Econ 509, *Introduction to Mathematical Economics I*
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Lecture notes based mostly on Chiang and Wainwright, *Fundamental Methods of Mathematical Economics*.

1 Mathematical economics

Why describe the world with mathematical models, rather than use verbal theory and logic? After all, this was the state of economics until not too long ago (say, 1950s).

1. Math is a **concise, parsimonious** language, so we can describe a lot using fewer words.
2. Math contains many **tools and theorems** that help making **general statements**.
3. Math forces us to **explicitly state all assumptions**, and help preventing us from failing to acknowledge implicit assumptions.
4. **Multi dimensionality** is easily described.

Math has become a **common language** for most economists. It facilitates communication between economists. **Warning:** despite its usefulness, if math is the only language for economists, then we are restricting not only communication among us, but more importantly we are restricting our understanding of the world.

Mathematical models make strong assumptions and use theorems to deliver insightful conclusions. But, remember the **A-A' C-C' Theorem:**

- *Let C be the set of conclusions that follow from the set of assumptions A. Let A' be a small perturbation of A. There exists such A' that delivers a set of conclusions C' that is disjoint from C.*

Thus, the insightfullness of C depends critically on the plausibility of A.

The plausibility of A depends on empirical validity, which needs to be established, usually using econometrics. On the other hand, sometimes theory informs us on how to look at existing data, how to collect new data, and which tools to use in its analysis. Thus, there is a **constant discourse** between theory and empirics. Neither can be without the other (see the inductivism vs. deductivism debate).

Theory is an abstraction of the world. You focus on the most important relationships that you consider important a priori to understanding some phenomenon. This may yield an economic model.

2 Economic models

Some useful notation: $\forall$ for all, $\exists$ exists, $\exists!$ exists and is unique. If we cross any of these, or prefix by $\neg$ or $-$, then it means "not": e.g., $\nexists$, $\neg\exists$ and $-\exists$ all mean "does not exist".

2.1 Ingredients of mathematical models

1. Equations:

$$\begin{aligned} \text{Definitions/Identities} &: \pi = R - C \\ &: Y = C + I + G + X - M \\ &: K_{t+1} = (1-\delta) K_t + I_t \\ &: Mv = PY \end{aligned}$$

$$\begin{aligned} \text{Behavioral/Optimization} &: q^d = \alpha - \beta p \\ &: MC = MR \\ &: MC = P \end{aligned}$$

$$\text{Equilibrium} : q^d = q^s$$

2. Parameters: e.g. α, β, δ from above.

3. Variables: exogenous, endogenous.

Parameters and functions govern relationships between variables. Thus, any complete mathematical model can be written as

$$F(\theta, Y, X) = 0 \ ,$$

where F is a set of functions (e.g., demand, supply and market clearing conditions), θ is a set of parameters (e.g., elasticities), Y are endogenous variables (e.g., price and quantity) and X are exogenous, predetermined variables (e.g., income, weather). Some models will not have explicit X variables. Moving from a "partial equilibrium" model closer to a "general equilibrium" model involves treating more and more exogenous variables as endogenous.

Models typically have the following ingredients: a sense of time, model population (who makes decisions), technology and preferences.

2.2 From chapter 3: equilibrium analysis

One general definition of a model's equilibrium is "a constellation of **selected**, **interrelated** variables so adjusted to one another that **no inherent tendency to change** prevails in the model which they constitute".

- **Selected**: there may be other variables. This implies a choice of what is endogenous and what is exogenous, but also the overall set of variables that are explicitly considered in the model. Changing the set of variables that is discussed, and the partition to exogenous and endogenous will likely change the equilibrium.
- **Interrelated**: The value of each variable must be consistent with the value of all other variables. Only the relationships within the model determine the equilibrium.
- **No inherent tendency to change**: all variables must be simultaneously in a state of rest, given the exogenous variables and parameters are all fixed.

Since all variables are at rest, an equilibrium is often called a **static**. Comparing equilibria is called therefore **comparative statics** (there is different terminology for dynamic models).

An equilibrium can be defined as Y^* that solves

$$F(\theta, Y, X) = 0 \ ,$$

for given θ and X. This is one example for the usefulness of mathematics for economists: see how much is described by so little notation.

We are interested in finding an equilibrium for $F(\theta, Y, X) = 0$. Sometimes, there will be no solution. Sometimes it will be unique and sometimes there will be multiple equilibria. Each of these situations is interesting in some context. In most cases, especially when policy is involved, we want a model to have a unique equilibrium, because it implies a function from (θ, X) to Y (the implicit function theorem). But this does not necessarily mean that reality follows a unique equilibrium; that is only a feature of a model. **Warning**: models with a unique equilibrium are useful for many theoretical purposes, but it takes a leap of faith to go from model to reality—as if the unique equilibrium pertains to reality.

Students should familiarize themselves with the rest of chapter 3 on their own.

2.3 Numbers

- Natural, $\mathbb{N}$: $0, 1, 2...$ or sometimes $1, 2, 3, ...$
- Integers, $\mathbb{Z}$: $... - 2, -1, 0, 1, 2, ...$
- Rational, $\mathbb{Q}$: n/d where both n and d are integers and d is not zero. n is the numerator and d is the denominator.
- Irrational numbers: cannot be written as rational numbers, e.g., π, e, $\sqrt{2}$.

- Real, $\mathbb{R}$: rational and irrational. The real line: $(-\infty, \infty)$. This is a special set, because it is dense. There are just as many real numbers between 0 and 1 (or any other two real numbers) as on the entire real line.

- Complex: an extension of the real numbers, where there is an additional dimension in which we add to the real numbers imaginary numbers: $x + iy$, where $i = \sqrt{-1}$.

2.4 Sets

We already described some sets above ($\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$). A set S contains elements e:

$$S = \{e_1, e_2, e_3, e_4\} \ ,$$

where e_i may be numbers or objects (say: car, bus, bike, etc.). We can think of sets in terms of the number of elements that they contain:

- Finite: $S = \{e_1, e_2, e_3, e_4\}$.

- Countable: there is a mapping between the set and $\mathbb{N}$. Trivially, a finite set is countable.

- Infinite and countable: $\mathbb{Q}$. Despite containing infinitely many elements, they are countable.

- Uncountable: $\mathbb{R}$, $[0, 1]$.

Membership and relationships between sets:

- $e \in S$ means that the element e is a member of set S.

- Subset: $S_1 \subset S_2$: $\forall e \in S_1, \ e \in S_2$. Sometimes denoted as $S_1 \subseteq S_2$. Sometimes a strict subset is defined as $\forall e \in S_1, \ e \in S_2$ and $\exists e \in S_2, \ e \notin S_1$.

- Equal: $S_1 = S_2$: $\forall e \in S_1, \ e \in S_2$ and $\forall e \in S_2, \ e \in S_1$.

- The null set, $\varnothing$, is a subset of any set, including itself, because it does not contain any element that is not in any subset (it is empty).

- Cardinality: there are 2^n subsets of any set of magnitude $n = |S|$.

- Disjoint sets: S_1 and S_2 are disjoint if they do not share common elements, i.e. if $\nexists e$ such that $e \in S_1$ and $e \in S_2$.

Operations on sets:

- Union (or): $A \cup B = \{e | e \in A \text{ or } e \in B\}$.

- Intersection (and): $A \cap B = \{e | e \in A \text{ and } e \in B\}$.
- Complement: define Ω as the universe set. Then $\bar{A}$ or $A^c = \{e | e \in \Omega \text{ and } e \notin A\}$.
- Minus: for $B \subset A$, $A \backslash B = \{e | e \in A \text{ and } e \notin B\}$. E.g., $\bar{A} = \Omega \backslash A$.

Rules:

- Commutative:

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{aligned}$$

- Association:

$$\begin{aligned} (A \cup B) \cup C &= A \cup (B \cup C) \\ (A \cap B) \cap C &= A \cap (B \cap C) \end{aligned}$$

- Distributive:

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned}$$

Do Venn diagrams.

2.5 Relations and functions

Ordered pairs: whereas $\{x, y\} = \{y, x\}$ because they are sets, but not ordered, $(x, y) \neq (y, x)$ unless $x = y$ (think of the two dimensional plane $\mathbb{R}^2$). Similarly, one can define ordered triplets, quadruples, etc.

Let X and Y be two sets. The **Cartesian product** of X and Y is a set S that is given by

$$S = X \times Y = \{(x, y) \,| x \in X, y \in Y\} \ .$$

For example, $\mathbb{R}^n$ is a Cartesian product

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R} = \{(x_1, x_2, ... x_n) \,| x_i \in \mathbb{R}\} \ .$$

Cartesian products are **relations** between sets:

$$\forall x \in X, \ \exists y \in Y \text{ such that } (x, y) \in X \times Y \ ,$$

so that the set Y is related to the set X. Any subset of a Cartesian product also has this trait. Note that each $x \in X$ may have more than one $y \in Y$ related to it (and vice versa). Thus the relation may assign to any $x \in X$ a set of values in Y, $S_x \in Y$. (Analysis of the shape of these sets in the context of relations will be useful when discussing dynamic programming.)

If

$$\forall x \in X, \ \exists! y \in Y \text{ such that } (x, y) \in S \subset X \times Y \ ,$$

then y is a **function** of x. We write this in shorthand notation as

$$y = f(x)$$

or

$$f : X \to Y \ .$$

The second term is also called **mapping**, or **transformation**. Note that although for y to be a function of x we must have $\forall x \in X, \ \exists! y \in Y$, it is not necessarily true that $\forall y \in Y, \ \exists! x \in X$. In fact, there need not exist any such x at all. For example, $y = a + x^2$, $a > 0$.

In $y = f(x)$, y is the **value** or **dependent variable**; x is the **argument** or **independent variable**. The set of all permissible values of x is called **domain**. For $y = f(x)$, y is the **image** of x. The set of all possible images is called the **range**, which is a subset of Y.

2.6 Functional forms

Students should familiarize themselves with polynomials, exponents, logarithms, "rectangular hyperbolic" functions (unit elasticity), etc. See Chapter 2.5 in CW.

2.7 Functions of more than one variable

$z = f(x, y)$ means that

$$\forall (x, y) \in \text{domain} \subset X \times Y, \ \exists! z \in Z \text{ such that } (x, y, z) \in S \subset X \times Y \times Z \ .$$

This is a function from a plane in $\mathbb{R}^2$ to $\mathbb{R}$ or a subset of it. $y = f(x_1, x_2, ...x_n)$ is a function from the $\mathbb{R}^n$ hyperplane or hypersurface to $\mathbb{R}$ or a subset of it.

3 Equilibrium analysis

Students cover independently. Conceptual points are reported above in Section 2.2.

4 Matrix algebra

4.1 Definitions

- **Matrix:**

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}] \quad i = 1, 2, ...m, \;\; j = 1, 2, ...n \; .$$

  Notation: usually matrices are denoted in upper case; m and n are called the **dimensions**.

- **Vector:**

$$x_{m\times 1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} .$$

  Notation: usually lowercase. Sometimes called a **column vector**. A **row vector** is

$$x' = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} .$$

4.2 Matrix operations

- **Equality:** $A = B$ iff $a_{ij} = b_{ij}$ $\forall ij$. Clearly, the dimensions of A and B must be equal.
- **Addition/subtraction:** $A \pm B = C$ iff $a_{ij} \pm b_{ij} = c_{ij}$ $\forall ij$.
- **Scalar multiplication:** $B = cA$ iff $b_{ij} = c \cdot a_{ij}$ $\forall ij$.
- **Matrix multiplication:** Let $A_{m\times n}$ and $B_{k\times l}$ be matrices.
  - if $n = k$ then the product $A_{m\times n}B_{n\times l}$ exists and is equal to a matrix $C_{m\times l}$ of dimensions $m \times l$.
  - if $m = l$ then the product $B_{k\times m}A_{m\times n}$ exists and is equal to a matrix $C_{k\times n}$ of dimensions $k \times n$.

– If product exists, then

$$
\begin{aligned}
A_{m\times n}B_{n\times l} &= \left[\begin{array}{cccc} & & \rightarrow & \\ a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array}\right] \left[\downarrow \begin{array}{cccc} b_{11} & b_{12} & \dots & b_{1l} \\ b_{21} & b_{22} & & b_{2l} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nl} \end{array}\right] \\
&= \left[c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}\right] \quad i = 1, 2, ...m,\ j = 1, 2, ...l\ .
\end{aligned}
$$

- **Transpose:** Let $A_{m\times n} = [a_{ij}]$. Then $A'_{n\times m} = [a_{ji}]$. Also denoted A^T. Properties:
  - $(A')' = A$
  - $(A + B)' = A' + B'$
  - $(AB)' = B'A'$

- Operation rules
  - **Commutative addition:** $A + B = B + A$.
  - **Distributive addition:** $(A + B) + C = A + (B + C)$.
  - **NON commutative multiplication:** $AB \neq BA$, even if both exist.
  - **Distributive multiplication:** $(AB)\,C = A\,(BC)$.
  - **Association:** premultiplying $A\,(B + C) = AB + AC$ and postmultiplying $(A + B)\,C = AC + BC$.

4.3 Special matrices

- **Identity matrix:**

$$
I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} .
$$

  $AI = IA = A$ (dimensions must conform).

- **Zero matrix:** all elements are zero. $0 + A = A$, $0A = A0 = 0$ (dimensions must conform).

- **Idempotent matrix:** $AA = A$. $A^k = A$, $k = 1, 2, ...$
  Example: the linear regression model is $y_{n\times 1} = X_{n\times k}\beta_{k\times 1} + \varepsilon_{n\times 1}$. The estimated model by OLS is $y = Xb + e$, where $b = (X'X)^{-1}X'y$. Therefore we have predicted values

$\widehat{y} = Xb = X\left(X'X\right)^{-1}X'y$ and residuals $e = y - \widehat{y} = y - Xb = y - X\left(X'X\right)^{-1}X'y = \left[I - X\left(X'X\right)^{-1}X'\right]y$. We can define the projection matrix as $P = X\left(X'X\right)^{-1}X'$ and the residual generating matrix as $R = [I - P]$. Both P and R are idempotent. What does it mean that P is idempotent? And that R is idempotent? What is the product PR, and what does that imply?

- **Singular matrices:** even if $AB = 0$, this does NOT imply that $A = 0$ or $B = 0$. E.g.,

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}.$$

  Likewise, $CD = CE$ does NOT imply $D = E$. E.g.,

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}.$$

  This is because A, B and C are singular: there is one (or more) row or column that is a linear combination of the other rows or columns, respectively. (More on this to come).

- **Nonsingular matrix**: a square matrix that has an inverse. (More on this to come).

- **Diagonal matrix**

$$D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}.$$

- **Upper triangular matrix**. Matrix U is upper triangular if $u_{ij} = 0$ for all $i > j$, i.e. all elements below the diagonal are zero. E.g.,

$$\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}.$$

- **Lower triangular matrix**. Matrix W is lower triangular if $w_{ij} = 0$ for all $i < j$, i.e. all elements above the diagonal are zero. E.g.,

$$\begin{bmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & i \end{bmatrix}.$$

- **Symmetric matrix**: $A = A'$.

- **Permutation matrix**: a matrix of 0s and 1s in which each row and each column contains exactly one 1. E.g.,

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} .$$

Multiplying a conformable matrix by a permutation matrix changes the order of the rows or the columns (unless it is the identity matrix). For example,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} .$$

- **Partitioned matrix**: a matrix with elements that are matrices themselves, e.g.,

$$\begin{bmatrix} A_{g\times h} & B_{g\times i} & C_{g\times j} \\ D_{k\times h} & E_{k\times h} & F_{k\times h} \end{bmatrix}_{(g+k)\times(h+i+j)} .$$

Note that the dimensions of the **sub matrices** must conform.

4.4 Vector products

- **Scalar multiplication:** Let $x_{m\times 1}$ be a vector. The scalar product cx is

$$cx_{m\times 1} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_m \end{bmatrix} .$$

- **Inner product:** Let $x_{m\times 1}$ and $y_{m\times 1}$ be vectors. The inner product is a scalar

$$x'y = \sum_{i=1}^{m} x_i y_i .$$

This is useful for computing correlations.

- **Outer product:** Let $x_{m\times 1}$ and $y_{n\times 1}$ be vectors. The outer product is a matrix

$$xy' = \begin{bmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_n \\ x_2y_1 & x_2y_2 & & x_2y_n \\ \vdots & & & \vdots \\ x_my_1 & x_my_2 & \dots & x_my_n \end{bmatrix}_{m\times n} .$$

This is useful for computing the variance/covariance matrix.

- **Geometric interpretations:** do in 2 dimensions. All extends to n dimensions.
  - Scalar multiplication.
  - Vector addition.
  - Vector subtraction.
  - Inner product and orthogonality ($xy = 0$ means $x \perp y$).

4.5 Linear independence

Definition 1: a set of k vectors $x_1, x_2, ...x_k$ are linearly independent iff neither one can be expressed as a linear combination of all or some of the others. Otherwise, they are linearly dependent.
Definition 2: a set of k vectors $x_1, x_2, ...x_k$ are linearly independent iff $\neg\exists$ a set of scalars $c_1, c_2, ...c_k$ such that $c_i \neq 0$ for some or all i and $\sum_{i=1}^{k} c_i x_i = 0$. Otherwise, they are linearly dependent. I.e., if such set of scalars exists, then the vectors are linearly dependent.

Consider $\mathbb{R}^2$:

- All vectors that are multiples are linearly dependent. If two vectors cannot be expressed as multiples then they are linearly independent.
- If two vectors are linearly independent, then any third vector can be expressed as a linear combination of the two.
- It follows that any set of $k > 2$ vectors in $\mathbb{R}^2$ must be linearly dependent.

4.6 Vector spaces and metric spaces

The complete set of vectors of n dimensions is a space, a **vector space**. If all elements of these vectors are real numbers ($\in \mathbb{R}$), then this space is $\mathbb{R}^n$.

- *Any* set of n linearly independent vectors is a **base** for $\mathbb{R}^n$.
- A base **spans** the space to which it pertains. This means that any vector in $\mathbb{R}^n$ can be expressed as a linear combination of the base (it is spanned by the base).
- Bases are not unique.
- Bases are minimal: they contain the smallest number of vectors that span the space.

Example: **unit vectors.** Consider the vector space $\mathbb{R}^3$. Then

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is a base. Indeed, e_1, e_3, e_3 are linearly independent.

- **Distance metric:** Let $x, y \in S$, some set. Define the distance between x and y by a function d: $d = d(x, y)$, which has the following properties:
  - $d(x, y) \geq 0$.
  - $d(x, y) = d(y, x)$.
  - $d(x, y) = 0 \;\Leftrightarrow\; x = y$.
  - $d(x, y) > 0 \;\Leftrightarrow\; x \neq y$.
  - $d(x, y) \leq d(x, z) + d(z, y) \;\; \forall x, y, z$ (triangle inequality).

A **metric space** is given by a vector space + distance metric. The **Euclidean space** is given by $\mathbb{R}^n$ + the following distance function

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{(x - y)' (x - y)} \; .$$

Other distance metrics give rise to different metric spaces.

4.7 Inverse matrix

Definition: if for some square $(n \times n)$ matrix A there exists a matrix B such that $AB = I$, then B is the inverse of A, and is denoted A^{-1}, i.e. $AA^{-1} = I$.

Properties:

- Not all square matrices have an inverse. If A^{-1} does not exist, then A is **singular**. Otherwise, A is **nonsingular**.
- A is the inverse of A^{-1} and vice versa.
- The inverse is square.
- The inverse, if it exists, is **unique**. Proof: suppose not, i.e. $AB = I$ and $B \neq A^{-1}$. Then $A^{-1}AB = A^{-1}I$, $IB = B = A^{-1}$, a contradiction $\blacksquare$

Operation rules:

- $\left(A^{-1}\right)^{-1} = A$. Proof: suppose not, i.e. $\left(A^{-1}\right)^{-1} = B$ and $B \neq A$. Then $AA^{-1} = I \;\Rightarrow\; \left(AA^{-1}\right)^{-1} = I^{-1} \;\Rightarrow\; \left(A^{-1}\right)^{-1} A^{-1} = I \;\Rightarrow\; BA^{-1}A = IA \;\Rightarrow\; BI = B = A$, a contradiction $\blacksquare$

- $(AB)^{-1} = B^{-1}A^{-1}$, but only if both B^{-1} and A^{-1} exist. Proof: Let $(AB)^{-1} = C$. Then $(AB)^{-1}(AB) = I = C(AB) = CAB \Rightarrow CABB^{-1} = CA = IB^{-1} = B^{-1} \Rightarrow CAA^{-1} = C = B^{-1}A^{-1}$ ■

  Note that in the linear regression model above $P = X(X'X)^{-1}X'$, but unless X is square we CANNOT write $(X'X)^{-1} = X^{-1}X'^{-1}$. (If we could, then $P = I$ but then there are no degrees of freedom: The model fits exactly the data, but the data are not very informative, because they are only one sample drawn from the population).

- $(A')^{-1} = (A^{-1})'$. Proof: Let $(A')^{-1} = B$. Then $(A')^{-1}A' = I = BA' \Rightarrow (BA')' = AB' = I' = I \Rightarrow A^{-1}AB' = A^{-1}I \Rightarrow B' = A^{-1} \Rightarrow B = (A^{-1})'$ ■

Conditions for nonsingularity:

- Necessary condition: matrix is square.

- Given square matrix, a sufficient condition is that the rows or columns are linearly independent. It does not matter whether we use the row or column criterion because matrix is square.

$$\underbrace{A \text{ is square} + \text{linear independence}}_{\text{necessary and sufficient conditions}} \Leftrightarrow A \text{ is nonsingular} \Leftrightarrow \exists A^{-1}$$

How do we find the inverse matrix? Soon... Why do we care? See next section.

4.8 Solving systems of linear equations

We seek a solution x to the system $Ax = c$

$$A_{n\times n}x_{n\times 1} = c_{n\times 1} \Rightarrow x = cA^{-1} ,$$

where A is a nonsingular matrix and c is a vector. Each row of A gives coefficients to the elements of x:

$$\begin{aligned} \text{row } 1 &: \sum_{i=1}^{n} a_{1i}x_i = c_1 \\ \text{row } 2 &: \sum_{i=1}^{n} a_{2i}x_i = c_2 \end{aligned}$$

Many linear (or linearized) models can be solved this way. We will learn clever ways to compute the solution to this system. We care about singularity of A because (given c) it tells us something about the solution x.

4.9 Markov chains

We introduce this through an example. Let x denote a vector of employment and unemployment rates: $x' = \begin{bmatrix} e & u \end{bmatrix}$, where $e + u = 1$ and $e, u \geq 0$. Define the matrix P as a transition matrix that gives the conditional probabilities for transition from the state today to a state next period,

$$P = \begin{bmatrix} p_{ee} & p_{eu} \\ p_{ue} & p_{uu} \end{bmatrix} ,$$

where $p_{ij} = \Pr(\text{state } j \text{ tomorrow}|\text{state } i \text{ today})$. Each row of P sums to unity: $p_{ee} + p_{eu} = 1$ and $p_{ue} + p_{uu} = 1$; and since these are probabilities, $p_{ij} \geq 0 \ \forall ij$. Now add a time dimension to x: $x'_t = \begin{bmatrix} e_t & u_t \end{bmatrix}$.

We ask: What is the employment and unemployment rates going to be in $t+1$ given x_t? Answer:

$$x'_{t+1} = x'_t P = \begin{bmatrix} e_t & u_t \end{bmatrix} \begin{bmatrix} p_{ee} & p_{eu} \\ p_{ue} & p_{uu} \end{bmatrix} = \begin{bmatrix} e_t p_{ee} + u_t p_{ue} & e_t p_{eu} + u_t p_{uu} \end{bmatrix} .$$

What will they be in $t + 2$? Answer: $x'_{t+2} = x'_{t+1} P = x'_t P^2$. More generally, $x'_{t_0+k} = x'_{t_0} P^k$.

A **transition matrix,** sometimes called **stochastic matrix**, is defined as a square matrix whose elements are non negative and all rows sum to 1. This gives you conditional transition probabilities starting from each state, where each row is a starting state and each column is the state in the next period.

Steady state: a situation in which the distribution over the states is not changing over time. How do we find such a state, if it exists?

- Method 1: Start with some initial condition x_0 and iterate forward $x'_k = x'_0 P^k$, taking $k \to \infty$.
- Method 2: define x as the steady state value. Solve $x' = x'P$. Or $P'x = x$.

5 Matrix algebra continued and linear models

5.1 Rank

Definition: The number of linearly independent rows (or, equivalently, columns) of a matrix A is the rank of A: $r = rank\,(A)$.

- If $A_{m\times n}$ then $rank\,(A) \leq \min\{m,n\}$.
- If a square matrix $A_{n\times n}$ has rank n, then we say that A is **full rank**.
- Multiplying a matrix A by a another matrix B that is full rank does not reduce the rank of the product relative to the rank of A.
- If $rank\,(A) = r_A$ and $rank\,(B) = r_B$, then $rank\,(AB) = \min\{r_A, r_B\}$.

Finding the rank: the **echelon matrix method**. First define **elementary operations**:

1. Multiply a row by a non zero scalar: $c \cdot R_i$, $c \neq 0$.
2. Adding c times of one row to another: $R_i + cR_j$.
3. Interchanging rows: $R_i \leftrightarrow R_j$.

All these operations alter the matrix, but do not change its rank (in fact, they can all be expressed by multiplying matrices, which are all full rank).

Define: **echelon matrix.**

1. Zero rows appear at the bottom.
2. For non zero rows, the first element on the left is 1.
3. The first element of each row on the left (which is 1) appears to the left of the row directly below it.

The number of non zero rows in the echelon matrix is the rank.

We use the elementary operations in order to change the subject matrix into an echelon matrix, which has as many zeros as possible. A good way to start the process is to concentrate zeros at the bottom. Example:

$$A = \begin{bmatrix} 0 & -11 & -4 \\ 2 & 6 & 2 \\ 4 & 1 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3 : \begin{bmatrix} 4 & 1 & 0 \\ 2 & 6 & 2 \\ 0 & -11 & -4 \end{bmatrix} \quad \frac{1}{4}R_1 : \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 2 & 6 & 2 \\ 0 & -11 & -4 \end{bmatrix}$$

$$R_2 - 2R_1 : \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 5\frac{1}{2} & 2 \\ 0 & -11 & -4 \end{bmatrix} \quad R_3 + 2R_2 : \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 5\frac{1}{2} & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \frac{2}{11}R_2 : \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 1 & 4/11 \\ 0 & 0 & 0 \end{bmatrix}$$

There is a row of zeros: $rank\,(A) = 2$. So A is singular.

5.2 Determinants and nonsingularity

Denote the determinant of a square matrix as $|A_{n\times n}|$. This is not absolute value. If the determinant is zero then the matrix is singular.

1. $|A_{1\times 1}| = a_{11}$.

2. $|A_{2\times 2}| = a_{11}a_{22} - a_{12}a_{21}$.

3. Determinants for higher order matrices. Let $A_{k\times k}$ be a square matrix. The i-j **minor** $|M_{ij}|$ is the determinant of the matrix given by erasing row i and column j from A. Example:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \ , \ |M_{11}| = \begin{vmatrix} e & f \\ h & i \end{vmatrix} \ .$$

The **Laplace Expansion of row i** gives the determinant of A:

$$|A_{k\times k}| = \sum_{j=1}^{k} (-1)^{i+j}\, a_{ij}\, |M_{ij}| = \sum_{j=1}^{k} a_{ij} C_{ij} \ ,$$

where $C_{ij} = (-1)^{i+j} |M_{ij}|$ is called the **cofactor** of a_{ij} (or the i-j^{th} cofactor). Example: expansion by row 1

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= aC_{11} + bC_{12} + cC_{13} \\ &= a\,|M_{11}| - b\,|M_{12}| + c\,|M_{13}| \\ &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a\,(ei - fh) - b\,(di - fg) + c\,(dh - eg) \ . \end{aligned}$$

In doing this, it is useful to choose the expansion with the row that has the most zeros.

Properties of determinants

1. $|A'| = |A|$

2. Interchanging rows or columns flips the sign of the determinant.

3. Multiplying a row or column by a scalar c multiplies the determinant by c.

4. $R_i + cR_j$ does not change the determinant.

5. If a row or a column are multiples of another row or column, respectively, then the determinant is zero: linear dependence.

6. Changing the minors in the Laplace expansion by **alien minors,** i.e. using $|M_{nj}|$ instead of $|M_{ij}|$ for row $i \neq n$, will give zero:

$$\sum_{j=1}^{k} a_{ij} (-1)^{i+j} |M_{nj}| = 0 \ , \quad i \neq n \ .$$

   This is like forcing linear dependency by repeating elements. $\sum_{j=1}^{k} a_{ij} (-1)^{i+j} |M_{nj}|$ is the determinant of some matrix. That matrix can be reverse engineered from the last expression. If you do this, you will find that that reverse-engineered matrix has linear dependent columns (try a 3×3 example).

Determinants and singularity: $|A| \neq 0$

$$\begin{aligned}
&\Leftrightarrow \quad A \text{ is nonsingular} \\
&\Leftrightarrow \quad \text{columns and rows are linearly independent} \\
&\Leftrightarrow \quad \exists A^{-1} \\
&\Leftrightarrow \quad \text{for } Ax = c \ , \ \exists! x = A^{-1}c \\
&\Leftrightarrow \quad \text{the column (or row) vectors of } A \text{ span the vector space.}
\end{aligned}$$

5.3 Finding the inverse matrix

Let A be a nonsingular matrix,

$$A_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} .$$

The **cofactor matrix** of A is C_A:

$$C_A = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & & C_{2n} \\ \vdots & & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix} ,$$

where $C_{ij} = (-1)^{i+j} |M_{ij}|$. The **adjoint matrix** of A is $\text{adj}A = C'_A$:

$$\text{adj}A = C'_A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & & C_{n2} \\ \vdots & & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} .$$

Consider AC'_A:

$$\begin{aligned} AC'_A &= \begin{bmatrix} \sum_{j=1}^n a_{1j}C_{1j} & \sum_{j=1}^n a_{1j}C_{2j} & \dots & \sum_{j=1}^n a_{1j}C_{nj} \\ \sum_{j=1}^n a_{2j}C_{1j} & \sum_{j=1}^n a_{2j}C_{2j} & & \sum_{j=1}^n a_{2j}C_{nj} \\ \vdots & & & \vdots \\ \sum_{j=1}^n a_{nj}C_{1j} & \sum_{j=1}^n a_{nj}C_{2j} & \dots & \sum_{j=1}^n a_{nj}C_{nj} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^n a_{1j}C_{1j} & 0 & \dots & 0 \\ 0 & \sum_{j=1}^n a_{2j}C_{2j} & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & \sum_{j=1}^n a_{nj}C_{nj} \end{bmatrix} \\ &= \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & |A| \end{bmatrix} = |A|\, I \ , \end{aligned}$$

where the off diagonal elements are zero due to alien cofactors. It follows that

$$\begin{aligned} AC'_A &= |A|\, I \\ AC'_A \frac{1}{|A|} &= I \\ A^{-1} &= C'_A \frac{1}{|A|} = \frac{\text{adj}A}{|A|} \ . \end{aligned}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \ , \ C_A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \ , \ C'_A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \ , \ |A| = -2 \ , \ A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \ .$$

And you can verify this.

5.4 Cramer's rule

For the system $Ax = c$ and nonsingular A, we have

$$x = A^{-1}c = \frac{\text{adj}A}{|A|} c \ .$$

Denote by A_j the matrix A with column j replaced by c. Then it turns out that

$$x_j = \frac{|A_j|}{|A|} \ .$$

To see why, note that each row of $C'_A c$ is c times row of C'_A, i.e. each row r is $\sum_j C_{jr} c_j$, which is a Laplace Expansion by row r of some matrix. That matrix is A_j and the Laplace expansion gives the determinant of A_j.

5.5 Homogenous equations: $Ax = 0$

Let the system of equations be homogenous: $Ax = 0$.

- If A is nonsingular, then only $x = 0$ is a solution. Recall: if A is nonsingular, then its columns are linearly independent. Denote the columns of A by A_i. Then $Ax = \sum_{i=1}^{n} x_i A_i = 0$ implies $x_i = 0 \ \forall i$ by linear independence of the columns.
- If A is singular, then there are infinite solutions, including $x = 0$.

5.6 Summary of linear equations: $Ax = c$

For nonsingular A:

1. $c \neq 0 \ \Rightarrow \ \exists! x \neq 0$
2. $c = 0 \ \Rightarrow \ \exists! x = 0$

For singular A:

1. $c \neq 0 \ \Rightarrow \ \exists x$, infinite solutions $\neq 0$.
    - If there is inconsistency—linear dependency in A, the elements of c do not follow the same linear combination—there is no solution.
2. $c = 0 \ \Rightarrow \ \exists x$, infinite solutions, including 0.

One can think of the system $Ax = c$ as defining a relation between c and x. If A is nonsingular, then there is a function (mapping/transformation) between c and x. In fact, when A is nonsingular, this transformation is invertible.

5.7 Inverse of partitioned matrix (not covered in CW)

Let A be a partitioned matrix such that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

Sufficient conditions for nonsingularity of A are that A_{11} and A_{22} are **square**, **nonsingular matrices**. In that case

$$A^{-1} = \begin{bmatrix} B_{11} & -B_{11}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}B_{11} & A_{22}^{-1} + A_{22}^{-1}A_{21}B_{11}A_{12}A_{22}^{-1} \end{bmatrix}, \tag{1}$$

where $B_{11} = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}$, or alternatively

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22} \\ -B_{22}A_{21}A_{11}^{-1} & B_{22} \end{bmatrix}, \tag{2}$$

where $B_{22} = \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}$. (This is useful for econometrics.)

To prove the above start with $AB = I$ and figure out what the partitions of B need to be. To get (1) you must assume (and use) A_{22} nonsingular; and to get (2) you must assume (and use) A_{11} nonsingular.

Note that A_{11} and A_{22} being nonsingular are *not necessary conditions* in general. For example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is nonsingular but does not meet the sufficient conditions. However if A is positive definite (we will define this below; a bordered Hessian is *not* positive definite), then A_{11} and A_{22} being nonsingular is also a necessary condition.

5.8 Leontief input/output model

We are interested in computing the level of output that is required from each industry in an economy that is required to satisfy *final* demand. This is not a trivial question, because output of all industries (depending on how narrowly you define an industry) are inputs for other industries, while also being consumed in final demand. These inter-industry relationships constitute input/output linkages.

Assume

1. Each industry produces one homogenous good.

2. Inputs are used in fixed proportions.

3. Constant returns to scale.

This gives rise to the Leontief (fixed proportions) production function. The second assumption can be relaxed, depending on the interpretation of the model. If you only want to use the framework for accounting purposes, then this is not critical.

- Define a_{io} as the **unit requirement** of inputs from industry i used in the production of output o. I.e., in order to produce one unit of output o you need a_{io} units of i. If some industry o does not require its own output for production, then $a_{oo} = 0$.

- For n industries $A_{n\times n} = [a_{io}]$ is a **technology matrix**. Each *column* tells you how much of each input is required to produce one unit of all outputs. Alternatively, each *row* tells you the input requirements to produce one unit of the industry of that row.

- If all industries were used as inputs as well as output, then there would be no **primary inputs** (i.e. time, labor, entrepreneurial talent, natural resources, land). To accommodate primary inputs, we add an **open sector**. If the a_{io} are denominated in monetary values—i.e., in order to produce \$1 in industry o you need $\$a_{io}$ of input i—then we must have $\sum_{i=1}^{n} a_{io} \leq 1$, because the revenue from producing output o is \$1. And if there is an open sector, then we must have $\sum_{i=1}^{n} a_{io} < 1$. This means that the cost of intermediate inputs required to produce \$1 of revenue is less than \$1. By CRS and competitive economy, we have the zero profit condition, which means that all revenue is paid out to inputs. So primary inputs receive $(1 - \sum_{i=1}^{n} a_{io})$ dollars from each dollar produced by industry o.

Equilibrium implies

$$\begin{aligned} \text{supply} &= \text{demand} \\ &= \text{demand for intermediate inputs} + \text{final demand} . \end{aligned}$$

In matrix notation

$$x = Ax + d .$$

And so

$$x - Ax = (I - A)\,x = d .$$

Let A'_o be the o^{th} row vector of A. Then for some output o (row) we have

$$\begin{aligned} x_o &= A'_o x + d_o \\ &= \sum_{i=1}^{n} a_{oi} x_i + d_o \\ &= \underbrace{a_{o1}x_1 + a_{o2}x_2 + ... + a_{on}x_n}_{\text{intermediate inputs}} + \underbrace{d_o}_{\text{final}} \ . \end{aligned}$$

For example, $a_{o2}x_2$ is the amount of output o that is required by industry 2, because you need a_{o2} units of o to produce each unit of industry 2 and x_2 units of industry 2 are produced. This implies

$$-a_{o1}x_1 - a_{o2}x_2 + ... (1 - a_{oo}) x_o - a_{o,o+1}x_{o+1} - ... - a_{on}x_n = d_o \ .$$

In matrix notation

$$\begin{bmatrix} (1-a_{11}) & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & (1-a_{22}) & -a_{23} & \cdots & -a_{2n} \\ -a_{31} & -a_{32} & (1-a_{33}) & \cdots & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & (1-a_{nn}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix} \ .$$

Or

$$(I - A) x = d \ .$$

$(I - A)$ is the **Leontief matrix**. This implies that you need to produce more than just final demand because some x are used as intermediate inputs (loosely speaking, "$I - A < I$").

If $(I - A)$ is nonsingular, then we can solve for x:

$$x = (I - A)^{-1} d \ .$$

But even then the solution to x may not be positive. While in reality this must be trivially satisfied in the data, we wish to find theoretical restrictions on the technology matrix to satisfy a non-negative solution for x.

5.8.1 Existence of non negative solution

Consider

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \ .$$

Define

- **Principal minor:** the determinant of the matrix that arises from deleting the i-th row and

i-th column. E.g.

$$|M_{11}| = \begin{vmatrix} e & f \\ h & i \end{vmatrix} \ , \ |M_{22}| = \begin{vmatrix} a & c \\ g & i \end{vmatrix} \ , \ |M_{33}| = \begin{vmatrix} a & b \\ d & e \end{vmatrix} \ .$$

- **k-th order principal minor:** is a principal minor of dimensions $k \times k$. If the dimensions of the original matrix are $n \times n$, then a k-th order principal minor is obtained after deleting the same $n-k$ rows and columns. E.g., the 1-st order principal minors of A are

  $$|a| \ , \ |e| \ , \ |i| \ .$$

  The 2-nd order principal minors are $|M_{11}|$, $|M_{22}|$ and $|M_{33}|$ given above.

- **Leading principal minors:** these are the 1^{st}, 2^{nd}, 3^{rd} (etc.) order principal minors, where we keep the upper most left corner of the original matrix in each one. E.g.

  $$|M_1| = |a| \ , \ |M_2| = \begin{vmatrix} a & b \\ d & e \end{vmatrix} \ , \ |M_3| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \ .$$

Simon-Hawkins Condition (Theorem): consider the system of equations $Bx = d$. If
(1) all off-diagonal elements of $B_{n\times n}$ are non positive, i.e. $b_{ij} \leq 0, \ \forall i \neq j$;
(2) all elements of $d_{n\times 1}$ are non negative, i.e. $d_i \geq 0, \ \forall i$;
Then $\exists x \geq 0$ such that $Bx = d$ iff
(3) all leading principal minors are strictly positive, i.e. $|M_i| > 0, \ \forall i$.

