**1.138J/2.062J/18.376J, WAVE PROPAGATION**

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**CHAPTER FOUR. WAVES IN WATER**

# 1 Governing equations for waves on the sea surface

In this chapter we shall model the water as an inviscid and incompressible fluid, and consider waves of infinitesimal amplitude so that the linearized approximation suffices.

Recall in the first chapter that when compressibility is included the velocity potential defined by $\mathbf{u} = \nabla\Phi$ is governed by the wave equation:

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \tag{1.1}$$

where $c = \sqrt{dp/d\rho}$ is the speed of sound. Consider the ratio

$$\frac{\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}}{\nabla^2\Phi} \sim \frac{\omega^2/k^2}{c^2}$$

As will be shown later, the phase speed of the fastest wave is $\omega/k = \sqrt{gh}$ where $g$ is the gravitational acceleration and $h$ the sea depth. Now $h$ is at most 4000 m in the ocean, and the sound speed in water is $c = 1400$ m/sec$^2$, so that the ratio above is at most

$$\frac{40000}{1400^2} = \frac{1}{49} \ll 1$$

We therefore approximate (1.1) by

$$\nabla^2\Phi = 0 \tag{1.2}$$

Let the free surface be $z = \zeta(x, y, t)$. Then for a gently sloping free surface the vertical velocity of the fluid on the free surface must be equal to the vertical velocity of the surface itself. i.e.,

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\Phi}{\partial z}, \quad z = 0. \tag{1.3}$$

Having to do with the velocity only, this is called the kinematic condition.

For small amplitude motion, the linearized momentum equation reads

$$\rho\frac{\partial \mathbf{u}}{\partial t} = -\nabla P - \rho g \mathbf{e}_z \tag{1.4}$$

Now let the total pressure be split into static and dynamic parts

$$P = p_o + p \tag{1.5}$$

where $p_o$ is the static pressure

$$p_o = -\rho g z \tag{1.6}$$

which satisfies

$$0 = -\nabla p_o + -\rho g \mathbf{e}_z \tag{1.7}$$

It follows that

$$\rho\frac{\partial \mathbf{u}}{\partial t} = \rho\frac{\partial \nabla\Phi}{\partial t} = -\nabla p \tag{1.8}$$

so that

$$p = -\rho\frac{\partial \Phi}{\partial t} \tag{1.9}$$

which relates the dynamic pressure to the velocity potential.

Let us assume that the air above the sea surface is essentially stagnant. Because of its very small density we ignore the dynamic effect of air and assume the air pressure to be constant, which can be taken to be zero without loss of generality. If surface tension is ignored, continuity of pressure requires that

$$p = p_o + p = 0, \quad z = \zeta.$$

to the leading order of approximation, we have, therefore

$$\rho g\zeta + \rho\frac{\partial \Phi}{\partial t} = 0, \quad z = 0. \tag{1.10}$$

Being a statement on forces, this is called the dynamic boundary condition. The two conditions (1.3) and (1.10) can be combined to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0, \quad z = 0 \tag{1.11}$$

If surface tension is also included then we adopt the model where there is a thin film covering the water surface with tension $T$ per unit length. Consider a horizontal rectangle $dxdy$ on the free surface. The net vertical force from four sides is

$$\left(T\frac{\partial\zeta}{\partial x}\bigg|_{x+dx} - T\,\frac{\partial\zeta}{\partial x}\bigg|_{x}\right)dy + \left(T\,\frac{\partial\zeta}{\partial y}\bigg|_{y+dy} - T\,\frac{\partial\zeta}{\partial x}\bigg|_{y}\right)dx = T\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right)dx\,dy$$

Continuity of vertical force on an unit area of the surface requires

$$p_o + p + T\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right) = 0.$$

Hence

$$-\rho g\zeta - \rho\frac{\partial\Phi}{\partial t} + T\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right) = 0, \quad z = 0. \tag{1.12}$$

which can be combined with the kinematic condition (1.3) to give

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} - \frac{T}{\rho}\nabla^2\frac{\partial\Phi}{\partial\partial z} = 0, \quad z = 0 \tag{1.13}$$

When viscosity is neglected, the normal fluid velocity vanishes on the rigid seabed,

$$\mathbf{n}\cdot\nabla\mathbf{\Phi} = \mathbf{0} \tag{1.14}$$

Let the sea bed be $z = -h(x,y)$ then the unit normal is

$$\mathbf{n} = \frac{(\mathbf{h_x}, .\mathbf{h_y}, \mathbf{1})}{\sqrt{\mathbf{1} + \mathbf{h_x^2} + \mathbf{h_y^2}}} \tag{1.15}$$

Hence

$$\frac{\partial\Phi}{\partial z} = -\frac{\partial h}{\partial x}\frac{\partial\Phi}{\partial x} - \frac{\partial h}{\partial y}\frac{\partial\Phi}{\partial y}, \qquad z = -h(x,y) \tag{1.16}$$

# 2 Progressive waves on a sea of constant depth

## 2.1 The velocity potential

Consider the simplest case of constant depth and sinusoidal waves with infinitively long crests parallel to the $y$ axis. The motion is in the vertical plane $(x,z)$. Let us seek a solution representing a wavetrain advancing along the $x$ direction with frequency $\omega$ and wave number $k$,

$$\Phi = f(z)e^{ikx-i\omega t} \tag{2.1}$$

In order to satisfy (1.2), (1.13) and (1.16) we need

$$f'' + k^2 f = 0, \quad -h < z < 0 \tag{2.2}$$

$$-\omega^2 f + g f' + \frac{T}{\rho} k^2 f' = 0, \quad z = 0, \tag{2.3}$$

$$f' = 0, \quad z = -h \tag{2.4}$$

Clearly solution to (2.2) and (2.4) is

$$f(z) = B \cosh k(z+h)$$

implying

$$\Phi = B \cosh k(z+h) e^{ikx - i\omega t} \tag{2.5}$$

In order to satisfy (2.3) we require

$$\omega^2 = \left(gk + \frac{T}{\rho} k^3\right) \tanh kh \tag{2.6}$$

which is the dispersion relation between $\omega$ and $k$. From (1.3) we get

$$\frac{\partial \zeta}{\partial t} = \left.\frac{\partial \Phi}{\partial z}\right|_{z=0} = (Bk \sinh kh) e^{ikx - i\omega t} \tag{2.7}$$

Upon integration,

$$\zeta = A e^{ikx - i\omega t} = \frac{Bk \sinh kh}{-i\omega} e^{ikx - i\omega t} \tag{2.8}$$

where $A$ denotes the surface wave amplitude, it follows that

$$B = \frac{-i\omega A}{k \sinh kh}$$

and

$$\begin{aligned} \Phi &= \frac{-i\omega A}{k \sinh kh} \cosh k(z+h) e^{ikx - i\omega t} \\ &= \frac{-igA}{\omega} \left(1 + \frac{Tk^2}{g\rho}\right) \frac{\cosh k(z+h)}{\cosh kh} e^{ikx - i\omega t} \end{aligned} \tag{2.9}$$

## 2.2 The dispersion relation

Let us first examine the dispersion relation (2.6), where three lengths are present : the depth $h$, the wavelength $\lambda = 2\pi/k$, and the length $\lambda_m = 2\pi/k_m$ with

$$k_m = \sqrt{\frac{g\rho}{T}}, \quad \lambda_m = \frac{2\pi}{k_m} = 2\pi\sqrt{\frac{T}{g\rho}} \tag{2.10}$$

For reference we note that on the air-water interface, $T/\rho = 74\, cm^3/s^2, g = 980\, cm/s^2$, so that $\lambda_m = 1.73cm$. The depth of oceanographic interest ranges from O(10cm) to thousand of meters. The wavelength ranges from a few centimeters to hundreds of meters.

Let us introduce

$$\omega_m^2 = 2gk_m = 2g\sqrt{\frac{g\rho}{T}} \tag{2.11}$$

then (2.6) is normalized to

$$\frac{\omega^2}{\omega_m^2} = \frac{1}{2}\frac{k}{k_m}\left(1 + \frac{k^2}{k_m^2}\right)\tanh kh \tag{2.12}$$

Consider first waves of length of the order of $\lambda_m$. For depths of oceanographic interest, $h \gg \lambda$, or $kh \gg 1$, $\tanh kh \approx 1$. Hence

$$\frac{\omega^2}{\omega_m^2} = \frac{1}{2}\frac{k}{k_m}\left(1 + \frac{k^2}{k_m^2}\right) \tag{2.13}$$

or, in dimensional form,

$$\omega^2 = gk + \frac{Tk^3}{\rho} \tag{2.14}$$

The phase velocity is

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\left(1 + \frac{Tk^2}{g\rho}\right)} \tag{2.15}$$

Defining

$$c_m = \frac{\omega_m}{k_m} \tag{2.16}$$

the preceding equation takes the normalized form

$$\frac{c}{c_m} = \sqrt{\frac{1}{2}\left(\frac{k_m}{k} + \frac{k}{k_m}\right)} \tag{2.17}$$

Clearly

$$c \approx \sqrt{\frac{Tk}{\rho}}, \quad \text{if } k/k_m \gg 1, \text{ or } \lambda/\lambda_m \ll 1 \tag{2.18}$$

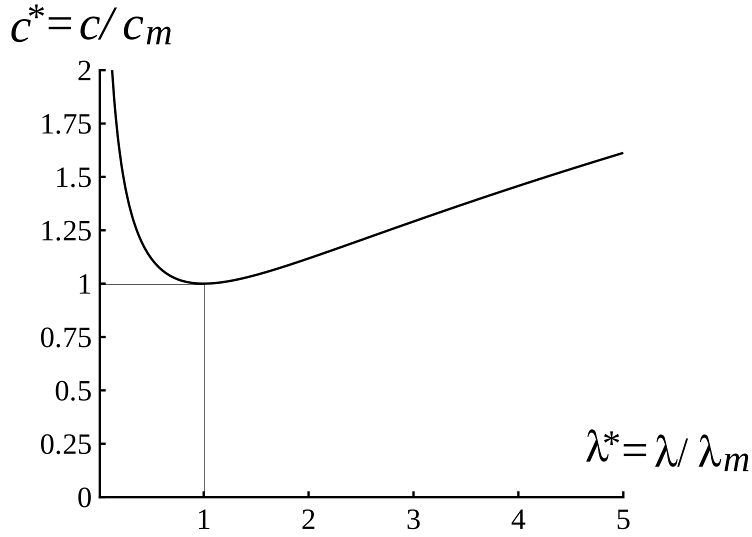


Figure 1: Phase speed of capillary-gravity waves in water much deeper than $\lambda_m$.

Thus for wavelengths much shorter than 1.7 cm, capillarity alone is important, These are called the capillary waves. On the other hand

$$c \approx \sqrt{\frac{g}{k}}, \quad \text{if} \quad k/k_m \ll 1, \quad \text{or} \quad \lambda/\lambda_m \ll 1 \tag{2.19}$$

Thus for wavelength much longer than 1.73 cm, gravity alone is important; these are called the gravity waves. Since in both limits, $c$ becomes large, there must be a minimum for some intermediate $k$. From

$$\frac{dc^2}{dk} = -\frac{g}{k^2} + \frac{T}{\rho} = 0$$

the minimum $c$ occurs when

$$k = \sqrt{\frac{g\rho}{T}} = k_m, \quad \text{or } \lambda = \lambda_m \tag{2.20}$$

The smallest value of $c$ is $c_m$. For the intermediate range where both capillarity and gravity are of comparable importance; the dispersion relation is plotted in figure (1).

