1. First-order Ordinary Differential Equations

1.1 Basic concepts and ideas

- Equations

  $3y^2 + y - 4 = 0 \Rightarrow y = ?$

  where y is an unknown.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1+48}}{6} = 1,\ -\frac{4}{3}$$

- Functions

  $f(x) = 2x^3 + 4x$,

  where x is a variable.

$x = -2$, $f(x) = -24$
$x = -1$, $f(x) = -6$
$x = 0$, $f(x) = 0$
$x = 1$, $f(x) = 6$
: :

- Differential equations

  A differential equation is an equation contains one or several derivative of unknown functions (or dependent variables). For example,

$$\frac{d^2 y}{d x^2} + e^x \left[\frac{dy}{dx}\right]^2 + x^2 \frac{dy}{dx} = xy\,, \quad \Leftarrow \text{(ordinary differential equation)}$$

$$\frac{\partial z^2}{\partial x \partial y} + \frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} + y^3 z = xyz\,. \quad \Leftarrow \text{(partial differential equation)}$$

- There are several kinds of differential equations
- An ordinary differential equation (*ODE*) is an equation that contains one independent variable and one or several derivatives of an unknown function (or dependent variable), which we call $y(x)$ and we want to determine from the equation. For example,

  $$\frac{dy}{dx} = \cos x \quad (i.e., y' = \cos x)$$

  $$x^2 y''' y' + 2e^x y'' = (x^2 + 2) y^2$$

  where y is called dependent variable and
  x is called independent variable.
- If a differential equation contains one dependent variable and two or more independent variables, then the equation is a partial differential equation (*PDE*).
- If differential equations contain two or more dependent variable and one independent variable, then the set of equations is called a system of differential equations.

- Summary
  A differential equation contains
  (1) one dependent variable and one independent variable ⇨ an ordinary differential equation.
  (2) one dependent variable and two or more independent variable ⇨ a partial differential equation.
  (3) Two or more dependent variable and one independent variable ⇨ a system of differential equations.

  $$\begin{cases} y_1'(x) = 2\, y_1(x) - 4\, y_2(x) \\ y_2'(x) = y_1(x) - 3\, y_2(x) \end{cases} \quad \Rightarrow \quad \begin{aligned} y_1(x) &= c_1 4 e^x + c_2 e^{-2x} \\ y_2(x) &= c_1 e^x + c_2 e^{-2x} \end{aligned}$$

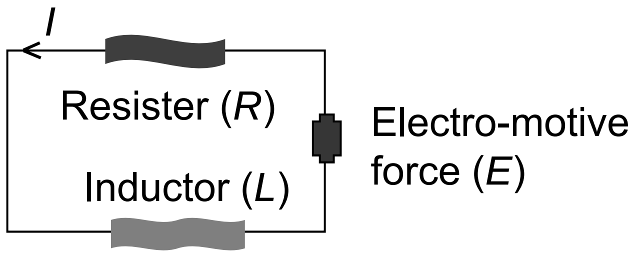
  (4) Two or more dependent variable and two or more independent variable ⇨ a system of partial differential equations. (rarely to see)

- What is the purpose of differential equations ?

Many physical laws and relations appear mathematically in the form of such equations. For example, electronic circuit, falling stone, vibration, *etc.*

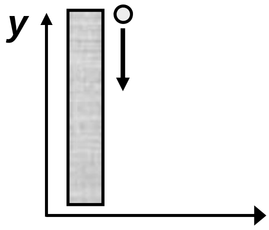
(1) Current I in an RL-circuit

$$LI' + RI = E.$$



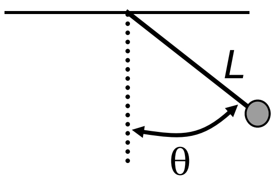
(2) Falling stone

$$y'' = g = \text{constant}.$$



(3) Pendulum

$$L\theta'' + g\sin\theta = 0.$$



- Any physical situation involved motion or measure rates of change can be described by a mathematical model, the model is just a differential equation.

The transition from the physical problem to a corresponding mathematical model is called modeling.

In this course, we shall pay our attention to solve differential equations and don't care of modeling.

That is, the purposes of this course are that given a differential equation

1. How do we know whether there is a solution ?
2. How many solutions might there be for a *DE*, and how are they related?
3. How do we find a solution ?
4. If we can't find a solution, can we approximate one numerically?

- A first-order *ODE* is an equation involving one dependent variable, one independent variable, and the first-order derivative. For example,

  $y' + xy^2 - 4x^3 = 0$

  $(y')^{3/2} + x^2 - \cos(xy') = 0.$

- A solution of a first-order *ODE* is a function which satisfies the equation. For example,

  $y(x) = e^{2x}$ is a solution of $y' - 2y = 0$.

  $y(x) = x^2$ is a solution of $xy' = 2y$.

- A solution which appears as an implicit function, given in the form $H(x, y) = 0$, is called an implicit solution;
  for example $x^2 + y^2 - 1 = 0$ is an implicit solution of *DE* $yy' = -x$.

  In contrast to an explicit solution with the form of $y = f(x)$;
  for example, $y = x^2$ is an explicit solution of $xy' = 2y$.

- A general solution is a solution containing one arbitrary constant;
  for example, $y = \sin x + c$ is a general solution of $y' = \cos x$.

  A particular solution is a solution making a specific choice of constant on the general solution. Usually, the choice is made by some additional constraints.
  For example, $y = \sin x - 2$ is a particular solution of $y' = \cos x$ with the condition $y(0) = -2$.

- A differential equation together with an initial condition is called an initial value problem. For example,

  $y' = f(x, y), \quad y(x_0) = y_0,$

  where x_0 and y_0 are given values.

- For example, $xy' = 3y$, $y(-4) = 16 \Rightarrow y = cx^3 \Rightarrow c = \dfrac{-1}{4} \Rightarrow y = \dfrac{-x^3}{4}.$

- Problem of Section 1.1.

1.2 Geometrical meaning of $y' = f(x, y)$; Direction fields

- Purpose

  To sketch many solution curves of a given *DE* without actually solving the differential equation.

- Method of direction fields

  The method applies to any differential equation $y' = f(x, y)$.

  Assume $y(x)$ is a solution of a given *DE*.

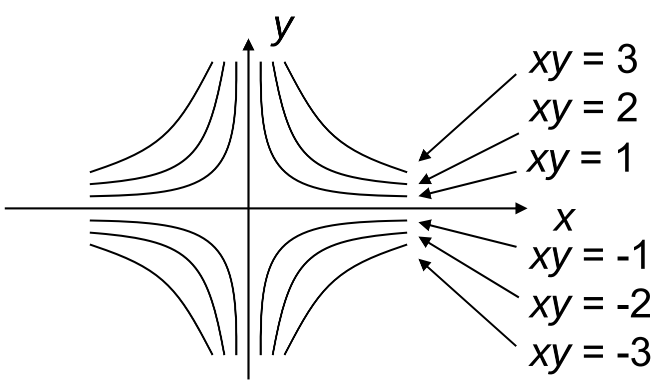
  $y(x)$ has slope $y'(x_0) = f(x_0, y_0)$ at (x_0, y_0).

