OUTLINE

1. INTRODUCTION TO BUSINESS CYCLES AND DSGE MODELS
2. INTRODUCTORY LEVEL MATH
3. SOLUTION AND ESTIMATION OF DSGE MODELS

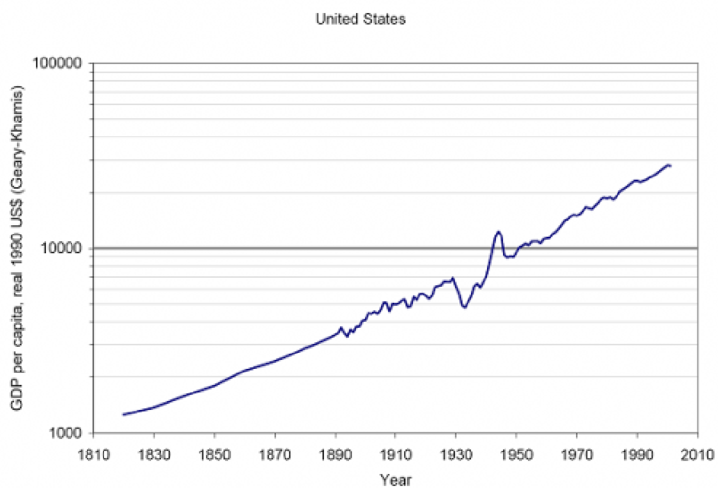
1- INTRODUCTION TO BUSINESS CYCLES AND DSGE MODELS

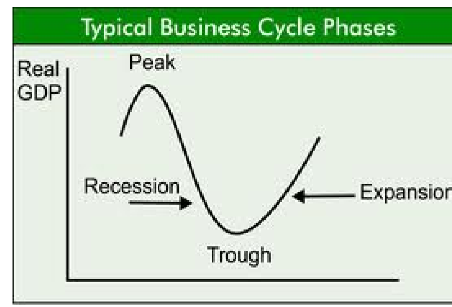
OUTLINE

- What is a Business Cycle?
- Extracting BCs from Raw Data
- What do Economic Theories say about It?
  - Classical Theory
  - Keynesian Theory
- Some Definitions
- Two DSGE Models: A Real Business Cycle (RBC) and a New-Keynesian Model

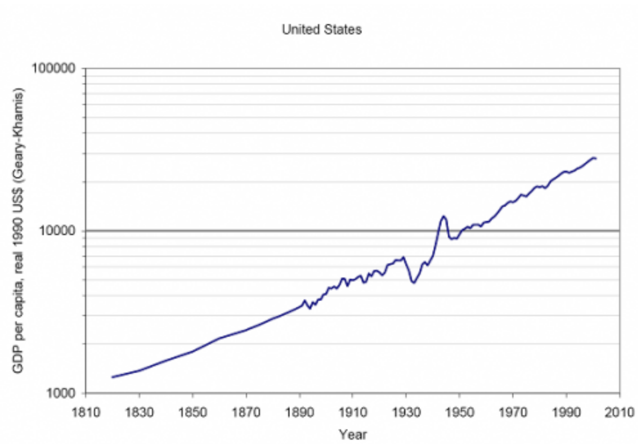
1- What are Business Cycles?

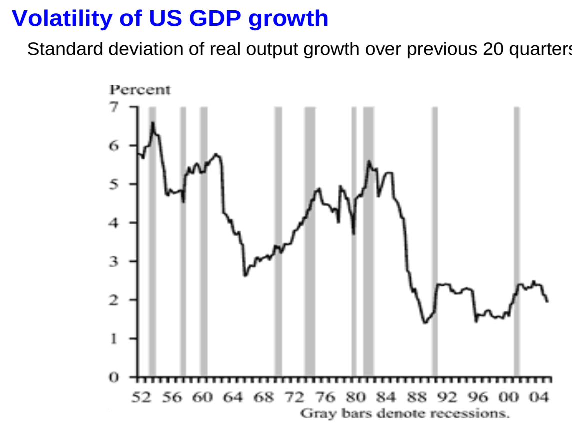
- They are fluctuations in the main macroeconomic variables of a country (GDP, consumption, employment rate, ...) that may have period of three months to a couple of years





- The figure on the right measures the volatility of Real GDP Growth by the standard deviation of the growth rate of the variable.





Summarizing Business Cycle Facts

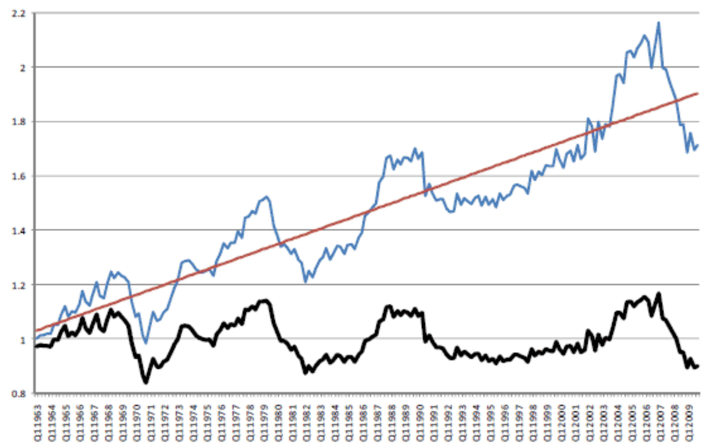
Business Cycle Statistics for the U.S. Economy

	Standard Deviation	Relative Standard Deviation	First Order Auto-correlation	Contemporaneous Correlation with Output
Y	1.81	1.00	0.84	1.00
C	1.35	0.74	0.80	0.88
I	5.30	2.93	0.87	0.80
N	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	-0.35
A	0.98	0.54	0.74	0.78

- Our goal is to write down an economic model that generates fluctuations consistent with the data

2- Extracting BCs from Raw Data

- Suppose the blue line shows the Real GDP. We can think of the data in two components. Growth component (red line: increasing and straight) and the volatile component (black line: volatile and not increasing), which we are interested in.



- The deviation of output from its long term can be interpreted as business cycles and can be found by *detrending* the data. However, detrending assumes GDP (or its log) changes at a constant rate

- Moreover, detrending collects all the irregular movements in the data left over after throwing away the growth component. These irregular movements (cycles) include cycles of high frequency (having short period) and the cycles of shorter frequency. However, in the Economics, cycles of 6 to 24-32 quarters is considered of crucial importance. So to extract Business Cycles of only this length, *filtering* methods are commonly used

- An alternative method is to convert trending variable to a stationary form. For the calse illustrated above, use the growth rate of the Real GDP instead of its level

3- What do Economic Theories Say about Business Cycles?

- Formally, *Business Cycle (or economic cycle)* is an irregular up-and-down movements in the economic activity, often measured by the fluctuations in the growth rate of income (Real GDP). These fluctuations are undesirable since we want economies to have stable growth rate. (However, just because they are undesirable, it does not mean that governments and central banks should respond to them. This decision depends how they see the cause of a fluctuation)
  - We know that equilibrium outcome (income) of the economy occurs at the point where the demand for goods and services is equal to supply. Hence, the change in equilibrium occurs either as a result of a change in the aggregate demand (Keynesian View), or a change in aggregate supply (Classical View)

- We will provide a very brief review of *History of Macroeconomic Thought*. But before doing that let's take a look at some definitions
  - *Procyclical:* Any economic quantity that is positively correlated with the overall state of the economy is said to be procyclical
  - *Countercyclical:* It is opposite of procyclical
  - *Acyclical:* Moving independent of the overall state of an economy
  - *Full Employment:* When there is no unemployment (I refer to the emlpoyment of labor, not capital)
  - *Natural Level of Unemployment:* The level of unemployment that is caused by the permanent problems in the supply side of the economy, such as frictional and structural unemployment

    * *Frictional (Search) Unemployment:* It takes time when a worker is searching for a job, or transitioning from one job to another
    * *Structural Unemployment:* It may result from wage rigidity, job rationing, or the unemployed workers may lack the skills needed for the job
        - The labour market is in equilibrium with the natural level of unemployment
- *Natural Level of Output (Potential Amount):* This is the amount of production when the unemployment is at its natural level
- *Output Gap:* The difference between the actual output and potential (natural level of) output

Early Business Cycle Theory

- *(Early) Classical Theory:* This theory is based on the Say's Law and the belief that prices, wages, and interest rates are flexible
  - *Say's Law:* When an economy produces a certain level of real GDP, it also generates the income needed to purchase that level of real GDP. Hence, the economy is always capable of achieving the natural level of real GDP (the long-run level of GDP)
  - *Flexible prices, wages, and interest rates:* It starts with Adam Smith's writing of the Wealth of Nations in 1776. This analysis suggest that self adjusting prices equates the demand to the supply. (so called invisible hand). For example, during a recession, wages and prices would decline to restore full employment. So economic fluctuations cannot be explained by the demand shocks, but supply

shocks (i.e. they have an external cause, like a change in oil prices). And fluctuations are the optimal response of an economy to these external (exogenous) changes. Thus, there is not much to do about them by using monetary or fiscal policies, or by regulation

* Classical economists also accept that there is an unemployment more than natural level of unemployment. However, they consider it as voluntary unemployment. Voluntarily unemployed workers are unemployed because they refuse to accept lower wages, or maybe wages are held too high by social and political forces, such as minimum wage laws, not because of market imperfections

- *(Early) Keynesian Theory:* States that there are imperfections in the markets (for goods and services, for labor, and for capital) so that wages, prices and the interest rate are not fully flexible and due to this fact in these markets demand does not come into balance with supply immediately. Also that private sector decisions sometimes lead to inefficient macroeconomic outcomes. For example, in recession, falling prices and wages depresses people's incomes and affects the expectations of consumers for the future state of the economy, which further affect aggregate demand and would lead to inefficient outcomes (Inspired by Great Depression in the US, around 1930). This theory is not consistent with Say's Law since it suggests that change in demand can lead to inefficient outcomes. This theory suggests that (i) a change in aggregate demand may cause fluctuations as well (ii) such fluctuations can be confronted with fiscal and monetary policies

– Denying or supporting Say's law is referred to as the "general glut" debate. In macroeconomics, a general glut is when supply exceeds demand. This exhibits itself in a general recession or depression; underutilization of resources. The Great Depression is often cited as an example of a general glut