Next we consider longer gravity waves where the depth effects are essential.

$$\omega = \sqrt{gk \tanh kh} \tag{2.21}$$

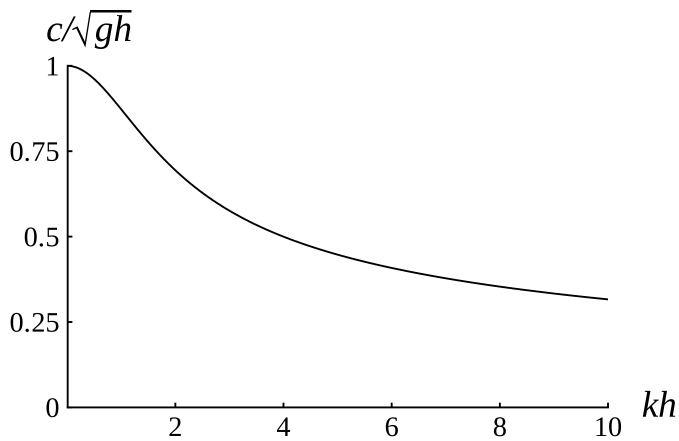


Figure 2: Phase speed of capillary-gravity waves in water of constant depth

For gravity waves on deep water, $kh \gg 1$, $\tanh kh \to 1$. Hence

$$\omega \approx \sqrt{gk}, \quad c \approx \sqrt{\frac{g}{k}} \tag{2.22}$$

Thus longer waves travel faster. These are also called short gravity waves. If however the waves are very long or the depth very small so that $kh \ll 1$, then $\tanh kh \sim kh$ and

$$\omega \approx k\sqrt{gh}, \quad c \approx \sqrt{gh} \tag{2.23}$$

Form intermediate values of $kh$, the phase speed decreases monotonically with increasing $kh$. All long waves with $kh \ll 1$ travel at the same maximum speed limited by the depth, $\sqrt{gh}$, hence there are non-dispersive. The dispersion relation is plotted in figure (**??**).

## 2.3 The flow field

For arbitrary $k/k_m$ and $kh$, the velocities and dynamic pressure are easily found from the potential (2.9) as follows

$$u = \frac{\partial \Phi}{\partial x} = \frac{gkA}{\omega}\left(1 + \frac{Tk^2}{g\rho}\right)\frac{\cosh k(z+h)}{\cosh kh} e^{ikx - i\omega t} \tag{2.24}$$

$$w = \frac{\partial \Phi}{\partial z} = \frac{-igkA}{\omega}\left(1 + \frac{Tk^2}{g\rho}\right)\frac{\sinh k(z+h)}{\cosh kh} e^{ikx - i\omega t} \tag{2.25}$$

$$p = -\rho\frac{\partial \Phi}{\partial t} = \rho g A\left(1 + \frac{Tk^2}{g\rho}\right)\frac{\cosh k(z+h)}{\cosh kh} e^{ikx - i\omega t} \tag{2.26}$$

Note that all these quantities decay monotonically in depth.

In deep water, $kh \gg 1$,

$$u = \frac{gkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)e^{kz}e^{ikx-i\omega t} \quad (2.27)$$

$$w = \frac{\partial\Phi}{\partial z} = \frac{-igkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)e^{kz}e^{ikx-i\omega t} \quad (2.28)$$

$$p = -\rho\frac{\partial\Phi}{\partial t} = \rho gA\left(1+\frac{Tk^2}{g\rho}\right)e^{kz}e^{ikx-i\omega t} \quad (2.29)$$

All dynamical quantities diminish exponentially to zero as $kz \to -\infty$. Thus the fluid motion is limited to the surface layer of depth $O(\lambda)$. Gravity and capillary-gravity waves are therefore surface waves.

For pure gravity waves in shallow water, $T = 0$ and $kh \ll 1$, we get

$$u = \frac{gkA}{\omega}e^{ikx-i\omega t} \quad (2.30)$$

$$w = 0, \quad (2.31)$$

$$p = -\rho\frac{\partial\Phi}{\partial t} = \rho gAe^{ikx-i\omega t} = \rho g\zeta \quad (2.32)$$

Note that the horizontal velocity is uniform in depth while the vertical velocity is negligible. Thus the fluid motion is essentially horizontal. The total pressure

$$P = p_o + p = \rho g(\zeta - z) \quad (2.33)$$

is hydrostatic and increases linearly with depth from the free surface.

## 2.4 The particle orbit

In fluid mechanics there are two ways of describing fluid motion. In the Lagrangian scheme, one follows the trajectory $x, z$ of all fluid particles as functions of time. Each fluid particle is identified by its static or initial position $x_o, z_o$. Therefore the instantaneous position at time $t$ depends parametrically on $x_o, z_o$. In the Eulerian scheme, the fluid motion at any instant $t$ is described by the velocity field at all fixed positions $x, z$. As the fluid moves, the point $x, z$ is occupied by different fluid particles at different times. At a particular time $t$, a fluid particle originally at $(x_o, z_o)$ arrives at $x, z$, hence its particle velocity must coincide with the fluid velocity there,

$$\frac{dx}{dt} = u(x, z, t), \quad \frac{dz}{dt} = w(x, z, t) \quad (2.34)$$

Once $u, w$ are known for all $x, z, t$, we can in principle integrate the above equations to get the particle trajectory. This Euler-Lagrange problem is in general very difficult.

In small amplitude waves, the fluid particle oscillates about its mean or initial position by a small distance. Integration of (2.34) is relatively easy. Let

$$x(x_o, z_o, t) = x_o + x'(x_o, z_o, t), \quad \text{and} z(x_o, z_o, t) = z_o + x'(x_o, z_o, t) \tag{2.35}$$

then $x' \ll x, z' \ll z$ in general. Equation (2.34) can be approximated by

$$\frac{dx'}{dt} = u(x_o, z_o, t), \quad \frac{dz'}{dt} = w(x_o, z_o, t) \tag{2.36}$$

From (2.24) and (2.25), we get by integration,

$$\begin{aligned} x' &= \frac{gkA}{-i\omega^2}\left(1 + \frac{Tk^2}{g\rho}\right)\frac{\cosh k(z_o + h)}{\cosh kh} e^{ikx_o - i\omega t} \\ &= -\frac{gkA}{\omega^2}\left(1 + \frac{Tk^2}{g\rho}\right)\frac{\cosh k(z_o + h)}{\cosh kh} \sin(kx_o - \omega t) && (2.37) \\ & && (2.38) \\ z' &= \frac{gkA}{\omega^2}\left(1 + \frac{Tk^2}{g\rho}\right)\frac{\sinh k(z_o + h)}{\cosh kh} e^{ikx_o - i\omega t} \\ &= \frac{gkA}{\omega^2}\left(1 + \frac{Tk^2}{g\rho}\right)\frac{\sinh k(z_o + h)}{\cosh kh} \cos(kx_o - \omega t) && (2.39) \\ & && (2.40) \end{aligned}$$

Letting

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{gkA}{\omega^2 \cosh kh}\left(1 + \frac{Tk^2}{g\rho}\right)\begin{pmatrix} \cosh k(z_o + h) \\ \sinh k(z_o + h) \end{pmatrix} \tag{2.41}$$

we get

$$\frac{x'^2}{a^2} + \frac{z'^2}{b^2} = 1 \tag{2.42}$$

The particle trajectory at any depth is an ellipse. Both horizontal (major) and vertical (minor) axes of the ellipse decrease monotonically in depth. The minor axis diminishes to zero at the seabed, hence the ellipse collapses to a horizontal line segment. In deep water, the major and minor axes are equal

$$a = b = \frac{gkA}{\omega^2}\left(1 + \frac{Tk^2}{g\rho}\right)e^{kz_o}, \tag{2.43}$$

therefore the orbits are circles with the radius diminishing exponentially with depth.

Also we can rewrite the trajectory as

$$x' = \frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z_o+h)}{\cosh kh}\sin(\omega t - kx_o) \tag{2.44}$$

$$z' = \frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\sinh k(z_o+h)}{\cosh kh}\sin(\omega t - kx_o - \frac{\pi}{2}) \tag{2.45}$$

When $\omega t - kx_o = 0$, $x' = 0$ and $z' = b$. A quarter period later, $\omega t - k_o = \pi/2$, $x' = a$ and $z' = 0$. Hence as time passes, the particle traces the elliptical orbit in the clockwise direction.

## 2.5 Energy and Energy transport

Beneath a unit length of the free surface, the time-averaged kinetic energy density is

$$\bar{E}_k = \frac{\rho}{2}\int_{-h}^{0} dz\left(\overline{u^2} + \overline{w^2}\right) \tag{2.46}$$

whereas the instantaneous potential energy density is

$$E_p = \frac{1}{2}\rho g\zeta^2 + T\frac{(ds-dx)}{dx} = \frac{1}{2}\rho g\zeta^2 + T\left(\sqrt{1+\zeta_x^2}-1\right) = \frac{1}{2}\rho g\zeta^2 + T\zeta_x^2 \tag{2.47}$$

Hence the time-average is

$$\bar{E}_p = \frac{1}{2}\rho g\overline{\zeta^2} + \frac{T}{2}\overline{\zeta_x^2} \tag{2.48}$$

Let us rewrite (2.24) and (2.25) in (2.48):

$$u = \Re\left\{\frac{gkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z+h)}{\cosh kh}e^{ikx}\right\}e^{-i\omega t} \tag{2.49}$$

$$w = \Re\left\{\frac{-igkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\sinh k(z+h)}{\cosh kh}e^{ikx}\right\}e^{-i\omega t} \tag{2.50}$$

Then

$$\begin{aligned}
\bar{E}_k &= \frac{\rho}{4}\left(\frac{gkA}{\omega}\right)^2\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{1}{\cosh^2 kh}\int_{-h}^{0} dz\left[\cosh^2 k(z+h) + \sinh^2 k(z+h)\right] \\
&= \frac{\rho}{4}\left(\frac{gkA}{\omega}\right)^2\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{\sinh 2kh}{2k\cosh^2 kh} = \frac{\rho}{4}\left(\frac{gkA}{\omega}\right)^2\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{\sinh kh}{k\cosh kh} \\
&= \frac{\rho g A^2}{4}\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{gk\tanh kh}{\omega^2} = \frac{\rho g A^2}{4}\left(1+\frac{Tk^2}{g\rho}\right)
\end{aligned} \tag{2.51}$$

after using the dispersion relation. On the other hand,

$$\bar{E}_p = \frac{\rho g A^2}{4}\left(1 + \frac{Tk^2}{\rho g}\right) \tag{2.52}$$

Hence the total energy density is

$$\bar{E} = \bar{E}_k + \bar{E}_p = \frac{\rho g A^2}{2}\left(1 + \frac{Tk^2}{\rho g}\right) = \frac{\rho g A^2}{2}\left(1 + \frac{k^2}{k_m^2}\right) = \frac{\rho g A^2}{2}\left(1 + \frac{\lambda_m^2}{\lambda^2}\right) \tag{2.53}$$

Note that the total energy is equally divided between kinetic and potential energies; this is called the equipartition of energy.