  (*i*) draw the curves $f(x, y) = k$, k is a real constant. These curves are called isoclines of the original *DE*.

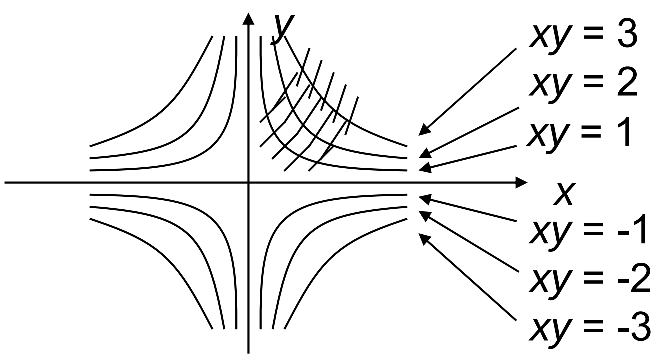
  (*ii*) along each isocline, draw a number of short line segments (called lineal element) of slope k to construct the direction field of the original *DE*. (That is, the direction field is just the set of all connected lineal elements.)

- Ex.1. Graph the direction field of the 1st-order *DE* $y' = xy$.

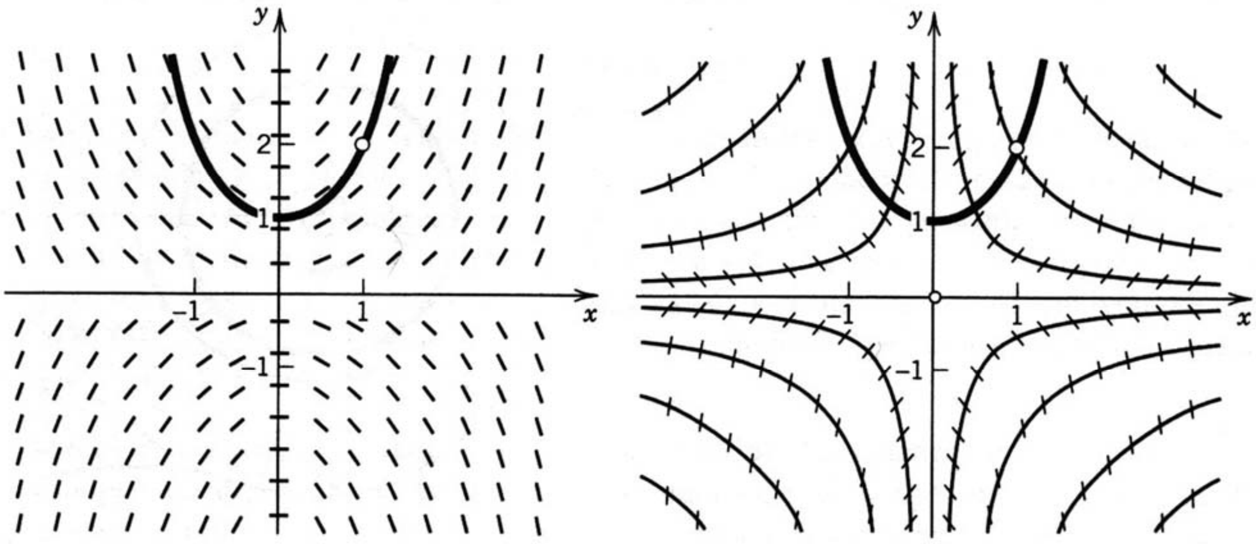
  (*i*) draw the curves (isoclines) $xy = \ldots -2, -1, 0, 1, 2, \ldots$



  (*ii*) draw lineal elements on each isocline,



(*iii*) connect the related lineal elements to form the direction field.



(a) By computer. (b) By hand.

Direction field of *y*’ = *xy*.

- Problems of Section 1.2.

1.3 Separable differential equations

- A *DE* is called separable if it can be written in the form of

  $g(y)\, y' = f(x)$ or $g(y)\, dy = f(x)\, dx$

  To solve the equation by integrate both sides with *x*,

$$\int g(y)y'dx = \int f(x)dx + c$$

$$\Rightarrow \int g(y)\frac{dy}{dx}dx = \int f(x)dx + c$$

$$\Rightarrow \int g(y)dy = \int f(x)dx + c\,.$$

- Ex.1. Solve $9yy' + 4x = 0$.

$$\Rightarrow\ 9ydy = -4xdx \ \Rightarrow\ \frac{9}{2}y^2 = -2x^2 + c'$$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = c \quad \text{with } c = c'/18\,.$$

❁ Ex.1. Solve $y' = 1 + y^2$

$$\Rightarrow \frac{dy}{1+y^2} = dx$$

$$\Rightarrow \tan^{-1}(y) = x + c$$

$$\Rightarrow y = \tan(x+c).$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

❁ Ex. Initial value problem

$y' + 5x^4y^2 = 0$ with initial condition $y(0)=1$.

$$\Rightarrow \frac{dy}{dx} = -5x^4y^2 \Rightarrow \frac{dy}{y^2} = -5x^4dx$$

$$\Rightarrow -\frac{1}{y} = -x^5 + c \Rightarrow y = \frac{1}{x^5 - c}.$$

Since $y(0)=1$ and $y(0) = \frac{1}{-c} \Rightarrow c = -1$

$$\Rightarrow y = \frac{1}{x^5+1}.$$

❁ Ex. Solve $y' = \frac{x}{y}$, $y(1) = 3$.

$$\Rightarrow ydy = xdx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + c \Rightarrow y = \pm\sqrt{x^2+2c}.$$

Since $y(1) = 3 = \sqrt{1+2c}$

$$\Rightarrow c = 4 \Rightarrow \text{ solution } y(x) = \sqrt{x^2+8}.$$

❁ Ex.3. Solve $y' = -2xy$, $y(0) = 0.8$.

$$\frac{dy}{y} = -2xdx \Rightarrow \ln y = -x^2 + c' \Rightarrow y = ce^{-x^2}.$$

Since $y(0) = ce^0 = c = 1.8 \Rightarrow$ solution $y(x) = 1.8e^{-x^2}$.

❁ Ex.4. Solve $y' = ky$, $y(0) = y_0$.

$$\frac{dy}{y} = kdt \Rightarrow \ln|y| = kt + c' \Rightarrow y = y_0e^{kt}.$$

❁ Example of no separable *DE* $(x-1)y' = 3x^2 + y$.

❁ Note: There is no nice test to determine easily whether or not a 1st-order equation is separable.

Reduction to separable forms

- Certain first-order differential equation are not separable but can be made separable by a simple change of variables (dependent variable)

The equation of the form $y' = g\left(\frac{y}{x}\right)$ can be made separable; and the form is called the *R*-1 formula.

step 1. Set $\frac{y}{x} = u$, then $y = ux$ (change of variables).

step 2. Differential $y' = u + xu'$ (product differentiation formula).

step 3. The original *DE* $y' = g\left(\frac{y}{x}\right) \Rightarrow u + xu' = g(u)$

$$\Rightarrow xu' = g(u) - u \Rightarrow \frac{du}{dx} = \frac{g(u)-u}{x} \Rightarrow \frac{du}{g(u)-u} = \frac{dx}{x}.$$

step 4. integrate both sides of the equation.

step 5. replace *u* by *y/x.*

- Ex.8. Solve $2xyy' = y^2 - x^2$.

