Braids, Knots and String Things
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1 Knots

Last week, we defined a knot as a closed piece of string, and represented them with diagrams, which were their projections onto the plane. In these diagrams, we drew arcs between points and noted which string passed over/under at each crossing. Let's warm up with some simple problems about knots.

Problem 1.1. Could you convey the information about a trefoil to someone else sitting at your table without drawing?

Problem 1.2. Draw a knot. Can you color the regions inside the knot black and white such that no two adjacent regions have the same coloring (i.e, a checkerboard coloring)? Can you do this for all knots?

Problem 1.3. What is the smallest number of crossings a knot can have and be non-trivial?

2 Colorings

We called two knots "equivalent" if we could find a smooth deformation from one knot to another. We used 3 moves to deform the knots and called them Reidemeister Moves (see Figure 1); they were the only moves that were required to get from any one knot diagram to another.

We began looking at properties of knots, including three-coloring of knots. A knot diagram was three-colorable if it satisfied the following properties:

1. Every arc was colored red, blue or yellow

2. Every crossing had all three colors appear, or only one color appear.
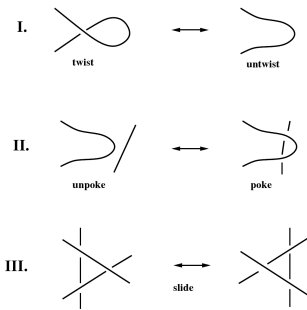
3. All colors were used
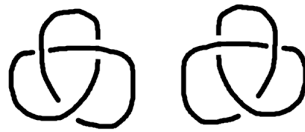
Figure 1: The three Reidemeister moves.

Figure 2: The Right and Left Trefoil in all of their glory.

We also showed that the Reidemeister moves were compatabile with three colorability. This invariant could tell us that the trefoil knot was not a trivial knot.

Problem 2.1. Can you find a knot that is not trivial, but not three colorable? Why does this make the three-coloring knot invariant less-than-ideal?

Problem 2.2. Suppose we loosen the definition of three coloring by dropping condition 3. We can then look at the three-coloring number, the number of ways to three-color a knot. What is the three-coloring-number of the trefoil? What about the unknot?

Problem 2.3. Compute the three-coloring number of 5_1 (drawn on the board)

Problem 2.4. We define a link to be a several closed pieces of string and we also represent it with a projection. Can you find a link that has the same three-coloring-number as the trefoil? Is this link three-colorable?

Problem 2.5. Can the three coloring number or the three coloring invariant distinguish between the right and left trefoil? (see Figure 2)

3   Bridge Number

An arc that begins at an undercrossing, goes through several overcrossings, and ends at an undercrossing is called a bridge. The Bridge number of a diagram is the number of bridges that are present in it. However, the bridge number is not necessarily the same as the number of crossings in the diagram. For instance, the bridge number of Figure 3 is 1, but the number of crossings is 5.

Problem 3.1. Which is bigger, the bridge number or the crossing number of a diagram? When are they equal?

The bridge number of a diagram, like the crossing number of a diagram, is not invariant under Reidemeister moves, so it is not such a wonderful knot invariant. We can force this property by looking at the minimal bridge number over all diagrams.

We define the Bridge Number of a knot to be the minimal bridge number of all possible diagrams.

Problem 3.2. What is the smallest possible bridge number. Can a knot with this bridge number be non-trivial?

Problem 3.3. What is the smallest possible bridge number for a non-trivial knot? How many knots have this bridge number?

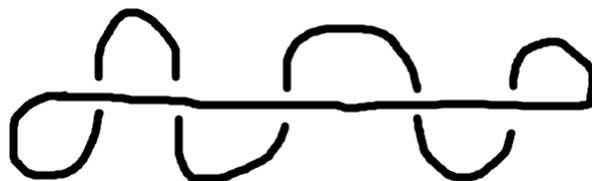*Problem 3.4.* Compute the bridge number for the trefoil knot.