We leave it as an exercise to show that the power flux (rate of energy flux) across a station $x$ is

$$\frac{d\bar{E}}{dt} = \int_{-h}^{0} \overline{pu}\, dz - T\overline{\zeta_x \zeta_t} = -\rho \int_{-h}^{0} \overline{\Phi_t \Phi_x}\, dz - T\overline{\zeta_x \zeta_t} = \bar{E} c_g \tag{2.54}$$

where $c_g$ is the speed of energy transport , or the group velocity

$$c_g = \frac{d\omega}{dk} = \frac{c}{2}\left\{\frac{\frac{k_m^2}{k^2} + 3}{\frac{k_m^2}{k^2} + 1} + \frac{2kh}{\sinh 2kh}\right\} = \frac{c}{2}\left\{\frac{\frac{\lambda^2}{\lambda_m^2} + 3}{\frac{\lambda^2}{\lambda_m^2} + 1} + \frac{2kh}{\sinh 2kh}\right\} \tag{2.55}$$

For pure gravity waves, $k/k_m \ll 1$ so that

$$c_g = \frac{c}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{2.56}$$

where the phase velocity is

$$c = \sqrt{\frac{g}{k}\tanh kh} \tag{2.57}$$

In very deep water $kh \gg 1$, we have

$$c_g = \frac{c}{2} = \frac{1}{2}\sqrt{\frac{g}{k}} \tag{2.58}$$

The shorter the waves the smaller the phase and group velocities. In shallow water $kh \ll 1$,

$$c_g = c = \sqrt{gh} \tag{2.59}$$

Long waves are the fastest and no longer dispersive.

For capillary-gravity waves with $kh \gg 1$, we have

$$c_g = \frac{c}{2}\left\{\frac{\frac{k_m^2}{k^2} + 3}{\frac{k_m^2}{k^2} + 1}\right\} = \frac{c}{2}\left\{\frac{\frac{\lambda^2}{\lambda_m^2} + 3}{\frac{\lambda^2}{\lambda_m^2} + 1}\right\}, \quad k_m = \frac{2\pi}{\lambda_m}\sqrt{\frac{\rho g}{T}} \tag{2.60}$$

where

$$c = \sqrt{\frac{g}{k} + \frac{Tk^3}{\rho}} \tag{2.61}$$

Note that $c_g = c$ when $k = k_m$, and

$$c_g \gtrless c, \quad \text{if} \quad k \gtrless k_m \tag{2.62}$$

In the limit of pure capillary waves of $k \gg k_m$, $c_g = 3c/2$. For pure gravity waves $c_g = c/2$ as in (2.58).

# 3 Wave resistance of a two-dimensional obstacle

Ref: Lecture notes on *Surface Wave Hydrodynamics* Theodore T.Y. WU, Calif. Inst.Tech.

As an application of the information gathered so far, let us examine the wave resistance on a two dimensional body steadily advancing parallel to the free surface. Let the body speed be $U$ from right to left and the sea depth be constant.

Due to two-dimensionality, waves generated must have crests parallel to the axis of the body ($y$ axis). After the steady state is reached, waves that keep up with the ship must have the phase velocity equal to the body speed. In the coordinate system fixed on the body, the waves are stationary. Consider first capillary -gravity waves in deep water $\lambda_* = \lambda/\lambda_m = O(1)$ and $kh \gg 1$. Equating $U = c$ we get from the normalized dispersion relation

$$U_*^2 = c_*^2 = \frac{1}{2}\left(\lambda_* + \frac{1}{\lambda_*}\right) \tag{3.1}$$

where $U_* \equiv U/c_m$. Hence

$$\lambda_*^2 - 2c_*^2\lambda_* + 1 = 0 = (\lambda_* - \lambda_{*1})(\lambda_* - \lambda_{*2})$$

which can be solved to give

$$\begin{bmatrix} \lambda_{*1} \\ \lambda_{*2} \end{bmatrix} = c_*^2 \pm \left(c_*^4 - 1\right)^{1/2} \tag{3.2}$$

and

$$\lambda_{*1} = \frac{1}{\lambda_{*2}} \tag{3.3}$$

Thus, as long as $c_* = U_* > 1$ two wave trains are present: the longer gravity wave with length $\lambda_{*1}$, and the shorter capillary wave with length $\lambda_{*2}$. Since $c_{g_1} < c = U$ and $c_{g_2} > c = U$, and energy must be sent from the body, the longer gravity waves must follow, while the shorter capillary waves stay ahead of, the body.

Balancing the power supply by the body and the power flux in both wave trains, we get

$$Rc = (c - c_{g_1})\bar{E}_1 + (c_{g_2} - c)\bar{E}_2 \tag{3.4}$$

Recalling that

$$\frac{c_g}{c} = \frac{1}{2}\frac{\lambda_*^2 + 3}{\lambda_*^2 + 1}$$

we find,

$$1 - \frac{c_g}{c} = 1 - \frac{1}{2}\left(1 + \frac{2}{\lambda_*^2 + 1}\right) = \frac{1}{2} - \frac{1/\lambda_*}{\lambda_* + 1/\lambda_*} = \frac{1}{2} - \frac{1/\lambda_*}{2c^2}$$

For the longer wave we replace $c_g/c$ by $c_{g*1}/c_*$ and $\lambda_*$ by $\lambda_{*1}$ in the preceding equation, and use (3.2), yielding

$$1 - \frac{c_{g*1}}{c_*} = \left(1 - c_*^{-4}\right)^{1/2} \tag{3.5}$$

Similarly we can show that

$$\frac{c_{g*2}}{c_*} - 1 = \left(1 - c_*^{-4}\right)^{1/2} = 1 - \frac{c_{g*1}}{c_*} \tag{3.6}$$

Since

$$\bar{E}_1 = \frac{\rho g A_1^2}{2}\left(1 + \frac{1}{\lambda_{*1}^2}\right) = \frac{\rho g A_1^2}{2}\frac{1}{\lambda_{*1}}\left(\lambda_{*1} + \frac{1}{\lambda_{*1}}\right) = \rho g A_1^2 \lambda_{*2} c_*^2, \tag{3.7}$$

we get finally

$$R = \frac{1}{2}\rho g\left(\lambda_{*2}A_1^2 + \lambda_{*1}A_2^2\right)(c_*^4 - 1)^{1/2} = \frac{1}{2}\rho g\left(\lambda_{*2}A_1^2 + \lambda_{*1}A_2^2\right)(U_*^4 - 1)^{1/2} \tag{3.8}$$

Note that when $U_* = 1$, the two waves become the same; no power input from the body is needed to maintain the single infinite train of waves; the wave resistance vanishes. When $U_* < 1$, no waves are generated; the disturbance is purely local and there is also no wave resistance. To get the magnitude of $R$ one must solve the boundary value problem for the wave amplitudes $A_1, A_2$ which are affected by the size (relative to the wavelengths), shape and depth of submergence.

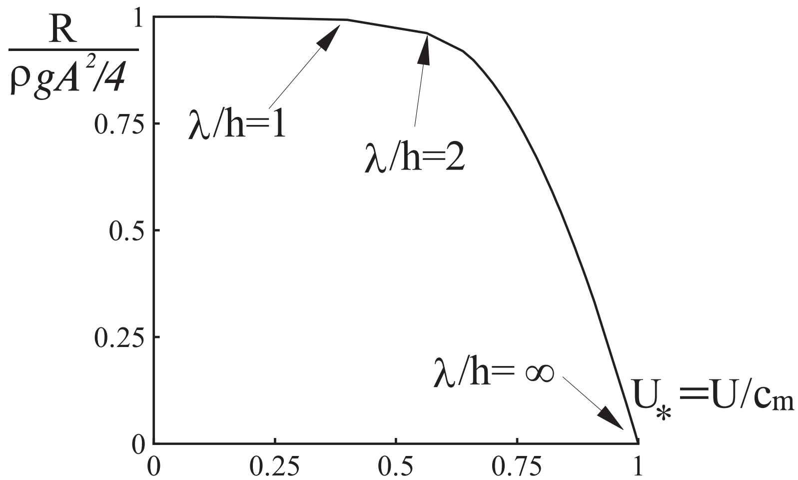


Figure 3: Dependence of wave resistance on speed for pure gravity waves

When the speed is sufficiently high, pure gravity waves are generated behind the body. Power balance then requires that

$$R = \left(1 - \frac{c_g}{U}\right)\bar{E} = \frac{\rho g A^2}{2}\left(\frac{1}{2} - \frac{kh}{\sinh 2kh}\right) \tag{3.9}$$

The wavelength generated by the moving body is given implicitly by

$$\frac{U}{\sqrt{gh}} = \left(\frac{\tanh kh}{kh}\right)^{1/2} \tag{3.10}$$

When $U \approx \sqrt{gh}$ the waves generated are very long, $kh \ll 1$, $c_g \to c = \sqrt{gh}$, and the wave resistance drops to zero. When $U \ll \sqrt{gh}$, the waves are very short, $kh \gg 1$,

$$R \approx \frac{\rho g A^2}{4} \tag{3.11}$$

For intermediate speeds the dependence of wave resistance on speed is plotted in figure (3).

# 4 Narrow-banded dispersive waves in general

In this section let us discuss the superposition of progressive sinusoidal waves with the amplitudes spread over a narrow spectrum of wave numbers

$$\zeta(x,t) = \int_0^\infty |\mathcal{A}(k)| \cos(kx - \omega t - \theta_A) dk = \Re \int_0^\infty \mathcal{A}(k) e^{ikx - i\omega t} dk \tag{4.1}$$

where$\mathcal{A}(k)$ is complex denotes the dimensionless amplitude spectrum of dimension (length)$^2$. The component waves are dispersive with a general nonlinear relation $\omega(k)$. Let $\mathcal{A}(k)$ be different from zero only within a narrow band of wave numbers centered at $k_o$. Thus the integrand is of significance only in a small neighborhood of $k_o$. We then approximate the integral by expanding for small $\Delta k = k - k_o$ and denote $\omega_o = \omega(k_o)$, $\omega_o' = \omega'(k_o)$, and $\omega_o'' = \omega''(k_o)$,

$$\begin{aligned}
\zeta &= \Re\left\{e^{ik_o x - i\omega_o t} \int_0^\infty \mathcal{A}(k)\, e^{i\Delta k x - i(\omega - \omega_o)t} dk\right\} \\
&= \Re\left\{e^{ik_o x - i\omega_o t} \int_0^\infty dk\, \mathcal{A}(k) \exp\left[i\Delta k x - i\left(\omega_o' \Delta k + \frac{1}{2}\omega_o''(\Delta k)^2\right) t + \cdots\right]\right\} \\
&= \Re\left\{A(x,t) e^{ik_o x - i\omega_o t}\right\} \qquad (4.2)
\end{aligned}$$

where

$$A(x,t) = \int_0^\infty dk\, \mathcal{A}(k)\, \exp\left[i\Delta k x - i\left(\omega_o' \Delta k + \frac{1}{2}\omega_o''(\Delta k)^2\right) t + \cdots\right] \tag{4.3}$$

Although the integration is formally extends from 0 to $\infty$, the effective range is only from $k_o - (\Delta k)_m$ to $k_o + (\Delta k)_m$, i,.e., the total range is $O((\Delta k)_m)$, where $(\Delta k)_m$ is the bandwidth. Thus the total wave is almost a sinusoidal wavetrain with frequency $\omega_o$ and wave number $k_o$, and amplitude $A(x,t)$ whose local value is slowly varying in space and time. $A(x,t)$ is also called the envelope. How slow is its variation?