– One of the fundamental distinction between Classical and Keynesian view is that Classical Economists tend to accept dichotomy between nominal and real sectors (i.e. they are distinct), while Keynesian Economists believe that money can influence real sector and monetary intervention, just like government spending, can be used as a policy tool to stabilize demand

- *Neo Classical Synthesis:* It a "synthesis" of Neoclassical and Keynesian theory. The conclusions of the model in the "long or medium run" or in a "perfectly working" IS-LM system are Neoclassical, but in the "short-run" or "imperfectly working" IS-LM system, Keynesian conclusions held. This synthesis is what you are used to see in undergraduate macro textbooks

- *(Early) Monetary Theory:* The distinction between Keynes and Monetarists (like Milton Friedman and Anna Schwartz) is that in the era of great Depression Keynes proposed government spending to stimulate aggregate demand, whereas Monetarist thought that the Great Depression was caused by a massive contraction of the money supply and remedy is steadily increase it. Keynes believed that especially during severe recession in which people stock money no matter how much the central bank tries to expand the money supply

Criticisms on the Early Business Cycle Literature

- *Lucas Critique* had led Neo-classical and Neo-Keynesian models into so called New-Classical and New-Keynesian models. In doing so, he offered the use of the *rational expectation assumption* and *microfounded models*

- Before Lucas, expectations about the future were used to be formed based on what has happened in the past (*Adaptive Expectations*)

  – Think about an increase in the money supply. Based on the previous experience, people may expect aggregate demand to increase, which increases prices as well (producers increase the supply too). New equilibrium is obtained where there is higher price level and higher output (this historical negative correlation between inflation and unemployment known as the Phillips Curve)

- Lucas argued that such linear relationships between various aggregate macroeconomic quantities over time. As a result, it may give wrong results to predict the effects of a change in economic policy entirely on the basis of relationships observed in historical data
  - Think about the previous example for the effect of an increase in money supply and an increase in output. Lucas Critique suggests that this negative relation between inflation and unemployment could break down if the monetary authorities attempted to exploit it. Permanently raising inflation would eventually cause firms' inflation forecasts to rise, not altering their employment decisions. Eventually, all that the government can do is raise the inflation rate, not employment

- Lucas argued that instead of using aggregate variables, economists should built models based on *Microeconomic Foundations* (like preferences, technology, and budget constraints) that should be unaffected by policy changes. The term microfoundations refers to the analysis of the behavior of individual agents such as households or firms to understand the dynamics of the macroeconomic variables

- Lucas also argued that instead of looking what happened in the past (adaptive expectations), these agents use all available information to make optimal forecasts about the future (*Rational Expectations*). It is assumed that agents' expectations may be individually wrong, but are correct on average.

  – The theory of rational expectations implies that the actual data will only deviate from the expectation if there is an 'information shock' caused by information unforeseeable at the time expectations were formed

- For instance the Lucas supply function with rational expectations implies that only unanticipated changes in the money supply affect real output. Anticipated changes in the money supply affect only the price level leaving real output equal to potential
  * Note: The above statement is only true when prices and wages are perfectly flexible. If prices are sticky, anticipated changes in the money supply have an effect on real output. This is to say even with Rational Expectations, by using market imperfections it is possible to construct Keynesian or Monetary models

Recent Business Cycle Theory: DSGE Models

The models that address Lucas Critique and use microfounded macroeconomic models based on rational choice are called dynamic stochastic general equilibrium (DSGE) models, whether the model is New-Classical, New-Keynesian or Monetarist one that are explained below

- There are also Dynamic General Equilibrium Models, used for the Growth Theory. For instance, the Ramsey Model uses both households and firms that maximize their utilities and profits. The Ramsey is deterministic, meaning there are no shocks to economy. The stochastic models (models that allow shocks) are called Dynamic Stochastic General Equilibrium Models). If models are stochastic, agents needs to make some forecasts for the future state of their economies

- DSGE models *often* assume that all agents of a given type are identical (i.e. there is a 'representative household' and a 'representative firm'), thereby avoiding aggregation problems. However, this is a simplifying assumption, and is not essential for the DSGE methodology.

- DSGE models use (real or nominal) shocks in general equilibrium setting to analyze the effect of deviation of variables from their expected values, which create Business Cycles.

- *New-Classical View:* There are different New-Classical Models. Their common property is use a DSGE model and flexible prices, or not to use prices at all (like the RBC Model below)

  - *Real Business Cycle (RBC) Theory:* Pioneering work by Kydland and Prescott (1982). It argues that it is mistake to call them 'fluctuations' around the trend of the economic growth. Trend itself would be volatile and those fluctuations could be just part of the data. Remember that technological development is the basis for growth. So this discussion implies that technology is a volatile process. To model the economy, they use constant growth rate for technology, like in the Neo-Classical Growth Model of the Ramsey, but they also use technology shocks (also called supply or productivity shocks) that causes technology to deviate from its trend and explains short run fluctuations (usually in the form of labor productivity). Since these are real shocks, as they affect the production, it is called Real Business Cycle Theory. Productivity shocks—that can be measured using Solow's 1958 growth accounting approach—could generate time series with the same complex patterns of persistence, comovement, and volatilities as complex economies. As fluctuations arise from the optimal response of agents to the real shocks, there is no need for active policy response to stabilize output over the business cycle, which is a general inference can be made from Classical Models.

– Optimization in DSGE models yields non-linear behavior, ruling out analytical solutions in general cases. The common approach is to linearize the model around the steady state of the system and consider an approximate solution

- *New-Keynesian Approach*: Demand management is in the heart of Keynesian Theory. New Keynesianism provide market imperfection in the form of sticky prices to justify demand management. This market imperfection forms a barrier in front of self adjusting mechanism of the economy assumed by Classical Economists. Besides sticky prices, another market imperfection built into most New Keynesian models is the assumption that firms are monopolistic competitors, which gives some monopoly power to firms. This modelling is necessary because it allow firms to use their market power to maintain their prices above marginal cost, so that even if they fail to set prices optimally they will remain profitable (RBC models use competitive markets instead)

4- Some Definitions: Stochastic version of Samuelson's (1939) Classical Model (An Early Macro Model)

$$y_t = c_t + i_t \tag{1}$$

$$c_t = \alpha y_{t-1} + \epsilon_{ct} \tag{2}$$

$$i_t = \gamma(c_t - c_{t-1}) + \epsilon_{it} \tag{3}$$

- Is this a model or system of equations?
  - Both. It is a model with three equations.
- Is this a dynamic or static model?
  - A dynamic one (Some past variables affect the current variables. In the static model everything occurs within the same period)

- What are ϵ's above?
  - They are disturbances (shocks). These are necessary because we cannot expect the real data of variables to move one-to-one in accordance with the relations put by these equations
    - Putting these terms together, this is a Dynamic Stochastic Model
- Is this a structural model or a reduced form one?
  - It is structural because it explains endogenous variables (that are on the left hand side, model attempts to explain) with current realizations of other endogenous variables. A reduced form model explains endogenous variables with their and other endogenous variables' lags, also current and past values of exogenous variables

- How can we understand if it is a good or useless model?
  - By looking at its fit to the data (the goodness of fit). There are different methods of estimating parameters of a model and measuring its goodness of fit. For each estimation method, we define an objective function, and try to find a set of model parameters that minimizes that. For example in the regression analysis, the objective function is minimizing sum of squared residuals

- Is it a (partial or general) equilibrium model?
  - This is not an equilibrium framework. It doesn't explain the underlying fundamental relation between the variables by aggregating the behavior of individuals and firms. i.e. it does not explain the behavior of supply, demand and prices in a whole economy with several markets by seeking to prove that equilibrium prices for goods exist. It just expresses a theoretical link between aggregate variables, which Lucas criticized
  - The next section gives two examples of Dynamic Stochastic Model with General Equilibrium framework

5- Some DSGE Models: RBC Models

- The utility of a representative consumer

$$\max E[\sum_{t=1}^{\infty} \beta^t U(C, 1 - N_t)] \qquad \beta < 1$$

where N is the amount of labor employed in the production by the consumer. S/he is endowed with 1 unit of labor and $1 - N$ is the leisure. The leisure in the utility function means that s/he dislikes working and gets utility from leisure as well as consumption

This equation is subject to the resource constraint

$$C_t + I_t = Y_t$$

dynamic equation for capital (capital accumulation)

$$K_t = I_t + (1 - \delta)K_{t-1}$$

the (neoclassical) production function

$$Y_t = Z_t F(K_{t-1}, A_t N_t)$$

exogenous processes for technology

$$A_t = \gamma A_{t-1} \quad \gamma > 1$$

$$\log Z_t = (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t \qquad \psi < 1 \qquad \epsilon_t \sim N(0, \sigma^2)$$

- This problem is different than the Ramsey Economy in two respects
  - There is Z_t in the production function, which governs short term movements in the technology, whilst A_t stands for the growth component of technology
  - The leisure (or labor) affects utility

* When this is the case, agents optimize their labor decision in response to technological shocks. Thus, the model is able to explain fluctuations in the employment that we observe in the real economies

- This is the problem of benevolent social planner (like in the ideal socialist economy). S/he tries to maximize the utility of citizens given the resource constraints in the economy. We could have written the same problem by defining consumers trying to maximize their utility and firms trying to maximize their profits (as in the case of free markets). As long as there is a perfect competition on the side of firms, competitive markets give the same solution with the social planner

- Referring to the (neo-classical) production function

$$Y_t = Z_t F(K_{t-1}, A_t N_t)$$

exogenous process for technologies

$$A_t = \gamma A_{t-1} \quad \gamma > 1 \qquad \text{and} \qquad \log Z_t = (1-\psi) \log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t$$

  - Timing: Realize that current output (Y_t) depends on the amount of capital that is decided previous period (K_{t-1}). This is just a dating convention and not an obligation. Since in the real economy it takes some time to make an investment and increase the stock of capital, in this model the current output is the output that can be produced with the capital in hand

- Technology: A_t is the technological process used to explain economic growth. This technology grows at a constant rate (γ) at each period, so does the economy
    * Z_t is technological process as well but it is only used to explain the short run deviations in the data. It is the heart of RBC theory, as it assumes that fluctuations in the economy comes from the supply (production) side
    * Z_t has a steady state value ($\bar{Z}$), which is subject to shocks (ϵ_t). The shocks first affect Z_t, but their effect lasts more than one period (because Z_{t+1} is affected by Z_t, also Z_{t+2} is affected by Z_{t+1}, and so on)