In our case, $B = I - A$, the Leontief matrix. Conditions (1) and (2) are satisfied. To illustrate the economic meaning of SHC, use a 2×2 example:

$$I - A = \begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix} \ .$$

Condition (3) requires $|M_1| = |1 - a_{11}| = 1 - a_{11} > 0$, i.e. $a_{11} < 1$. This means that less than the total output of x_1 is used to produce x_1, i.e. **viability**. Next, condition (3) also requires

$$\begin{aligned} |M_2| &= |I - A| \\ &= (1 - a_{11})(1 - a_{22}) - a_{12}a_{21} \\ &= 1 - a_{11} - a_{22} + a_{11}a_{22} - a_{12}a_{21} > 0 \end{aligned}$$

Rearranging terms we have

$$\underbrace{(1 - a_{11})\, a_{22}}_{\geq 0} + a_{11} + a_{12}a_{21} < 1$$

and therefore

$$\underbrace{a_{11}}_{\text{direct use}} + \underbrace{a_{12}a_{21}}_{\text{indirect use}} < 1$$

This means that the total amount of x_1 demanded (for production of x_1 and for production of x_2) is less than the amount produced (=1), i.e. the **resource constraint** is kept.

5.8.2 Closed model version

The closed model version treats the primary sector as any industry. Suppose that there is only one primary input: labor. The interpretation is that each good is consumed in fixed proportions (Leontief preferences). In the case when a_{ij} represents value, then the interpretation is that expenditure on each good is in fixed proportions (these preferences can be represented by a Cobb-Douglas utility function).

In this model final demand, as defined above, must equal zero. Since income accrues to primary inputs (think of labor) and this income is captured in x, then it follows that the d vector must be equal to zero. Since final demand equals income, then if final demand was positive, then we would have to have an open sector to pay for that demand (from its income). I.e. we have a homogenous system:

$$(I - A)\,x = 0$$

$$\begin{bmatrix} (1-a_{00}) & -a_{01} & -a_{02} \\ -a_{10} & (1-a_{11}) & -a_{12} \\ -a_{20} & -a_{21} & (1-a_{22}) \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where 0 denotes the primary sector (there could be more than one).

Each column o in the technology matrix A must sum to 1, i.e. $a_{0o} + a_{1o} + a_{2o} + ... + a_{no} = 1,\ \forall o$, because all of the revenue is exhausted in payments for inputs (plus consumption). Then each column in $I - A$ sums to zero. It follows that $I - A$ is singular, and therefore x is not unique (albeit not necessarily zero)! This implies that you can scale up or down the economy with no effect. In fact, this is a general property of CRS economies with no outside sector or endowment. One way to pin down the economy is to set some x_i to some level as an endowment and, accordingly, to set $x_{ii} = 0$ (you don't need land to produce land).

6 Derivatives and limits

Teaching assistant covers. See Chapter 6 in CW.

7 Differentiation and use in comparative statics

7.1 Differentiation rules

1. If $y = f(x) = c$, a constant, then $\frac{dy}{dx} = 0$
2. $\frac{d}{dx} ax^n = anx^{n-1}$
3. $\frac{d}{dx} \ln x = \frac{1}{x}$
4. $\frac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x)$
5. $\frac{d}{dx} [f(x) g(x)] = f'(x) g(x) + f(x) g'(x) = [f(x) g(x)] \frac{f'(x)}{f(x)} + [f(x) g(x)] \frac{g'(x)}{g(x)}$
6. $\frac{d}{dx} \left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \frac{f(x)}{g(x)} \frac{f'(x)}{f(x)} - \frac{f(x)}{g(x)} \frac{g'(x)}{g(x)}$
7. $\frac{d}{dx} f[g(x)] = \frac{df}{dg} \frac{dg}{dx}$ (the chain rule)
8. Inverse functions. Let $y = f(x)$ be strictly monotone (there are no "flats"). Then an inverse function $x = f^{-1}(y)$ exists and

$$\frac{dx}{dy} = \frac{df^{-1}(y)}{dy} = \frac{1}{dy/dx} = \frac{1}{df(x)/dx} ,$$

where x and y map one into the other, i.e. $y = f(x)$ and $x = f^{-1}(y)$.

- Strictly monotone means that $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ (strictly increasing) or $f(x_1) < f(x_2)$ (strictly decreasing). It implies that there is an inverse function $x = f^{-1}(y)$ because $\forall y \in$Range $\exists! x \in$ domain (recall: $\forall x \in$ domain $\exists! y \in$Range defines $f(x)$).

7.2 Partial derivatives

Let $y = f(x_1, x_2, ... x_n)$. Define the partial derivative of f with respect to x_i:

$$\frac{\partial y}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_i + \Delta x_i, \overline{x}_{-i}) - f(x_i, \overline{x}_{-i})}{\Delta x_i} .$$

Operationally, you derive $\partial y / \partial x_i$ just as you would derive dy/dx_i, while treating all other x_{-i} as constants.

Example. Consider the following production function

$$y = z \left[\alpha k^{\varphi} + (1 - \alpha) l^{\varphi}\right]^{1/\varphi} , \ \varphi \leq 1 .$$

Define the elasticity of substitution as the percent change in relative factor intensity (k/l) in response to a 1 percent change in the relative factor returns (r/w). What is the elasticity of substitution? If factors are paid their marginal product (which is a partial derivative in this case), then

$$\begin{aligned} y_k &= \frac{1}{\varphi} z \left[\cdot\right]^{\frac{1}{\varphi}-1} \varphi \alpha k^{\varphi-1} = r \\ y_l &= \frac{1}{\varphi} z \left[\cdot\right]^{\frac{1}{\varphi}-1} \varphi \left(1-\alpha\right) l^{\varphi-1} = w \ . \end{aligned}$$

Thus

$$\frac{r}{w} = \frac{\alpha}{1-\alpha} \left(\frac{k}{l}\right)^{\varphi-1}$$

and then

$$\frac{k}{l} = \left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{1-\varphi}} \left(\frac{r}{w}\right)^{-\frac{1}{1-\varphi}} \ .$$

The elasticity of substitution is $\sigma = \frac{1}{1-\varphi}$ and it is constant. This production function exhibits constant elasticity of substitution, denoted a CES production function. A 1 percent increase in r/w decreases k/l by σ percent.

7.3 Gradients

$$y = f\left(x_1, x_2, ...x_n\right)$$

The gradient is defined as

$$\nabla f = \left(f_1, f_2, ...f_n\right) \ ,$$

where

$$f_i = \frac{\partial f}{\partial x_i} \ .$$

We can use this in first order approximations:

$$\begin{aligned} \Delta f|_{x_0} &\approx \nabla f\left(x_0\right) \Delta x \\ f\left(x\right) - f\left(x_0\right) &\approx \left(f_1, f_2, ...f_n\right)|_{x_0} \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix} \right) \ . \end{aligned}$$

Application to open input/output model:

$$\begin{aligned} \left(I-A\right) x &= d \\ x &= \left(I-A\right)^{-1} d = Vd \\ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &= \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \ . \end{aligned}$$

Think of x as a function of d:

$$\nabla x_i = \begin{pmatrix} v_{i1} & v_{i2} & \cdots & v_{in} \end{pmatrix}$$

$$v_{ij} = \frac{\partial x_i}{\partial d_j} \ .$$

And more generally,

$$\Delta x = \nabla x \cdot \Delta d = V \Delta d \ .$$

7.4 Jacobian and functional dependence

Let there be two functions

$$\begin{aligned} y_1 &= f(x_1, x_2) \\ y_2 &= g(x_1, x_2) \ . \end{aligned}$$

The **Jacobian determinant** is

$$|J| = \left| \frac{\partial y}{\partial x'} \right| = \left| \frac{\partial \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}{\partial (x_1, x_2)} \right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} \ .$$

Theorem (functional dependence): $|J| = 0 \ \forall x$ iff the functions are dependent.

Example: $y_1 = x_1 x_2$ and $y_2 = \ln x_1 + \ln x_2$.

$$|J| = \begin{vmatrix} x_2 & x_1 \\ \frac{1}{x_1} & \frac{1}{x_2} \end{vmatrix} = 0 \ .$$

Example: $y_1 = x_1 + 2x_2^2$ and $y_2 = \ln\left(x_1 + 2x_2^2\right)$.

$$|J| = \begin{vmatrix} 1 & 4x_2 \\ \frac{1}{x_1+2x_2^2} & \frac{4x_2}{x_1+2x_2^2} \end{vmatrix} = 0 \ .$$

Another example: $x = Vd$,

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} v_{11} & \cdots & v_{13} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} \sum v_{1i} d_i \\ \vdots \\ \sum v_{ni} d_i \end{bmatrix} \ .$$

So $|J| = |V|$. It follows that linear dependence is equivalent to functional dependence for a system of linear equations. If $|V| = 0$ then there are ∞ solutions for x and the relationship between d and x cannot be inverted.

8 Total differential, total derivative and the implicit function theorem

8.1 Total derivative

Often we are interested in the total rate of change in some variable in response to a change in some other variable or some parameter. If there are indirect effects, as well as direct ones, you want to take this into account. Sometimes the indirect effects are due to general equilibrium constraints and can be very important.

Example: consider the utility function $u(x, y)$ and the budget constraint $p_x x + p_y y = I$. Then the total effect of a small change in x on utility is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \ .$$

Here $dy/dx = -p_x/p_y$ (if you sell one unit of x you get p_x, which can be used to buy y at the rate of p_y).

More generally: $F(x_1, ...x_n)$

$$\frac{dF}{dx_i} = \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \cdot \frac{dx_j}{dx_i} \ ,$$

where we know that $dx_i/dx_i = 1$.

Example: $z = f(x, y, u, v)$, where $x = x(u, v)$ and $y = y(u, v)$ and $v = v(u)$.

$$\frac{dz}{du} = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}\frac{dv}{du}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\frac{dv}{du}\right) + \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\frac{dv}{du} \ .$$

If we want to impose that v is not directly affected by u, then all terms that involve dv/du are zero:

$$\frac{dz}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du} + \frac{\partial f}{\partial u} \ .$$

Alternatively, we can impose that v is constant; in this case the derivative is denoted as $\frac{dz}{du}\big|_{\overline{v}}$ and the result is the same as above.

8.2 Total differential

Now we are interested in the change (not rate of...) in some variable or function if *all* its arguments change a bit, i.e. they are all perturbed. For example, if the saving function for the economy is $S = S(y, r)$, then

$$dS = \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial r}dr \ .$$

More generally, $y = F(x_1, ...x_n)$

$$dy = \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} dx_j \ .$$

One can view the total differential as a linearization of the function around a specific point, because $\partial F / \partial x_j$ must be evaluated at some point.

The same rules that apply to derivatives apply to differentials; just simply add dx after each partial derivative:

1. $dc = 0$ for constant c.
2. $d(cu^n) = cnu^{n-1}du = \frac{\partial(cu^n)}{\partial u}du$.
3. $d(u \pm v) = du \pm dv = \frac{\partial(u \pm v)}{\partial u}du + \frac{\partial(u \pm v)}{\partial v}dv$.
   - $d(u \pm v \pm w) = du \pm dv \pm dw = \frac{\partial(u \pm v \pm w)}{\partial u}du + \frac{\partial(u \pm v \pm w)}{\partial v}dv + \frac{\partial(u \pm v \pm w)}{\partial w}dw$.
4. $d(uv) = vdu + udv = \frac{\partial(uv)}{\partial u}du + \frac{\partial(uv)}{\partial v}dv = (uv)\frac{du}{u} + (uv)\frac{dv}{v}$.
   - $d(uvw) = vwdu + uwdv + uvdw = \frac{\partial(uvw)}{\partial u}du + \frac{\partial(uvw)}{\partial v}dv + \frac{\partial(uvw)}{\partial w}dw$.
5. $d(u/v) = \frac{vdu - udv}{v^2} = \frac{\partial(u/v)}{\partial u}du + \frac{\partial(u/v)}{\partial v}dv = \left(\frac{u}{v}\right)\frac{du}{u} - \left(\frac{u}{v}\right)\frac{dv}{v}$.

Example: suppose that you want to know how much utility, $u(x, y)$, changes if x and y are perturbed. Then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \ .$$

Now, if you imposed that utility is not changing, i.e. you are interested in an isoquant (the indifference curve), then this implies that $du = 0$ and then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

and hence

$$\frac{dy}{dx} = -\frac{\partial u / \partial x}{\partial u / \partial y} \ .$$

This should not be understood as a derivative, but rather as a ratio of perturbations. Soon we will characterize conditions under which this is actually a derivative of an implicit function (the implicit function theorem).

Log linearization. Suppose that you want to log-linearize $z = f(x, y)$ around some point, say (x^*, y^*, z^*). This means finding the percent change in z in response to a percent change in x and y. We have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \ .$$

Divide through by z^* to get

$$\begin{aligned}
\frac{dz}{z^*} &= \frac{x^*}{z^*}\frac{\partial z}{\partial x}\left(\frac{dx}{x^*}\right) + \frac{y^*}{z^*}\frac{\partial z}{\partial y}\left(\frac{dy}{y^*}\right) \\
\widehat{z} &= \frac{x^*}{z^*}\frac{\partial z}{\partial x}\widehat{x} + \frac{y^*}{z^*}\frac{\partial z}{\partial y}\widehat{y} \ ,
\end{aligned}$$

where

$$\widehat{z} = \frac{dz}{z^*} \approx d \ln z$$

is approximately the percent change.

Another example:

$$\begin{aligned}
Y &= C + I + G \\
dY &= dC + dI + dG \\
\frac{dY}{Y} &= \frac{C}{Y}\frac{dC}{C} + \frac{I}{Y}\frac{dI}{I} + \frac{G}{Y}\frac{dG}{G} \\
\widehat{Y} &= \frac{C}{Y}\widehat{C} + \frac{I}{Y}\widehat{I} + \frac{G}{Y}\widehat{G} \ .
\end{aligned}$$

8.3 The implicit function theorem

This is a useful tool to study the behavior of an equilibrium in response to a change in an exogenous variable.

Consider

$$F(x, y) = 0 \ .$$

We are interested in characterizing the implicit function between x and y, if it exists. We already saw one implicit function when we computed the utility isoquant (indifference curve). In that case, we had

$$u(x, y) = \overline{u}$$

for some constant level of $\overline{u}$. This can be rewritten in the form above as

$$u(x, y) - \overline{u} = 0 \ .$$

From this we derived a dy/dx slope. But this can be more general and constitute a function.

Another example: what is the slope of a tangent line at any point on a circle?

$$\begin{aligned}
x^2 + y^2 &= r^2 \\
x^2 + y^2 - r^2 &= 0 \\
F(x, y) &= 0
\end{aligned}$$

Taking the total differential

$$\begin{aligned} F_x dx + F_y dy &= 2xdx + 2ydy = 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \ , \ y \neq 0 \ . \end{aligned}$$

For example, the slope at $\left(r/\sqrt{2}, r/\sqrt{2}\right)$ is -1.

The implicit function theorem: Let the function $F(x,y) \in C^1$ on some open set and $F(x,y) = 0$. Then there exists a (implicit) function $y = f(x) \in C^1$ that satisfies $F(x, f(x)) = 0$, such that

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

on this open set.

More generally, if $F(y, x_1, x_2, ...x_n) \in C^1$ on some open set and $F(y, x_1, x_2, ...x_n) = 0$, then there exists a (implicit) function $y = f(x_1, x_2, ...x_n) \in C^1$ that satisfies $F(f(x), x) = 0$, such that

$$dy = \sum_{i=1}^{n} f_i dx_i \ .$$

This gives us the relationship between small perturbations of the x's and perturbation of y.

If we allow only one specific x_i to be perturbed, then $f_i = \frac{\partial y}{\partial x_i} = -F_{x_i}/F_y$. From $F(y, x_1, x_2, ...x_n) = 0$ and $y = f(x_1, x_2, ...x_n)$ we have

$$\frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial x_1} dx_1 + ... + \frac{\partial F}{\partial x_n} dx_n = 0$$

$$dy = f_1 dx_1 + ... + f_n dx_n$$

so that

$$\frac{\partial F}{\partial y}(f_1 dx_1 + ... + f_n dx_n) + F_{x_1} dx_1 + ... + F_{x_n} dx_n = (F_{x_1} + F_y f_1) dx_1 + ... + (F_{x_n} + F_y f_n) dx_n = 0 \ .$$

This gives us a relationship between perturbations of the x's. If we only allow x_i to be perturbed, $dx_{-i} = 0$, then $(F_{x_i} + F_y f_i) = 0$ and so $f_i = -F_{x_i}/F_y$, as above.

8.4 General version of the implicit function theorem

Implicit Function Theorem: Let $F(x,y) = 0$ be a set of $\boldsymbol{n}$ functions where $x_{m\times 1}$ (**exogenous**) and $y_{\boldsymbol{n}\times 1}$ (**endogenous**). Note that there are $\boldsymbol{n}$ equations in $\boldsymbol{n}$ unknown endogenous variables. If

1. $F \in C^1$ and

2. $|J| = \left|\frac{\partial F}{\partial y'}\right| \neq 0$ at some point (x_0, y_0) (no functional dependence),

then $\exists y = f(x)$, a set of n functions in a neighborhood of (x_0, y_0) such that $f \in C^1$ and $F(x, f(x)) = 0$ in that neighborhood of (x_0, y_0).

We further develop this. From $F(x, y) = 0$ we have

$$\left[\frac{\partial F}{\partial y'}\right]_{n\times n} dy_{n\times 1} + \left[\frac{\partial F}{\partial x'}\right]_{n\times m} dx_{m\times 1} = 0 \quad \Rightarrow \quad \left[\frac{\partial F}{\partial y'}\right] dy = -\left[\frac{\partial F}{\partial x'}\right] dx \ . \tag{3}$$

Since $|J| = |\partial F/\partial y'| \neq 0$, then $[\partial F/\partial y']^{-1}$ exists and we can write

$$dy = -\left[\frac{\partial F}{\partial y'}\right]^{-1} \left[\frac{\partial F}{\partial x'}\right] dx \ . \tag{4}$$

So there is a mapping from dx to dy.

From $y = f(x)$ we have

$$dy_{n\times 1} = \left[\frac{\partial y}{\partial x'}\right]_{n\times m} dx_{m\times 1}$$

Combining into (3) we get

$$\left[\frac{\partial F}{\partial y'}\right]_{n\times n} \left[\frac{\partial y}{\partial x'}\right]_{n\times m} dx_{m\times 1} = -\left[\frac{\partial F}{\partial x'}\right]_{n\times m} dx_{m\times 1} \ .$$

Now suppose that only x_1 is perturbed, so that $dx' = \begin{bmatrix} dx_1 & 0 & \cdots & 0 \end{bmatrix}$. Then we get only the first column in the set of equations above:

$$\begin{array}{ll} \text{row } 1 & : \left[\frac{\partial F^1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial F^1}{\partial y_2}\frac{\partial y_2}{\partial x_1} + ... + \frac{\partial F^1}{\partial y_n}\frac{\partial y_n}{\partial x_1}\right] dx_1 = -\frac{\partial F^1}{\partial x_1} dx_1 \\ & \vdots \\ \text{row } n & : \left[\frac{\partial F^n}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial F^n}{\partial y_2}\frac{\partial y_2}{\partial x_1} + ... + \frac{\partial F^n}{\partial y_n}\frac{\partial y_n}{\partial x_1}\right] dx_1 = -\frac{\partial F^n}{\partial x_1} dx_1 \end{array}$$

By eliminating the dx_1 terms we get

$$\begin{array}{ll} \text{row } 1 & : \left[\frac{\partial F^1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial F^1}{\partial y_2}\frac{\partial y_2}{\partial x_1} + ... + \frac{\partial F^1}{\partial y_n}\frac{\partial y_n}{\partial x_1}\right] = -\frac{\partial F^1}{\partial x_1} \\ & \vdots \\ \text{row } n & : \left[\frac{\partial F^n}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial F^n}{\partial y_2}\frac{\partial y_2}{\partial x_1} + ... + \frac{\partial F^n}{\partial y_n}\frac{\partial y_n}{\partial x_1}\right] = -\frac{\partial F^n}{\partial x_1} \end{array}$$

and thus, stacking together

$$\left[\frac{\partial F}{\partial y'}\right]_{n\times n} \left[\frac{\partial y}{\partial x_1}\right]_{n\times 1} = -\left[\frac{\partial F}{\partial x_1}\right]_{n\times 1} \ .$$

Since we required $|J| = \left|\frac{\partial F}{\partial y'}\right| \neq 0$ it follows that the $\left[\frac{\partial F}{\partial y'}\right]_{n\times n}$ matrix is nonsingular, and thus $\exists! \left[\frac{\partial y}{\partial x_1}\right]_{n\times 1}$, a solution to the system. This can be obtained by Cramer's rule:

$$\frac{\partial y_j}{\partial x_1} = \frac{|J_j|}{|J|} \ ,$$

where $|J_j|$ is obtained by replacing the j^{th} column in $|J_j|$ by $\left[\frac{\partial F}{\partial x_1}\right]$. In fact, we could have jumped directly to here from (4).

Why is this useful? We are often interested in how a model behaves around some point, usually an equilibrium or a steady state. But models are typically nonlinear and the behavior is hard to characterize without implicit functions. Think of x as exogenous and y as endogenous. So this gives us a method for evaluating how several endogenous variables respond to a small change in one an exogenous variable or policy – while holding all other x's constant. This describes a lot of what we do in economics.

A fuller description of what's going on:

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{bmatrix} + \begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \frac{\partial F^1}{\partial x_2} & \cdots & \frac{\partial F^1}{\partial x_m} \\ \frac{\partial F^2}{\partial x_1} & \frac{\partial F^2}{\partial x_2} & \cdots & \frac{\partial F^2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \frac{\partial F^n}{\partial x_2} & \cdots & \frac{\partial F^n}{\partial x_m} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \frac{\partial F^1}{\partial x_2} & \cdots & \frac{\partial F^1}{\partial x_m} \\ \frac{\partial F^2}{\partial x_1} & \frac{\partial F^2}{\partial x_2} & \cdots & \frac{\partial F^2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \frac{\partial F^n}{\partial x_2} & \cdots & \frac{\partial F^n}{\partial x_m} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix}$$

$$\begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{bmatrix} = \begin{bmatrix} \frac{\partial y^1}{\partial x_1} & \frac{\partial y^1}{\partial x_2} & \cdots & \frac{\partial y^1}{\partial x_m} \\ \frac{\partial y^2}{\partial x_1} & \frac{\partial y^2}{\partial x_2} & \cdots & \frac{\partial y^2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y^n}{\partial x_1} & \frac{\partial y^n}{\partial x_2} & \cdots & \frac{\partial y^n}{\partial x_m} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_m} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \frac{\partial F^1}{\partial x_2} & \cdots & \frac{\partial F^1}{\partial x_m} \\ \frac{\partial F^2}{\partial x_1} & \frac{\partial F^2}{\partial x_2} & \cdots & \frac{\partial F^2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \frac{\partial F^n}{\partial x_2} & \cdots & \frac{\partial F^n}{\partial x_m} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix}$$

8.5 Example: demand-supply system

8.5.1 Using the implicit function theorem

$$\begin{aligned} \text{demand} &: \quad q^d = d(\overset{-}{p}, \overset{+}{y}) \\ \text{supply} &: \quad q^s = s(\overset{+}{p}) \\ \text{equilibrium} &: \quad q^d = q^s \ . \end{aligned}$$

Let $d, s \in C^1$. By eliminating q we get

$$s(\overset{+}{p}) - d(\overset{-}{p}, \overset{+}{y}) = 0 \ ,$$

which is an implicit function

$$F(p, y) = 0 \ ,$$

where p is endogenous and y is exogenous.

We are interested in how the endogenous price responds to income. By the implicit function theorem $\exists p = p(y)$ such that

$$\frac{dp}{dy} = -\frac{F_y}{F_p} = -\frac{-d_y}{s_p - d_p} = \frac{d_y}{s_p - d_p} > 0$$

because $d_p < 0$. An increase in income unambiguously increases the price.

To find how quantity changes we apply the total derivative approach to the demand function:

$$\frac{dq}{dy} = \underbrace{\frac{\partial d}{\partial p}\frac{dp}{dy}}_{\text{"substitution effect"}<0} + \underbrace{\frac{\partial d}{\partial y}}_{\text{"income effect"}>0}$$

so the sign here is ambiguous. The income effect is the shift outwards of the demand curve. If supply did not respond to price (infinite elasticity), then that would be it. The substitution effect is the shift along the (shifted) demand curve that is invoked by the increase in price. But we can show that dq/dy is positive by using the supply side:

$$\frac{dq}{dy} = \frac{\partial s}{\partial p}\frac{dp}{dy} > 0 \ .$$

- Draw demand-supply system.

This example is simple, but the technique is very powerful, especially in nonlinear general equilibrium models.

8.5.2 Using the implicit function theorem in a system of two equations

Now consider the system by writing it as a system of two implicit functions:

$$F(p, q; y) = 0$$

$$\begin{aligned} F^1(p, q, y) &= d(p, y) - q = 0 \\ F^2(p, q, y) &= s(p) - q = 0 \ . \end{aligned}$$

Apply the general theorem. Check for functional dependence in the endogenous variables:

$$|J| = \left| \frac{\partial F}{\partial (p, q)} \right| = \begin{vmatrix} d_p & -1 \\ s_p & -1 \end{vmatrix} = -d_p + s_p > 0 \ .$$

So there is no functional dependence. Thus $\exists p = p(y)$ and $\exists q = q(y)$. We now wish to compute the derivatives with respect to the exogenous argument y. Since $dF = 0$ we have

$$\begin{aligned} \frac{\partial F^1}{\partial p} dp + \frac{\partial F^1}{\partial q} dq + \frac{\partial F^1}{\partial y} dy &= 0 \\ \frac{\partial F^2}{\partial p} dp + \frac{\partial F^2}{\partial q} dq + \frac{\partial F^2}{\partial y} dy &= 0 \end{aligned}$$

Thus

$$\begin{bmatrix} \frac{\partial F^1}{\partial p} & \frac{\partial F^1}{\partial q} \\ \frac{\partial F^2}{\partial p} & \frac{\partial F^2}{\partial q} \end{bmatrix} \begin{bmatrix} dp \\ dq \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial y} dy \\ \frac{\partial F^2}{\partial y} dy \end{bmatrix}$$

Use the following

$$\begin{aligned} dp &= \frac{\partial p}{\partial y} dy \\ dq &= \frac{\partial q}{\partial y} dy \end{aligned}$$

to get

$$\begin{bmatrix} \frac{\partial F^1}{\partial p} & \frac{\partial F^1}{\partial q} \\ \frac{\partial F^2}{\partial p} & \frac{\partial F^2}{\partial q} \end{bmatrix} \begin{bmatrix} \frac{\partial p}{\partial y} dy \\ \frac{\partial q}{\partial y} dy \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial y} dy \\ \frac{\partial F^2}{\partial y} dy \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial F^1}{\partial p} & \frac{\partial F^1}{\partial q} \\ \frac{\partial F^2}{\partial p} & \frac{\partial F^2}{\partial q} \end{bmatrix} \begin{bmatrix} \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial y} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial y} \\ \frac{\partial F^2}{\partial y} \end{bmatrix}$$

Using the expressions for F^1 and F^2 we get

$$\begin{bmatrix} \frac{\partial d}{\partial p} & -1 \\ \frac{\partial s}{\partial p} & -1 \end{bmatrix} \begin{bmatrix} \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{\partial d}{\partial y} \\ 0 \end{bmatrix} \ .$$

We seek a solution for $\frac{\partial p}{\partial y}$ and $\frac{\partial q}{\partial y}$. This is a system of equations, which we solve using Cramer's rule:

$$\frac{\partial p}{\partial y} = \frac{|J_1|}{|J|} = \frac{\begin{vmatrix} -\frac{\partial d}{\partial y} & -1 \\ 0 & -1 \end{vmatrix}}{|J|} = \frac{\frac{\partial d}{\partial y}}{|J|} > 0$$

and

$$\frac{\partial q}{\partial y} = \frac{|J_2|}{|J|} = \frac{\begin{vmatrix} \frac{\partial d}{\partial p} & -\frac{\partial d}{\partial y} \\ \frac{\partial s}{\partial p} & 0 \end{vmatrix}}{|J|} = \frac{\frac{\partial d}{\partial y}\frac{\partial s}{\partial p}}{|J|} > 0 \ .$$

- Try this with three functions for three endogenous variables, i.e. $F\left(p, q^s, q^d; y\right) = 0$.

8.5.3 Using the total derivative approach

Now we use the total derivative approach. We have

$$s\left(p\right) - d\left(p, y\right) = 0 \ .$$

Take the total derivative with respect to y:

$$\frac{\partial s}{\partial p}\frac{dp}{dy} - \frac{\partial d}{\partial p}\frac{dp}{dy} - \frac{\partial d}{\partial y} = 0$$

Thus

$$\frac{dp}{dy}\left[\frac{\partial s}{\partial p} - \frac{\partial d}{\partial p}\right] = \frac{\partial d}{\partial y}$$

and so

$$\frac{dp}{dy} = \frac{\frac{\partial d}{\partial y}}{\frac{\partial s}{\partial p} - \frac{\partial d}{\partial p}} > 0 \ .$$

9 Optimization with one variable and Taylor expansion

A function may have many local minima and maxima. A function may have only one global minimum and maximum, if it exists.

9.1 Local maximum, minimum

First order necessary conditions (FONC): Let $f \in C^1$ on some open convex set (will be defined properly later) around x_0. If $f'(x_0) = 0$, then x_0 is a **critical point**, i.e. it could be either a maximum or minimum—or neither.

1. x_0 is a **local maximum** if $f'(x_0)$ changes from positive to negative as x increases around x_0.
2. x_0 is a **local minimum** if $f'(x_0)$ changes from negative to positive as x increases around x_0.
3. Otherwise, x_0 is an **inflection point** (not max nor min).

Second order sufficient conditions (SOC): Let $f \in C^2$ on some open convex set around x_0. If $f'(x_0) = 0$ (FONC satisfied) then:

1. x_0 is a **local maximum** if $f''(x_0) < 0$ around x_0.
2. x_0 is a **local minimum** if $f''(x_0) > 0$ around x_0.
3. Otherwise ($f''(x_0) = 0$) we cannot be sure.

Extrema at the boundaries: if the domain of $f(x)$ is bounded, then the boundaries may be extrema without satisfying any of the conditions above.

- Draw graphs for all cases.

Example:

$$y = x^3 - 12x^2 + 36x + 8$$

FONC:

$$f'(x) = 3x^2 - 24x + 36 = 0$$

$$\begin{aligned} x^2 - 8x + 12 &= 0 \\ x^2 - 2x - 6x + 12 &= 0 \\ x(x-2) - 6(x-2) &= 0 \\ (x-6)(x-2) &= 0 \end{aligned}$$

$x_1 = 6,\ x_2 = 2$ are critical points and both satisfy the FONC.

$$f''(x) = 6x - 24$$

$$f''(2) = -12 \ \Rightarrow \ \text{maximum}$$

$$f''(6) = +12 \ \Rightarrow \ \text{minimum}$$

9.2 The N^{th} derivative test

If $f'(x_0) = 0$ and the first non zero derivative at x_0 is of order n, $f^{(n)}(x_0) \neq 0$, then

1. If n is **even** and $f^{(n)}(x_0) < 0$ then x_0 is a local maximum.
2. If n is **even** and $f^{(n)}(x_0) > 0$ then x_0 is a local minimum.
3. Otherwise n is odd and x_0 is an inflection point.

Example:

$$f(x) = (7-x)^4 \ .$$

$$f'(x) = -4(7-x)^3 \ ,$$

so $x = 7$ is a critical point (satisfies the FONC).

$$f''(x) = -12(7-x)^2 \ , \quad f''(7) = 0$$

$$f'''(x) = -24(7-x) \ , \quad f'''(7) = 0$$

$$f''''(x) = 24 > 0 \ ,$$

so $x = 7$ is a minimum: $f^{(4)}$ is the first non zero derivative. 4 is even. $f^{(4)} > 0$.

The N^{th} derivative test is based on **Maclaurin expansion** and **Taylor expansion**.

9.3 Maclaurin expansion

Terms of art:

- Expansion: express a function as a polynomial.
- Around x_0: in a small neighborhood of x_0.

Consider the following polynomial

$$\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n \\
f^{(1)}(x) &= a_1 + 2a_2 x + 3a_3 x^2 + ... + n a_n x^{n-1} \\
f^{(2)}(x) &= 2a_2 + 2 \cdot 3a_3 x + ... + (n-1) n a_n x^{n-2} \\
&\vdots \\
f^{(n)}(x) &= 1 \cdot 2 \cdot ... (n-1) n a_n \ .
\end{aligned}$$

Evaluate at $x = 0$:

$$\begin{aligned}
f(0) &= a_0 = 0! a_0 \\
f^{(1)}(0) &= a_1 = 1! a_1 \\
f^{(2)}(0) &= 2a_2 = 2! a_2 \\
&\vdots \\
f^{(n)}(0) &= 1 \cdot 2 \cdot ... (n-1) n a_n = n! a_n \ .
\end{aligned}$$

Therefore

$$a_n = \frac{f^{(n)}}{n!} \ .$$

Using the last results gives the **Maclaurin expansion** around 0:

$$f(x)|_{x=0} = \frac{f(0)}{0!} + \frac{f^{(1)}(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x + ... \frac{f^{(n)}(0)}{n!} x^n \ .$$

9.4 Taylor expansion

Example: quadratic equation.

$$f(x) = a_0 + a_1 x + a_2 x^2 \ .$$

Define $x = x_0 + \delta$, where we fix x_0 as an anchor and allow δ to vary. This is essentially relocating the origin to $(x_0, f(x_0))$. Define

$$g(\delta) \equiv a_0 + a_1 (x_0 + \delta) + a_2 (x_0 + \delta)^2 = f(x) \ .$$

Note that

$$g(\delta) = f(x) \quad \text{and} \quad g(0) = f(x_0) \ .$$

Taking derivatives

$$\begin{aligned} g'(\delta) &= a_1 + 2a_2(x_0+\delta) = a_1 + 2a_2x_0 + 2a_2\delta \\ g''(\delta) &= 2a_2 \ . \end{aligned}$$

Use Maclaurin's expansion for $g(\delta)$ around $\delta = 0$:

$$g(\delta)|_{\delta=0} = \frac{g(0)}{0!} + \frac{g^{(1)}(0)}{1!}\delta + \frac{g^{(2)}(0)}{2!}\delta^2 \ .$$

Using $\delta = x - x_0$ and the fact that $x = x_0$ when $\delta = 0$, we get a Maclaurin expansion for $f(x)$ around $x = x_0$:

$$f(x)|_{x=x_0} = \frac{f(x_0)}{0!} + \frac{f^{(1)}(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 \ .$$

More generally, we have the **Taylor expansion** for an arbitrary C^n function:

$$\begin{aligned} f(x)|_{x=x_0} &= \frac{f(x_0)}{0!} + \frac{f^{(1)}(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n \\ &= P_n + R_n \ , \end{aligned}$$

where R_n is a **remainder (Theorem):**

- As we choose higher n, then R_n will be smaller and in the limit vanish.
- As x is farther away from x_0 R_n *may* grow.

The **Lagrange form** of R_n: for some point $p \in [x_0, x]$ (if $x > x_0$) or $p \in [x, x_0]$ (if $x < x_0$) we have

$$R_n = \frac{1}{(n+1)!} f^{(n+1)}(p)(x-x_0)^{n+1} \ .$$

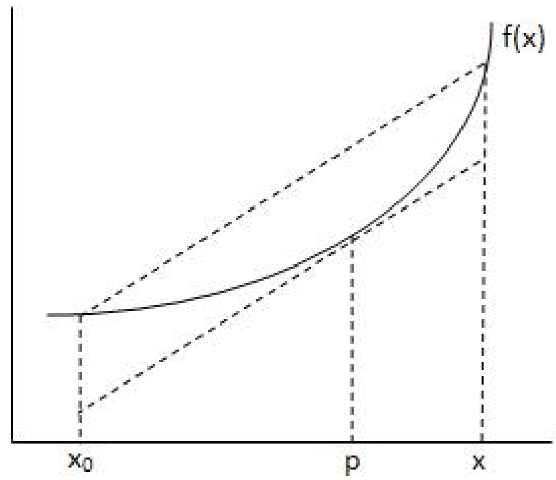
Example: for $n = 0$ we have

$$f(x)|_{x=x_0} = \frac{f(x_0)}{0!} + R_n = f(x_0) + R_n = f(x_0) + f'(p)(x-x_0) \ .$$

Rearranging this we get

$$f(x) - f(x_0) = f'(p)(x-x_0)$$

for some point $p \in [x_0, x]$ (if $x > x_0$) or $p \in [x, x_0]$ (if $x < x_0$). This is the Mean Value Theorem:



9.5 Taylor expansion and the N-th derivative test

Define: x_0 is a maximum (minimum) of $f(x)$ if the change in the function, $\Delta f \equiv f(x) - f(x_0)$, is negative (positive) in a neighborhood of x_0, *both on the right and on the left of* x_0.

The Taylor expansion helps determining this.

$$\Delta f = f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \underbrace{\frac{1}{(n+1)!} f^{(n+1)}(p)(x - x_0)^{n+1}}_{\text{remainder}} .$$

1. Consider the case that $f'(x_0) \neq 0$, i.e. the first non zero derivative at x_0 is of order 1. Choose $n = 0$, so that the remainder will be of the same order of the first non zero derivative and evaluate
$$\Delta f = f'(p)(x - x_0) .$$
Using the fact that p is very close to x_0, so close that $f'(p) \neq 0$, we have that Δf changes signs around x_0, because $(x - x_0)$ changes sign around x_0.

2. Consider the case of $f'(x_0) = 0$ and $f''(x_0) \neq 0$. Choose $n = 1$, so that the remainder will be of the same order of the first non zero derivative (2) and evaluate
$$\Delta f = f'(x_0)(x - x_0) + \frac{f''(p)}{2}(x - x_0)^2 = \frac{1}{2} f''(p)(x - x_0)^2 .$$
Since $(x - x_0)^2 > 0$ always and $f''(p) \neq 0$ we get Δf is either positive (minimum) or negative (maximum) around x_0.

3. Consider the case of $f'(x_0) = 0$, $f''(x_0) = 0$ and $f'''(x_0) \neq 0$. Choose $n = 2$, so that the

remainder will be of the same order of the first non zero derivative (3) and evaluate

$$\Delta f = f'(x_0)(x - x_0) + \frac{f''(p)}{2}(x - x_0)^2 + \frac{f'''(p)}{6}(x - x_0)^3 = \frac{1}{6} f'''(p)(x - x_0)^3 \ .$$

Since $(x - x_0)^3$ changes signs around x_0 and $f'''(p) \neq 0$ we get Δf is changing signs and therefore not an extremum.

4. In the general case $f'(x_0) = 0$, $f''(x_0) = 0$, ... $f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. Choose $n - 1$, so that the remainder will be of the same order of the first non zero derivative (n) and evaluate

$$\begin{aligned}\Delta f &= f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2}(x - x_0)^2 + ... + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + \frac{1}{n!} f^{(n)}(p)(x - x_0)^n \\ &= \frac{1}{n!} f^{(n)}(p)(x - x_0)^n \ .\end{aligned}$$

In all cases $f^{(n)}(p) \neq 0$.

If n is odd, then $(x - x_0)^n$ changes signs around x_0 and Δf changes signs and therefore not an extremum.

If n is even, then $(x - x_0)^n > 0$ always and Δf is either positive (minimum) or negative (maximum).

- **Warning:** in all the above we need $f \in C^n$ at x_0. For example,

$$f(x) = \begin{cases} e^{-\frac{1}{2}x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not C^1 at 0, and yet $x = 0$ is the minimum.

10 Exponents and logs

These are used a lot in economics due to their useful properties, some of which have economic interpretations, in particular in dynamic problems that involve time.

10.1 Exponent function

$$y = f(t) = b^t \ , \quad b > 1 \ .$$

(the case of $0 < b < 1$ can be dealt with similarly.)

