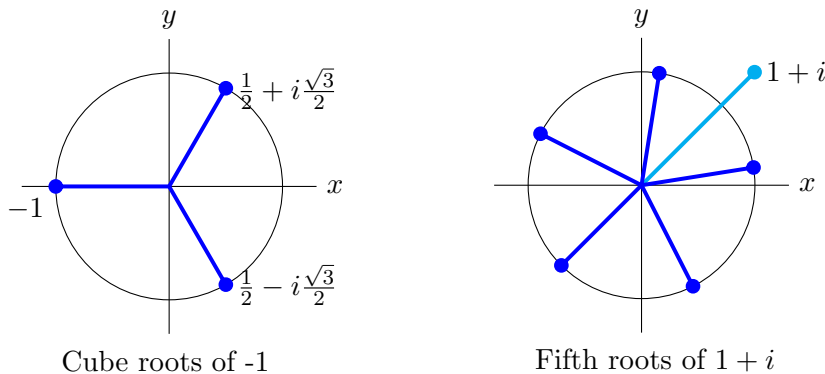
1.7   Inverse Euler formula

Euler's formula gives a complex exponential in terms of sines and cosines. We can turn this around to get the inverse Euler formulas.

Euler's formula says:

$$e^{it} = \cos(t) + i\sin(t) \quad \text{and} \quad e^{-it} = \cos(t) - i\sin(t).$$

By adding and subtracting we get:

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \quad \text{and} \quad \sin(t) = \frac{e^{it} - e^{-it}}{2i}.$$

Please take note of these formulas we will use them frequently!

1.8   de Moivre's formula

For positive integers n we have de Moivre's formula:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

Proof. This is a simple consequence of Euler's formula:

$$(\cos(\theta) + i\sin(\theta))^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

The reason this simple fact has a name is that historically de Moivre stated it before Euler's formula was known. Without Euler's formula there is not such a simple proof.

1.9   Representing complex multiplication as matrix multiplication

Consider two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ and their product

$$z_1 z_2 = (a + bi)(c + id) = (ac - bd) + i(bc + ad) =: w \tag{3}$$

Now let's define two matrices

$$Z_1 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad Z_2 = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

Note that these matrices store the same information as z_1 and z_2, respectively. Let's compute their matrix product

$$Z_1 Z_2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{bmatrix} := W.$$

Comparing W just above with w in Equation 3, we see that W is indeed the matrix corresponding to the complex number $w = z_1 z_2$. Thus, we can represent any complex number z equivalently by the matrix

$$Z = \begin{bmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Re} z \end{bmatrix}$$

and complex multiplication then simply becomes matrix multiplication. Further note that we can write

$$Z = \operatorname{Re} z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \operatorname{Im} z \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

i.e., the imaginary unit i corresponds to the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $i^2 = -1$ becomes

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Polar form (decomposition). Writing $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, we find

$$Z = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

corresponding to a stretch factor r multiplied by a 2D rotation matrix. In particular, multiplication by i corresponds to the rotation with angle $\theta = \pi/2$ and $r = 1$.

We will not make a lot of use of the matrix representation of complex numbers, but later it will help us remember certain formulas and facts.

1.10   The exponential function

We have Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. We can extend this to the complex exponential function e^z.

Definition. For $z = x + iy$ the complex exponential function is defined as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

In this definition e^x is the usual exponential function for a real variable x.

It is easy to see that all the usual rules of exponents hold:

1. $e^0 = 1$

2. $e^{z_1+z_2} = e^{z_1}e^{z_2}$

3. $(e^z)^n = e^{nz}$ for positive integers n.

4. $(e^z)^{-1} = e^{-z}$

5. $e^z \neq 0$

   It will turn out that the property $\dfrac{de^z}{dz} = e^z$ also holds, but we can't prove this yet because we haven't defined what we mean by the complex derivative $\dfrac{d}{dz}$.

   Here are some more simple, but extremely important properties of e^z. You should become fluent in their use and know how to prove them.
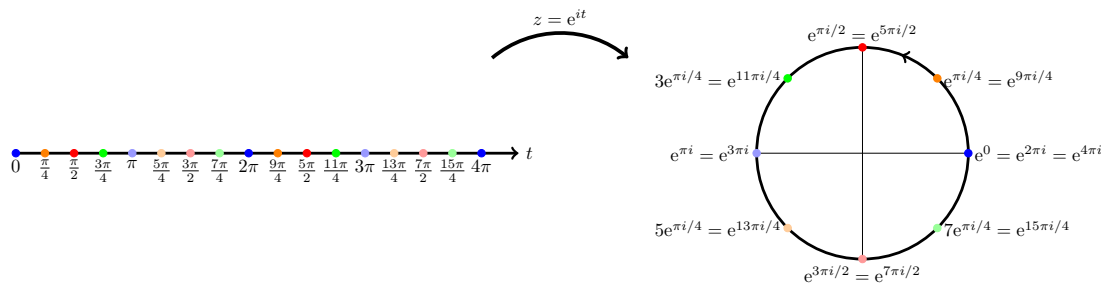
6. $|e^{i\theta}| = 1$

   Proof.
   $$|e^{i\theta}| = |\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

7. $|e^{x+iy}| = e^x$ (as usual $z = x + iy$ and x, y are real).

   Proof. You should be able to supply this. If not: ask a teacher or TA.

8. The path e^{it} for $0 < t < \infty$ wraps counterclockwise around the unit circle. It does so infinitely many times. This is illustrated in the following picture.

The map $t \to e^{it}$ wraps the real axis around the unit circle.

1.11   Complex functions as mappings

A complex function $w = f(z)$ is hard to graph because it takes 4 dimensions: 2 for z and 2 for w. So, to visualize them we will think of complex functions as mappings. That is we will think of $w = f(z)$ as taking a point in the complex z-plane and mapping (sending) it to a point in the complex w-plane.

We will use the following terms and symbols to discuss mappings.

- A function $w = f(z)$ will also be called a mapping of z to w.