Dividing by x^2, we have

$$2\frac{y}{x}y' - \left(\frac{y}{x}\right)^2 + 1 = 0 \Rightarrow 2u(u + u'x) - u^2 + 1 = 0 \quad \text{by setting } u = \frac{y}{x}$$

$$\Rightarrow 2u^2 + 2uxu' - u^2 + 1 = 0 \Rightarrow 2uxu' + u^2 + 1 = 0$$

$$\Rightarrow -2uxu' = u^2 + 1 \Rightarrow \frac{2udu}{u^2+1} = \frac{-dx}{x}$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\Rightarrow \ln(1+u^2) = -\ln|x| + c^* \Rightarrow 1 + u^2 = \frac{c}{x}$$

$$\Rightarrow 1 + \left(\frac{y}{x}\right)^2 = \frac{c}{x} \Rightarrow x^2 + y^2 = cx \Rightarrow \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

❁ Ex. Solve initial value problem

$$y' = \frac{y}{x} + \frac{2x^3 \cos(x^2)}{y}, \quad y\left(\sqrt{\pi}\right) = 0.$$

Change of variable $u = \frac{y}{x}$

$$\Rightarrow xu' + u = u + \frac{2x^2 \cos(x^2)}{u} \Rightarrow uu' = 2x \cos(x^2)$$

$$\Rightarrow \frac{u^2}{2} = \sin\left(x^2\right) + c \Rightarrow y = ux = x\sqrt{2\sin\left(x^2\right) + 2c}.$$

Since $y\left(\sqrt{\pi}\right) = 0 \Rightarrow c = 0 \Rightarrow y = x\sqrt{2\sin\left(x^2\right)}.$

❁ Ex. Solve $(2x - 4y + 5)\, y' + x - 2y + 3 = 0$.

If we set $u = y/x$, then the equation will become no-separable.

One way by setting $x - 2y = v$. Then $y' = \frac{1}{2}(1 - v')$.

$$\Rightarrow (2v + 5)\frac{1}{2}(1 - v') + v + 3 = 0 \Rightarrow v + \frac{5}{2} - vv' - \frac{5}{2}v' + v + 3 = 0$$

$$\Rightarrow 2v + 5 - 2vv' - 5v' + 2v + 6 = 0 \Rightarrow (2v + 5)v' = 4v + 11$$

$$\Rightarrow \frac{2v + 5}{4v + 11}dv = dx \Rightarrow \frac{1}{2}\left(1 - \frac{1}{4v + 11}\right)dv = dx$$

$$\Rightarrow \left(1 - \frac{1}{4v + 11}\right)dv = 2dx \Rightarrow v - \frac{1}{4}\ln\left|4v + 11\right| = 2x + c^*.$$

Since $v = x - 2y$

$$\boxed{\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c}$$

$$\Rightarrow x - 2y - \frac{1}{4}\ln\left|4x - 8y + 11\right| = 2x + c^*$$

$$\Rightarrow 4x - 8y - \ln\left|4x - 8y + 11\right| = 8x - c$$

$$\Rightarrow 4x + 8y + \ln\left|4x - 8y + 11\right| = c.$$

R-2 formula

- Now we want to handle differential equations of the form
  $\frac{dy}{dx}=f\left(\frac{ax+by+c}{gx+ey+h}\right)$, where *a, b, c, g, e,* and *h* are constants.

  It implies that $\frac{dy}{dx}=f\left(\frac{a+b\,(y/x)+(c/x)}{g+e\,(y/x)+(h/x)}\right)$,

  which is *R*-1 formula when $c = h = 0$, and

  R-2 formula when $c \neq 0$ or $h \neq 0$.

- There are two ways to solve the equation:
  i. *R*-2 formula $\Rightarrow$ *R*-1 formula $\Rightarrow$ separable or
  ii. *R*-2 formula $\Rightarrow$ separable (directly).

- *Case* 1. Suppose that $ae - bg \neq 0$.

  Change variables $x = X + \alpha$

  $y = Y + \beta$ to eliminate the effect of *c* and *h* ,

  where *X* and *Y* are two new variables; α and β are two constants.

  $$\frac{dy}{dx}=\frac{dY}{dX}$$

  The differential equation becomes

  $$\frac{dY}{dX}=f\left(\frac{a\,(X+\alpha)+b\,(Y+\beta)+c}{g\,(X+\alpha)+e\,(Y+\beta)+h}\right)=f\left(\frac{a\,X+b\,Y+(a\alpha+b\beta+c)}{g\,X+e\,Y+(g\alpha+e\beta+h)}\right).$$

  Now we choose α and β such that

  $$\begin{cases} a\alpha+b\beta+c=0 \\ g\alpha+e\beta+h=0 \end{cases}$$

  Since $ae - bg \neq 0$, then exist α and β satisfying these equations

  Such that $\frac{dY}{dX}=f\left(\frac{aX+bY}{gX+eY}\right)=f\left(\frac{a+b\left(\frac{Y}{X}\right)}{g+e\left(\frac{Y}{X}\right)}\right)$. It is a *R*-1 formula.

❁ Ex.

$$\frac{dy}{dx}=\left(\frac{2x+y-1}{x-2}\right)^2, \quad \text{where } ae-bg = 2 * 0 - 1 * 1 \neq 0.$$

Let $x = X + \alpha$ and $y = Y + \beta$ to get

$$\frac{dY}{dX}=\left(\frac{2X+Y+(2\alpha+\beta-1)}{X+(\alpha-2)}\right)^2$$

Solving the system of linear equations

$$\begin{cases} 2\alpha+\beta-1=0 \\ \alpha-2=0 \end{cases} \Rightarrow \alpha=2 \text{ and } \beta=-3.$$

Then the equation becomes

$$\frac{dY}{dX}=\left(\frac{2X+Y}{X}\right)^2=\left(2+\frac{Y}{X}\right)^2.$$

Let $u = Y/X \Rightarrow Y = Xu$.

$$\frac{dY}{dX}=u+X\frac{du}{dX}=(2+u)^2 \Rightarrow X\frac{du}{dX}=u^2+3u+4$$

$$\Rightarrow \frac{du}{u^2+3u+4}=\frac{dX}{X} \Rightarrow \frac{du}{\left(u+\frac{3}{2}\right)^2+\frac{7}{4}}=\frac{dX}{X}$$

$$\boxed{\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c.}$$

$$\Rightarrow \ln|X|=\frac{1}{\frac{\sqrt{7}}{2}}\tan^{-1}\frac{u+\frac{3}{2}}{\frac{\sqrt{7}}{2}}+c \Rightarrow \ln|X|=\frac{2}{\sqrt{7}}\tan^{-1}\left[\frac{2}{\sqrt{7}}\left(u+\frac{3}{2}\right)\right]+c.$$

Since $u = Y/X$,

$$\ln|X|=\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2Y+3X}{\sqrt{7}X}\right)+c \Rightarrow Y(X)=\frac{1}{2}\left[\sqrt{7}X\tan\left(\frac{\sqrt{7}}{2}(\ln|X|-c)\right)-3X\right].$$

Since $X = x - 2$ and $Y = y + 3$.

$$y(x)=\frac{1}{2}\left[\sqrt{7}(x-2)\tan\left(\frac{\sqrt{7}}{2}(\ln|x-2|-c)\right)-3(x-2)\right]-3.$$

❁ *Case* 2. Suppose that $ae - bg = 0$.