- $f(t) \in C^\infty$.
- $f(t) > 0 \ \forall t \in \mathbb{R}$ (since $b > 1 > 0$).
- $f'(t) > 0$, $f''(t) > 0$, therefore strictly increasing and so $\exists t = f^{-1}(y) = \log_b y$, where $y \in \mathbb{R}_{++}$.
- Any $y > 0$ can be expressed as an exponent of many bases. Make sure you know how to convert bases:
$$\log_b y = \frac{\log_a y}{\log_a b} \ .$$

10.2 The constant e

The expression

$$y = Ae^{rt}$$

describes constantly growing processes.

$$\begin{aligned} \frac{d}{dt} e^t &= e^t \\ \frac{d}{dt}\left(Ae^{rt}\right) &= rAe^{rt} \ . \end{aligned}$$

It turns out that

$$\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m = \lim_{n\to 0}(1+n)^{1/n} = e = 2.71828...$$

Think of $1/m = n$ as time. To see this, use a Taylor expansion of e^x and evaluate it around zero:

$$\begin{aligned} e^x &= e^0 + \frac{1}{1!}\left.(e^x)'\right|_{x=0}(x-0) + \frac{1}{2!}\left.(e^x)''\right|_{x=0}(x-0)^2 + \frac{1}{3!}\left.(e^x)'''\right|_{x=0}(x-0)^3 + ... \\ &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ... \end{aligned}$$

Evaluate this at $x = 1$:

$$e^1 = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... = 2.71828...$$

10.3 Examples

10.3.1 Interest compounding

Suppose that you are offered an interest rate r on your savings after a year. Then the return after one year is $1 + r$. If you invested A, then at the end of the year you have

$$A\,(1 + r) \ .$$

Now suppose that an interest of $\left(\frac{r}{m}\right)$ is offered for each $1/m$ of a year. In that case you get a $\left(\frac{r}{m}\right)$ return compounded m times throughout the year. In that case an investment of A will be worth at the end of the year

$$A\left(1 + \frac{r}{m}\right)^m = A\left[\left(1 + \frac{r}{m}\right)^{m/r}\right]^r \ .$$

Now suppose that you get a **instant rate of interest** r for each instant (a period of length $1/m$, where $m \to \infty$), compounded $m \to \infty$ times throughout the year. In that case an investment of A will be worth at the end of the year

$$\lim_{m\to\infty} A\left(1 + \frac{r}{m}\right)^m = \lim_{m\to\infty} A\left[\left(1 + \frac{r}{m}\right)^{m/r}\right]^r = A\left[\lim_{m\to\infty}\left(1 + \frac{r}{m}\right)^{m/r}\right]^r = A\left[\lim_{u=r/m\to 0}(1+u)^{1/u}\right]^r = Ae^r \ .$$

Thus, r is the instantaneous rate of return.

Suppose that we are interested in an arbitrary period of time, t, where, say $t = 1$ is a year (but this is arbitrary). Then the same kind of math will lead us to find the value of an investment A after t time to be

$$A\left(1 + \frac{r}{m}\right)^{mt} = A\left[\left(1 + \frac{r}{m}\right)^{m/r}\right]^{rt} \ .$$

If m is finite, then that is it. if we get continuous compounding ($m \to \infty$), then the value of the investment A after t time will be

$$Ae^{rt} \ .$$

10.3.2 Growth rates

The interest rate example tells you how much the investment is worth when it grows at a constant, instantaneous rate:

$$\text{growth rate} = \frac{dV/dt}{V} = \frac{rAe^{rt}}{Ae^{rt}} = r \text{ per instant } (dt).$$

Any discrete growth rate can be described by a continuous growth rate:

$$A(1+i)^t = Ae^{rt} ,$$

where

$$(1+i) = e^r .$$

10.3.3 Discounting

The value today of X t periods in the future is

$$\text{PV} = \frac{X}{(1+i)^t} ,$$

where $1/(1+i)^t$ is the discount factor. This can also be represented by continuous discounting

$$\text{PV} = \frac{X}{(1+i)^t} = Xe^{-rt} ,$$

where the same discount factor is $1/(1+i)^t = (1+i)^{-t} = e^{-rt}$.

10.4 Logarithms

Log is the inverse function of the exponent. For $b > 1$, $t \in \mathbb{R}$, $y \in \mathbb{R}++$

$$y = b^t \quad \Leftrightarrow \quad t = \log_b y .$$

This is very useful, e.g. for regressions analysis.

E.g.,

$$\begin{aligned} 2^4 &= 16 \quad \Leftrightarrow \quad 4 = \log_2 16 . \\ 5^3 &= 125 \quad \Leftrightarrow \quad 3 = \log_5 125 . \end{aligned}$$

Also, note that

$$y = b^{\log_b y} .$$

Convention:

$$\log_e x = \ln x .$$

Rules:

- $\ln(uv) = \ln u + \ln v$
- $\ln(u/v) = \ln u - \ln v$

- $\ln\left(au^b\right) = \ln a + b\ln u$
- $\log_b x = \frac{\log_a x}{\log_a b}$, where $a, b, x > 0$
  - Corollary: $\log_b e = \frac{\ln e}{\ln b} = \frac{1}{\ln b}$

Some useful properties of logs:

1. Log differences approximate growth rates:

$$\ln X_2 - \ln X_1 = \ln \frac{X_2}{X_1} = \ln\left(\frac{X_2}{X_1} - 1 + 1\right) = \ln\left(1 + \frac{X_2 - X_1}{X_1}\right) = \ln(1+x) \ ,$$

   where x is the growth rate of X. Take a first order Taylor approximation of $\ln(1+x)$ around $\ln(1)$:

$$\ln(1+x) \approx \ln(1) + (\ln(1))'(1+x-1) = x \ .$$

   So we have

$$\ln X_2 - \ln X_1 \approx x \ .$$

   This approximation is good for small percent changes. Beware: large log differences give much larger percent changes (e.g., a log difference of 1=100% is 2.7=270%).

2. Logs "bend down" their image relative to the argument below the 45 degree line. Exponents do the opposite.

3. The derivative of log is always positive, but ever diminishing: $(\log x)' > 0$, $(\log x)'' < 0$.

4. Nevertheless, $\lim\limits_{x\to\infty} \log_b x = \infty$. Also, $\lim\limits_{x\to 0} \log_b x = -\infty$. Therefore the range is $\mathbb{R}$.

5. Suppose that $y = Ae^{rt}$. Then $\ln y = \ln A + rt$. Therefore

$$t = \frac{\ln y - \ln A}{r} \ .$$

   This answers the question: how long will it take to grow from A to y, if growth is at an instantaneous rate of r.

6. Converting $y = Ab^{ct}$ into $y = Ae^{rt}$: $b^c = e^r$, therefore $c\ln b = r$, therefore $y = Ae^{rt} = y = Ae^{(c\ln b)t}$.

10.5 Derivatives of exponents and logs

$$\begin{aligned} \frac{d}{dt}\ln t &= \frac{1}{t} \\ \frac{d}{dt}\log_b t &= \frac{d}{dt}\frac{\ln t}{\ln b} = \frac{1}{t\ln b} \end{aligned}$$

$$\frac{d}{dt}e^t = e^t$$

Let $y = e^t$, so that $t = \ln y$:

$$\frac{d}{dt}e^t = \frac{d}{dt}y = \frac{1}{dt/dy} = \frac{1}{1/y} = y = e^t \ .$$

By chain rule:

$$\begin{aligned} \frac{d}{dt}e^u &= e^u \frac{du}{dt} \\ \frac{d}{dt}\ln u &= \frac{du/dt}{u} \end{aligned}$$

Higher derivatives:

$$\frac{d^n}{(dt)^n}e^t = e^t$$

$$\frac{d}{dt}\ln t = \frac{1}{t} \ , \ \frac{d^2}{(dt)^2}\ln t = -\frac{1}{t^2} \ , \ \frac{d^3}{(dt)^3}\ln t = \frac{2}{t^3} ...$$

$$\frac{d}{dt}b^t = b^t \ln b \ , \ \frac{d^2}{(dt)^2}b^t = b^t (\ln b)^2 \ , \ \frac{d^3}{(dt)^3}b^t = b^t (\ln b)^3 ...$$

10.6 Application: optimal timing

The value of k bottles of wine is given by

$$V(t) = ke^{\sqrt{t}} \ .$$

Discounting: $D(t) = e^{-rt}$. The present value of $V(t)$ today is

$$PV = D(t) V(t) = e^{-rt} ke^{\sqrt{t}} = ke^{\sqrt{t}-rt} \ .$$

Choosing t to maximize $PV = ke^{\sqrt{t}-rt}$ is equivalent to choosing t to maximize $\ln PV = \ln k + \sqrt{t} - rt$.
FONC:

$$\begin{aligned} 0.5t^{-0.5} - r &= 0 \\ 0.5t^{-0.5} &= r \end{aligned}$$

Marginal benefit to wait one more instant = marginal cost of waiting one more instant. $t^* = 1/\left(4r^2\right)$.
SOC:

$$-0.25t^{-1.5} < 0$$

so t^* is a maximum.

10.7 Growth rates again

Denote

$$\frac{d}{dt}x = \dot{x} \ .$$

So the growth rate at some point in time is

$$\frac{dx/dt}{x} = \frac{\dot{x}}{x} \ .$$

So in the case $x = Ae^{rt}$, we have

$$\frac{\dot{V}}{V} = r \ .$$

And since $x(0) = Ae^{r0} = A$, we can write without loss of generality $x(t) = x_0 e^{rt}$.

Growth rates of combinations:

1. For $y(t) = u(t)\, v(t)$ we have

$$\begin{aligned} \frac{\dot{y}}{y} &= \frac{\dot{u}}{u} + \frac{\dot{v}}{v} \\ g_y &= g_u + g_v \end{aligned}$$

   Proof:

$$\begin{aligned} \ln y(t) &= \ln u(t) + \ln v(t) \\ \frac{d}{dt}\ln y(t) &= \frac{d}{dt}\ln u(t) + \frac{d}{dt}\ln v(t) \\ \frac{1}{y(t)}\frac{dy}{dt} &= \frac{1}{u(t)}\frac{du}{dt} + \frac{1}{v(t)}\frac{dv}{dt} \end{aligned}$$

2. For $y(t) = u(t)/v(t)$ we have

$$\begin{aligned} \frac{\dot{y}}{y} &= \frac{\dot{u}}{u} - \frac{\dot{v}}{v} \\ g_y &= g_u - g_v \end{aligned}$$

   Proof: similar to above.

3. For $y(t) = u(t) \pm v(t)$ we have

$$g_y = \frac{u}{u \pm v} g_u \pm \frac{u}{u \pm v} g_v$$

10.8 Elasticities

An elasticity of y with respect to x is defined as

$$\sigma_{y,x} = \frac{dy/y}{dx/x} = \frac{dy}{dx}\frac{x}{y} \ .$$

Since

$$d\ln x = \frac{\partial \ln x}{\partial x}dx = \frac{dx}{x}$$

we get

$$\sigma_{y,x} = \frac{d\ln y}{d\ln x} \ .$$

11 Optimization with more than one choice variable

11.1 The differential version of optimization with one variable

This helps developing concepts for what follows. Let $z = f(x) \in C^1$, $x \in \mathbb{R}$. Then

$$dz = f'(x)\, dx \ .$$

- FONC: an extremum may occur when $dz = 0$, i.e. when $f'(x) = 0$. Think of this condition as a situation when small **arbitrary** perturbations of x do not affect the value of the function; therefore $dx \neq 0$ in general. No perturbation of the argument ($dx = 0$) will trivially not induce perturbation of the image.

- SOC:
$$d^2 z = d\,[dz] = d\left[f'(x)\, dx\right] = f''(x)\, dx^2 \ .$$
A maximum occurs when $f''(x) < 0$ or equivalently when $d^2 z < 0$.
A minimum occurs when $f''(x) > 0$ or equivalently when $d^2 z > 0$.

11.2 Extrema of a function of two variables

Let $z = f(x, y) \in C^1$, $x, y \in \mathbb{R}$. Then

$$dz = f_x dx + f_y dy \ .$$

FONC: $dz = 0$ for **arbitrary** values of dx and dy, not both equal to zero. A necessary condition that gives this is

$$f_x = 0 \text{ and } f_y = 0 \ .$$

As before, this is not a sufficient condition for an extremum, not only because of inflection points, but also due to **saddle points**.

- Note: in matrix notation
$$dz = \left[\frac{\partial f}{\partial (x, y)}\right] \begin{bmatrix} dx \\ dy \end{bmatrix} = \nabla f dx = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = f_x dx + f_y dy \ .$$
If $x \in \mathbb{R}^n$ then
$$dz = \left[\frac{\partial f}{\partial x'}\right] dx = \nabla f dx = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = \sum_{i=1}^{n} f_i dx_i \ .$$

Define

$$
\begin{aligned}
f_{xx} &= \frac{\partial^2 f}{\partial x^2} \\
f_{yy} &= \frac{\partial^2 f}{\partial y^2} \\
f_{xy} &= \frac{\partial^2 f}{\partial x \partial y} \\
f_{yx} &= \frac{\partial^2 f}{\partial y \partial x}
\end{aligned}
$$

Young's Theorem: If both f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.

Now we apply this

$$
\begin{aligned}
d^2 z &= d[dz] = d[f_x dx + f_y dy] = d[f_x dx] + d[f_y dy] \\
&= f_{xx} dx^2 + f_{yx} dx dy + f_{xy} dy dx + f_{yy} dy^2 \\
&= f_{xx} dx^2 + 2 f_{xy} dx dy + f_{yy} dy^2 \ .
\end{aligned}
$$

(The $d[dx]$ and $d[dy]$ terms drop out. The reason is that we are considering arbitrarily small arbitrary dx and dy, so the second order differential is nil.) In matrix notation

$$
d^2 z = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \ .
$$

And more generally, if $x \in \mathbb{R}^n$ then

$$
d^2 z = dx' \underbrace{\left[\frac{\partial^2 f}{\partial x \partial x'} \right]}_{\text{Hessian}} dx \ .
$$

SONC (second order **necessary** conditions): for arbitrary values of dx and dy

- $d^2 z \leq 0$ gives a maximum.
- $d^2 z \geq 0$ gives a minimum.

SOSC (second order **sufficient** conditions): for arbitrary values of dx and dy

- $d^2 z < 0$ gives a maximum. In the two variable case

$$
d^2 z < 0 \text{ iff } f_{xx} < 0,\ f_{yy} < 0 \text{ and } f_{xx} f_{yy} > f_{xy}^2 \ .
$$

- $d^2 z > 0$ gives a minimum. In the two variable case

$$
d^2 z > 0 \text{ iff } f_{xx} > 0,\ f_{yy} > 0 \text{ and } f_{xx} f_{yy} > f_{xy}^2 \ .
$$

Comments:

- SONC is necessary but not sufficient, while SOSC are not necessary.
- If $f_{xx}f_{yy} = f_{xy}^2$ a point can be an extremum nonetheless.
- If $f_{xx}f_{yy} < f_{xy}^2$ then this is a saddle point.
- If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then $f_{xx}f_{yy} > f_{xy}^2 \geq 0$ implies $\text{sign}(f_{xx}) = \text{sign}(f_{yy})$.

11.3 Quadratic form and sign definiteness

This is a tool to help analyze SOCs. Relabel terms for convenience:

$$\begin{aligned} z &= f(x_1, x_2) \\ d^2 z &= q \ , \quad dx_1 = d_1 \ , \quad dx_2 = d_2 \\ f_{11} &= a \ , \quad f_{22} = b \ , \quad f_{12} = h \end{aligned}$$

Then

$$\begin{aligned} d^2 z &= f_{11} dx_1^2 + 2 f_{12} dx_1 dx_2 + f_{22} dx_2^2 \\ q &= a d_1^2 + 2 h d_1 d_2 + b d_2^2 \\ &= \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} . \end{aligned}$$

This is the quadratic form.

- Note: d_1 and d_2 are variables, not constants, as in the FONC. We require the SOCs to hold $\forall d_1, d_2$, and in particular $\forall d_1, d_2 \neq 0$.

Denote the **Hessian** by

$$H = \left[\frac{\partial^2 f}{\partial x \partial x'} \right]$$

The quadratic form is

$$q = d' H d$$

Define

$$q \text{ is } \left\{ \begin{array}{c} \text{positive definite} \\ \text{positive semidefinite} \\ \text{negative semidefinite} \\ \text{negative definite} \end{array} \right\} \text{ if } q \text{ is invariably } \left\{ \begin{array}{c} > 0 \\ \geq 0 \\ \leq 0 \\ < 0 \end{array} \right\} ,$$

regardless of values of d. Otherwise, q is indefinite.

Consider the determinant of H, $|H|$, which we call here the **discriminant** of H:

$$q \text{ is } \left\{ \begin{array}{c} \text{positive definite} \\ \text{negative definite} \end{array} \right\} \text{ iff } \left\{ \begin{array}{c} |a| > 0 \\ |a| < 0 \end{array} \right\} \text{ and } |H| > 0 \ .$$

$|a|$ is (the determinant of) the first ordered minor of H. In the simple two variable case, $|H|$ is (the determinant of) the second ordered minor of H. In that case

$$|H| = ab - h^2 \ .$$

If $|H| > 0$, then a and b must have the same sign, since $ab > h^2 > 0$.

11.4 Quadratic form for n variables and sign definiteness

$$q = d'Hd = \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} d_i d_j \ .$$

- q is **positive definite** iff all (determinants of) the principal minors are positive

$$|H_1| = |h_{11}| > 0, \ \ |H_2| = \left| \begin{array}{cc} h_{11} & h_{12} \\ h_{21} & h_{22} \end{array} \right| > 0, ... |H_n| = |H| > 0 \ .$$

- q is **negative definite** iff (determinants of) the odd principal minors are negative and the even ones are positive:

$$|H_1| < 0, \ \ |H_2| > 0, \ \ |H_3| < 0, ...$$

11.5 Characteristic roots test for sign definiteness

Consider some $n \times n$ matrix $H_{n \times n}$. We look for a **characteristic root** r (scalar) and **characteristic vector** $x_{n \times 1}$ $(n \times 1)$ such that

$$Hx = rx \ .$$

Developing this expression:

$$Hx = rIx \quad \Rightarrow \quad (H - rI)\, x = 0 \ .$$

Define $(H - rI)$ as the **characteristic matrix**:

$$(H - rI) = \left[\begin{array}{cccc} h_{11} - r & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} - r & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \ldots & h_{nn} - r \end{array} \right]$$

If $(H - rI)\, x = 0$ has a non trivial solution $x \neq 0$, then $(H - rI)$ must be singular, so that $|H - rI| = 0$. This is an equation that we can solve for r. The equation $|H - rI| = 0$ is the

characteristic equation, and is an n degree polynomial in r, with n non trivial solutions (some of the solutions can be equal). Some properties:

- If H is symmetric, then we will have $r \in \mathbb{R}$. This is useful, because many applications in economics will deal with symmetric matrices, like Hessians and variance-covariance matrices.
- For each characteristic root that solves $|H - rI| = 0$ there are many characteristic vectors x such that $Hx = rx$. Therefore we normalize: $x'x = 1$. Denote the normalized characteristic vectors as v. Denote the characteristic vectors (**eigenvector**) of the characteristic root (**eigenvalue**) as v_i and r_i.
- The set of eigenvectors is **orthonormal**, i.e. orthogonal and normalized: $v_i'v_j = 0 \ \forall i \neq j$ and $v_i'v_i = 1$.

11.5.1 Application to quadratic form

Let $V = (v_1, v_2, ...v_n)$ be the set of eigenvectors of the matrix H. Define the vector y that solves $d = Vy$. We use this in the quadratic form

$$q = d'Hd = y'V'HVy = y'Ry \ ,$$

where $V'HV = R$. It turns out that

$$R = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & r_n \end{bmatrix}$$

Here is why:

$$\begin{aligned} V'HV &= V' \begin{bmatrix} Hv_1 & Hv_2 & \cdots & Hv_n \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} \begin{bmatrix} r_1v_1 & r_2v_2 & \cdots & r_nv_n \end{bmatrix} \\ &= \begin{bmatrix} r_1v_1'v_1 & r_1v_1'v_2 & \cdots & r_1v_1'v_n \\ r_2v_2'v_1 & r_2v_2'v_2 & \cdots & r_2v_2'v_n \\ \vdots & \vdots & \ddots & \vdots \\ r_nv_n'v_1 & r_nv_n'v_2 & \cdots & r_nv_n'v_n \end{bmatrix} = R \ , \end{aligned}$$

where the last equality follows from $v_i'v_j = 0 \ \forall i \neq j$ and $v_i'v_i = 1$. It follows that $\text{sign}(q)$ depends only on the characteristic roots: $q = y'Ry = \sum_{i=1}^n r_i y_i^2$.

11.5.2 Characteristic roots test for sign definiteness

$$q \text{ is } \left\{ \begin{array}{c} \text{positive definite} \\ \text{positive semidefinite} \\ \text{negative semidefinite} \\ \text{negative definite} \end{array} \right\} \text{ iff all } r_i \left\{ \begin{array}{c} > 0 \\ \geq 0 \\ \leq 0 \\ < 0 \end{array} \right\},$$

regardless of values of d. Otherwise, q is indefinite.

- When n is large, finding the roots can be hard, because it involves finding the roots of a polynomial of degree n. But the computer can do it for us.

11.6 Global extrema, convexity and concavity

We seek conditions for a **global** maximum or minimum. If a function has a "hill shape" over its entire domain, then we do not need to worry about boundary conditions and the local extremum will be a global extremum. Although the global maximum can be found at the boundary of the domain, this will not be detected by the FONC.

- If f is **strictly concave**: the global maximum is unique.
- If f is **concave**, but not strictly: this allows for flat regions, so the global maximum may not be unique (the argument may take many values, which all have the same maximal image).

Let $z = f(x) \in C^2$, $x \in \mathbb{R}^n$.

$$\text{If } d^2z \text{ is } \left\{ \begin{array}{c} \text{positive definite} \\ \text{positive semidefinite} \\ \text{negative semidefinite} \\ \text{negative definite} \end{array} \right\} \forall x \text{ in the domain, then } f \text{ is } \left\{ \begin{array}{c} \text{strictly convex} \\ \text{convex} \\ \text{concave} \\ \text{strictly concave} \end{array} \right\},$$

When an objective function is general, then we must *assume* convexity or concavity. If a specific functional form is used, we can check whether it is convex or concave.

11.7 Convexity and concavity defined

Definition 1: A function f is **concave** iff $\forall f(x), f(y) \in$ graph of f the line between $f(x)$ and $f(y)$ lies on or below the graph.

- If $\forall x \neq y$ the line lies strictly below the graph, then f is **strictly concave**.
- For convexity replace "below" with "above".

Definition 2: A function f is **concave** iff $\forall x, y \in$ domain of f, which is assumed to be a **convex set** (see below), and $\forall \theta \in (0,1)$ we have

$$\theta f(x) + (1-\theta) f(y) \leq f[\theta x + (1-\theta) y] \ .$$

- For strict concavity replace "$\leq$" with "$<$" and add $\forall x \neq y$.
- For convexity replace "$\leq$" with "$\geq$" and "$<$" with "$>$".

The term $\theta x + (1-\theta) y$, $\theta \in (0,1)$ is called a **convex combination**.

Properties:

1. If f is linear, then f is both concave and convex, but not strictly.
2. If f is (strictly) concave, then $-f$ is (strictly) convex.
   - Proof: f is concave. Therefore $\forall x, y \in$ domain of f and $\forall \theta \in (0,1)$ we have

$$\begin{aligned} \theta f(x) + (1-\theta) f(y) &\leq f[\theta x + (1-\theta) y] \ \ / \ \times(-1) \\ \theta[-f(x)] + (1-\theta)[-f(y)] &\geq -f[\theta x + (1-\theta) y] \ \ \blacksquare \end{aligned}$$

3. If f and g are concave functions, then $f + g$ is also concave. If one of the concave functions is strictly concave, then $f + g$ is strictly concave.
   - Proof: f and g are concave, therefore

$$\begin{aligned} \theta f(x) + (1-\theta) f(y) &\leq f[\theta x + (1-\theta) y] \\ \theta g(x) + (1-\theta) g(y) &\leq g[\theta x + (1-\theta) y] \\ \theta[f(x) + g(x)] + (1-\theta)[f(y) + g(y)] &\leq f[\theta x + (1-\theta) y] + g[\theta x + (1-\theta) y] \\ \theta[(f+g)(x)] + (1-\theta)[(f+g)(y)] &\leq (f+g)[\theta x + (1-\theta) y] \ \ \blacksquare \end{aligned}$$

   The proof for strict concavity is identical.

11.7.1 Example

Is $z = x^2 + y^2$ concave or convex? Consider first the LHS of the definition:

$$\text{(i)} : \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2) = \theta\left(x_1^2 + y_1^2\right) + (1-\theta)\left(x_2^2 + y_2^2\right) \ .$$

Now consider the RHS of the definition:

$$\begin{aligned} \text{(ii)} \quad &: \quad f\left[\theta x_1 + (1-\theta)\, x_2, \theta y_1 + (1-\theta)\, y_2\right] = \left[\theta x_1 + (1-\theta)\, x_2\right]^2 + \left[\theta y_1 + (1-\theta)\, y_2\right]^2 \\ &= \quad \theta^2 \left(x_1^2 + y_1^2\right) + (1-\theta)^2 \left(x_2^2 + y_2^2\right) + 2\theta\,(1-\theta)\,(x_1 x_2 + y_1 y_2) \ . \end{aligned}$$

Now subtract (i)$-$(ii):

$$\theta\,(1-\theta)\left(x_1^2 + y_1^2 + x_2^2 + y_2^2\right) - 2\theta\,(1-\theta)\,(x_1 x_2 + y_1 y_2) = \theta\,(1-\theta)\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right] \geq 0 \ .$$

So this is a convex function. Moreover, it is **strictly convex**, since $\forall x_1 \neq x_2$ and $\forall y_1 \neq y_2$ we have (i)$-$(ii)> 0.

Using similar steps, you can verify that $-\left(x^2 + y^2\right)$ is strictly concave.

11.7.2 Example

Is $f\,(x, y) = (x+y)^2$ concave or convex? Use the same procedure from above.

$$\text{(i)} : \theta f\,(x_1, y_1) + (1-\theta)\, f\,(x_2, y_2) = \theta\,(x_1 + y_1)^2 + (1-\theta)\,(x_2 + y_2)^2 \ .$$

Now consider

$$\begin{aligned} \text{(ii)} \quad &: \quad f\left[\theta x_1 + (1-\theta)\, x_2, \theta y_1 + (1-\theta)\, y_2\right] = \left[\theta x_1 + (1-\theta)\, x_2 + \theta y_1 + (1-\theta)\, y_2\right]^2 \\ &= \quad \left[\theta\,(x_1 + y_1) + (1-\theta)\,(x_2 + y_2)\right]^2 \\ &= \quad \theta^2\,(x_1 + y_1)^2 + 2\theta\,(1-\theta)\,(x_1 + y_1)\,(x_2 + y_2) + (1-\theta)^2\,(x_2 + y_2)^2 \ . \end{aligned}$$

Now subtract (i)$-$(ii):

$$\begin{aligned} &\theta\,(1-\theta)\left[(x_1 + y_1)^2 + (x_2 + y_2)^2\right] - 2\theta\,(1-\theta)\,(x_1 + y_1)\,(x_2 + y_2) \\ = \quad &\theta\,(1-\theta)\left[(x_1 + y_1) - (x_2 + y_2)\right]^2 \geq 0 \ . \end{aligned}$$

So convex but not strictly. Why not strict? Because when $x + y = 0$, i.e. when $y = -x$, we get $f\,(x, y) = 0$. The shape of this function is a hammock, with the bottom at $y = -x$.

11.8 Differentiable functions, convexity and concavity

Let $f\,(x) \in C^1$ and $x \in \mathbb{R}$. Then f is concave iff $\forall x^1, x^2 \in$ domain of f

$$f\left(x^2\right) - f\left(x^1\right) \leq f'\left(x^1\right)\left(x^2 - x^1\right) \ .$$

For convex replace "$\geq$" with "$\leq$".

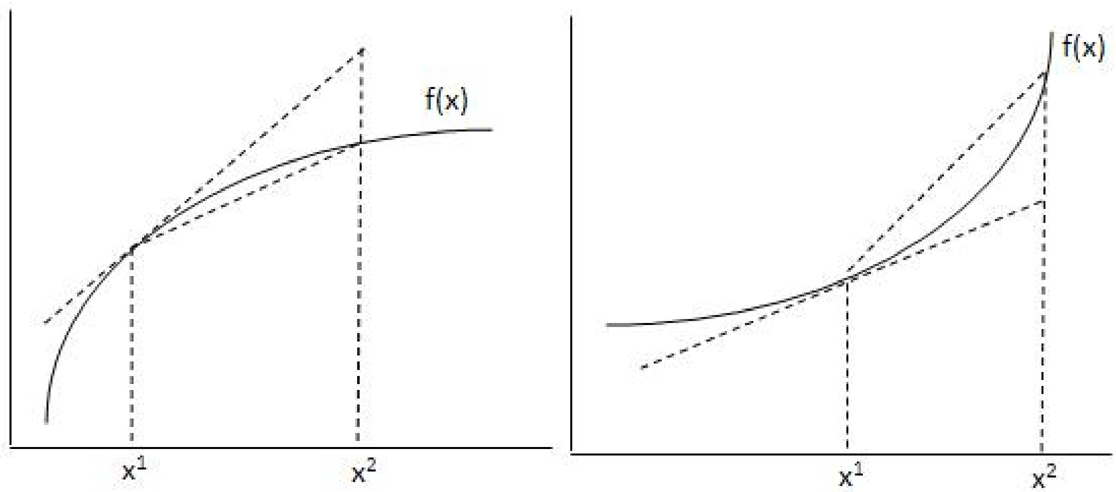
When $x^2 > x^1$ and both $x^2, x^1 \in \mathbb{R}$ we can divide through by $\left(x^2 - x^1\right)$ without changing the direction of the inequality to get

$$f'\left(x^1\right) \geq \frac{f\left(x^2\right)-f\left(x^1\right)}{x^2-x^1} \quad , \quad x^2>x^1 \ .$$

I.e. the slope from x^1 to x^2 is smaller than the derivative at x^1. Think of x^1 as the point of reference and x^2 as a target point. When $x^2<x^1$ we can divide through by $\left(x^2-x^1\right)$ but must change the direction of the inequality to get

$$f'\left(x^1\right) \leq \frac{f\left(x^2\right)-f\left(x^1\right)}{x^2-x^1}=\frac{f\left(x^1\right)-f\left(x^2\right)}{x^1-x^2} \quad , \quad x^2<x^1 \ .$$

I.e. the slope is larger than the derivative at x^1.



Derivative Condition for Concave Function   Derivative Condition for Convex Function

If $x \in \mathbb{R}^n$, then $f \in C^1$ is concave iff $\forall x^1, x^2 \in$ domain of f

$$f\left(x^2\right)-f\left(x^1\right) \leq \nabla f\left(x^1\right)\left(x^2-x^1\right)$$

For convex replace "$\leq$" with "$\geq$".

Let $z=f(x) \in C^2$ and $x \in \mathbb{R}^n$. Then f is concave iff $\forall x \in$ domain of f we have d^2z is negative semidefinite. If d^2z is negative definite, then f is strictly concave (but not "only if"). Replace "negative" with "positive" for convexity.

11.9 Global extrema, convexity and concavity again

Suppose a point x_0 satisfies the FONC: you have found a critical point of the function f. Then you examine the SOC: if $q=d^2z$ is negative (positive) definite, then x_0 is at a local maximum (minimum), i.e. x_0 is a local maximizer (minimizer). This implies examining the Hessian at x_0.

But if you know something about the concavity/convexity properties of f, then you know something more. If f is concave (convex), then you know that if x_0 satisfies the FONC, then x_0 is at a global maximum (minimum), i.e. x_0 is a global maximizer (minimizer). And if f is strictly concave (convex), then you know that x_0 is at a unique global maximum (minimum), i.e. x_0 is a unique global maximizer (minimizer).

Determining concavity/convexity (strict or not) of a function f implies examining the Hessian at all points of its domain. As noted above, sign definiteness of d^2z is determined by the sign definiteness of the Hessian. Thus

$$\text{If } H \text{ is } \left\{ \begin{array}{c} \text{positive definite} \\ \text{positive semidefinite} \\ \text{negative semidefinite} \\ \text{negative definite} \end{array} \right\} \forall x \text{ in the domain, then } f \text{ is } \left\{ \begin{array}{c} \text{strictly convex} \\ \text{convex} \\ \text{concave} \\ \text{strictly concave} \end{array} \right\} .$$

11.10 Convex sets in $\mathbb{R}^n$

This is related, but distinct from convex and concave functions.

Define: convex set in $\mathbb{R}^n$. Let the set $S \subset \mathbb{R}^n$. If $\forall x, y \in S$ and $\forall \theta \in [0, 1]$ we have

$$\theta x + (1 - \theta) y \in S$$

then S is a convex set. (This definition holds in other spaces as well.) Essentially, a set is convex if it has no "holes" (no doughnuts) and the boundary is not "dented" (no bananas).

11.10.1 Relation to convex functions 1

The concavity condition $\forall x, y \in$ domain of f and $\forall \theta \in (0, 1)$ we have

$$\theta f(x) + (1 - \theta) f(y) \leq f[\theta x + (1 - \theta) y]$$

assumes that the domain is convex: $\forall x, y \in$ domain of f and $\forall \theta \in (0, 1)$

$$\theta x + (1 - \theta) y \in \text{domain of } f \ ,$$

because $f[\theta x + (1 - \theta) y]$ must be defined.

11.10.2 Relation to convex functions 2

Necessary condition for convex function: if f is a convex function, then $\forall k \in \mathbb{R}$ the set

$$S = \{x : f(x) \leq k\}$$

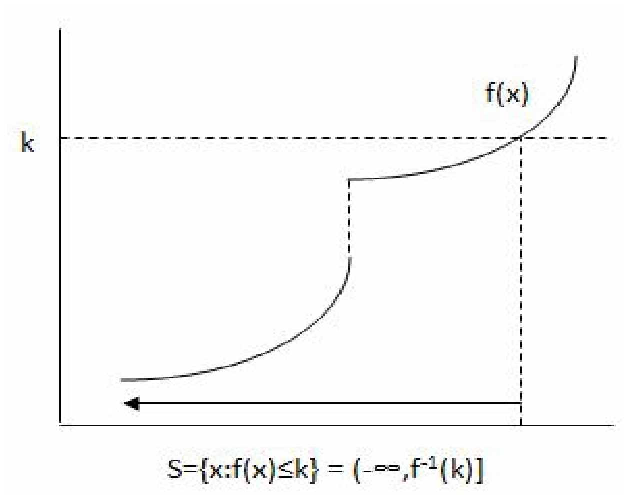
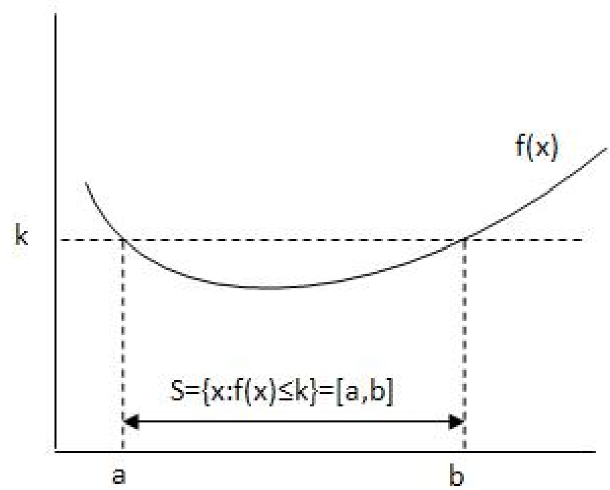


Figure 1: Convex set, but function is not convex

is a convex set.

This is **NOT** a sufficient condition, i.e. the causality runs from convexity of f to convexity of S, but not vice versa. Convexity of S does not necessarily imply convexity of f. But violation of convexity of S implies non-convexity of f.

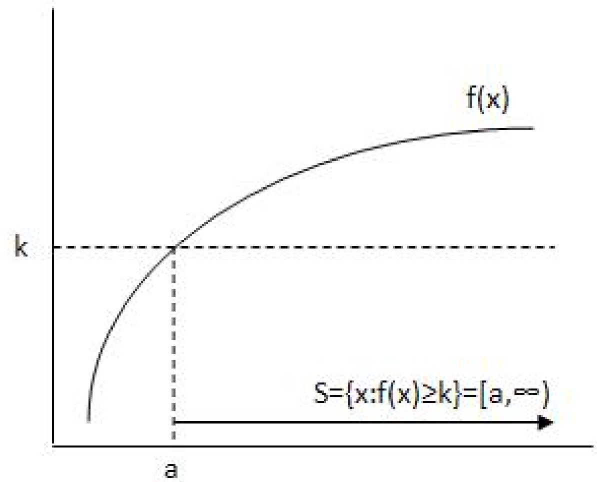


Convex set, convex function

If f is a concave function, then the set

$$S = \{x : f(x) \geq k\} \ , \quad k \in \mathbb{R}$$

is a convex set. This is **NOT** a sufficient condition, i.e. the causality runs from concavity of f to convexity of S, but not vice versa. Convexity of S does not necessarily imply concavity of f.



Convex set, concave function

- This is why there is an intimate relationship between convex preferences and concave utility functions.

11.11 Example: input decisions of a firm

$$\pi = R - C = pq - wl - rk \ .$$

Let p, w, r be given, i.e. the firm is a price taker in a competitive economy. To simplify, let output, q, be the numeraire, so that $p = 1$ and everything is then denominated in units of output:

$$\pi = q - wl - rk \ .$$

Production function with decreasing returns to scale:

$$q = k^{\alpha} l^{\alpha} \ , \quad \alpha < 1/2$$

so that

$$\pi = k^{\alpha} l^{\alpha} - wl - rk \ .$$

Choose $\{k, l\}$ to maximize π. FONC:

$$\begin{aligned} \frac{\partial \pi}{\partial k} &= \alpha k^{\alpha-1} l^{\alpha} - r = 0 \\ \frac{\partial \pi}{\partial k} &= \alpha k^{\alpha} l^{\alpha-1} - w = 0 \ . \end{aligned}$$

SOC: check properties of the Hessian

$$H=\left[\frac{\partial^2\pi}{\partial\begin{pmatrix}k\\ l\end{pmatrix}\partial\begin{pmatrix}k & l\end{pmatrix}}\right]=\begin{bmatrix}\alpha(\alpha-1)k^{\alpha-2}l^{\alpha} & \alpha^2k^{\alpha-1}l^{\alpha-1}\\ \alpha^2k^{\alpha-1}l^{\alpha-1} & \alpha(\alpha-1)k^{\alpha}l^{\alpha-2}\end{bmatrix}.$$

$|H_1|=\alpha(\alpha-1)k^{\alpha-2}l^{\alpha}<0\ \forall k,l>0$. $|H_2|=|H|=\alpha^2(1-2\alpha)k^{2(\alpha-1)}l^{2(\alpha-1)}>0\ \forall k,l$. Therefore π is a strictly concave function and the extremum will be a maximum.

