

10.4 Hyperbolas

What you should learn

- Write equations of hyperbolas in standard form.
- Find asymptotes of and graph hyperbolas.
- Use properties of hyperbolas to solve real-life problems.
- Classify conics from their general equations.

Why you should learn it

Hyperbolas can be used to model and solve many types of real-life problems. For instance, in Exercise 42 on page 761, hyperbolas are used in long distance radio navigation for aircraft and ships.



AP/Wide World Photos

Introduction

The third type of conic is called a **hyperbola**. The definition of a hyperbola is similar to that of an ellipse. The difference is that for an ellipse the *sum* of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola the *difference* of the distances between the foci and a point on the hyperbola is fixed.

Definition of Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane, the difference of whose distances from two distinct fixed points (**foci**) is a positive constant. See Figure 10.29.

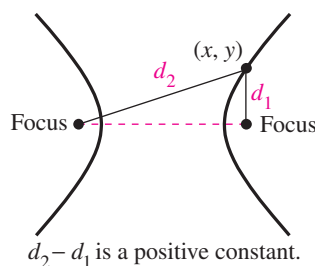


FIGURE 10.29

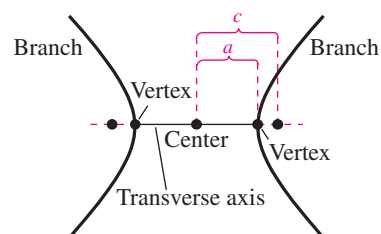


FIGURE 10.30

The graph of a hyperbola has two disconnected **branches**. The line through the two foci intersects the hyperbola at its two **vertices**. The line segment connecting the vertices is the **transverse axis**, and the midpoint of the transverse axis is the **center** of the hyperbola. See Figure 10.30. The development of the standard form of the equation of a hyperbola is similar to that of an ellipse. Note in the definition below that a , b , and c are related differently for hyperbolas than for ellipses.

Standard Equation of a Hyperbola

The **standard form of the equation of a hyperbola** with center (h, k) is

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.}$$

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1. \quad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$. If the center of the hyperbola is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{Transverse axis is vertical.}$$

Figure 10.31 shows both the horizontal and vertical orientations for a hyperbola.

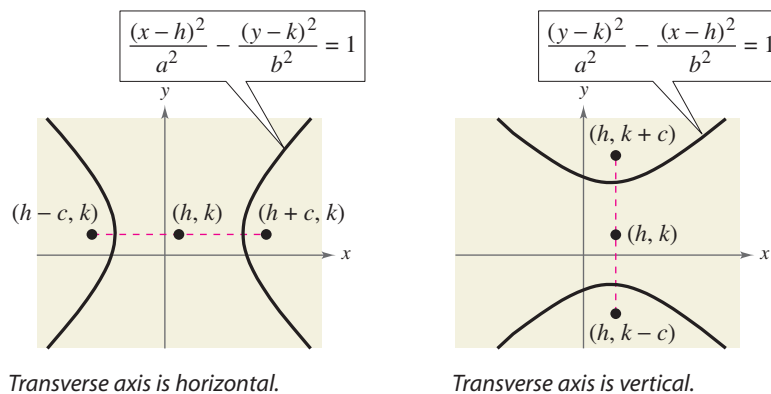


FIGURE 10.31

STUDY TIP

When finding the standard form of the equation of any conic, it is helpful to sketch a graph of the conic with the given characteristics.

Example 1 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with foci $(-1, 2)$ and $(5, 2)$ and vertices $(0, 2)$ and $(4, 2)$.

Solution

By the Midpoint Formula, the center of the hyperbola occurs at the point $(2, 2)$. Furthermore, $c = 5 - 2 = 3$ and $a = 4 - 2 = 2$, and it follows that

$$b = \sqrt{c^2 - a^2} = \sqrt{3^2 - 2^2} = \sqrt{9 - 4} = \sqrt{5}.$$

So, the hyperbola has a horizontal transverse axis and the standard form of the equation is

$$\frac{(x-2)^2}{2^2} - \frac{(y-2)^2}{(\sqrt{5})^2} = 1. \quad \text{See Figure 10.32.}$$

This equation simplifies to

$$\frac{(x-2)^2}{4} - \frac{(y-2)^2}{5} = 1.$$

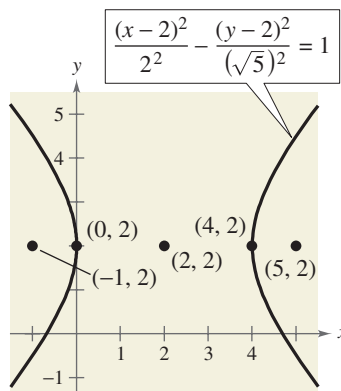


FIGURE 10.32



CHECKPOINT

Now try Exercise 27.