- How to solve a RBC model?
  - We can reduce equations into two

$$\max E[\sum_{t=1}^{\infty} \beta^t U(C, 1 - N_t)]$$

and

$$Y_t + (1 - \delta)K_{t-1} = C_t + K_t$$

- Now write it in Lagrangian form and take FOCs w.r.t. C_t, N_t, K_t

$$\max_{\{C_t,K_t,N_t\}_{t=0}^{\infty}} E[\sum_{t=1}^{\infty} \beta^t U(C_t, 1-N_t) + \lambda_t(Y_t + (1-\delta)K_{t-1} - C_t - K_t)]$$

$$\frac{\partial}{\partial C_t}: \quad \beta^t U'_C(C_t, 1-N_t) - \lambda_t = 0 \tag{1}$$

$$\frac{\partial}{\partial N_t}: \quad -\beta^t U'_{1-N}(C_t, 1-N_t) + \lambda_t \frac{\partial Y_t}{\partial N_t} = 0 \tag{2}$$

$$\frac{\partial}{\partial K_t}: \quad -\lambda_t + \lambda_{t+1}[\frac{\partial Y_{t+1}}{\partial K_t} + (1-\delta)] = 0 \tag{3}$$

- Combining equations (1) and (2)

$$U'_{1-N}(C_t, 1-N_t) = U'_C(C_t, 1-N_t)\frac{\partial Y_t}{\partial N_t}$$

this equation says that at the equilibrium the consumer is indifferent between having a leisure and get the utility of $U'_{1-N}(C_t, 1-N_t)$ and getting the utility of $U'_C(C_t, 1-N_t)$ from consuming each unit of good that s/he affords from producing $\partial Y_t/\partial N_t$, which, under competitive markets equal to the wage

- Combining equations (1) and (3)

$$U'_C(C_t, 1 - N_t) = \beta U'_C(C_{t+1}, 1 - N_{t+1}) R_{t+1}$$

where

$$R_t = \frac{\partial Y_t}{\partial K_{t-1}} + (1 - \delta)$$

the second equation is the gross return on capital. This is equal to the marginal product of capital, $\partial Y_t / \partial K_{t-1}$, plus the proportion of capital left at the end of capital $(1 - \delta)$. The left hand side of the first equation is marginal utility of consumption. This equation suggests that agents are indifferent between consuming today, or use it for saving and consume the return tomorrow

Example: Hansen's RBC Model

- The utility of a representative consumer

$$\max E[\sum_{t=1}^{\infty} \beta^t (\ln C_t + a(1 - N_t)]$$

subject to the resource constraint and dynamic equation for capital

$$C_t + I_t = Y_t \qquad\qquad K_t = I_t + (1-\delta)K_{t-1}$$

production function

$$Y_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho}$$

exogenous process for technology

$$\log Z_t = (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t$$

- In Lagrangian form

$$\max_{\{C_t,K_t,N_t\}_{t=0}^{\infty}} E[\sum_{t=1}^{\infty}\beta^t(\ln C_t+A(1-N_t))+\lambda_t(Z_tK_{t-1}^{\rho}N_t^{1-\rho}+(1-\delta)K_{t-1}-C_t-K_t)]$$

$$\frac{\partial}{\partial C_t}: \quad \beta^t\frac{1}{C_t}-\lambda_t=0 \tag{1}$$

$$\frac{\partial}{\partial N_t}: \quad -\beta^t A+\lambda_t(1-\rho)\frac{Y_t}{N_t}=0 \tag{2}$$

$$\frac{\partial}{\partial K_t}: \quad -\lambda_t+\lambda_{t+1}[\rho\frac{Y_{t+1}}{K_t}+(1-\delta)]=0 \tag{3}$$

  - Combining equations (1) & (2)

$$a=\frac{1}{C_t}(1-\rho)\frac{Y_t}{N_t} \tag{4}$$

– Combining equations (1) & (3)

$$\frac{1}{C_t} = \beta \frac{1}{C_{t+1}} R_{t+1} \tag{5}$$

where

$$R_t = \rho \frac{Y_t}{K_{t-1}} + (1 - \delta) \tag{6}$$

How can we use this model to analyze fluctuations in an economy?

1 First we need to find the steady state of the model that can explain the economy in its steady state. The economy is its steady state when there is no shock to economy. The model is its steady state when there is no shock to the model.

– As we eliminated growth from the model, the steady state of consumption satisfies that $C_t = C_{t+1} = ... = \bar{C}$. The case is similar for other variables.

* In this model there are many unknowns ($\bar{C}$, $\bar{Y}$, $\bar{K}$, $\bar{Z}$, $\bar{R}$, $\bar{I}$, $\bar{N}$, a, ρ, β, δ, σ). We can use the macro data to find some of these steady state values (the long-term average of variables) and use studies of micro data to calibrate some parameters. Then we can calculate the rest from the model

2 Then we should analyze the responses of the model to the shocks and see if the responses of the model match the fluctuations in the real economy. There are two common ways of doing that

2a Use growth accounting and find the technological progress in the data

$$\frac{\Delta Z}{\bar{Z}} = \frac{\Delta Y}{\bar{Y}} - \rho\frac{\Delta K}{\bar{K}} - (1-\rho)\frac{\Delta N}{N}$$

then supply economic model with these shocks (so called Solow Residuals) and see if it fits the data; i.e. if it matches the data and generates the same complex patterns (correlation among variables, autocorrelation and variance of each variable) with the actual data

2b Or (more common case) go directly to the economic model, and supply it with artificial shocks, ϵ_t. Then check if the generated data matches the second moments of the data (again correlation among variables, autocorrelation and variance of each variable)

3 Once you have a good model you have an idea of how the economy works. You can also use the model for policy analysis. For example you may change the parameters of the model and see how the economy (both its steady state and the way it responds to technological shocks) responds to that change

1- The Steady State

- Equation (4), (5) and (6) implies that

$$a = \frac{1}{\bar{C}}(1-\rho)\frac{\bar{Y}}{\bar{N}} \qquad 1 = \beta\bar{R} \qquad \bar{R} = \rho\frac{\bar{Y}}{\bar{K}}+(1-\delta) \quad \text{where} \quad \bar{Y} = \bar{Z}\bar{K}^{\rho}\bar{N}^{1-\rho}$$

  - To find some of A, $\bar{C}$, $\bar{Y}$, $\bar{K}$, $\bar{Z}$, $\bar{R}$, $\bar{I}$, $\bar{N}$, ρ, β, δ, σ, we can refer to the macro and micro data and use
  - $\bar{Z} = 1$ (Normalization)
  - $\rho = .36$ (Capital Share in the production)
  - $\bar{N} = 1/3$ (Steady state employment is a third of total time endowment)
  - $\delta = 0.025$ (Depreciation rate for capital)
  - $\bar{R} = 1.01$ (One percent real interest per quarter)

- $\psi = 0.95$ (Autocorrelation of technology shock)

- Then we can calculate the rest
  - $\beta = 1/\bar{R} = 0.99$
  - $\bar{Y}/\bar{K} = (\bar{R} + \delta - 1)/\rho$
  - $\bar{K} = (\bar{Y}/\bar{K} * 1/\bar{Z})\hat{}(1/(\rho - 1)) * \bar{N}$
  - $\bar{I} = \delta * \bar{K}$
  - $\bar{Y} = \bar{Y}/\bar{K} * \bar{K}$
  - $\bar{C} = \bar{Y} - \delta\bar{K}$
  - $a = 1/\bar{C} * (1 - \rho) * \bar{Y}/\bar{N}$ (Parameter in utility function)

- There is one final parameter left, but we are free to set it to any value
  - $\sigma = 0.712$; % Standard deviation of technology shock

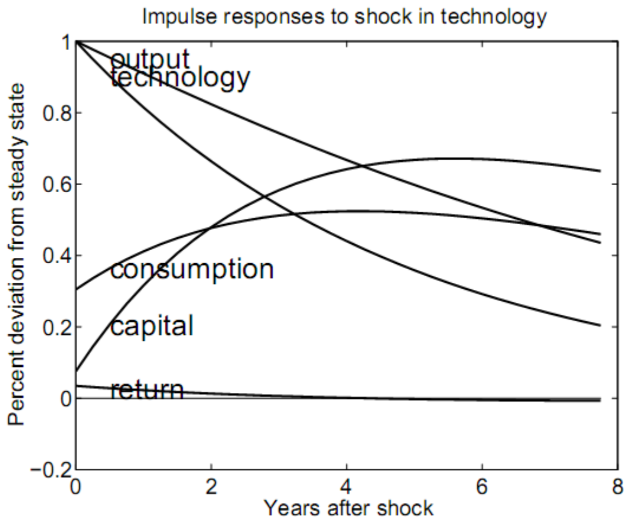
2 - Linearization around the steady state (will come to this)

3 - Evaluating the Model

Standard deviations in percent (a) and correlations with output economies.

	Quarterly U.S. time series[a] (55,3–84,1)		Economy with divisible labor[b]	
Series	(a)	(b)	(a)	(b)
Output	1.76	1.00	1.35 (0.16)	1.00 (0.00)
Consumption	1.29	0.85	0.42 (0.06)	0.89 (0.03)
Investment	8.60	0.92	4.24 (0.51)	0.99 (0.00)
Capital stock	0.63	0.04	0.36 (0.07)	0.06 (0.07)
Hours	1.66	0.76	0.70 (0.08)	0.98 (0.01)
Productivity	1.18	0.42	0.68 (0.08)	0.98 (0.01)

4 - Impulse Responses from the Model and Policy Analysis



Impulse responses show us the effect of a temporary technology shock (ϵ_t) on the variables of the model, i.e. how they deviate from their steady state values. This figure is another way to see the information in the third and fourth column of the previous table

- Upon a temporary technology shock, output and marginal return to capital (and the interest rate) increase. This stimulates investment on capital. The increase in output induces people to consume more as well. After some periods, the economy goes back to its steady state

INTRODUCTORY LEVEL MATH

- *Difference equation* expresses the value of a variable as a function of its own lagged values, time, and other variables
- Linear difference equation of order n:

$$y_t = a_0 + a_1 y_{t-1} + ... + a_n y_{t-n} + \varepsilon_t$$

- *Time-series econometrics* is concerned with the estimation of difference equations containing stochastic components
- *Ordinary Differential Eq. (ODE):* Depends one independent variable
- *Partial Differential Eq. (PDE):* Depends multiple independent variables