From the FONC:

$$\begin{aligned}\alpha k^{\alpha-1}l^{\alpha} &= \alpha\frac{q}{k}=r\\ \alpha k^{\alpha}l^{\alpha-1} &= \alpha\frac{q}{l}=w\end{aligned}$$

so that $rk=wl=\alpha q$. Thus

$$\begin{aligned}k &= \frac{\alpha q}{r}\\ l &= \frac{\alpha q}{w}.\end{aligned}$$

Using this in the production function:

$$q=k^{\alpha}l^{\alpha}=\left(\frac{\alpha q}{r}\right)^{\alpha}\left(\frac{\alpha q}{w}\right)^{\alpha}=\alpha^{2\alpha}q^{2\alpha}\left(\frac{1}{rw}\right)^{\alpha}=\alpha^{\frac{2\alpha}{1-2\alpha}}\left(\frac{1}{rw}\right)^{\frac{\alpha}{1-2\alpha}},$$

so that

$$\begin{aligned}k &= \alpha^{\frac{1}{1-2\alpha}}\left(\frac{1}{r}\right)^{\frac{1-\alpha}{1-2\alpha}}\left(\frac{1}{w}\right)^{\frac{\alpha}{1-2\alpha}}\\ l &= \alpha^{\frac{1}{1-2\alpha}}\left(\frac{1}{r}\right)^{\frac{\alpha}{1-2\alpha}}\left(\frac{1}{w}\right)^{\frac{1-\alpha}{1-2\alpha}}.\end{aligned}$$

12 Optimization under equality constraints

12.1 Example: the consumer problem

$$\begin{aligned}\text{Objective} &: \text{ Choose } \{x, y\} \text{ to maximize } u(x, y) \\ \text{Constraint(s)} &: \text{ s.t. } (x, y) \in B = \{(x, y) : x, y \geq 0,\ xp_x + yp_y \leq I\}\end{aligned}$$

(draw the budget set, B). Under some conditions, which we will explore soon, we will get the result that the consumer chooses a point on the budget line, $xp_x + yp_y = I$ (nonsatiation and quasi-concavity of u). Additional conditions ensure that that $x, y \geq 0$ is trivially satisfied. So we state a simpler problem:

$$\begin{aligned}\text{Objective} &: \text{ Choose } \{x, y\} \text{ to maximize } u(x, y) \\ \text{Constraint(s)} &: \text{ s.t. } xp_x + yp_y = I \ .\end{aligned}$$

The **optimum** will be denoted (x^*, y^*). The **value of the problem** is $u(x^*, y^*)$. Constraints can only hurt the unconstrained value (although they may not). This will happen when the unconstrained optimum point is not in the constraint set. E.g.,

$$\text{Choose } \{x, y\} \text{ to maximize } x - x^2 + y - y^2$$

has a maximum at $(x^*, y^*) = (1/2, 1/2)$, but this point is not on the line $x + y = 2$, so applying this constraint will move us away from the unconstrained optimum and hurt the objective.

12.2 Lagrange method: one constraint, two variables

Let $f, g \in C^1$. Suppose that (x^*, y^*) is the solution to

$$\text{Choose } \{x, y\} \text{ to maximize } z = f(x, y)\,, \text{ s.t. } g(x, y) = c$$

and that (x^*, y^*) is not a critical point of $g(x, y)$, i.e. **not both** $g_x \neq 0$ and $g_y \neq 0$ at (x^*, y^*). Then there exists a number λ^* such that (x^*, y^*, λ^*) is a critical point of

$$\mathcal{L} = f(x, y) + \lambda [c - g(x, y)] \ ,$$

i.e.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \lambda} &= c - g(x,y) = 0 \\
\frac{\partial \mathcal{L}}{\partial x} &= f_x - \lambda g_x = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= f_y - \lambda g_y = 0 \ .
\end{aligned}$$

From this it follows that at (x^*, y^*, λ^*)

$$\begin{aligned}
g(x^*, y^*) &= c \\
\lambda^* &= f_x / g_x \\
\lambda^* &= f_y / g_y \ .
\end{aligned}$$

- The last equations make it clear why we must check the **constraint qualifications,** that not both $g_x \neq 0$ and $g_y \neq 0$ at (x^*, y^*), i.e. check that (x^*, y^*) is not a critical point of $g(x,y)$. For linear constraints this will be automatically satisfied.

- **Always write** $+\lambda[c - g(x,y)]$.

If the constraint qualification fails then this means that we cannot freely search for an optimum. It implies that the theorem does not apply; it does not imply that there is no optimum. Recall that the gradient $\nabla g(x,y)$ is a vector that tells you in which direction to move in order to increase g as much as possible at some point (x,y). But if both $g_x = 0$ and $g_y = 0$ at (x^*, y^*), then this means that we are not free to search in any direction.

Recall that for unconstrained maximum, we must have

$$dz = f_x dx + f_y dy = 0 \ ,$$

and thus

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

In the constrained problem this still holds—as we will see below—except that now dx and dy are **not arbitrary**: they must satisfy the constraint, i.e.

$$g_x dx + g_y dy = 0 \ .$$

Thus

$$\frac{dy}{dx} = -\frac{g_x}{g_y} \ .$$

From both of these we obtain

$$\frac{g_x}{g_y} = \frac{f_x}{f_y} ,$$

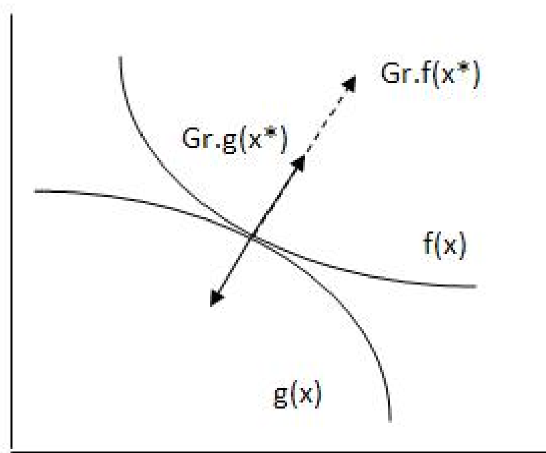
i.e. the **objective and the constraint are tangent at the optimum**. This follows from

$$\frac{f_y}{g_y} = \lambda = \frac{f_x}{g_x} .$$

A graphic interpretation. Think of the gradient as a vector that points in a particular direction. This direction is where to move in order to increase the function the most, and is perpendicular to the isoquant of the function (because, by definition, movement on the isoquant does not change the value). Notice that we have

$$\begin{aligned} \nabla f(x^*, y^*) &= \lambda^* \nabla g(x^*, y^*) \\ (f_{x^*}, f_{y^*}) &= \lambda^* (g_{x^*}, g_{y^*}) . \end{aligned}$$

This means that the constraint and the isoquant of the objective at the optimal value are parallel. They may point in the same direction if $\lambda > 0$ or in opposite directions if $\lambda < 0$.



Gradient Condition for Optimization

In the figure above: the upper curve is given by the isoquant $f(x) = f(x^*)$ and the lower curve is given by $g(x) = c$.

12.3 λ is the shadow cost of the constraint

λ tells you how much f would increase if we relax the constraint by one unit, i.e. increase or decrease c by 1 (for equality constraints, it will be either-or). For example, if the objective is utility and the constraint is your budget in euros, then λ is in terms of utils/euro. It tells you how many more utils you would get if you had one more euro.

Write the system of equations that define the optimum as **identities, evaluated at** (λ^*, x^*, y^*)

$$\begin{aligned} F^1(\lambda, x, y) &= c - g(x, y) = 0 \\ F^2(\lambda, x, y) &= f_x - \lambda g_x = 0 \\ F^2(\lambda, x, y) &= f_y - \lambda g_y = 0 \ . \end{aligned}$$

This is a system of functions of the form $F(\lambda, x, y, c) = 0$. If all these functions are C^1 and $|J| \neq 0$ at (λ^*, x^*, y^*), where

$$|J| = \left| \frac{\partial F}{\partial (\lambda \ x \ y)} \right| = \left| \begin{array}{ccc} 0 & -g_x & -g_y \\ -g_x & f_{xx} - \lambda g_{xx} & f_{xy} - \lambda g_{xy} \\ -g_y & f_{xy} - \lambda g_{xy} & f_{yy} - \lambda g_{yy} \end{array} \right| ,$$

then by the implicit function theorem there exits a set of functions $\lambda^* = \lambda(c)$, $x^* = x(c)$ and $y^* = y(c)$ with well defined derivatives (they are differentiable). It follows that there is a sense in which $d\lambda^*/dc$ is meaningful.

Now consider the value of the Lagrangian

$$\mathcal{L}^* = \mathcal{L}(\lambda^*, x^*, y^*) = f(x^*, y^*) + \lambda^* [c - g(x^*, y^*)] \ ,$$

where we remember that (x^*, y^*, λ^*) is a critical point. Take the total derivative w.r.t. c:

$$\begin{aligned} \frac{d\mathcal{L}^*}{dc} &= f_x \frac{dx^*}{dc} + f_y \frac{dy^*}{dc} + \frac{d\lambda^*}{dc}[c - g(x^*, y^*)] + \lambda^* \left[1 - g_x \frac{dx^*}{dc} - g_y \frac{dy^*}{dc}\right] \\ &= \frac{dx^*}{dc}[f_x - \lambda^* g_x] + \frac{dy^*}{dc}[f_y - \lambda^* g_y] + \frac{d\lambda^*}{dc}[c - g(x^*, y^*)] + \lambda^* \\ &= \lambda^* \ . \end{aligned}$$

Therefore

$$\frac{d\mathcal{L}^*}{dc} = \lambda^* = \frac{\partial \mathcal{L}^*}{\partial c} \ .$$

This is a manifestation of **the envelope theorem** (see below). But we also know that at the optimum we have

$$c - g(x^*, y^*) = 0 \ .$$

So at the optimum we have

$$\mathcal{L}(x^*, y^*, \lambda^*) = f(x^*, y^*) \ ,$$

and therefore

$$\frac{d\mathcal{L}^*}{dc} = \frac{df^*}{dc} = \lambda^* \ .$$

12.4 The envelope theorem

Let x^* be a critical point of $f(x,\theta)$. Then

$$\frac{df(x^*,\theta)}{d\theta} = \frac{\partial f(x^*,\theta)}{\partial \theta} .$$

Proof: since at x^* we have $f_x(x^*,\theta) = 0$, we have

$$\frac{df(x^*,\theta)}{d\theta} = \frac{\partial f(x^*,\theta)}{\partial x}\frac{dx}{d\theta} + \frac{\partial f(x^*,\theta)}{\partial \theta} = \frac{\partial f(x^*,\theta)}{\partial \theta} \blacksquare$$

- Drawing of an "envelope" of functions and optima for $f(x^*,\theta_1)$, $f(x^*,\theta_2)$, ...

12.5 Lagrange method: one constraint, many variables

Let $f(x), g(x) \in C^1$ and $x \in \mathbb{R}^n$. Suppose that x^* is the solution to

$$\text{Choose } x \text{ to maximize } f(x), \text{ s.t. } g(x) = c .$$

and that x^* is not a critical point of $g(x) = c$. Then there exists a number λ^* such that (x^*, λ^*) is a critical point of

$$\mathcal{L} = f(x) + \lambda [c - g(x)] ,$$

i.e.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \lambda} &= c - g(x,y) = 0 \\
\frac{\partial \mathcal{L}}{\partial x_i} &= f_i - \lambda g_i = 0 , \quad i = 1,2,...n .
\end{aligned}$$

- The constraint qualification is similar to above:

$$\nabla g^* = (g_1(x^*), g_2(x^*), ...g_n(x^*)) \neq 0 .$$

12.6 Lagrange method: many constraints, many variables

Let $f(x), g^j(x) \in C^1$ $j = 1,2,...m$, and $x \in \mathbb{R}^n$. Suppose that x^* is the solution to

$$\text{Choose } x \text{ to maximize } f(x), \text{ s.t. } g^1(x) = c_1,\ g^2(x) = c_2,\ ...g^m(x) = c_m .$$

and that x^* satisfies the constraint qualifications. Then there exists m numbers $\lambda_1^*, \lambda_2^*, ...\lambda_m^*$ such that (x^*, λ^*) is a critical point of

$$\mathcal{L} = f(x) + \sum_{j=1}^{m} \lambda_j \left[c_j - g^j(x)\right] ,$$

i.e.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \lambda_j} &= c_j - g^j(x) = 0 \ , \quad j = 1, 2, ...m \\ \frac{\partial \mathcal{L}}{\partial x_i} &= f_i - \lambda g_i = 0 \ , \quad i = 1, 2, ...n \ .\end{aligned}$$

- The constraint qualification now requires that

$$rank \left[\frac{\partial g}{\partial x'}\right]_{m \times n} = m \ ,$$

  which is as large as it can possibly be. This means that we must have $m \leq n$, because otherwise the maximal rank would be $n < m$. This constraint qualification, as all the others, means that there exists a $n - m$ dimensional tangent hyperplane (a $\mathbb{R}^{n-m}$ vector space). Loosely speaking, it ensures that we can construct tangencies freely enough.

12.7 Constraint qualifications in action

This example shows that when the constraint qualification is not met, the Lagrange method does not work.

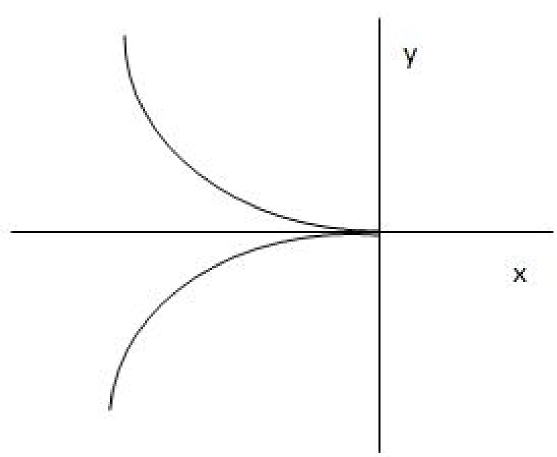
$$\text{Choose } \{x, y\} \text{ to maximize } x, \text{ s.t. } x^3 + y^2 = 0 \ .$$

The constraint set is given by

$$y^2 = -x^3 \quad \Rightarrow \quad y = \pm x^{3/2} \text{ for } x \leq 0 \ ,$$

i.e.

$$C = \left\{(x, y) : x \leq 0 \text{ and } (y = x^{3/2} \text{ or } y = -x^{3/2})\right\}$$



Notice that $(0,0)$ is the maximum point. Evaluate the gradient of g at the optimum:

$$\begin{aligned}
\nabla g &= \begin{pmatrix} 3x^2 & 2y \end{pmatrix} \\
\nabla g\,(0,0) &= (0,0) \ .
\end{aligned}$$

This violates the constraint qualifications, since $(0,0)$ is a critical point of $g\,(x,y)$.

Now check the Lagrangian

$$\begin{aligned}
\mathcal{L} &= x + \lambda\left(-x^3 - y^2\right) \\
\mathcal{L}_\lambda &= -x^3 - y^2 = 0 \\
\mathcal{L}_x &= 1 - \lambda 3x^2 = 0 \quad \Rightarrow \quad \lambda = 1/3x^2 \\
\mathcal{L}_y &= -\lambda 2y = 0 \quad \Rightarrow \quad \text{either } \lambda = 0 \text{ or } y = 0 \ .
\end{aligned}$$

- Suppose $x = 0$. Then $\lambda = \infty$ – not admissible.
- Suppose $x \neq 0$. Then $\lambda > 0$ and thus $y = 0$. But then from the constraint set $x = 0$ – a contradiction.

Comment: This method of trial and error is general, as you will see below in other examples.

12.8 Constraint qualifications and the Fritz-John Theorem

Let $f\,(x)\,, g\,(x) \in C^1$, $x \in \mathbb{R}^n$. Suppose that x^* is the solution to

$$\text{Choose } x \text{ to maximize } f\,(x)\,, \text{ s.t. } g\,(x) = c$$

Then there exists two numbers λ_0^* and λ_1^* such that (λ_1^*, x^*) is a critical point of

$$\mathcal{L} = \lambda_0 f\,(x) + \lambda_1\,[c - g\,(x)] \ ,$$

i.e.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \lambda} &= c - g\,(x) = 0 \\
\frac{\partial \mathcal{L}}{\partial x_i} &= \lambda_0 f_i - \lambda_1 g_i = 0 \ , \quad i = 1, 2, ...n
\end{aligned}$$

and

$$\begin{aligned}
\lambda_0^* &\in \{0, 1\} \\
\{\lambda_0^*, \lambda_1^*\} &\neq (0,0) \ .
\end{aligned}$$

This generalizes to multi constraint problems.

12.9 Second order conditions

We want to know whether d^2z is negative or positive definite **on the constraint set**. Using the Lagrange method we find a critical point (x^*, λ^*) of the problem

$$\mathcal{L} = f(x) + \lambda [c - g(x)] \ .$$

But this is not a maximum of the $\mathcal{L}$ problem. In fact, (x^*, λ^*) is a **saddle point:** perturbations of x around x^* will hurt the objective, while perturbations of λ around λ^* will help the objective. If (x^*, λ^*) is a critical point of the $\mathcal{L}$ problem, and we wish to maximize $f(x)$ on the constraint set, then: holding λ^* constant, x^* maximizes the value of the problem; and holding x^* constant, λ^* minimizes the value of the problem. This makes sense: lowering the shadow cost of the constraint as much as possible at the point that maximizes the value. And vice versa if we wish to minimize $f(x)$ on the constraint set.

This complicates characterizing the second order conditions, to distinguish maxima from minima. We want to know whether d^2z is negative or positive definite on the constraint set.

Consider the two variables case

$$dz = f_x dx + f_y dy \ .$$

This holds for any dx and dy. From $g(x, y) = c$ we have

$$g_x dx + g_y dy = 0 \ ,$$

i.e. dx and dy are not simultaneously arbitrary. We can treat dy **as a function of** x **and** y

$$dy = -\frac{g_x}{g_y} dx$$

and use this when we differentiate dz the second time:

$$\begin{aligned}
d^2z &= d(dz) = \frac{\partial (dz)}{\partial x} dx + \frac{\partial (dz)}{\partial y} dy \\
&= \frac{\partial}{\partial x} [f_x dx + f_y dy] dx + \frac{\partial}{\partial y} [f_x dx + f_y dy] dy \\
&= \left[f_{xx} dx + f_{yx} dy + f_y \frac{\partial (dy)}{\partial x} \right] dx + \left[f_{xy} dx + f_{yy} dy + f_y \frac{\partial (dy)}{\partial y} \right] dy \\
&= f_{xx} dx^2 + 2 f_{xy} dx dy + f_{yy} dy^2 + f_y d^2 y \ ,
\end{aligned}$$

where we use

$$d^2 y = d(dy) = \frac{\partial (dy)}{\partial x} dx + \frac{\partial (dy)}{\partial y} dy \ .$$

This is not a quadratic form, but we use $g(x, y) = c$ again to transform it into one, by eliminating

d^2y. Differentiate

$$dg = g_x dx + g_y dy = 0 \ ,$$

using dy **as a function of** x **and** y again:

$$\begin{aligned}
d^2g &= d(dg) = \frac{\partial (dg)}{\partial x}dx + \frac{\partial (dg)}{\partial y}dy \\
&= \frac{\partial}{\partial x}\left[g_x dx + g_y dy\right] dx + \frac{\partial}{\partial y}\left[g_x dx + g_y dy\right] dy \\
&= \left[g_{xx}dx + g_{yx}dy + g_y \frac{\partial (dy)}{\partial x}\right] dx + \left[g_{xy}dx + g_{yy}dy + g_y \frac{\partial (dy)}{\partial y}\right] dy \\
&= g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2 + g_y d^2y \\
&= 0 \ .
\end{aligned}$$

Thus

$$d^2y = -\frac{1}{g_y}\left[g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2\right] \ .$$

Use this in the expression for d^2z to get

$$d^2z = \left(f_{xx} - f_y\frac{g_{xx}}{g_y}\right)dx^2 + 2\left(f_{xy} - f_y\frac{g_{xy}}{g_y}\right)dxdy + \left(f_{yy} - f_y\frac{g_{yy}}{g_y}\right)dy^2 \ .$$

From the FONCs we have

$$\lambda = \frac{f_y}{g_y}$$

and by differentiating the FONCs we get

$$\begin{aligned}
\mathcal{L}_{xx} &= f_{xx} - \lambda g_{xx} \\
\mathcal{L}_{yy} &= f_{yy} - \lambda g_{yy} \\
\mathcal{L}_{xy} &= f_{xy} - \lambda g_{xy} \ .
\end{aligned}$$

We use all this to get

$$d^2z = \mathcal{L}_{xx}dx^2 + 2\mathcal{L}_{xy}dxdy + \mathcal{L}_{yy}dy^2 \ .$$

This is a quadratic form, but not a standard one, because, dx and dy are not arbitrary. As before, we want to know the sign of d^2z, but unlike the unconstrained case, dx and dy must satisfy $dg = g_x dx + g_y dy = 0$. Thus, we have second order necessary conditions (SONC):

- If d^2z is negative semidefinite s.t. $dg = 0$, then maximum.
- If d^2z is positive semidefinite s.t. $dg = 0$, then minimum.

The second order sufficient conditions are (SOSC):

- If d^2z is negative definite s.t. $dg = 0$, then maximum.
- If d^2z is positive definite s.t. $dg = 0$, then minimum.

These are less stringent conditions relative to unconstrained optimization, where we required conditions on d^2z for all values of dx and dy. Here we consider only a subset of those values, so the requirement is less stringent, although slightly harder to characterize.

12.10 Bordered Hessian and constrained optimization

Using the notations we used before for a Hessian,

$$H = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

(except that here it will be the Hessian of $\mathcal{L}$, not of f) we can write

$$d^2z = \mathcal{L}_{xx}dx^2 + 2\mathcal{L}_{xy}dxdy + \mathcal{L}_{yy}dy^2$$

as

$$d^2z = adx^2 + 2hdxdy + bdy^2 \ .$$

We also rewrite

$$g_x dx + g_y dy = 0$$

as

$$\alpha dx + \beta dy = 0 \ .$$

The second order conditions involve the sign of

$$\begin{aligned} d^2z &= adx^2 + 2hdxdy + bdy^2 \\ \text{s.t. } 0 &= \alpha dx + \beta dy \ . \end{aligned}$$

Eliminate dy using

$$dy = -\frac{\alpha}{\beta}dx$$

to get

$$d^2z = \left[a\beta^2 - 2h\alpha\beta + b\alpha^2\right]\frac{dx^2}{\beta^2} \ .$$

The sign of d^2z depends on the square brackets. For a maximum we need it to be negative. It turns out that

$$\left[a\beta^2 - 2h\alpha\beta + b\alpha^2\right] = -\begin{vmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix} \equiv -\left|\overline{H}\right| \ .$$

Notice that $\overline{H}$ contains the Hessian, and is **bordered** by the gradient of the constraint. Thus, the term "**bordered Hessian**".

The $\boldsymbol{n}$-dimensional case with one constraint

Let $f(x), g(x) \in C^2$, $x \in \mathbb{R}^n$. Suppose that x^* is a critical point of the Lagrangian problem. Let

$$H_{n\times n} = \left[\frac{\partial^2 \mathcal{L}}{\partial x \partial x'}\right]$$

be the Hessian of $\mathcal{L}$ evaluated at $(\lambda^*\ x^*)$. Let ∇g be a linear constraint on $d_{n\times 1}$ $(= dx_{n\times 1})$, evaluated at x^*:

$$\nabla g(x^*)d = 0\ .$$

We want to know what is the sign of

$$d^2 z = q = d' H d$$

such that

$$\nabla g(x^*)d = 0\ .$$

The sign definiteness of the quadratic form q depends on the following bordered Hessian

$$\overline{H}_{(n+1)\times(n+1)} = \begin{bmatrix} 0 & \nabla g_{1\times n} \\ \nabla g'_{n\times 1} & H_{n\times n} \end{bmatrix}\ .$$

Recall that sign definiteness of a matrix depends on the signs of the determinants of the leading principal minors. Therefore

$$d^2 z \text{ is } \left\{ \begin{array}{c} \text{positive definite (min)} \\ \text{negative definite (max)} \end{array} \right\} \text{ s.t. } dg = 0 \text{ iff } \left\{ \begin{array}{c} \left|\overline{H}_3\right|,\ \left|\overline{H}_4\right|, \ldots\ \left|\overline{H}_n\right| < 0 \\ \left|\overline{H}_3\right| > 0,\ \left|\overline{H}_4\right| < 0,\ \left|\overline{H}_5\right| > 0, \ldots \end{array} \right\}\ ,$$

- Note that in the Chiang and Wainwright text they start from $\left|\overline{H}_2\right|$, which they define as the *third* leading principal minor and is an abuse of notation. We have one consistent way to define leading principal minors of a matrix and we should stick to that.

The general case

Let $f(x), g^j(x) \in C^2$ $j = 1, 2, \ldots m$, and $x \in \mathbb{R}^n$. Suppose that $(\lambda^*\ x^*)$ is a critical point of the Lagrangian problem. Let

$$H_{n\times n} = \left[\frac{\partial^2 \mathcal{L}}{\partial x \partial x'}\right]$$

be the Hessian of $\mathcal{L}$ evaluated at $(\lambda^*\ x^*)$. Let

$$A_{m\times n} = \left[\frac{\partial g}{\partial x'}\right]$$

be the set of linear constraints on $d_{n\times 1}$ $(= dx_{n\times 1})$, evaluated at x^*:

$$Ad = 0 \ .$$

We want to know the sign of

$$d^2 z = q = d' H d$$

such that

$$Ad = 0 \ .$$

The sign definiteness of the quadratic form q depends on the **bordered Hessian**

$$\overline{H}_{(m+n)\times(m+n)} = \begin{bmatrix} 0_{m\times m} & A_{m\times n} \\ A'_{n\times m} & H_{n\times n} \end{bmatrix} \ .$$

The sign definiteness of $\overline{H}$ depends on the signs of the determinants of the leading principal minors.

- For a maximum ($d^2 z$ negative definite) we require that $\left|\overline{H}_{2m}\right|, \ \left|\overline{H}_{2m+1}\right| ... \ \left|\overline{H}_{m+n}\right|$ alternate signs, where $\operatorname{sign}\left(\left|\overline{H}_{2m}\right|\right) = (-1)^m$ (Dixit). Note that we require $m < n$, so that $2m < m+n$.
- An alternative formulation for a maximum ($d^2 z$ negative definite) requires that the last $n-m$ leading principal minors alternate signs, where $\operatorname{sign}\left(\left|\overline{H}_{n+m}\right|\right) = (-1)^n$ (Simon and Blume).
- The formulation in the Chiang and Wainwright text is wrong.
- For a minimum...? We know that searching for a minimum of f is like searching for a maximum of $-f$. So one could set up the problem that way and just treat it like a maximization problem.

12.11 Quasiconcavity and quasiconvexity

This is a less restrictive condition on the objective function.

- **Definition:** a function f is **quasiconcave** iff $\forall x^1, x^2 \in$ domain of f, which is assumed to be a convex set, and $\forall \theta \in (0,1)$ we have

  $$f\left(x^2\right) \geq f\left(x^1\right) \quad \Rightarrow \quad f\left[\theta x^1 + (1-\theta)\, x^2\right] \geq f\left(x^1\right) \ .$$

  For **strict quasiconcavity** replace the second inequality with a strict inequality, but not the first. More simply put

  $$f\left[\theta x^1 + (1-\theta)\, x^2\right] \geq \min\left\{f\left(x^2\right), f\left(x^1\right)\right\} \ .$$

  In words: the image of the convex combination is larger than the lower of the two images.

- **Definition:** a function f is **quasiconvex** iff $\forall x^1, x^2 \in$ domain of f, which is assumed to be a convex set, and $\forall \theta \in (0,1)$ we have

$$f\left(x^2\right) \geq f\left(x^1\right) \quad \Rightarrow \quad f\left[\theta x^1 + (1-\theta)\, x^2\right] \leq f\left(x^2\right) \ .$$

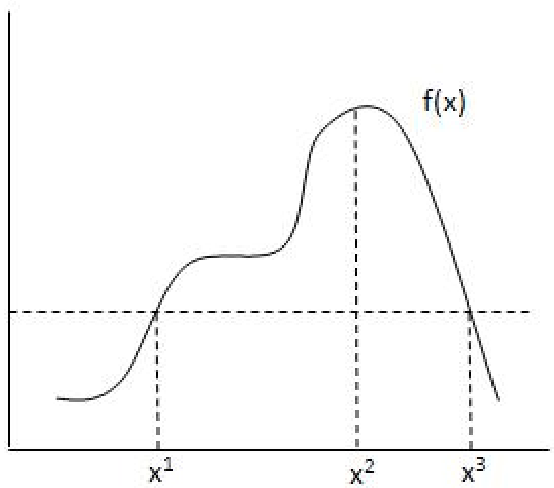
  For **strict quasiconvexity** replace the second inequality with a strict inequality, but not the first. More simply put

$$f\left[\theta x^1 + (1-\theta)\, x^2\right] \leq \max\left\{f\left(x^2\right), f\left(x^1\right)\right\} \ .$$

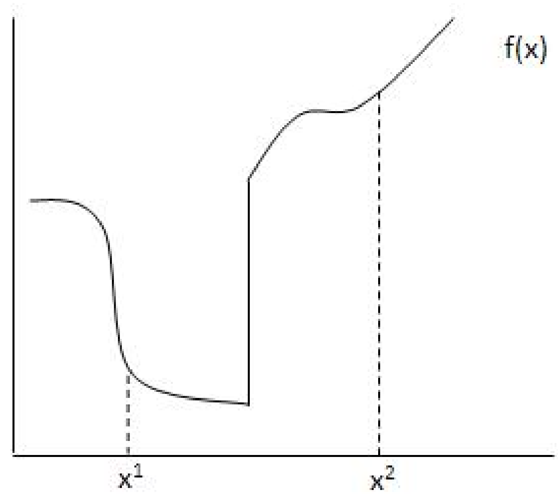
  In words: the image of the convex combination is smaller than the higher of the two images.

- Strict quasiconcavity and strict quasiconvexity rule out flat segments.

- $\theta \notin \{0,1\}$.

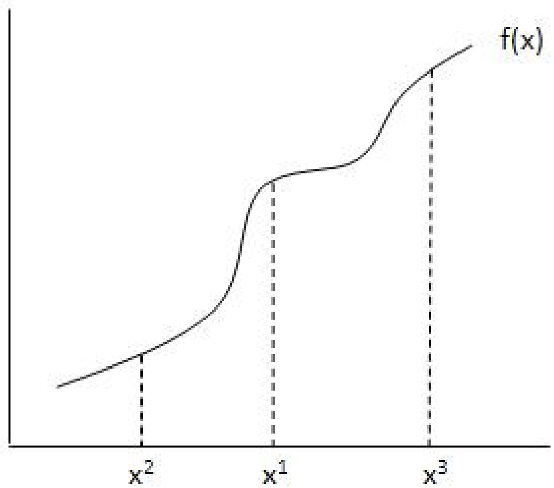


Quasiconcave Function, not Strictly



Strictly Quasiconvex Function

Due to the flat segment, the function on the left is not strictly quasiconcave. Note that neither of these functions is convex nor concave. Thus, this is a weaker restriction. The following function, while not convex nor concave, is both quasiconcave and quasiconvex.



Quasiconcave and Quasiconvex

- Compare definition of quasiconcavity to concavity and then compare graphically.

Properties:

1. If f is linear, then it is both quasiconcave and quasiconvex.
2. If f is (strictly) quasiconcave, then $-f$ is (strictly) quasiconvex.
3. If f is concave (convex), then it is quasiconcave (quasiconvex)—but not only if.

- Note that unlike concave functions, the sum of two quasiconcave functions is **NOT** necessarily quasiconcave. Similarly for quasiconvex functions.

Alternative "set" definitions: Let $x \in \mathbb{R}^n$.

- f is quasiconcave **iff** $\forall k \in \mathbb{R}$ the set

$$S_k^+ = \{x : f(x) \geq k\} \ , \quad k \in \mathbb{R}$$

is a convex set (for concavity it is "only if", not "iff").

- f is quasiconvex **iff** $\forall k \in \mathbb{R}$ the set

$$S_k^- = \{x : f(x) \leq k\} \ , \quad k \in \mathbb{R}$$

is a convex set (for convexity it is "only if", not "iff").

These may be easier to verify and more useful. Think of S_k^+ in the context of utility: S_k^+ is the set of objects that give at least as much utility as k.

Recall that for concavity and convexity the conditions above were necessary, but not sufficient. Here, these are "set" definitions, so they are necessary and sufficient conditions (iff).

Consider a continuously differentiable function $f(x) \in C^1$ and $x \in \mathbb{R}^n$. Then f is

- **quasiconcave** iff $\forall x^1, x^2 \in$ domain of f, which is assumed to be a convex set, we have

  $$f\left(x^2\right) \geq f\left(x^1\right) \quad \Rightarrow \quad \nabla f\left(x^1\right)\left(x^2 - x^1\right) \geq 0 \ .$$

  In words: the function does not change the sign of the slope more than once.

- **quasiconvex** iff $\forall x^1, x^2 \in$ domain of f, which is assumed to be a convex set, we have

  $$f\left(x^2\right) \geq f\left(x^1\right) \quad \Rightarrow \quad \nabla f\left(x^2\right)\left(x^2 - x^1\right) \geq 0 \ .$$

  In words: the function does not change the sign of the slope more than once.

- For strictness, change the second inequality to a strict one, which rules out flat regions.

Consider a *twice* continuously differentiable function $f(x) \in C^2$ and $x \in \mathbb{R}^n$. The Hessian of f is denoted H and the gradient as ∇f. Define

$$B = \begin{bmatrix} 0_{1\times 1} & \nabla f_{1\times n} \\ \nabla f'_{n\times 1} & H_{n\times n} \end{bmatrix}_{(n+1)\times(n+1)} .$$

Conditions for quasiconcavity and quasiconvexity **in the positive orthant**, $x \in \mathbb{R}_+^n$ involve the leading principal minors of B.

Necessary condition: f is **quasiconcave on** $\mathbb{R}_+^n$ if (but not only if) $\forall x \in \mathbb{R}_+^n$, the leading principal minors of B follow this pattern

$$|B_2| \leq 0, \ \ |B_3| \geq 0, \ \ |B_4| \leq 0, ...$$

Sufficient condition: f is **quasiconcave on** $\mathbb{R}_+^n$ only if $\forall x \in \mathbb{R}_+^n$, the leading principal minors of B follow this pattern

$$|B_2| < 0, \ \ |B_3| > 0, \ \ |B_4| < 0, ...$$

Finally, there are also **explicitly quasiconcave** functions.

- **Definition:** a function f is **explicitly quasiconcave** if $\forall x^1, x^2 \in$ domain of f, which is assumed to be a convex set, and $\forall \theta \in (0,1)$ we have

$$f\left(x^2\right) > f\left(x^1\right) \quad \Rightarrow \quad f\left[\theta x^1 + (1-\theta)\, x^2\right] > f\left(x^1\right) \ .$$

  This rules out flat segments, except at the top of the hill.

Ranking of concavity, from strongest to weakest:

1. strictly concave
2. concave
3. strictly quasiconcave

$$f\left(x^2\right) \geq f\left(x^1\right) \quad \Rightarrow \quad f\left[\theta x^1 + (1-\theta)\, x^2\right] > f\left(x^1\right) \ .$$

   (no flat regions)

4. explicitly quasiconcave

$$f\left(x^2\right) > f\left(x^1\right) \quad \Rightarrow \quad f\left[\theta x^1 + (1-\theta)\, x^2\right] > f\left(x^1\right) \ .$$

   (only one flat region allowed, at the top)

5. quasiconcave

$$f\left(x^2\right) \geq f\left(x^1\right) \quad \Rightarrow \quad f\left[\theta x^1 + (1-\theta)\, x^2\right] \geq f\left(x^1\right) \ .$$

12.12 Why is quasiconcavity important? Invariance to positive monotone transformation

Quasiconcavity is important because it allows arbitrary cardinality in the utility function, while maintaining ordinality. Concavity imposes decreasing marginal utility, which is not necessary for characterization of convex preferences, convex indifference sets and convex upper contour sets. Only when dealing with risk do we need to impose concavity. We do not need concavity for global extrema.

Quasiconcave functions preserve quasi concavity under any positive monotone transformation. Suppose that some utility function u is quasiconcave, i.e. the set $S = \{x : u(x) \geq k\}$ is a convex set $\forall k \in \mathbb{R}$. This means that any $x \in S$ is at least as good as any $x \notin S$. Now consider a positive monotone transformation of $u(x)$, denoted $t(u(x))$. Then the set $T = \{x : t(u(x)) \geq t(k)\}$ is still convex $\forall k \in \mathbb{R}$. Moreover, $T = S$, i.e. the same $x \in S$ that are at least as good as any $x \notin S$ are the same $x \in T$ that are at least as good as any $x \notin T$. A corollary is that if I find a maximizer of $u(x)$, it is also a maximizer of $t(u(x))$.

Concave functions DO NOT preserve concavity under all positive monotone transformations. Proof: By example. $-x^2$ is concave (make sure that you know how to prove this). But e^{-x^2}—a positive monotone transformation of $-x^2$—is not concave (it has the shape of the normal distribution function) ■

Note that since concave functions are also quasiconcave, then any positive monotone transformation will keep it at least quasiconcave, like in the example above: e^{-x^2} is indeed quasiconcave.

12.13 Why is quasiconcavity important? Global maximum

Suppose that x^* is the solution to

$$\text{Choose } x \text{ to maximize } f(x), \text{ s.t. } g(x) = c .$$

If

1. the set $\{x : g(x) = c\}$ is a convex set, and
2. f is *explicitly* quasiconcave,

then $f(x^*)$ is a global (constrained) maximum.

If f is *strictly* quasiconcave, then this global maximum is **unique**.

- This doesn't apply for a quasi-concave function (not explicitly, not strictly) because then we can have several flat regions, not only at the top. This will not allow distinguishing local maxima from global maximum based only on the properties of f.

12.14 Application: cost minimization

We like apples (a) and bananas (b), but want to reduce the cost of any (a, b) bundle for a given level of utility ($U(\overset{+}{a}, \overset{+}{b})$) (or fruit salad, if we want to use a production metaphor).

$$\text{Choose } \{a, b\} \text{ to minimize cost } C = ap_a + bp_b, \text{ s.t. } U(a, b) = u$$

Set up the appropriate Lagrangian

$$\mathcal{L} = ap_a + bp_b + \lambda [u - U(a, b)] .$$

Here λ is in units of \$/util: it tells you how much an additional util will cost. If U was a production function for salad, then λ would be in units of \$ per unit of salad, i.e. the price of one unit of salad.

FONC:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \lambda} &= u - U(a,b) = 0 \\
\frac{\partial \mathcal{L}}{\partial a} &= p_a - \lambda U_a = 0 \quad \Rightarrow \quad p_a/U_a = \lambda > 0 \\
\frac{\partial \mathcal{L}}{\partial b} &= p_b - \lambda U_b = 0 \quad \Rightarrow \quad p_b/U_b = \lambda > 0 \ .
\end{aligned}$$

Thus

$$MRS = \frac{U_a}{U_b} = \frac{p_a}{p_b}$$

So we have tangency. Let the value of the problem be

$$C^* = a^* p_a + b^* p_b \ .$$

Take a total differential at the optimum to get

$$dC = p_a da + p_b db = 0 \quad \Rightarrow \quad \frac{db}{da} = -\frac{p_a}{p_b} < 0 \ .$$

We could also obtain this result from the implicit function theorem, since $C(a,b), U(a,b) \in C^1$ and $|J| \neq 0$. Yet another way to get this is to see that since $U(a,b) = u$, a constant,

$$dU(a,b) = U_a da + U_b db = 0 \quad \Rightarrow \quad \frac{db}{da} = -\frac{U_a}{U_b} < 0 \ .$$

At this stage all we know is that the isoquant for utility slopes downward, and that it is tangent to the isocost line at the optimum, if the optimum exists.

SOC:

$$\left[\overline{H}\right] = \begin{bmatrix} 0 & U_a & U_b \\ U_a & -\lambda U_{aa} & -\lambda U_{ab} \\ U_b & -\lambda U_{ab} & -\lambda U_{bb} \end{bmatrix} \ .$$

We need positive definiteness of d^2C^* for a minimum—which requires negative definiteness of $\overline{H}$—so we need $\left|\overline{H}_2\right| < 0$ and $\left|\overline{H}_3\right| = \left|\overline{H}\right| < 0$.

$$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & U_a \\ U_a & -\lambda U_{aa} \end{vmatrix} = -U_a^2 < 0 \ ,$$

so this is good (in fact, $\left|\overline{H}_2\right|$ is always ≤ 0, just not necessarily < 0). But

$$\begin{aligned}
\left|\overline{H}_3\right| &= 0 - U_a\left[U_a(-\lambda U_{bb}) - (-\lambda U_{ab})U_b\right] + U_b\left[U_a(-\lambda U_{ab}) - (-\lambda U_{aa})U_b\right] \\
&= U_a^2 \lambda U_{bb} - U_a \lambda U_{ab} U_b - U_b U_a \lambda U_{ab} + U_b^2 \lambda U_{aa} \\
&= \lambda\left(U_a^2 U_{bb} - 2U_a U_{ab} U_b + U_b^2 U_{aa}\right)
\end{aligned}$$

Without further conditions on U, we do not know whether the expression in the parentheses is negative or not ($\lambda > 0$).

