1 Fundamentals of open-channel flow

Open channels are natural or manmade conveyance structures that normally have an open top, and they include rivers, streams and estuaries. An important characteristic of open-channel flow is that it has a free surface at atmospheric pressure. Open-channel flow can occur also in conduits with a closed top, such as pipes and culverts, provided that the conduit is flowing partially full. For example, the flow in most sanitary and storm sewers has a free surface, and is therefore classified as open-channel flow.

1.1 GEOMETRIC ELEMENTS OF OPEN CHANNELS

A channel section is defined as the cross-section taken perpendicular to the main flow direction. Referring to Figure 1.1, the geometric elements of an open channel are defined as follows:

Flow depth, y	Vertical distance from the channel bottom to the free surface.
Depth of flow section, d	Flow depth measured perpendicular to the channel bottom. The relationship between d and y is $d = y\cos\theta$. For most manmade and natural channels $\cos\theta \approx 1.0$, and therefore $y \approx d$. The two terms are used interchangeably.
Top width, T	Width of the channel section at free surface.
Wetted perimeter, P	Length of the interface between the water and the channel boundary.
Flow area, A	Cross-sectional area of the flow.
Hydraulic depth, D	Flow area divided by top width, $D = A/T$.
Hydraulic radius, R	Flow area divided by wetted perimeter, $R = A/P$.
Bottom slope, S_0	Longitudinal slope of the channel bottom, $S_0 = \tan\theta \approx \sin\theta$.

Table 1.1 presents the relationship between various section elements. A similar, more detailed table was previously presented by Chow (1959).

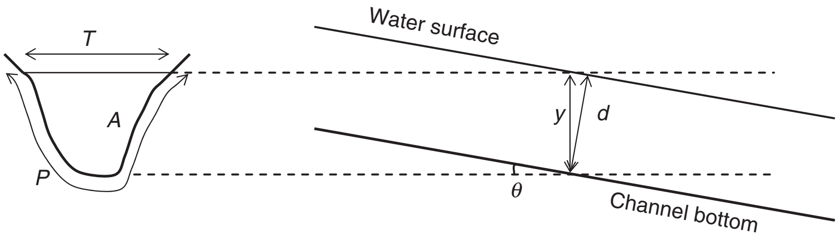


FIGURE 1.1
Definition sketch for section elements

1.2 VELOCITY AND DISCHARGE

At any point in an open channel, the flow may have velocity components in all three directions. For the most part, however, open-channel flow is assumed to be one-dimensional, and the flow equations are written in the main flow direction. Therefore, by *velocity* we usually refer to the velocity component in the main flow direction. The velocity varies in a channel section due to the friction forces on the boundaries and the presence of the free-surface. We use the term *point velocity* to refer to the velocity at different points in a channel section. Figure 1.2 shows a typical distribution of point velocity, v, in a trapezoidal channel.

The volume of water passing through a channel section per unit time is called the *flow rate* or *discharge*. Referring to Figure 1.3, the incremental discharge, dQ, through an incremental area, dA, is

$$dQ = vdA \tag{1.1}$$

where v = point velocity.

Then by definition,

$$Q = \int_A dQ = \int_A vdA \tag{1.2}$$

where Q = discharge.

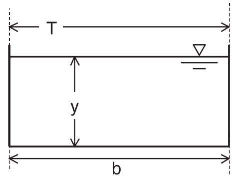
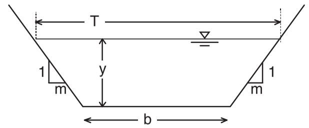
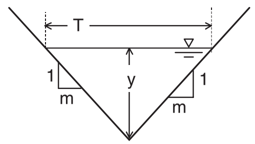
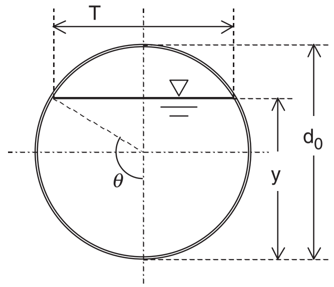
In most open-channel flow applications we use the *cross-sectional average velocity*, V, defined as

$$V = \frac{Q}{A} = \frac{1}{A}\int_A vdA \tag{1.3}$$

1.3 HYDROSTATIC PRESSURE

Pressure represents the force the water molecules push against other molecules or any surface submerged in water. The molecules making up the water are in

TABLE 1.1 Geometric elements of channel sections

Section type	Area A	Wetted perimeter P	Hydraulic radius R	Top width T	Hydraulic depth D
Rectangular 	by	$b+2y$	$\frac{by}{b+2y}$	b	y
Trapezoidal 	$(b+my)y$	$b+2y\sqrt{1+m^2}$	$\frac{(b+my)y}{b+2y\sqrt{1+m^2}}$	$b+2my$	$\frac{(b+my)y}{b+2my}$
Triangular 	my^2	$2y\sqrt{1+m^2}$	$\frac{my}{2\sqrt{1+m^2}}$	$2my$	$\frac{y}{2}$
Circular 	$\frac{1}{8}(2\theta-\sin 2\theta)d_0^2$ $\theta=\pi-\text{arcCos}\left[\left(y-\frac{d_0}{2}\right)/(d_0/2)\right]$	θd_0	$\frac{1}{4}\left(1-\frac{\sin 2\theta}{2\theta}\right)d_0$	$(\sin\theta)d_0$ or $2\sqrt{y(d_0-y)}$	$\frac{1}{8}\left(\frac{2\theta-\sin 2\theta}{\sin\theta}\right)d_0$

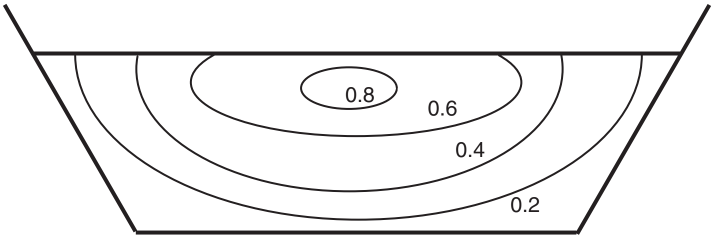


FIGURE 1.2 Velocity distribution in a trapezoidal channel section

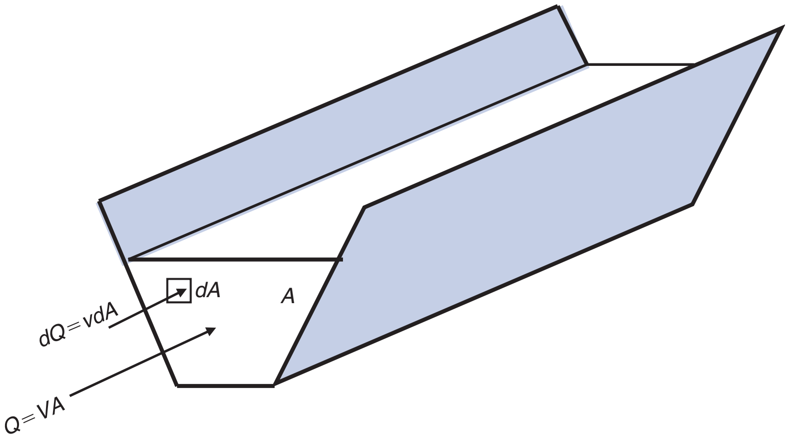


FIGURE 1.3 Definition of discharge

constant motion even when a body of water is at rest in the macroscopic sense. The pressure results from the collisions of these molecules with one another and with any submerged surface like the walls of a container holding a water body. Because, the molecular motion is random, the resulting pressure is the same in every direction at any point in water.