Set $v = \dfrac{ax + by}{a}$ .... (1)

$\Rightarrow av = ax + by \Rightarrow a(v - x) = by$

Since $ae = bg$ (*i.e.*, $a = \dfrac{bg}{e}$)

$\Rightarrow \dfrac{bg}{e}(v - x) = by \Rightarrow g(v - x) = ey$

$\Rightarrow gv = gx + ey \Rightarrow v = \dfrac{gx + ey}{g}$ .... (2)

by Eq.(1) $\dfrac{dv}{dx} = 1 + \dfrac{b}{a}\dfrac{dy}{dx}$

So that $\dfrac{dy}{dx} = \dfrac{a}{b}\left(\dfrac{dv}{dx} - 1\right).$ .... (3)

Then the original equation $\dfrac{dy}{dx} = f\left(\dfrac{ax + by + c}{gx + ey + h}\right)$

can be derived from Eqs.(1), (2), and (3).

$\Rightarrow \dfrac{a}{b}\left(\dfrac{dv}{dx} - 1\right) = f\left(\dfrac{av + c}{gv + h}\right) \Rightarrow \dfrac{dv}{dx} = 1 + \dfrac{b}{a} f\left(\dfrac{av + c}{gv + h}\right)$

$\Rightarrow \dfrac{dv}{1 + \dfrac{b}{a} f\left(\dfrac{av + c}{gv + h}\right)} = dx.$ It is a separable equation.

❁ Ex.

$\dfrac{dy}{dx} = \dfrac{2x + y - 1}{4x + 2y - 4}$, where $ae - bg = 2 * 2 - 4 * 1 = 0.$

Let $v = \dfrac{2x + y}{2}$.

The differential equation becomes

$$\frac{dv}{dx} = 1 + \frac{1}{2}\left(\frac{2v-1}{4v-4}\right)$$

$$\Rightarrow \left(\frac{8v-8}{10v-9}\right)dv = dx \Rightarrow \left(\frac{8v - \frac{36}{5} - \frac{4}{5}}{10v-9}\right)dv = dx$$

$$\Rightarrow \left(\frac{4}{5} - \frac{4}{5}\left(\frac{1}{10v-9}\right)\right)dv = dx \Rightarrow \frac{4v}{5} - \frac{2}{25}\ln|10v-9| + c = x$$

$$\Rightarrow \frac{2}{5}(2x+y) - \frac{2}{25}\ln|10x+5y-9| + c = x$$

$$\Rightarrow -\frac{x}{5} + \frac{2y}{5} - \frac{2}{25}\ln|10x+5y-9| + c = 0.$$

❁ Problems of Section 1.3.

1.4 Exact differential equations

❁ Now we want to consider a *DE* as

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}.$$

That is, $M(x, y)dx + N(x, y)dy = 0$.

❁ The solving principle can be
method 1: transform this equation to be separable or *R*-1;
method 2: to find a function $u(x, y)$ such that
the total differential du is equal to $Mdx + Ndy$.

❁ In the latter strategy, if u exists, then equation $Mdx + Ndy = 0$ is called exact, and $u(x, y)$ is called a potential function for this differential equation.

We know that "$du = 0 \Rightarrow u(x, y) = c$";

it is just the general solution of the differential equation.

- How to find such an u ?

since $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = Mdx + Ndy$,

$\boxed{\frac{\partial u}{\partial x} = M}$ and $\boxed{\frac{\partial u}{\partial y} = N}$.

> To find u, u is regarded as a function of two independent variables x and y.

*step*1. to integrate M *w.r.t.* x or integrate N *w.r.t.* y to obtain u. Assume u is obtained by integrating M, then

$$u(x, y) = \int M dx + k(y).$$

*step*2. partial differentiate u *w.r.t.* y (*i.e.*, $\frac{\partial u}{\partial y}$), and to compare with N to find k function.

- How to test $Mdx + Ndy = 0$ is exact or not ?

Proposition (Test for exactness)

If M, N, $\frac{\partial N}{\partial x}$, and $\frac{\partial M}{\partial y}$ are continuous over a rectangular region R, then "$Mdx + Ndy = 0$ is exact for (x, y) in R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in R ".

- Ex. Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$.

1*st step*: (testing for exactness)

$M = x^3 + 3xy^2$, $N = 3x^2y + y^3$

$$\frac{\partial M}{\partial y} = 6xy = \frac{\partial N}{\partial x}$$

It implies that the equation is exact.

2*nd step*:

$$u = \int M dx + k(y) = \int (x^3 + 3xy^2)dx + k(y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + k(y)$$

3*rd step*:

Since $\frac{\partial u}{\partial y} = N \Rightarrow 3x^2y + k'(y) = 3x^2y + y^3$,

$k'(y) = y^3$. That is $k(y) = \frac{1}{4}y^4 + c^*$.

Thus $u(x, y) = \frac{1}{4}(x^4 + 6x^2y^2 + y^4) + c^*$.

The solution is then $\frac{1}{4}(x^4 + 6x^2y^2 + y^4) = c$.

This is an implicit solution to the original *DE*.

4th step: (checking solution for $Mdx + Ndy = 0$)

$$\frac{d}{dx}\left[\frac{1}{4}(x^4+6x^2y^2+y^4)\right]=\frac{dc}{dx}$$

$$\Rightarrow \frac{1}{4}(4x^3 + 12xy^2 + 12x^2yy' + 4y^3y') = 0$$

$$\Rightarrow (x^3 + 3xy^2) + (3x^2y + y^3)y' = 0$$

$$\Rightarrow (x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0. \quad QED$$

- Ex.2. Solve $(\sin x \cosh y)dx - (\cos x \sinh y)dy = 0$, $y(0) = 3$.

Answer. $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$M = \sin x \cosh y$, $N = -\cos x \sinh y$

$\dfrac{\partial M}{\partial y} = \sin x \sinh y = \dfrac{\partial N}{\partial x}$. The *DE* is exact.

If $u = \int \sin x \cosh y\, dx + k(y) = -\cos x \cosh y + k(y)$

$\dfrac{\partial u}{\partial y} = -\cos x \sinh y \Rightarrow k$ = constant $\Rightarrow$ Solution is $\cos x \cosh y = c$.

Since $y(0) = 3$, $\cos 0 \cosh 3 = c \Rightarrow \cos x \cosh y = \cosh 3$.

- Ex.3. (non-exact case)

$ydx - xdy = 0$

$M = y$, $N = -x$

step 1: $\dfrac{\partial M}{\partial y} = 1 \neq \dfrac{\partial N}{\partial x} = -1.$

If you solve the equation by the same method.

step 2: $u = \int Mdx + k(y) = xy + k(y)$

step 3: $\dfrac{\partial u}{\partial y} = x + k'(y) = N = -x$

$\Rightarrow k'(y) = -2x.$

Since $k(y)$ depends only on y; we can not find the solution.