The curvature of the utility isoquant is given by

$$\begin{aligned}
\frac{d}{da}\left(\frac{db}{da}\right) &= \frac{d^2b}{da^2} = \frac{d}{da}\left(-\frac{U_a}{U_b}\right) = -\frac{d}{da}\left(\frac{U_a(a,b)}{U_b(a,b)}\right) = \\
&= -\frac{\left(U_{aa} + U_{ab}\frac{db}{da}\right)U_b - U_a\left(U_{bb}\frac{db}{da} + U_{ab}\right)}{U_b^2} \\
&= -\frac{\left[U_{aa} + U_{ab}\left(-\frac{U_a}{U_b}\right)\right]U_b - U_a\left[U_{bb}\left(-\frac{U_a}{U_b}\right) + U_{ab}\right]}{U_b^2} \\
&= -\frac{U_{aa}U_b - U_aU_{ab} + U_a^2U_{bb}/U_b - U_aU_{ab}}{U_b^2} \\
&= -\frac{U_{aa}U_b^2 - 2U_aU_{ab}U_b + U_a^2U_{bb}}{U_b^3} \\
&= -\frac{1}{U_b^3}\left(U_{aa}U_b^2 - 2U_aU_{ab}U_b + U_a^2U_{bb}\right) \ .
\end{aligned}$$

This involves the same expression in the parentheses. If the indifference curve is convex, then $\frac{d^2b}{da^2} \geq 0$ and thus the expression in the parentheses must be negative. This coincides with the positive *semi*-definiteness of d^2C^*. Thus, convex isoquants and existence of a global minimum in this case come together. This would ensure a global minimum, although not a unique one. If $\frac{d^2b}{da^2} > 0$, then the isoquant is strictly convex and the global minimum is unique, as dC^* is positive definite.

- **If U is strictly quasiconcave**, then indeed the isoquant is strictly convex and the global minimum is unique.

12.15 Related topics

12.15.1 Expansion paths

Consider the problem described above in Section 12.14. Let $a^*(p_a, p_b, u)$ and $b^*(p_a, p_b, u)$ be the optimal quantities of apples and bananas chosen given prices and a level of promised utility (AKA "demand"). The expansion path is the function $b(a)$ that is given by changes in u, when prices are fixed.

One way to get this is to notice that the FONCs imply

$$\begin{aligned}
\frac{U_a(a^*, b^*)}{U_b(a^*, b^*)} &= \frac{p_a}{p_b} \\
U(a^*, b^*) &= u \ ,
\end{aligned}$$

which define a system of equations, which can be written as

$$F\left(a^*, b^*, u, p_a, p_b\right) = 0 \ .$$

Fix prices. By applying the implicit function theorem we get $a^*(u)$ and $b^*(u)$. Each level of u defines a unique level of demand. The expansion path $b(a)$ is the function of all the unique combinations of $a^*(u)$ and $b^*(u)$ at all levels of u.

12.15.2 Homotheticity

This is when the expansion path $b(a)$ is a ray (a straight line starting at the origin). Equivalently, homotheticity implies that—and is implied by—the ratio $\frac{b^*(u)}{a^*(u)} \equiv \left(\frac{b^*}{a^*}\right)(u)$ is constant, not affected by u.

This is useful when you wish to aggregate over different individuals that may have different levels of utility. If all have identical and homothetic preferences (i.e., the same expansion path for all, with the same slope), then *relative* demand does not depend on levels of utility, only on relative prices. This is used a lot in international trade theory, where relative prices are what matter for the patterns of trade, while the levels of trade flows are given in the end by market clearing conditions.

12.15.3 Elasticity of substitution

An elasticity η is defined as the percent change in y that is invoked by a percent change in x:

$$\eta_{y,x} = \frac{dy/y}{dx/x} = \frac{dy}{dx}\frac{x}{y} = \frac{d\ln y}{d\ln x} \ .$$

an elasticity of substitution is usually referred to as an elasticity that does not change the level of some function. For example, if

$$F(y, p) = c \ ,$$

then the elasticity of substitution is the percent change in y that is invoked by a percent change in p. By the implicit function theorem

$$\frac{dy}{dp} = -\frac{F_p}{F_y}$$

and

$$\eta_{y,p} = -\frac{F_p}{F_y}\frac{p}{y} \ .$$

Elasticities of substitution often arise in the context of optimization. For example, consider the problem described above in Section 12.14. Let $a^*(p_a, p_b, u)$ and $b^*(p_a, p_b, u)$ be the optimal quantities of apples and bananas chosen given prices and a level of promised utility (AKA "demand").

The elasticity of substitution in demand (between a and b) is given by

$$\frac{d\left(a^*/b^*\right)/\left(a^*/b^*\right)}{d\left(p_a/p_b\right)/\left(p_a/p_b\right)} .$$

This number tells you how much the relative intensity of consumption of a (relative to b) changes with the relative price of a (relative to b).

12.15.4 Constant elasticity of substitution and relation to Cobb-Douglas

A general production function that exhibits constant elasticity of substitution (CES) is

$$q = z\left[\alpha k^{\varphi} + (1-\alpha)\, l^{\varphi}\right]^{1/\varphi} . \tag{5}$$

q is the quantity of output and k and l are inputs. $\alpha \in [0,1]$ is called the distribution parameter. z is a level shifter ("productivity" in the output context). The function is CES because a 1 percent change in the marginal products implies $-\sigma$ percent change in the input ratio:

$$\frac{d\left(k/l\right)/\left(k/l\right)}{d\left(MP_k/MP_l\right)/\left(MP_k/MP_l\right)} = -\sigma = -\frac{1}{1-\varphi} . \tag{6}$$

To see this, note that

$$\begin{aligned} MP_k &= \frac{\partial q}{\partial k} = \frac{1}{\varphi} z\left[\alpha k^{\varphi} + (1-\alpha)\, l^{\varphi}\right]^{1/\varphi - 1} \varphi \alpha k^{\varphi - 1} \\ MP_l &= \frac{\partial q}{\partial l} = \frac{1}{\varphi} z\left[\alpha k^{\varphi} + (1-\alpha)\, l^{\varphi}\right]^{1/\varphi - 1} \varphi (1-\alpha)\, l^{\varphi - 1} \end{aligned}$$

so that

$$\frac{MP_k}{MP_l} = \frac{\alpha}{1-\alpha}\left(\frac{k}{l}\right)^{\varphi - 1} = \frac{\alpha}{1-\alpha}\left(\frac{k}{l}\right)^{-1/\sigma} .$$

Taking the total differential we get

$$d\left(\frac{MP_k}{MP_l}\right) = -\frac{1}{\sigma}\frac{\alpha}{1-\alpha}\left(\frac{k}{l}\right)^{-1/\sigma - 1} d\left(\frac{k}{l}\right) .$$

Dividing through by MP_k/MP_l and rearranging, we get (6).

This general form can be applied as a utility function as well, where q represents a level of utility and where k and l represent quantities of different goods in consumption.

Now, it follows that when $\varphi = 0$ we have $\sigma = 1$. But you cannot simply plug $\varphi = 0$ into (5) because

$$q = \lim_{\varphi \to 0} z\left[\alpha k^{\varphi} + (1-\alpha)\, l^{\varphi}\right]^{1/\varphi} = \text{"}0^{\infty}\text{"} .$$

In order to find out the expression for q when $\varphi = 0$ rewrite (5) as

$$\ln(q/z) = \frac{\ln\left[\alpha k^{\varphi} + (1-\alpha) l^{\varphi}\right]}{\varphi} \ .$$

Now take the limit

$$\lim_{\varphi \to 0} \ln(q/z) = \lim_{\varphi \to 0} \frac{\ln\left[\alpha k^{\varphi} + (1-\alpha) l^{\varphi}\right]}{\varphi} = \frac{0}{0} \ .$$

Now apply **L'Hopital's Rule**:

$$\lim_{\varphi \to 0} \frac{\ln\left[\alpha k^{\varphi} + (1-\alpha) l^{\varphi}\right]}{\varphi} = \lim_{\varphi \to 0} \frac{\alpha k^{\varphi} \ln k + (1-\alpha) l^{\varphi} \ln l}{1 \cdot \left[\alpha k^{\varphi} + (1-\alpha) l^{1-\varphi}\right]} = \alpha \ln k + (1-\alpha) \ln l \ .$$

So that

$$\lim_{\varphi \to 0} \ln(q/z) = \alpha \ln k + (1-\alpha) \ln l$$

or

$$q = z k^{\alpha} l^{1-\alpha} \ , \qquad (7)$$

which is the familiar Cobb-Douglas production function. It follows that (7) is a particular case of (5) with $\sigma = 1$.

Note: to get this result we had to have the distribution parameter α. Without it, you would not get this result.

13 Optimization with inequality constraints

13.1 One inequality constraint

Let $f(x), g(x) \in C^1$, $x \in \mathbb{R}^n$. The problem is

$$\text{Choose } x \text{ to maximize } f(x), \text{ s.t. } g(x) \leq c \ .$$

Write the constraint in a "**standard way**"

$$g(x) - c \leq 0 \ .$$

Suppose that x^* is the solution to

$$\text{Choose } x \text{ to maximize } f(x), \text{ s.t. } g(x) - c \leq 0$$

and that **if the** $g(x) - c \leq 0$ **constraint binds at** x^*—i.e., $g(x^*) = c$—then x^* is not a critical point of $g(x)$, i.e., the **constraint qualifications** are not violated and $\nabla g(x^*) \neq 0$. Write

$$\mathcal{L} = f(x) + \lambda [c - g(x)] \ .$$

Then there exists a number λ^* such that

$$\begin{aligned}
(1) &: \frac{\partial \mathcal{L}}{\partial x_i} = f_i - \lambda^* g_i = 0 \ , \quad i = 1, 2, ...n \\
(2) &: \lambda^* [c - g(x, y)] = 0 \\
(3) &: \lambda^* \geq 0 \\
(4) &: g(x) \leq c \ .
\end{aligned}$$

- The **standard way**: write $g(x) - c \leq 0$ and then flip it in the Lagrangian function $\lambda [c - g(x)]$.
- Conditions 2 and 3 are called **complementary slackness conditions**. If the constraint is not binding, then changing c a bit will not affect the value of the problem; in that case $\lambda = 0$.

Conditions 1-4 in the Chiang and Wainwright text are written differently, although they are an equivalent representation:

$$\begin{aligned}
(\text{i}) &: \frac{\partial \mathcal{L}}{\partial x_i} = f_i - \lambda g_i = 0 \ , \quad i = 1, 2, ...n \\
(\text{ii}) &: \frac{\partial \mathcal{L}}{\partial \lambda} = [c - g(x, y)] \geq 0 \\
(\text{iii}) &: \lambda \geq 0 \\
(\text{iv}) &: \lambda [c - g(x, y)] = 0 \ .
\end{aligned}$$

Notice that from (ii) we get $g(x) \leq c$. If $g(x) < c$, then $\mathcal{L}_\lambda > 0$.

13.2 One inequality constraint and one non-negativity constraint

There is really nothing special about this problem, but it is worthwhile setting it up, for practice. Let $f(x), g(x) \in C^1$, $x \in \mathbb{R}^n$. The problem is

$$\text{Choose } x \text{ to maximize } f(x)\,, \text{ s.t. } g(x) \leq c \text{ and } x \geq 0\ .$$

Rewrite this as

$$\text{Choose } x \text{ to maximize } f(x)\,, \text{ s.t. } g(x) - c \leq 0 \text{ and } -x - 0 \leq 0\ .$$

Suppose that x^* is the solution to this problem and that x^* does not violate the constraint qualifications, i.e. it is not a critical point of the constraint set (to be defined below). Write down the Lagrangian function

$$\mathcal{L} = f(x) + \lambda[c - g(x)] + \varphi[x]\ .$$

Then there exist two numbers λ^* and φ^* such that

$$\begin{aligned}
(1) &: \quad \frac{\partial \mathcal{L}}{\partial x_i} = f_i - \lambda g_i + \varphi = 0\ , \quad i = 1, 2, ...n \\
(2) &: \quad \lambda[c - g(x, y)] = 0 \\
(3) &: \quad \lambda \geq 0 \\
(4) &: \quad g(x) \leq c \\
(5) &: \quad \varphi[x] = 0 \\
(6) &: \quad \varphi \geq 0 \\
(7) &: \quad -x \leq 0 \quad \Leftrightarrow x \geq 0\ .
\end{aligned}$$

- The **constraint qualification** is that x^* is not a critical point of the constraints that bind. If only $g(x) = c$ binds, then we require $\nabla g(x^*) \neq 0$. See the general case below.

The text gives again a different—and I argue less intuitive—formulation. The Lagrangian is set up without explicitly mentioning the non-negativity constraints

$$\mathcal{Z} = f(x) + \varphi[c - g(x)]\ .$$

In the text the FONCs are written as

$$
\begin{aligned}
\text{(i)} &: \frac{\partial \mathcal{Z}}{\partial x_i} = f_i - \varphi g_i \leq 0 \\
\text{(ii)} &: x_i \geq 0 \\
\text{(iii)} &: x_i \frac{\partial \mathcal{Z}}{\partial x_i} = 0 \ , \quad i = 1, 2, ...n \\
\text{(iv)} &: \frac{\partial \mathcal{Z}}{\partial \varphi} = [c - g(x)] \geq 0 \\
\text{(v)} &: \varphi \geq 0 \\
\text{(vi)} &: \varphi \frac{\partial \mathcal{Z}}{\partial \varphi} = 0 \ .
\end{aligned}
$$

The unequal treatment of different constraints is confusing. My method treats all constraints consistently. A non-negativity constraint is just like any other.

13.3 The general case

Let $f(x), g^j(x) \in C^1$, $x \in \mathbb{R}^n$, $j = 1, 2, ...m$. The problem is

$$\text{Choose } x \text{ to maximize } f(x) \text{, s.t. } g^j(x) \leq c^j \ , \ j = 1, 2, ...m \ .$$

Write the the problem in the standard way

$$\text{Choose } x \text{ to maximize } f(x) \text{, s.t. } g^j(x) - c^j \leq 0 \ , \ j = 1, 2, ...m \ .$$

Write down the Lagrangian function

$$\mathcal{L} = f(x) + \sum_{j=1}^{m} \lambda_j \left[c^j - g^j(x) \right] \ .$$

Suppose that x^* is the solution to the problem above and that x^* does not violate the constraint qualifications (see below). Then there exists m numbers λ_j^*, $j = 1, 2, ...m$ such that

$$
\begin{aligned}
(1) &: \frac{\partial \mathcal{L}}{\partial x_i} = f_i - \sum_{j=1}^{m} \lambda_j g_i^j(x) = 0 \ , \quad i = 1, 2, ...n \\
(2) &: \lambda_j \left[c^j - g^j(x) \right] = 0 \\
(3) &: \lambda_j \geq 0 \\
(4) &: g^j(x) \leq c^j \ , \quad j = 1, 2, ...m \ .
\end{aligned}
$$

- The **constraint qualifications** are as follows. Consider all the binding constraints. Count them by $j_b = 1, 2, ...m_b$. Then we must have that the rank of

$$\left[\frac{\partial g^B(x^*)}{\partial x'}\right] = \begin{bmatrix} \frac{\partial g^1(x^*)}{\partial x'} \\ \frac{\partial g^2(x^*)}{\partial x'} \\ \vdots \\ \frac{\partial g^{m_b}(x^*)}{\partial x'} \end{bmatrix}_{m_b \times n}$$

is m_b, as large as possible.

13.4 Minimization

It is worthwhile to consider minimization separately, although minimization of f is just like maximization of $-f$. We compare to maximization.

Let $f(x), g(x) \in C^1$, $x \in \mathbb{R}^n$. The problem is

$$\text{Choose } x \text{ to maximize } f(x), \text{ s.t. } g(x) \leq c\ .$$

Rewrite as

$$\text{Choose } x \text{ to maximize } f(x), \text{ s.t. } g(x) - c \leq 0$$

Write down the Lagrangian function

$$\mathcal{L} = f(x) + \lambda\left[c - g(x)\right]\ .$$

FONCs

$$\begin{aligned} (1) &: \frac{\partial \mathcal{L}}{\partial x_i} = f_i - \lambda g_i = 0\ , \quad i = 1, 2, ...n \\ (2) &: \lambda\left[c - g(x, y)\right] = 0 \\ (3) &: \lambda \geq 0 \\ (4) &: g(x) \leq c\ . \end{aligned}$$

Compare this to

$$\text{Choose } x \text{ to minimize } h(x), \text{ s.t. } g(x) \geq c\ .$$

Rewrite as

$$\text{Choose } x \text{ to minimize } h(x), \text{ s.t. } g(x) - c \geq 0$$

Write down the Lagrangian function

$$\mathcal{L} = h(x) + \lambda\left[c - g(x)\right]\ .$$

FONCs

$$
\begin{aligned}
(1) &: \frac{\partial \mathcal{L}}{\partial x_i} = h_i - \lambda g_i = 0 \ , \quad i = 1, 2, ...n \\
(2) &: \lambda \left[c - g\left(x, y\right) \right] = 0 \\
(3) &: \lambda \geq 0 \\
(4) &: g\left(x\right) \geq c \ .
\end{aligned}
$$

Everything is the same. Just pay attention to the inequality setup correctly. This will be equivalent.

Consider the problem

$$
\text{Choose } x \text{ to maximize } -h\left(x\right), \text{ s.t. } g\left(x\right) \geq c \ .
$$

Rewrite as

$$
\text{Choose } x \text{ to maximize } -h\left(x\right), \text{ s.t. } c - g\left(x\right) \leq 0 \ .
$$

and set up the proper Lagrangian function for maximization

$$
\mathcal{L} = -h\left(x\right) + \lambda \left[g\left(x\right) - c \right] \ .
$$

This will give the same FONCs as above.

13.5 Example

$$
\text{Choose } \{x, y\} \text{ to maximize } \min\{ax, by\}, \text{ s.t. } xp_x + yp_y \leq I \ ,
$$

where $a, b, p_x, p_y > 0$. Convert this to the following problem

$$
\text{Choose } \{x, y\} \text{ to maximize } ax, \text{ s.t. } ax \leq by, \ xp_x + yp_y - I \leq 0 \ .
$$

This is equivalent, because given a level of y, we will never choose $ax > by$, nor can the objective exceed by by construction.

$$
\text{Choose } \{x, y\} \text{ to maximize } ax, \text{ s.t. } ax - by \leq 0, \ xp_x + yp_y - I \leq 0 \ .
$$

Set up the Lagrangian

$$
\mathcal{L} = ax + \lambda \left[I - xp_x - yp_y \right] + \varphi \left[by - ax \right] \ .
$$

FONC:

$$
\begin{aligned}
1 &: \mathcal{L}_x = a - \lambda p_x - a\varphi = 0 \\
2 &: \mathcal{L}_y = -\lambda p_y + b\varphi = 0 \\
3 &: \lambda \left[I - xp_x - yp_y\right] = 0 \\
4 &: \lambda \geq 0 \\
5 &: xp_x + yp_y \leq I \\
6 &: \varphi \left[by - ax\right] = 0 \\
7 &: \varphi \geq 0 \\
8 &: ax \leq by\ .
\end{aligned}
$$

The solution process is a trial and error process. The best way is to start by checking which constraints do not bind.

1. Suppose $\varphi = 0$. Then from 2: $-\lambda p_y = 0 \;\Rightarrow\; \lambda = 0 \;\Rightarrow\;$ from 1: $a - a\varphi = 0 \;\Rightarrow\; \varphi = 1 > 0$ – a contradiction. Therefore $\boldsymbol{\varphi > 0}$ must hold. Then from 6: $ax = by \;\Rightarrow\; y = ax/b$.

2. Suppose $\lambda = 0$ (while $\varphi > 0$). Then from 2: $b\varphi = 0 \;\Rightarrow\; \varphi = 0$ – a contradiction (even if $\varphi = 0$, we would reach another contradiction from 1: $a = 0$). Therefore $\boldsymbol{\lambda > 0}$. Then $xp_x + yp_y = I$ $\Rightarrow\; xp_x + axp_y/b = I \;\Rightarrow\; x\left(p_x + ap_y/b\right) = I \;\Rightarrow\; x^* = I/\left(p_x + ap_y/b\right)$, $y^* = aI/\left(bp_x + ap_y\right)$.

Solving for the multipliers (which is an integral part of the solution) involves solving 1 and 2:

$$
\begin{aligned}
\lambda p_x + a\varphi &= a \\
\lambda p_y - b\varphi &= 0\ .
\end{aligned}
$$

This can be written in matrix notation

$$
\begin{bmatrix} p_x & a \\ p_y & -b \end{bmatrix} \begin{bmatrix} \lambda \\ \varphi \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}\ .
$$

The solution requires nonsingular matrix:

$$
\begin{vmatrix} p_x & a \\ p_y & -b \end{vmatrix} = -bp_x - ap_y < 0\ .
$$

Solving by Cramer's Rule:

$$\lambda^* = \frac{\begin{vmatrix} a & a \\ 0 & -b \end{vmatrix}}{-bp_x - ap_y} = \frac{ab}{bp_x + ap_y} > 0$$

$$\varphi^* = \frac{\begin{vmatrix} p_x & a \\ p_y & 0 \end{vmatrix}}{-bp_x - ap_y} = \frac{ap_y}{bp_x + ap_y} > 0 \ .$$

Try to build economic interpretations for the shadow costs:

$$\lambda^* = \frac{ab}{bp_x + ap_y} = \frac{a}{p_x + \frac{a}{b}p_y} \ .$$

λ^* tells you how many more utils you would get if income (I) increased by one unit. What do you do with this additional unit of income? You spend it on x and y—optimally: for each x you buy, you also buy a/b units of y. How much does this cost you? Exactly the denominator of λ^*. So you get $1/\left(p_x + \frac{a}{b}p_y\right)$ additional units of x for each unit of income (at the margin). And each one gives you an additional a utils.

Finally, we check the constraint qualifications. Since both constraints bind ($\lambda^* > 0$, $\varphi^* > 0$), we must have a rank of two for the matrix

$$\frac{\partial \begin{bmatrix} xp_x + yp_y - I \\ ax - by \end{bmatrix}}{\partial \begin{bmatrix} x & y \end{bmatrix}} = \begin{bmatrix} p_x & p_y \\ a & -b \end{bmatrix} \ .$$

In this case we can verify that the rank is two by the determinant, since this is a square 2×2 matrix:

$$\begin{vmatrix} p_x & p_y \\ a & -b \end{vmatrix} = -bp_x - ap_y < 0 \ .$$

It is no accident that the determinant is the same as above.

13.6 Another example

$$\text{Choose } \{x, y\} \text{ to maximize } x^2 + x + 4y^2, \text{ s.t. } 2x + 2y \le 1 \ , \ x, y \ge 0$$

Rewrite as

$$\text{Choose } \{x, y\} \text{ to maximize } x^2 + x + 4y^2, \text{ s.t. } 2x + 2y - 1 \le 0 \ , \ -x \le 0 \ , \ -y \le 0$$

Consider the Jacobian of the constraints

$$\frac{\partial \begin{bmatrix} 2x+2y \\ -x \\ -y \end{bmatrix}}{\partial \begin{bmatrix} x & y \end{bmatrix}} = \begin{bmatrix} 2 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} .$$

This has rank 2 for any submatrix $\forall (x, y) \in \mathbb{R}^2$, and since at most two constaraints can bind, the constraint qualifications are never violated. The constraint set is a triangle: all the constraints are linear and independent, so the constraint qualification will not fail.

Set up the Lagrangian function

$$\mathcal{L} = x^2 + x + 4y^2 + \lambda [1 - 2x - 2y] + \varphi [x] + \beta [y]$$

FONCs

$$\begin{aligned} \mathcal{L}_x &= 2x + 1 - 2\lambda + \varphi = 0 \\ \mathcal{L}_y &= 8y - 2\lambda + \beta = 0 \end{aligned}$$

$$\begin{array}{ccc} \lambda [1 - 2x - 2y] = 0 & \lambda \geq 0 & 2x + 2y \leq 1 \\ \varphi x = 0 & \varphi \geq 0 & x \geq 0 \\ \beta y = 0 & \beta \geq 0 & y \geq 0 \end{array}$$

1. From $\mathcal{L}_x = 0$ we have

   $$2x + 1 + \varphi = 2\lambda > 0$$

   with strict inequality, because $x \geq 0$ and $\varphi \geq 0$. Thus $\boldsymbol{\lambda > 0}$ and the constraint

   $$2x + 2y = 1$$

   binds, so that

   $$y = 1/2 - x \;\text{ or }\; x = 1/2 - y \; .$$

2. Suppose $\varphi > 0$. Then $x = 0 \;\Rightarrow\; y = 1/2 \;\Rightarrow\; \beta = 0 \;\Rightarrow\; \lambda = 2 \;\Rightarrow\; \varphi = 3$. A candidate solution is $(x^*, y^*) = (0, 1/2)$.

3. Suppose $\varphi = 0$. Then

   $$2x + 1 = 2\lambda \; .$$

   From $\mathcal{L}_y = 0$ we have

   $$8y + \beta = 2\lambda \; .$$

Combining the two we get

$$\begin{aligned} 2x+1 &= 8y+\beta \\ 2(1/2-y)+1 &= 8y+\beta \\ 2-2y &= 8y+\beta \\ 10y+\beta &= 2\ . \end{aligned}$$

The last result tells us that we cannot have both $\beta=0$ and $y=0$, because we would get $0=2$ – a contradiction (also because then we would get $\lambda=0$ from $\mathcal{L}_y=0$). So either $\beta=0$ or $y=0$ but not both.

(a) Suppose $y=0$. Then $x=1/2 \;\Rightarrow\; \lambda=1 \;\Rightarrow\; \beta=2$. A candidate solution is $(x^*,y^*)=(1/2,0)$.

(b) Suppose $y>0$. Then $\beta=0 \;\Rightarrow\; y=0.2 \;\Rightarrow\; x=0.3 \;\Rightarrow\; \lambda=0.8$. A candidate solution is $(x^*,y^*)=(0.3,0.2)$.

Eventually, we need to evaluate the objective function with each candidate to see which is the global maximizer.

13.7 The Kuhn-Tucker sufficiency theorem

Let $f(x), g^j(x) \in C^1$, $j=1,2,...m$. The problem is

$$\begin{aligned} \text{Choose } x &\in \mathbb{R}^n \text{ to maximize } f(x)\,, \\ \text{s.t. } x &\geq 0 \text{ and } g^j(x) \leq c^j\ ,\ j=1,2,...m\ . \end{aligned}$$

Theorem: if

1. f is concave on $\mathbb{R}^n$,
2. g^j are convex on $\mathbb{R}^n$,
3. x^* satisfies the FONCs of the Lagrangian

then x^* is a global maximum—not necessarily unique.

- We know: if $g^j(x)$ are convex then $\left\{x : g^j(x) \leq c^j\right\}$ are convex sets. One can show that the intersection of convex sets is also a convex set, so that the constraint set is also convex. The theorem says that trying to maximize a concave function on a convex set give a global maximum, if it exists. Whether it exists on the border or not, the FONCs will detect it.

- Also note that if f is concave, then the set $\{x : f(x) \geq k\}$ is convex. In particular, the set $\{x : f(x) \geq f(x^*)\}$ is also convex: the upper contour set is convex.

But these are strong conditions on our objective and constraint functions. The next theorem relaxes things quite a bit.

13.8 The Arrow-Enthoven sufficiency theorem

Let $f(x), g^j(x) \in C^1$, $j = 1, 2, ...m$. The problem is

$$\begin{aligned} \text{Choose } x &\in \mathbb{R}^n \text{ to maximize } f(x), \\ \text{s.t. } x &\geq 0 \text{ and } g^j(x) \leq c^j \ , \ j = 1, 2, ...m \ . \end{aligned}$$

Theorem: If

1. f is *quasi*concave on $\mathbb{R}^n_+$,
2. g^j are *quasi*convex on $\mathbb{R}^n_+$,
3. x^* satisfies the FONCs of the Kuhn-Tucker Lagrangian,
4. Any one of the following conditions on f holds:
   (a) $\exists i$ such that $f_i(x^*) < 0$.
   (b) $\exists i$ such that $f_i(x^*) > 0$ and $x_i^* > 0$ ($x_i \geq 0$ does not bind in the i^{th} dimension).
   (c) $\nabla f(x^*) \neq 0$ and $f \in C^2$ around x^*.
   (d) $f(x)$ is concave.

then x^* is a global maximum, not necessarily unique.

Arrow-Enthoven constraint qualification test for a maximization problem: If

1. $g^j(x) \in C^1$ are *quasi*convex,
2. $\exists x_0 \in \mathbb{R}^n_+$ such that all constraints are slack,
3. Any one of the following holds:
   (a) $g^j(x)$ are convex.
   (b) $\partial g(x) / \partial x' \neq 0 \ \forall x$ in the constraint set.

then the constraint qualification is not violated.

13.9 Envelope theorem for constrained optimization

Recall the envelope theorem for unconstrained optimization: if x^* is a critical point of $f(x,\theta)$. Then

$$\frac{df(x^*,\theta)}{d\theta} = \frac{\partial f(x^*,\theta)}{\partial\theta} .$$

This was due to $\frac{\partial f(x^*,\theta)}{\partial x} = 0$.

Now we face a more complicated problem:

$$\text{Choose } x \in \mathbb{R}^n \text{ to maximize } f(x,\theta), \text{ s.t. } g(x,\theta) = c .$$

For a problem with inequality constraints we simply use only those constraints that bind. We will consider small perturbations of θ, so small that they will not affect which constraint binds. Set up the Lagrangian function

$$\mathcal{L} = f(x,\theta) + \lambda[c - g(x,\theta)] .$$

FONCs

$$\begin{aligned} \mathcal{L}_\lambda &= c - g(x,\theta) = 0 \\ \mathcal{L}_{x_i} &= f_i(x,\theta) - \lambda g_i(x,\theta) = 0 , \quad i = 1,2,...n \end{aligned}$$

We apply the implicit function theorem to this set of equations to get $\exists x^*(\theta)$ and $\lambda^*(\theta)$ for which there well defined derivatives around (λ^*, x^*). We know that at the optimum we have that the value of the problem is the value of the Lagrangian function

$$\begin{aligned} f(x^*,\theta) &= \mathcal{L}^* = f(x^*,\theta) + \lambda^*[c - g(x^*,\theta)] \\ &= f(x^*(\theta),\theta) + \lambda^*(\theta)[c - g(x^*(\theta),\theta)] . \end{aligned}$$

Define the value of the problem as

$$v(\theta) = f(x^*,\theta) = f(x^*(\theta),\theta) .$$

Take the derivative with respect to θ to get

$$\begin{aligned} \frac{dv(\theta)}{d\theta} &= \frac{d\mathcal{L}^*}{d\theta} = f_x^*\frac{dx^*}{d\theta} + f_\theta^* + \frac{d\lambda^*}{d\theta}[c - g(x^*(\theta),\theta)] - \lambda^*\left[g_x^*\frac{dx^*}{d\theta} + g_\theta^*\right] \\ &= [f_x^* - \lambda^* g_x^*]\frac{dx^*}{d\theta} + \frac{d\lambda^*}{d\theta}[c - g(x^*(\theta),\theta)] + f_\theta^* - \lambda^* g_\theta^* \\ &= f_\theta^* - \lambda^* g_\theta^* . \end{aligned}$$

Of course, we could have just applied this directly using the envelope theorem:

$$\frac{dv(\theta)}{d\theta} = \frac{d\mathcal{L}^*}{d\theta} = \frac{\partial \mathcal{L}^*}{\partial \theta} = f_\theta^* - \lambda g_\theta^* \ .$$

13.10 Duality

We will demonstrate the duality of utility maximization and cost minimization. But the principles here are more general than consumer theory.

The **primal problem** is

$$\text{Choose } x \in \mathbb{R}^n \text{ to maximize } u(x) \text{, s.t. } p'x = I \ .$$

(this should be stated with $\leq$ but we focus on preferences with nonsatiation and strict convexity—and therefore strict concavity of u—so the solution lies on the budget line and $x > 0$ is also satisfied). The Lagrangian function is

$$\mathcal{L} = u(x) + \lambda \left[I - p'x \right]$$

FONCs:

$$\begin{aligned} \mathcal{L}_{x_i} &= u_i - \lambda p_i = 0 \quad \Rightarrow \quad \lambda = u_i / p_i \ , \ i = 1, ...n \\ \mathcal{L}_\lambda &= \left[I - p'x \right] = 0 \end{aligned}$$

Recall: λ tells you how many utils we get for one additional unit of income.

Apply the implicit function theorem to this set of equations to get **Marshallian demand**

$$x_i^m = x_i^m(p, I)$$

and

$$\lambda^m = \lambda^m(p, I)$$

for which there are well defined derivatives around (λ^*, x^*). Define the **indirect utility function**

$$v(p, I) = u\left[x^m(p, I)\right] \ .$$

The **dual problem** is

$$\text{Choose } x \in \mathbb{R}^n \text{ to minimize } p'x \text{ s.t. } u(x) = u \ ,$$

where u is a level of promised utility (this should be stated with $u(x) \geq u$ but we assume that u is strictly increasing in x, and since the objective is also strictly increasing in x, the solution must lie at $u(x) = u$). The Lagrangian function is

$$Z = p'x + \varphi [u - u(x)] \ .$$

FONCs:

$$\begin{aligned} Z_{x_i} &= p_i - \varphi u_i = 0 \quad \Rightarrow \quad \varphi = p_i/u_i \ , \ i = 1, ...n \\ Z_{\varphi} &= u - (x) = 0 \end{aligned}$$

Recall: φ tells you how much an additional util will cost.

Apply the implicit function theorem to this set of equations to get **Hicksian demand**

$$x_i^h = x_i^h(p, u)$$

and

$$\varphi^h = \varphi^h(p, u)$$

for which there are well defined derivatives around (φ^*, x^*). Define the **expenditure function**

$$e(p, u) = p'x^h(p, u) \ .$$

Duality: all FONCs imply the same thing:

$$\frac{u_i}{u_j} = \frac{p_i}{p_j} \ ,$$

Thus, **at the optimum**

$$\begin{aligned} x_i^m(p, I) &= x_i^h(p, u) \\ e(p, u) &= I \\ v(p, I) &= u \ . \end{aligned}$$

Moreover,

$$\varphi = \frac{1}{\lambda}$$

and this makes sense given the interpretation of φ and λ.

- Duality relies on **unique global extrema**. We need to have all the preconditions for that.
- Make drawing.

13.11 Roy's identity

$$v(p,I) = u(x^m) + \lambda^m \left(I - p'x^m\right) \ .$$

Taking the derivative with respect to a price,

$$\begin{aligned}
\frac{\partial v}{\partial p_i} &= \sum_{j=1}^{n} u_j \frac{\partial x_j^m}{\partial p_i} + \frac{\partial \lambda}{\partial p_i}\left(I - p'x^m\right) - \lambda \left[\sum_{j=1}^{n} p_j \frac{\partial x_j^m}{\partial p_i} + x_i^m\right] \\
&= \sum_{j=1}^{n} \left(u_j - \lambda p_j\right) \frac{\partial x_j^m}{\partial p_i} + \frac{\partial \lambda}{\partial p_i}\left(I - p'x^m\right) - \lambda x_i^m \\
&= -\lambda x_i^m \ .
\end{aligned}$$

An increase in p_i will lower demand by x_i^m, which decreases the value of the problem, as if by decreasing income by x_i^m times λ utils/\$ per dollar of lost income. In other words, income is now worth x_i^m less, and this taxes the objective by λx_i^m. Taking the derivative with respect to income,

$$\begin{aligned}
\frac{\partial v}{\partial I} &= \sum_{j=1}^{n} u_j \frac{\partial x_j^m}{\partial I} + \frac{\partial \lambda}{\partial I}\left(I - p'x^m\right) + \lambda \left[1 - \sum_{j=1}^{n} p_j \frac{\partial x_j^m}{\partial I}\right] \\
&= \sum_{j=1}^{n} \left(u_j - \lambda p_j\right) \frac{\partial x_j^m}{\partial I} + \frac{\partial \lambda}{\partial I}\left(I - p'x^m\right) + \lambda \\
&= \lambda \ .
\end{aligned}$$

An increase in income will increase our utility by λ, which is the standard result.

- In fact, we could get these results applying the envelope theorem directly:

$$\begin{aligned}
\frac{\partial v}{\partial p_i} &= -\lambda x_i^m \\
\frac{\partial v}{\partial I} &= \lambda \ .
\end{aligned}$$

Roy's identity is thus

$$-\frac{\partial v / \partial p_i}{\partial v / \partial I} = x_i^m \ .$$

Why is this interesting? Because this is the amount of income needed to compensate consumers for (that will leave them indifferent to) an increase in the price of some good x_i. To see this, first consider

$$v(p,I) = u \ ,$$

where u is a level of promised utility (as in the dual problem). By the implicit function theorem $\exists I(p_i)$ **in a neighborhood of** x^m, which has a well defined derivative dI/dp. This function is defined at the optimal bundle x^m. Now consider the total differential of v, evaluated at the optimal bundle x^m:

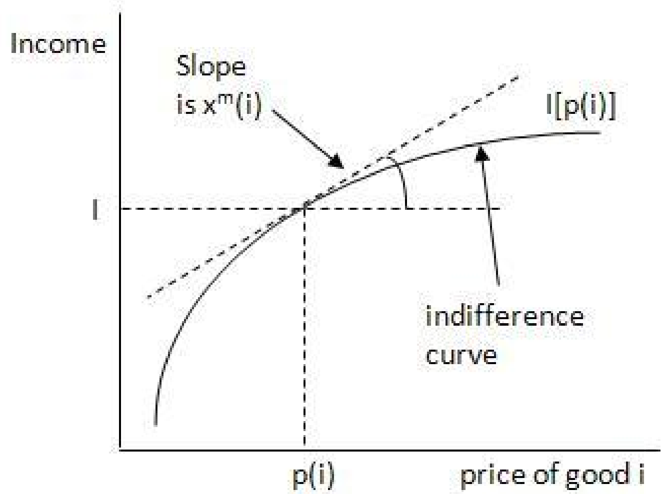
$$v_{p_i} dp_i + v_I dI = 0 \ .$$

This differential does not involve other partial derivatives because it is evaluated at the the optimal bundle x^m (i.e. the envelope theorem once again). And we set this differential to zero, because we are considering keeping the consumer exactly indifferent, i.e. her promised utility and optimal bundle remain unchanged. Then we have

$$\frac{dI}{dp_i} = -\frac{v_{p_i}}{v_I} = -\frac{\partial v/\partial p_i}{\partial v/\partial I} = x_i^m \ .$$

This result tells you that if you are to keep the consumer indifferent to a small change in the price of good i, i.e. not changing the optimally chosen bundle, then you must compensate the consumer by x_i^m units of income. We will see this again in the dual problem, using Shephard's lemma, where keeping utility fixed is explicit. We will see that $\frac{\partial e}{\partial p_i} = x_i^h = x_i^m$ is exactly the change in expenditure that results from keeping utility fixed, while increasing the price of good i.

To see this graphically, consider a level curve of utility. The slope of the curve at (p, I) (more generally, the gradient) is x^m.