The water surface in an open channel is exposed to the atmosphere. Millions of collisions take place every second between the molecules making up the atmosphere and the water surface. As a result, the atmosphere exerts some pressure on the water surface. This pressure is called *atmospheric pressure*, and it is denoted by p_{atm}.

The pressure occurring in a body of water at rest is called *hydrostatic pressure.* In Figure 1.4, consider a column of water extending from the water surface to point B at depth of Y_B. Let the horizontal cross-sectional area of the column be A_0. This column of water is pushed downward at the surface by a force equal to $p_{atm}A_0$ due to the atmospheric pressure and upward at the bottom by a force $(p_{abs})_B A_0$ due to the absolute water pressure, $(p_{abs})_B$ at point B. In addition, the weight of the water column, a downward force, is $W = \gamma Y_B A_0$ where $\gamma =$ specific weight of water. Because the water column is in equilibrium,

$$(p_{abs})_B A_0 = p_{atm} A_0 + \gamma Y_B A_0$$

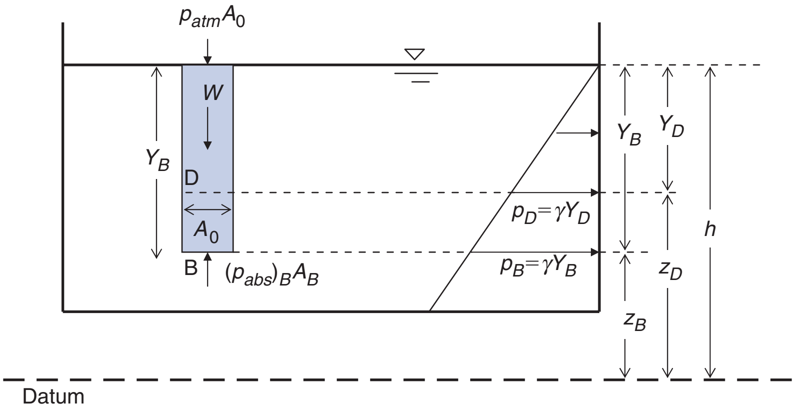


FIGURE 1.4
Hydrostatic pressure distribution

or

$$(p_{\text{abs}})_B - p_{\text{atm}} = \gamma Y_B$$

Pressure is usually measured using atmospheric pressure as base. Therefore, the difference between the absolute pressure and the atmospheric pressure is usually referred to as *gage pressure*. In this text we will use the term *pressure* interchangeably with gage pressure. Denoting the gage pressure or pressure by p,

$$p_B = (p_{\text{abs}})_B - p_{\text{atm}} = \gamma Y_B \tag{1.4}$$

In other words, the hydrostatic pressure at any point in the water is equal to the product of the specific weight of water and the vertical distance between the point and the water surface. Therefore, the hydrostatic pressure distribution over the depth of water is triangular as shown in Figure 1.4.

Let the elevation of point B be z_B above a horizontal datum as shown in Figure 1.4. Let us now consider another point D, which is a distance z_D above the datum and Y_D below the water surface. The pressure at this point is $p_D = \gamma Y_D$. Thus, $Y_D = p_D/\gamma$. An inspection of Figure 1.4 reveals that

$$z_B + \frac{p_B}{\gamma} = z_D + \frac{p_D}{\gamma} = h \tag{1.5}$$

where h is the elevation of the water surface above the datum. As we will see later, $(z + p/\gamma)$ is referred to as *piezometric head*. Equation 1.5 indicates that the piezometric head is the same at any point in a vertical section if the pressure distribution is hydrostatic.

The hydrostatic pressure distribution is valid even if there is flow as long as the flow lines are horizontal. Without any vertical acceleration, the sum of the vertical forces acting on a water column should be zero. Then, the derivation given above for the hydrostatic case is valid for horizontal flow as well. If the flow lines are inclined but parallel to the channel bottom, we can show that

$$p_B = \gamma Y_B \cos^2 \theta \tag{1.6}$$

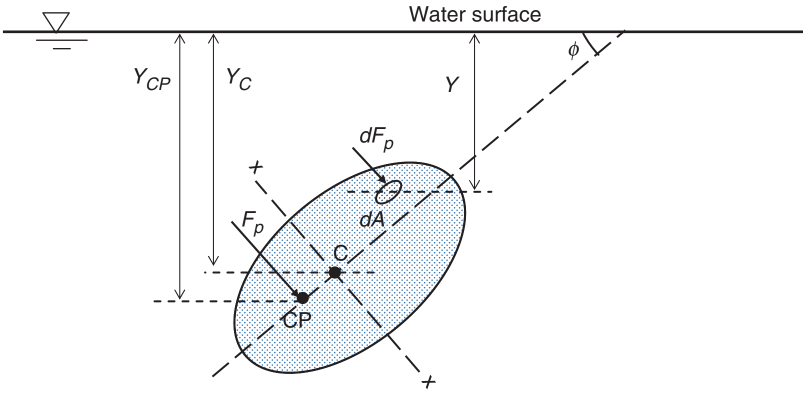


FIGURE 1.5
Hydrostatic pressure force

where θ = angle between the horizontal and the bottom of the channel. Therefore, strictly speaking, the pressure distribution is not hydrostatic when the flow lines are inclined. However, for most manmade and natural open channels θ is small and $\cos\theta \approx 1$. We can assume that the pressure distribution is hydrostatic as long as θ is small and the flow lines are parallel.

The hydrostatic forces resulting from the hydrostatic pressure act in a direction normal to a submerged surface. Consider a submerged, inclined surface as shown in Figure 1.5. Let C denote the centroid of the surface. The pressure force acting on the infinitesimal area dA is $dF_p = pdA$ or $dF_p = \gamma Y dA$. To find the total hydrostatic force, we integrate dF_p over the total area A of the surface. Thus

$$F_p = \int_A \gamma Y dA \tag{1.7}$$

Noting that γ is constant, and recalling the definition of the centroid (point C in Figure 1.5) as

$$Y_C = \frac{\int_A Y dA}{A} \tag{1.8}$$

we obtain

$$F_p = \gamma Y_C A \tag{1.9}$$

In other words, the hydrostatic pressure force acting on a submerged surface, vertical, horizontal, or inclined, is equal to the product of the specific weight of water, area of the surface, and the vertical distance from the free surface to the centroid of the submerged surface. Again, the direction of the hydrostatic force is normal to the submerged surface. The point of application of the resultant hydrostatic force is called the *center of pressure* (point CP in Figure 1.5). The location of the center of pressure can be found by equating the moment of the resultant F_p around the centroidal horizontal axis (axis xx in Figure 1.5) to that of dF_p integrated over the area. This will result in the relationship

$$Y_{CP} = Y_C + \frac{I_x(\sin\phi)^2}{AY_C} \tag{1.10}$$

where ϕ = angle between the water surface and the plane of the submerged surface, and I_x = moment of inertia of the surface with respect to the centroidal horizontal axis.