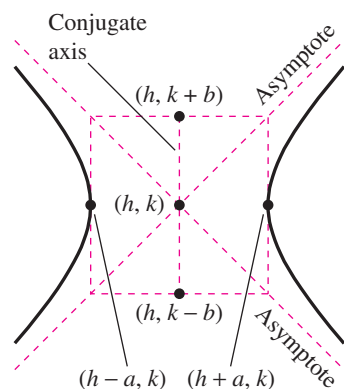


FIGURE 10.33

Asymptotes of a Hyperbola

Each hyperbola has two **asymptotes** that intersect at the center of the hyperbola, as shown in Figure 10.33. The asymptotes pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k) . The line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ [or $(h + b, k)$ and $(h - b, k)$] is the **conjugate axis** of the hyperbola.

Asymptotes of a Hyperbola

The equations of the asymptotes of a hyperbola are

$$y = k \pm \frac{b}{a}(x - h) \quad \text{Transverse axis is horizontal.}$$

$$y = k \pm \frac{a}{b}(x - h). \quad \text{Transverse axis is vertical.}$$

Example 2 Using Asymptotes to Sketch a Hyperbola

Sketch the hyperbola whose equation is $4x^2 - y^2 = 16$.

Solution

Divide each side of the original equation by 16, and rewrite the equation in standard form.

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1 \quad \text{Write in standard form.}$$

From this, you can conclude that $a = 2$, $b = 4$, and the transverse axis is horizontal. So, the vertices occur at $(-2, 0)$ and $(2, 0)$, and the endpoints of the conjugate axis occur at $(0, -4)$ and $(0, 4)$. Using these four points, you are able to sketch the rectangle shown in Figure 10.34. Now, from $c^2 = a^2 + b^2$, you have $c = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$. So, the foci of the hyperbola are $(-2\sqrt{5}, 0)$ and $(2\sqrt{5}, 0)$. Finally, by drawing the asymptotes through the corners of this rectangle, you can complete the sketch shown in Figure 10.35. Note that the asymptotes are $y = 2x$ and $y = -2x$.

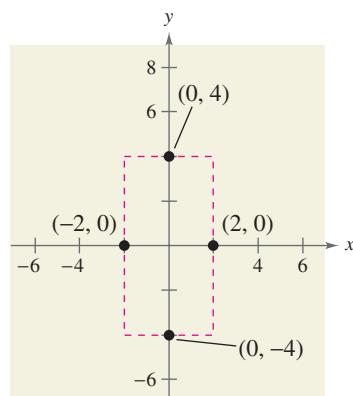


FIGURE 10.34

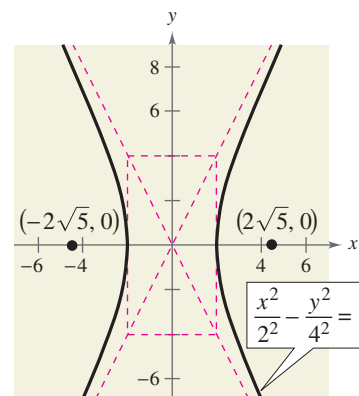


FIGURE 10.35

CHECKPOINT Now try Exercise 7.

Example 3 Finding the Asymptotes of a Hyperbola

Sketch the hyperbola given by $4x^2 - 3y^2 + 8x + 16 = 0$ and find the equations of its asymptotes and the foci.

Solution

$$4x^2 - 3y^2 + 8x + 16 = 0$$

Write original equation.

$$(4x^2 + 8x) - 3y^2 = -16$$

Group terms.

$$4(x^2 + 2x) - 3y^2 = -16$$

Factor 4 from x -terms.

$$4(x^2 + 2x + 1) - 3y^2 = -16 + 4$$

Add 4 to each side.

$$4(x + 1)^2 - 3y^2 = -12$$

Write in completed square form.

$$-\frac{(x + 1)^2}{3} + \frac{y^2}{4} = 1$$

Divide each side by -12 .

$$\frac{y^2}{2^2} - \frac{(x + 1)^2}{(\sqrt{3})^2} = 1$$

Write in standard form.