Roy's Identity

13.12 Shephard's lemma

$$e\,(p, u) = p' x^h + \varphi^h \left[u - u\left(x^h \right) \right] \ .$$

Taking the derivative with respect to a price,

$$\begin{aligned}
\frac{\partial e}{\partial p_i} &= x_i^h + \sum_{j=1}^{n} p_j \frac{\partial x_j^p}{\partial p_i} + \frac{\partial \varphi}{\partial p_i}\left[u - u\left(x^h\right)\right] - \varphi \sum_{j=1}^{n} u_j \frac{\partial x_j^h}{\partial p_i} \\
&= \sum_{j=1}^{n} \left(p_j - \varphi u_j\right) \frac{\partial x_j^h}{\partial p_i} + \frac{\partial \varphi}{\partial p_i}\left[u - u\left(x^h\right)\right] + x_i^h \\
&= x_i^h \ .
\end{aligned}$$

An increase in p_i will increases cost by x_i^h while keeping utility fixed at u (remember that this is a minimization problem so increasing the value of the problem is "bad"). Note that this is exactly the result of Roy's Identity. Taking the derivative with respect to promised utility,

$$\begin{aligned}
\frac{\partial e}{\partial u} &= \sum_{j=1}^{n} p_j \frac{\partial x_j^h}{\partial u} + \frac{\partial \varphi}{\partial u}\left[u - u\left(x^h\right)\right] + \varphi\left[1 - \sum_{j=1}^{n} u_j \frac{\partial x_j^h}{\partial u}\right] \\
&= \sum_{j=1}^{n} \left(p_j - \varphi u_j\right) \frac{\partial x_j^h}{\partial u} + \frac{\partial \varphi}{\partial u}\left[u - u\left(x^h\right)\right] + \varphi \\
&= \varphi \ .
\end{aligned}$$

An increase in utility will increase expenditures by φ, which is the standard result.

- In fact, we could get these results applying the envelope theorem directly:

$$\begin{aligned}
\frac{\partial e}{\partial p_i} &= x_i^h \\
\frac{\partial e}{\partial u} &= \varphi \ .
\end{aligned}$$

This is used often with cost functions in the context of production. Let e be the lowest cost to produce u units of output (with $u(x)$ serving as the production function that takes the vector of inputs x and where p are their prices). Then taking the derivative of the cost function e w.r.t. p gives you demand for inputs. And taking the derivative of the cost function e w.r.t. the quantity produced (u) gives you the cost (price) of producing one additional unit of output.

13.13 Mundlak (1968) REStud example

The following is based on Mundlak (1968).

Let $y = f(x) \in C^2$, $x \in \mathbb{R}^n$ be the strictly concave production function of a competitive firm.

The firm solves the cost minimization problem

$$\begin{aligned} \text{choose } x \text{ to minimize } c &= p'x \ , \\ \text{s.t.} \quad f(x) &\geq y \ . \end{aligned}$$

Setting y as numeraire normalizes $p_y = 1$, so that the factor prices are interpreted as real prices, in terms of units of output. The Lagrangian can be written as (with appropriate interpretation of the multiplier λ):

$$\mathcal{L} = p'x + \frac{1}{\lambda}\left[y - f(x)\right]$$

If (x^*, λ^*) is a critical point then it satisfies the FONCs

$$\begin{aligned} f_i(x^*) &= \lambda^* p_i \quad , \quad i = 1, 2, ...n \\ f(x^*) &= y \ . \end{aligned}$$

The values of x and λ are optimal, so that small perturbations of prices will have no indirect effect on the objective through them (the envelope theorem). To ease notation I will not carry on the asterisks in what follows.

We treat the FONCs as identities and differentiate around the optimal choice

$$\begin{aligned} \sum_j f_{ij} dx_j &= d\lambda p_i + \lambda dp_i \\ &= \frac{d\lambda}{\lambda} \lambda p_i + \lambda dp_i \\ &= \widehat{\lambda} f_i + \lambda dp_i \ \text{ for } \ i = 1, 2, ...n \ , \end{aligned}$$

where $\widehat{\lambda} = d\lambda / \lambda = d \ln \lambda$. Rearranging gives the endogenous perturbations of the optimal values induced by perturbations of the exogenous prices

$$\sum_j f_{ij} dx_j - \widehat{\lambda} f_i = \lambda dp_i \ \text{ for } \ i = 1, 2, ...n$$

and in matrix notation

$$H dx - \nabla f \widehat{\lambda} = \lambda \left[dp\right] \ .$$

I am writing the gradient as a *column* vector, rather than the conventional *row* vector of partials. This can be written as

$$\begin{bmatrix} \nabla f & H \end{bmatrix} \begin{bmatrix} -\widehat{\lambda} \\ dx \end{bmatrix} = \lambda \left[dp\right] \ .$$

The other differential is

$$dy = \sum_i f_i dx_i = \nabla f' dx \ .$$

Define[1]

$$dz \equiv \frac{1}{\lambda} dy \ ,$$

which is how tightening the constraint affects the objective (value of the problem). Stacking this on top of the previous matrices gives

$$\begin{bmatrix} 0 & \nabla f' \\ \nabla f & H \end{bmatrix} \begin{bmatrix} -\widehat{\lambda} \\ dx \end{bmatrix} = \lambda \begin{bmatrix} dz \\ dp \end{bmatrix} \ .$$

The first matrix on the LHS is just the bordered Hessian of the problem. Given regularity conditions (and conditions for a minimum), the bordered Hessian is invertible so we can write

$$\begin{bmatrix} -\widehat{\lambda} \\ dx \end{bmatrix} = \lambda \begin{bmatrix} 0 & \nabla f' \\ \nabla f & H \end{bmatrix}^{-1} \begin{bmatrix} dz \\ dp \end{bmatrix} \ .$$

Mundlak defines

$$\begin{aligned} K &= \begin{bmatrix} K_{00} & K_{0j} \\ K_{i0} & K_{ij} \end{bmatrix} \equiv \lambda \begin{bmatrix} 0 & \nabla f' \\ \nabla f & H \end{bmatrix}^{-1} \\ &= \lambda \begin{bmatrix} -\left(\nabla f' H^{-1} \nabla f\right)^{-1} & \left(\nabla f' H^{-1} \nabla f\right)^{-1} f_i' f_{ij}^{-1} \\ H^{-1} \nabla f \left(\nabla f' H^{-1} \nabla f\right)^{-1} & H^{-1} - H^{-1} \nabla f \left(\nabla f' H^{-1} \nabla f\right)^{-1} \nabla f' H^{-1} \end{bmatrix} , \end{aligned}$$

(see Section 5.7 for inverting partitioned matrices) so that

$$\begin{bmatrix} -\widehat{\lambda} \\ dx \end{bmatrix} = \begin{bmatrix} K_{00} & K_{0j} \\ K_{i0} & K_{ij} \end{bmatrix} \begin{bmatrix} dz \\ dp \end{bmatrix} \ . \tag{8}$$

Furthermore,

$$\begin{aligned} \nabla f' K_{i0} &= \lambda \\ \nabla f' K_{ij} &= 0 \\ K_{ij} \nabla f &= 0 \ . \end{aligned}$$

I.e. K_{ij} is a singular matrix. Finally, it is useful to write (8) without its first row:

$$\begin{aligned} dx &= \left[K_{ij}\right] dp + \left[K_{i0}\right] dz \\ \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix} &= \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & & k_{2n} \\ \vdots & & \ddots & \vdots \\ k_{n1} & \cdots & & k_{nn} \end{bmatrix} \begin{bmatrix} dp_1 \\ dp_2 \\ \vdots \\ dp_n \end{bmatrix} + \begin{bmatrix} k_{10} \\ k_{20} \\ \vdots \\ k_{n0} \end{bmatrix} dz \end{aligned}$$

[1] In other optimization problems, e.g. constant cost and constant marginal cost, dz will take a different form.

Along the output isoquant ($dy = 0 \Rightarrow dz = 0$) we have

$$\frac{\partial x_i}{\partial p_j} = k_{ij} \ . \tag{9}$$

Since the objective is $c = x'p$, the envelope theorem gives (on the output isoquant) (Shephard's lemma):

$$\frac{\partial c}{\partial p_j} = x_j \ . \tag{10}$$

Writing this as an elasticity gives

$$\frac{\partial \ln c}{\partial \ln p_j} = \frac{\partial c}{\partial p_j}\frac{p_j}{c} = \frac{p_j x_j}{c} = v_j \ , \tag{11}$$

which is the cost share.

Remark 1 *The cost elasticity w.r.t. the price of a particular input is the cost share of that input. A corollary is that the sum of the elasticities of cost w.r.t. all prices is unity.*

Now writing (9) as a price elasticity, we have

$$\eta_{ij} \equiv \frac{\partial \ln x_i}{\partial \ln p_j} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i} = k_{ij}\frac{p_j}{x_i} = k_{ij}\frac{c}{x_i x_j}\frac{p_j x_j}{c} = \sigma_{ij} v_j \ , \tag{12}$$

where k_{ij} is the i-j element of $[K_{ij}]$, σ_{ij} is the (Allen partial) elasticity of substitution and v_j is defined in (11).

Remark 2 *The elasticity of input i w.r.t. the price of input j is equal to the (Allen partial) elasticity of substitution of input i w.r.t. input j times the cost share of input j.*

Note that for a particular input i

$$\sum_j \eta_{ij} = \sum_j k_{ij}\frac{p_j}{x_i} = \frac{1}{\lambda x_i}\sum_j \lambda p_j k_{ij} = \frac{1}{\lambda x_i}\sum_j f_j k_{ij} = 0 \ , \tag{13}$$

since the summation is just one column of $\nabla f' K_{ij} = 0$.

Remark 3 *The sum of price elasticities (12) for any particular input i is zero.*

From (13) we have

$$\sum_j \eta_{ij} = \sum_j \sigma_{ij} v_j = \sigma_{ii} v_i + \sum_{j \neq i} \sigma_{ij} v_j = 0$$

and therefore

$$\sigma_{ii} = -\frac{1}{v_i}\sum_{j \neq i} \sigma_{ij} v_j < 0 \ .$$

This result holds for any factor i that is actually used in production ($v_i > 0$, i.e. no corner solution for i) under the following conditions: there is at least one factor $j \neq i$ that exhibits some substitutability with i, and which is actually used in production, i.e. $\exists j \neq i$ s.t. $\sigma_{ij} > 0$ and $v_j > 0$ ($v_j \geq 0$ and $\sigma_{ij} \geq 0$ for all j for concave CRS production functions). While the object σ_{ii} does not have a clear interpretation (at least not to me), it helps sign the own-price elasticity:

$$\eta_{ii} = \sigma_{ii} v_i = -\sum_{j \neq i} \sigma_{ij} v_j < 0 \ .$$

Remark 4 *The own-price elasticity is negative.*

For example, Allen (1938) shows (page 342) that in the case of two inputs

$$\sigma_{12} = \frac{f_1 f_2 \left(x_1 f_1 + x_2 f_2\right)}{-x_1 x_2 \left(f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2\right)} = \frac{f_1 f_2}{-\left(f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2\right)} \frac{\left(x_1 f_1 + x_2 f_2\right)}{x_1 x_2} \ .$$

Substituting the FONCs, we have

$$\begin{aligned} \sigma_{12} &= \frac{f_1 f_2}{-\left(f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2\right)} \frac{\left(x_1 \lambda p_1 + x_2 \lambda p_2\right)}{x_1 x_2} \\ &= \underbrace{\lambda \frac{f_1 f_2}{-\left(f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2\right)}}_{k_{12}} \frac{\left(x_1 p_1 + x_2 p_2\right)}{x_1 x_2} = k_{12} \frac{c}{x_1 x_2} \ . \end{aligned}$$

So that

$$\eta_{12} = \sigma_{12} v_2 = k_{12} \frac{c}{x_1 x_2} \frac{x_2 p_2}{c} = k_{12} \frac{p_2}{x_1}$$

as above.

References

Allen, R. G. D. (1938): *Mathematical Analysis for Economists.* Macmillan, London, First edition, 1967 reprint.

Mundlak, Y. (1968): "Elasticities of Substitution and the Theory of Derived Demand," *The Review of Economic Studies*, 53(2), 225–236.

14 Integration

14.1 Preliminaries

Consider a continuous differentiable function

$$x = x(t)$$

and its derivative with respect to time

$$\frac{dx}{dt} \equiv \dot{x} \ .$$

This is how much x changes during a very short period dt. Suppose that you know $\dot{x}$ at any point in time. We can write down how much x changed from some initial point in time, say $t = 0$, until period t as follows:

$$\int_0^t \dot{x} dt \ .$$

This is the sum of all changes in x from period 0 to t. The term of art is **integration**, i.e. we are integrating all the increments. But you cannot say what $x(t)$ is, unless you have the value of x at the initial point. This is the same as saying that you know what the growth rate of GDP is, but you do not know the level. But given $x_0 = x(0)$ we can tell what $x(t)$ is:

$$x(t) = x_0 + \int_0^t \dot{x} dt \ .$$

E.g.

$$\begin{aligned} \dot{x} &= t^2 \\ \int_0^t \dot{x} dt &= \int_0^t u^2 du = \frac{1}{3} t^3 + c \ . \end{aligned}$$

The constant c is arbitrary and captures the fact that we do not know the level.

Suppose that the instant growth rate of y is a constant r, i.e.

$$\frac{\dot{y}}{y} = r \ .$$

This can be written as

$$\dot{y} - ry = 0 \ ,$$

which is an **ordinary differential equation**. We know that $y = e^{rt}$ gives $\dot{y}/y = r$. But so does $y = ke^{rt}$. So once again, without having additional information about the value of y at some initial point, we cannot say what $y(t)$ is.

14.2 Indefinite integrals

Denote

$$f(x) = \frac{dF(x)}{dx} \ .$$

Therefore,

$$dF(x) = f(x)\,dx \ .$$

Summing over all small increments we get

$$\int dF(x) = \int f(x)\,dx = F(x) + c \ ,$$

where the **constant of integration**, c, denotes that the integral is correct up to an indeterminate constant. This is so because knowing the sum of increments does not tell you the level. Another way to see this is

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left[F(x) + c\right] \ .$$

Integration is the opposite operation of differentiation. Instead of looking at small perturbations, or increments, we look for the sum of all increments.

Commonly used integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
2. $\int f'(x)\,e^{f(x)}dx = e^{f(x)} + c$, $\int e^x dx = e^x + c$, $\int f'(x)\,b^{f(x)}dx = \frac{b^{f(x)}}{\ln b} + c$
3. $\int \frac{f'(x)}{f(x)}dx = \ln\left[f(x)\right] + c$, $\int \frac{1}{x}dx = \int \frac{dx}{x} = \ln x + c$

Operation rules

1. Sum: $\int \left[f(x) + g(x)\right]dx = \int f(x)\,dx + \int g(x)\,dx$
2. Scalar multiplication: $k\int f(x)\,dx = \int kf(x)\,dx$
3. Substitution/change of variables: Let $u = u(x)$. Then

$$\int f(u)\,u'dx = \int f(u)\,\frac{du}{dx}dx = \int f(u)\,du = F(u) + c \ .$$

E.g.

$$\int 2x\left(x^2+1\right)dx = 2\int \left(x^3 + x\right)dx = 2\int x^3 dx + 2\int x dx = \frac{1}{2}x^4 + x^2 + c$$

Alternatively, define $u = x^2+1$, hence $u' = 2x$, and so

$$\begin{aligned}
\int 2x\left(x^2+1\right)dx &= \int \frac{du}{dx}udx = \int udu = \frac{1}{2}u^2 + c' \\
&= \frac{1}{2}\left(x^2+1\right)^2 + c' = \frac{1}{2}\left(x^4+2x^2+1\right) + c' \\
&= \frac{1}{2}x^4 + x^2 + \frac{1}{2} + c' = \frac{1}{2}x^4 + x^2 + c\ .
\end{aligned}$$

4. Integration by parts: Since

$$d\left(uv\right) = udv + vdu$$

we have

$$\int d\left(uv\right) = uv = \int udv + \int vdu\ .$$

Thus the integration by part formula is

$$\int udv = uv - \int vdu\ .$$

To reduce confusion denote

$$\begin{aligned}
V &= V\left(x\right)\ ,\quad v\left(x\right) = dV\left(x\right)/dx \\
U &= U\left(x\right)\ ,\quad u\left(x\right) = dU\left(x\right)/dx
\end{aligned}$$

Then we write the formula as

$$\begin{aligned}
\int U\left(x\right)dV\left(x\right) &= U\left(x\right)V\left(x\right) - \int V\left(x\right)dU\left(x\right) \\
\int U\left(x\right)v\left(x\right)dx &= U\left(x\right)V\left(x\right) - \int u\left(x\right)V\left(x\right)dx\ .
\end{aligned}$$

E.g., let $f\left(x\right) = \varphi e^{-\varphi x}$. Then

$$\int x\varphi e^{-\varphi x}dx = -xe^{-\varphi x} - \int -e^{-\varphi x}dx$$

In the notation above, we have

$$\int \underbrace{x}_{U}\cdot\underbrace{\varphi e^{-\varphi x}}_{v}dx = \underbrace{x}_{U}\cdot\underbrace{-e^{-\varphi x}}_{V} - \int \underbrace{1}_{u}\cdot\underbrace{\left(-e^{-\varphi x}\right)}_{V}dx$$

14.3 Definite integrals

The area under the f curve for a continuous f on $[a, b]$, i.e. between the f curve and the horizontal axis, from a to b is

$$\int_a^b f(x)\,dx = F(b) - F(a) \ .$$

This is also called the **fundamental theorem of calculus**. Note that this area may be positive or negative, depending on whether the area lies more above the horizontal axis or below it.

The **Riemann Integral**: create n rectangles that lie under the curve, that take the minimum of the heights: r_i, $i = 1, 2...n$. Then create n rectangles with height the maximum of the heights: R_i, $i = 1, 2...n$. As the number of these rectangles increases, the sums of the rectangles may converge. If they do, then we say that f is **Reimann-integrable**. I.e. if

$$\lim_{n\to\infty} \sum_{i=1}^{n} r_i = \lim_{n\to\infty} \sum_{i=1}^{n} R_i$$

then

$$\int_a^b f(x)\,dx$$

exists and is well defined.

Properties of definite integrals:

1. Minus/switching the integration limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx = F(b) - F(a) = -[F(a) - F(b)]$
2. Zero: $\int_a^a f(x)\,dx = F(a) - F(a) = 0$
3. Partition: for all $a < b < c$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \ .$$

4. Scalar multiplication: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, $\forall k \in \mathbb{R}$
5. Sum: $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
6. By parts: $\int_a^b U(x)\,v(x)\,dx = U(x)\,V(x)\,|_a^b - \int_a^b u(x)\,V(x)\,dx = U(b)\,V(b) - U(a)\,V(b) - \int_a^b u(x)\,V(x)\,dx$
7. Substitution/change of variables: Let $u = u(x)$. Then

$$\int_a^b f(u)\,u' dx = \int_a^b f(u)\,\frac{du}{dx} dx = \int_{u(a)}^{u(b)} f(u)\,du = F(u) + c \ .$$

Suppose that we wish to integrate a function from some initial point x_0 until some indefinite point x. Then

$$\int_{x_0}^{x} f(t)\,dt = F(x) - F(x_0) \ .$$

and so

$$F(x) = F(x_0) + \int_{x_0}^{x} f(t)\,dx \ .$$

14.4 Leibnitz's Rule

Let $f \in C^1$ (i.e. $F \in C^2$). Then

$$\frac{\partial}{\partial\theta}\int\limits_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx = f(b(\theta),\theta)\frac{\partial b(\theta)}{\partial\theta} - f(a(\theta),\theta)\frac{\partial a(\theta)}{\partial\theta} + \int\limits_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta} f(x,\theta)\,dx \ .$$

Proof: let $f(x,\theta) = dF(x,\theta)/dx$. Then

$$\begin{aligned}
\frac{\partial}{\partial\theta}\int\limits_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx &= \frac{\partial}{\partial\theta}\left[F(x,\theta)\right|_{a(\theta)}^{b(\theta)} \\
&= \frac{\partial}{\partial\theta}\left[F(b(\theta),\theta) - F(a(\theta),\theta)\right] \\
&= F_x(b(\theta),\theta)\frac{\partial b(\theta)}{\partial\theta} + F_\theta(b(\theta),\theta) - F_x(a(\theta),\theta)\frac{\partial a(\theta)}{\partial\theta} - F_\theta(a(\theta),\theta) \\
&= f(b(\theta),\theta)\frac{\partial b(\theta)}{\partial\theta} - f(a(\theta),\theta)\frac{\partial a(\theta)}{\partial\theta} + \left[F_\theta(b(\theta),\theta) - F_\theta(a(\theta),\theta)\right] \\
&= f(b(\theta),\theta)\frac{\partial b(\theta)}{\partial\theta} - f(a(\theta),\theta)\frac{\partial a(\theta)}{\partial\theta} + \int\limits_{a(\theta)}^{b(\theta)} \frac{d}{dx}F_\theta(x,\theta)\,dx \\
&= f(b(\theta),\theta)\frac{\partial b(\theta)}{\partial\theta} - f(a(\theta),\theta)\frac{\partial a(\theta)}{\partial\theta} + \int\limits_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta} f(x,\theta)\,dx \ .
\end{aligned}$$

The last line follows from **Young's Theorem**: for a continuously differentiable F,

$$\frac{\partial^2 F(x,y)}{\partial x\partial y} = \frac{\partial}{\partial x}\frac{\partial F(x,y)}{\partial y} = \frac{\partial}{\partial y}\frac{\partial F(x,y)}{\partial x} = \frac{\partial^2 F(x,y)}{\partial y\partial x} \ .$$

If the integration limits do not depend on θ, then

$$\frac{\partial}{\partial\theta}\int\limits_{a}^{b} f(x,\theta)\,dx = \int\limits_{a}^{b} \frac{\partial}{\partial\theta} f(x,\theta)\,dx \ ,$$

and if f does not depend on θ, then

$$\frac{\partial}{\partial\theta}\int\limits_{a(\theta)}^{b(\theta)} f(x)\,dx = f(b(\theta))\frac{\partial b(\theta)}{\partial\theta} - f(a(\theta))\frac{\partial a(\theta)}{\partial\theta} \ .$$

14.5 Improper integrals

14.5.1 Infinite integration limits

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx = \lim_{b\to\infty} F(b) - F(a) \ .$$

E.g., $X \sim$exp(φ): $F(x) = 1 - e^{-\varphi x}$, $f(x) = \varphi e^{-\varphi x}$ for $x \geq 0$.

$$\int_0^\infty \varphi e^{-\varphi x} dx = \lim_{b\to\infty}\int_0^b \varphi e^{-\varphi x} dx = \lim_{b\to\infty} - e^{-\varphi b} + e^{-\varphi 0} = 1 \ .$$

Also

$$\begin{aligned} E(x) &= \int_0^\infty x f(x)\, x = \int_0^\infty x\varphi e^{-\varphi x} dx = \left[-xe^{-\varphi x}\right|_0^\infty - \int_0^\infty -e^{-\varphi x} dx \\ &= \text{"} - \infty e^{-\varphi\infty}\text{"} + 0e^{-\varphi 0} + \left[-\frac{1}{\varphi}e^{-\varphi x}\right|_0^\infty = 0 - \frac{1}{\varphi}e^{-\varphi\infty} + \frac{1}{\varphi}e^{-\varphi 0} \\ &= \frac{1}{\varphi} \ . \end{aligned}$$

E.g.

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}dx = \left[\ln(x)\right|_1^\infty = \ln(\infty) - \ln(1) = \infty - 0 = \infty \ .$$

14.5.2 Infinite integrand

E.g., sometimes the integral is **divergent**, even though the integration limits are finite:

$$\int_0^1 \frac{1}{x}dx = \lim_{b\to 0}\int_b^1 \frac{1}{x}dx = \left[\ln(x)\right|_0^1 = \ln(1) - \ln(0) = 0 + \infty = \infty \ .$$

Suppose that for some $p \in (a, b)$

$$\lim_{x\to p} f(x) = \infty \ .$$

Then the integral from a to b is convergent iff the partitions are also convergent:

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx \ .$$

E.g.

$$\lim_{x\to 0}\frac{1}{x^3} = \infty \ .$$

Therefore, the integral

$$\int_{-1}^{1} \frac{1}{x^3} dx = \int_{-1}^{0} \frac{1}{x^3} dx + \int_{0}^{1} \frac{1}{x^3} dx = \left[-\frac{1}{2x^2} \right|_{-1}^{0} + \left[-\frac{1}{2x^2} \right|_{0}^{1}$$

does not exist, because neither integral converges.

14.6 Example: investment and capital formation

In discrete time we have the capital accumulation equation

$$K_{t+1} = (1-\delta)\, K_t + I_t \ ,$$

where I_t is gross investment at time t. Rewrite as

$$K_{t+1} - K_t = I_t - \delta K_t \ .$$

We want to rewrite this in continuous time. In this context, investment, I_t, is instantaneous and capital depreciates at an instantaneous rate of δ. Consider a period of length Δ. The accumulation equation is

$$K_{t+\Delta} - K_t = \Delta I_t - \Delta \delta K_t \ .$$

Divide by Δ to get

$$\frac{K_{t+\Delta} - K_t}{\Delta} = I_t - \delta K_t \ .$$

Now take $\Delta \to 0$ to get

$$\dot{K}_t = I_t - \delta K_t \ ,$$

where it is understood that I_t is instantaneous investment at time t, and K_t is the amount of capital available at that time. δK_t is the amount of capital that vanishes due to depreciation. Write

$$\dot{K}_t = I_t^n \ ,$$

where I_t^n is net investment. Given a functional form for I_t^n we can tell how much capital is around at time t, given an initial amount at time 0, K_0.

Let $I_t^n = t^a$. then

$$K_t - K_0 = \int_0^t \dot{K} dt = \int_0^t I_u^n du = \int_0^t u^a du = \left[\frac{u^{a+1}}{a+1} \right|_0^t = \frac{t^{a+1}}{a+1} \ .$$

14.7 Domar's growth model

Domar was interested in answering: what must investment be in order to satisfy the equilibrium condition at all times.

Structure of the model:

1. Fixed saving rate: $I_t = sY_t$, $s \in (0,1)$. Therefore $\dot{I} = s\dot{Y}$. And so

$$\dot{Y} = \frac{1}{s}\dot{I} \ ,$$

i.e. there is a multiplier effect of investment on output.

2. Potential output is given by a CRS production function

$$\pi_t = \rho K_t \ ,$$

therefore

$$\dot{\pi} = \rho \dot{K} = \rho I \ .$$

3. Long run equilibrium is given when potential output is equal to actual output

$$\pi = Y \ ,$$

therefore

$$\dot{\pi} = \dot{Y} \ .$$

We have three equations:

$$\begin{aligned} \text{(i) output demand} &: \quad \dot{Y} = \frac{1}{s}\dot{I} \\ \text{(ii) potential output} &: \quad \dot{\pi} = \rho I \\ \text{(iii) equilibrium} &: \quad \dot{\pi} = \dot{Y} \ . \end{aligned}$$

Use (iii) in (ii) to get

$$\rho I = \dot{Y}$$

and then use (i) to get

$$\rho I = \frac{1}{s}\dot{I} \ ,$$

which gives

$$\frac{\dot{I}}{I} = \rho s \ .$$

Now integrate in order to find the level of investment at any given time:

$$\begin{aligned} \int \frac{\dot{I}}{I} dt &= \int \rho s dt \\ \ln I &= \rho s t + c \\ I_t &= e^{(\rho s)t + c} = e^{(\rho s)t} e^c = I_0 e^{(\rho s)t} \ . \end{aligned}$$

The larger is productivity, ρ, and the higher the saving rate, s, the more investment is *required*. This is the amount of investment needed for output to keep output in check with potential output.

Now suppose that output is not equal to its potential, i.e. $\pi \neq Y$. This could happen if the investment is not growing at the correct rate of ρs. Suppose that investment is growing at rate a, i.e.

$$I_t = I_0 e^{at} \ .$$

Define the **utilization rate**

$$u = \lim_{t\to\infty} \frac{Y_t}{\pi_t} \ .$$

Compute what the capital stock is at any moment:

$$K_t - K_0 = \int_0^t \dot{K} d\tau + \int_0^t I_\tau d\tau = \int_0^t I_0 e^{a\tau} d\tau = \frac{1}{a} I_0 e^{at}$$

(the constant of integration is absorbed in K_0.) Now compute

$$u = \lim_{t\to\infty} \frac{Y_t}{\pi_t} = \lim_{t\to\infty} \frac{\frac{1}{s} I_t}{\rho K_t} = \frac{1}{\rho s} \lim_{t\to\infty} \frac{I_t}{K_t} = \frac{1}{\rho s} \lim_{t\to\infty} \frac{I_0 e^{at}}{\frac{1}{a} I_0 e^{at} + K_0} = \frac{a}{\rho s} \lim_{t\to\infty} \frac{I_0 e^{at}}{I_0 e^{at} + a K_0} = \frac{a}{\rho s} \ .$$

The last equality can be derived using **L'Hopital's rule**, or by simply noting that $\frac{I_0 e^{at}}{I_0 e^{at} + aK_0} = \frac{1}{1 + aK_0 I_0 e^{-at}} \to 1$ as $t \to \infty$. If $a > \rho s$ then $u > 1$ there is a shortage of capacity, excess demand. If $a < \rho s$ then $u < 1$ there is a excess of capacity, excess supply. Thus, in order to keep output demand equal to output potential we must have $a = \rho s$ and thus $u = 1$.

In fact, this holds at any point in time:

$$\dot{I} = \frac{d}{dt} I_0 e^{at} = a I_0 e^{at} \ .$$

Therefore

$$\begin{aligned} \dot{Y} &= \frac{1}{s}\dot{I} = \frac{a}{s} I_0 e^{at} \\ \dot{\pi} &= \rho I = \rho I_0 e^{at} \ . \end{aligned}$$

So

$$\frac{\dot{Y}}{\dot{\pi}} = \frac{\frac{a}{s} I_0 e^{at}}{\rho I_0 e^{at}} = \frac{a}{s\rho} = u \ .$$

If the utilization rate is too high $u > 1$, then demand growth outstrips supply, $\dot{Y} > \dot{\pi}$. If the utilization rate is too low $u < 1$, then demand growth lags behind supply, $\dot{Y} < \dot{\pi}$.

Thus, the **razor edge condition**: only $a = s\rho$ keeps us at a sustainable equilibrium path:

- If $u > 1$, i.e. $a > s\rho$, there is excess demand, investment is too high. Entrepreneurs will try to invest even more to increase supply, but this implies an even larger gap between the two.

- If $u < 1$, i.e. $a < s\rho$, there is excess supply, investment is too low. Entrepreneurs will try to cut investment to lower demand, but this implies an even larger gap between the two.

This model is clearly unrealistic and highly stylized.

15 First order differential equations

We deal with equations that involve $\dot{y}$. The general form is

$$\dot{y} + u(t)\,y(t) = w(t)$$

The goal is to characterize $y(t)$ in terms of $u(t)$ and $w(t)$. Note that this can be written as

$$\dot{y} = f(y,t) \ .$$

- First order means $\frac{dy}{dt}$, not $\frac{d^2y}{dt^2}$.
- No products: $\dot{y} \cdot y$ is not permitted.

In principle, we can have $d^n y/dt^n$, where n is the order of the differential equation. In the next chapter we will deal with up to d^2y/dt^2.

15.1 Fundamental theorem of differential equations

Consider solving for $y(t)$ in

$$\begin{aligned} \dot{y} &= f(y,t) \ , \\ \text{where } \ y(t_0) &= y_0 \ \text{ is known.} \end{aligned} \tag{14}$$

Suppose that f is a continuous function at (t_0, y_0). Then there exists a C^1 function $y : I \to \mathbb{R}$ on the open interval $I = (t_0 - a, t_0 + a)$ such that $y(t_0) = y_0$ and $\dot{y}(t) = f(y(t), t)$ for all $t \in I$, i.e. $y(t)$ solves (14). If in addition $f \in C^1$ at (t_0, y_0), then the solution $y(t)$ is unique; any two solutions of (14) must be equal to each other on the intersection of their domains.

Most differential equations in economic applications will have $f \in C^1$, so all solutions will be unique. But in the following case

$$\dot{y} = 3y^{2/3}$$

there are multiple solutions because $3y^{2/3}$ is not differentiable at 0 (the derivative is ∞ there). Since $3y^{2/3}$ is nonetheless continuous at 0, a solution exists but it is not unique. For example, both $y(t) = 0$ and $y(t) = t^3$ solve the differential equation.

15.2 Constant coefficients

15.2.1 Homogenous case

$$\dot{y} + ay = 0$$

This gives rise to

$$\frac{\dot{y}}{y} = -a$$

which has solution

$$y(t) = y_0 e^{-at} \ .$$

We need an additional condition to pin down y_0.

15.2.2 Non homogenous case

$$\dot{y} + ay = b \ ,$$

where $b \neq 0$. The solution method involves splitting the solution into two:

$$y(t) = y_c(t) + y_p(t) \ ,$$

where $y_p(t)$ is a **particular solution** and $y_c(t)$ is a **complementary function**.

- $y_c(t)$ solves the homogenous equation

  $$\dot{y} + ay = 0 \ ,$$

  so that

  $$y_c(t) = Ae^{-at} \ .$$

- $y_p(t)$ solves the original equation for a stationary solution, i.e. $\dot{y} = 0$, which implies that y is constant and thus $y = b/a$, where $a \neq 0$. The solution is thus

  $$y = y_c + y_p = Ae^{-at} + \frac{b}{a} \ .$$

  Given an initial condition $y(0) = y_0$, we have

  $$y_0 = Ae^{-a0} + \frac{b}{a} = A + \frac{b}{a} \quad \Rightarrow \quad A = y_0 - \frac{b}{a} \ .$$

The general solution is

$$y(t) = \left(y_0 - \frac{b}{a}\right) e^{-at} + \frac{b}{a} = y_0 e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right) \ .$$

One way to think of the solution is a linear combination of two points: the initial condition y_0 and the particular, stationary solution b/a. (If $a > 0$, then for $t \geq 0$ we have $0 \leq e^{-at} \leq 1$, which yields a convex combination). Verify this solution:

$$\begin{aligned}\dot{y} &= -a\left(y_0-\frac{b}{a}\right)e^{-at} = -a\left[\underbrace{\left(y_0-\frac{b}{a}\right)e^{-at}+\frac{b}{a}}_{y}-\frac{b}{a}\right] = -ay+b \\ &\Rightarrow \quad \dot{y}+ay=b\ .\end{aligned}$$

Yet a different way to look at the solution is

$$\begin{aligned}y(t) &= \left(y_0-\frac{b}{a}\right)e^{-at}+\frac{b}{a} \\ &= ke^{-at}+\frac{b}{a}\ ,\end{aligned}$$

for some arbitrary point k. In this case

$$\dot{y} = -ake^{-at}\ ,$$

and we have

$$\dot{y}+ay = -ake^{-at}+a\left(ke^{-at}+\frac{b}{a}\right) = b\ .$$

- When $a=0$, we get

  $$\dot{y}=b$$

  so

  $$y=y_0+bt\ .$$

  This follows directly from

  $$\begin{aligned}\int \dot{y}dt &= \int bdt \\ y &= bt+c\ ,\end{aligned}$$

  where $c=y_0$. We can also solve this using the same technique as above. y_c solves $\dot{y}=0$, so that this is a constant $y_c=A$. y_p should solve $0=b$, but this does not work unless $b=0$. So try a different particular solution, $y_p=kt$, which requires $k=b$, because then $\dot{y}_p=k=b$. So the general solution is

  $$y=y_c+y_p=A+bt\ .$$

  Together with a value for y_0, we get $A=y_0$.

E.g.

$$\dot{y} + 2y = 6 \ .$$

y_c solves $\dot{y} + 2y = 0$, so

$$y_c = Ae^{-2t} \ .$$

y_p solves $2y = 6$ ($\dot{y} = 0$), so

$$y_p = 3 \ .$$

Thus

$$y = y_c + y_p = Ae^{-2t} + 3 \ .$$

Together with $y_0 = 10$ we get $10 = Ae^{-2 \cdot 0} + 3$, so that $A = 7$. This completes the solution:

$$y = 7e^{-2t} + 3 \ .$$

Verifying this solution:

$$\dot{y} = -14e^{-2t}$$

and

$$\dot{y} + 2y = -14e^{-2t} + 2\left(7e^{-2t} + 3\right) = 6 \ .$$

15.3 Variable coefficients

The general form is

$$\dot{y} + u(t)\, y(t) = w(t) \ .$$

15.3.1 Homogenous case

$w(t) = 0$:

$$\dot{y} + u(t)\, y(t) = 0 \quad \Rightarrow \quad \frac{\dot{y}}{y} = -u(t) \ .$$

Integrate both sides to get

$$\begin{aligned}
\int \frac{\dot{y}}{y} dt &= \int -u(t)\, dt \\
\ln y + c &= -\int u(t)\, dt \\
y &= e^{-c-\int u(t)dt} = Ae^{-\int u(t)dt} \ ,
\end{aligned}$$

where $A = e^{-c}$. Thus, the general solution is

$$y = Ae^{-\int u(t)dt} \ .$$

Together with a value for y_0 and a functional form for $u(t)$ we can solve explicitly.

E.g.

$$\begin{aligned}
\dot{y} + 3t^2 y &= 0 \\
\dot{y} + \left(3t^2\right) y &= 0 \ .
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\dot{y}}{y} &= -3t^2 \\
\int \frac{\dot{y}}{y} dt &= \int -3t^2 dt \\
\ln y + c &= -\int 3t^2 dt \\
y &= e^{-c-\int 3t^2 dt} = Ae^{-t^3} \ .
\end{aligned}$$

15.3.2 Non homogenous case

$w(t) \neq 0$:

$$\dot{y} + u(t) y(t) = w(t) \ .$$

The solution is

$$y = e^{-\int u(t)dt} \left[A + \int w(t) e^{\int u(t)dt} dt \right] \ .$$

Obtaining this solution requires some footwork. But first, see that it works: e.g.,

$$\dot{y} + t^2 y = t^2 \quad \Rightarrow \quad u(t) = t^2, \ w(t) = t^2 \ .$$

$$\int u(t)\, dt = \int t^2 dt = \frac{1}{3} t^3$$

$$\int w(t) e^{\int u(t)dt} dt = \int t^2 e^{t^3/3} dt = e^{t^3/3} \ ,$$

since

$$\int f'(y) e^{f(y)} dy = e^{f(y)} \ .$$

Thus

$$y = e^{-t^3/3} \left[A + e^{t^3/3} \right] = Ae^{-t^3/3} + 1 \ .$$

Verifying this solution:

$$\dot{y} = -t^2 A e^{-t^3/3}$$

so

$$\begin{aligned}
\dot{y} + u(t)\,y(t) &= -t^2 A e^{-t^3/3} + \left(t^2\right)\left[A e^{-t^3/3} + 1\right] \\
&= -t^2 A e^{-t^3/3} + t^2 A e^{-t^3/3} + t^2 \\
&= t^2 \\
&= w(t) \ .
\end{aligned}$$

15.4 Solving exact differential equations

Suppose that the primitive differential equation can be written as

$$F(y,t) = c$$

so that

$$dF = F_y dy + F_t dt = 0 \ .$$

We use the latter total differential to obtain $F(y,t)$, from which we obtain $y(t)$. We set $F(y,t) = c$ to get initial conditions.

Definition: the differential equation

$$M dy + N dt = 0$$

is an **exact differential equation** iff $\exists F(y,t)$ such that $M = F_y$ and $N = F_t$.

If such a function $F(y,t)$ exists, then by Young's theorem we have

$$\frac{\partial M}{\partial t} = \frac{\partial^2 F}{\partial t \partial y} = \frac{\partial N}{\partial y} \ .$$

And this relationship is what we will be checking in practice to verify that a differential equations is indeed exact.

E.g., let $F(y,t) = y^2 t = c$. Then

$$dF = F_y dy + F_t dt = 2yt dy + y^2 dt = 0 \ .$$

Set

$$M = 2yt \ , \quad N = y^2 \ .$$

Check:

$$\begin{aligned}\frac{\partial^2 F}{\partial t \partial y} &= \frac{\partial M}{\partial t} = 2y \\ \frac{\partial^2 F}{\partial t \partial y} &= \frac{\partial N}{\partial y} = 2y\end{aligned}$$

So this is an exact differential equation.