- Alternatively we will write $z \mapsto w$ or $z \mapsto f(z)$. This is read as "z maps to w".

- We will say that "w is the image of z under the mapping" or more simply "w is the image of z".

- If we have a set of points in the z-plane we will talk of the image of that set under the mapping. For example, under the mapping $z \mapsto iz$ the image of the imaginary z-axis is the real w-axis.
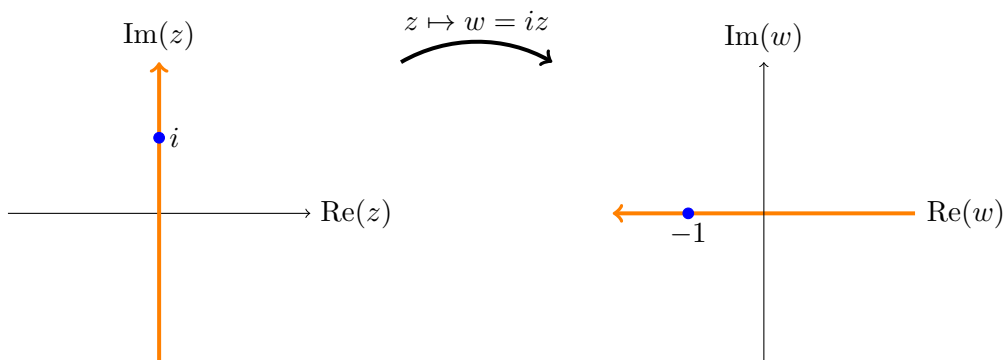
The image of the imaginary axis under $z \mapsto iz$.

Next, we'll illustrate visualizing mappings with some examples:

Example 1.14. The mapping $w = z^2$. We visualize this by putting the z-plane on the left and the w-plane on the right. We then draw various curves and regions in the z-plane and the corresponding image under z^2 in the w-plane.

In the first figure we show that rays from the origin are mapped by z^2 to rays from the origin. We see that

1. The ray L_2 at $\pi/4$ radians is mapped to the ray $f(L_2)$ at $\pi/2$ radians.

2. The rays L_2 and L_6 are both mapped to the same ray. This is true for each pair of diametrically opposed rays.

3. A ray at angle θ is mapped to the ray at angle 2θ.
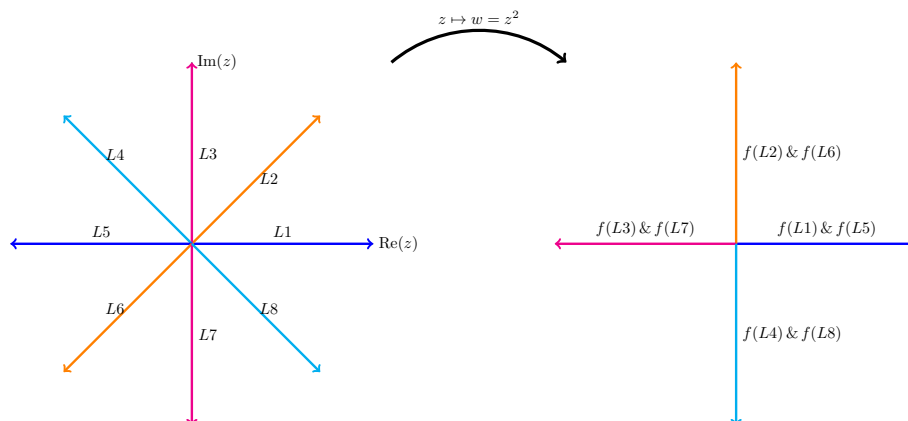

$f(z) = z^2$ maps rays from the origin to rays from the origin.

The next figure gives another view of the mapping. Here we see vertical stripes in the first quadrant are mapped to parabolic stripes that live in the first and second quadrants.
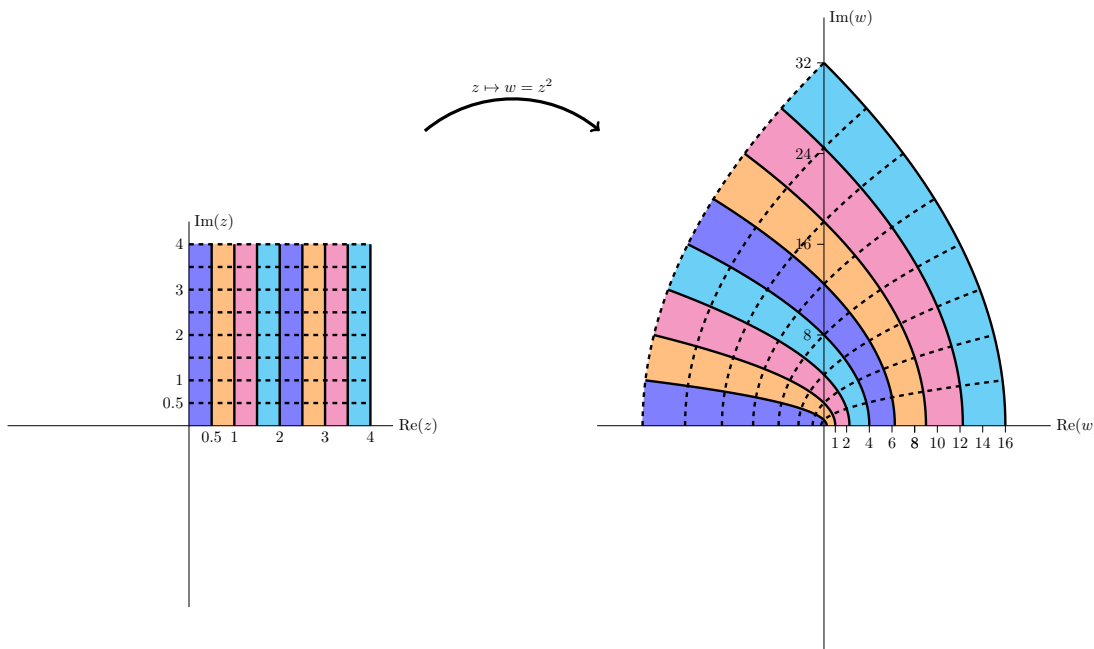
$z^2 = (x^2 - y^2) + i2xy$ maps vertical lines to left facing parabolas.

