1 Mathematical Concepts

Mathematics is the language of geophysical fluid dynamics. Thus, in order to interpret and communicate the motions of the atmosphere and oceans. While a thorough discussion of the detailed mathematics needed to fully describe fluid motions will not be undertaken here, we begin by reviewing and introducing some of the key mathematical tools that will be used extensively throughout the course.

1.1 Notation

As with most languages, there are multiple ways to communicate an idea in mathematics. Instead of attempting to be internally consistent throughout the course, we will instead use a variety of ways to describe a mathematical idea, in hopes that you will become familiar and comfortable with all of them. For example, here are three ways one might describe the time derivative of a quantity x:

$$\frac{dx}{dt}, \quad \dot{x}, \quad x_t \tag{1.1}$$

Discuss handout *Mathematical Concepts & Notation, Appendix A*

1.2 Vector Calculus

1.2.1 Scalar vs. vector fields

A *scalar field*, e.g. ψ, is a field that associates a scalar value (a single number) at every point in space. An interesting aspect of a scalar field is that it is coordinate-independent, meaning no matter what your reference frame or your coordinate system, as long as you are looking at the same point in each case, you will always get the same answer. As an example, temperature (T) is a scalar field - it is a single number at each point in space. A scalar field at a particular point (x, y, z) is straightforward to describe:

$$T = T(x, y, z). \tag{1.2}$$

A *vector field*, e.g. $\mathbf{u}$, on the other hand, is a field that associates a vector quantity (magnitude and direction) at every point in space. Thus, at every point (x, y) there will be multiple numbers in multiple directions specifying both the magnitude and direction of the vector field there. Unlike a scalar field, a vector field is not coordinate-independent, since the description of the vector assumes a particular coordinate system. As an example, wind ($\mathbf{v}$) is a vector field - it has both a magnitude and a direction. Note, however, that wind speed is a scalar field. Recall that a vector field at a particular point (x, y, z) is typically described by coordinates:

$$\mathbf{v}(x, y, z) = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}, \tag{1.3}$$

where $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are the 3 unit vectors describing the 3 orthogonal (perpendicular) directions in 3-dimensional space.

1.2.2 Operators

Appendix A of these notes details important vector operators that will be used extensively in this course. An operator is just what it sounds like, it is a mathematical *operation* that acts on a field. For example, the most important operator used in this course is the *vector differential operator*, ∇ (pronounced "nabla").

- When you take the cross-product of ∇ with a vector field ($\nabla \times \mathbf{u}$), it is called the *curl* of the vector field.
- When you take the dot-product of ∇ with a vector field ($\nabla \cdot \mathbf{u}$), it is called the *divergence* of the vector field.
- When you apply ∇ to a scalar field (∇T), it is called the *gradient* of the field.

An important thing to remember about vector operators is that how they operate is dependent on the coordinate system. Most of you are probably most familiar with Cartesian Coordinates, where

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{1.4}$$

Using this relationship, and your knowledge from vector calculus of a derivative, cross-product and dot-product, you can figure out the cartesian expansion of the gradient, curl and divergence.

Discuss handout *Mathematical Concepts & Notation, Appendix B*

1.3 Lagrangian vs. Eulerian perspectives (Vallis 1.1, Marshall & Plumb 6.1)

Before we can discuss the behaviours of fluids, we must first all agree on what a "fluid" is. [1] In this course we will operate under the *Continuum Hypothesis: fluid is assumed a continuous distribution of matter that flows and deforms*.

[1] In this course, we will be focusing on Newtonian fluids (e.g. water, air, juice), rather than non-Newtonian fluids, who's viscosity depends on the shear rate (e.g. ketchup, toothpaste, shampoo).

While Newtonian physics applies just as well to fluids as it does to solid objects, there are multiple ways to describe the motions and we want to choose the way that is the most compact and useful for our fluid discussion. For example, if you have two toy trains that smash into one another, you would likely describe the forces at play by separating the variables into the forces felt by train 1 and the forces felt by train 2. Such a procedure in fluid dynamics would require that you describe every interaction of the fluid in terms of tiny *fluid parcels* or *fluid elements*, each given their own number, and then you would track these fluid elements through time and space. Thus, you would travel with each fluid parcel, describing the forces acting on it and tabulating its location at every moment in time. This would be a *Lagrangian* description of the flow.