- *First Order PDE:* That involves only the first derivatives of the variables

$$F(x_1, ..., x_n, U, U_{x_1}, ..., U_{x_n}) = 0$$

where $U(x_1, ..., x_n)$ is an unknown function

- *Recursive Problem:* The decisions made today and realization of uncertainty determine the state variable in the next sequential peirod

- In a typical DSGE model, k_t is called as the *state variable* (embodies the prior and current information of the economy), c_t is the *control variable*, and z_t is the *exogenous state variable*

- We consumption in terms of state variables at time: $c_t(k_t, z_t)$, called the *policy function*

Solving First Order Homogeneous Difference Equations

- Consider the equation

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$$

- The homogeneous part of this equation is

$$y_t^h = a_1 y_{t-1}^h$$

- The homogeneous parts have a solution of the form

$$y_t^h = \alpha^t y_0$$

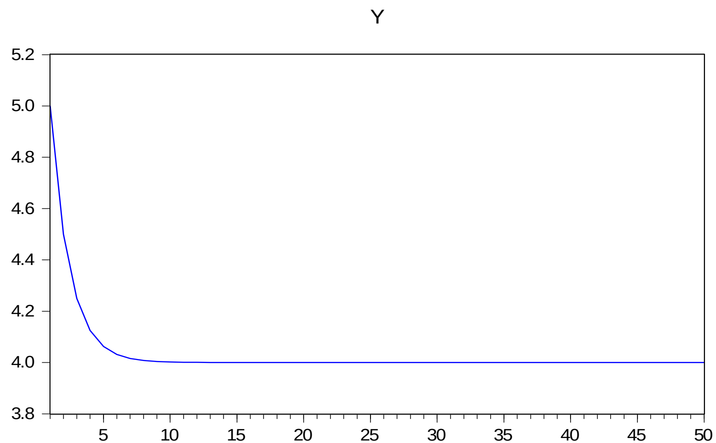
- Combining the last two equations finds

$$\alpha = a_1$$

- Stability requires that $\mid a_1 \mid < 1$. In that case, y will be a convergent series and its long term mean is

$$\bar{y} = a_0 + a_1 \bar{y} \qquad \Rightarrow \qquad \bar{y} = \frac{a_0}{1 - a_1}$$

- *Ex:* $y_t = 2 + 0.5 y_{t-1}$, where $y_0 = 5$



Solving Second Order Homogeneous Difference Equations

- Consider the homogeneous equation

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} = 0$$

- Its solution has the form

$$y_t^h = \alpha^t y_0$$

combining with equation (45)

$$\alpha^t y_0 - a_1 \alpha^{t-1} y_0 - a_2 \alpha^{t-2} y_0 = 0$$

dividing by α^{t-2}

$$\alpha^2 - a_1 \alpha - a_2 = 0 \tag{47}$$

- Solving this quadratic equation yields two characteristic roots:

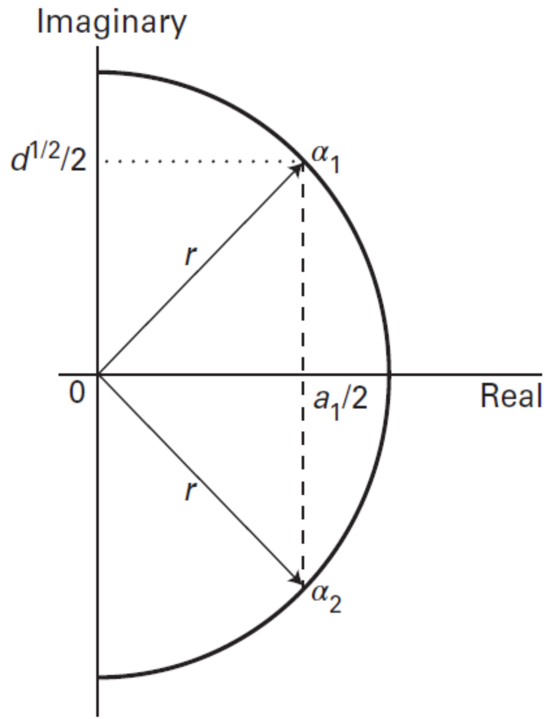
$$\alpha_1, \alpha_2 = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

- If $a_1^2 + 4a_2 \geq 0$
  - There will be real characteristic roots
- If $a_1^2 + 4a_2 < 0$
  - In this case the characteristic roots have both real and imaginary parts

$$\alpha_1, \alpha_2 = (a_1 \pm i\sqrt{-d})/2$$

where $d = a_1^2 + 4a_2$ and $i = \sqrt{-1}$

- Consider the following semicircle



- Real numbers are measured on the horizontal axis and imaginary numbers are measured on the vertical axis
- In the time-series literature, it is simply stated that
  - If all characteristics roots lie within the unit circle, then the equation and its solution are stable
  - If at least one characteristics root lie outside the unit circle, then the equation and its solution are unstable
  - If at least one characteristics root lie on the unit circle, then the equation is unstable and contains a unit root

Solving Second Order Homogeneous Difference Equations with Lag Operators

- Consider once again the homogeneous equation

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} = 0$$

which can be written as

$$y_t - a_1 L y_t - a_2 L^2 y_t = 0$$

which can be simplified as

$$1 - a_1 L - a_2 L^2 = 0$$

we can either try to solve this quadratic equation, or multiply it by L^{-2}, which finds

$$L^{-2} - a_1 L^{-1} - a_2 = 0$$

when we compare it with Equation (47)

$$\alpha^2 - a_1\alpha - a_2 = 0$$

we see that $\alpha = L^{-1}$

- Hence, when written in lag operators, stability requires that
  - If all characteristics roots lie outside the unit circle, then the equation and its solution are stable
  - If at least one characteristics root lie inside the unit circle, then the equation and its solution are unstable
  - If at least one characteristics root lie on the unit circle, then the equation is unstable and contains a unit root

(Log) Linear Approximation

- Non-linear system of equations hardly has a closed analytical solution
- They are converted to a linear ones around a non-stochastic s.s.
- Taylor Approximation of x around x^*:

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{f''(x^*)}{2}(x - x^*) + ..$$

Ex: Cobb-Douglas Production Function

$$Y = K^{\alpha} L^{1-\alpha}$$

- First order linear Taylor Approximation

$$Y = Y^{*} + \alpha \frac{Y^{*}}{K^{*}}(K - K^{*}) + (1-\alpha)\frac{Y^{*}}{L^{*}}(L - L^{*})$$

that is

$$Y - Y^{*} = \text{constant} * (K - K^{*}) + \text{constant} * (L - L^{*})$$

where the terms inside the paranthesis are linear and show deviations from s.s. in levels.

- When divided by Y^{*}

$$\frac{Y - Y^{*}}{Y^{*}} = \alpha \frac{K - K^{*}}{K^{*}} + (1-\alpha)\frac{L - L^{*}}{L^{*}}$$

- First order log-linear Taylor Approximation
  - First take the logs

$$\ln(Y) = \alpha \ln(K) + (1-\alpha)\ln L$$

  - then apply Taylor approximation

$$\ln(Y) = \ln(Y^*) + \alpha \frac{1}{K^*}(K - K^*) + (1-\alpha)\frac{1}{L^*}(L - L^*)$$

  - Both $(K - K^*)/K^*$ and $\ln(Y) - \ln(Y^*)$ are percentage deviations from their steady states; hence,

$$\widetilde{y} = \alpha \widetilde{k} + (1-\alpha)\widetilde{l}$$

- An easy way of taking first order log-linearization (Uhlig, 1999)

$$X_t = Xe^{x_t}$$

where x_t is the log deviation from steady state.

$$\text{Proof:} \qquad Xe^{x_t} = Xe^{\ln(X_t)-\ln(X)} = Xe^{\ln(X_t/X)} = X\frac{X_t}{X} = X_t$$

- Some useful transformations

$$e^{ax_t+by_t} = 1 + ax_t + by_t$$

$$x_t * y_t \approx 0$$

- Ex: Cobb-Douglas Production Function

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$

make the transformation

$$Ye^{y_t} = (Ke^{k_t})^{\alpha}(Le^{l_t})^{1-\alpha}$$

as $Y = K^{\alpha}L^{1-\alpha}$, the capital letters cancel out

$$e^{y_t} = e^{\alpha k_t + (1-\alpha) l_t}$$

which finds

$$y_t = \alpha k_t + (1-\alpha) l_t$$

- Ex:

$$Y_t = C_t + I_t$$

make the transformation

$$Ye^{y_t} = Ce^{c_t} + Ie^{i_t}$$

$$Y(1 + y_t) = C(1 + c_t) + I(1 + i_t)$$

$$Y + Yy_t = C + Cc_t + I + Ii_t$$

note that $Y = C + I$; hence

$$y_t = \frac{C}{Y}c_t + \frac{I}{Y}i_t$$

- Another way of taking first order log-linear Taylor Approximation

$$\ln(Y_t) = \alpha \ln(K_t) + (1-\alpha)\ln L_t$$

take derivative w.r.t. time

$$\frac{\dot{Y}}{Y} = \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L}$$

where $\dot{Y}/Y$ is the percentage change from the s.s. value, that is

$$y_t = \alpha k_t + (1-\alpha) l_t$$

- This method becomes messy with more complicated equations

First and Second Order (Log) Linear Approximations

- First order approximations are linear functions of deviations in the variables
$$y = f(x) = f(x^*) + f'(x^*)(x - x^*) \qquad \text{(F)}$$
whereas second order approximations are not
$$y = f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{f''(x^*)}{2}(x - x^*)^2 \qquad \text{(S)}$$

- For linear functions
$$E(f(y)) = f(E(y)));$$
hence, with (F), the expected value of y is equal to its s.s. value
$$E(y) = f(x^*) + f'(x^*)[E(x) - x^*)] = f(x^*)$$

whereas with (S) it also depends on the curvature of the function and the variance of x

$$E(y) = f(x^*) + \frac{f'(x^*)}{2} var(x)$$

- In order to characterize impulse response and second moments of variables it suffices to analyze first order properties of the model

  – With (F)

$$var(y) = [f'(x^*)]^2 var(x)$$

  – With (S)

$$var(y) = [f'(x^*)]^2 var(x) + [\frac{f''(x^*)}{2}]^2 [var(x)]^2$$

  where the second term can be ignored

- In general, according to Schmitt-Grohe and Uribe (2004), for the solution of discrete time rational expectations models, where:

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = 0$$

  where y_t is a vector of non-predetermined variables and x_t is a vector predetermined endogenous and exogenous state variables

  – Up to first order approximation, the policy function is, $c_t(\cdot)$, is independent of the variance of shocks (certainty equivalance)

  – Up to second order approximation, the policy function is, $c_t(\cdot)$, is affected from the size of the variance of shocks. However, there is no interaction term between this variance and other state variables

- First-order accurate approximations are insufficient in many economic examples, in particular for comparing welfare across policies which do not have any first-order effects on the model's non-stochastic steady state

- To conclude, both a second order approximation of utility and a second order approximation of the economic model may be necessary. In other words, the solution of interesting problems requires perturbation methods

- See Woodford (1999) for a discussion of conditions under which it is correct up to second order to approximate the level of welfare using first-order approximations to the policy function.