The next figure is similar to the previous one, except in this figure we look at vertical stripes in both the first and second quadrants. We see that they map to parabolic stripes that live in all four quadrants.
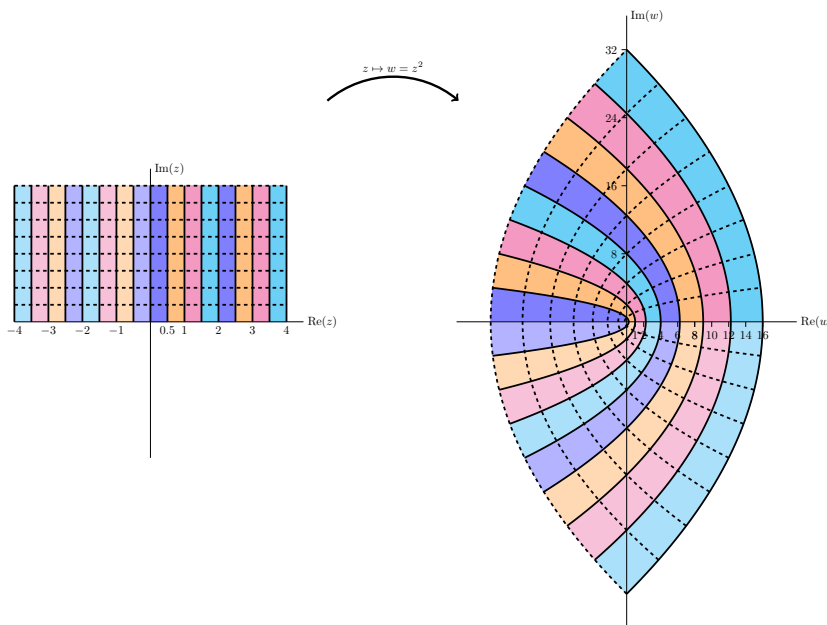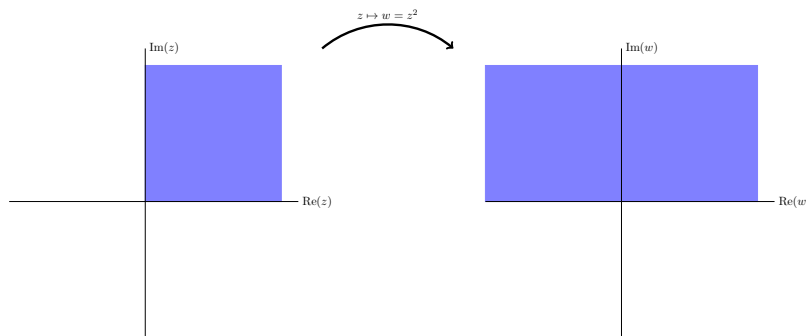
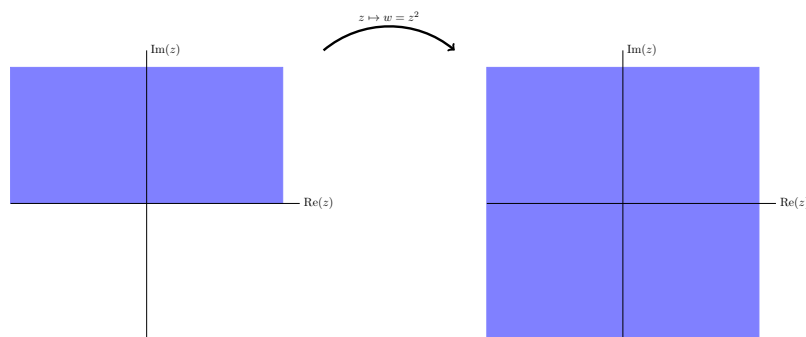$f(z) = z^2$ maps the first two quadrants to the entire plane.

The next figure shows the mapping of stripes in the first and fourth quadrants. The image map is identical to the previous figure. This is because the fourth quadrant is minus the second quadrant, but $z^2 = (-z)^2$.
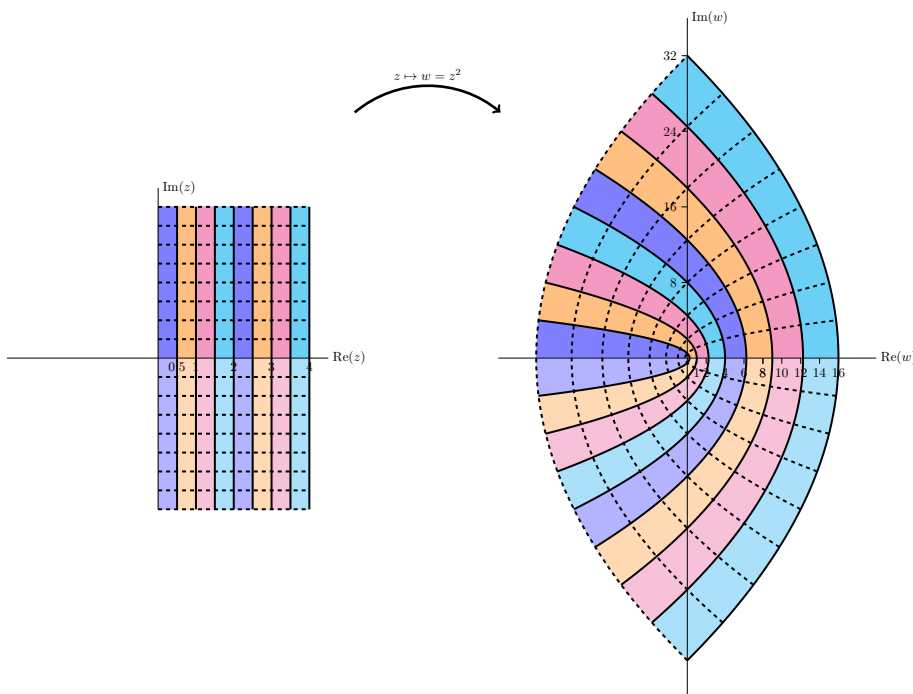
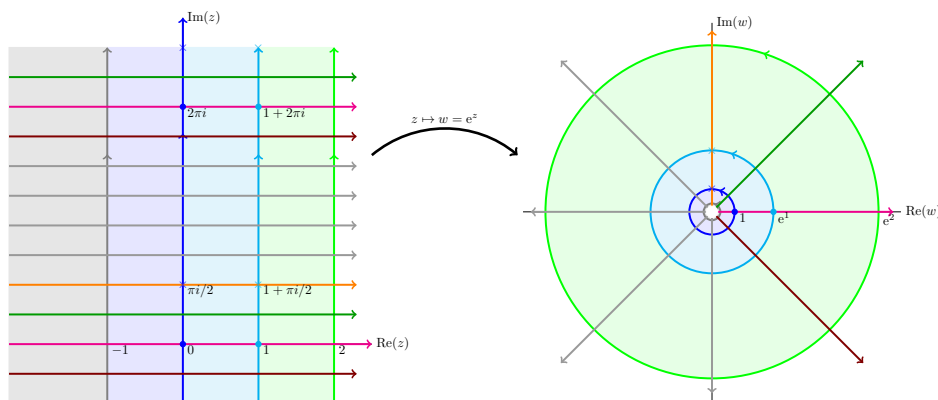Vertical stripes in quadrant 4 are mapped identically to vertical stripes in quadrant 2.