While the Lagrangian perspective is certainly correct and useful, it is much more than one normally needs, and would be incredibly complicated to implement! Instead, in fluid dynamics we often instead sit at a fixed point in space and watch the fluid flow over us, describing the fluid at that point as it changes in time. This would be a *Eulerian* description of the flow. Since the fluid is a continuum, this knowledge is the same as the Lagrangian perspective and one can convert between the two.

1.3.1 Properties of a fluid at a fixed point (Eulerian)

If you are interested in a particular property of a fluid ϕ. From an Eulerian perspective, you would want to describe the fluid motions from a particular frame of reference, that is, at each point in space. Thus, you would look for evolution equations of the form

$$\frac{\partial}{\partial t}\phi(x, y, z, t) = G(\phi, x, y, z, t), \tag{1.5}$$

where G is some operator that includes thermodynamics and Newton's laws...that is, the physics of the system. Thus, at some point (x, y, z), you wish to know how ϕ evolves in time.

1.3.2 Properties of a fluid moving with a parcel (Lagrangian)

In the Lagrangian perspective, instead of thinking about a fluid property field ϕ, you would instead think of the value of that property for each of the millions (or more) of fluid elements, each with a particular label, say, ϕ_n. Thus, you would be interested in evolution equations of the form

$$\frac{d}{dt}\phi_n(t) = H(\phi(n), t), \tag{1.6}$$

where H now represents the thermodynamics and Newtonian physics applied to each fluid element. Note that we use $\frac{d}{dt}$ rather than $\frac{\partial}{\partial t}$ since $\phi_n(t)$ is only a function of time.

1.3.3 Material derivative

As previously alluded to, the Lagrangian and Eulerian perspectives can be related to one another. Imagine we wish to describe the property of a fluid, say density ρ, as a function of time and in three space dimensions: $\rho = \rho(x, y, z, t)$. Along a trajectory $\vec{x}(t)$, the change in ρ following a parcel is

$$\delta\rho = \frac{\partial\rho}{\partial t}\delta t + \frac{\partial\rho}{\partial x}\delta x + \frac{\partial\rho}{\partial y}\delta y + \frac{\partial\rho}{\partial z}\delta z. \tag{1.7}$$

The total time derivative would then be given by

$$\frac{d\rho}{dt} = \rho_t + \frac{dx}{dt}\rho_x + \frac{dy}{dt}\rho_y + \frac{dz}{dt}\rho_z \tag{1.8}$$

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \nabla\rho \tag{1.9}$$

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + (\mathbf{v} \cdot \nabla)\,\rho, \tag{1.10}$$

where $(\mathbf{v} \cdot \nabla)$ is the *advective operator*. The rate of change of a fluid property following a parcel is called the *material derivative*. Since we will use this formulation over and over again, we will give it its own symbol:

$$\boxed{\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + (\mathbf{v} \cdot \nabla)\,\rho} \tag{1.11}$$

The left-hand-side (l.h.s.) of 1.11 is the rate of change of ρ following a parcel, that is, the Lagrangian perspective. The right-hand-side (r.h.s.) of 1.11 is written in terms of the field and how it varies in space, that is, it is the Eulerian perspective.

1.3.4 Streamlines, pathlines, and streaklines

We take a moment to discuss some important concepts related to fluid flow that become vitally important when trying to convert between the Eulerian and Lagrangian perspectives. To begin, we define three descriptors of fluid flow:[2]

- **Streamlines** are a family of curves that are instantaneously tangent to the velocity vector of the flow. These show the direction a massless fluid element will travel in at any point in time.

- **Pathlines** are the trajectories that individual fluid particles follow. These can be thought of as "recording" the path of a fluid element in the flow over a certain period. The direction the path takes will be determined by the streamlines of the fluid at each moment in time.

[2]definitions taken from Wikipedia: https://en.wikipedia.org/wiki/Streamlines,_streaklines,_and_pathlines

- **Streaklines** are the loci of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline.

If a flow is *steady*, that is, not changing in time (although it can change in space), then *streamlines*, *pathlines* and *streaklines* will all coincide. Perhaps, more importantly though, if the flow is not steady, then these three quantities can be very different. This means that the advective operator in 1.11 is also a function of time, and must be re-evaluated at each point in time if the flow ($\mathbf{v}$) is not steady. Finally, note that in laboratory experiments, one typically observes streaklines, by emitting dye at a fixed location and watching it evolve in time.

Show Wikipedia movie: https://en.wikipedia.org/wiki/File:Streaklines_and_pathlines_animation.gif