ECON 2015-1

Exam 1

Carlos Hurtado
Game Theory

1 Multiple Choice

[7 pt] Indicate the most correct answers in each of the following questions. Please note that there will be negative marking for incorrect answers.

1.1) The question: Do zero-sum games have a solution? was first answered in a general context (more than five strategies) by:
a. French economist Antoine Augustine Cournot in 1838.
b. Hungarian Mathematician Von Neumann and the Austrian economist Morgenstern in 1944.
c. French mathematician Emile Borel in 1921.
d. Hungarian mathematician John Von Neumann in 1928.

1.2) An example of a game that challenges the assumption of rationality is:
a. centipede game.
b. predation game.
c. ultimatum game.
d. none of the above.

1.3) Game theory is concerned with:
a. predicting the results of bets placed on games like roulette.
b. the choice of an optimal strategy in conflict situations.
c. utility maximization by firms in perfectly competitive markets.
d. the way in which a player can win every game.

1.4) Which one of the following is a part of every game theory model?
a. Players
b. Payoffs
c. Probabilities
d. Strategies

1.5) In game theory, a situation in which one firm can gain only what another firm loses is called a:
a. nonzero-sum game.
b. prisoners' dilemma.
c. zero-sum game.
d. Predation game.

1.6) Which of the following circumstances will result in a Nash equilibrium?
a. All players have a dominated strategy and each player chooses its dominated strategy.
b. All players have a dominated strategy, but only some choose to follow it.
c. All players have a dominated strategy, and none choose it.
d. None of the above is correct.

1.7) Which of the following is a nonzero-sum game?
a. Prisoners' dilemma
b. Chess
c. Competition among duopolists when market share is the payoff
d. All of the above.

1.8) Which of the following is a zero-sum game?
a. Prisoners' dilemma
b. Chess
c. A cartel member's decision regarding whether or not to cheat
d. All of the above.

1.9) A game that involves multiple moves in a series of identical situations is called a:
a. sequential game.
b. repeated game.
c. zero-sum game.
d. nonzero-sum game.

1.10) A prisoners' dilemma is a game with all of the following characteristics except one. Which one is not present in a prisoners' dilemma?
a. Players cooperate as their strategy.
b. Both players have a dominant strategy.
c. Both players would be better off if neither chose their dominant strategy.
d. The payoff from a strategy depends on the choice made by the other player.

2 Nash Equilibrium (MWG exercise 8.D.6)

[5 pt] Consider any two-player game of the following form (where letters indicate arbitrary payoffs), with the usual convention of first component of payoffs for player 1 and second for player 2:

$1/2$	b_1	b_2
a_1	u, v	l, m
a_2	w, x	y, z

Assume further that $u \neq w$, $l \neq y$, $v \neq m$ and $x \neq z$. Show that a mixed strategy Nash equilibrium (MSNE) always exist in this game.

Hint: Define player 1's strategy to be his probability of choosing action a_1 and players 2's to be his probability of choosing b_1, then examine the best-response of the two players

Answer:

Case 1: $u > w$ and $l > y$. In this case player 1 always plays his dominant strategy a_1. Player 2 will play his best response to this strategy, that is, b_1 if $v > m$, or b_2 if $v < m$; otherwise player 2 will play a 50-50 lottery between $b_1$and b_2.

Case 2: $u < w$ and $l < y$. In this case player 1 always plays his dominant strategy a_2. Player 2 will play his best response to this strategy, that is, b_1 if $x > z$, or b_2 if $x < z$; otherwise player 2 will play a 50-50 lottery between $b_1$and b_2.

Case 3: $v > m$ and $x > z$. In this case player 2 plays his dominant strategy b_1. Player 1 will play his best response to this strategy, that is, a_1 if $u > w$, or $a_2$if $u < w$; otherwise player 1 will play a 50-50 lottery between $a_1$and a_2.

Case 4: $v < m$ and $x < z$. In this case player 2 plays his dominant strategy b_2. Player 1 will play his best response to this strategy, that is, a_1 if $l > m$, or $a_2$if $l < m$; otherwise player 1 will play a 50-50 lottery between $a_1$and a_2.