If we ignore terms of $(\Delta k)^2$ in the integrand, (4.3) reduces to

$$A(x,t) = \int_0^\infty dk\, \mathcal{A}(k)\, \exp\left[i\Delta k(x - \omega_o' t)\right] \tag{4.4}$$

Clearly $A = A(x - \omega_o' t)$. Thus the envelope itself is a wave traveling at the speed $\omega_o'$. This speed is called the group velocity,

$$c_g(k_o) = \left.\frac{d\omega}{dk}\right|_{k_o} \tag{4.5}$$

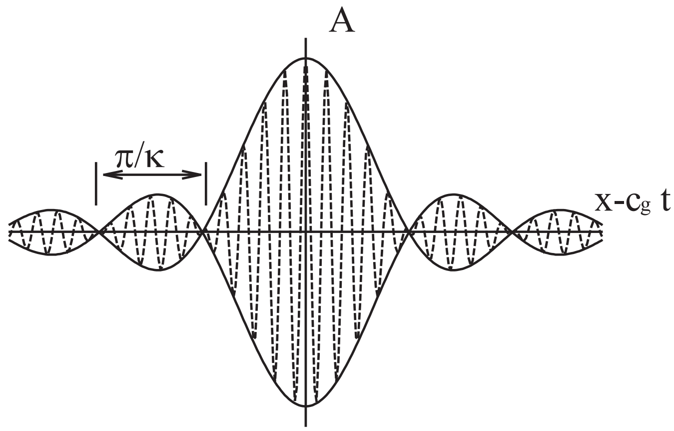


Figure 4: Envelope of waves with a rectangular band of wavenumbers

Note that the characteristic length and time scales are $(\Delta k_m)^{-1}$ and $(\omega_o'\Delta k_m)^{-1}$ respectively, therefore much longer than those of the component waves : $k_o^{-1}$ and $\omega_o^{-1}$. In other words, (4.3) is adequate for the slow variation of $A_e$ in the spatial range of $\Delta k_m\, x = O(1)$ and the time range of $\omega_o'\Delta k_m\, t = O(1)$.

As a specific example we let the amplitude spectrum be a real constant within the narrow band of $k_o - \kappa, k_o + \kappa$,

$$\zeta = \mathcal{A}\int_{k_o-\kappa}^{k_o+\kappa} e^{ikx-i\omega(k)t}dk, \quad \kappa \ll k_o \tag{4.6}$$

then

$$\begin{aligned}\zeta &= k_o\mathcal{A}e^{ik_ox-i\omega_ot}\int_{-\kappa}^{\kappa} d\xi e^{ik_o\xi(x-c_gt)} + \cdots \\ &= \frac{2\mathcal{A}\,\sin\kappa(x-c_gt)}{x-c_gt}e^{ik_ox-i\omega_ot} = Ae^{ik_ox-i\omega_ot}\end{aligned} \tag{4.7}$$

where $\xi = k - k_o/k_o$ and

$$A = \frac{2\mathcal{A}\,\sin\kappa(x-c_gt)}{(x-c_gt)} \tag{4.8}$$

as plotted in figure (4).

By differentiation, it can be verified that

$$\frac{\partial A}{\partial t} + c_g\frac{\partial A}{\partial x} = 0, \tag{4.9}$$

Multiplying (4.9) by $A^*$,

$$A^* \frac{\partial A}{\partial t} + c_g A^* \frac{\partial A}{\partial x} = 0,$$

and adding the result to its complex conjugate,

$$A \frac{\partial A^*}{\partial t} + c_g A \frac{\partial A^*}{\partial x} = 0,$$

we get

$$\frac{\partial |A|^2}{\partial t} + c_g \frac{\partial |A|^2}{\partial x} = 0 \tag{4.10}$$

We have seen that for a monochromatic wave train the energy density is proportional to $|A|^2$. Thus the time rate of change of the *local* energy density is balanced by the net flux of energy by the group velocity.

Now let us examine the more accurate approximation (4.3). By straightforward differentiation, we find

$$\begin{aligned}
\frac{\partial A}{\partial t} &= \int_0^\infty \left[ -i\omega'(k_o)\Delta k - \frac{i\omega''(k_o)}{2}(\Delta k)^2 \right] \mathcal{A}(k) e^{iS} dk \\
\frac{\partial A}{\partial x} &= \int_0^\infty (i\Delta k) \mathcal{A}(k) e^{iS} dk \\
\frac{\partial^2 A}{\partial x^2} &= \int_0^\infty \left( -(\Delta k)^2 \right) \mathcal{A}(k) e^{iS} dk
\end{aligned}$$

where

$$S = \Delta k\, x - \omega_o' \Delta k\, t - \frac{1}{2}\omega_o''(\Delta k)^2\, t \tag{4.11}$$

is the phase function. It can be easily verified that

$$\frac{\partial A}{\partial t} + \omega_o' \frac{\partial A}{\partial x} = \frac{i\omega_o''}{2} \frac{\partial^2 A}{\partial x^2} \tag{4.12}$$

By keeping the quadratic term in the expansion, (4.12) is now valid for a larger spatial range of $(\Delta k)^2 x = O(1)$. In the coordinate system moving at the group velocity, $\xi = x - c_g t, \tau = t$, we easily find

$$\frac{\partial A(\xi, \tau)}{\partial t} = \frac{\partial A}{\partial \tau} - c_g \frac{\partial A}{\partial \xi}, \quad \frac{\partial A(\xi, \tau)}{\partial x} = \frac{\partial A}{\partial x}$$

so that (4.12) simplifies to the Schrödinger equation:

$$\frac{\partial A}{\partial \tau} = \frac{i\omega_o''}{2} \frac{\partial^2 A}{\partial \xi^2} \tag{4.13}$$

By manipulations similar to those leading to (4.10), we get

$$\frac{\partial |A|^2}{\partial \tau} = \frac{i\omega_o''}{2}\frac{\partial}{\partial \xi}\left(A^*\frac{\partial A}{\partial \xi} - A\frac{\partial A^*}{\partial \xi}\right) \tag{4.14}$$

Thus the local energy density is not conserved over a long distance of propagation. Higher order effects of dispersion redistribute energy to other parts of the envelope. For either a wave packet whose envelope has a finite length ( $A(\pm\infty) = 0$), or for a periodically modulated envelope ($A(x) = A(x + L)$), we can integrate (4.14) to give

$$\frac{\partial}{\partial \tau}\int |A|^2 d\xi = 0 \tag{4.15}$$

where the integration extends over the entire wave packet or the group period. Thus the total energy in the entire wave packet or in a group period is conserved.

# 5 Radiation of surface waves forced by an oscillating pressure

We demonstrate the reasoning which is typical in many similar radiation problems.

The governing equations are

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0, \quad -\infty < z < 0. \tag{5.1}$$

with the kinematic boundary condition

$$\phi_z = \zeta_t, \quad z = 0 \tag{5.2}$$

and the dynamic boundary condition

$$\frac{p_a}{\rho} + \phi_t + g\zeta = 0 \tag{5.3}$$

where $p_a$ is the prescribed air pressure. Eliminating the free surface displacement we get

$$\phi_{tt} + g\phi_z = -\frac{(p_a)_t}{\rho}, \quad z = 0. \tag{5.4}$$

Let us consider only sinusoidal time dependence:

$$p_a = P(x)e^{-i\omega t} \tag{5.5}$$

and assume

$$\phi(x,z,t) = \Phi(x,z)e^{-i\omega t}, \quad \zeta(x,t) = \eta(x)e^{-i\omega t} \tag{5.6}$$

then the governing equations become

$$\nabla^2\Phi = \Phi_{xx} + \Phi_{yy} = 0, \qquad -\infty < z < 0. \tag{5.7}$$

$$\Phi_z = -i\omega\eta, \quad z = 0 \tag{5.8}$$

and

$$\Phi_z - \frac{\omega^2}{g}\Phi = \frac{i\omega}{\rho g}P(x), \qquad z = 0. \tag{5.9}$$

Define the Fourier transform and its inverse by

$$\bar{f}(\alpha) = \int_{-\infty}^{\infty} dx\, e^{-i\alpha x} f(x), \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\, e^{i\alpha x}\bar{f}(\alpha), \tag{5.10}$$

We then get the transforms of (5.1) and (5.4)

$$\bar{\Phi}_{zz} - \alpha^2\bar{\Phi} = 0, \qquad z < 0 \tag{5.11}$$

subject to

$$\bar{\Phi}_z - \frac{\omega^2}{g}\bar{\Phi} = \frac{i\omega}{\rho g}\bar{P}(\alpha), \qquad z = 0. \tag{5.12}$$

The solution finite at $z \sim -\infty$ for all $\alpha$ is

$$\bar{\Phi} = Ae^{|\alpha|z}$$

To satisfy the free surface condition

$$|\alpha|A - \frac{\omega^2}{g}A = \frac{i\omega\bar{P}(\alpha)P(\alpha)}{\rho g}$$

hence

$$A = \frac{\frac{i\omega\bar{P}(\alpha)}{\rho g}}{|\alpha| - \omega^2/g}$$

or

$$\begin{aligned}
\Phi &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha e^{i\alpha x}e^{|\alpha|z}\frac{\frac{i\omega\bar{P}(\alpha)}{\rho g}}{|\alpha| - \omega^2/g} \\
&= \frac{i\omega}{\rho g}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha e^{i\alpha x}e^{|\alpha|z}\int_{-\infty}^{\infty} dx'\, e^{-i\alpha x'}P(x')\frac{1}{|\alpha| - \omega^2/g}, \\
&= \frac{i\omega}{\rho g}\int_{-\infty}^{\infty} dx'\, P(x')\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\, e^{i\alpha(x-x')}e^{|\alpha|z}\frac{1}{|\alpha| - \omega^2/g}
\end{aligned} \tag{5.13}$$

Let

$$k = \frac{\omega^2}{g} \tag{5.14}$$

we can rewrite (5.13) as

$$\Phi = \frac{i\omega}{\rho g} \int_{-\infty}^{\infty} dx'\, P(x') \frac{1}{\pi} \int_0^{\infty} d\alpha\, e^{\alpha z} \frac{\cos(\alpha(x - x'))}{\alpha - k} \tag{5.15}$$

The final formal solution is

$$\phi = \frac{i\omega}{\rho g} e^{-i\omega t} \int_{-\infty}^{\infty} dx'\, P(x') \frac{1}{\pi} \int_0^{\infty} d\alpha\, e^{\alpha z} \frac{\cos(\alpha(x - x'))}{\alpha - k} \tag{5.16}$$