1.4 Mass, Momentum and Energy Transfer in Open-Channel Flow

1.4.1 MASS TRANSFER

The *mass* of an object is the quantity of matter contained in the object. The *volume* of an object is the space it occupies. The *density*, ρ, is the mass per unit volume. Water is generally assumed to be incompressible in open-channel hydraulics, and the density is constant for incompressible fluids. The *mass transfer rate* or *mass flux* in open-channel flow is the rate with which the mass is transferred through a channel section. Recalling that Q = *discharge* is the volume transfer rate, we can write

$$\text{Rate of } \textit{mass transfer} = \rho Q \qquad (1.11)$$

1.4.2 MOMENTUM TRANSFER

Momentum or *linear momentum* is a property only moving objects have. An object of mass M moving with velocity V_M has a momentum equal to MV_M. In the absence of any external forces acting on the object in (or opposite to) the direction of the motion, the object will continue to move with the same velocity. From everyday life, we know that it is more difficult to stop objects that are moving faster or that are heavier (that is objects with higher momentum). Thus we can loosely define the *momentum* as a numerical measure of the tendency of a moving object to keep moving in the same manner.

The rate of mass transfer at any point in a channel section through an incremental area dA (as in Figure 1.3) is $\rho dQ = \rho v dA$, and therefore the momentum transfer rate is $\rho v^2 dA$. Integrating this over the area A, we obtain the momentum transfer rate through the section as

$$\text{Rate of momentum transfer} = \rho \int_A v^2 dA \qquad (1.12)$$

We often express the momentum transfer rate in terms of the average cross-sectional velocity, V, as

$$\text{Rate of momentum transfer} = \beta \rho V^2 A = \beta \rho Q V \qquad (1.13)$$

where β = *momentum coefficient* (or *momentum correction coefficient*) introduced to account for the non-uniform velocity distribution within the channel section.

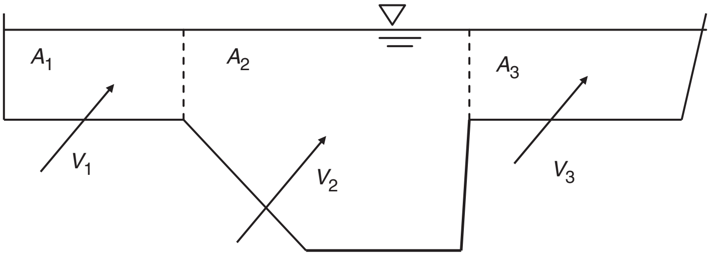


FIGURE 1.6
Compound channel

Then, from Equations 1.12 and 1.13, we obtain

$$\beta = \frac{\int_A v^2 dA}{V^2 A} \tag{1.14}$$

For regular channels β is often set equal to 1.0 for simplicity. For compound channels, as in Figure 1.6, it can be substantially higher. For a compound channel as in Figure 1.6, we can evaluate β by using

$$\beta = \frac{V_1^2 A_1 + V_2^2 A_2 + V_3^2 A_3}{V^2 A} \tag{1.15}$$

in which $A = A_1 + A_2 + A_3$ and V is obtained as

$$V = \frac{V_1 A_1 + V_2 A_2 + V_3 A_3}{A_1 + A_2 + A_3}.2 \tag{1.16}$$

Note that if $V_1 = V_2 = V_3$, Equation 1.15 yields $\beta = 1.0$.

1.4.3 ENERGY TRANSFER

Energy is generally defined as a measure of an object's capability to perform work. It can be in different forms. For open-channel flow problems, potential energy, kinetic energy, and internal energy are of interest. We will define the *total energy* as the sum of these three forms.

In the earth's gravitational field, every object has *potential energy*, or capability to perform work due to its position (elevation). The potential energy cannot be defined as an absolute quantity; it is defined as a relative quantity. For example, with respect to a horizontal datum (a reference elevation), the potential energy of an object of mass M is Mgz_C where g = gravitational acceleration and z_C = elevation of the center of mass of the object above the datum. In open channel flow, Q = rate of volume transfer, and ρQ = rate of mass transfer. Therefore, we can define the rate of potential energy transfer through a channel section as

$$\text{Rate of potential energy transfer} = \rho Q g z_C \tag{1.17}$$

where z_C = the elevation of the center of gravity or center of mass (the same as the centroid, since ρ is constant) of the channel section above the datum.

A moving object has the capability of performing work because of its motion. *Kinetic energy* is a measure of this capability. The kinetic energy of a mass M traveling with velocity V_M is defined as $M(V_M)^2/2$. In open-channel flow, we are concerned with the rate of kinetic energy transfer or the kinetic energy transfer through a channel section per unit time. The mass rate at any point in a channel section through an incremental area dA (as in Figure 1.3) is $\rho dQ = \rho v dA$. Therefore, the kinetic energy transfer per unit time through the incremental area is $\rho v^3 dA/2$. Integrating over the section area, and assuming ρ is constant for an incompressible fluid like water, we obtain

$$\text{Rate of kinetic energy transfer} = \frac{\rho}{2}\int_A v^3 dA \tag{1.18}$$

Note that in the above equation v stands for the point velocity, which varies over the channel section. In practice, we work with the average cross-sectional velocity, V. We define the rate of kinetic energy transfer in terms of the average cross-sectional velocity as

$$\text{Rate of kinetic energy transfer} = \alpha\frac{\rho}{2}V^3 A = \alpha\frac{\rho}{2}QV^2 \tag{1.19}$$

where α = *energy coefficient* (or *kinetic energy correction coefficient*) to account for the non-uniform point velocity distribution within a section. From Equations 1.18 and 1.19 we obtain

$$\alpha = \frac{\int_A v^3 dA}{V^3 A} \tag{1.20}$$

For regular channels, α is usually set equal to 1.0. However, in compound channels, like an overflooded river with a main channel and two overbank channels, α can be substantially higher. For the case for Figure 1.6, Equation 1.20 can be approximated using

$$\alpha = \frac{V_1^3 A_1 + V_2^3 A_2 + V_3^3 A_3}{V^3 A} \tag{1.21}$$

where $A = A_1 + A_2 + A_3$ and V is as defined by Equation 1.16. As expected, Equation 1.21 yields $\alpha = 1.0$ if $V_1 = V_2 = V_3$.