Case 5: all other cases: Let μ be the probability that player 1 assigns to the strategy a_1. Then, $E_2\left[b_1\right] = v \cdot \mu + x \cdot (1-\mu)$ and $E_2\left[b_2\right] = m \cdot \mu + z \cdot (1-\mu)$. In a MSNE, player 2 must be indifferent between any strategy. Hence,

$$v \cdot \mu + x \cdot (1-\mu) = m \cdot \mu + z \cdot (1-\mu) \implies \mu = \frac{(z-x)}{(z-x)+(v-m)}.$$

Additionally, let β be the probability that player 2 assigns to the strategy b_1. Then, $E_1\left[a_1\right] = u \cdot \beta + l \cdot (1-\beta)$ and, $E_1\left[a_2\right] = w \cdot \beta + y \cdot (1-\beta)$. In a MSNE, player 2 must be indifferent between any strategy. Hence,

$$u \cdot \beta + l \cdot (1-\beta) = w \cdot \beta + y \cdot (1-\beta) \implies \beta = \frac{(y-l)}{(y-l)+(u-w)}.$$

Then, the MSNE is

$$\begin{aligned} \text{player1} &: \left(\frac{(z-x)}{(z-x)+(v-m)}, \frac{(v-m)}{(z-x)+(v-m)}\right) \\ \text{player 2} &: \left(\frac{(y-l)}{(y-l)+(u-w)}, \frac{(u-w)}{(y-l)+(u-w)}\right) \end{aligned}$$

3 Studying for the Exam

Consider the following game: $n > 1$ students would like to get together and discuss exercises to prepare for the game theory exam. Each student i should choose simultaneously the number of hours $s_i \in [0, n]$ that should study before the meeting. The preparation for the exam is proportional to the number of hours that the n students study and, each student has a cuadraticaly increasing disutility by the hours spend preparing the meeting. That can be summarized by the following utility function:

$$u_i(s_i, s_{-i}) = \sum_{j=1}^{n} s_j - \frac{s_i^2}{2} = \sum_{j \neq i} s_j + s_i - \frac{s_i^2}{2}$$

Let $T_{-i} = \sum_{j \neq i} s_j$ be the total number of hours spend by the other students studying. The utility of player i can be reduced to

$$u_i(s_i, s_{-i}) = T_{-i} + s_i - \frac{s_i^2}{2}$$

a. [2 pt] Is this game solvable by iterated elimination of (strictly) dominated strategies? Justify your answer. If so, find the solution.

b. [2 pt] Find the pure strategy Nash equilibrium (PSNE) of the game. Justify your answer.

c. [1 pt] Find the social optimum of the game, that is, the profile of strategies $s = (s_i, s_{-i})$ that solvesmax$_{s_i, s_{-i}} \sum_{i=1}^{n} u_i(s_i, s_{-i})$. Explain why is this different from the Nash equilibrium.

Answer:

a. Note that, player i takes $T_{-i} \in [0, (n-1) \cdot n]$ as given. Moreover, $\frac{\partial u_i}{\partial s_i} = 1 - s_i$, that is, if player i plays $\tilde{s}_i < 1$ then $T_{-i} + \tilde{s}_i - \frac{\tilde{s}_i^2}{2} < T_{-i} + \frac{1}{2}$. This implies that player i strategy $s_i = 1$ strictly dominates $s_i < 1$. Moreover, if player i plays $\tilde{s}_i > 1$ then $T_{-i} + \tilde{s}_i - \frac{\tilde{s}_i^2}{2} < T_{-i} + \frac{1}{2}$. Hence, player i strategy $s_i = 1$ strictly dominates $s_i > 1$. In summary, the game is solvable by iterated elimination of strictly dominated strategies and the solution is $s = (1, 1, \cdots, 1)$. In this solution all players (students) choose to study one hour to prepare the meeting.

b. Given that the iterated delation of strictly dominated strategies reaches a unique solution, this is the unique PSNE, by the theorem proved in class.

c. The social optimum is reached by maximizing

$$\sum_{i=1}^{n} u_i(s_i, s_{-i}) = n \sum_{i=1}^{n} s_i - \frac{1}{2} \sum_{i=1}^{n} s_i^2$$

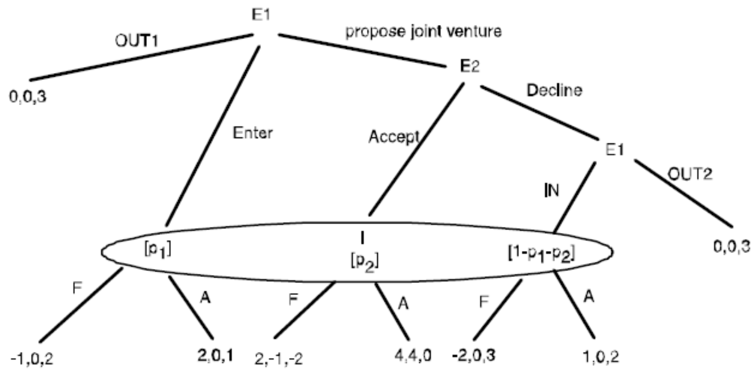
Then, the F.O.C. are

$$n - s_i = 0$$

The previous implies that social optimum is reached when each player chooses $s_i = n$. The difference between this solution and the one reached using the concept of Nash equilibrium is due to the fact that in the social optimum the social utility aggregates the effort (study) that each player (student) chooses whereas in the Nash equilibrium each player (student) maximizes his strategy, regardless of others strategies.

