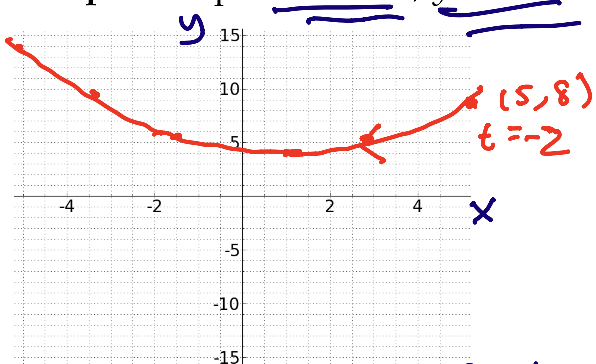
§10.1 - Parametric Equations

$y = x^2$

Definition. A **cartesian equation** for a curve is an equation in terms of x and y only.
Definition. **Parametric equations** for a curve give both x and y as functions of a third variable (usually t). The third variable is called the **parameter**.
Example. Graph $x = 1 - 2t$, $y = t^2 + 4$



arrow is direction of increasing t-values

t	x	y
-2	5	8
-1	3	5
0	1	4
1	-1	5
2	-3	8
3	-5	13

$x = 1 - 2t$ $\qquad$ $y = t^2 + 4$

solve for t

$2t = 1 - x \Rightarrow t = \frac{1-x}{2}$

Find a Cartesian equation for this curve.

$$y = \left(\frac{1-x}{2}\right)^2 + 4$$

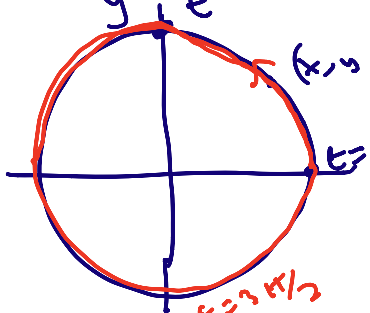
parametric to Cartesian: ① solve for t & plug in
② use relationships like $\cos^2 t+\sin^2 t=1$

Example. Plot each curve and find a Cartesian equation:

$\cos^2 t+\sin^2 t=1 \Rightarrow x^2+y^2=1$

A. $x = \cos(t)$, $y = \sin(t)$, for $0 \le t \le 2\pi$

t angle

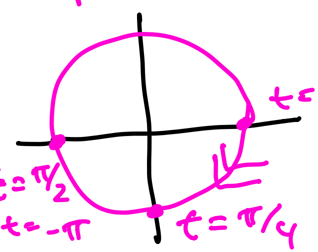


$x^2+y^2=1$

B. $x = \cos(-2t)$, $y = \sin(-2t)$, for $0 \le t \le 2\pi$

$\cos^2(-2t)+\sin^2(-2t)=1$

$x^2+y^2=1$



C. $x = \cos^2(t)$, $y = \cos(t)$

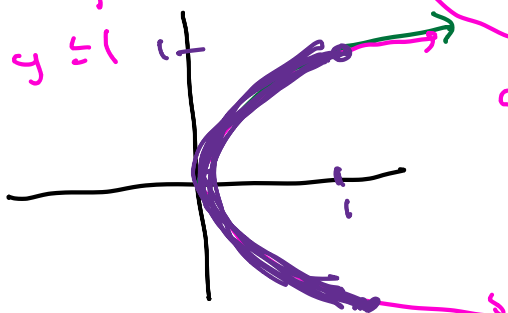
~~$y=\sqrt{x}$~~

$x=y^2$

$-1\le y\le 1$

$x\ge 0$ $y<0$ or $y>0$ is possible

don't need this

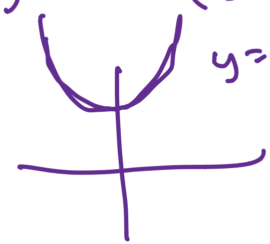


D. $x = 5\sqrt{t}$, $y = 3 + t^2$

$t=\frac{x^2}{5}$

$y=3+\left(\frac{x^2}{5}\right)^2$

$y=3+\frac{x^4}{25}$



Cartesean to parametric ① copycat parametrization

Example. Write the following in parametric equations:

1. $y = \sqrt{x^2 - x}$ for $x \leq 0$ and $x \geq 1$

$x = t$ $t \leq 0,\ t \geq 1$ copycat

$y = \sqrt{t^2 - t}$

Aside $x = 3e^y$

$y = t$

$x = 3e^t$

2. $25x^2 + 36y^2 = 900$

$$\left.\begin{aligned} 36y^2 &= 900 - 25x^2 \\ y^2 &= \frac{900 - 25x^2}{36} \\ y &= \pm\sqrt{\frac{900 - 25x^2}{36}} \end{aligned}\right\} \text{awkward}$$