Try $u = \int Ndy + k(x)$ also gets the same contradiction.

Truly, the *DE* is separable.

Integrating factors

- If a *DE* $\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$ (or $M(x, y)dx + N(x, y)dy = 0$) is not exact, then we can sometimes find a nonzero function $F(x, y)$ such that $F(x, y)M(x, y)dx + F(x, y)N(x, y)dy = 0$ is exact.

  We call $F(x, y)$ an integrating factor for $Mdx + Ndy = 0$.

- Note

  1. Integrating factor is not unique.
  2. The integrating factor is independent of the solution.

- Ex.4. Solve $ydx - xdy = 0$ (non-exact)

  Assume there is an integrating factor $F = \frac{1}{x^2}$, then the original *DE* becomes exact,

$$\frac{\partial(\frac{y}{x^2})}{\partial y} = \frac{1}{x^2} = \frac{\partial(\frac{-1}{x})}{\partial x}$$

  There are several differential factors: $\frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}, \cdots$

  (conclusion: Integrating factor is not unique)

- Ex. Solve $2\sin(y^2)\,dx + xy\cos(y^2)\,dy = 0$, Integrating factor $F(x, y) = x^3$.

  $FM = 2x^3\sin(y^2)$

  $FN = x^4 y\cos(y^2)$

$$\frac{\partial(FM)}{\partial y} = 4x^3 y\cos(y^2) = \frac{\partial(FN)}{\partial x}$$

  Then we can solve the equation by the method of exact equation.

- How to find integrating factors ?
  there are no better method than inspection or “try and error”.

- How to “try and error” ?
  Since (FM) dx + (FN) dy = 0 is exact,

$$\frac{\partial(FM)}{\partial y}=\frac{\partial(FN)}{\partial x} \text{ ; that is, } F\frac{\partial M}{\partial y}+M\frac{\partial F}{\partial y}=F\frac{\partial N}{\partial x}+N\frac{\partial F}{\partial x} \quad \text{.............} \quad (1)$$

Let us consider three cases:

Case 1. Suppose $F = F(x)$ or $F = F(y)$

Theorem 1. If $F = F(x)$, then $\frac{\partial F}{\partial y} = 0$.

It implies that Eq.(1) becomes

$$F\frac{\partial M}{\partial y}=F\frac{\partial N}{\partial x}+N\frac{dF}{dx}$$

$$\Rightarrow F\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=N\frac{dF}{dx}$$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx=\frac{1}{F}dF.$$

$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ must be only a function of x only;

thus the *DE* becomes separable.

$$\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx+c^*=\ln F$$

$$\Rightarrow F=exp\left[\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx\right].$$

Theorem 2. If $F = F(y)$, then $\frac{\partial F}{\partial x} = 0$.

It implies that *PDE* (1) becomes

$$F\frac{\partial M}{\partial y} + M\frac{\partial F}{\partial y} = F\frac{\partial N}{\partial x}$$

$$\Rightarrow M\frac{\partial F}{\partial y} = F(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$$

$$\Rightarrow \frac{1}{F}dF = \frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dy.$$

$\frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$ must be only a function of y only; thus the *DE* becomes separable and $F(y) = exp\left[\int \frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dy\right]$.

Case 2. Suppose $F(x, y) = x^a y^b$ and attempt to solve coefficients a and b by substituting F into Eq.(1).

Case 3. If cases 1 and 2 both fail, you may try other possibilities, such as $e^{ax + by}$, $x^a e^{by}$, $e^{ax} y^b$, and so on.

❁ Ex. (Example for case 1)

Solve the initial value problem

$2xydx + (4y + 3x^2)dy = 0$, $y(0.2) = -1.5$

$M = 2xy$, $N = 4y + 3x^2$

$$\frac{\partial M}{\partial y} = 2x \neq \frac{\partial N}{\partial x} = 6x \quad \text{(non-exact)}$$

Testing whether $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ depends only on x or not.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{4y + 3x^2}(2x - 6x)$$ depends on both x and y.

testing whether $\frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$ depends only on *y* or not.

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{2xy}(6x - 2x) = \frac{2}{y}$$ depends only on *y*.

Thus $F(y) = exp\int \frac{2}{y} dy = y^2$.

The original *DE* becomes

$2xy^3dx + (4y^3 + 3x^2y^2)dy = 0$ (exact)

$u = \int 2xy^3dx + k(y) = x^2y^3 + k(y)$

$\frac{\partial u}{\partial y} = 3x^2y^2 + k'(y) = 4y^3 + 3x^2y^2$

$\Rightarrow k'(y) = 4y^3$

$\Rightarrow k(y) = y^4 + c^*$

$\Rightarrow u = x^2y^3 + y^4 + c^* = c'$

$\Rightarrow x^2y^3 + y^4 = c$.

Since $y(0.2) = -1.5 \Rightarrow c = 4.9275$.

Solution $x^2y^3 + y^4 = 4.9275$.

❁ Ex. (Example for case 2)

$(2y^2 – 9xy)dx + (3xy – 6x^2)dy = 0$ (non-exact)

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{1}{3xy - 6x^2}(4y - 9x - 3y + 12x) = \frac{y + 3x}{3x(y - 2x)}$$ depends on *x* & *y*.

$$\frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) = \frac{1}{2y^2 - 9xy}(3y - 12x - 4y + 9x) = \frac{-3x - y}{y(2y - 9x)}$$ depends on *x* & *y*.

Take $F = x^a y^b$, then

$$\frac{\partial(FM)}{\partial y} = \frac{\partial(FN)}{\partial x}$$

$$\Rightarrow \frac{\partial(2y^2x^ay^b - 9xyx^ay^b)}{\partial y} = \frac{\partial(3xyx^ay^b - 6x^2x^ay^b)}{\partial x}$$

$$\Rightarrow 2(2+b)y^{b+1}x^a – 9(b+1)x^{a+1}y^b = 3(a+1)x^ay^{b+1} – 6(a+2)x^{a+1}y^b$$

$$\Rightarrow \begin{cases} 2(2+b) = 3(a+1) \\ 9(b+1) = 6(a+2) \end{cases}$$

$$\Rightarrow \begin{cases} 3a - 2b - 1 = 0 \\ 6a - 9b + 3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a = 1 \\ b = 1 \end{cases}$$

$$\Rightarrow F(x, y) = xy.$$

❁ Problems of Section 1.4.

1.5 Linear differential equation and Bernoulli equation

❁ A first-order *DE* is said to be linear if it can be written

$$y' + p(x)y = r(x).$$

If $r(x) = 0$, the linear *DE* is said to be homogeneous, if $r(x) \neq 0$, the linear *DE* is said to be nonhomogeneous.