Internal energy results from the random motion of the molecules making up an object and the mutual attraction between these molecules. Denoting the internal energy per unit mass of water by e, the rate of internal energy transfer through an incremental area dA (as in Figure 1.3) is $\rho e v dA$. Integrating this over the area, and assuming e is distributed uniformly,

$$\text{Rate of internal energy transfer} = \rho e V A = \rho e Q \tag{1.22}$$

1.5 Open-Channel Flow Classification

Open-channel flow is classified in various ways. If time is used as the criterion, open-channel flow is classified into steady and unsteady flows. If, at a given flow section, the flow characteristics remain constant with respects to time, the flow is said to be *steady*. If flow characteristics change with time, the flow is said to be *unsteady*. If space is used as a criterion, flow is said to be *uniform* if flow characteristics remain constant along the channel. Otherwise the flow is said to be *non-uniform*. A non-uniform flow can be classified further into *gradually-varied* and *rapidly-varied* flows, depending on whether the variations along the channel are gradual or rapid. For example, the flow is gradually varied between Sections 1 and 2 and 2 and 3 in Figure 1.7. It is rapidly varied between 3 and 4 and uniform between 4 and 5. Usually, the pressure distribution can be assumed to be hydrostatic for uniform and gradually-varied flows.

Various types of forces acting on open-channel flow affect the hydraulic behavior of the flow. The Reynolds Number, R_e, defined as

$$R_e = \frac{4VR}{\nu} \tag{1.23}$$

where $\nu =$ kinematic viscosity of water, represents the ratio of inertial to viscous forces acting on the flow. At low Reynolds numbers, say $R_e < 500$, the flow region appears to consist of an orderly series of fluid laminae or layers conforming generally to the boundary configuration. This type of flow is called laminar flow. If we inject dye into a uniform laminar flow, the dye will flow along a straight line. Any disturbance introduced to laminar flow, due to irregular boundaries for instance, is eventually dampened by viscous forces. For $R_e > 12\,500$, the viscous forces are not sufficient to dampen the disturbances introduced to the flow. Minor disturbances are always present in moving water, and at high Reynolds numbers such disturbances will grow and spread throughout the entire zone of motion. Such flow is called turbulent, and water particles in turbulent flow follow irregular paths that are not continuous. A transitional state exists between the laminar and turbulent states. We should point out that the limits for the different states are by no means precise. Under laboratory conditions, for instance, laminar flow can be maintained for Reynolds numbers much higher than 500.

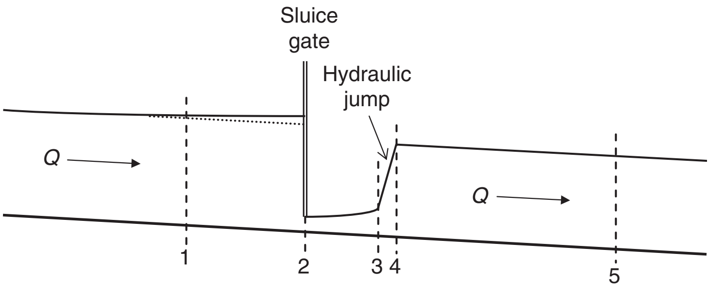


Figure 1.7 Various flow types

However, under most natural and practical open-channel flow conditions, the flow is turbulent.

The ratio of the inertial to gravitational forces acting on the flow is represented by the dimensionless *Froude number*, F_r, defined as

$$F_r = \frac{V}{\sqrt{gD}} \tag{1.24}$$

where g = gravitational acceleration. The flow is said to be at the *critical* state when $F_r = 1.0$. The flow is *subcritical* when $F_r < 1.0$, and it is *supercritical* when $F_r > 1.0$. The hydraulic behavior of open-channel flow varies significantly depending on whether the flow is critical, subcritical, or supercritical.

1.6 Conservation Laws

The laws of conservation of mass, momentum, and energy are the basic laws of physics, and they apply to open-channel flow. Rigorous treatment of the conservation laws for open-channel flow can be found in the literature (e.g. Yen, 1973). A simplified approach is presented herein.

1.6.1 CONSERVATION OF MASS

Consider a volume element of an open channel between an upstream section U and a downstream section D, as shown in Figure 1.8. The length of the element along the flow direction is Δx, and the average cross-sectional area is A. The mass of water present in the volume element is then $\rho A \Delta x$. Suppose water enters the volume element at section U at a mass transfer rate of ρQ_U (see Equation 1.11) and leaves the element at section D at a rate ρQ_D. Over a finite time increment, Δt, we can write that

$$\text{Rate of change of mass of water in the element} = \frac{\Delta(\rho A \Delta x)}{\Delta t}$$

$$\text{Net rate of mass transfer into element} = \rho Q_U - \rho Q_D$$

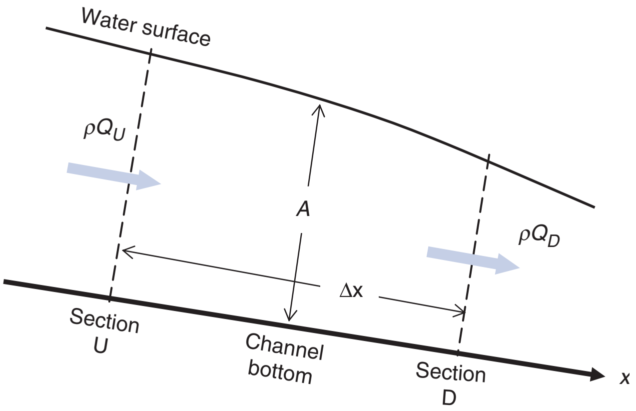


Figure 1.8 Definition sketch for conservation of mass principle

The principle of conservation of mass requires that

$$\text{(Rate of change of mass of water in the element)}$$
$$= \text{(Net rate of mass transfer into elelment)}$$

therefore

$$\frac{\Delta(\rho A \Delta x)}{\Delta t} = \rho Q_U - \rho Q_D \tag{1.25}$$

Water is considered to be an incompressible fluid, and therefore ρ is constant. Equation 1.25 can then be written as

$$\frac{\Delta A}{\Delta t} + \frac{Q_D - Q_U}{\Delta x} = 0 \tag{1.26}$$

For gradually-varied flow A and Q are continuous in space and time, and as Δx and Δt approach zero Equation 1.26 becomes

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{1.27}$$

where t = time, and x = displacement in the main flow direction. We usually refer to Equation 1.27 as the *continuity equation*.