Solution and Estimation of DSGE Models

1. Write down the model
2. Solve for Equilibrium Conditions
   - Take the FOCs, which, together with model constraints, form a system of non-linear stochastic difference equations
   - Dynamic Optimization in Discrete time: Dynamic Programming (Bellman Equation)
   - Dynamic Optimization in Continuos Time: Optimal Control Theory (Hamiltonian)

3. Find the steady state ($k_t = k_{t-1} = k$, $\varepsilon_t = 0$) by parametrization
   - Suppose a steady state condition is: $1 = \beta \bar{R}$
     - If one fixes β, it will imply value for $\bar{R}$
     - If one can fixes $\bar{R}$, it will imply value for β
4. Solve for the policy (decision) rules, such as $c(states, var(\varepsilon_t))$ or $k(states, var(\varepsilon_t))$
   - Approximation around s.s.
     - First order (log) linearization
       * Neglects $var(\varepsilon_t)$ (certainty equivalance)
       * Finds a system of linear difference equations in state-space form, which can be solved via *the Method of Undertermined Coeffcients*, or via *Blanchard and Kahn (1980)*

- Linear-Quadratic Approximation
- Higher orders approximations
  - Approximations can be carried out via Perturbation
- Solution may require the solution of matrix quadratic equations
- Method of Undetermined Coefficients
- State-Space Approach
- Blanchard and Kahn (1980)
- Projection methods?

• Value function iteration (it is slow)

5. Model Evaluation (to be compared with the real data)

- Use simulation (via employing shocks to the model)
  - Good for small sample properties that is required for testing
- or Frequency Domain Methods (theoretical solution)
- to obtain
  - Impulse Response Functions
  - Correlations and Relative Variances between variables
    - HP filter is typically applied before calculate the moments
- Compare the results with the real data

Parameters of a DSGE Model

- Calibration
- Estimation
  - Frequentist (Classical) Approach
    - GMM
    - Indirect Inference (Matching Impulse Response Functions)
    - Maximum Likelihood (Dynare)
- Bayesian Estimation (Calibration +Likelihood) (Dynare)

- What is likelihood?

- *Ex:* Consider a random sample of the following 10 observations from a Poisson distribution: 5, 0, 1, 1, 0, 3, 2, 3, 4, and 1. Given θ, probability of observing x_i is
$$f(x_i|\theta) = \frac{e^{-\theta}\theta^{x_i}}{x_i!}$$

- Because the observations in the example are independent, their joint density, which is the likelihood, is as follows
$$f(x_1, x_2, ..., x_{10}|\theta) = \prod_{i=1}^{10} f(x_i|\theta) = \frac{e^{-10\theta}\theta^{20}}{207,360}$$

- The last result gives the probability of observing this particular sample for every θ, assuming that a Poisson distribution

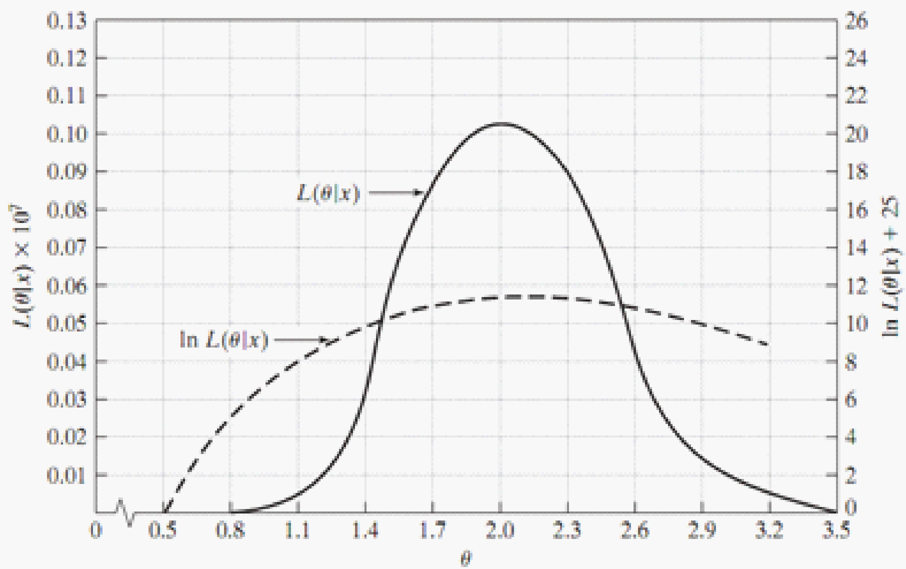


FIGURE 14.1 Likelihood and Log-Likelihood Functions for a Poisson Distribution.

$$p(x|\theta) = L(\theta|x)$$

- How do you calculate the likelihood for DSGE models?
  - Put your model into state-space form

$$x_t = Fx_{t-1} + Bz_t \tag{1}$$

$$y_t = Hx_t + z_t \tag{2}$$

    where (1) is the state transition equation and (2) is the observation equation
  - Assume a wide range of parameter ($\theta = F, B, H$) values
  - Kalman filter gives the probability of observing data for each parameter set: $L(\theta|Y)$
  - Maximum Likelihood Estimation: Finds the θ that maximizes the likelihood

- Bayesian Estimation: Given prior belief about parameters, $P(\theta)$, it calculates the posterior distribution: $L(\theta|Y) * P(\theta) = f(\theta)$
  - It is hard to find a functional form for these probabilities (as a function of the parameter set). Employ Markov chain Monte Carlo (MCMC) methods, such as
    - Gibbs Sampling
    - Metropolis-Hastings Algorithm

Summary of the Process

- The posterior is a mixture (multiplication) of the prior information and the "current information" that is, the data.

- The next step is evaluating the likelihood function. When the model is linear in terms of the structural parameters in θ, we can use maximum likelihood principles. However, in DSGE models often this is not the case. Yet, since DSGE models are linear in terms of the variables, the likelihood may be evaluated with a linear prediction error algorithm like the Kalman filter.

- The prediction errors of the Kalman filter are normally distributed, which gives the probability of observing any data (including the real observation itself) given the set of parameters: $p(Y|\theta) = L(\theta|Y)$

- Kalman filter is calculated for every θ, and the likelihood is nonlinear and complicated function of θ's. Thus, we cannot obtain an explicit form for it. Instead, we use Metropolis-Hastings algorithm.

- We draw some parameter set, say θ_A, from the priors and calculate $L(\theta|Y)P(\theta)$ $[= f(\theta)]$. Then assuming that θ_A is the mean of a normal distribution, pick another set of parameters, say θ_B, from this distribution and calculate f(θ). If θ_B improves $f(\theta)$ compared to θ_A, reset the new mean of a normal distribution to the θ_B, and pick new draws, and again calculate $f(\theta)$. Otherwise use θ_A as a mean and again continue to pick new draws. When you make enough sampling, build a histogram of the retained values. This "smoothed histogram" will eventually be the posterior distribution; that is $f(\theta)$.

Solving Recursive Stochastic Linear Systems with the Method of Undetermined Coefficients

- We know that eventually we will obtain the following policy rules as a solution of a log-linearized model

$$k_t = v_{kk} k_{t-1} + v_{kz} z_t$$

$$c_t = v_{ck} k_{t-1} + v_{cz} z_t$$

- To solve for the coeffcients v_{kk}, v_{kz}, v_{ck}, v_{cz}, substitute the equations given above into the equilibirum conditions of your model until only k_{t-1} and z_t remain.

- Then compare coeffcients on the RHS and LHS

- *For larger models solving for everything by hand is cumbercome. This task can be avoided by applying directly the theorems discussed below (Uhlig, 1999)
  - Brute Force
  - Sensitivity

Solving Recursive Stochastic Linear Systems with the Method of Undetermined Coefficients: BRUTE FORCE

- x_t is a vector
- z_t is a vector of exogenous variables that are given at date t, i.e. which cannot be changed at date t

$$0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t] \tag{5.1}$$

$$z_{t+1} = Nz_t + \epsilon_{t+1}; \qquad E_t[\epsilon_{t+1}] = 0 \tag{5.2}$$

For some equations it may be that $F = L = 0$

- What one is looking for a linear function (the recursive equilibrium law of motion)

$$x_t = Px_{t-1} + Qz_t \tag{5.3}$$

- Theorem: If there is a recursive equilibrium law of motion solving equations (5.1), and (5.2), then the following must be true
  - Calculation of P and Q from the coefficient matrices is straightforward
  - However, P satisfies the (matrix) quadratic equation

$$0 = FP^2 + GP + H$$

  And the equilibrium described by (5.3) and (5.2) is stable if all eigenvalues of P are smaller than unity in absolute value

Solving Recursive Stochastic Linear Systems with the Method of Undetermined Coefficients: SENSITIVITY

- x_t is a vector of endogenous states (size m)
- y_t is a vector of other endogenous variables (size n)
- z_t is a vector of exogenous stochastic processes (size k)
- This time we separate equations that do not involve taking expectations (the number of which is l)

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t \tag{5.7}$$

$$0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t] \tag{5.8}$$

$$z_{t+1} = Nz_t + \epsilon_{t+1}; \qquad E_t[\epsilon_{t+1}] = 0 \tag{5.9}$$

- What one is looking for a linear function (the recursive equilibrium law of motion)

$$x_t = Px_{t-1} + Qz_t \tag{5.10}$$

$$y_t = Rx_{t-1} + Sz_t \tag{5.11}$$

where it is assumed that C is of size $l * n$ and $l \geq n$

- Calculation of P, R, Q and S from the coefficient matrices is straightforward

- Depending on whether $l > n$ or $l = n$
  - P satisfies different (matrix) quadratic equations

$$0 = \ldots$$

  The equilibrium described by (5.11), (5.12) and (5.9) is stable if all eigenvalues of P are smaller than unity in absolute value