Simplified view of the first quadrant being mapped to the first two quadrants.

Simplified view of the first two quadrants being mapped to the entire plane.

Example 1.15. The mapping $w = e^z$. Here we present a series of plots showing how the exponential function maps z to w.


Notice that vertical lines are mapped to circles and horizontal lines to rays from the origin.

The next four figures all show essentially the same thing: the exponential function maps horizontal stripes to circular sectors. Any horizontal stripe of width 2π gets mapped to the entire plane minus the origin,

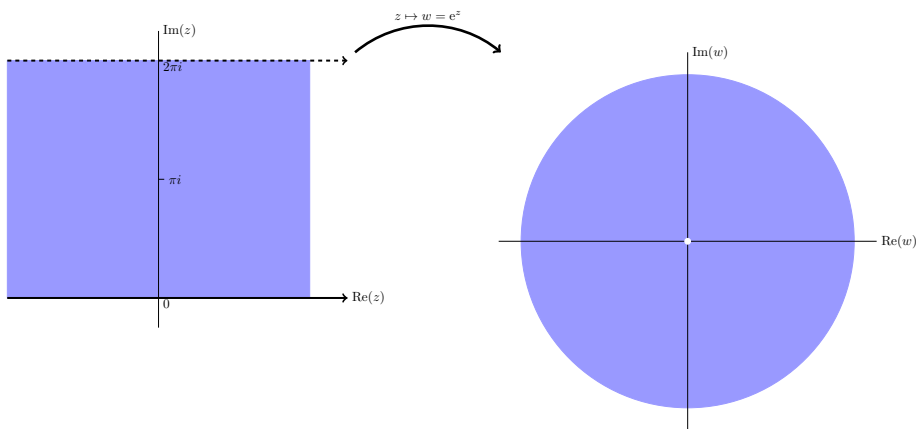Because the plane minus the origin comes up frequently we give it a name:

Definition. The punctured plane is the complex plane minus the origin. In symbols we can write it as $\mathbf{C} - \{0\}$ or $\mathbf{C}/\{0\}$.
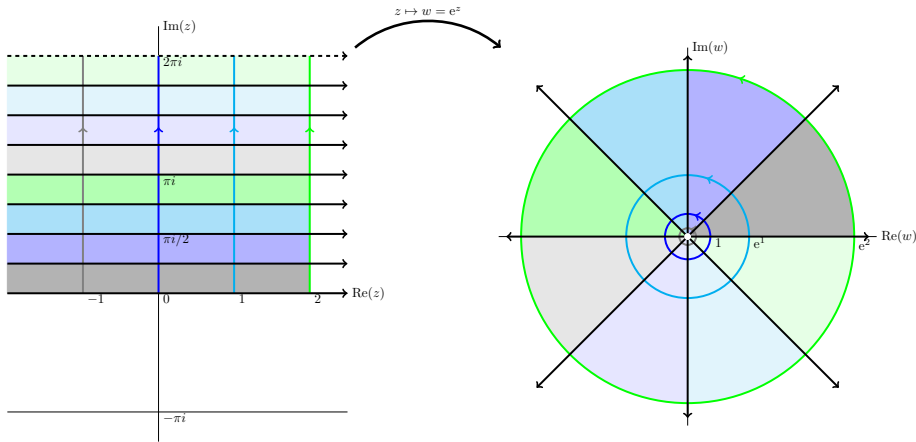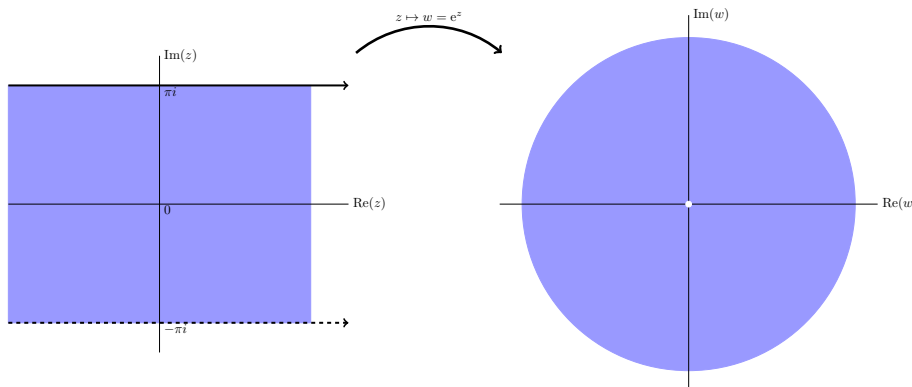
The horizontal strip $0 \leq y < 2\pi$ is mapped to the punctured plane
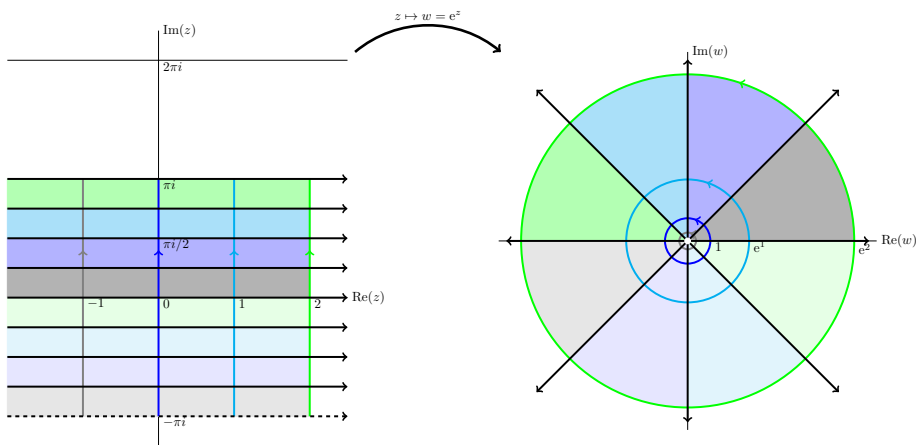
The horizontal strip $-\pi < y \leq \pi$ is mapped to the punctured plane