If we chose

$$P(x') = P_o \delta(x') \tag{5.17}$$

then

$$\Phi \to \mathcal{G}(x, z) = \frac{i\omega P_o}{\rho g} \frac{1}{\pi} \int_0^{\infty} d\alpha\, e^{\alpha z} \frac{\cos(\alpha x)}{\alpha - k} \tag{5.18}$$

is clearly the response to a concentrated surface pressure and the response to a pressure distribution (5.16) can be written as a superposition of concentrated loads over the free surface,

$$\phi = \int_{-\infty}^{\infty} dx'\, P(x') \mathcal{G}(x - x', z). \tag{5.19}$$

where

$$\mathcal{G}(x, z, t) = \frac{i\omega P_o}{\rho g} e^{-i\omega t} \frac{1}{\pi} \int_0^{\infty} d\alpha\, e^{\alpha z} \frac{\cos(\alpha x)}{\alpha - k} \tag{5.20}$$

In these results, e.g., (5.20), the Fourier integral is so far undefined since the integrand has a real pole at $\alpha = k$ which is on the path of integration. To make it mathematically defined we can chose the principal value, deform the contour from below or from above the pole as shown in figure (5). This indefiniteness is due to the assumption of quasi

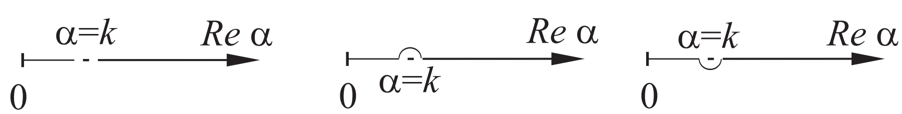


Figure 5: Possible paths of integration

steady state where the influence of the initial condition is no longer traceable. We must now impose the radiation condition that waves must be outgoing as $x \to \infty$. This

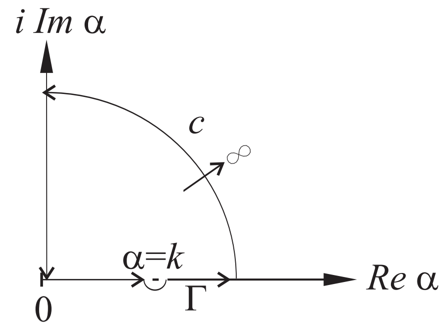


Figure 6: Closed contour in the upper half plane

condition can only be satisfied if we deform the contour from below. Denoting this contour by $\Gamma$, we now manipulate the integral to exhibit the behavior at infinity, and to verify the choice of path. For simplicity we focus attention on $\mathcal{G}$. Due to symmetry, it suffices to consider $x > 0$. Rewriting,

$$\begin{aligned}\mathcal{G}(x,z,t) &= \frac{i\omega P_o}{\rho g} e^{-i\omega t} \frac{1}{2\pi}(I_1 + I_2) \\ &= \frac{i\omega P_o}{\rho g} e^{-i\omega t} \frac{1}{2\pi} \int_\Gamma d\alpha\, e^{\alpha z} \left[\frac{e^{i\alpha x}}{\alpha - k} + \frac{e^{-i\alpha x}}{\alpha - k}\right]\end{aligned} \tag{5.21}$$

Consider the first integral in (5.21). In order that the first integral converges for large $|\alpha|$, we close the contour by a large circular arc in the upper half plane, as shown in figure (6), where $\Im\alpha > 0$ along the arc. The term

$$e^{i\alpha x} = e^{i\Re\alpha x} e^{-\Im\alpha x}$$

is exponentially small for positive x. Similarly, for the second integral we must chose the contour by a large circular arc in the lower half plane as shown in figure (7).

Back to the first integral in (5.21)

$$I_1 = \int_\Gamma d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} \tag{5.22}$$

The contour integral is

$$\begin{aligned}\oint d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} &= \int_\Gamma d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} + \int_C d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} + \int_{i\infty}^0 d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} \\ &= I_1 + 0 + \int_{i\infty}^0 d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k}\end{aligned}$$

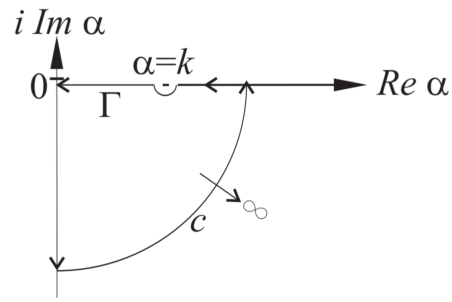


Figure 7: Closed contour in the lower half plane

The contribution by the circular arc $C$ vanishes by Jordan's lemma. The left hand side is

$$LHS = 2\pi i e^{ikx} e^{kz} \tag{5.23}$$

by Cauchy's residue theorem. By the change of variable $\alpha = i\beta$, the right hand side becomes

$$RHS = I_1 + i \int_\infty^0 d\beta \frac{e^{-\beta x} e^{i\beta z}}{i\beta - k}$$

Hence

$$I_1 = 2\pi i e^{ikx} e^{kz} + i \int_0^\infty d\beta \frac{e^{-\beta x} e^{i\beta z}}{i\beta - k} \tag{5.24}$$

Now consider $I_2$

$$I_2 = \int_\Gamma d\alpha \frac{e^{-i\alpha x} e^{\alpha z}}{\alpha - k} \tag{5.25}$$

and the contour integral along the contour closed in the lower half plane,

$$-\oint d\alpha \frac{e^{-i\alpha x} e^{\alpha z}}{\alpha - k} = I_2 + 0 + \int_0^\infty d\alpha \frac{e^{-i\alpha x} e^{\alpha z}}{\alpha - k}$$

Again no contribution comes from the circular arc $C$. Now the pole is outside the contour hence $LHS = 0$. Let $\alpha = -i\beta$ in the last integral we get

$$I_2 = -i \int_0^\infty d\beta \frac{e^{-\beta x} e^{-i\beta y}}{-i\beta - k} \tag{5.26}$$

Adding the results (5.24) and (5.26).,

$$\begin{aligned} I_1 + I_2 &= 2\pi i e^{ikx} e^{kz} + \int_0^\infty d\beta \left( \frac{i e^{-\beta x} e^{i\beta z}}{i\beta - k} - \frac{i e^{-\beta x} e^{-i\beta z}}{-i\beta - k} \right) \\ &= 2\pi i e^{ikx} e^{kz} + 2 \int_0^\infty d\beta \frac{e^{-\beta x}}{\beta^2 + k^2} \left( \beta \cos \beta y + k \sin \beta y \right) \end{aligned} \tag{5.27}$$

Finally, the total potential is, on the side of $x > 0$,

$$\begin{aligned}\mathcal{G}(x,z,t) &= -\frac{\omega}{\rho g}e^{-i\omega t}\left(\frac{1}{2\pi i}(I_1+I_2)\right)e^{-i\omega t}\\ &= -\frac{\omega}{\rho g}e^{-i\omega t}\left\{e^{ikx}e^{kz}+\frac{1}{\pi}\int_0^\infty d\beta\frac{e^{-\beta x}}{\beta^2+k^2}\left(\beta\cos\beta z+k\sin\beta z\right)\right\}\end{aligned} \tag{5.28}$$

The first term gives an outgoing waves. For a concentrated load with amplitude $P_o$, the displacement amplitude is $P_o/\rho g$. The integral above represent local effects important only near the applied pressure. If the concentrated load is at $x = x'$, one simply replaces $x$ by $x - x'$ everywhere.

# 6 The Kelvin pattern of ship wave

The action of the ship's propeller
Has a thrust pattern
To which the ship reacts by moving forward,
Which also results secondarily,
In the ship's bow elevated waves,
And its depressed transverse stern wave,
Which wave disturbances of the water
Are separate from the propeller's thrust waves.

–R.Buckminster Fuller, *Intuition- Metaphysical Mosaic.* 1972.

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Anyone flying over a moving ship must be intrigued by the beautiful pattern in the ship's wake. The theory behind it was first completed by Lord Kelvin, who invented the method of stationary phase for the task. Here we shall give a physical/geometrical derivation of the key results (lecture notes by T. Y. Wu, Caltech).

Consider first two coordinate systems. The first $\mathbf{r} = (x, y, z)$ moves with ship at the uniform horizontal velocity $\mathbf{U}$. The second $\mathbf{r}' = (x', y', z)$ is fixed on earth so that water is stationary while the ship passes by at the velocity $\mathbf{U}$. The two systems are related by the Galilean transformation,

$$\mathbf{r}' = \mathbf{r} + \mathbf{U}t \tag{6.29}$$

A train of simple harmonic progressive wave

$$\zeta = \Re\left\{A\exp[i(\mathbf{k}\cdot\mathbf{r}' - \omega t)]\right\} \tag{6.30}$$

in the moving coordinates should be expressed as

$$\begin{aligned}\zeta &= \Re\left\{A\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{U}t) - i\omega t]\right\} = \Re\left\{A\exp[i\mathbf{k}\cdot\mathbf{r} - i(\omega - \mathbf{k}.\mathbf{U})t]\right\} \\ &= \Re\left\{A\exp[i\mathbf{k}\cdot\mathbf{r} - i\sigma t]\right\}\end{aligned} \tag{6.31}$$

in the stationary coordinates. Therefore the apparent frequency in the moving coordinates is

$$\sigma = \omega - \mathbf{k}\cdot\mathbf{U} \tag{6.32}$$

The last result is essentially the famous Doppler's effect. To a stationary observer, the whistle from an approaching train has an increasingly high pitch, while that from a leaving train has a decreasing pitch.

If a ship moves in very deep water at the constant speed $-\mathbf{U}$ in stationary water, then relative to the ship, water appears to be washed downstream at the velocity $\mathbf{U}$. A stationary wave pattern is formed in the wake. Once disturbed by the passing ship, a fluid parcel on the ship's path radiates waves in all directions and at all frequencies. Wave of frequency $\omega$ spreads out radially at the phase speed of $c = g/\omega$ according to the dispersion relation. Only those parts of the waves that are stationary relative to the ship will form the ship wake, and they must satisfy the condition

$$\sigma = 0, \tag{6.33}$$

i.e.,

$$\omega = \mathbf{k}\cdot\mathbf{U}, \text{or} \quad c = \frac{\omega}{k} = \frac{\mathbf{k}}{k}\cdot\mathbf{U} \tag{6.34}$$

Referring to figure 8, let $O, (x = 0)$ represents the point ship in the ship-bound coordinates. The current is in the positive $x$ direction. Any point $x_1$ is occupied by a fluid parcel $Q_1$ which was disturbed directly by the passing ship at time $t_1 = x_1/U$ earlier. This disurbed parcel radiates waves of all frequencies radially. The phase of wave at the frequency $\omega$ reaches the circle of radius $ct_1$ where c=$g/\omega$ by the deep water dispersion relation. Along the entire circle however only the point that satisfies (6.34)

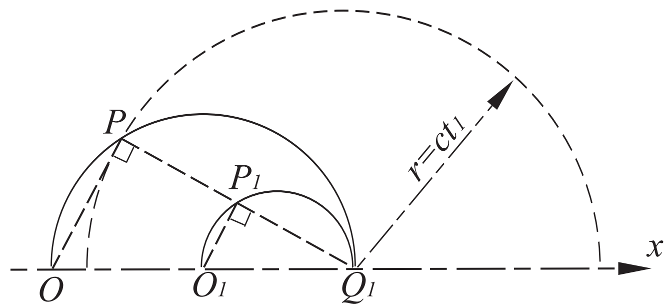


Figure 8: Waves radiated from disturbed fluid parcel

can contribute to the stationary wave pattern, as marked by $P$. Since $OQ_1 = x_1 = Ut_1$, $Q_1P = ct_1$ and $OP = \mathbf{Ut_1} \cdot \mathbf{k}/k$, where $\mathbf{k}$ is in the direction of $\vec{Q_1P}$. It follows that $\triangle OPQ_1$ is a right triangle, and $P$ lies on a semi circle with diameter $OQ_1$. Accounting for the radiated waves of all frequencies, hence all $c$, every point on the semi circle can be a part of the stationary wave phase formed by signals emitted from $Q_1$. Now this argument must be rectified because wave energy only travels at the group velocity which is just half of the phase velocity in deep water. Therefore stationary crests due to signals from $Q_1$ can only lie on the semi-circle with the diameter $O_1Q_1 = OQ_1/2$. Thus $P_1$ instead of $P$ is one of the points forming a stationary crest in the ship's wake, as shown in figure 8.