Solving Recursive Stochastic Linear Systems: BLANCHARD and KAHN (1980)

- The method of undetermined coefficients assumes the solution exists, this one does not

- Consider the following linear stochastic difference equations under rational expectations, which can be written in state-space form

$$A_0 E_t Y_{t+1} = A_1 Y_t + B_0 \varepsilon_{t+1} \tag{1}$$

  where Y_t denotes a vector of all endogenous and ε_t a vector of all exogenous variables

- The standard method of eigenvector-eigenvalue decomposition (separating– or decoupling–the system into stable and unstable variables)

- Steps of the Algorithm
  - Multiply Equation (1) by A_0^{-1}

$$E_t Y_{t+1} = AY_t + B\varepsilon_{t+1}$$

  - The vector Y_t is partitioned into predetermined variables k_t (with size m) and non-predetermined variables y_t (with size n)

$$\begin{bmatrix} k_t \\ E_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} k_t \\ y_t \end{bmatrix} + B\varepsilon_{t+1} \tag{2}$$

  - A Jordan decomposition of the matrix A is given by

$$A = P\Lambda P^{-1} \tag{3}$$

    where P is a matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues. Assume number of unstable eigenvectors is l

– The Blanchard-Kahn condition for determinacy:

  * The solution exists and unique if and only if $l = n$
  * if $l < n$, multiple equilibria (the model yields indeterminacy)
  * If $l > n$, either there is no solution or explosive paths

– (2) and (3) are used together to obtain the following policy functions

$$\begin{bmatrix} k_{t+1} \\ y_t \end{bmatrix} = \begin{bmatrix} . & . \\ . & 0 \end{bmatrix} \begin{bmatrix} k_t \\ \varepsilon_{t+1} \end{bmatrix}$$

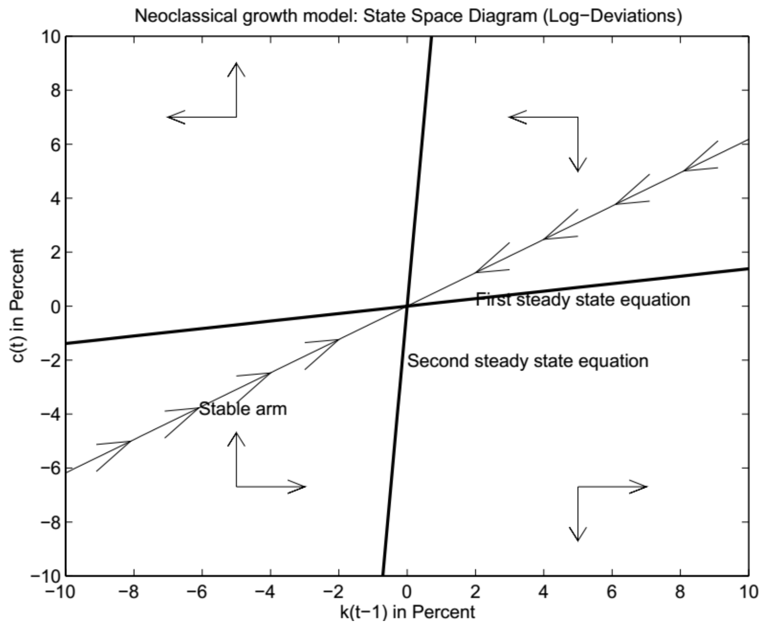
State-Space Approach

- Find the steady state of the log-linearized model, which finds

$$k = f(c)$$

$$c = g(k)$$

- These two steady state equations describe two curves in the two dimensional (k_{t-1}, c_t) plane cutting that plane into four quadrants


Neoclassical growth model: State Space Diagram (Log−Deviations)
c(t) in Percent
k(t−1) in Percent
First steady state equation
Second steady state equation
Stable arm
10
8
6
4
2
0
−2
−4
−6
−8
−10
−10
−8
−6
−4
−2
0
2
4
6
8
10

- Use the equilibirum conditions of your log-linearized model to find the following conditions

$$k_t - k_{t-1} = ...$$

$$c_t - c_{t-1} = ...$$

- Next, calculate the changes $k_t - k_{t-1}$ and $c_t - c_{t-1}$ when starting from any point on the plane. The signs of these changes depend on the quadrant in which the point lies

- The stable arm is the function $c_t = v_{ck} k_{t-1}$ which was derived with the method of undetermined coefficients

Second order difference equations

- Combine the equilibirum conditions of your log-linearized model for $c_t(\cdot)$ and $k_t(\cdot)$ to find the following the second order difference equation

$$0 = k_{t+1} - \gamma k_t - \frac{1}{\beta} k_{t-1}$$

- Solving this homogeneous equation leads to the following general solution in terms of characteristics roots of the equation

$$k_t = a v_1^t + b v_2^t$$

- If only one root is stable, $v_1 = v_{kk}$ which was derived with the method of undetermined coefficients. If both roots are stable, then the solution obtained with the method of undetermined coeffcients is no longer valid

PERTURBATION METHODS (DYNARE)

- Perturbation methods solve the functional equation problem:

$$H(d) = 0$$

  by specifying a Taylor series expansion

- *Ex.* Finding the negative root of the cubic equation

$$x^3 - 4.1x + 0.2 = 0$$

- The first step includes the transformation of the problem into a perturbed problem indexed by a small perturbation parameter ε

$$x^3 - (4 + \varepsilon)x + 2\varepsilon = 0$$

  where $\varepsilon = 0.1$

- We know that the solutions x will depend on the perturbation parameter, i.e. $x = g(\varepsilon)$.

$$g(\varepsilon)^3 - (4 + \varepsilon)g(\varepsilon) + 2\varepsilon = 0$$

- The solution for a particular choice of one of the perturbation parameters, namely for $\varepsilon = 0$, is quite easy to obtain. In this case the equation $x^3 - 4x = 0$ has to be solved, implying the roots given by -2, 0, 2. As a result, $g(0) = -2$.

- Taking the derivative

$$3g(\varepsilon)^2 g'(\varepsilon) - g(\varepsilon) - (4 + \varepsilon)g'(\varepsilon) + 2 = 0$$

finds $g'(0) = -1/2$

- Taking the second derivative finds $g''(0) = 1/4$.

- Then we can calculate x using Taylor approximation

$$x = g(0) + \sum_{n=1}^{\infty} \frac{g^{(n)}(0)}{n!} \varepsilon^n$$

- Table 1 summarizes the results up to second order

Table 1: The problem of the cubic equation

order	Taylor approximation	$\varepsilon \equiv 0.1$	$x^3 - 4.1x + 0.2$
0th	$x \simeq -2$	$x = -2$	0.4
1st	$x \simeq -2 - \frac{1}{2}\varepsilon$	$x = -2.05$	-0.010125
2nd	$x \simeq -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2$	$x = -2.04875$	0.00049976758...

- A first order perturbation is the same with linearization

Interpreting the Results for the Policy Rules

$$x_t = Px_{t-1} + Qz_t$$

$$y_t = Rx_{t-1} + Sz_t$$

$$z_{t+1} = Nz_t + \epsilon_{t+1}$$

- Since x_t, y_t and z_t are log-deviations, the entries in P, Q, R, S and N can be understood as elasticities

- Impulse responses to a particular shock can be calculated by setting $x_0 = 0; y_0 = 0$ and $z_0 = 0$, as well as $\epsilon_t = 0$ for $t \geqslant 2$, and recursively calculating z_t and then x_t and y_t, given x_{t-1}, z_{t-1} and t for $t = 1; :::; T$ with the recursive equilibrium law of motion and the law of motion for z_t

- Use simulation (via employing shocks to the model)
  - Good for small sample properties that is required for testing
- or Frequency Domain Methods (theoretical solution)
- to obtain
  - Impulse Response Functions
  - Correlations and Relative Variances between variables
    - Hodrick-Prescott filter is typically applied before calculate the moments

DYNAMIC PROGRAMMING AND VALUE FUNCTION ITERATION

- As an example I use a centralized economy. Social planner maximizes the utility function of the representative consumer

  $$\max \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t u(c_t)$$

  where utility function is strictly increasing ($u\prime(c) > 0$) and strictly concave ($u''(c) < 0$)

  subject to the resource constraint of the economy

  $$k_{t+1} \leq y_t + (1-\delta)k_t - c_t$$

  and the restriction that the capital stock cannot fall below zero

  $$k_t \geq 0 \quad \forall t$$

- *Value Function* is defined as follows

$$V(k_0) = \sum_{t=0}^{\infty} (\frac{1}{1+\rho})^t u(c_t^*(k_0))$$

  it gives the discounted utility obtained with optimal consumption sequence for each possible value of k_0

- At the final period T, instead of $u(c_T(k_T))$, we write $V_T(k_T)$

- As a result, the consumer problem in period $T-1$ can be written as

$$\max_{c_{T-1}} (u(c_{T-1}) + \frac{1}{1+\rho} V_T(k_T))$$

- Choosing $c_{T-1}(k_{T-1})$ determines k_T. Hence, everything can be written in terms of k_{T-1}. In other words, the maximization problem given above is equal to

$$V_{T-1}(k_{T-1}) = \max_{c_{T-1}}(u(c_{T-1}) + \frac{1}{1+\rho}V_T(k_T))$$

- In general, the problem can be written in terms of *the Bellman equation*

$$V_t(k_t) = \max_{c_t}(u(c_t) + \frac{1}{1+\rho}V_{t+1}(k_{t+1}))$$

Ex: The Cake-Eating Problem

Consider the discrete time optimal growth problem, where

$$f(k) = k$$

This problem is commonly called a "cake-eating" problem. The consumer starts with a certain amount of capital, and "eats" it over time. The planner's problem is to maximize:

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to the constraints:

$$k_{t+1} \leq k_t - c_t$$
$$k_t \geq 0$$
$$k_0 > 0$$

where log is the natural (base e) logarithm function. We will use this problem to see the solution methods in dynamic programming.

a-) Write down Bellman's equation for this problem.

Solution: Bellman's equation is:

$$V(k) = \max_{c \in [0,k]} \{\log c + \beta V(k - c)\}$$

b-) First we perform policy function iteration. We start with guessing the optimal policy (control variable as a function of state variable) as

$$c_t(k) = (1-\beta)k_t$$

Write down the value of k_t (for any $t > 0$) as a function of k_0, β, and t if this policy is followed.