Simplified view showing e^z maps the horizontal stripe $0 \leq y < 2\pi$ to the punctured plane.

Simplified view showing e^z maps the horizontal stripe $-\pi < y \leq \pi$ to the punctured plane.

1.12   The function $\arg(z)$

1.12.1   Many-to-one functions

The function $f(z) = z^2$ maps $\pm z$ to the same value, e.g. $f(2) = f(-2) = 4$. We say that $f(z)$ is a 2-to-1 function. That is, it maps 2 different points to each value. (Technically, it only maps one point to 0, but we will gloss over that for now.) Here are some other examples of many-to-one functions.

Example 1.16. $w = z^3$ is a 3-to-1 function. For example, 3 different z values get mapped to $w = 1$:

$$1^3 = \left(\frac{-1 + \sqrt{3}\,i}{2}\right)^3 = \left(\frac{-1 - \sqrt{3}\,i}{2}\right)^3 = 1$$

Example 1.17. The function $w = e^z$ maps infinitely many points to each value. For example

$$e^0 = e^{2\pi i} = e^{4\pi i} = \ldots = e^{2n\pi i} = \ldots = 1$$
$$e^{i\pi/2} = e^{i\pi/2 + 2\pi i} = e^{i\pi/2 + 4\pi i} = \ldots = e^{i\pi/2 + 2n\pi i} = \ldots = i$$

In general, $e^{z + 2n\pi i}$ has the same value for every integer n.

1.12.2   Branches of $\arg(z)$

Important note. You should master this section. Branches of $\arg(z)$ are the key that really underlies all our other examples. Fortunately it is reasonably straightforward.

The key point is that the argument is only defined up to multiples of $2\pi i$ so every z produces infinitely many values for $\arg(z)$. Because of this we will say that $\arg(z)$ is a multiple-valued function.

Note. In general a function should take just one value. What that means in practice is that whenever we use such a function will have to be careful to specify which of the possible values we mean. This is known as specifying a branch of the function.

Definition. By a branch of the argument function we mean a choice of range so that it becomes single-valued. By specifying a branch we are saying that we will take the single value of $\arg(z)$ that lies in the branch.

Let's look at several different branches to understand how they work:

(i) If we specify the branch as $0 \le \arg(z) < 2\pi$ then we have the following arguments.

$$\arg(1) = 0; \quad \arg(i) = \pi/2; \quad \arg(-1) = \pi; \quad \arg(-i) = 3\pi/2$$

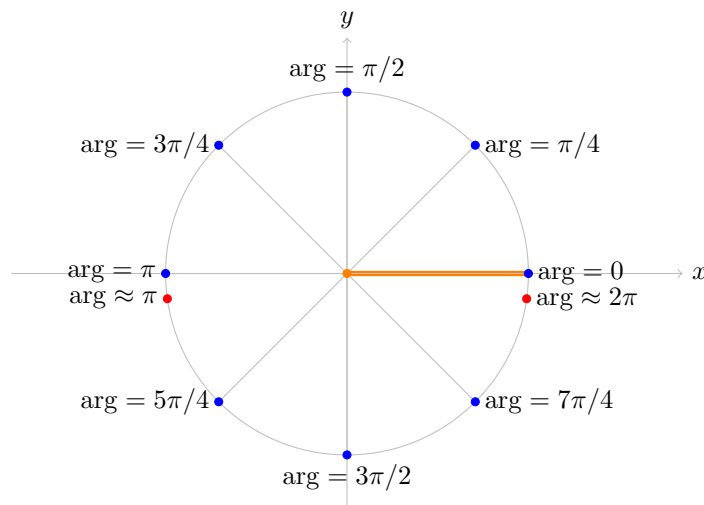This branch and these points are shown graphically in Figure (i) below.

Figure (i): The branch $0 \le \arg(z) < 2\pi$ of $\arg(z)$.

Notice that if we start at $z = 1$ on the positive real axis we have $\arg(z) = 0$. Then $\arg(z)$ increases as we move counterclockwise around the circle. The argument is continuous until we get back to the positive real axis. There it jumps from almost 2π back to 0.

There is no getting around (no pun intended) this discontinuity. If we need $\arg(z)$ to be continuous we will need to remove (cut) the points of discontinuity out of the domain. The branch cut for this branch of $\arg(z)$ is shown as a thick orange line in the figure. If we make the branch cut then the domain for $\arg(z)$ is the plane minus the cut, i.e. we will only consider $\arg(z)$ for z not on the cut.