$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$

$(\quad)^2 + (\quad)^2 = 1$

$\frac{25x^2}{900} + \frac{36y^2}{900} = 1 \Rightarrow \frac{x^2}{36} + \frac{y^2}{25} = 1$

$\cos t = \frac{x}{6}$ $\sin t = \frac{y}{5}$

$\Rightarrow x = 6\cos t$ $y = 5\sin t$

Example. Describe a circle with radius r and center (h, k):

a) with a Cartesian equation

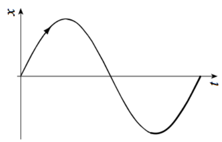
b) with parametric equations

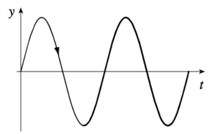
Example. Find parametric equations for a line through the points $(2, 5)$ and $(6, 8)$.

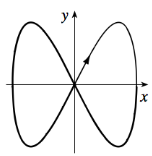
1. any way you want.

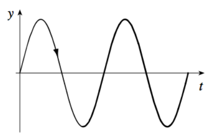
2. so that the line is at $(2, 5)$ when $t = 0$ and at $(6, 8)$ when $t = 1$.

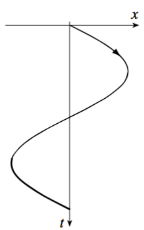
Example. Lissajous figure: $x = \sin(t)$, $y = \sin(2t)$



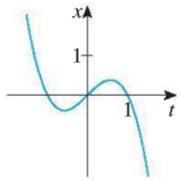


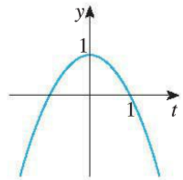






Example. Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch a graph of y in terms of x.

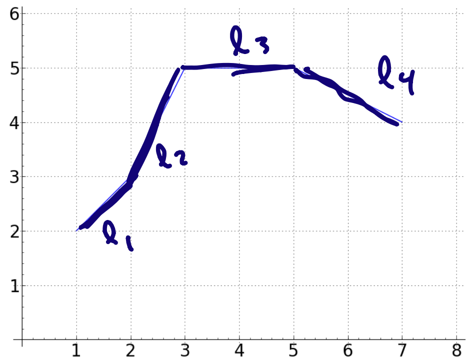


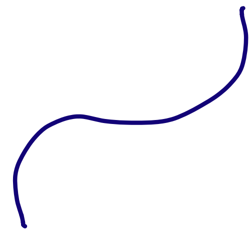


§10.2 Calculus using Parametric Equations

ARC LENGTH

Example. Find the length of this curve.





$x = f(t)$
$y = g(t)$

$\ell_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\ell_2 = \sqrt{1^2 + 2^2} = \sqrt{5}$

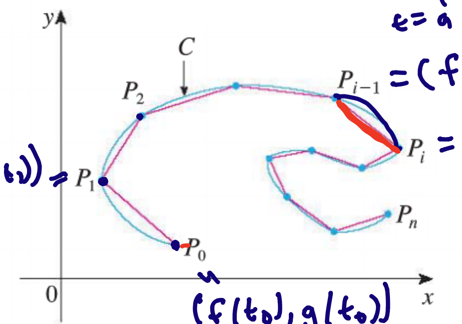
$\ell_3 = 2$

$\ell_4 = \sqrt{2^2 + 1^2} = \sqrt{5}$

Length $= \sqrt{2} + \sqrt{5} + 2 + \sqrt{5}$

Note. In general, it is possible to approximate the length of a curve $x = f(t)$, $y = g(t)$ between $t = a$ and $t = b$ by dividing it up into n small pieces and approximating each curved piece with a line segment.

$t=a \quad t_1 \quad t_2 \quad t_3 \quad \cdots \quad t=b$



$(f(t_1), g(t_1)) = P_1$

$P_{i-1} = (f(t_{i-1}), g(t_{i-1}))$

$P_i = (f(t_i), g(t_i))$

$P_0 = (f(t_0), g(t_0))$

length of line segment $\overline{P_{i-1}P_i}$

$$= \sqrt{\Delta x^2 + \Delta y^2}$$

$$= \sqrt{(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2}$$

total length of all line segments

$$= \sum_{i=1}^{n} \sqrt{(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2}\, \frac{\Delta t}{\Delta t}$$

$$= \sum_{i=1}^{n} \sqrt{\left(\frac{f(t_i) - f(t_{i-1})}{(\Delta t)}\right)^2 + \left(\frac{g(t_i) - g(t_{i-1})}{(\Delta t)}\right)^2}\, \Delta t$$

$$= \sum_{i=1}^{n} \sqrt{(f'(t_i^*))^2 + (g'(t_i^*))^2}\, \Delta t$$

distance

Arc length is given by the formula:

limit of this Riemann sum

$$\text{arc length} = \int^{b} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

on $\boxed{\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$

ds

Set up an integral to express the arclength of the Lissajous figure

$$x = \cos(t), y = \sin(2t)$$

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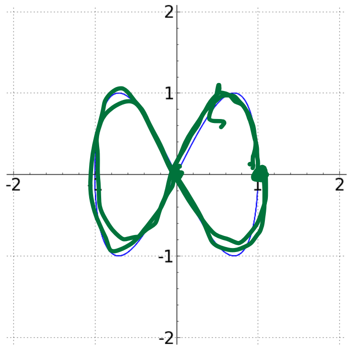