Solution: If this control policy is followed:

$$\begin{aligned}
k_1 &= k_0 - c_0 = k_0 - (1-\beta)k_0 = \beta k_0 \\
k_2 &= \beta k_1 \\
&\ldots \\
k_t &= \beta^t k_0
\end{aligned}$$

c-) Write down the value of c_t as a function of k_0, β, and t if this policy is followed.

Solution:

$$c_t = (1-\beta)k_t = (1-\beta)\beta^t k_0$$

d-) What is the value function if the equation above describes the optimal policy? (Just to keep things simple, feel free to drop any constant term)

Solution: The value function is calculated by simply substituting the sequence $\{c_t\}$ into the utility function:

$$\begin{aligned} V(k) &= \sum_{t=0}^{\infty} \beta^t \log((1-\beta)\beta^t k) \\ &= \sum_{t=0}^{\infty} \beta^t (\log((1-\beta)\beta^t) + \log k) \\ &= \sum_{t=0}^{\infty} \beta^t (\log((1-\beta)\beta^t) + \sum_{t=0}^{\infty} \beta^t \log k \end{aligned}$$

$$V(k) = \text{constant} + \frac{1}{1-\beta} \log k$$

e-) The next step is to calculate the optimal policy under the new value function. This will give you a new policy function, which we will call $c_1(k)$. Find $c_1(k)$.

Solution: We can re-write the Bellman equation with the value function as

$$V(k) = \{\log c + \beta[\frac{1}{1-\beta}\log(k-c)]\}$$

There is a specific value of current consumption, $c_1(k)$, that maximizes

$$c_1(k) = \arg\max_{c\in[0,k]}\{\log c + \beta[\frac{1}{1-\beta}\log(k-c)]\}$$

Taking the first order condition, the new policy function can be found as

$$0 = \frac{1}{c} - \frac{\beta}{1-\beta}\frac{1}{k-c} \qquad \Rightarrow \qquad c_1(k) = (1-\beta)k$$

f-) To find the true optimal policy, you keep applying these two steps until your policy function stops changing i.e., $c_i(k) = c_{i+1}(k)$. What is the true optimal policy function?

Solution: Since $c_1(k)$ and $c_0(k)$ are identical, we do not need any further iteration. We have already found the optimal policy function as

$$c(k) = (1 - \beta)k$$

g-) Next we will try value function iteration. First we guess at the form of the value function.

Suppose that your initial guess for the value function is:

$$V_0(k) = \log k$$

Next, we calculate a new value function according to the formula:

$$V_{i+1}(k) = \max_{c \in [0,k]} \{\log c + \beta V_i(k - c)\}$$

Calculate V_1, V_2, and V_3. (Feel free to throw out any constant terms.)

Solution: First we find V_1 by solving the maximization problem:

$$V_1(k) = \max_{c \in [0,1]} \{\log c + \beta V_0(k-c)\} = \max_{c \in [0,1]} \{\log c + \beta \log(k-c)\}$$

The first order conditions are:

$$\frac{1}{c} = \frac{\beta}{k-c} \quad \Rightarrow \quad c = \frac{1}{1+\beta} k$$

Then we substitute back in to get V_1

$$V_1(k) = \log(\frac{1}{1+\beta} k) + \beta \log(\frac{\beta}{1+\beta} k) = \text{constant} + (1+\beta)\log k$$

Since V_1 and V_0 are not identical, we continue. Skipping through the algebra:

$$\begin{aligned} V_2(k) &= \text{constant} + (1+\beta+\beta^2)\log k \\ V_3(k) &= \text{constant} + (1+\beta+\beta^2+\beta^3)\log k \end{aligned}$$

h-) If you've done it right, you should see a pattern. Use this pattern to discern V_i for an arbitrary integer i.

Solution: The pattern should be clear at this point:

$$V_i(k) = \text{constant} + \sum_{j=0}^{i} \beta_j \log k$$

i-) What is the limit of this as $i \to \infty$?

Solution: The limit is:

$$V(k) = \text{constant} + \frac{1}{1-\beta} \log k$$

j-) What is the optimal policy function when this is the value function?

Solution:

$$V_1(k) = \max_{c\in[0,1]}\{\log c + \beta V_0(k-c)\} = \max_{c\in[0,1]}\{\log c + \frac{\beta}{1-\beta}\log(k-c)\}$$

The first order conditions are

$$\frac{1}{c} = \frac{\beta}{1-\beta}\frac{1}{k-c}$$

Not too surprisingly, the optimal policy under this value function is:

$$c(k) = (1-\beta)k$$

k-) Try to solve the same problem by assuming the same functional form for the value function with the utility function

Solution: The Bellman equation is

$$V_t(k_t) = \max_c \{\log c_t + \beta V_{t+1}(k_{t+1})\}$$

subject to

$$k_{t+1} = k_t - c_t$$

- Some additional Information:
  - All CRRA, CARA, and quadratic utility functions are the class of HRRA (Hyperbolic absolute risk aversion) utility function
  - Merton shows that the value function is of the same functional form as the utility function for the HRRA utility functions if labor

income is fully diversified, or where there is no labor income at all and all income is derived from tradable wealth.

Hence, assuming the same functional form for the value function with the utility function

$$V_t(k_t) = A \log k_t + B$$

the Bellman equation reduces to

$$A \log k_t + B = \log \max_{c_t} \{\log c_t + \beta(A \log k_{t+1} + B)\}$$

FOC wrt c_t

$$0 = \frac{1}{c_t} + A\beta \frac{-1}{k_t - c_t} \tag{1}$$

FOC wrt k_t

$$\frac{1}{k_t} = \beta \frac{1}{k_t - c_t} \tag{2}$$

(1) and (2) imply that $c_t = k_t/A$. Using this equality in (2)

$$\frac{1}{k_t} = \beta \frac{1}{k_t - Ak_t} \quad \Rightarrow \quad A = 1/(1-\beta) \quad \Rightarrow \quad c(k) = (1-\beta)k$$

DYNAMIC OPTIMIZATION IN DISCRETE TIME: DYNAMIC PROGRAMMING APPROACH

- I use a centralized economy. Social planner maximizes the utility function of the representative consumer

$$\max \sum_{t=0}^{\infty} (\frac{1}{1+\rho})^t u(c_t)$$

where utility function is strictly increasing ($u\prime(c) > 0$) and strictly concave ($u''(c) < 0$)

subject to the resource constraint of the economy

$$k_{t+1} \leq y_t + (1-\delta)k_t - c_t$$

and the restriction that the capital stock cannot fall below zero

$$k_t \geq 0 \quad \forall t$$

- Solving the Bellman equation:

$$V_t(k_t) = \max_{c_t}(u(c_t) + \frac{1}{1+\rho}V_{t+1}(k_{t+1}))$$

subject to

$$k_{t+1} \leq f(k_t) + (1-\delta)k_t - c_t$$

  – FOC w.r.t. c_t gives

$$u'(c_t) = \frac{1}{1+\rho}V'_{t+1}(k_{t+1})$$

  – FOC w.r.t. k_t gives

$$V'_t(k_t) = \frac{1+f'(k_t)-\delta}{1+\rho}V'_{t+1}(k_{t+1})$$

    * Actually there are two additional terms coming from the derivative of c_t w.r.t. k_t, but they cancel out due to the first FOC

– Combining FOCs

$$V_t'(k_t) = (1 + f'(k_t) - \delta)u'(c_t)$$

– Inserting this equation into the first FOCw.r.t. c_t, we find the Euler Equation

$$u'(c_t) = \frac{1 + f'(k_{t+1}) - \delta}{1 + \rho} u'(c_{t+1})$$

and notice that in the market solution $f'(k_{t+1}) + 1 = R$ and $f'(k_{t+1}) - \delta = r$. In that case the final equation reduces to

$$u'(c_t) = \frac{1 + r}{1 + \rho} u'(c_{t+1})$$

Notes:

- Value Function transforms infinite horizon problem into the sequential problem
- Their differentiability can be put to good use both in the economic analysis of the problem and in the design of numerical methods
- However,
  - If the model does not satisfy the two fundamental welfare theorems, we cannot easily move between the social planner's problem and the competitive equilibrium
  - In that case, also, the value function of the household and firms will require laws of motion for individual and aggregate state variables that can be challenging to characterize

DYNAMIC OPTIMIZATION IN CONTINUOS TIME

- Social planner maximizes utility of the consumer s.t. the resource constraint of the economy

$$\max \int_0^\infty u(c(t))e^{-\rho t}dt \qquad s.t. \quad k^{\cdot} = f(k) - c - \delta k$$

(Control variable: c, State variable: a)

- When ρ is small, $e^{-\rho t}$ can be written as $(1-\rho)$. And this can approximated also by $1/(1+\rho)$ (you can use $\rho = 0.02$ to see this)

- We use a constraint coming from credit markets (actually we do not need this constraint in this setting)):

$$\boxed{\lim_{t\to\infty} k(t)\exp(-\int_0^t [f'(k(v)) - \delta]dv) \geqslant 0}$$

which implies that at the steady state $f'(k) - \delta = r > 0$.