1.6.2 CONSERVATION OF MOMENTUM

Momentum is a vector quantity, and separate equations are needed if there are flow components in more than one direction. However, open-channel flow is usually treated as being one-dimensional, and the momentum equation is written in the main flow direction. Consider a volume element of an open channel between an upstream section U and a downstream section D as shown in Figure 1.9. Let the element have an average cross-sectional area of A, flow

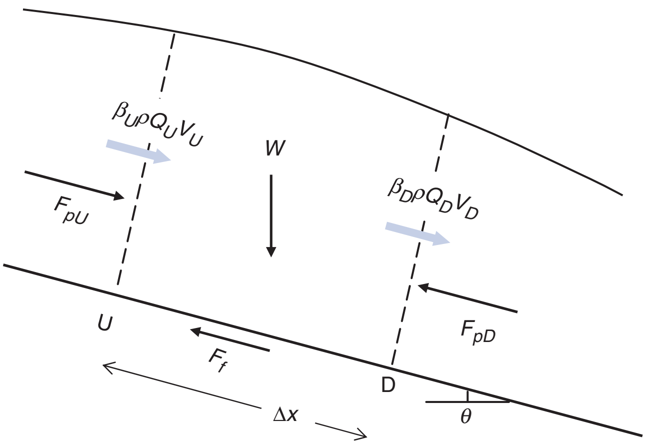


FIGURE 1.9 Definition sketch for conservation of momentum principle

velocity V, and length Δx. The momentum within this element is $\rho A \Delta x V$. The momentum is transferred into the element at section U at a rate $\beta_U \rho Q_U V_U$ (see Equation 1.13) and out of the element at section D at rate $\beta_D \rho Q_D V_D$. The external forces acting on this element in same direction as the flow are the pressure force at section U, $F_{pU} = \gamma Y_{CU} A_U$ (see Equation 1.9) and the weight component $W \sin\theta = \gamma A \Delta x \sin\theta$. The external forces acting opposite to the flow direction are the pressure force at section D, $F_{pD} = \gamma Y_{CD} A_D$, friction force on the channel bed, F_f, and any other external force, F_e, opposite to the flow direction (like a force exerted by the channel walls at a contracted section).

Therefore, we can write that

Time rate of change of the momentum accumulated within the element

$$= \Delta(\rho A \Delta x V)/\Delta t = \rho \Delta x (\Delta Q/\Delta t)$$

Net rate of momentum transfer into the element

$$= (\beta_U \rho Q_U V_U - \beta_D \rho Q_D V_D)$$

Sum of the external forces in the flow direction

$$= \gamma Y_{CU} A_U + \gamma A \Delta x \sin\theta - \gamma Y_{CD} A_D - F_f - F_e$$

The law of conservation of momentum requires that

(Time rate of change of the momentum accumulated within the element)

= (Net rate of momentum transfer into the element)

+ (Sum of the external forces in the flow direction)

Thus

$$\rho \Delta x (\Delta Q/\Delta t) = (\beta_U \rho Q_U V_U - \beta_D \rho Q_D V_D) + (\gamma Y_{CU} A_U - \gamma Y_{CD} A_D) + \gamma A \Delta x \sin\theta - F_f - F_e \quad (1.28)$$

Dividing both sides of the equation by $\rho \Delta x$, assuming $F_e = 0$, noting $S_0 =$ longitudinal channel bottom slope $= \sin\theta$, and introducing S_f = friction slope = boundary friction force per unit weight of water as

$$S_f = \frac{F_f}{\gamma A \Delta x} \quad (1.29)$$

we obtain

$$\frac{\Delta Q}{\Delta t} + \frac{(\beta_D Q_D V_D - \beta_U Q_U V_U)}{\Delta x} + \frac{g(Y_{CD} A_D - Y_{CU} A_U)}{\Delta x} + gAS_f - gAS_0 = 0 \quad (1.30)$$

For gradually-varied flow, all the flow variables are continuous in time and space. Therefore, as Δx and Δt approach zero, Equation 1.30 becomes

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(\beta Q V) + gA\frac{\partial y}{\partial x} + gAS_f - gAS_0 = 0 \quad (1.31)$$

Note that in arriving at Equation 1.31 from Equation 1.30 we have used

$$\frac{g(Y_{CD}A_D - Y_{CU}A_U)}{\Delta x} = g\frac{\partial(AY_C)}{\partial x} = gA\frac{\partial y}{\partial x} \tag{1.32}$$

as Δx approaches zero. This equality is not obvious. However, it can be proven mathematically using the Leibnitz rule if the changes in the channel width are negligible (see Problem P.1.15). A more rigorous analysis presented by Chow *et al*. (1988) demonstrates that Equation 1.32 is valid even if the changes in channel width are not negligible.

Noting that $Q = AV$, we can expand Equation 1.31 as

$$V\frac{\partial A}{\partial t} + A\frac{\partial V}{\partial t} + \beta Q\frac{\partial V}{\partial x} + \beta V\frac{\partial Q}{\partial x} + QV\frac{\partial \beta}{\partial x} + gA\frac{\partial y}{\partial x} + gAS_f - gAS_0 = 0 \tag{1.33}$$

or

$$V\left(\frac{\partial A}{\partial t} + \beta\frac{\partial Q}{\partial x}\right) + A\frac{\partial V}{\partial t} + \beta Q\frac{\partial V}{\partial x} + QV\frac{\partial \beta}{\partial x} + gA\frac{\partial y}{\partial x} + gAS_f - gAS_0 = 0 \tag{1.34}$$

For $\beta \approx 1$ and $\partial\beta/\partial x \approx 0$, substituting Equation 1.27 into 1.34, and dividing both sides by gA, we obtain

$$\frac{1}{g}\frac{\partial V}{\partial t} + \frac{V}{g}\frac{\partial V}{\partial x} + \frac{\partial y}{\partial x} + S_f - S_0 = 0 \tag{1.35}$$

1.6.3 CONSERVATION OF ENERGY

Consider a volume element of an open channel between an upstream section U and a downstream section D as shown in Figure 1.10. Let the element have an average cross-sectional area of A, flow velocity V, and length Δx. Suppose the elevation of the center of gravity of the element above a reference datum is z_C.

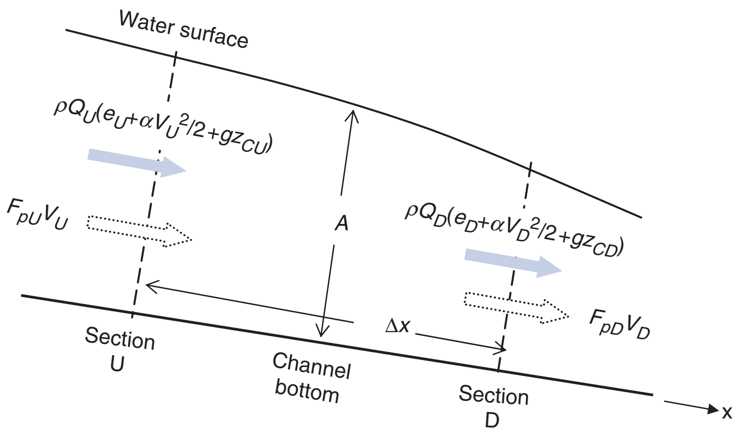


FIGURE 1.10 Definition sketch for conservation of energy principle

The total energy stored within this element is $[gz_C + (V^2/2) + e]\rho A\Delta x$. The energy is transferred into the element at section U at a rate $\rho Q_U[gz_{CU} + \alpha_U(V_U{}^2/2) + e_U]$ (see Equations 1.17, 1.19, and 1.22) and out of the element at section D at rate $\rho Q_D[gz_{CD} + \alpha_D(V_D{}^2/2) + e_D]$. The rate of work (or power) the surroundings perform on the volume element due to the hydrostatic pressure force at section U is $F_{pU}V_U$. The rate of work (or power) the volume element performs on the surroundings due the hydrostatic pressure force, which is opposing the flow at section D, is $F_{pD}V_D$. Referring to Equation 1.9, and noting $\gamma = \rho g$, we have $F_{pU}V_U = \rho g Y_{CU}A_U V_U$ and $F_{pD}V_D = \rho g Y_{CD}A_D V_D$.