For future reference you should note that, on this branch, $\arg(z)$ is continuous near the negative real axis, i.e. the arguments of nearby points are close to each other.

(ii) If we specify the branch as $-\pi < \arg(z) \le \pi$ then we have the following arguments:

$$\arg(1) = 0; \quad \arg(i) = \pi/2; \quad \arg(-1) = \pi; \quad \arg(-i) = -\pi/2$$

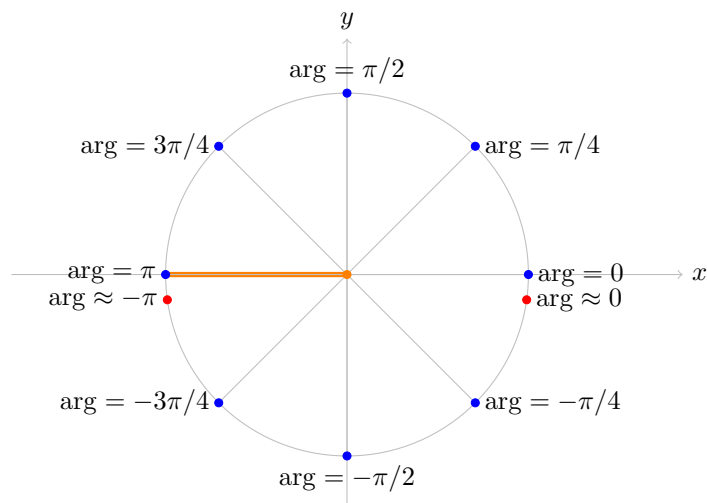This branch and these points are shown graphically in Figure (ii) below.

Figure (ii): The branch $-\pi < \arg(z) \leq \pi$ of $\arg(z)$.

Compare Figure (ii) with Figure (i). The values of $\arg(z)$ are the same in the upper half plane, but in the lower half plane they differ by 2π.

For this branch the branch cut is along the negative real axis. As we cross the branch cut the value of $\arg(z)$ jumps from π to something close to $-\pi$.

(iii) Figure (iii) shows the branch of $\arg(z)$ with $\pi/4 \leq \arg(z) < 9\pi/4$.
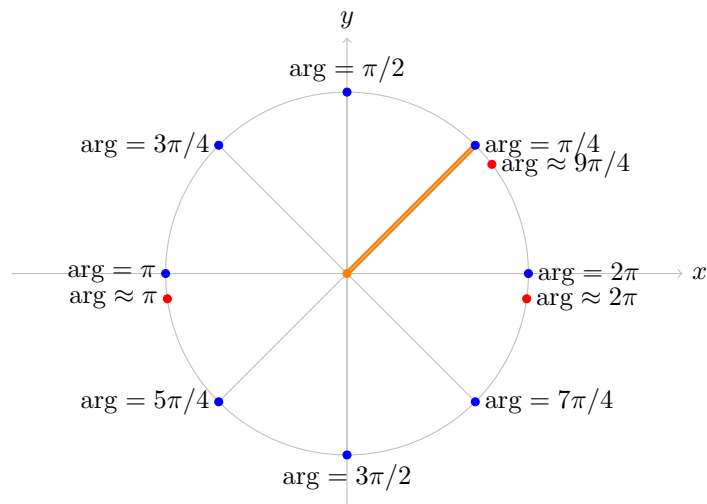

Figure (iii): The branch $\pi/4 \leq \arg(z) < 9\pi/4$ of $\arg(z)$.

Notice that on this branch $\arg(z)$ is continuous at both the positive and negative real axes. The jump of 2π occurs along the ray at angle $\pi/4$.

(iv) Obviously, there are many many possible branches. For example,

$$42 < \arg(z) \leq 42 + 2\pi.$$

(v) We won't make use of this in 18.04, but, in fact, the branch cut doesn't have to be a straight line. Any curve that goes from the origin to infinity will do. The argument will be continuous except for a jump by 2π when z crosses the branch cut.

1.12.3   The principal branch of $\arg(z)$

Branch (ii) in the previous section is singled out and given a name:

Definition. The branch $-\pi < \arg(z) \le \pi$ is called the principal branch of $\arg(z)$. We will use the notation $\text{Arg}(z)$ (capital A) to indicate that we are using the principal branch. (Of course, in cases where we don't want there to be any doubt we will say explicitly that we are using the principal branch.)

1.12.4   Continuity of $\arg(z)$

The examples above show that there is no getting around the jump of 2π as we cross the branch cut. This means that when we need $\arg(z)$ to be continuous we will have to restrict its domain to the plane minus a branch cut.

1.13   Concise summary of branches and branch cuts

We discussed branches and branch cuts for $\arg(z)$. Before talking about $\log(z)$ and its branches and branch cuts we will give a short review of what these terms mean. You should probably scan this section now and then come back to it after reading about $\log(z)$.

Consider the function $w = f(z)$. Suppose that $z = x + iy$ and $w = u + iv$.

Domain. The domain of f is the set of z where we are allowed to compute $f(z)$.

Range. The range (image) of f is the set of all $f(z)$ for z in the domain, i.e. the set of all w reached by f.

Branch. For a multiple-valued function, a branch is a choice of range for the function. We choose the range to exclude all but one possible value for each element of the domain.

Branch cut. A branch cut removes (cuts) points out of the domain. This is done to remove points where the function is discontinuous.

1.14   The function $\log(z)$

Our goal in this section is to define the log function. We want $\log(z)$ to be the inverse of e^z. That is, we want $e^{\log(z)} = z$. We will see that $\log(z)$ is multiple-valued, so when we use it we will have to specify a branch.

We start by looking at the simplest example which illustrates that $\log(z)$ is multiple-valued.

Example 1.18. Find $\log(1)$.

Solution: We know that $e^0 = 1$, so $\log(1) = 0$ is one answer.