Any other fluid parcel $Q_2$ at $x_2$ must have been disturbed by the passing ship at time $t_2 = x_2/U$ earlier. Its radiated signals contribute to the stationary wave pattern only along the semi circle with diameter $O_2Q_2 = OQ_2/2$. Combining the effects of all fluid parcels along the $+x$ axis, stationary wave pattern must be confined inside the wedge which envelopes all these semi circles. The half apex angle $\beta_o$ of the wedge, which defines the wake, is given by

$$\sin\beta_o = \frac{Ut/4}{3Ut/4} = 1/3, \tag{6.35}$$

hence $\beta_o = \sin^{-1} 1/3 = 19.5°$, see figure 9.

Now any point $P$ inside the wedge is on two semicircles tangent to the boundary of the wedge, i.e., there are two segments of the wave crests intersecting at $P$: one perpendicular to $PQ_1$ and one to $PQ_2$, as shown in figure 9.

Another way of picturing this is to examine an interior ray from the ship. In figure

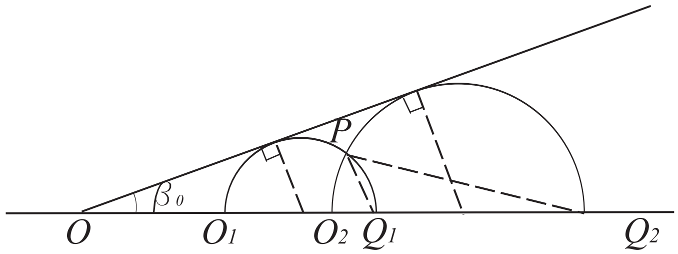


Figure 9: Wedge angle of the ship wake

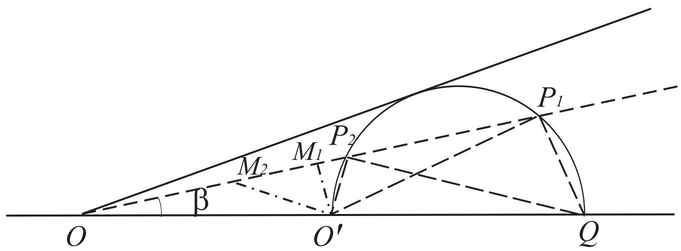


Figure 10: Geometrical relation to find Points of dependence

(10), draw a semi circle with the diameter $O'Q = OQ/2$, then at the two intersections $P_1$ and $P_2$ with the ray are the two segments of the stationary wave crests. In other words, signals originated from $Q$ contribute to the stationary wave pattern only at the two points $P_1$ and $P_2$, as shown in figure 10. Point $Q$ can be called the point of dependence for points $P_1$ and $P_2$ on the crests.

For any interior point $P$ there is a graphical way of finding the two points of dependence $Q_1$ and $Q_2$. Referring to figure 10, $\triangle O'QP_1$ and $\triangle O'QP_2$ are both right triangles. Draw $O_1M_1 \parallel QP_1$ and $O_2M_2 \parallel QP_2$ where $M_1$ and $M_2$ lie on the ray inclined at the angle $\beta$. it is clear that $OM_1 = OP_1/2$ and $OM_2 = OP_2/2$, and $\triangle M_1O'P_1$ and $\triangle M_2O'P_2$ are both right triangles. Hence $O'$ lies on two semi circles with diameters $M_1P_1$ and $M_2P_2$.

We now reverse the process, as shown in figure 11. For any point $P$ on an interior ray, let us mark the mid point $M$ of $OP$ and draw a semi circle with diameter $MP$. The semi circle intersects the $x$ axis at two points $S_1$ and $S_2$. We then draw from $P$ two lines parallel to $MS_1$ and $MS_2$, the two points of intersection $Q_1$ an $Q_2$ on the $x$ axis are just the two points of dependence.

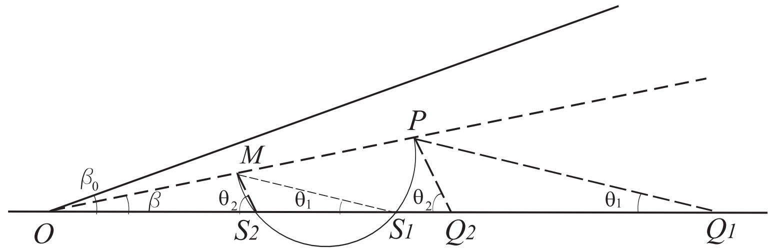


Figure 11: Points of dependence

Let $\angle PQ_1O = \angle MS_1O = \theta_1$ and $\angle PQ_2O = \angle MS_2O = \theta_2$. then

$$\tan(\theta_i + \beta) = \frac{PS_i}{MS_i} = \frac{PS_i}{PQ_i/2} = 2\tan\theta_i \quad i = 1, 2.$$

hence

$$2\tan\theta_i = \frac{\tan\theta_i + \tan\beta}{1 - \tan\theta_i \tan\beta}$$

which is a quadratic equation for $\theta_i$, with two solutions:

$$\left\{ \begin{array}{c} \tan\theta_1 \\ \tan\theta_2 \end{array} \right\} = \frac{1 \pm \sqrt{1 - 8\tan^2\beta}}{4\tan\beta} \tag{6.36}$$

They are real and distinct if

$$1 - 8\tan^2\beta > 0 \tag{6.37}$$

These two angles define the local stationary wave crests crossing $P$, and they must be perpendicular to $PQ_1$ and $PQ_2$. There are no solutions if $1 - 8\tan^2\beta < 0$, which corresponds to $\sin\beta > 1/3$ or $\beta > 19.5°$, i.e., outside the wake. At the boundary of the wake, $\beta = 19,5°$ and $\tan\beta = \sqrt{1/8}$, the two angles are equal

$$\theta_1 = \theta_2 = \tan^{-1}\frac{\sqrt{2}}{2} = 55°. \tag{6.38}$$

By connecting these segments at all points in the wedge, one finds two systems of wave crests, the *diverging waves* and the *transverse waves*, as shown in figure div-trans.

A beautiful photograph is shown in Figure 13

Knowing that waves are confined in a wedge, we can estimate the behavior of the wave amplitude by balancing in order of magnitude work done by the wave drag $R$ and the steady rate of energy flux

$$RU = (\bar{E}c_g)r \sim (|A|^2 c_g)r \tag{6.39}$$

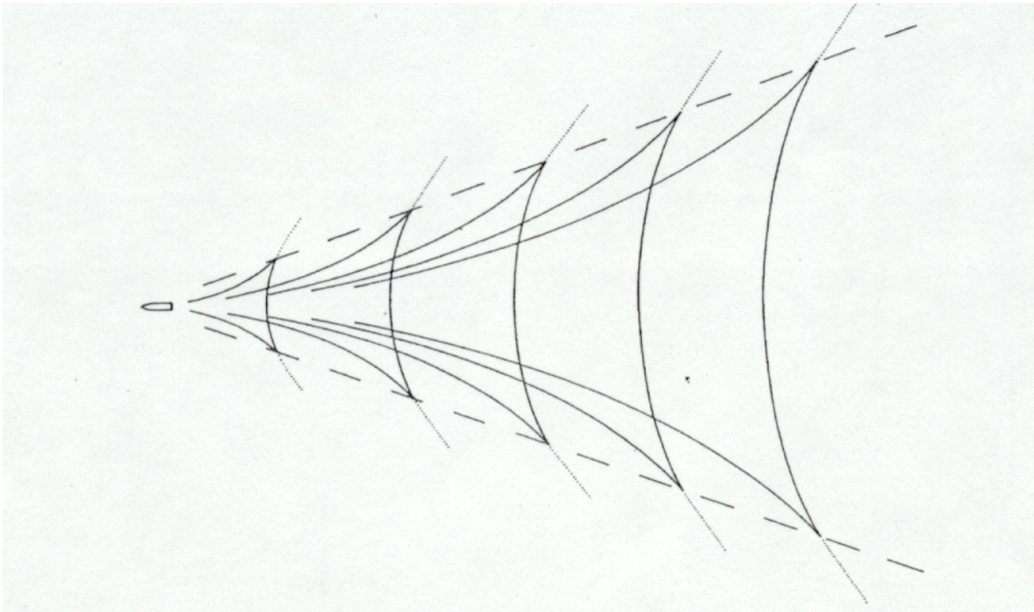

Figure 12: Diverging and transverse waves in a ship wake

hence

$$A \sim r^{1/2} \tag{6.40}$$

This estimate is valid throughout the wedge except near the outer boundaries, where

$$A \sim r^{-1/3} \tag{6.41}$$

by a more refined analysis (Stoker, 1957, or Wehausen & Laitone, 1960).

# 7 Basic theory for two-dimensional Internal waves in a stratified fluid

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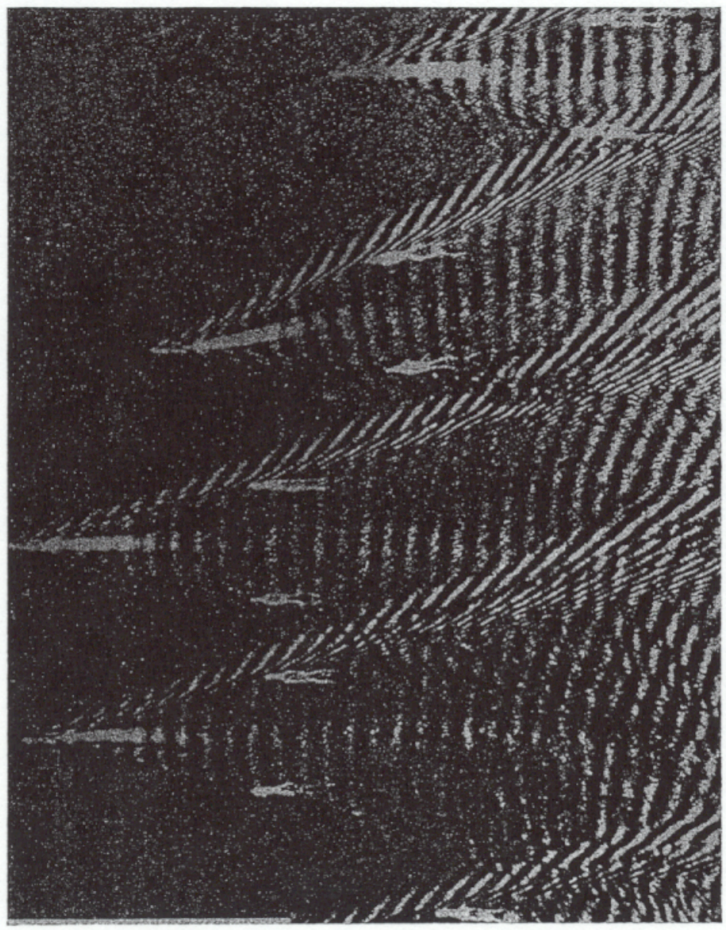

Figure 13: Ships in a straight course. From Stoker, 1957.p. 280.