4 Weak Perfect Bayesian Nash Equilibria -WPBNE- (MWG Example 9.C.2)

Consider the following game of imperfect information played by a first Entrant (E_1), a second Entrant (E_2), and an Incumbent (I), where the payoffs of the player E_1 are in the first component, the payoffs of player E_2 are in the second, and the payoffs of the player I are in the third component:



a. [2 pt] For what values of p_1 and p_2 does the incumbent I choose F over A?

b. [1 pt] Is there a weak perfect Bayesian equilibrium in pure strategies in which E_1 chooses OUT1 to start the game? If so, then fully describe the equilibrium.

c. [1 pt] Fully describe a weak perfect Bayesian equilibrium in pure strategies in which E_1 proposes the joint venture and E_2 accepts.

d. [1 pt] Is your answer to c a sequential equilibrium? Explain.

Answer:

a. $E_I[F|p_1, p_2] = 2p_1 - 2p_2 + 3(1 - p_1 - p_2) > p_1 + 0p_2 + 2(1 - p_1 - p_2) = E_I[A|p_1, p_2]$. Then,

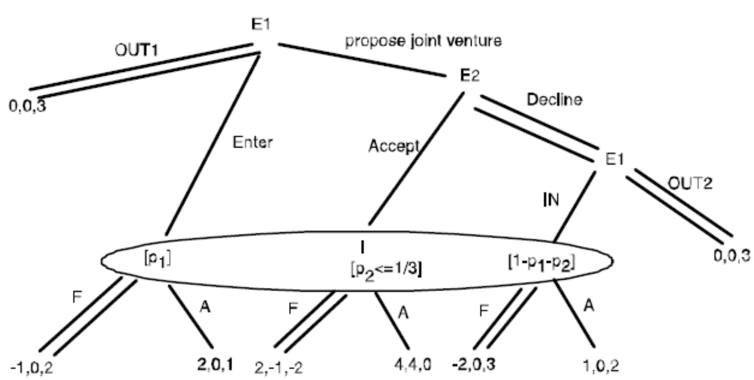
$$\begin{aligned} 3 - p_1 - 5p_2 &> 2 - p_1 - 2p_2 \\ 1 &> 3p_2 \\ \frac{1}{3} &> p_2 \end{aligned}$$

Hence, for $p_2 \in [0, 1/3)$ and $p_1 \in [1/3, 1]$.

b. If E1 is to choose OUT1 to start the game, then it must be that E2 would decline E1's offer if he proposed a joint venture (both of E1's payoffs are positive if E2 accepts his proposal). We must have $p_2 < 1/3$ so that I chooses F over A; if I instead chooses A over F, then E1 would choose Enter over OUT1. E1 chooses OUT2 at his right hand node, which completes the definition of the equilibrium as follows:

$$\begin{aligned} \text{player } E_1 &: (OUT1, OUT2) \\ \text{player } E_2 &: (Decline) \\ \text{player } I &: (F) \textit{ and } p_2 \in [0, 1/3) \textit{ and } p_1 \in [1/3, 1] \end{aligned}$$

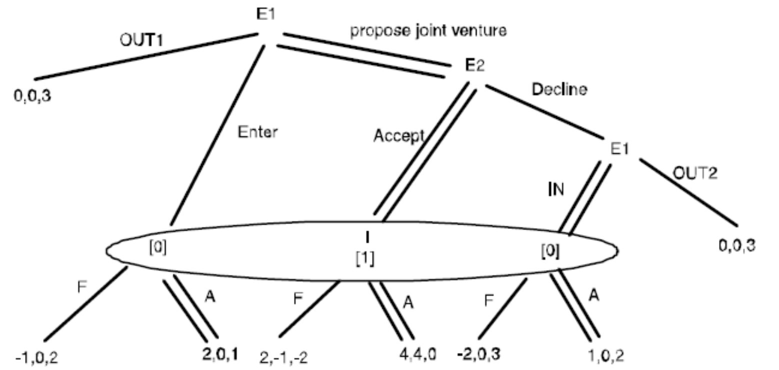
The proposed equilibrium is thus as follows:



c. a WPBNE in pure strategies in which E1 proposes the joint venture and E2 accepts is

$$\begin{aligned} \text{player } E_1 &: (Joint\ Venture, IN) \\ \text{player } E_2 &: (Accept) \\ \text{player } I &: (A) \textit{ and } p_2 = 1 \end{aligned}$$