We also know that $e^{2\pi i} = 1$, so $\log(1) = 2\pi i$ is another possible answer. In fact, we can choose any multiple of $2\pi i$:

$$\log(1) = 2n\pi i, \quad \text{where } n \text{ is any integer}$$

This example leads us to consider the polar form for z as we try to define $\log(z)$. If $z = re^{i\theta}$ then one possible value for $\log(z)$ is

$$\log(z) = \log(re^{i\theta}) = \log(r) + i\theta,$$

here $\log(r)$ is the usual logarithm of a real positive number. For completeness we show explicitly that with this definition $e^{\log(z)} = z$:

$$e^{\log(z)} = e^{\log(r)+i\theta} = e^{\log(r)}e^{i\theta} = re^{i\theta} = z.$$

Since $r = |z|$ and $\theta = \arg(z)$ we have arrived at our definition.

Definition. The function $\log(z)$ is defined as

$$\log(z) = \log(|z|) + i\arg(z),$$

where $\log(|z|)$ is the usual natural logarithm of a positive real number.

Remarks.

1. Since $\arg(z)$ has infinitely many possible values, so does $\log(z)$.

2. $\log(0)$ is not defined. (Both because $\arg(0)$ is not defined and $\log(|0|)$ is not defined.)

3. Choosing a branch for $\arg(z)$ makes $\log(z)$ single valued. The usual terminology is to say we have chosen a branch of the log function.

4. The principal branch of log comes from the principal branch of arg. That is,

$$\log(z) = \log(|z|) + i\arg(z), \quad \text{where} \quad -\pi < \arg(z) \le \pi \quad \text{(principal branch)}.$$

Example 1.19. Compute all the values of $\log(i)$. Specify which one comes from the principal branch.

Solution: We have that $|i| = 1$ and $\arg(i) = \dfrac{\pi}{2} + 2n\pi$, so

$$\log(i) = \log(1) + i\frac{\pi}{2} + i2n\pi = i\frac{\pi}{2} + i2n\pi, \text{ where } n \text{ is any integer.}$$

The principal branch of $\arg(z)$ is between $-\pi$ and π, so $\text{Arg}(i) = \pi/2$. Therefore, the value of $\log(i)$ from the principal branch is $i\pi/2$.

Example 1.20. Compute all the values of $\log(-1 - \sqrt{3}\,i)$. Specify which one comes from the principal branch.

Solution: Let $z = -1 - \sqrt{3}\,i$. Then $|z| = 2$ and in the principal branch $\text{Arg}(z) = -2\pi/3$. So all the values of $\log(z)$ are

$$\log(z) = \log(2) - i\frac{2\pi}{3} + i2n\pi.$$

The value from the principal branch is $\log(z) = \log(2) - i2\pi/3$.

1.14.1 Figures showing $w = \log(z)$ as a mapping

The figures below show different aspects of the mapping given by $\log(z)$.

In the first figure we see that a point z is mapped to (infinitely) many values of w. In this case we show $\log(1)$ (blue dots), $\log(4)$ (red dots), $\log(i)$ (blue cross), and $\log(4i)$ (red

cross). The values in the principal branch are inside the shaded region in the w-plane. Note that the values of $\log(z)$ for a given z are placed at intervals of $2\pi i$ in the w-plane.
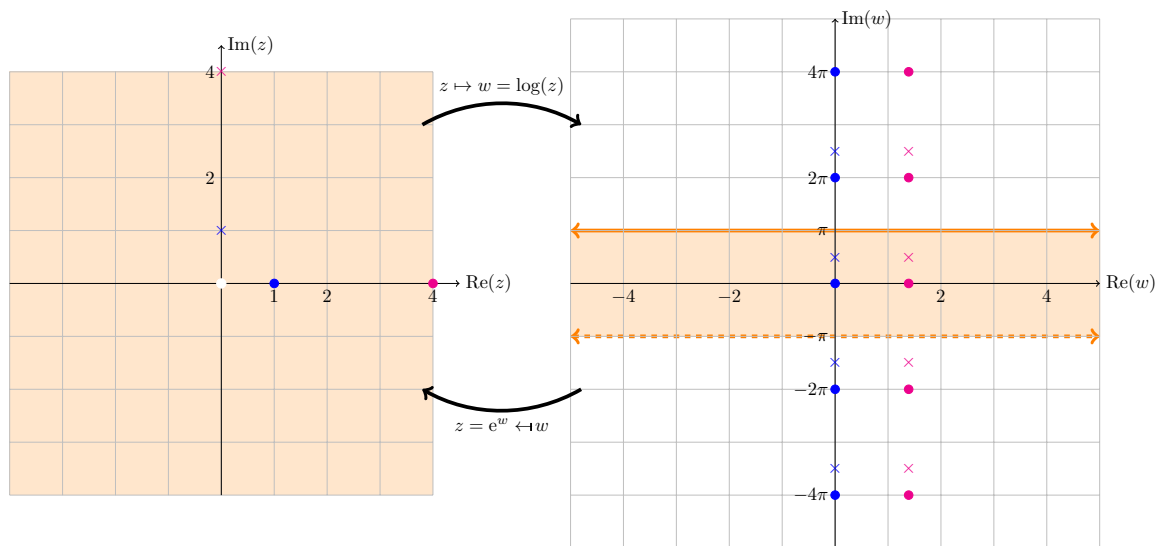