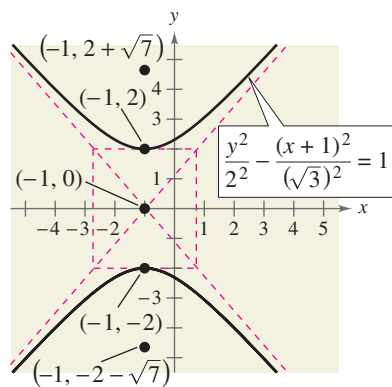


FIGURE 10.36

From this equation you can conclude that the hyperbola has a vertical transverse axis, centered at $(-1, 0)$, has vertices $(-1, 2)$ and $(-1, -2)$, and has a conjugate axis with endpoints $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$. To sketch the hyperbola, draw a rectangle through these four points. The asymptotes are the lines passing through the corners of the rectangle. Using $a = 2$ and $b = \sqrt{3}$, you can conclude that the equations of the asymptotes are

$$y = \frac{2}{\sqrt{3}}(x + 1) \quad \text{and} \quad y = -\frac{2}{\sqrt{3}}(x + 1).$$

Finally, you can determine the foci by using the equation $c^2 = a^2 + b^2$. So, you have $c = \sqrt{2^2 + (\sqrt{3})^2} = \sqrt{7}$, and the foci are $(-1, -2 - \sqrt{7})$ and $(-1, -2 + \sqrt{7})$. The hyperbola is shown in Figure 10.36.

**CHECKPOINT**

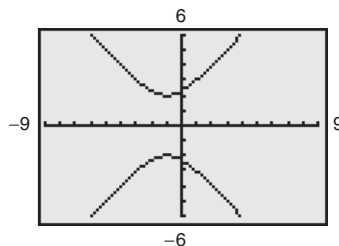
Now try Exercise 13.

Technology

You can use a graphing utility to graph a hyperbola by graphing the upper and lower portions in the same viewing window. For instance, to graph the hyperbola in Example 3, first solve for y to get

$$y_1 = 2\sqrt{1 + \frac{(x + 1)^2}{3}} \quad \text{and} \quad y_2 = -2\sqrt{1 + \frac{(x + 1)^2}{3}}.$$

Use a viewing window in which $-9 \leq x \leq 9$ and $-6 \leq y \leq 6$. You should obtain the graph shown below. Notice that the graphing utility does not draw the asymptotes. However, if you trace along the branches, you will see that the values of the hyperbola approach the asymptotes.



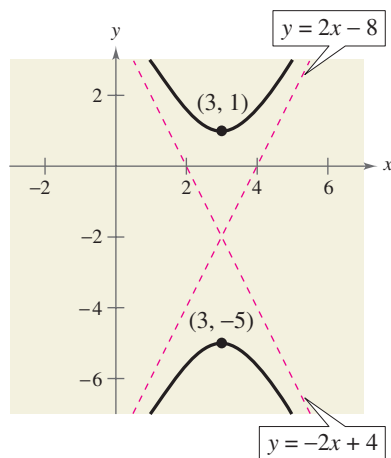


FIGURE 10.37

Example 4 Using Asymptotes to Find the Standard Equation

Find the standard form of the equation of the hyperbola having vertices $(3, -5)$ and $(3, 1)$ and having asymptotes

$$y = 2x - 8 \quad \text{and} \quad y = -2x + 4$$

as shown in Figure 10.37.

Solution

By the Midpoint Formula, the center of the hyperbola is $(3, -2)$. Furthermore, the hyperbola has a vertical transverse axis with $a = 3$. From the original equations, you can determine the slopes of the asymptotes to be

$$m_1 = 2 = \frac{a}{b} \quad \text{and} \quad m_2 = -2 = -\frac{a}{b}$$

and, because $a = 3$ you can conclude

$$2 = \frac{a}{b} \quad \Rightarrow \quad 2 = \frac{3}{b} \quad \Rightarrow \quad b = \frac{3}{2}.$$

So, the standard form of the equation is

$$\frac{(y + 2)^2}{3^2} - \frac{(x - 3)^2}{\left(\frac{3}{2}\right)^2} = 1.$$

CHECKPOINT Now try Exercise 35.

As with ellipses, the *eccentricity* of a hyperbola is

$$e = \frac{c}{a} \quad \text{Eccentricity}$$

and because $c > a$, it follows that $e > 1$. If the eccentricity is large, the branches of the hyperbola are nearly flat, as shown in Figure 10.38. If the eccentricity is close to 1, the branches of the hyperbola are more narrow, as shown in Figure 10.39.