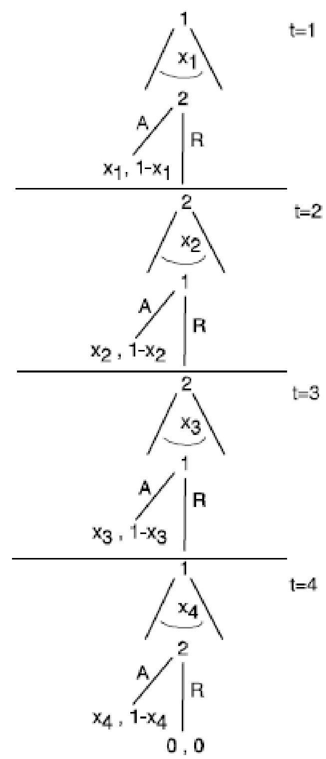
The proposed equilibrium is thus as follows:



d. Yes. I's beliefs at his information set are determined by E's mixed strategy. That is: actions are consistent with beliefs and beliefs are consistent with actions.

5 The One Million Dollar Question

Consider the following 4 stage bargaining game in which \$1 million is to be split between players 1 and 2. Notice that player 1 makes the first and last offers, while player 2 makes the second and third offers. Each player i has discount factor $\delta_i \in (0,1)$:



a. [1 pt] Determine the subgame perfect equilibrium of this game.

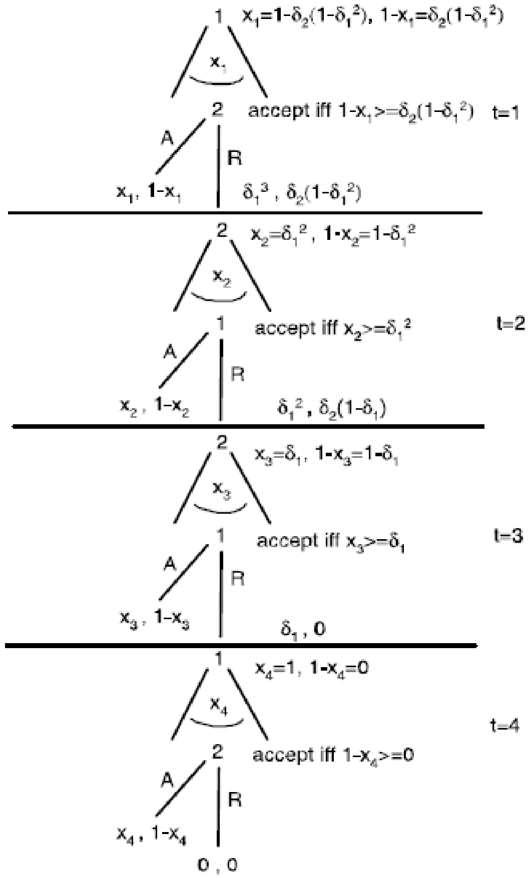
b. [1 pt] Devise a Nash equilibrium in which the million is split in stage $t = 3$, with player 1 receiving 1/3 of the million and player 2 receiving 2/3 of the million. Be sure to specify all aspects of the equilibrium.

c. [1 pt] Explain why your answer to b is not subgame perfect. Hint: Your answer need not be a lengthy discourse on subgame perfection; focus on criticizing your answer to b.

Answer:

a.

Using the strategies described in each node of the following game tree:



b. There are many possible answers to this problem. One answer is as follows:

In all stages other than stage 3, the person who makes the proposal offers 0 to the other player and proposes to keep the million dollar for himself. The responder follows the rule of rejecting all offers. In stage 3, player 2 chooses $x_3 = 1/3$ and player 1 adopts the rule, "Accept x_3 iff $x_3 \geq 1/3$." This is clearly a Nash equilibrium.

c. The flaw of the equilibrium concerns has to do with the path behavior: if player 1 were to offer player 2 to split with $1 - x_4 > 0$ in stage 4, for instance, the strategies given in b. require that 2 reject this amount in favor of a payoff equal to 0, which is not a best response in this subgame.