Solving exact differential equations:

Before solving, one must always check that the equation is indeed exact.

- **Step 1:** Since
$$dF = F_y dy + F_t dt$$
we could integrate both sides, but this does not lead to right answer. Instead, integrate only F_y over y and add a residual function of t alone:
$$\begin{aligned}F(y,t) &= \int F_y dy + \varphi(t) \\ &= \int M dy + \varphi(t) \ ,\end{aligned}$$
where $\varphi(t)$ is a residual function.

- **Step 2:** Take the derivative of $F(y,t)$ from step 1 w.r.t. t, N, and equate it to F_t from the original differential function. This identifies $\varphi(t)$.

- **Step 3:** Solve for $y(t)$, taking into account $F(y,t) = c$.

Example:

$$\underbrace{2yt}_{M} dy + \underbrace{y^2}_{N} dt = 0 \ .$$

Step 1:

$$F(y,t) = \int M dy + \varphi(t) = \int 2yt dy + \varphi(t) = y^2 t + \varphi(t) \ .$$

Step 2:

$$\frac{\partial F(y,t)}{\partial t} = \frac{\partial}{\partial t}\left[y^2 t + \varphi(t)\right] = y^2 + \varphi'(t) \ .$$

Since $N = y^2$ we must have $\varphi'(t) = 0$, i.e. $\varphi(t)$ is a constant function, $\varphi(t) = k$, for some k. Thus

$$F(y,t) = y^2 t + k = c \ ,$$

so we can ignore the constant k and write

$$F(y,t) = y^2 t = c \ .$$

Step 3: We can now solve for $y(t)$:

$$y(t) = \pm c^{1/2} t^{-1/2} \ .$$

The solution for $\pm c$ will be given by an initial condition.

Example:

$$(t+2y)\,dy + \left(y+3t^2\right) dt = 0 \ .$$

So that

$$\begin{aligned} M &= (t+2y) \\ N &= \left(y+3t^2\right) \ . \end{aligned}$$

Check that this equation is exact:

$$\frac{\partial M}{\partial t} = 1 = \frac{\partial N}{\partial y} \ ,$$

so this is indeed an exact differential equation.

Step 1:

$$F(y,t) = \int M dy + \varphi(t) = \int (t+2y)\,dy + \varphi(t) = ty + y^2 + \varphi(t) \ .$$

Step 2:

$$\frac{\partial F(y,t)}{\partial t} = \frac{\partial}{\partial t}\left[ty + y^2 + \varphi(t)\right] = y + \varphi'(t) = N = y + 3t^2 \ ,$$

so that

$$\varphi'(t) = 3t^2$$

and

$$\varphi(t) = \int \varphi'(t)\,dt = \int 3t^2 dt = t^3 \ .$$

Thus

$$F(y,t) = ty + y^2 + \varphi(t) = ty + y^2 + t^3 \ .$$

Step 3: we cannot solve this analytically for $y(t)$, but using the implicit function theorem, we can characterize it.

Example:

Let $T \sim F(t)$ be the time until some event occurs, $T \geq 0$. Define the hazard rate as

$$h(t) = \frac{f(t)}{1 - F(t)} \ ,$$

which is the "probability" that the event occurs at time t, given that it has not occurred by time t.

We can write

$$h(t) = -\frac{R'(t)}{R(t)} \ ,$$

where $R(t) = 1 - F(t)$. We know how to solve such differential equations:

$$R'(t) + h(t) R(t) = 0 \ .$$

$$R(t) = Ae^{-\int^t h(s)ds} \ .$$

Since $R(0) = 1$ (the probability that the event occurs at all), then we have $A = 1$:

$$R(t) = e^{-\int^t h(s)ds} \ .$$

It follows that

$$f(t) = -R'(t) = -e^{-\int^t h(s)ds} \frac{\partial}{\partial t}\left[-\int^t h(s)\,ds\right] = -e^{-\int^t h(s)ds}\left[-h(t)\right] = h(t)\,e^{-\int^t h(s)ds} \ .$$

Suppose that the hazard rate is constant:

$$h(t) = \alpha \ .$$

In that case

$$f(t) = \alpha e^{-\int^t \alpha ds} = \alpha e^{-\alpha t} \ ,$$

which is the p.d.f. of the exponential distribution.

Now suppose that the hazard rate is not constant, but

$$h(t) = \alpha\beta t^{\beta-1} \ .$$

In that case

$$f(t) = \alpha\beta t^{\beta-1} e^{-\int^t \alpha\beta s^{\beta-1} ds} = \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}} \ ,$$

which is the p.d.f. of the Weibull distribution. This is useful if you want to model an increasing hazard ($\beta > 1$) or decreasing hazard ($\beta < 1$). When $\beta = 1$ or we get the exponential distribution.

15.5 Integrating factor and the general solution

Sometimes we can turn a non exact differential equation into an exact one. For example,

$$2tdy + ydt = 0$$

is not exact:

$$\begin{aligned} M &= 2t \\ N &= y \end{aligned}$$

and

$$M_t = 2 \neq N_y = 1 \ .$$

But if we multiply the equation by y, we get an exact equation:

$$2ytdy + y^2 dt = 0 \ ,$$

which we saw above is exact.

15.5.1 Integrating factor

We have the general formulation

$$\dot{y} + uy = w \ ,$$

where all variables are functions of t and we wish to solve for $y(t)$. Write the equation above as

$$\begin{aligned} \frac{dy}{dt} + uy &= w \\ dy + uydt &= wdt \\ dy + (uy - w)\, dt &= 0 \ . \end{aligned}$$

The **integrating factor** is

$$e^{\int^t u(s)ds} \ .$$

If we multiply the equation by this factor we always get an exact equation:

$$e^{\int^t u(s)ds} dy + e^{\int^t u(s)ds} (uy - w)\, dt = 0 \ .$$

To verify this, write

$$\begin{aligned} M &= e^{\int^t u(s)ds} \\ N &= e^{\int^t u(s)ds} (uy - w) \end{aligned}$$

and

$$\begin{aligned}\frac{\partial M}{\partial t} &= \frac{\partial}{\partial t} e^{\int^t u(s)ds} = e^{\int^t u(s)ds} u(t) \\ \frac{\partial N}{\partial y} &= \frac{\partial}{\partial y} e^{\int^t u(s)ds} (uy - w) = e^{\int^t u(s)ds} u(t) \ .\end{aligned}$$

So $\partial M / \partial t = \partial N / \partial y$.

This form can be recovered from the **method of undetermined coefficients.** We seek some A such that

$$\underbrace{A}_{M} dy + \underbrace{A (uy - w)}_{N} dt = 0$$

and

$$\begin{aligned}\frac{\partial M}{\partial t} &= \frac{\partial A}{\partial t} = \dot{A} \\ \frac{\partial N}{\partial y} &= \frac{\partial}{\partial y} [A (uy - w)] = Au\end{aligned}$$

are equal. This means

$$\begin{aligned}\dot{A} &= Au \\ \dot{A}/A &= u \\ A &= e^{\int^t u(s)ds} \ .\end{aligned}$$

15.5.2 The general solution

We have some equation that is written as

$$\dot{y} + uy = w \ .$$

Rewrite as

$$dy + (uy - w)\, dt = 0 \ .$$

Multiply by the integrating factor to get an exact equation

$$\underbrace{e^{\int^t u(s)ds}}_{M} dy + \underbrace{e^{\int^t u(s)ds} (uy - w)}_{N} dt = 0 \ .$$

- **Step 1:**

$$F(y,t) = \int M dy + \varphi(t) = \int e^{\int^t u(s)ds} dy + \varphi(t) = y e^{\int^t u(s)ds} + \varphi(t) \ .$$

- **Step 2:**
$$\frac{\partial F}{\partial t}=\frac{\partial}{\partial t}\left[ye^{\int^t u(s)ds}+\varphi(t)\right]=ye^{\int^t u(s)ds}u(t)+\varphi'(t)=N\ .$$
Using N from above we get
$$N=ye^{\int^t u(s)ds}u(t)+\varphi'(t)=e^{\int^t u(s)ds}(uy-w)\ ,$$
so that
$$\varphi'(t)=-e^{\int^t u(s)ds}w$$
and so
$$\varphi(t)=\int -e^{\int^t u(s)ds}w dt\ .$$
Now we can write
$$F(y,t)=ye^{\int^t u(s)ds}-\int e^{\int^t u(s)ds}w dt=c$$

- **Step 3,** solve for y:
$$y=e^{-\int^t u(s)ds}\left[c+\int e^{\int^t u(s)ds}w dt\right]\ .$$

15.6 First order nonlinear differential equations of the 1st degree

In general,
$$\dot{y}=h(y,t)$$
will yield an equation like this
$$f(y,t)\,dy+g(y,t)\,dt=0\ .$$
In principle, y and t can appear in any degree.

- First order means $\dot{y}$, not $y^{(n)}$.
- First degree means $\dot{y}$, not $(\dot{y})^n$.

15.6.1 Exact differential equations

See above.

15.6.2 Separable variables

$$f(y)\,dy+g(t)\,dt=0\ .$$

Then just integrate
$$\int f(y)\,dy=-\int g(t)\,dt$$

and solve for $y(t)$.

Example:

$$3y^2dy - tdt = 0$$

$$\begin{aligned}
\int 3y^2dy &= \int tdt \\
y^3 &= \frac{1}{2}t^2 + c \\
y(t) &= \left(\frac{1}{2}t^2 + c\right)^{1/3} .
\end{aligned}$$

An initial condition will pin down c.

Example:

$$2tdy - ydt = 0$$

$$\begin{aligned}
\frac{dy}{y} &= \frac{dt}{2t} \\
\int \frac{dy}{y} &= \int \frac{dt}{2t} \\
\ln y &= \frac{1}{2}\ln t + c \\
y &= e^{c+\frac{1}{2}\ln t} = e^{c+\ln t^{1/2}} = e^c t^{1/2} = At^{1/2} .
\end{aligned}$$

An initial condition will pin down A.

15.6.3 Reducible equations

Suppose that

$$\dot{y} = h(y,t)$$

can be written as

$$\dot{y} + Ry = Ty^m , \tag{15}$$

where

$$\begin{aligned}
R &= R(t) \\
T &= T(t)
\end{aligned}$$

are functions only of t and

$$m \neq 0, 1 .$$

- When $m = 0$ we get $\dot{y} + Ry = T$, which we know how to solve.

- When $m = 1$ we get $\dot{y} + Ry = Ty$, and then we solve $\dot{y}/y = (T - R)$.

Equation (15) is a **Bernoulli equation**, which can be **reduced to a linear equation by changing variables** and solved as such. Once the solution is found, we can back out the original function $y\,(t)$. Here's how:

$$\begin{aligned}\dot{y} + Ry &= Ty^m \\ \frac{\dot{y}}{y^m} + Ry^{1-m} &= T\end{aligned}$$

Use a change of variables

$$z = y^{1-m}$$

so that

$$\begin{aligned}\dot{z} &= (1-m)\,y^{-m}\dot{y} \\ \frac{\dot{y}}{y^m} &= \frac{\dot{z}}{1-m}\ .\end{aligned}$$

Plug this in the equation to get

$$\begin{aligned}\frac{\dot{z}}{1-m} + Rz &= T \\ dz + \Bigg[\underbrace{(1-m)\,R}_{u}z - \underbrace{(1-m)\,T}_{w}\Bigg]\,dt &= 0 \\ dz + [uz + w]\,dt &= 0\ .\end{aligned}$$

This is something we know how to solve:

$$z\,(t) = e^{-\int^t u(s)ds}\left[A + \int e^{\int^t u(s)ds} w dt\right]\ .$$

from which we get the original

$$y\,(t) = z\,(t)^{\frac{1}{1-m}}\ .$$

An initial condition will pin down A.

Example:

$$\dot{y} + ty = 3ty^2$$

In this case

$$\begin{aligned} R &= t \\ T &= 3t \\ m &= 2 \ . \end{aligned}$$

Divide by y^2 and rearrange to get

$$y^{-2}\dot{y} + ty^{-1} - 3t = 0 \ .$$

Change variables

$$\begin{aligned} z &= y^{-1} \\ \dot{z} &= -y^{-2}\dot{y} \end{aligned}$$

so that we get

$$\begin{aligned} -\dot{z} + tz - 3t &= 0 \\ dz + (-tz + 3t)\, dt &= 0 \ . \end{aligned}$$

Note that this differential equation is not exact: $M = 1$, $N = (-tz + 3t)$, and $\partial M \partial t = 0 \neq \partial N \partial z = -t$. Set

$$\begin{aligned} u &= -t \\ w &= -3t \ . \end{aligned}$$

Using the formula we get

$$\begin{aligned} z(t) &= e^{-\int^t u(s)ds} \left[A + \int e^{\int^t u(s)ds} w dt \right] \\ &= e^{\int^t sds} \left[A - 3 \int e^{-\int^t sds} t dt \right] \\ &= e^{t^2/2} \left[A - 3 \int e^{-t^2/2} t dt \right] \\ &= e^{t^2/2} \left[A + 3e^{-t^2/2} \right] \\ &= Ae^{t^2/2} + 3 \ . \end{aligned}$$

So that

$$y(t) = \frac{1}{z} = \left(Ae^{t^2/2} + 3 \right)^{-1} \ .$$

An initial condition will pin down A.

Example:

$$\dot{y} + y/t = y^3 \ .$$

In this case

$$\begin{aligned} R &= 1/t \\ T &= 1 \\ m &= 3 \ . \end{aligned}$$

Divide by y^3 and rearrange to get

$$y^{-3}\dot{y} + t^{-1}y^{-2} - 1 = 0 \ .$$

Change variables

$$\begin{aligned} z &= y^{-2} \\ \dot{z} &= -2y^{-3}\dot{y} \end{aligned}$$

so that we get

$$\begin{aligned} -\frac{\dot{z}}{2} + \frac{z}{t} - 1 &= 0 \\ \dot{z} + -2\frac{z}{t} + 2 &= 0 \\ dz + \left(-2\frac{z}{t} + 2\right) dt &= 0 \ . \end{aligned}$$

so that we set

$$\begin{aligned} u &= -2/t \\ w &= -2 \ . \end{aligned}$$

Using the formula we get

$$
\begin{aligned}
z(t) &= e^{-\int^t u(s)ds}\left[A + \int e^{\int^t u(s)ds} w dt\right] \\
&= e^{2\int^t t^{-1}ds}\left[A - 2\int e^{-2\int^t t^{-1}ds} dt\right] \\
&= e^{2\ln t}\left[A - 2\int e^{-2\ln t} dt\right] \\
&= t^2\left[A - 2t^{-2}\right] \\
&= At^2 - 2 \ .
\end{aligned}
$$

So that

$$y(t) = \frac{1}{z^2} = \left(At^2 - 2\right)^{-2} \ .$$

An initial condition will pin down A.

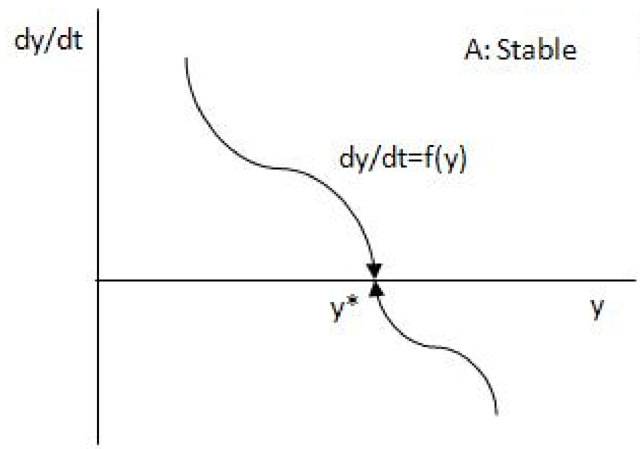
15.7 The qualitative graphic approach

Given

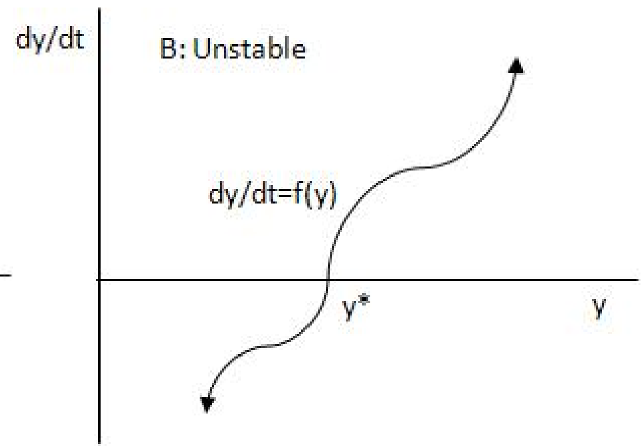
$$\dot{y} = f(y) \ ,$$

we can plot $\dot{y}$ on the vertical axis against y on the horizontal axis. This is called a **phase diagram**. This is an autonomous differential equation, since t does not appear explicitly as an argument. We have three cases:

1. $\dot{y} > 0$: y is growing, so we shift to the right.
2. $\dot{y} < 0$: y is decreasing, so we shift to the left.
3. $\dot{y} = 0$: y is stationary, an equilibrium.



Stable Differential Equation



Unstable Differential Equation

- System A is dynamically stable: the $\dot{y}$ curve is downward sloping; any movement away from the stationary point y^* will bring us back there.
- System B is dynamically *un*stable: the $\dot{y}$ curve is upward sloping; any movement away from the stationary point y^* take farther away.

For example, consider

$$\dot{y} + ay = b$$

with solution

$$\begin{aligned} y(t) &= \left[y_0 - \frac{b}{a}\right] e^{-at} + \frac{b}{a} \\ &= \left(e^{-at}\right) y_0 + \left(1 - e^{-at}\right) \frac{b}{a} \ . \end{aligned}$$

This is a linear combination between the initial point y_0 and b/a.

- System A happens when $a > 0$: $\lim_{t\to\infty} e^{-at} \to 0$, so that $\lim_{t\to\infty} y(t) \to b/a = y^*$.
- System B happens when $a < 0$: $\lim_{t\to\infty} e^{-at} \to \infty$, so that $\lim_{t\to\infty} y(t) \to \pm\infty$.

15.8 The Solow growth model (no long run growth version)

The model has three ingredients:

1. Supply: CRS production function

$$\begin{aligned} Y &= F(K,L) \\ y &= f(k) \end{aligned}$$

where $y = Y/L$, $k = K/L$. Given $F_K > 0$ and $F_{KK} < 0$ we have $f' > 0$ and $f'' < 0$.

2. Demand: Constant saving rate sY (consumption is $(1-s)Y$).

3. Constant labor force growth: $\dot{L}/L = n$.

Given these, we can characterize the law of motion for capital *per capita*, k. The law of motion for capital is $\dot{K} = I - \delta K$. In a closed economy savings equal investment, so $I = sY$. Therefore,

$$\begin{aligned} \dot{K} &= sF(K,L) - \delta K = sLf(k) - \delta K \\ \frac{\dot{K}}{L} &= sf(k) - \delta k \ . \end{aligned}$$

Since

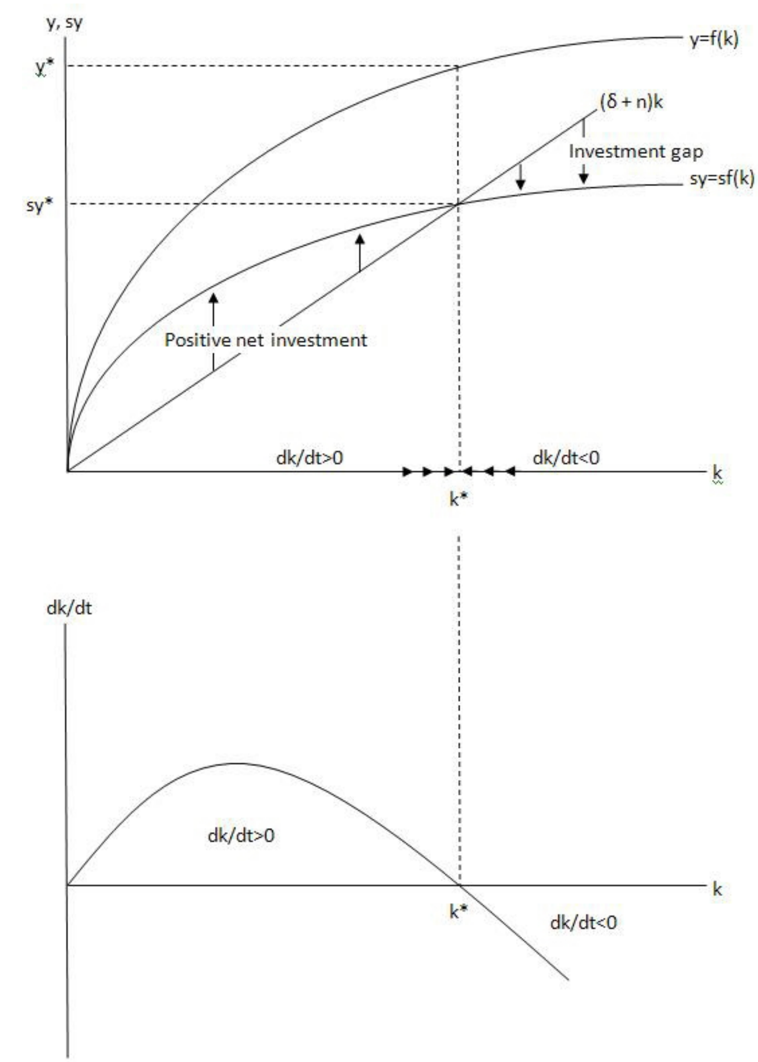
$$\dot{k} = \frac{d}{dt}\left(\frac{K}{L}\right) = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} = \frac{\dot{K}}{L} - kn \ ,$$

we get

$$\dot{k} = sf(k) - (n+\delta)k \ .$$

This is an autonomous differential equation in k.

Since $f' > 0$ and $f'' < 0$ we know that $\exists k$ such that $sf(k) < (n+\delta)k$. And given the Inada conditions ($f'(0) = \infty$ and $f(0) = 0$), then $\exists k$ such that $sf(k) > (n+\delta)k$. Therefore, $\dot{k} > 0$ for low levels of k; and $\dot{k} < 0$ for high levels of k. Given the continuity of f we know that $\exists k^*$ such that $\dot{k} = 0$, i.e. the system is stable.


y, sy
y=f(k)
y*
(δ + n)k
Investment gap
sy=sf(k)
sy*
Positive net investment
dk/dt>0
dk/dt<0
k
k*
dk/dt
dk/dt>0
k
k*
dk/dt<0

Solow Model

16 Higher order differential equations

We will discuss here only second order, since it is very rare to find higher order differential equations in economics. The methods introduced here can be extended to higher order differential equations.

In fact the fundamental theorem of differential equations, described above in Section 15.1 extends to j^{th} order differential equations, provided that we specify j initial conditions:

$$y(t_0) = y_0,\ y'(t_0) = y_1,\ y''(t_0) = y_2,\ \dots\ y^{[j-1]}(t_0) = y_{j-1}\ .$$

16.1 Second order, constant coefficients

$$y'' + a_1 y' + a_2 y = b\ ,$$

where

$$\begin{aligned} y &= y(t) \\ y' &= dy/dt \\ y'' &= d^2y/dt^2\ , \end{aligned}$$

and a_1, a_2, and b are constants. The solution will take the form

$$y = y_p + y_c\ ,$$

where the particular solution, y_p, characterizes a stable point and the complementary function, y_c, characterizes dynamics/transitions.

The particular solution. We start with the simplest solution possible; if this fails, we move up in the degree of complexity.

- If $a_2 \neq 0$, then $y_p = b/a_2$ is solution, which implies a stable point.
- If $a_2 = 0$ and $a_1 \neq 0$, then $y_p = \frac{b}{a_1}t$.
- If $a_2 = 0$ and $a_1 = 0$, then $y_p = \frac{b}{2}t^2$.

In the latter solutions, the "stable point" is moving. Recall that this solution is too restrictive, because it constrains the dynamics of y. That is why we add the complementary function.

The complementary function solves the homogenous equation

$$y'' + a_1 y' + a_2 y = 0\ .$$

We "guess"

$$y = Ae^{rt}$$

which implies

$$\begin{aligned} y' &= rAe^{rt} \\ y'' &= r^2 Ae^{rt} \end{aligned}$$

and thus

$$y'' + a_1 y' + a_2 y = A\left(r^2 + a_1 r + a_2\right) e^{rt} = 0 \ .$$

Unless $A = 0$, we must have

$$r^2 + a_1 r + a_2 = 0 \ .$$

This is sometimes called the **characteristic equation**. The roots are

$$r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \ .$$

For each root r_i there is a potentially different A_i coefficient. So there are two possible solutions for the complementary function:

$$\begin{aligned} y_1 &= A_1 e^{r_1 t} \\ y_2 &= A_2 e^{r_2 t} \ . \end{aligned}$$

We cannot just chose one of the two solutions because this would restrict the dynamics of y. Thus, we have

$$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t} \ .$$

Given two initial conditions on y and y' at some point in time we can pin down A_1 and A_2.

There are **three options for the composition of the roots**:

- **Two distinct real roots:** $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$. This will give us values for A_1 and A_2, given two conditions on y.
$$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t} \ .$$

- **Repeated real root:** $r_1 = r_2 \in \mathbb{R}$, $r = -a_1/2$. It might seem that we can just add up the solution as before, but this actually not general enough because it restricts the dynamics of y. Moreover, if we used $y_c = (A_1 + A_2)\, e^{rt}$, then we cannot separately identify A_1 from A_2.

We guess again:

$$\begin{aligned} y_1 &= A_1 e^{rt} \\ y_2 &= A_2 \cdot t \cdot e^{rt} \ . \end{aligned}$$

This turns out to work, because both solve the homogenous equation. You can check this. Thus for repeated real root the complementary function is

$$y_c = A_1 e^{r_1 t} + A_2 t e^{r_2 t} \ .$$

- **Complex roots:** $r_{1,2} = r \pm bi$, $i = \sqrt{-1}$, $a_1^2 < 4a_2$. This gives rise to oscillating dynamics

$$y_c = e^{rt} \left[A_1 \cos(bt) + A_2 \sin(bt) \right]$$

We do not discuss in detail here.

Stability: does $y_c \to 0$?

- $r_1, r_2 \in \mathbb{R}$: need both $r_1, r_2 < 0$.
- $r_1 = r_2 = r \in \mathbb{R}$: need $r < 0$.
- $r_{1,2} = r \pm bi$ complex roots: need $r < 0$.

Why do we need both $A_1 e_2^{r_1 t}$ and $A_2 e^{r_2 t}$ in $y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$ when there are two distinct real roots? Denote $y_1 = A_1 e^{r_1 t}$ and $y_2 = A_2 e^{r_2 t}$. First, note that since both y_1 and y_2 solve the homogenous equation and because the homogenous equation is *linear*, then $y_1 + y_2$ also solves the homogenous equation and therefore $y_c = y_1 + y_2$ is a solution. To see why the *general solution* must have this form, denote $y(t_0) = y_0$ and $y'(t_0) = y_1$ as the initial conditions for the problem and suppose that *w.o.l.g.* $t_0 = 0$. Then

$$\begin{aligned} y(0) &= A_1 e^{r_1 0} + A_2 e^{r_2 0} = A_1 + A_2 = y_0 \\ y'(0) &= r_1 A_1 e^{r_1 0} + r_2 A_2 e^{r_2 0} = r_1 A_1 + r_2 A_2 = y_1 \ , \end{aligned}$$

which implies

$$\underbrace{\begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix}}_{R} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

and since $r_1 \neq r_2$, the matrix R is nonsingular so that given any values for y_0 and y_1 there is a unique solution for A_1 and A_2. They may be equal, but we do not want to impose this *ex ante*.

When there is only one real root the matrix R is singular and therefore this general form for the solution does not work: there would be an infinite number of solutions. That is why we use $y_1 = A_1 e^{rt}$ and $y_2 = A_2 t e^{rt}$. As before, since each one solves the homogenous equation and because the homogenous equation is *linear*, then $y_1 + y_2$ also solves the homogenous equation and therefore $y_c = y_1 + y_2$ is a solution. Following the steps from above,

$$\begin{aligned} y(0) &= A_1 e^{r0} + A_2 0 e^{r0} = A_1 = y_0 \\ y'(0) &= r A_1 e^{r0} + A_2 \left[e^{r0} + r0 e^{r0} \right] = r A_1 + A_2 = y_1 \ , \end{aligned}$$

which implies

$$\underbrace{\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}}_{R} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \ .$$

Now the matrix R is nonsingular so that given any values for y_0 and y_1 there is a unique solution for A_1 and A_2.

16.2 Differential equations with moving constant

$$y'' + a_1 y' + a_2 y = b(t) \ ,$$

where a_1 and a_2 are constants. We require that $b(t)$ takes a form that combines a finite number of "elementary functions", e.g. kt^n, e^{kt}, etc. We find y_c in the same way as above, because we consider the homogenous equation where $b(t) = 0$. We find y_p by using some educated guess and verify our guess by using the **method of undetermined coefficients**. There is no general solution procedure for any type of $b(t)$.

Example: polynomial $b(t)$:

$$y'' + 5y' + 3y = 6t^2 - t - 1 \ .$$

Guess:

$$y_p = \varphi_2 t^2 + \varphi_1 t + \varphi_0 \ .$$

This implies

$$\begin{aligned} y_p' &= 2\varphi_2 t + \varphi_1 \\ y_p'' &= 2\varphi_2 \ . \end{aligned}$$

Plug this into y_p to get

$$\begin{aligned} y'' + 5y' + 3y &= 2\varphi_2 + 5(2\varphi_2 t + \varphi_1) + 3\left(\varphi_2 t^2 + \varphi_1 t + \varphi_0\right) \\ &= 3\varphi_2 t^2 + (10\varphi_2 + 3\varphi_1) t + (2\varphi_2 + 5\varphi_1 + 3\varphi_0) \ . \end{aligned}$$

we need to solve

$$\begin{aligned} 3\varphi_2 &= 6 \\ 10\varphi_2 + 3\varphi_1 &= -1 \\ 2\varphi_2 + 5\varphi_1 + 3\varphi_0 &= -1 \ . \end{aligned}$$

This gives $\varphi_2 = 2$, $\varphi_1 = -7$, $\varphi_0 = 10$. Thus

$$y_p = 2t^2 - 7t + 10 \ .$$

But this may not always work. For instance, if

$$y'' + a_1 y' + a_2 y = t^{-1} \ .$$

Then no guess of the type $y_p = \varphi t^{-1}$ or $y_p = \varphi \ln t$ will work.

Example: missing $y(t)$ and polynomial $b(t)$

$$y'' + 5y' = 6t^2 - t - 1 \ .$$

The former type of guess,

$$y_p = \varphi_2 t^2 + \varphi_1 t + \varphi_0 \ ,$$

will not work, because φ_0 will never show up in the equation, so cannot be recovered. Instead, try

$$y_p = t\left(\varphi_2 t^2 + \varphi_1 t + \varphi_0\right) \ .$$

If this fails, try

$$y_p = t^2\left(\varphi_2 t^2 + \varphi_1 t + \varphi_0\right) \ ,$$

and so on.

Example: exponential $b(t)$

$$y'' + a_1 y' + a_2 y = Be^{rt} \ .$$

Guess:

$$y_p = Ate^{rt}$$

with the same r and look for solutions for A. The guess $y_p = Ae^{rt}$ will not work. E.g.

$$y'' + 3y' - 4y = 2e^{-4t} \ .$$

Guess:

$$\begin{aligned} y_p &= Ate^{-4t} \\ y_p' &= Ae^{-4t} + -4Ate^{-4t} = Ae^{-4t}(1-4t) \\ y_p'' &= -4Ae^{-4t}(1-4t) + -4Ae^{-4t} = Ae^{-4t}(-8+16t) \ . \end{aligned}$$

Plug in the guess

$$\begin{aligned} y'' + 3y' - 4y &= Ae^{-4t}(-8+16t) + 3Ae^{-4t}(1-4t) + -4Ate^{-4t} \\ &= Ae^{-4t}(-8+16t+3-12t-4t) \\ &= -5Ae^{-4t} \end{aligned}$$

We need to solve

$$-5Ae^{-4t} = 2e^{-4t}$$

so $A = -0.4$ and

$$y_p = -0.4te^{-4t} \ .$$

17 First order difference equations

$$y_{t+1} + ay_t = c \ .$$

As with differential equations, we wish to trace out a path for some variable y over time, i.e. we seek $y(t)$. But now time is discrete, which gives rise to some peculiarities. Define

$$\Delta y_t \equiv y_{t+1} - y_t \ ,$$

(not the standard notation) which is like

$$\frac{\Delta y_t}{\Delta t} = \frac{y_{t+\Delta t} - y_t}{\Delta t} \ ,$$

where $\Delta t = 1$ ($\dot{y}$ is when $\Delta t \to 0$).

17.1 Backward iteration

1. $\Delta y_t = y_{t+1} - y_t = c$.

$$\begin{aligned} y_1 &= y_0 + c \\ y_2 &= y_1 + c = y_0 + c + c = y_0 + 2c \\ y_3 &= y_2 + c = y_0 + 2c + c = y_0 + 3c \\ &\vdots \\ y_t &= y_0 + ct \ . \end{aligned}$$

2. $ay_{t+1} - by_t = 0$, $a \neq 0$. Then $y_{t+1} = ky_t$, where $k = b/a$.

$$\begin{aligned} y_1 &= ky_0 \\ y_2 &= ky_1 = k^2 y_0 \\ &\vdots \\ y_t &= k^t y_0 \ . \end{aligned}$$

17.2 General solution

$$y_{t+1} + ay_t = c \ ,$$

where $a \neq 0$. The solution method involves splitting the solution into two:

$$y(t) = y_c(t) + y_p(t) \ ,$$

where $y_p(t)$ is a **particular solution** and $y_c(t)$ is a **complementary function**.

- $y_c(t)$ solves the homogenous equation

$$y_{t+1} + ay_t = 0 \ .$$

Guess

$$y_t = Ab^t$$

so that $y_{t+1} + ay_t = 0$ implies

$$\begin{aligned} Ab^{t+1} + aAb^t &= 0 \\ b + a &= 0 \\ b &= -a \ . \end{aligned}$$

$$y_c(t) = A(-a)^t \ ,$$

where $a \neq 0$.

- $a \neq -1$. $y_p(t)$ solves the original equation for a stationary solution, $y_t = k$, a constant. This implies

$$\begin{aligned} k + ak &= c \\ k &= \frac{c}{1+a} \ . \end{aligned}$$

So that

$$y_p = \frac{c}{1+a} \ , \quad a \neq -1 \ .$$

- $a = -1$. Guess $y_p(t) = kt$. This implies

$$\begin{aligned} k(t+1) - kt &= c \\ k &= c \ . \end{aligned}$$

So that

$$y_p = ct \ , \quad a = -1 \ .$$

The general solution is

$$y_t = y_c(t) + y_p(t) = \begin{cases} A(-a)^t + \frac{c}{1+a} & \text{if} \quad a \neq -1 \\ A + ct & \text{if} \quad a = -1 \end{cases} \ .$$

Given an initial condition $y(0) = y_0$, then

- for $a \neq -1$

$$y_0 = A + \frac{c}{1+a} \quad \Rightarrow \quad A = y_0 - \frac{c}{1+a} \ .$$

- for $a = -1$

$$y_0 = A \ .$$

The general solution is

$$y_t = \begin{cases} \left[y_0 - \frac{c}{1+a}\right](-a)^t + \frac{c}{1+a} & \text{if} \quad a \neq -1 \\ y_0 + ct & \text{if} \quad a = -1 \end{cases} .$$

For $a \neq -1$ we have

$$y_t = y_0 (-a)^t + \left[1 - (-a)^t\right] \frac{c}{1+a} \ ,$$

which is a linear combination of the initial point and the stationary point $\frac{c}{1+a}$. And if $a \in (-1, 1)$, then this process is stable. Otherwise it is not. For $a = -1$ and $c \neq -1$ the process is never stable.

Example:

$$y_{t+1} - 5y_t = 1 \ .$$

First, notice that $a \neq -1$ and $a \neq 0$. y_c solves

$$y_{t+1} - 5y_t = 0 \ .$$

Let $y_c(t) = Ab^t$, so that

$$\begin{aligned} Ab^{t+1} - 5Ab^t &= 0 \\ Ab^t (b-5) &= 0 \\ b &= 5 \ , \end{aligned}$$

so that

$$y_c(t) = A5^t \ .$$

$y_p = k$ solves

$$\begin{aligned} k - 5k &= 1 \\ k &= -1/4 \ , \end{aligned}$$

so that $y_p = -1/4$.

$$y_t = y_c(t) + y_p(t) = A5^t - 1/4 \ .$$

Given $y_0 = 7/4$ we have $A = 2$, which completes the solution.

17.3 Dynamic stability

Given

$$y_t = \left[y_0 - \frac{c}{1+a}\right] b^t + \frac{c}{1+a} \ ,$$

the dynamics are governed by $b\,(= -a)$.

1. $b < 0$ will give rise to oscillating dynamics.

   - $-1 < b < 0$: oscillations diminish over time. In the limit we converge on the stationary point $\frac{c}{1+a}$.
   - $b = -1$: constant oscillations.
   - $b < -1$: growing oscillations over time. The process is divergent.

2. $b = 0$ and $b = 1$: no oscillations, but this is degenerate.

   - $b = 0$ means $a = 0$, so $y_t = c$.
   - $b = 1$ means $a = -1$, so $y_t = y_0 + ct$.

3. $0 < b < 1$ gives convergence to the stationary point $\frac{c}{1+a}$.

4. $b > 1$ gives divergent dynamics.

Only $|b| < 1$ gives convergent dynamics.

17.4 Application: cobweb model

This is an early model of agriculture markets. Farmers determined supply last year based on the prevailing price at that time. Consumers determine demand based on current prices. Thus, three equations complete the description of this model

$$\begin{aligned}
\text{supply} &: \quad q^s_{t+1} = s\,(p_t) = -\gamma + \delta p_t \\
\text{demand} &: \quad q^d_{t+1} = d\,(p_{t+1}) = \alpha - \beta p_{t+1} \\
\text{equilibrium} &: \quad q^s_{t+1} = q^d_{t+1} \ ,
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta > 0$. Imposing equilibrium:

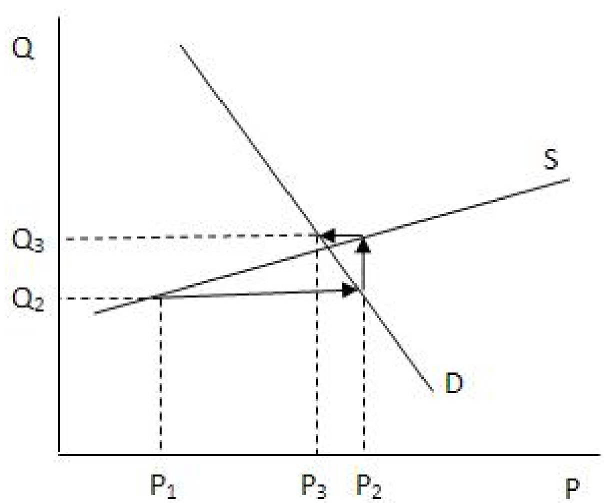
$$\begin{aligned}
-\gamma + \delta p_t &= \alpha - \beta p_{t+1} \\
p_{t+1} + \underbrace{\left(\frac{\delta}{\beta}\right)}_{a} p_t &= \underbrace{\frac{\alpha + \gamma}{\beta}}_{c} \ .
\end{aligned}$$

The solution to this difference equation is

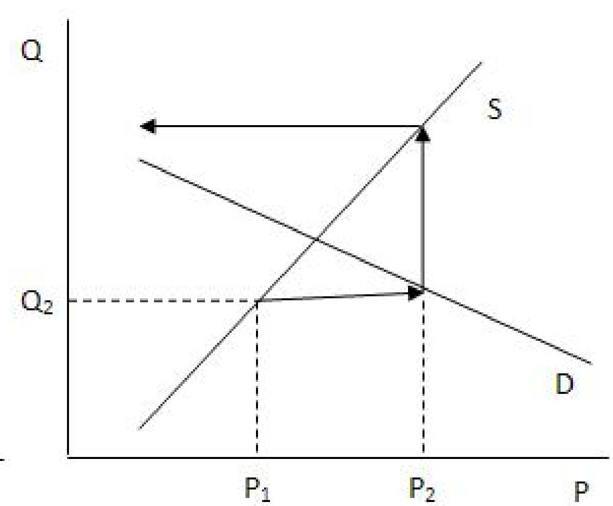
$$p_t = \left[p_0 - \frac{\alpha+\gamma}{\beta+\delta}\right]\left(-\frac{\delta}{\beta}\right)^t + \frac{\alpha+\gamma}{\beta+\delta} \ .$$

The process is convergent (stable) iff $|\delta| < |\beta|$. Since both are positive, we need $\delta < \beta$.