❁ Solving the *DE*

(*a*) For homogeneous equation (⇒ separable)

$$y' + p(x)y = 0$$

$$\Rightarrow \frac{dy}{dx} = -p(x)y$$

$$\Rightarrow \frac{1}{y}\,dy = -\,p(x)dx$$

$$\Rightarrow \ln|y| = -\int p(x)dx + c^*$$

$$\Rightarrow y = ce^{-\int p(x)dx}.$$

(b) For nonhomogeneous equation

$(py - r)dx + dy = 0$

since $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = p$ is a function of x only,

we can take an integrating factor

$$F(x) = exp\left[\int \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})dx\right] = e^{\int pdx}$$

such that the original DE $y' + py = r$ becomes

$$e^{\int pdx}(y' + py) = (e^{\int pdx}y)' = e^{\int pdx}r$$

Integrating with respect to x,

$$e^{\int pdx}y = \int e^{\int pdx}r\,dx + c$$

$$\Rightarrow y(x) = e^{-\int pdx}[\int e^{\int pdx}rdx + c] .$$

- The solution of the homogeneous linear *DE* is a special case of the solution of the corresponding non-homogeneous linear *DE*.

- Ex. Solve the linear *DE*

$$y' - y = e^{2x}$$

Solution.

$p = -1,\ r = e^{2x},\ \int pdx = -x$

$$y(x) = e^{x}[\int e^{-x} e^{2x} dx + c]$$
$$= e^{x}[\int e^{x}dx + c]$$
$$= e^{2x} + ce^{x}.$$

- Ex. Solve the linear *DE*

$$y' + 2y = e^{x}(3 \sin 2x + 2 \cos 2x)$$

Solution.

$p = 2,\ r = e^{x}(3 \sin 2x + 2 \cos 2x),\ \int pdx = 2x$

$$y(x) = e^{-2x}[\int e^{2x} e^{x}(3 \sin 2x + 2 \cos 2x)\,dx + c]$$
$$= e^{-2x}[e^{3x} \sin 2x + c]$$
$$= c\,e^{-2x} + e^{x} \sin 2x .$$

Bernoulli equation

- The Bernoulli equation is formed of

  $y' + p(x)\, y = r(x)\, y^a$, where a is a real number.

  If $a = 0$ or $a = 1$, the equation is linear.

- Bernoulli equation can be reduced to a linear form by change of variables.

  We set $u(x) = [y(x)]^{1-a}$,

  then differentiate the equation and substitute y' from Bernoulli equation

  $u' = (1 - a)\, y^{-a}\, y' = (1 - a)\, y^{-a}\, (r y^a - py)$

  $= (1 - a)\, (r - py^{1-a})$

  $= (1 - a)\, (r - pu)$

  $\Rightarrow u' + (1 - a)\, pu = (1 - a)\, r$ (This is a linear *DE* of *u*.)

- Ex. 4.

  $y' - Ay = - By^2$

  $a = 2,\ u = y^{-1}$

  $u' = -y^{-2}\, y' = -y^{-2}\, (-By^2 + Ay) = B - Ay^{-1} = B - Au$

  $\Rightarrow u' + Au = B$

  $u = e^{-\int pdx}\, [\, \int e^{\int pdx}\, r\, dx + c\,]$

  $= e^{-Ax}\, [\, \int Be^{Ax}\, dx + c\,]$

  $= e^{-Ax}\, [\, B/A\, e^{Ax} + c\,]$

  $= B/A + c\, e^{-Ax}$

  $\Rightarrow\ y = 1/u = 1/(B/A + ce^{-Ax})$

- Problems of Section 1.5.

Riccati equation (problem 44 on page 40)

$y' = p(x)\, y^2 + q(x)\, y + r(x)$ is a Riccati equation.

❁ Solving strategy

If we can some how (often by observation, guessing, or trial and error) produce one specific solution $y = s(x)$, then we can obtain a general solution as follows:

Change variables from y to z by setting

$y = s(x) + 1/z$

$\Rightarrow\ y' = s'(x) - (1/z^2)\, z'$

Substitution into the Riccati equation given us

$$s'(x) - (1/z^2)\, z' = [\, p(x)\, s(x)^2 + q(x)\, s(x) + r(x)\,] + [\, p(x)\,(1/z^2) + 2p(x)s(x)\,(1/z) + q(x)\,(1/z)\,]$$

Since $s(x)$ is a solution of original equation.

$\Rightarrow\ -(1/z^2)\, z' = p\,(1/z^2) + 2\, p\, s\,(1/z) + q\,(1/z)$

multiplying through by $-z^2$

$\Rightarrow\ z' + (2\, p\, s + q)\, z = -p$,

which is a linear *DE* for z and can be found the solution.

$\Rightarrow\ z = c/u(x) + [1/u(x) \int -p(x)u(x)dx]$,

where $u(x) = e^{\int [2ps + q]\, dx}$

$\Rightarrow\ z = e^{-\int [2ps + q]\, dx}\, [\, \int -e^{\int (2ps + q)\, dx}\, p\, dx + c\,]$

Then, $y = s(x) + 1/z$ is a general solution of the Riccate equation.

❁ There are two difficulties for solving Riccati equations:

(1) one must first find a specific solution $y = s(x)$.

(2) one must be able to perform the necessary integrations.

❁ Ex. $y' = (1/x)\, y^2 + (1/x)\, y - 2/x$, $s(x) = 1$.

Solution. $y(x) = (2x^3 + c)/(c - x^3)$.

Summary for 1st order *DE*

1. Separable $f(x)\,dx = g(y)\,dy$ [separated integration]

2. *R*-1 formula $dy/dx = f(y/x)$ [change variable $u = y/x$]

3. *R*-2 $dy/dx = f((ax + by + c)/(gx + ey + h))$, $c \neq 0$ or $h \neq 0$.
   with two cases
   - *i*. $ae - bg \neq 0$ [$x = X + \alpha$, $y = Y + \beta \Rightarrow$ *R*-1 $\Rightarrow$ separable]
   - *ii*. $ae - bg = 0$ [$v = (ax+by)/a = (gx+ey)/g \Rightarrow$ separable]

4. Exact $dy/dx = -M(x, y)/N(x, y)$ [$\partial M/\partial y = \partial N/\partial x \Rightarrow$ exact]
   ($M\,dx + N\,dy = 0$) [deriving $u \Rightarrow du = Mdx + Ndy$]

5. Integrating factor $dy/dx = -M/N$ [find $F \Rightarrow (FM)dx + (FN)dy = 0$ is exact]
   try some factors
   - *i*. $(\partial M/\partial y - \partial N/\partial x)/N = F(x)$ or $(\partial N/\partial x - \partial M/\partial y)/M = F(y)$
   - *ii*. $F = x^a y^b$
   - *iii*. $F = e^{ax+by}$, $x^a e^{by}$, $e^{ax} y^b$, …

6. Linear 1st-order *DE* $y' + p(x)y = r(x)$ integrating factor $F(x) = e^{\int p\,dx}$
   $$y = e^{-\int p(x)\,dx}\left[\int r(x)\, e^{\int p(x)\,dx}\,dx + c\right]$$

7. Bernoulli equation $y' + p(x)\,y = r(x)\,y^a$
   set $u(x) = [y(x)]^{1-a}$, $\Rightarrow u' + (1-a)\,pu = (1-a)\,r$ (linear *DE*)

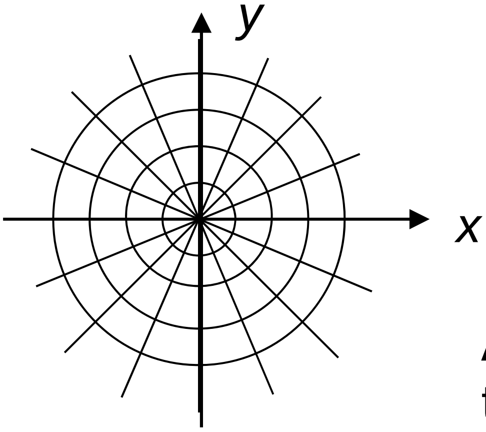
8. Riccati equation $y' = p(x)y^2 + q(x)y + r(x)$
   (1) guess a specific solution $s(x)$
   (2) change variable $y = s(x) + 1/z$
   (3) to derive a linear *DE* $z' + (2ps + q)\,z = -p$.