Mapping $\log(z)$: $\log(1)$, $\log(4)$, $\log(i)$, $\log(4i)$

The next figure illustrates that the principal branch of log maps the punctured plane to the horizontal strip $-\pi < \operatorname{Im}(w) \le \pi$. We again show the values of $\log(1)$, $\log(4)$, $\log(i)$ and $\log(4i)$. Since we've chosen a branch, there is only one value shown for each log.
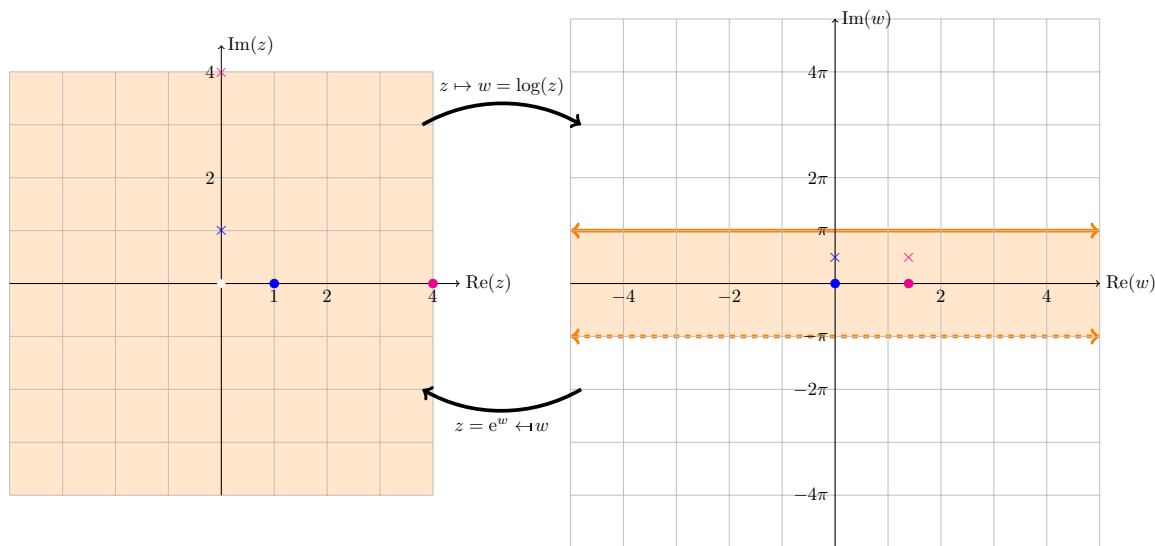
Mapping $\log(z)$: the principal branch and the punctured plane

The third figure shows how circles centered on 0 are mapped to vertical lines, and rays from the origin are mapped to horizontal lines. If we restrict ourselves to the principal branch the circles are mapped to vertical line segments and rays to a single horizontal line in the principal (shaded) region of the w-plane.
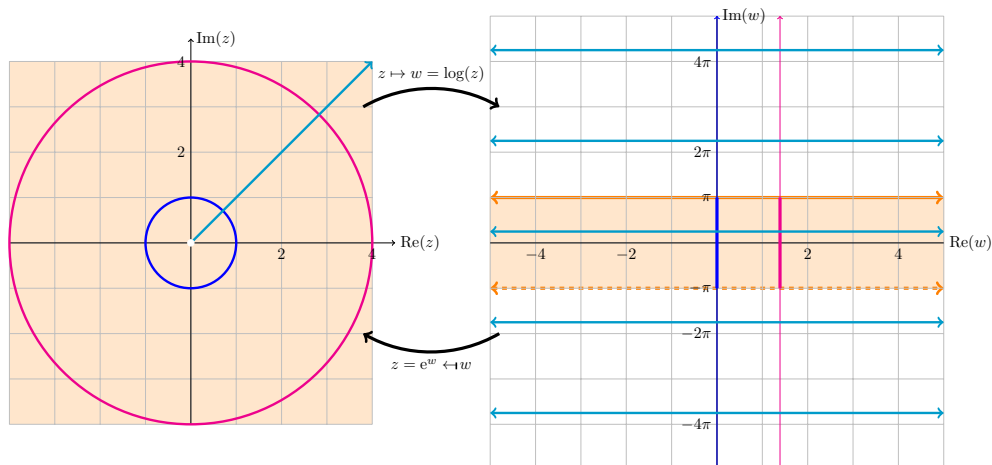
Mapping $\log(z)$: mapping circles and rays

1.14.2   Complex powers

We can use the log function to define complex powers.

Definition. Let z and a be complex numbers then the power z^a is defined as

$$z^a = e^{a \log(z)}.$$

This is generally multiple-valued, so to specify a single value requires choosing a branch of $\log(z)$.

Example 1.21. Compute all the values of $\sqrt{2i}$. Give the value associated to the principal branch of $\log(z)$.

Solution: We have

$$\log(2i) = \log(2e^{\frac{i\pi}{2}}) = \log(2) + i\frac{\pi}{2} + i2n\pi.$$

So,

$$\sqrt{2i} = (2i)^{1/2} = e^{\frac{\log(2i)}{2}} = e^{\frac{\log(2)}{2} + \frac{i\pi}{4} + in\pi} = \sqrt{2}e^{\frac{i\pi}{4} + in\pi}.$$

(As usual n is an integer.) As we saw earlier, this only gives two distinct values. The principal branch has $\text{Arg}(2i) = \pi/2$, so

$$\sqrt{2i} = \sqrt{2}e^{\left(\frac{i\pi}{4}\right)} = \sqrt{2}\frac{(1+i)}{\sqrt{2}} = 1 + i.$$

The other distinct value is when $n = 1$ and gives minus the value just above.

Example 1.22. Cube roots: Compute all the cube roots of i. Give the value which comes from the principal branch of $\log(z)$.

Solution: We have $\log(i) = i\frac{\pi}{2} + i2n\pi$, where n is any integer. So,

$$i^{1/3} = e^{\frac{\log(i)}{3}} = e^{i\frac{\pi}{6} + i\frac{2n\pi}{3}}$$

This gives only three distinct values

$$e^{i\pi/6}, \quad e^{i5\pi/6}, \quad e^{i9\pi/6}$$

On the principal branch $\log(i) = i\dfrac{\pi}{2}$, so the value of $i^{1/3}$ which comes from this is

$$\mathrm{e}^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{i}{2}.$$

Example 1.23. Compute all the values of 1^i. What is the value from the principal branch?

Solution: This is similar to the problems above. $\log(1) = 2n\pi i$, so

$$1^i = \mathrm{e}^{i\log(1)} = \mathrm{e}^{i2n\pi i} = \mathrm{e}^{-2n\pi}, \ \text{ where } n \text{ is an integer.}$$

The principal branch has $\log(1) = 0$ so $1^i = 1$.