- This is called *Complementary-Slackness condition*, and means that, in the long run a country's debt per person (negative values of $k(t)$) cannot grow as fast as $r(t)$. It rules out Ponzi schemes for debt

- Secondly, we write the problem as *Hamiltonian* (on the following slide). When we do that, the above constraint takes the form of

$$\lim_{t \to \infty} \gamma(t) k(t) = 0$$

- This is called *Transversality Condition*
  - $\gamma(t)$ is the (present) value of $k(t)$ at time t. It says that as t goes to infinity, either $k(t)$ should approaches to 0, or if $k(t)$ is positive, then its price, $\gamma(t)$, should be 0 (so that it is harmless to left any positive capital)
  - With the equality sign this equation tells us that it would be sub-optimal for households to accumulate positive assets forever at the rate r or higher because utility would increase if these assets were instead consumed in finite time

- This equation can be solved by Hamiltonian

$$\mathtt{H} = e^{-\rho t}u(c) + \lambda(f(k) - c - \delta k)$$

where λ is called *shadow price of capital (or income). It is* the value of an extra unit of capital at time t in units of utility at time 0

- The first order conditions

$$\frac{\partial \mathtt{H}}{\partial c} = 0 \quad \Rightarrow \quad u'(c)e^{-\rho t} - \lambda = 0 \tag{1}$$

$$\frac{\partial \mathtt{H}}{\partial \hat{k}} = -\dot{\lambda} \quad \Rightarrow \quad -\dot{\lambda} = \lambda[f'(k) - \delta)] \tag{2}$$

- Taking derivative of (1) with respect to time

$$(1) \quad \Rightarrow \quad u''(c)\dot{c}e^{-\rho t} - u'(c)\rho e^{-\rho t} = \dot{\lambda}$$

substitute this back into (2)

$$(f'(k)-\delta)u'(c)e^{-\rho t} = u''(c)\dot{c}e^{-\rho t} - u'(c)\rho e^{-\rho t}$$

- which results in famous *Euler Equation* (where $u''(c) < 0$)

$$\Rightarrow \quad \frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c}(f'(k) - \delta - \rho)$$

- In the decentralized (market) economy, $f'(k) - \delta = r$
- **Note**: To see that the discrete time Euler equation is analogous to its continuos time version, use the Taylor Approximation:

$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x + ..$$

- Using the first order approximation for $u'(c_{t+1})$ in the discrete time Euler Equation

$$u'(c_t) = \frac{1 + f'(k) - \delta}{1 + \rho}[u'(c_t) + u''(c_t)(c_{t+1} - c_t)]$$

which can be simplified as

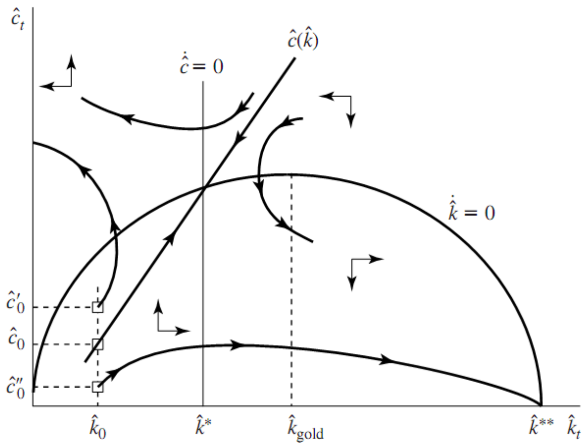
$$\frac{c_{t+1} - c_t}{c_t} = -\frac{u'(c_t)}{u''(c_t)c_t}(\frac{f'(k) - \delta - \rho}{1 + \rho})$$

which is, given that $1 + \rho \simeq 1$, identical to continuos time version of the Euler Equation, repeated below

$$\frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c}(f'(k) - \delta - \rho) \tag{5}$$

Summary of the Model Solution

- (12): $\frac{c^{\cdot}}{c} = -\frac{u'(c)}{u''(c)c}[f'(k) - \delta - \rho]$ [c is rising for $k < k^*$]
- (13): $k^{\cdot} = f(k) - c - \delta k$ [k is falling for c above the solid curve]



The $c^{\cdot} = 0$ and the $k^{\cdot} = 0$ lines cross three times. So there are three steady states (the first one is the origin$-c = 0$ & $k = 0-$, the second steady state is with k^* and c^*, and the third one involves a positive capital stock$-k^{**} > 0$). For any initial positive capital stock, $k(0)$, the only stable equilibria is the one on $c(k)$ locus that reaches k^* and c^*

- **Notes on the Euler Equation**:

$$\Rightarrow \quad \frac{\dot{c}}{c} = -\frac{u'(c)}{u''(c)c}(r - \rho)$$

- When $r = \rho$, the interest rate is equal to future discount rate, households would select a flat consumption profile with $(\frac{\dot{c}}{c} = 0)$
- If $r > \rho$, households give up consumption today for more consumption tomorrow $(\frac{\dot{c}}{c} > 0)$
- The more $-\frac{u''(c)c}{u'(c)}$, the less $\frac{\dot{c}}{c}$ responds to an increase in $r > \rho$

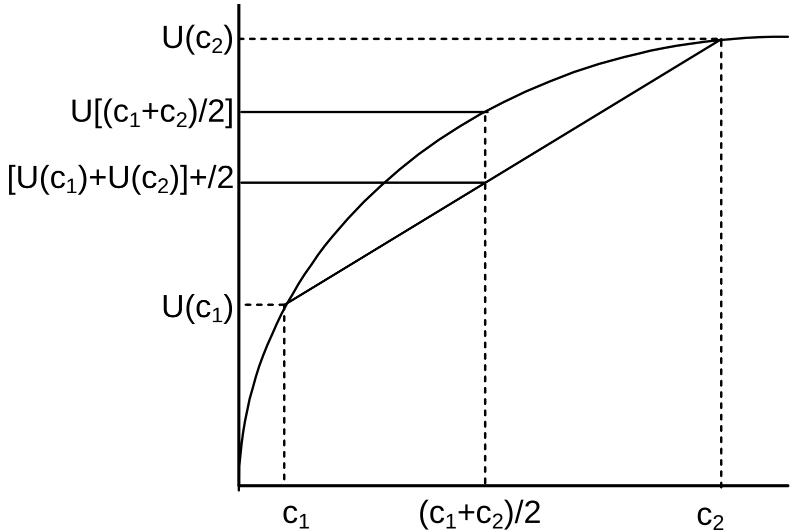
- **Notes on the Utility Function**:

- The utility function satisfies
$$u'(c) > 0 \qquad u''(c) < 0$$
$$\lim_{c\to 0} u'(c) = \infty \qquad \lim_{c\to\infty} u'(c) = 0$$

  – i.e. the utility households receive from an extra unit of consumption is always positive, but decreases with an increase in c (just like the neoclassical production function does in its inputs)

  – This utility function corresponds to households' desire to smooth their consumption pattern. This comes with its concaveness ($u''(c) < 0$). Instead of consuming c_1 and c_2 separately and having the average utility of $[U(c_1) + U(c_2)]/2$, households prefer to consume

$(c_1 + c_2)/2$ in both time periods



- Note: The term $-\dfrac{u''(c)c}{u'(c)}$ is the elasticity of $u'(c)$ with respect to c and can be written as

$$\frac{u''(c)c}{u'(c)} = \frac{\dfrac{\partial u'(c)}{\partial c}c}{u'(c)} = \frac{\dfrac{\partial u'(c)}{u'(c)}}{\dfrac{\partial c}{c}}$$

It is called *the coefficient of relative risk aversion*. So more elastic utility function, the more risk averse households are, meaning the less they would be willing to change their consumption pattern over their life time. Indeed, this term is also *1/elasticity of intertemporal substitution*. This means that the more elastic is the marginal utility, the less elastic is the intertemporal substitution

Ex.

- Assume the functional form $u(c) = \dfrac{c^{1-\theta} - 1}{1 - \theta} \quad \theta > 0$

$$u'(c) = (1-\theta)\frac{c^{-\theta}}{1-\theta} = c^{-\theta} \qquad u''(c) = -\theta c^{-\theta-1}$$

$$\Rightarrow \quad \frac{u''(c)c}{u'(c)} = \rho - \frac{-\theta c^{-\theta-1}c}{c^{-\theta}} = -\theta$$

  this utility function has constant relative risk aversion (CRRA), which is θ, and constant intertemporal elasticity of substitution, which is $1/\theta$

  - $\theta \uparrow \;\Rightarrow$ less willing households are to accept deviations from a uniform c
  - $\theta \to 0 \;\Rightarrow u(c)$ approaches a linear form in c (to see this use

the l'hopital rule), which makes households indifferent to timing of consumption if $r = \rho$

– $\theta \to 1 \;\Rightarrow u(c)$ approaches a log-utility form, which we will analyze later

- With this form of utility function: $$\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) \tag{5}$$

  – If $\theta \uparrow$, then $\frac{\dot{c}}{c}$ responds less to the gap between r & ρ

- **Note: Finding the Level of Consumption**

Households try to find the optimal consumption amount at each point in time

$$\max \int_0^\infty u(c(t))e^{-\rho t}dt$$

$$s.t. \quad \dot{a}_t = \omega_t + r_t a_t - c_t$$

(Control variable: c, State variable: a)

- The solution of the problem

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) \tag{5}$$

- This equation only defines consumption path from one period to the next. If we want to define consumption at time t in terms of the consumption at time 0, we can first use $\exp(-\int_0^t r(v)dv)$, which is the present value factor that converts a unit of income at time t to an equivalent unit of income at $t = 0$. Then we can define the average interest rate between 0 and t as

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(v)dv$$

- c can be written as

$$c_t = c_0 \exp\{\frac{1}{\theta}[\bar{r}(t) - \rho]\}t \tag{7}$$

- Our next question is: 'What is c_0?'
  - We have derived the consumption at any point in time w.r.t. consumption at time 0; hence, we can calculate the total consumption

at time 0 (in finite time case it is called life time consumption). Then we can equate the total life time consumption in terms of c_0 to the total (life time) wealth at time 0, and solve for the level of consumption. This is called (life time) *Budget Constraint* and can be written in a form of

$$\int_0^\infty c_t e^{-\bar{r}(t)t} dt = a(0) + \int_0^\infty \omega_t e^{-\bar{r}(t)t} dt \tag{8}$$

This equation says that total (life time) consumption (when it is brought to time 0) must be equal to initial wealth plus total (life time) labor income

– Plugging equation (7) into the equation (8) finds

$$c_0 = ([\int_0^\infty \exp[\frac{1-\theta}{\theta}\bar{r}(t) - \frac{\rho}{\theta}]dt) \cdot [a(0) + \int_0^\infty \omega_t e^{-\bar{r}(t)t} dt] \tag{9}$$

– This equation shows that (depending on θ) an increase in r may increase or decrease c_0. This is because r has two effects. One is what we see in equation (5). It changes consumption relatively between periods, which is called *substitution effect*. Due to this effect an increase in r leads to decline in c_0. The second effect is the *income effect* that we can realize from (life time) Budget Constraint. Due to this effect an increase in r leads to an increase in life time wealth (as it leads to future value of $a(0)$ to increase) and raises c_t at all dates. In fact there is also a third, *wealth effect*, as a result of which an increase in r decreases the current value of future labor income gains (the last term in equation (9)), but this just reinforces the substitution effect for c_0)

  * If $\theta < 1$, (small θ), consumers care relatively little about consumption smoothing, and substitution (intertemporal effect) is

large

* If $\theta > 1$, substitution effect is relatively weak compared to income effect

* If $\theta = 1$, which is the case with log utility, the two effects exactly cancel out each other