Figure 3: A poorly drawn trivial knot.

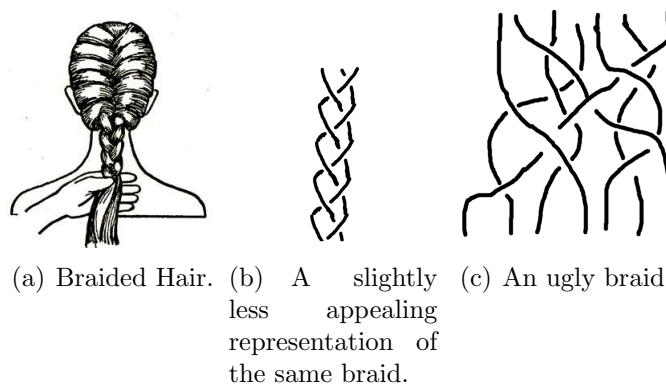(a) Braided Hair. (b) A slightly less appealing representation of the same braid. (c) An ugly braid.

Figure 4: Some Braids of various elegance.

4 Braids

We define a braid to be a collection of strings that run from one plane to another, such that no string doubles back on itself. Braids are a lot like knots, in that we represent them with diagrams as well. However, they have a far more accessible structure, as they run only "up and down". Both braids and knots have crossings with many of the same properties. We will, in fact, define relationship between braids using the Reidemeister moves as well.

Problem 4.1. Which of the Reidemeister moves can be applied to braids? Why not the other(s)?

Problem 4.2. Describe a braid diagram (figure 4(b)) to someone else at your table without drawing any pictures. Is this more difficult or easier than describing the structure of a knot?

Problem 4.3. Can you come up with a formal way to describe braids?

We will describe braids as elements of a group; a set where there is a "multiplication" that satisfies the usual properties. The set that we will be looking at is $\mathcal{B}_n$, the collection of all braids with n strings.

Problem 4.4. How many elements are in $\mathcal{B}_n$?

In this group we will be able to combine braids by "stacking" (figure 5). If we have two braids, β_1 and β_2, we will write the braid resulting from their stacking as $\beta_3 = \beta_1\beta_2$.

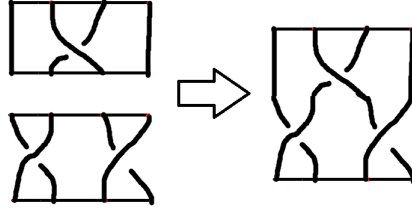For the following problems, consider the braids in $\mathcal{B}_4$.

Figure 5: A crudely drawn diagram of Braid Composition

Problem 4.5. Can you think of a braid β_I such that when it is combine with any other braid β, we have that $\beta_I \beta = \beta \beta_I = \beta$?

Problem 4.6. Given β_1 can you think of a braid β_1^{-1} such that $\beta_1^{-1} \beta_1 = \beta_1 \beta_1^{-1} = \beta_I$? Draw a few examples to see how it works.

Problem 4.7. Can you think of two braids β_1 and β_2 such that $\beta_1 \beta_2 = \beta_2 \beta_1$?

We can begin to look at all of the generators of braids. We will call them $\sigma_1, \sigma_2 \ldots \sigma_{n-1}$, and they will be the crossing of the i and $i+1$ strings. We can make any braid diagram by stringing these simplest braids together (Figure 6).

Problem 4.8. Can you represent Reidemeister's 2nd move in terms of braids?

Problem 4.9. Can you represent Reidemeister's 3rd move in terms of braids?

Problem 4.10. Do these represent all of the possible relationships between braids?
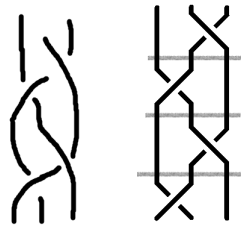

Figure 6: Braid on left represented as $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$.