Therefore, over a time increment Δt, we can write that

Time rate of change of total energy stored in the volume element

$$= \Delta\{(gz_C + (V^2/2) + e)\rho A\Delta x\}/\Delta t$$

Net rate of energy transfer into the element

$$= \rho Q_U\{gz_{CU} + \alpha_U(V_U^2/2) + e_U\} - \rho Q_D\{gz_{CD} + \alpha_D(V_D^2/2) + e_D\}$$

Net rate of energy added due to the work performed by the surroundings on the element $= \rho g Y_{CU}A_U V_U - \rho g Y_{CD}A_D V_D$

In the absence of energy added to the system due to external sources, the conservation of energy principle requires that

Time rate of change of total energy stored in the volume element
= Net rate of energy transfer into the element
+ Net rate of energy added due to the work performed by the surroundings on the element

Therefore

$$\frac{\Delta}{\Delta t}\left[\left(gz_C + \frac{V^2}{2} + e\right)\rho A\Delta x\right] = \rho\left[Q_U\left(gz_{CU} + \alpha_U\frac{V_U^2}{2} + e_U\right) - Q_D\left(gz_{CD} + \alpha_D\frac{V_D^2}{2} + e_D\right)\right] + \rho g(Q_U Y_{CU} - Q_D Y_{CD}) \quad (1.36)$$

Dividing both sides by Δx and rearranging gives

$$\frac{\Delta}{\Delta t}\left[\left(gz_C + \frac{V^2}{2} + e\right)\rho A\right] = \frac{\rho\left[Q_U\left(gz_{CU} + gY_{CU} + \alpha_U(V_U^2/2) + e_U\right) - Q_D\left(gz_{CD} + gY_{CD} + \alpha_D(V_D^2/2) + e_D\right)\right]}{\Delta x} \quad (1.37)$$

Let us define z_b = elevation of the channel bottom above the datum and recall that y = flow depth. Therefore, at any flow section $z_C + Y_C = z_b + y$. Then

$$\frac{\Delta}{\Delta t}\left[\left(e+\frac{V^2}{2}+gz_C\right)\rho A\right] = \frac{\rho\left[Q_U\left(gz_{bU}+gy_U+\alpha_U(V_U^2/2)+e_U\right)-Q_D\left(gz_{bD}+gy_D+\alpha_D(V_D^2/2)+e_D\right)\right]}{\Delta x} \tag{1.38}$$

As Δt and Δx approach zero Equation 1.38 becomes

$$\frac{\partial}{\partial t}\left[\left(e+\frac{V^2}{2}+gz_C\right)\rho A\right]+\frac{\partial}{\partial x}\left[\rho Q\left(gz_b+gy+\alpha\frac{V^2}{2}+e\right)\right]=0 \tag{1.39}$$

Now, substituting $z_C = z_b + y - Y_C$ we can write the first group of terms on the left side of Equation 1.39 as

$$\frac{\partial}{\partial t}\left[\left(e+\frac{V^2}{2}+gz_C\right)\rho A\right] = \rho\left[e\frac{\partial A}{\partial t}+A\frac{\partial e}{\partial t}\right]+\rho\left[\frac{V^2}{2}\frac{\partial A}{\partial t}+Q\frac{\partial V}{\partial t}\right]+\rho g\left[\frac{\partial(Az_b)}{\partial t}+\frac{\partial(Ay)}{\partial t}-\frac{\partial(AY_C)}{\partial t}\right] \tag{1.40}$$

By analogy to Equation 1.32,

$$-\frac{\partial(AY_C)}{\partial t} = -A\frac{\partial y}{\partial t} \tag{1.41}$$

Substituting Equation 1.41 into 1.40, noting that $\partial z_b/\partial t = 0$, and regrouping the terms:

$$\frac{\partial}{\partial t}\left[\left(e+\frac{V^2}{2}+gz_C\right)\rho A\right]=\rho\left(gz_b+gy+\frac{V^2}{2}+e\right)\frac{\partial A}{\partial t}+\rho A\frac{\partial e}{\partial t}+\rho Q\frac{\partial V}{\partial t} \tag{1.42}$$

Likewise,

$$\frac{\partial}{\partial x}\left[\rho Q\left(gz_b+gy+\alpha\frac{V^2}{2}+e\right)\right] = \rho\left(gz_b+gy+\alpha\frac{V^2}{2}+e\right)\frac{\partial Q}{\partial x}+\rho Q\frac{\partial}{\partial x}\left(gz_b+gy+\alpha\frac{V^2}{2}+e\right) \tag{1.43}$$

Substituting Equations 1.42 and 1.43 into 1.39 and assuming $\alpha = 1$,

$$\rho\left(gz_b+gy+\frac{V^2}{2}+e\right)\left(\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}\right) + \rho A\frac{\partial e}{\partial t}+\rho Q\frac{\partial V}{\partial t}+\rho Q\frac{\partial}{\partial x}\left(gz_b+gy+\frac{V^2}{2}+e\right)=0 \tag{1.44}$$

Substituting Equation 1.27 into 1.44 and dividing by ρQg, we obtain

$$\frac{1}{g}\frac{\partial V}{\partial t}+\frac{\partial}{\partial x}\left(z_b+y+\frac{V^2}{2g}\right)+\frac{1}{g}\left(\frac{1}{V}\frac{\partial e}{\partial t}+\frac{\partial e}{\partial x}\right)=0 \tag{1.45}$$

We will now define S_e = energy slope as

$$S_e = \frac{1}{g}\left(\frac{1}{V}\frac{\partial e}{\partial t} + \frac{\partial e}{\partial x}\right) = \frac{1}{g}\frac{de}{dx} \qquad (1.46)$$

Substituting Equation 1.46 into 1.45 and noting that $\partial z_b/\partial x = -S_0$,

$$\frac{1}{g}\frac{\partial V}{\partial t} + \frac{V}{g}\frac{\partial V}{\partial x} + \frac{\partial y}{\partial x} + S_e - S_0 = 0 \qquad (1.47)$$

If we recall that e = internal energy per unit mass of water, Equation 1.46 indicates that positive values of S_e represent an increase in the internal energy per unit weight of water per unit distance. However, because the total energy is conserved, this increase in the internal energy is accompanied by a decrease in the mechanical (potential and kinetic) energy. Because the mechanical energy is usable energy, any conversion of mechanical energy to internal energy is commonly viewed as 'energy loss', and the energy slope is defined as the energy loss per unit weight of water per unit distance. The procedure we adopted in this text to derive the energy equation does not explain how the mechanical energy is converted to internal energy. Another approach, based on the integration of the Navier-Stokes equations presented by Strelkoff (1969) and Yen (1973), clearly demonstrates that the losses in the mechanical energy are due to the work done by the internal stresses to overcome the velocity gradients. Turbulent exchange of molecules between different velocity zones sets up an internal friction force between adjacent layers since slow-moving molecules entering a higher-velocity layer will drag the faster-moving molecules. The energy dissipated to overcome these internal friction forces in the form of heat will increase the internal energy while causing a reduction in the mechanical energy.