Interpretation: what are β and δ? These are the slopes of the demand and supply curves, respectively. If follows that if the slope of the supply curve is lower than that of the demand curve, then the process if convergent. I.e., as long as the farmers do not "overreact" to current prices next year, the market will converge on a happy stable equilibrium price and quantity. Conversely, as long as consumers are not "insensitive" to prices, then...



Stable Cobweb Dynamics



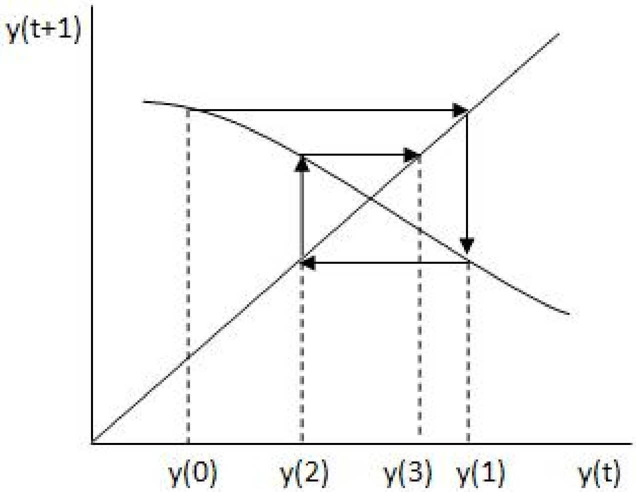
Unstable Cobweb Dynamics

17.5 Nonlinear difference equations

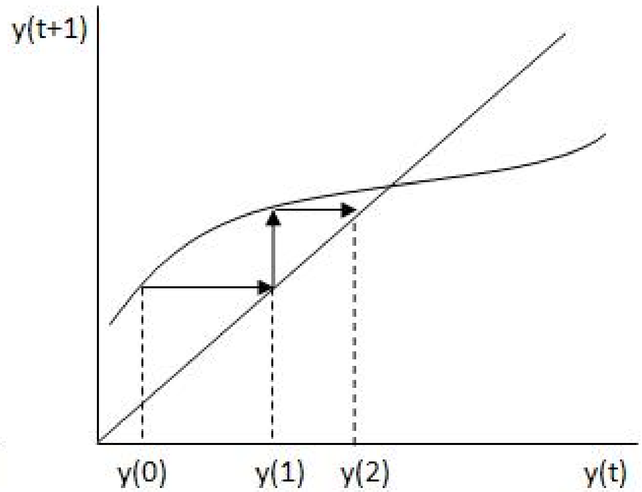
We will use only a qualitative/graphic approach and restrict to autonomous equations, in which t is not explicit. Let

$$y_{t+1} = \varphi(y_t) \ .$$

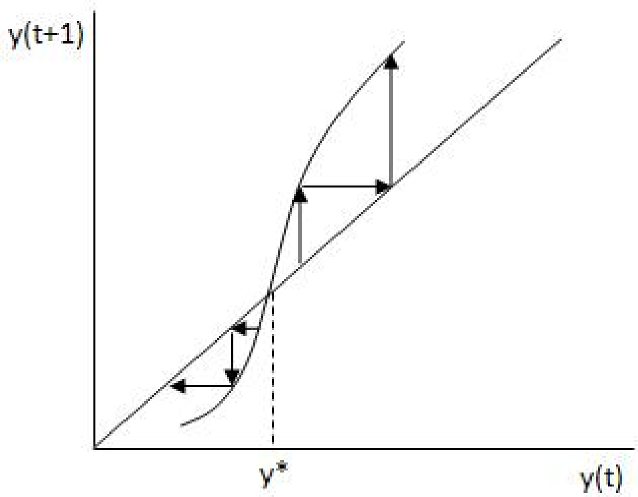
Draw a phase diagram with y_{t+1} on the vertical axis and y_t on the horizontal axis and the 45 degree ray starting from the origin. For simplicity, $y > 0$. A stationary point satisfies $y = \varphi(y)$. But sometimes the stationary point is not stable. If $|\varphi'(y)| < 1$ at the stationary point, then the process is stable. More generally, as long as $|\varphi'(y_t)| < 1$ the process is stable, i.e. it will converge to some stationary point. When $|\varphi'(y_t)| \geq 1$ the process will diverge.



Stable Nonlinear Cobweb Difference Equation



Stable Nonlinear Difference Equation



Unstable Nonlinear Difference Equation

- Example: Galor and Zeira (1993), REStud.

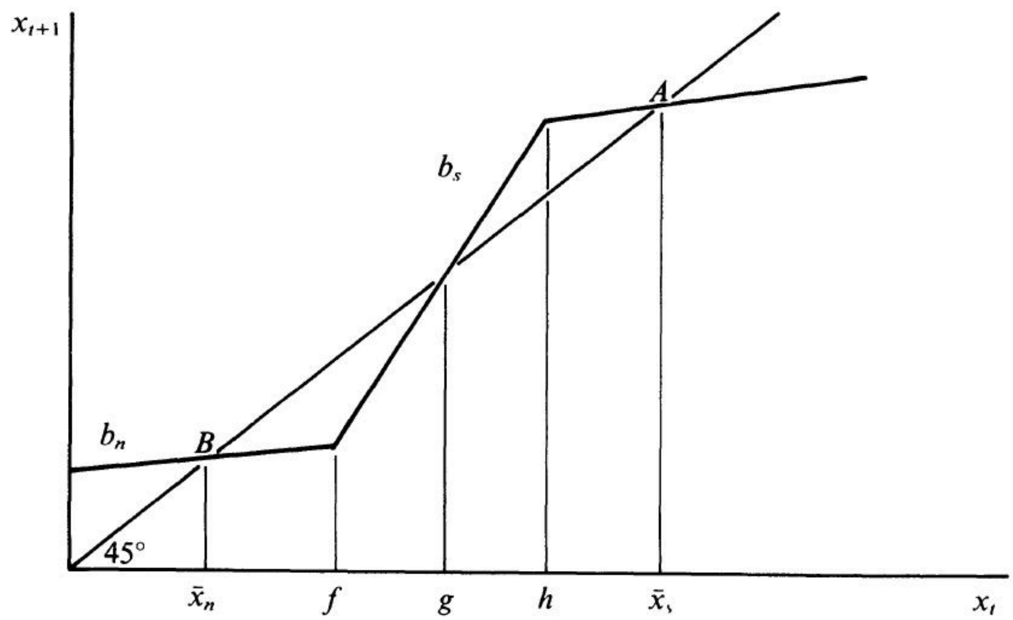


FIGURE 1

18 Phase diagrams with two variables (CW 19.5)

We now analyze a system of two autonomous differential equations:

$$\begin{aligned}\dot{x} &= F(x,y) \\ \dot{y} &= G(x,y)\ .\end{aligned}$$

First we find the $\dot{x}=0$ and $\dot{y}=0$ loci by setting

$$\begin{aligned}F(x,y) &= 0 \\ G(x,y) &= 0\ .\end{aligned}$$

Apply the implicit function theorem separately to the above, which gives rise to two (separate) functions:

$$\begin{aligned}\dot{x} &= 0 : y = f_{\dot{x}=0}(x) \\ \dot{y} &= 0 : y = g_{\dot{y}=0}(x)\ ,\end{aligned}$$

where

$$\begin{aligned}f' &= -\frac{F_x}{F_y} \\ g' &= -\frac{G_x}{G_y}\ .\end{aligned}$$

Now suppose that we have enough information about F and G to characterize f and g. And suppose that f and g intersect, which is the interesting case, because this gives rise to a stationary point (x^*, y^*), in which both x and y are constant:

$$f_{\dot{x}=0}(x^*) = g_{\dot{y}=0}(x^*) = y^*\ .$$

There are two interesting cases, although you can characterize the other ones, once you do this.

18.1 Case 1: dynamic stability

$$\begin{aligned}\dot{x} &: F_x < 0,\ F_y > 0 \\ \dot{y} &: G_x > 0,\ G_y < 0\ .\end{aligned}$$

Both f and g are upward sloping and f is steeper than g at the intersection: $f'(x^*, y^*) > g'(x^*, y^*)$.

Consider a point on the $f_{\dot{x}=0}$ locus. Now suppose that you move slightly above it or slightly below it. How does this affect the $\dot{x}$? And similarly for points slightly above or below the $\dot{y}$ locus.

By looking at the partial derivatives of F and G:

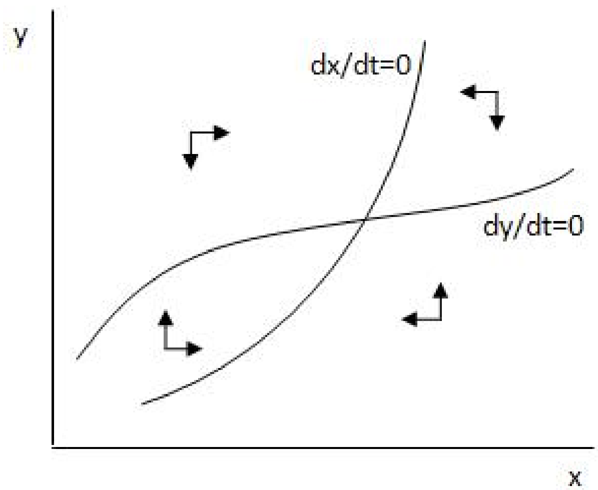
- at all points to the right of the $f_{\dot{x}=0}$ locus (or above the $f_{\dot{x}=0}$ locus) $\dot{x} < 0$ and in all points to the left of the $f_{\dot{x}=0}$ locus $\dot{x} > 0$ ($F_x < 0$).

- at all points above the $g_{\dot{y}=0}$ locus $\dot{y} < 0$ and in all points below the $g_{\dot{y}=0}$ locus $\dot{y} > 0$ ($G_y < 0$).

Given an intersection, this gives rise to four regions in the (x, y) space:

1. Below $f_{\dot{x}=0}$ and above $g_{\dot{y}=0}$: $\dot{x} < 0$ and $\dot{y} < 0$.
2. Above $f_{\dot{x}=0}$ and above $g_{\dot{y}=0}$: $\dot{x} > 0$ and $\dot{y} < 0$.
3. Above $f_{\dot{x}=0}$ and below $g_{\dot{y}=0}$: $\dot{x} > 0$ and $\dot{y} > 0$.
4. Below $f_{\dot{x}=0}$ and below $g_{\dot{y}=0}$: $\dot{x} < 0$ and $\dot{y} > 0$.

This gives rise to a stable system. From any point in the (x, y) space we converge to (x^*, y^*).



Dynamically Stable Phase Diagram

Given the values that $\dot{x}$ and $\dot{y}$ take (given the direction in which the arrows point in the figure), we can draw trajectories. In this case, all trajectories will eventually arrive at the stationary point at the intersection of $\dot{x} = 0$ and $\dot{y} = 0$.

- **Notice that at the point in which we cross the $\dot{x} = 0$ the trajectory is vertical. Similarly, at the point in which we cross the $\dot{y} = 0$ the trajectory is horizontal.** This will become important below.

18.2 Case 2: saddle point

$$\begin{aligned} \dot{x} &: \quad F_x > 0,\ F_y < 0 \\ \dot{y} &: \quad G_x < 0,\ G_y > 0\ . \end{aligned}$$

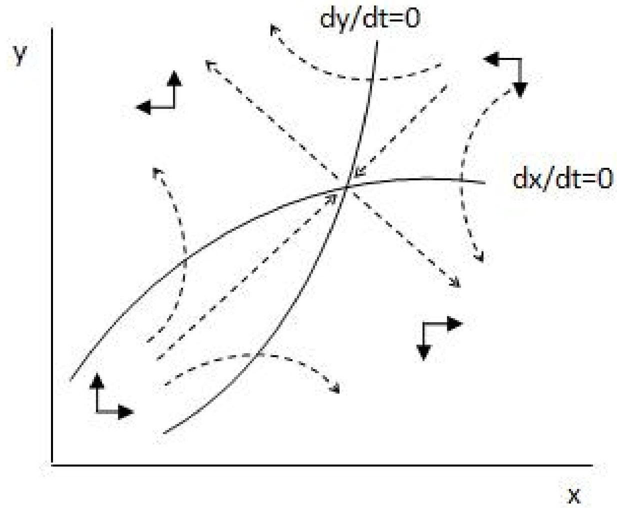
Both f and g are upward sloping and g is steeper than f at the intersection: $f'(x^*, y^*) < g'(x^*, y^*)$. Notice that

- in all points above $f_{\dot{x}=0}$ $\dot{x} < 0$ and in all points below $f_{\dot{x}=0}$ $\dot{x} > 0$.
- in all points above $g_{\dot{y}=0}$ $\dot{y} > 0$ and in all points below $g_{\dot{y}=0}$ $\dot{y} < 0$.

Given an intersection, this gives rise to four regions in the (x, y) space:

1. Below $f_{\dot{x}=0}$ and above $g_{\dot{y}=0}$: $\dot{x} > 0$ and $\dot{y} > 0$.
2. Above $f_{\dot{x}=0}$ and above $g_{\dot{y}=0}$: $\dot{x} < 0$ and $\dot{y} > 0$.
3. Above $f_{\dot{x}=0}$ and below $g_{\dot{y}=0}$: $\dot{x} < 0$ and $\dot{y} < 0$.
4. Below $f_{\dot{x}=0}$ and below $g_{\dot{y}=0}$: $\dot{x} > 0$ and $\dot{y} < 0$.

This gives rise to an *un*stable system. However, there is a stationary point at the intersection, (x^*, y^*). In order to converge to (x^*, y^*) there are only two trajectories that bring us there, one from the region above $f_{\dot{x}=0}$ and below $g_{\dot{y}=0}$, the other from the region below $f_{\dot{x}=0}$ and above $g_{\dot{y}=0}$. These trajectories are called **stable branches**. If we are not on those trajectories, then we are on *un*stable branches. Note that being in either region does not ensure that we are on a stable branch, as the figure illustrates.



Saddle Point Phase Diagram

19 Optimal control

Like in static optimization problems, we want to maximize (or minimize) an objective function. The difference is that the objective is the sum of a path of values at any instant in time; therefore, we must choose an entire path as a maximizer.[2]

The problem is generally stated as follows:

$$\begin{aligned}&\text{Choose } u(t) \text{ to maximize } \int_0^T F(y,u,t)\,dt \\ &\text{s.t.}\end{aligned}$$

$$\begin{aligned}\text{Law of motion} &: \dot{y} = g(y,u,t) \\ \text{Initial condition} &: y(0) = y_0 \\ \text{Transversality condition} &: y(T)\,e^{-\bar{r}T} \geq 0 \ .\end{aligned}$$

where $\bar{r}$ is some average discount rate that is relevant to the problem. To this we need to sometimes add

$$\begin{aligned}\text{Terminal condition} &: y(T) = y_T \\ \text{Constraints on control} &: u(t) \in U\end{aligned}$$

The function $y(t)$ is called the **state variable**. The function $u(t)$ is called the **control variable**. It is useful to think of the state as a stock (like capital) and the control as a flow (like investment or consumption). Usually we will have $F, g \in C^1$, but in principle we could do without differentiability with respect to u. I.e., we only need that the functions F and g are continuously differentiable with respect to y and t.

In a finite horizon problem $(T < \infty)$ $e^{-\bar{r}T} > 0$ so that the transversality condition immediately implies that $y(T) \geq 0$, but also something more: If this constraint binds, then $y(T) = 0$. Either way, this tells you that the value of y at the end of the problem cannot be negative. This will become clearer below, when we discuss the Lagrangian approach.

- If there is no law of motion for y, then we can solve the problem separately at any instant as a static problem. The value would just be the sum of those static values.
- There is no uncertainty here. To deal with uncertainty, wait for your next course in math.
- To ease notation we will omit time subscripts when there is no confusion.

[2] The theory behind this relies on "calculus of variations", which was first developed to compute trajectories of missiles (to the moon and elsewhere) in the U.S.S.R.

Example: the saving/investment problem for individuals.

1. Output: $Y = F(K, L)$.
2. Investment/consumption: $I = Y - C = F(K, L) - C$.
3. Capital accumulation: $\dot{K} = I - \delta K$.

We want to maximize the present value of instantaneous utility from now (at $t = 0$) till we die (at some distant time T). The problem is stated as

$$\text{Choose } C(t) \text{ to maximize } \int_0^T e^{-\rho t} U[C(t)]\, dt$$
s.t.

$$\begin{aligned} \dot{K} &= I - \delta K \\ K(0) &= K_0 \\ K(T) &= K_T \ . \end{aligned}$$

19.1 Pontryagin's maximum principle and the Hamiltonian function

Define the **Hamiltonian function:**

$$H(y, u, t, \lambda) = F(y, u, t) + \lambda g(y, u, t) \ .$$

The function $\lambda(t)$ is called the **co-state function** and also has a law of motion. Finding λ is part of the solution. The FONCs of this problem ensure that we maximize H at every point in time, and as a whole. If u^* is a maximizing plan then

$$\begin{aligned} \text{(i)} &: H(y, u^*, t, \lambda) \geq H(y, u, t, \lambda) \ \forall u \in U \\ \text{or} &: \frac{\partial H}{\partial u} = 0 \text{ if } F, g \in C^1 \\ \text{State equation (ii)} &: \frac{\partial H}{\partial \lambda} = \dot{y} \quad \Rightarrow \quad \dot{y} = g(y, u, t) \\ \text{Costate equation (iii)} &: \frac{\partial H}{\partial y} = -\dot{\lambda} \quad \Rightarrow \quad \dot{\lambda} + F_y + \lambda g_y = 0 \\ \text{Transversality condition (iv)} &: \lambda(T) = 0 \text{ or other (see below)} \ . \end{aligned}$$

Conditions (ii)+(iii) are a system of first order differential equations that can be solved explicitly if we have functional forms and two conditions: $y(0) = y_0$ and $\lambda(T) = 0$. But $\lambda(T) = 0$ is only one way to get a transversality/terminal condition.

- **Interpretation of the Hamiltonian**: u and y affect the value of the problem directly through F. But they also affect the value of the problem indirectly, through their effect on $\dot{y}$. This is captured by λg. So in this context λ is the cost/benefit of allowing y to grow a bit faster. So λ has the same interpretation as the Lagrange multiplier: it is the shadow cost of the constraint at any instant.

We adopt the convention that $y(0) = y_0$ is always given. There are a few way to introduce **terminal conditions,** which gives the following taxonomy

1. When T is fixed, i.e. the problem must end at T.

   (a) $\lambda(T) = 0$, $y(T)$ free.

   (b) $y(T) = y_T$, $\lambda(T)$ free.

   (c) $y(T) \geq y_{\min}$ (or $y(T) \leq y_{\max}$), $\lambda(T)$ free. Add the following complementary slackness conditions:

$$\begin{aligned} y(T) &\geq y_{\min} \\ \lambda(T) &\geq 0 \\ \lambda(T)(y(T) - y_{\min}) &= 0 \end{aligned}$$

2. T is free and $y(T) = y_T$, i.e. you finish whenever $y(T)$ hits y_T. Add $H(T) = 0$.

3. $T \leq T_{\max}$ (or $T \geq T_{\min}$) and $y(T) = y_T$, i.e. you finish whenever $y(T)$ hits y_T, but this must happen before $T_{\max}$ (or after $T_{\min}$). Add the following complementary slackness conditions:

$$\begin{aligned} H(T) &\geq 0 \\ T &\leq T_{\max} \\ H(T)(T_{\max} - T) &= 0 \end{aligned}$$

19.2 The Lagrangian approach

This is based on Barro and Sala-i-Martin (2001), *Economic Growth*, MIT Press, third edition.

The problem is

$$\begin{aligned} &\text{Choose } u(t) \text{ to maximize } \int_0^T F(y, u, t)\, dt \\ &\text{s.t.} \end{aligned}$$

$$
\begin{aligned}
\dot{y} &= g(y,u,t) \\
y(T)e^{-rT} &\geq 0 \\
y(0) &= y_0 \ .
\end{aligned}
$$

You can think of $\dot{y} = g(y,u,t)$ as an inequality $\dot{y} \leq g(y,u,t)$; this is the correct way to think about it when $F_y > 0$. We can write this up as a Lagrangian. For this we need Lagrange multipliers for the law of motion constraint at every point in time, as well as an additional multiplier for the transversality condition:

$$
\begin{aligned}
\mathcal{L} &= \int_0^T F(y,u,t)\,dt + \int_0^T \lambda(t)\left[g(y,u,t) - \dot{y}\right]dt + \theta y(T)e^{-rT} \\
&= \int_0^T \underbrace{\left[F(y,u,t) + \lambda(t)\,g(y,u,t)\right]}_{H(y,u,t,\lambda)}dt - \int_0^T \lambda(t)\,\dot{y}(t)\,dt + \theta y(T)e^{-rT} \ .
\end{aligned}
$$

In this context, both u and y are part of the "critical path" (paraphrasing critical point). The problem here is that we do not know how to take the derivative of $\dot{y}$ w.r.t. y. To avoid this, use integration by parts to get

$$
-\int \lambda\dot{y}dt = -\lambda y + \int \dot{\lambda} y dt
$$

so that

$$
\begin{aligned}
\mathcal{L} &= \int_0^T \left[F(y,u,t) + \lambda(t)\,g(y,u,t)\right]dt - \left[\lambda(t)\,y(t)\right|_0^T + \int_0^T \dot{\lambda}(t)\,y(t)\,dt + \theta y(T)e^{-rT} \\
&= \int_0^T \left[F(y,u,t) + \lambda(t)\,g(y,u,t)\right]dt - \lambda(T)\,y(T) + \lambda(0)\,y(0) + \int_0^T \dot{\lambda}(t)\,y(t)\,dt + \theta y(T)e^{-rT} \ .
\end{aligned}
$$

The FONCs for the Lagrangian are

$$
\begin{aligned}
\text{(i)} &: \ \mathcal{L}_u = F_u + \lambda g_u = 0 \\
\text{(ii)} &: \ \mathcal{L}_y = F_y + \lambda g_y + \dot{\lambda} = 0 \\
\text{(iii)} &: \ \dot{y} = g \ .
\end{aligned}
$$

These are consistent with

$$
\begin{aligned}
\text{(i)} &: \ H_u = F_u + \lambda g_u = 0 \\
\text{(ii)} &: \ H_y = F_y + \lambda g_y = -\dot{\lambda} \\
\text{(iii)} &: \ H_\lambda = g = \dot{y} \ ,
\end{aligned}
$$

which are the FONCs for the Hamiltonian

$$H(y, u, t, \lambda) = F(y, u, t) + \lambda(t)\, g(y, u, t) \ .$$

The requirement that $y(0) = y_0$ can also be captured in the usual way, as well as $y(T) = y_T$, if it is required. The transversality condition is captured by the complementary slackness conditions

$$\begin{aligned} y(T)\, e^{-rT} &\geq 0 \\ \theta &\geq 0 \\ \theta y(T)\, e^{-rT} &= 0 \ . \end{aligned}$$

We see here that if $y(T)\, e^{-rT} > 0$, then its value, θ, must be zero.

19.3 Autonomous problems

In these problems t is not an explicit argument.

$$\text{Choose } u \text{ to maximize } \int_0^T F(y, u)\, dt \text{ s.t. } \dot{y} = g(y, u)$$

plus boundary conditions. The Hamiltonian is thus

$$H(y, u, \lambda) = F(y, u) + \lambda g(y, u) \ .$$

These problems are easier to solve and are amenable to analysis by phase diagrams. This type of problem appears more often in economic applications than problems in which t is explicit (but see current value Hamiltonian below).

19.3.1 Example: the cake eating problem (with no discounting)

Objective: You want to eat your cake in an optimal way, maximizing your satisfaction from eating it, starting now ($t = 0$) and finishing before bedtime, at T.

- The cake starts at size S_0.
- When you eat cake, the size diminishes by the amount that you ate: $\dot{S} = -C$.
- Inada conditions: You like cake, but less so when you eat more: $U'(C) > 0$, $U''(C) < 0$. If you are not eating cake $U(0) = 0$, but you really want some: $U'(0) = \infty$. Eventually, too much cake is no longer beneficial: $U'(\infty) = 0$.

The problem is

$$\text{Choose } C \text{ to maximize } \int_0^T U(C)\, dt \text{ s.t.}$$

$$\begin{aligned}\dot{S} &= -C \\ S(0) &= S_0 \\ S(T) &\geq 0 \ .\end{aligned}$$

This is an autonomous problem. The Hamiltonian is

$$H(C, S, \lambda) = U(C) + \lambda[-C] \ .$$

FONCs:

$$\begin{aligned}\text{(i)} &: \quad \frac{\partial H}{\partial C} = U'(C) - \lambda = 0 \\ \text{(ii)} &: \quad \frac{\partial H}{\partial \lambda} = -C = \dot{S} \\ \text{(iii)} &: \quad \frac{\partial H}{\partial S} = 0 = -\dot{\lambda} \\ \text{(iv)} &: \quad S(T) \geq 0, \ \lambda(T) \geq 0, \ S(T)\lambda(T) = 0 \ .\end{aligned}$$

From (iii) it follows that λ is constant. From (i) we have $U'(C) = \lambda > 0$, and since λ is constant, C is constant too. Then given a constant C we get from (ii) that

$$S = A - Ct \ .$$

And given $S(0) = S_0$ we have

$$S = S_0 - Ct \ .$$

But we still do not know what is C, except that it is constant. So we solve for the complementary slackness conditions, i.e., will we leave leftovers?

Suppose $\lambda > 0$. Then $S(T) = 0$. Therefore

$$0 = S_0 - CT \ ,$$

which gives

$$C = \frac{S_0}{T} \ .$$

Suppose $\lambda = 0$. Then it is possible to have $S(T) > 0$. But then we get $U' = 0$—a contradiction.

The solution is thus

$$\begin{aligned}C(t) &= S_0/T \\ \lambda(t) &= U'(S_0/T) \\ S(t) &= S_0 - (S_0/T)\, t \ ,\end{aligned}$$

where only $S(t)$ evolves over time and C and λ are constants.

If we allowed a flat part in the utility function after some satiation point, then we could have a solution with leftovers $S(T) > 0$. In that case we would have more than one optimal path: all would be global because with one flat part U is still quasi concave.

- Try solving this problem with cake depreciation: $\dot{S} = -C - \delta S$. In order to solve completely you will need to make a functional form assumption on U, but even without this you can characterize the solution very accurately.

19.3.2 Anecdote: the value of the Hamiltonian is constant in autonomous problems

We demonstrate that on the optimal path the value of the Hamiltonian function is constant.

$$H(y, u, t, \lambda) = F(y, u, t) + \lambda g(y, u, t) \ .$$

The derivative with respect to time is

$$\frac{dH}{dt} = H_u \dot{u} + H_y \dot{y} + H_\lambda \dot{\lambda} + H_t \ .$$

The FONCs were

$$\begin{aligned} H_u &= 0 \\ H_y &= -\dot{\lambda} \\ H_\lambda &= \dot{y} \ . \end{aligned}$$

Plugging this into dH/dt gives

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \ .$$

This is a consequence of the envelope theorem. If time is not explicit in the problem, then $\frac{\partial H}{\partial t} = 0$, which implies the statement above. The interpretation is that since the optimal path gives a critical point of the Hamiltonian at every moment in time, the value does not change. This is slightly more complicated than the Lagrangian due to the dynamics, but the reasoning is the same. If time was explicit then this reasoning does not work, because we do not choose t optimally.

19.4 Current value Hamiltonian

Many problems in economics involve discounting, so the problem is not autonomous. However, usually the only place that time is explicit is in the discount factor,

$$\int_0^T F(y, u, t)\, dt = \int_0^T e^{-rt} \cdot G(y, u)\, dt \ .$$

You can try to solve those problems "as-is", but an easier way (especially if the costate is of no particular interest) is to use the **current value Hamiltonian:**

$$\widetilde{H} = e^{rt} H = G(y,u) + \varphi g(y,u) \ ,$$

where

$$\varphi = \lambda e^{rt} \ .$$

A maximizing plan u^* satisfies the following FONCs:

$$\begin{aligned}
\text{(i)} &: \widetilde{H}(y,u^*,\varphi) \geq \widetilde{H}(y,u,\varphi) \ \ \forall u \in U \\
\text{or} &: \frac{\partial \widetilde{H}}{\partial u} = 0 \text{ if } \widetilde{H}, g \in C^1 \\
\text{State equation (ii)} &: \frac{\partial \widetilde{H}}{\partial \varphi} = \dot{y} \quad \Rightarrow \quad \dot{y} = g(y,u) \\
\text{Costate equation (iii)} &: \frac{\partial \widetilde{H}}{\partial y} = -\dot{\varphi} + r\varphi \quad \Rightarrow \quad \dot{\varphi} - r\varphi + F_y + \lambda g_y = 0 \\
\text{Transversality condition (iv)} &: \varphi(T) = 0 \text{ or } \widetilde{H}(T) = 0 \text{ or other.}
\end{aligned}$$

Since $\varphi = \lambda e^{rt}$ we have

$$\dot{\varphi} = \dot{\lambda} e^{rt} + \lambda r e^{rt} = \dot{\lambda} e^{rt} + r\varphi \ .$$

Therefore

$$-\dot{\lambda} e^{rt} = -\dot{\varphi} + r\varphi \ ,$$

which is what

$$\frac{\partial \widetilde{H}}{\partial y} = \frac{\partial}{\partial y}\left[e^{rt} H\right] = e^{rt}\frac{\partial H}{\partial y} = -\dot{\lambda} e^{rt} \ .$$

implies.

You can derive the FONCs of the current value Hamiltonian $\widetilde{H}$ by multiplying the FONCs of the regular Hamiltonian H by e^{rt}. Multiplying an objective function by a positive constant (in this case, e^{rt}, or any other strictly positive monotone transformation) does not change the optimal solution, just the value of the problem.

19.4.1 Example: the cake eating problem with discounting

We now need to choose a functional form for the instantaneous utility function. The problem is

$$\text{Choose } C \text{ to maximize } \int_0^T e^{-rt} \ln(C)\, dt \text{ s.t.}$$

$$\begin{aligned} \dot{S} &= -C \\ S(0) &= S_0 \\ S(T) &\geq 0 \ . \end{aligned}$$

We write the present value Hamiltonian

$$\widetilde{H} = \ln C + \varphi [-C]$$

FONCs:

$$\begin{aligned} \text{(i)} &: \frac{\partial \widetilde{H}}{\partial C} = \frac{1}{C} - \varphi = 0 \\ \text{(ii)} &: \frac{\partial \widetilde{H}}{\partial \varphi} = -C = \dot{S} \\ \text{(iii)} &: \frac{\partial \widetilde{H}}{\partial S} = 0 = -\dot{\varphi} + r\varphi \\ \text{(iv)} &: S(T) \geq 0,\ \varphi(T) \geq 0,\ S(T)\varphi(T) = 0 \ . \end{aligned}$$

From (iii) we have

$$\frac{\dot{\varphi}}{\varphi} = r \ ,$$

hence

$$\varphi = Be^{rt} \ ,$$

for some B. From (i) we have

$$C = \frac{1}{\varphi} = \frac{1}{B} e^{-rt} \ .$$

Consumption is falling over time, at rate r. From (ii) we have

$$\begin{aligned} \dot{S} &= -C \\ \int_0^t \dot{S} dz &= \int_0^t -C dz \\ S(t) - S(0) &= \int_0^t -C dz \ , \end{aligned}$$

which, together with $S(0) = S_0$ implies

$$S(t) = S_0 - \int_0^t C dz \ ,$$

which makes sense. Now, using $C = B^{-1}e^{-rt}$ we get

$$\begin{aligned} S(t) &= S_0 - \int_0^t B^{-1}e^{-rz}dz \\ &= S_0 - B^{-1}\left[-\frac{1}{r}e^{-rz}\right|_0^t \\ &= S_0 - B^{-1}\left[-\frac{1}{r}e^{-rt} + \frac{1}{r}e^{-r0}\right] \\ &= S_0 - \frac{1}{rB}\left[1 - e^{-rt}\right] \end{aligned}$$

Suppose $\varphi(T) = 0$. Then from (i) $C(T) = \infty$, which is not possible. So $\varphi(T) > 0$, which implies $S(T) = 0$. Therefore

$$\begin{aligned} 0 &= S_0 - \frac{1}{rB}\left[1 - e^{-rT}\right] \\ B &= \frac{\left[1 - e^{-rT}\right]}{rS_0} \end{aligned}$$

Therefore

$$C = \frac{rS_0}{\left[1 - e^{-rT}\right]}e^{-rt} ,$$

which is decreasing, and

$$\varphi = \frac{\left[1 - e^{-rT}\right]}{rS_0}e^{rt} ,$$

which is increasing. And finally

$$S(t) = S_0\left[1 - \frac{1 - e^{-rt}}{1 - e^{-rT}}\right] .$$

This completes the characterization of the problem.

19.5 Infinite time horizon

When the problem's horizon is infinite, i.e. never ends, we need to modify the transversality condition. These are

$$\lim_{T\to\infty} \lambda(T)\, y(T) = 0$$

for the present value Hamiltonian, and

$$\lim_{T\to\infty} \varphi(T)\, e^{-rT} k(T) = 0$$

for the current value Hamiltonian.

19.5.1 Example: The neoclassical growth model

1. Preferences: $u(C)$, $u' > 0$, $u'' < 0$. Inada conditions: $u(0) = 0$, $u'(0) = \infty$, $u'(C) \to 0$ as $C \to \infty$.

2. Aggregate production function: $Y = F(K, L)$, CRS, $F_i > 0$, $F_{ii} < 0$. Given this we can write the per-worker version $y = f(k)$, where $f' > 0$, $f'' < 0$ and $y = Y/L$, $k = K/L$. Additional Inada conditions: $f(0) = 0$, $f'(0) = \infty$, $f'(\infty) = 0$ (these are stability conditions).

3. Capital accumulation: $\dot{K} = I - \delta K = Y - C - \delta K$. As we saw in the Solow model, we can write this in per worker terms $\dot{k} = f(k) - c - (n + \delta) k$, where n is the constant growth rate of labor force.

4. There cannot be negative consumption. In addition, once output is converted into capital, we cannot eat it. This can be summarized in $0 \le C \le F(K, L)$. This is an example of a restriction on the control variable.

5. A social planner chooses a consumption plan to maximize everyone's welfare, in equal weights. The objective function is

$$V = \int_0^\infty L_0 e^{nt} \cdot e^{-\rho t} u(c)\, dt = \int_0^\infty e^{-rt} u(c)\, dt \ ,$$

   where we normalize $L_0 = 1$ and we set $r = \rho - n > 0$, which ensures integrability. Notice that everyone gets the average level of consumption $c = C/L$.

The problem is

Choose c to maximize V s.t.

$$\begin{aligned} \dot{k} &= f(k) - c - (n + \delta) k \\ 0 &\le c \le f(k) \\ k(0) &= k_0 \end{aligned}$$

Write down the current value Hamiltonian

$$H = u(c) + \varphi \left[f(k) - c - (n + \delta) k \right] \ .$$

FONCs:

$$\begin{aligned} H_c &= u'(c) - \varphi = 0 \\ H_\varphi &= \left[f(k) - c - (n + \delta) k \right] = \dot{k} \\ H_k &= \varphi \left[f'(k) - (n + \delta) \right] = r\varphi - \dot{\varphi} \end{aligned}$$

$$\lim_{T\to\infty} \varphi(T)\, e^{-rT} k(T) = 0$$

Ignore for now $0 \le c \le f(k)$. The transversality condition here is a sufficient condition for a maximum, although in general this specific condition is not necessary. If this was a present value Hamiltonian the same transversality condition would be $\lim_{T\to\infty} \lambda(T)\, k(T) = 0$, which just means that the value of an additional unit of capital in the limit is zero.

From H_c we have $u'(c) = \varphi$. From H_k we have

$$\frac{\dot{\varphi}}{\varphi} = -\left[f'(k) - (n+\delta+r)\right] \ .$$

We want to characterize the solution qualitatively using a phase diagram. To do this, we need two equations: one for the state, k, and one for the control, c. Taking derivatives w.r.t. time of $\varphi = u'(c)$ we get

$$\dot{\varphi} = u''(c)\, \dot{c} \ ,$$

so

$$\frac{\dot{\varphi}}{\varphi} = \frac{u''(c)\, \dot{c}}{u'(c)} = -\left[f'(k) - (n+\delta+r)\right] \ .$$

Rearrange to get

$$\frac{\dot{c}}{c} = -\frac{u'(c)}{cu''(c)}\left[f'(k) - (n+\delta+r)\right] \ .$$

Notice that

$$-\frac{cu''(c)}{u'(c)}$$

is the coefficient of relative risk aversion. Let

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \ .$$

This is a class of constant relative relative risk aversion (CRRA) utility functions, with coefficient of RRA $= \sigma$.

Eventually, our two equations are

$$\begin{aligned} \dot{k} &= f(k) - c - (n+\delta)\, k \\ \frac{\dot{c}}{c} &= \frac{1}{\sigma}\left[f'(k) - (n+\delta+r)\right] \ . \end{aligned}$$

From this we derive

$$\begin{aligned} \dot{k} &= 0 : c = f(k) - (n+\delta)\, k \\ \dot{c} &= 0 : f'(k) = n+\delta+r \ . \end{aligned}$$

The $\dot{c} = 0$ locus is a vertical line in the (k, c) space. Given the Inada conditions and diminishing returns to capital, we have that the $\dot{k} = 0$ locus is hump shaped. The peak of the hump is found by maximizing c: Choose k to maximize $f(k) - (n + \delta) k$. The FONC implies

$$f'(k) = n + \delta \ .$$

Since $r > 0$, the peak of the hump is to the right of the vertical $\dot{c} = 0$ locus (if $f'(k)$ is higher, then k is lower, because $f'' < 0$).

The phase diagram features a saddle point, with two stable branches. If k is to the right of the $\dot{c} = 0$ locus, then $\dot{c} < 0$ and vice versa for k to the left of the $\dot{c} = 0$ locus. For c above the $\dot{k} = 0$ locus we have $\dot{k} < 0$ and vice versa for c below the $\dot{k} = 0$ locus. See textbook for figure.

Define the stationary point as (k^*, c^*). Suppose that we start with $k_0 < k^*$. Then the optimal path for consumption must be on the stable branch, i.e. c_0 is on the stable branch, and $c(t)$ will eventually go to c^*. The reason is that any other choice is not optimal. Higher consumption will eventually lead to depletion of the capital stock, which eventually leads to no output and therefore no consumption (U.S.A.). Too little consumption will lead first to an increase in the capital stock and an increase in output, but eventually this is not sustainable as the plan requires more and more consumption forgone to keep up with effective depreciation $(n + \delta)$ and eventually leads to zero consumption as well (U.S.S.R.).

One can do more than just analyze the phase diagram. First, given functional forms we can compute the exact paths for all dynamic variables. Second, we could linearize (a first order Taylor expansion) the system of differential equations around the saddle point to compute dynamics around that point (or any other point, for that matter).