Additional Example

Find the standard form of the equation of the hyperbola having vertices $(-4, 2)$ and $(1, 2)$ and having foci $(-7, 2)$ and $(4, 2)$.

$$\text{Answer: } \frac{\left(x + \frac{3}{2}\right)^2}{\left(\frac{5}{2}\right)^2} - \frac{(y - 2)^2}{(\sqrt{24})^2} = 1$$

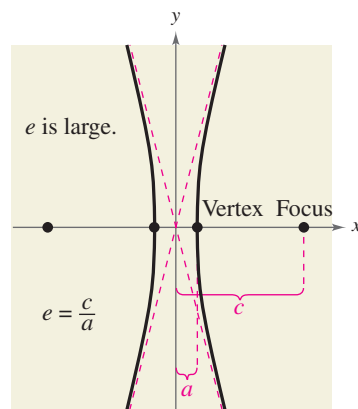


FIGURE 10.38

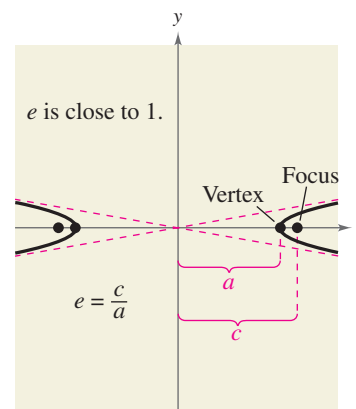


FIGURE 10.39

Applications

The following application was developed during World War II. It shows how the properties of hyperbolas can be used in radar and other detection systems.

Example 5 An Application Involving Hyperbolas

Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where did the explosion occur? (Assume sound travels at 1100 feet per second.)

Solution

Assuming sound travels at 1100 feet per second, you know that the explosion took place 2200 feet farther from B than from A, as shown in Figure 10.40. The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

$$c = \frac{5280}{2} = 2640$$

and

$$a = \frac{2200}{2} = 1100.$$

So, $b^2 = c^2 - a^2 = 2640^2 - 1100^2 = 5,759,600$, and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola

$$\frac{x^2}{1,210,000} - \frac{y^2}{5,759,600} = 1.$$

 **CHECKPOINT** Now try Exercise 41.

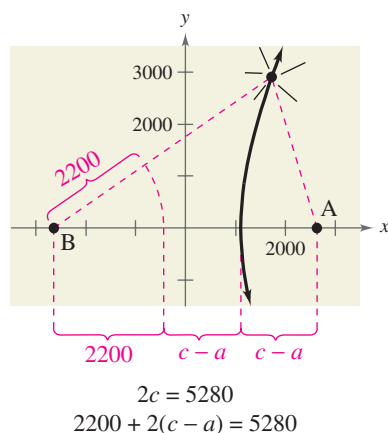


FIGURE 10.40

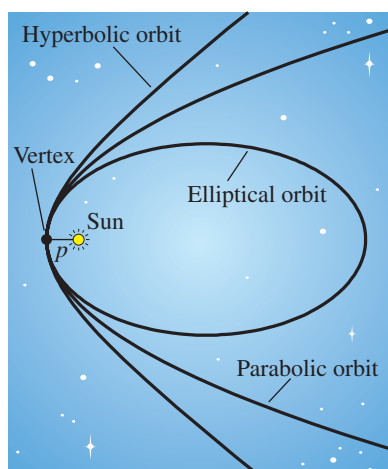


FIGURE 10.41

Another interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure 10.41. Undoubtedly, there have been many comets with parabolic or hyperbolic orbits that were not identified. We only get to see such comets *once*. Comets with elliptical orbits, such as Halley's comet, are the only ones that remain in our solar system.

If p is the distance between the vertex and the focus (in meters), and v is the velocity of the comet at the vertex (in meters per second), then the type of orbit is determined as follows.

1. Ellipse: $v < \sqrt{2GM/p}$
2. Parabola: $v = \sqrt{2GM/p}$
3. Hyperbola: $v > \sqrt{2GM/p}$

In each of these relations, $M = 1.989 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared (the universal gravitational constant).

Activities

1. Find the standard form of the equation of the hyperbola with asymptotes $y = \pm 2x$ and vertices $(0, \pm 2)$.

Answer: $\frac{y^2}{2^2} - \frac{x^2}{1^2} = 1$

2. Classify the graph of each equation.
- $3x^2 - 2y^2 + 4y - 3 = 0$
 - $2y^2 - 3x + 2 = 0$
 - $x^2 + 4y^2 - 2x - 3 = 0$
 - $x^2 - 2x + 4y - 1 = 0$

Answer: (a) Hyperbola, (b) Parabola, (c) Ellipse, (d) Parabola

General Equations of Conics**Classifying a Conic from Its General Equation**

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is one of the following.