Due to seasonal changes of temperature, the density of water or atmosphere can have significant variations in the vertical direction. Variation of salt content can also lead to density stratification. Freshwater from rivers can rest on top of the sea water. Due to the small diffusivity, the density contrast remains for a long time.

Consider a calm and stratified fluid with a static density distribution $\overline{\rho}_o(z)$ which decreases with height ($z$). If a fluid parcel is moved from the level $z$ upward to $z+\zeta$, it is surrounded by lighter fluid of density $\overline{\rho}(z+dz)$. The upward buoyancy force per unit volume is

$$g(\overline{\rho}(z+\zeta)-\overline{\rho}(z)) \approx g\frac{d\overline{\rho}}{dz}\zeta$$

and is negative. Applying Newton's law to the fluid parcel of unit volume

$$\overline{\rho}\frac{d^2\zeta}{dt^2} = g\frac{d\overline{\rho}}{dz}\zeta$$

or

$$\frac{d^2\zeta}{dt^2} + N^2\zeta = 0 \tag{7.1}$$

where

$$N = \left(-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}\right)^{1/2} \tag{7.2}$$

is called the Brunt-Väisälä frequency. This elementary consideration shows that once a fluid is displaced from its equilibrium position, gravity and density gradient provides restoring force to enable oscillations. In general there must be horizontal nonunifomities, hence waves are possible.

We start from the exact equations for an inviscid and incompressible fluid with variable density.

For an incompressible fluid the density remains constant as the fluid moves,

$$\rho_t + \mathbf{q}\cdot\nabla\rho = 0 \tag{7.3}$$

where $\mathbf{q} = (u, w)$ is the velocity vector in the vertical plane of $(x, z)$. Conservation of mass requires that

$$\nabla\cdot\mathbf{q} = 0 \tag{7.4}$$

The law of momentum conservation reads

$$\rho(\mathbf{q}_t + \mathbf{q}\cdot\nabla\mathbf{q}) = -\nabla p - \rho g\mathbf{e}_z \tag{7.5}$$

and $\mathbf{e}_z$ is the unit vector in the upward vertical direction.

## 7.1 Linearized equations

Consider small disturbances

$$p = \overline{p} + p', \quad \rho = \overline{\rho}(z) + \rho', \quad \vec{q} = (u', w') \tag{7.6}$$

with

$$\overline{\rho} \gg \rho', \quad \overline{p} \gg p' \tag{7.7}$$

and $u', v', w'$ are small. Linearizing by omitting quadratically small terms associated with the fluid motion, we get

$$\rho'_t + w' \frac{d\overline{\rho}}{dz} = 0. \tag{7.8}$$

$$u'_x + w'_z = 0 \tag{7.9}$$

$$\overline{\rho} u'_t = -p'_x \tag{7.10}$$

$$\overline{\rho} w'_t = -\overline{p}_z - p'_z - g\overline{\rho} - g\rho' \tag{7.11}$$

In the last equation the static part must be in balance

$$0 = -\overline{p}_z - g\overline{\rho}, \tag{7.12}$$

hence

$$\overline{p}(z) = \int_0^z \bar{\rho}(z) dz. \tag{7.13}$$

The remaining dynamically part must satisfy

$$\overline{\rho} w'_t = -p'_z - g\rho' \tag{7.14}$$

Upon eliminating $p'$ from the two momentum equations we get

$$\frac{d\overline{\rho}}{dz} u'_t + \overline{\rho}(u'_z - w'_x)_t = g\rho'_x \tag{7.15}$$

Eliminating $\rho'$ from (7.8) and (7.15) we get

$$\frac{d\overline{\rho}}{dz} u'_{tt} + \overline{\rho}(u'_z - w'_x)_{tt} = g\rho'_{xt} = -g\frac{d\overline{\rho}}{dz} w'_x \tag{7.16}$$

Let us introduce the disturbance stream function $\psi$:

$$u' = \psi_z, \quad w' = -\psi_x \tag{7.17}$$

It follows from (7.16) that

$$\overline{\rho}\,(\psi_{xx}+\psi_{zz})_{tt}=\frac{d\overline{\rho}}{dz}\,(g\psi_{xx}-\psi_{ztt}) \tag{7.18}$$

by virture of Eqns. (7.8) and (7.17). Note that

$$N=\sqrt{-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}} \tag{7.19}$$

is the Brunt-Väisälä frequency. In the ocean, density gradient is usually very small ( $N\sim 5\times 10^{-3}$ rad/sec). Hence $\overline{\rho}$ can be approximated by a constant reference value, say, $\rho_0=\overline{\rho}(0)$ in (7.10) and (7.14) without much error in the inertia terms. However density variation must be kept in the buoyancy term associated with gravity, which is the only restoring force responsible for wave motion. This is called the *Boussinesq approximation* and amounts to taking $\overline{\rho}$ to be constant in (Eq:17.1) only. With it (7.18) reduces to

$$(\psi_{xx}+\psi_{zz})_{tt}+N^2(z)\,\psi_{xx}=0. \tag{7.20}$$

Note that because of linearity, $u'$ and $w'$ satisfy Eqn. (7.20) also, i.e.,

$$(w'_{xx}+w'_{zz})_{tt}+N^2\,w'_{xx}=0 \tag{7.21}$$

etc.

## 7.2 Linearized Boundary conditions on the sea surface

Dynamic boundary condition : Total pressure is equal to the atmospheric pressure

$$(\overline{p}+p')_{z=\zeta}=0. \tag{7.22}$$

On the free surface $z=\zeta$, we have

$$\overline{p}\approx -g\int_0^\zeta \overline{\rho}(0)dz=-g\overline{\rho}(0)\zeta$$

Therefore,

$$-\overline{\rho}g\zeta+p'=0,\quad z=0, \tag{7.23}$$

implying

$$-\overline{\rho}g\zeta_{xxt}=-p'_{xxt},\quad z=0. \tag{7.24}$$

Kinematic condition :

$$\zeta_t = w, \quad z = 0. \tag{7.25}$$

The left-hand-side of (7.24) can be written as

$$-\overline{\rho} g \zeta_{xxt} = -\overline{\rho} g \, w'_{xx}$$

Using 7.10, the right-hand-side of 7.24 may be written,

$$-p_{xxt} = \overline{\rho} \, u'_{xtt} = -\overline{\rho} w'_{ztt}$$

hence

$$w'_{ztt} - g w'_{xx} = 0, \quad \text{on} \quad z = 0. \tag{7.26}$$

Since $w' = -\psi_x$, $\psi$ also satisfies the same boundary condition

$$\psi_{ztt} - g\psi_{xx} = 0, \quad \text{on} \quad z = 0. \tag{7.27}$$

On the seabed, $z = -h(x)$ the normal velocity vanishes. For a horizontal bottom we have

$$\psi(x, -h, t) = 0. \tag{7.28}$$

# 8 Internal waves modes for finite N

Consider a horizontally propagating wave beneath the sea surface. Let

$$\psi = F(z) \, e^{\pm ikx} e^{-i\omega t}. \tag{8.1}$$

From Eqn. (7.21),

$$-\omega^2 \left( \frac{d^2 F}{dz^2} - k^2 F \right) + N^2 \left( -k^2 \right) F = 0$$

or,

$$\frac{d^2 F}{dz^2} + \frac{N^2 - \omega^2}{\omega^2} \, k^2 F = 0 \quad z < 0. \tag{8.2}$$

On the (horizontal) sea bottom

$$F = 0 \qquad z = -h. \tag{8.3}$$

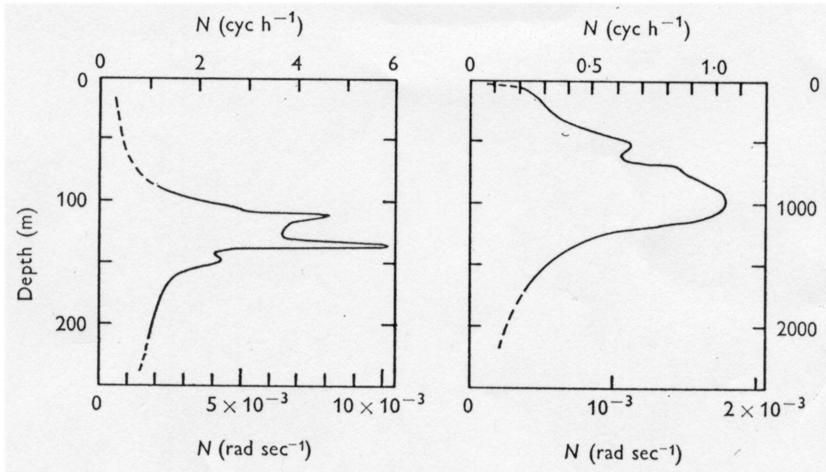


Figure 14: Typical variation of Brunt-Väisälä frequency in the ocean. From O. M. Phillips, 1977

From Eqn. (7.27),

$$\frac{dF}{dz} - g\,\frac{k^2}{\omega^2}\,F = 0 \qquad z = 0. \tag{8.4}$$

Equations (8.2), (8.3) and (8.4) constitute an eigenvalue condition.

If $\omega^2 < N^2$, then $F$ is oscillatory in $z$ within the thermocline. Away from the thermocline, $\omega^2 > N^2$, $W$ must decay exponentially. Therefore, the thermocline is a waveguide within which waves are trapped. Waves that have the greatest amplitude beneath the free surface is called internal waves.

Since for internal waves, $\omega < N$ while $N$ is very small in oceans, oceanic internal waves have very low natural frequencies. For most wavelengths of practical interests $\omega^2 \ll gk$ so that

$$F \cong 0 \quad \text{on } z = 0. \tag{8.5}$$

This is called the *rigid lid approximation*, which will be adopted in the following.