Although Equation 1.47 appears very similar to Equation 1.35, the two equations are fundamentally different. Momentum is a vector quantity and energy is a scalar quantity. The two equations look similar because they are both for one-dimensional flow. If we had flow components in, say, three directions, we would have three different momentum equations, while the energy approach would still yield a single equation. We assumed that $\beta = 1$ when we derived Equation 1.35 and $\alpha = 1$ for Equation 1.47. These two correction factors are actually different. The friction slope, S_f, appearing in Equation 1.35 corresponds to the (external) boundary friction forces, while the energy slope, S_e, in Equation 1.47 is related to the work done by the internal friction forces. Nevertheless, in most applications we do not differentiate between S_f and S_e and use the term *friction slope* for either.

1.6.4 STEADY FLOW EQUATIONS

The flow is said to be *steady* if the flow conditions do not vary in time. Therefore, the partial derivative terms with respect to time can be dropped from the continuity, momentum, and energy equations. As a result, we obtain

$$\frac{dQ}{dx} = 0 \tag{1.48}$$

$$\frac{V}{g}\frac{dV}{dx} + \frac{dy}{dx} + S_f - S_0 = 0 \tag{1.49}$$

and

$$\frac{V}{g}\frac{dV}{dx} + \frac{dy}{dx} + S_e - S_0 = 0 \tag{1.50}$$

Equation 1.48 shows that, under steady state conditions, the discharge is the same at any channel section. Also, Equations 1.49 and 1.50 can be rearranged to obtain

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - F_r^2} \tag{1.51}$$

$$\frac{dy}{dx} = \frac{S_0 - S_e}{1 - F_r^2} \tag{1.52}$$

For the volume element shown in Figure 1.9, Equation 1.28 can be written for steady state conditions as

$$\left(\beta_D \frac{Q_D^2}{gA_D} + Y_{CD}A_D\right) = \left(\beta_U \frac{Q_U^2}{gA_U} + Y_{CU}A_U\right) - \frac{F_f}{\gamma} - \frac{F_e}{\gamma} + \Delta x S_0 \frac{A_D + A_U}{2} \tag{1.53}$$

Equation 1.53 is valid regardless of whether the flow between the sections U and D is gradually or rapidly varied, as long as the pressure distribution is hydrostatic at sections U and D.

Likewise, we can obtain the steady state energy equation by discretizing Equation 1.50, reintroducing the energy coefficient α, defining h_f = head loss = energy loss per unit weight = $(\Delta x)S_e$, and rearranging the terms

$$\left(z_{bU} + y_U + \alpha_U \frac{V_U^2}{2g}\right) = \left(z_{bD} + y_D + \alpha_D \frac{V_D^2}{2g}\right) + \Delta x S_e \tag{1.54}$$

1.6.5 STEADY SPATIALLY-VARIED FLOW EQUATIONS

Flow in an open channel is said to be *spatially varied* if there is lateral flow into (or out of) the channel, as shown schematically in Figure 1.11. For steady spatially-varied flow, the continuity equation becomes

$$\frac{dQ}{dx} = q_L \tag{1.55}$$

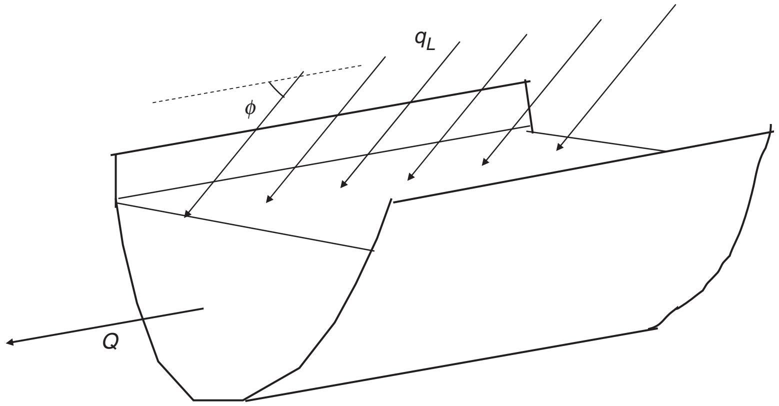


FIGURE 1.11 Definition sketch for spatially-varied flow

where q_L = lateral inflow rate per unit length of the channel. Note that the dimension of q_L is {length}^{2}/{time}.

As demonstrated by Yen and Wenzel (1970), for $\beta = 1$, the momentum equation for steady spatially-varied flow can be written as

$$\frac{dy}{dx} = \frac{S_0 - S_f - (q_L/gA)(2V - U_L \cos\phi)}{1 - F_r^2} \tag{1.56}$$

where U_L = velocity of lateral flow, and ϕ = angle between the lateral flow and channel flow directions. If lateral flow joins (or leaves) the channel in a direction perpendicular to the main flow direction, the equation becomes

$$\frac{dy}{dx} = \frac{S_0 - S_f - (2q_L V/gA)}{1 - F_r^2} \tag{1.57}$$

Yen and Wenzel (1970) also demonstrated that, for $\alpha = 1$, the energy equation can be written as

$$\frac{dy}{dx} = \frac{S_0 - S_e + (q_L/VA)\left((U_L^2/2g) - (3V^2/2g) + h_{LAT} - h\right)}{1 - F_r^2} \tag{1.58}$$

where $h = z_b + y$ = piezometric head of the main channel flow, and h_{LAT} = piezometric head of the lateral inflow. If $h = h_{LAT}$, and $V = U_L$, Equation 1.58 is simplified to obtain

$$\frac{dy}{dx} = \frac{S_0 - S_e - (q_L V/gA)}{1 - F_r^2} \tag{1.59}$$

Note that the third term in the numerator of the right side of Equation 1.59 is different from that of Equation 1.57 by a factor of 2.0. This discrepancy is due to the different assumptions involved in the two equations.

1.6.6 COMPARISON AND USE OF MOMENTUM AND ENERGY EQUATIONS

It should be clear to the reader by now that the momentum and the energy equations are obtained by using different laws of physics. Also, the friction slope, S_f, and the energy slope, S_e, appearing in these equations are fundamentally different. However, it is not practical to evaluate either S_f or S_e on the basis of their strict definitions. In practice, we employ the same empirical equations to evaluate S_f and S_e. Therefore, S_e in the energy equation is often replaced by S_f. If we also assume that $\alpha = 1$ and $\beta = 1$, then, for gradually-varied flow, the momentum and energy equation become identical (Equations 1.35 and 1.47 for unsteady flow and 1.49 and 1.50 for steady flow).