5 Braid Closures

We can start to make knots out of braids. We simply glue the bottom of the braid to the top; we call this operation the closure of the braid, and represent it as $\bar{\beta}$. The simplest way to understand this is to see a picture; the closure $\bar{\sigma_1^3}$ from the braid group $\mathcal{B}_2$ is shown in Figure 7. This gives us two new important invariants; the Braid index and Braid Word. We define the Braid index of a knot K to be the smallest n such that the closure of a braid from $\mathcal{B}_n$ is K. We then define the Braid Word to be a braid of shortest length (meaning having the least number of symbols) whose closure is K.

Problem 5.1. Is the Braid Index a good invariant? In particular, are their multiple different knots with the same braid index? Can a knot have multiple different values for the Braid Index?

Problem 5.2. How is the length of the Braid Word related to it's knot?

Problem 5.3. Use the braid word to prove that the right trefoil is not the same as the left trefoil.

Problem 5.4. What knots have braid index 2?

Problem 5.5. Where have we seen the braid index before?

6 Things to Think About

Problem 6.1. We can use our Braid Relationships to resolve Reidemeister's 2nd and 3rd moves in our braid word; is there a way to represent the 1st one?

Problem 6.2. Under what conditions is the closure of a braid a knot? A link on 2 strings? A link on n-strings?

Problem 6.3. It is easy to see that the closure of every braid is either a knot or a link. Is their a braid to represent every single knot or link?
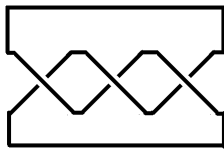

Figure 7: The closure of σ_1^3.

7 Finding the Braid Word

We can find the braid word of a knot diagram by the following method:
Assign an orientation to the knot diagram. This means put a direction on
a segment, and follow this direction around the diagram. We call the knot
diagram orientable if there exists a point inside of the digram for which all of
the arcs maintain the same orientation. For example, the trefoil is orientable
as shown in Figure 8.

Once we have an oriented diagram, we draw a ray from the point of orientation and "cut" knot along this ray. This gives us a braid on n strings,
where n is the number of strings we cut. We then read the braid word of the
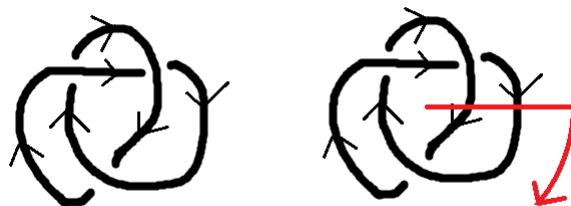diagram off of this braid.

Figure 8: An oriented Trefoil. Isn't it Pretty?

Problem 7.1. Orient the knots 5_1 and 8_{18} (drawn on the board).

Problem 7.2. Calculate the braid word of the knots 5_1 and 8_{18} (drawn on the
board).

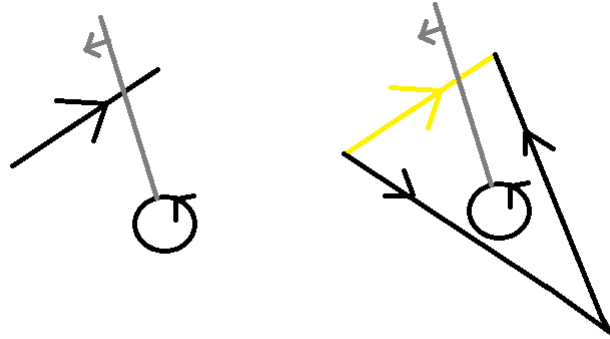*Problem 7.3.* Try calculating the braid word of 4_1 (drawn on the board).

Figure 9: The Orientation Changing Process.

What is the problem with this knot diagram? Can you change it using Reidimeister moves so you can find the braid word?

We will now explore a method for taking a non-oriented knot diagram and turning it into an oriented one.