1.6 Orthogonal trajectories of curves

- Purpose

  use differential equation to find curves that intersect given curves at right angles. The new curves are then called the orthogonal trajectories of the given curves.

- Example



Any blue line is orthogonal to any pink circle.

- Principle

  to represent the original curves by the general solution of a *DE* $y' = f(x, y)$, then replace the slope y' by its negative reciprocal, $-1/y'$, and solve the new *DE* $-1/y' = f(x, y)$.

- Family of curves

  If for each fixed value of c the equation $F(x, y, c) = 0$ represents a curve in the xy-plane and if for variable c it represents infinitely many curves, then the totality of these curves is called a one-parameter family of curves, and c is called the parameter of the family.

- Determination of orthogonal trajectories

  step 1. Given a family of curves $F(x, y, c) = 0$,

  to find their *DE* in the form $y' = f(x, y)$,

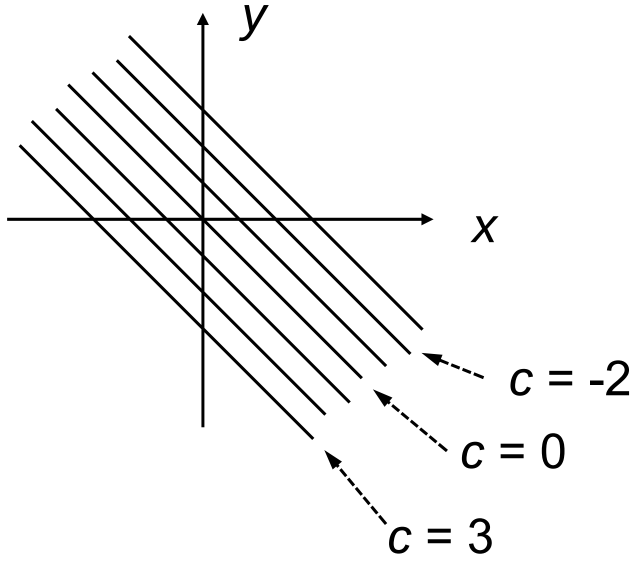
  step 2. Find the orthogonal trajectories by solving their *DE*

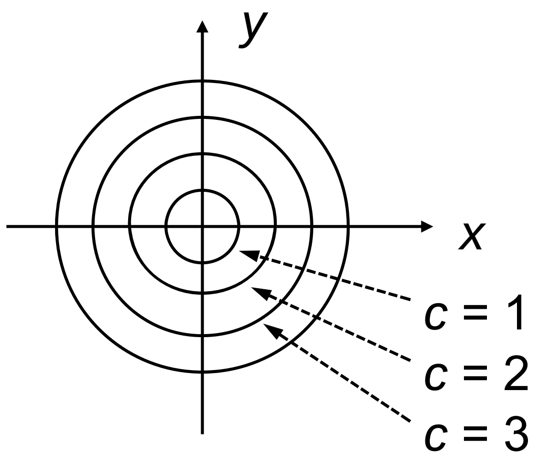
  $y' = -1/f(x, y)$.

Ex.

(1) The equation $F(x, y, c) = x + y + c = 0$ represents a family of parallel straight lines.

(2) The equation $F(x, y, c) = x^2 + y^2 - c^2 = 0$ represents a family of concentric circles of radius c with center at the original.





Ex.

(1) differentiating $x + y + c = 0$, gives the *DE* $y' = -1$.

(2) differentiating $x^2 + y^2 - c^2 = 0$,

gives the *DE* $2x + 2yy' = 0 \Rightarrow y' = -x/y$.

(3) differentiating the family of parabolas $y = cx^2$,

gives the *DE* $y' = 2cx$.

since $c = y/x^2$, $y' = 2y/x$.

Another method

$$c = yx^{-2}$$

$$\Rightarrow 0 = y'x^{-2} - 2yx^{-3}$$

$$\Rightarrow y'x = 2y$$

$$\Rightarrow y' = 2y/x .$$

❁ Ex.

Find the orthogonal trajectories of the parabolas $y = cx^2$.

Step 1. $y' = 2y/x$

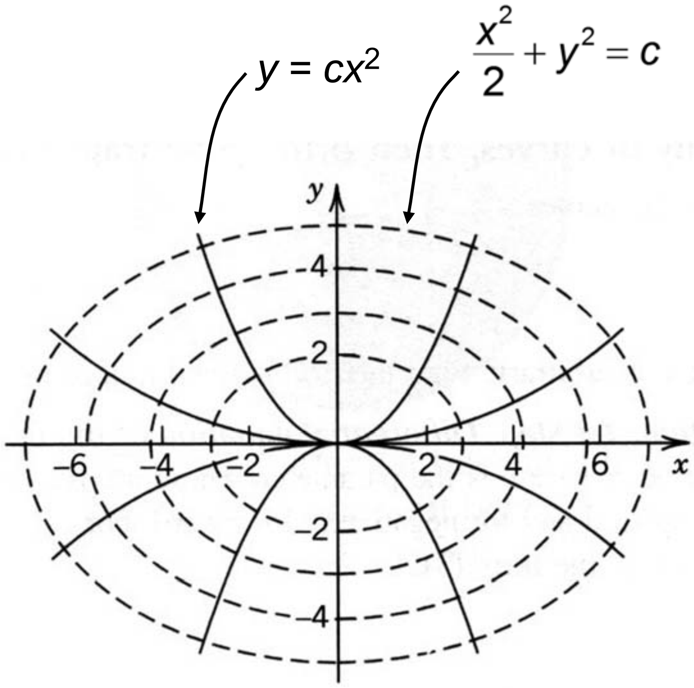
Step 2. solve

$$y' = -x/2y$$

$$\Rightarrow 2y\,dy = -x\,dx$$

$$\Rightarrow y^2 = -x^2/2 + c^*$$

$$\Rightarrow \frac{x^2}{2} + y^2 = c.$$



❁ Ex.