For spatially-varied flow, however, the momentum and the energy equations are different even if we assume $S_e = S_f$ and $\alpha = \beta = 1.0$. We can use the momentum equation, Equation 1.56, only if we know the direction of the lateral flow. If the lateral inflow joins a channel at an angle close to 90°, as in most natural and manmade systems, the use of Equation 1.57 is appropriate. The direction of the lateral flow is irrelevant in the energy equation, since energy is a scalar quantity. However, where lateral flow joins a main channel, some energy loss occurs due to the local mixing. This loss is not accounted for in Equation 1.59, so Equation 1.59 should not be used for lateral inflow situations. In cases involving lateral outflow, on the other hand, the assumptions of Equation 1.59 are satisfied for the most part, and the use of Equation 1.59 is allowed.

The open-channel flow is not always gradually varied. Rapid changes in the flow variables can occur near channel transitions or hydraulic structures. The momentum and energy equations given for a volume element, Equations 1.53 and 1.54, are still valid as long as the pressure distribution at sections U and D is hydrostatic. However, the term $S_e \Delta x$ in Equation 1.54 needs to be replaced by h_L = head loss, which would account for all the energy losses between the two sections. Considering Equations 1.53 and 1.54 only the former includes an external force term. Therefore, if the problem at hand involves calculation of an external force (like the force exerted by a sluice gate on the flow), the momentum equation is the only choice. The energy equation is particularly useful in situations where the energy loss between the upstream and downstream sections is negligible.

PROBLEMS

P.1.1 Derive expressions for the flow area, A, wetted perimeter, P, top width, T, hydraulic radius, R, and hydraulic depth, D, in terms of the flow depth, y, for the channel sections shown in Figure P.1.1.

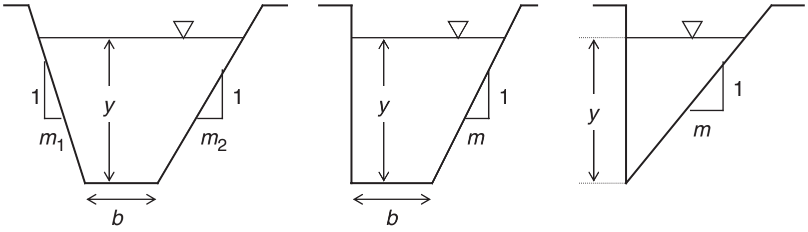


FIGURE P.1.1
Problem P.1.1

P.1.2 A nearly horizontal channel has a bottom width of 3 ft, and it carries a discharge of 60 cfs at a depth of 4 ft. Determine the magnitude and direction of the hydrostatic pressure force exerted on each of the sidewalls per unit length of the channel if

(a) the channel is rectangular with vertical sidewalls
(b) the channel is trapezoidal with each sidewall sloping outward at a slope 2 horizontal over 1 vertical, that is $m = 2$.

P.1.3 Let the point velocity in a wide rectangular channel be expressed as

$$v = 2.5 v_* \ln\left(\frac{30z}{k_s}\right)$$

where v = point velocity, $v_* = (\tau_o/\rho)^{1/2}$ = shear velocity, τ_o = average shear stress on channel bed, ρ = density, z = distance measured from channel bed, and k_s = length measure of bed roughness. The flow depth in the channel is y. Treating v_* and k_s as constants, derive an expression for the discharge per unit of the width of the channel.

Hint 1: $v = 0$ at $z = k_s/30$

Hint 2: $\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$.

P.1.4 Derive an expression for the average cross-sectional velocity, V, for the velocity distribution given in problem P.1.3.

Hint: $y \gg k_s$

P.1.5 At what z in Problem P.1.3 is the velocity maximum? Derive an expression for v_{max}.

P.1.6 For the channel of Problem P.1.3, show that

$$\frac{v_{max}}{V} - 1 = \frac{1}{\ln(30y/k_s - 1)}$$

P.1.7 For the velocity distribution given in Problem P.1.3, determine at what z the point velocity is equal to the average cross-sectional velocity. Often, a single velocity measurement taken at distance 0.6y from the free surface is used as an approximation to the cross-sectional velocity at a stream section. Is this a valid approximation?

P.1.8 Using the velocity distribution and the hints given in Problem P.1.3, show that

$$\beta = 1 + \left(\frac{V_{max}}{V} - 1\right)^2$$

P.1.9 Considering a unit width of the channel described in Problem P.1.3, determine the discharge, rate of momentum transfer, and rate of kinetic energy transfer if $y = 0.94$ m, $k_s = 0.001$ m, $\tau_o = 3.7$ N/m^2 and $\rho = 1000$ kg/m^3.

P.1.10 Determine the average cross-sectional velocity V and the discharge Q for the compound channel shown in Figure P.1.2.

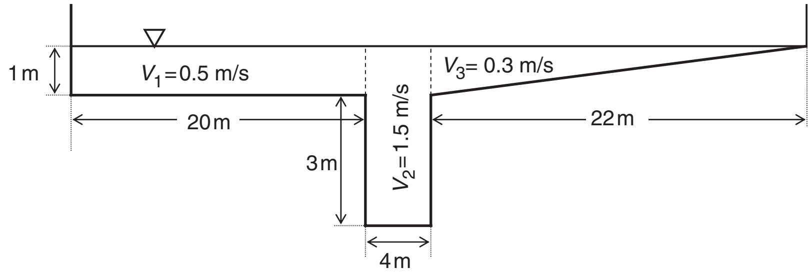


FIGURE P.1.2 Problems P.1.10 and P.1.11

P.1.11 Determine the rate of momentum transfer and the rate of kinetic energy transfer for the compound channel shown in Figure 1.P.2.

P.1.12 A trapezoidal channel with bottom width $b = 5$ ft and side slopes $m = 2$ (that is 2.0 horizontal over 1.0 vertical) carries $Q = 100$ cfs at depth $y = 3.15$. The water temperature is 60°F, and the kinematic viscosity at this temperature is $v = 1.217 \times 10^{-5}$ ft^2/s.

(a) Determine if the flow is turbulent or laminar.
(b) Determine if the flow is subcritical or supercritical.

P.1.13 Is the flow likely to be uniform or non-uniform:

(a) at a natural stream section partially blocked by a fallen tree?
(b) at a drainage channel just upstream of an undersized culvert?
(c) at a section of a long prismatic, delivery channel a far distance from upstream and downstream ends?
(d) in a tidal river during high tide?

P.1.14 Is the flow likely to be steady or unsteady:

(a) in a street gutter during a short storm event?
(b) in a laboratory flume fed constant discharge at upstream end?
(c) in a drainage ditch after a long dry period?

P.1.15 Using the Leibnitz rule given below, verify Equation 1.32.

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(\eta,x)d\eta = \int_{a(x)}^{b(x)} \frac{\partial f(\eta,x)}{\partial x}d\eta + f[b(x),x]\frac{\partial b}{\partial x} - f[a(x),x]\frac{\partial a}{\partial x}$$

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