With the rigid-lid approximation, if $N$=constant (if the total depth is relataively small compared to the vertical scale of stratification, the solution for $F$ is

$$F = A\sin\left(k(z+h)\frac{\sqrt{N^2-\omega^2}}{\omega}\right) \tag{8.6}$$

where

$$kh\frac{\sqrt{N^2-\omega^2}}{\omega} = n\pi, \quad n = 1, 2, 3... \tag{8.7}$$

This is an eigen-value condition. For a fixed wave number $k$, it gives the eigen-frequencies,

$$\omega_n = \frac{N}{\sqrt{1 + \left(\frac{n\pi}{kh}\right)^2}} \tag{8.8}$$

For a given wavenumber $k$, this dispersion relation gives the eigen-frequency $\omega_n$. For a given frequency $\omega$, it gives the eigen-wavenumbers $k_n$,

$$k_n = \frac{n\pi}{h} \frac{\omega}{\sqrt{N^2 - \omega^2}} \tag{8.9}$$

For a simple lake with vertical banks and length $L$, $0 < x < L$, we must impose the conditions :

$$u' = 0, \quad \text{hence} \quad \psi = 0, \quad x = 0, L \tag{8.10}$$

The solution is

$$\psi = A \sin k_m x \, \exp(-i\omega_{nm} t) \sin\left[k_m(z+h)\frac{\sqrt{N^2 - \omega_{nm}^2}}{\omega_{nm}}\right]. \tag{8.11}$$

with

$$k_m L = m\pi, \quad m = 1, 2, 3, ... \tag{8.12}$$

The eigen-frequencies are:

$$\omega_{nm} = \frac{N}{\sqrt{1 + \left(\frac{nL}{mh}\right)^2}} \tag{8.13}$$

# 9 Internal waves in a vertically unbounded fluid

Consider $N =$ constant (which is good if attention is limited to a small vertical extent), and denote by $(\alpha, \beta)$ the $(x, z)$ components of the wave number vector $\vec{k}$ Let the solution be a plane wave in the vertical plane

$$\psi = \psi_0 \, e^{i(\alpha x + \beta z - \omega t)}$$

Then

$$\omega^2 = N^2 \frac{\alpha^2}{\alpha^2 + \beta^2}$$

or

$$\omega = \pm N \frac{\alpha}{k} \tag{9.1}$$

$$k^2 = \alpha^2 + \beta^2 \tag{9.2}$$

This is the dispersion relation. Note that is

$$\frac{\omega}{N} = \pm \cos \theta' \tag{9.3}$$

where $\theta'$ is the inclination of $\vec{k}$ with repect to the $x$ axis. For a given frequency, there are two possible signs for $\alpha$. Since the above relation is also even in $\beta$, there are four possible inclinations for the wave crests and troughs with respect to the horizon; the angle of inclination is

$$|\theta'| = \cos^{-1} \frac{\omega}{N} \tag{9.4}$$

For $\omega > N$, there is no wave.

To under the physics better we note first that the phase velocity is

$$\vec{C} = \pm \frac{\omega}{k^2} (\alpha, \beta) \tag{9.5}$$

while the group velocity components are

$$\begin{aligned} C_{gx} &= \frac{\partial \omega}{\partial \alpha} = \pm N \left( \frac{1}{k} - \frac{\alpha}{k^2} \frac{\alpha}{k} \right) \\ &= \pm \frac{N}{k} \left( 1 - \frac{\alpha^2}{k^2} \right) = \pm \frac{N}{k^3} \beta^2 \end{aligned}$$

$$C_{gz} = \frac{\partial \omega}{\partial \beta} = \mp \frac{\alpha \beta}{k^3}. \tag{9.6}$$

Thus

$$\vec{C}_g = \pm N \frac{\beta}{k^2} \left( \frac{\beta}{k}, \frac{-\alpha}{k} \right). \tag{9.7}$$

Therefore, the group velocity is perpendicular to the phase velocity,

$$\vec{C}_g \cdot \vec{C} = 0. \tag{9.8}$$

Since

$$\vec{C} + \vec{C}_g = \pm \frac{N}{k^3} \left( \alpha^2 + \beta^2, 0 \right) = \pm \frac{N}{k^2} (k, 0) \tag{9.9}$$

the sum of $\vec{C}$ and $\vec{C}_g$ is a horizontal vector, as shown by any of the sketches in Figure 17. Note that when the phase velocity as an upward component, the group velocity has a downward component, and vice versa. Now let us consider energy transport. from

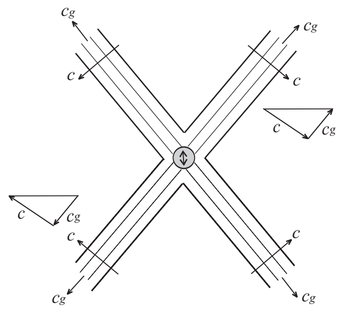


Figure 15: Phase and group velocities

(7.10) we get

$$-p'_x = \overline{\rho}\psi_{zt} = \overline{\rho}\omega\beta\psi_o e^{i(\alpha x+\beta z-\omega t)}$$

hence the dynamic pressure is

$$p' = i\omega\overline{\rho}\frac{\beta}{k}\psi_o e^{i(\alpha x+\beta z-\omega t)} \tag{9.10}$$

The fluid velocity is easily calculated

$$\vec{q}' = (u', v') = (\psi_z, -\psi_x) = i\overline{\rho}(\beta, -\alpha)\psi_o e^{i(\alpha x+\beta z-\omega t)} \tag{9.11}$$

The averaged rate of energy transport is therefore

$$\vec{E} = \frac{1}{2}\overline{\rho}^2|\psi|^2\frac{\beta}{\alpha}(\beta, \alpha) \tag{9.12}$$

which is in the same direction of the group velocity.

Energy must radiate outward from the oscillating source, hence the group velocity vectors must all be outward. Since there are 4 directions for $\vec{k}$. There are four radial beams parallel to $\vec{c}_g$, in four quadrants, forming *St. Andrews Cross*. The crests (phase lines) in the beam in the first quadrant must be in the south-easterly direction. Similarly the crests in all four beams must be outward and toward the horizontal axis. Let $\theta$ be the inclination of a beam with respect to the $x$ axis, then $\theta = \pi/2 - \theta'$ in the first quadrant. The dispersion relation can be written as

$$\frac{\omega}{N} = \pm\sin\theta \tag{9.13}$$

where $\theta$ is the inclination of a beam and not of the wavenumber vector.

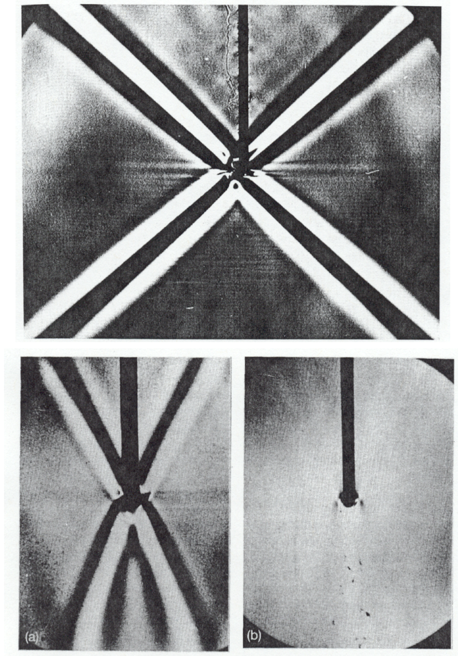


Figure 16: St Andrew's Cross in a stratified fluid. Top: $\omega/N = 0.7$; bottom left $\omega/N = 0.9$; bottom right: $\omega/N = 1.11$. From Mowbray & Rarity, 1965, JFM

Movie records indeed confirm these predictions. Within each of the four beams which have widths comparable to the cylinder diameter, only one or two wave lengths can be seen.

This unique property of anisotropy has been verified in dramatic experiments by Mowbray and Stevenson. By oscillating a long cylinder at various frequencies vertically in a stratified fluid, equal phase lines are only found along four beams forming St Andrew's Cross, see Figure (16) for $\omega/N = 0.7$, 0.9 and 1.11. It can be verified that angles are $|\theta| = 45°$ for $\omega/N = 0.7$, and $|\theta| = 64°$ for $\omega/N = 0.9$. In the last photo, $\omega/N = 1.11$. There is no wave. These results are all in accord with the condition (9.4).

Comparison between measured and predicted angles is plotted in Figure (17) for a wide range of $\omega/N$

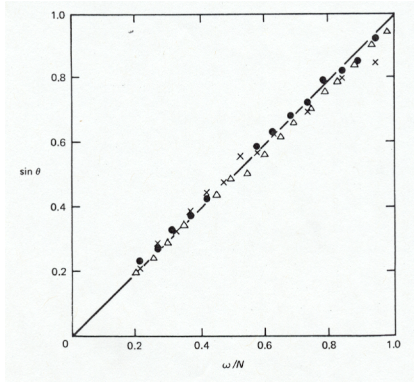


Figure 17: Comparison of measured and predicted angles of internal-wave beams. $\omega/N$ vs. $\sin\theta$. From Mowbray & Rarity, 1965, JFM

# 10 Reflection of internal waves at boundary

For another interesting feature, consider the reflection of an internal wave from a slope.

Recall that $\theta' = \pm\cos^{-1}\frac{\omega}{N}$, i.e., for a fixed frequency there are only two allowable directions with respect to the horizon. Relative to the sloping bottom inclined at $\theta_o$ the inclinations of the incident and reflected waves must be different, and are respectively $\theta' + \theta_o$ and $\theta' - \theta_o$, see Figure 18.

Let $\xi$ be along, and $\eta$ be normal to the slope. Since the slope must be a streamline, $\psi_i + \psi_r$ must vanish along $\eta = 0$ and be proportional to $e^{i(\alpha\xi-\omega t)}$; the total stream function must be of the form

$$\psi_i\, e^{i(k_t^{(i)}\xi-\omega t)} + \psi_r\, e^{i(k_t^{(r)}\xi-\omega t)} \propto \sin\beta\eta e^{i(\alpha\xi-\omega t)}.$$

In particular the wavenumber component along the slope must be equal,

$$k_t^{(i)} = k_t^{(r)} = \alpha$$

Therefore

$$k^{(i)}\,\cos(\theta' + \theta_o) = k^{(r)}\cos(\theta' - \theta_o),$$

which implies that

$$k^{(i)} \neq k^{(r)}. \tag{10.1}$$

as sketched in Figure 18. The incident wave and the reflected wave have different wavelengths! If $\theta' < \theta_o$, there is no reflection; refraction takes place instead.

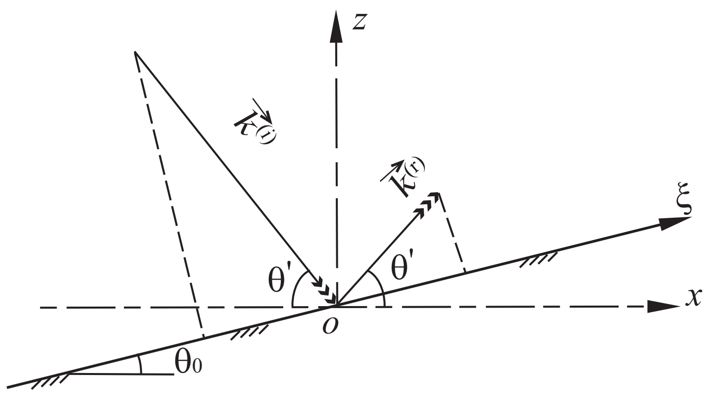


Figure 18: Internal wave reflected by in inclined surface.