1. Take the knot diagram and break it into line segments.

2. Assign an orientation to the line segments.

3. Pick an arbitrary point inside the knot diagram.

4. Draw a ray from this point, and assign an orientation to the ray.

5. Rotate the ray around the point: if you encounter a line segment with the wrong orientation, replace it with two line segments of proper orientation. Do this by the process shown in Figure 9.

When we finish this process, we have replaced every improperly oriented line segment with a properly oriented one. In Figure 7 we see the 4_1 knot oriented by this method.

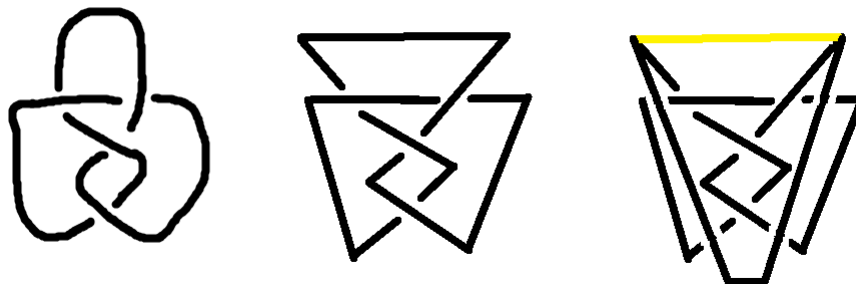*Problem 7.4.* Calculate the braid word for 6_2 (drawn on the board)

Figure 10: The transformation of 4_1 from un-oriented to oriented knot

8 Flat Braid Diagrams

In the same way that we represent knots by looking at their planar projections, we look at braids by looking at their planar projections. A braid diagram is the planar projection of arcs connecting a set of points $\{A, B, C \ldots N\}$ to $\{A', B', C', \ldots N'\}$ with each point associated with one arc. We also use Reidmeister moves to alter the diagrams of braids, but we don't allow arcs to travel through the points $\{A, B, C \ldots N\}$ or $\{A', B', C', \ldots N'\}$ as we deform them.

Suppose we allow arcs to travel over the points $\{A, B, C \ldots N\}$ or under the points $\{A', B', C', \ldots N'\}$. With these additional moves, we can take any braid diagram to a diagram that has no crossings. Such a diagram is called the flat braid diagram of a knot.

Problem 8.1. Draw both the standard and flat diagram of $\beta = \sigma_1 \sigma_2 \sigma_1^{-1}$ where the $\beta \in \mathcal{B}_3$.

Problem 8.2. Look at the braid in Figure 12. Name this braid, and draw it on you paper. Then draw the flat diagram of this braid.

Problem 8.3. We take the closure of a braid diagram by connecting $\{A, B, C \ldots N\}$ to $\{A', B', C', \ldots N'\}$ with arcs. What is interesting about the arcs drawn when we do this for a flat braid diagram? Do the flat braid diagram operations change the resulting knot?
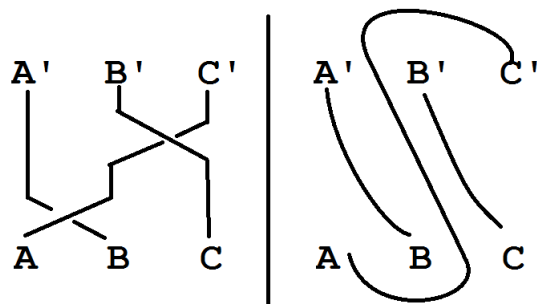
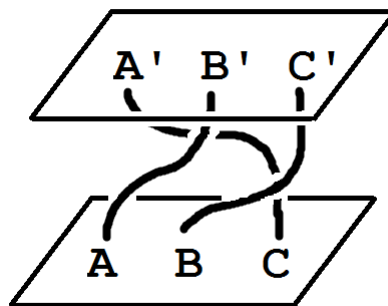Figure 11: The Braid $\sigma_1\sigma_2^{-1}$ in both standard and flat form

Figure 12: A three dimensional braid.