Find the orthogonal trajectories of the circles $x^2 + (y - c)^2 = c^2$.

step 1. Differentiating $x^2 + (y - c)^2$ to give $2x + 2(y - c)y' = 0$

$\Rightarrow y' = x/(c-y)$ (error)

Correct derivation $x^2 + (y - c)^2 = c^2$

$\Rightarrow x^2 + y^2 - 2cy = 0$

$\Rightarrow x^2 y^{-1} + y = 2c$

$\Rightarrow 2xy^{-1} - x^2 y^{-2} y' + y' = 0$

$\Rightarrow 2xy^{-1} = (x^2 y^{-2} - 1)\,y'$

$\Rightarrow 2xy = (x^2 - y^2)\,y'$

$\Rightarrow y' = 2xy/(x^2 - y^2)$

step 2. Solve $y' = (y^2 - x^2)/2xy$

$\Rightarrow y' = y/2x - x/2y$ (*R*-1 formula)

Solution. $(x - e)^2 + y^2 = e^2$,
where e is a constant.

❁ Problems of Section 1.6.

1.7 Existence and uniqueness of solutions

- Consider an initial value problem

  $y' = f(x, y)$, $y(x_0) = y_0$

  There are three possibilities of solution,

  (*i*) no solution; *e.g.*, $|y'| + |y| = 0$, $y(0) = 1$.
  0 is the only solution of the differential equation,
  the condition contradicts to the equation;
  moreover, y and y' are not continuous at $x = 0$.

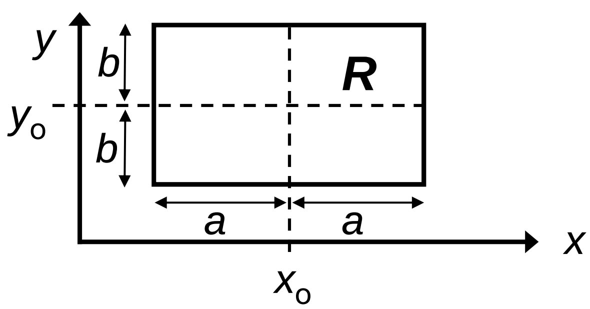
  (*ii*) unique solution; *e.g.*, $y' = x$, $y(0) = 1$, solution $y = \frac{1}{2}x^2 + 1$

  (*iii*) infinitely many solution; *e.g.*, $xy' = y - 1$, $y(0) = 1$,
  solution $y = 1 + cx$.

- Problem of existence
  Under what conditions does an initial value problem have at least one solution?

- Problem of uniqueness
  Under what conditions does that problem have at most one solution?

- Theorem 1 (Existence theorem)
  If $f(x, y)$ is continuous at all points (x, y) in some rectangle
  $R : |x - x_0| < a$, $|y - y_0| < b$ and bounded in R: $|f(x, y)| \le k$
  for all (x, y) in R, then
  the initial value problem
  "$y' = f(x, y)$, $y(x_0) = y_0$"
  has at least one solution $y(x)$.



❁ Theorem 2 (Uniqueness theorem)

If *f(x, y)* and ∂*f*/∂*y* are continuous for all (*x, y*) in that rectangle *R* and bounded,

(a) $|f| \le k$, (b) $\left|\frac{\partial f}{\partial y}\right| \le M$ for all (*x, y*) in *R*, then the

initial value problem has at most one solution *y*(*x*).

Hence, by Theorem 1, it has precisely one solution.

❁ The conditions in the two theorems are sufficient conditions rather than necessary ones and can be lessened.

For example, condition $\left|\frac{\partial f}{\partial y}\right| \le M$ may be replaced by the weaker

condition $|f(x, y_2) - f(x, y_1)| \le M\,|y_2 - y_1|$, where y_1 and y_2 are on the boundary of the rectangle ***R***. The later formula is known as a Lipschitz condition. However, continuity of *f*(*x*, *y*) is not enough to guarantee the uniqueness of the solution.

❁ Ex. 2. (Nonuniqueness)

The initial value problem

$$y' = \sqrt{|y|}, \qquad y(0) = 0.$$

has the two solutions

$$y = 0 \quad \text{and} \quad y = \begin{cases} x^2/4 & \text{if} \quad x \ge 0 \\ -x^2/4 & \text{if} \quad x < 0 \end{cases}$$

Although $f(x, y) = y' = \sqrt{|y|}$ is continuous for all *y*. The Lipschitz condition is violated in any region that include the line *y* = 0, because for $y_1 = 0$ and positive y_2, we have

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}}$$

and this can be made as large as we please by choose y_2 sufficiently small, whereas the Lipschitz condition requires that the quotient on the left side of the above equation should not exceed a fixed constant *M*.

Picards' iteration method

- Picards method gives approximate solutions of an initial value problem

  $y' = f(x, y),\ y(x_0) = y_0$

(*i*) The initial value problem can be written in the form

$$y(x) = y_0 + \int_{x_0}^{x} f[t, \underline{y(t)}]\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

unknown

(*ii*) Take an approximation

$$y_1(x) = y_0 + \int_{x_0}^{x} f[t, y_0(t)]\,dt$$

(*iii*) Substitute the function $y_1(x)$ in the same way to get

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t, y_1(t)]\,dt$$

$$\vdots$$

(*iv*) $y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\,dt$

Under some conditions, the sequence will converge to the solution $y(x)$ of the original initial value problem.

- Ex. Find approximate solutions to the initial value problem

  $y' = 1 + y^2,\ \ y(0) = 0\,.$

Solution.

$x_0 = 0\,,\ \ y_0 = 0\,,\ \ f(x, y) = 1 + y^2$

$$y_1(x) = 0 + \int_0^x [1 + 0]\,dt = x$$

$$y_2(x) = y_0 + \int_0^x f[t, y_1(t)]\,dt$$

$$= 0 + \int_0^x [1 + t^2]\,dt = x + \frac{1}{3}x^3$$

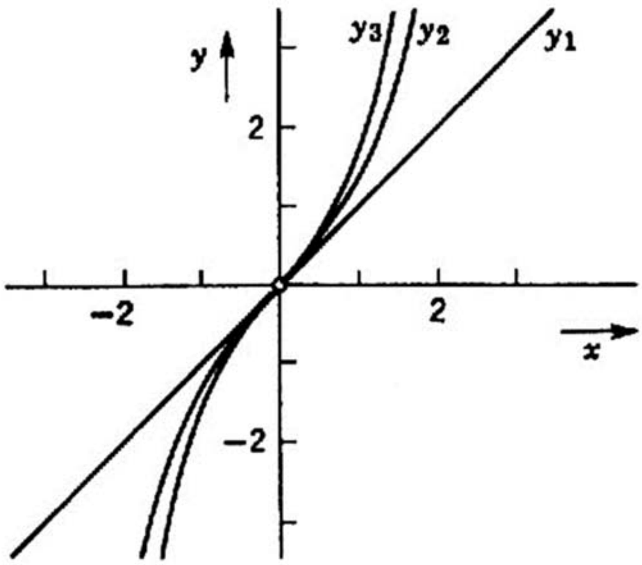
$$y_3(x) = y_0 + \int_0^x f[t, y_2(t)]\,dt$$

$$= 0 + \int_0^x \left[1 + (t + \frac{1}{3}t^3)^2\right]dt$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7$$

$$\vdots$$

Exact solution $y(x) = \tan(x)$.



❁ Problems of Section 1.7.

Laparoscopic surgical simulation