$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dt} = 2\cos(2t)$$

t	x	y
0	1	0
$\frac{\pi}{2}$	0	0
$\pi/4$	$\frac{\sqrt{2}}{2}$	1

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (2\cos 2t)^2}\, dt$$

⚠ bounds are in terms of t

Example. Write down an expression for the arc length of a curve given in Cartesian coordinates: $y = f(x)$.

$x = t$ $\quad \frac{dx}{dt} = 1$ $\qquad y = g(t)$ $\quad \frac{dy}{dt} = g'(t)$ $\qquad$ b/w $x = a$ & $x = b$

$$\text{arcLength} = \int_{t=a}^{t=b} \sqrt{1^2 + (g'(t))^2}\, dt$$

$$= \boxed{\int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Example. Find the arc length of the curve $y = \frac{1}{2}\ln(x) - \frac{x^2}{4}$ from $x = 1$ to $x = 3$.

Note: Clever trick to computing this integral

$$\frac{dy}{dx} = \frac{1}{2x} - \frac{x}{2}$$

$$\text{arclength} = \int_1^3 \sqrt{1 + \left(\frac{1}{2x} - \frac{x}{2}\right)^2}\, dx$$

looks hard to compute but there is a trick

$$= \int_1^3 \sqrt{1 + \frac{1}{4x^2} - 2\left(\frac{1}{2x}\right)\left(\frac{x}{2}\right) + \frac{x^2}{4}}\, dx$$

$$= \int_1^3 \sqrt{1 + \frac{4}{x^2} - \frac{1}{2} + \frac{x^2}{4}}\, dx$$

$$= \int_1^3 \sqrt{\frac{4}{x^2} + \frac{1}{2} + \frac{x^2}{4}}\, dx$$

notice that the $-\frac{1}{2}$ is now a $+\frac{1}{2}$ since $1 - \frac{1}{2} = \frac{1}{2}$

$$= \int_1^3 \sqrt{\left(\frac{1}{2x} + \frac{x}{2}\right)^2}\, dx$$

normally $\sqrt{(a)^2} = |a|$ not a, but since everything is positive, $\sqrt{a^2} = a$

$$= \int_1^3 \frac{1}{2x} + \frac{x}{2}\, dx$$

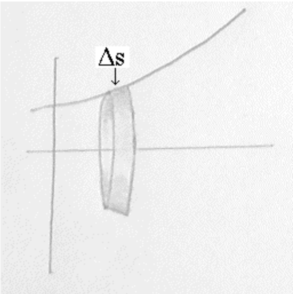
$$= \frac{1}{2}\ln|x| + \frac{x^2}{4} \Big|_1^3$$

$$= \left(\frac{1}{2}\ln 3 + \frac{9}{4}\right) - \left(\cancel{\frac{1}{2}\ln 1}^{\,0} + \frac{1}{4}\right)$$

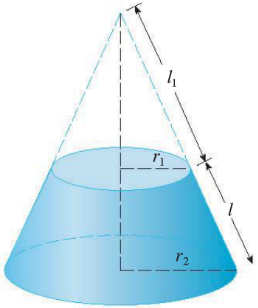
$$= \frac{1}{2}\ln 3 + 8$$

SURFACE AREA

To find the surface area of a surface of revolution, imagine approximating it with pieces of cones.



We will need a formula for the area of a piece of a cone.



The area of this piece of a cone is

$$A = 2\pi r\ell$$

where $r = \dfrac{r_1 + r_2}{2}$ is the average radius and ℓ is the length along the slant.

Use the formula for the area of a piece of cone $A = 2\pi r\ell$ to derive a formula for surface area of the surface formed by rotating a curve in parametric equations around the x-axis.

Example. Prove that the surface area of a sphere is $4\pi r^2$

Example. The infinite hotel:

1. You are hired to paint the interior surface of an infinite hotel which is shaped like the curve $y = \frac{1}{x}$ with $x \geq 1$, rotated around the x-axis. How much paint will you need? (Assume that a liter of paint covers 1 square meter of surface area, and x and y are in meters.)
2. Your co-worker wants to save time and just fill the hotel with paint to cover all the walls and then suck out the excess paint. How much paint is needed for your co-worker's scheme?