- Circle:** $A = C$
- Parabola:** $AC = 0$ $A = 0$ or $C = 0$, but not both.
- Ellipse:** $AC > 0$ A and C have like signs.
- Hyperbola:** $AC < 0$ A and C have unlike signs.

The test above is valid *if* the graph is a conic. The test does not apply to equations such as $x^2 + y^2 = -1$, whose graph is not a conic.

Example 6 Classifying Conics from General Equations

Classify the graph of each equation.

- $4x^2 - 9x + y - 5 = 0$
- $4x^2 - y^2 + 8x - 6y + 4 = 0$
- $2x^2 + 4y^2 - 4x + 12y = 0$
- $2x^2 + 2y^2 - 8x + 12y + 2 = 0$

Solution

- a. For the equation $4x^2 - 9x + y - 5 = 0$, you have

$$AC = 4(0) = 0. \quad \text{Parabola}$$

So, the graph is a parabola.

- b. For the equation $4x^2 - y^2 + 8x - 6y + 4 = 0$, you have

$$AC = 4(-1) < 0. \quad \text{Hyperbola}$$

So, the graph is a hyperbola.

- c. For the equation $2x^2 + 4y^2 - 4x + 12y = 0$, you have

$$AC = 2(4) > 0. \quad \text{Ellipse}$$

So, the graph is an ellipse.

- d. For the equation $2x^2 + 2y^2 - 8x + 12y + 2 = 0$, you have

$$A = C = 2. \quad \text{Circle}$$

So, the graph is a circle.

 **CHECKPOINT** Now try Exercise 49.

The Granger Collection

**Historical Note**

Caroline Herschel (1750–1848) was the first woman to be credited with detecting a new comet. During her long life, this English astronomer discovered a total of eight new comets.

WRITING ABOUT MATHEMATICS

Sketching Conics Sketch each of the conics described in Example 6. Write a paragraph describing the procedures that allow you to sketch the conics efficiently.

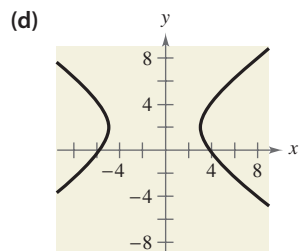
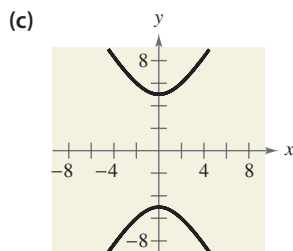
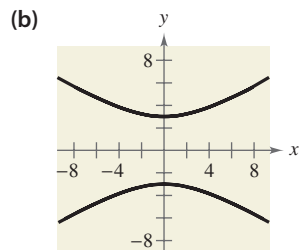
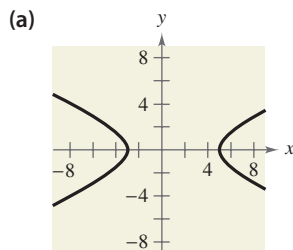
10.4 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. A _____ is the set of all points (x, y) in a plane, the difference of whose distances from two distinct fixed points, called _____, is a positive constant.
2. The graph of a hyperbola has two disconnected parts called _____.
3. The line segment connecting the vertices of a hyperbola is called the _____, and the midpoint of the line segment is the _____ of the hyperbola.
4. Each hyperbola has two _____ that intersect at the center of the hyperbola.
5. The general form of the equation of a conic is given by _____.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at www.Eduspace.com.

In Exercises 1–4, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



1. $\frac{y^2}{9} - \frac{x^2}{25} = 1$
2. $\frac{y^2}{25} - \frac{x^2}{9} = 1$
3. $\frac{(x-1)^2}{16} - \frac{y^2}{4} = 1$
4. $\frac{(x+1)^2}{16} - \frac{(y-2)^2}{9} = 1$

In Exercises 5–16, find the center, vertices, foci, and the equations of the asymptotes of the hyperbola, and sketch its graph using the asymptotes as an aid.

5. $x^2 - y^2 = 1$
6. $\frac{x^2}{9} - \frac{y^2}{25} = 1$
7. $\frac{y^2}{25} - \frac{x^2}{81} = 1$
8. $\frac{x^2}{36} - \frac{y^2}{4} = 1$
9. $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{1} = 1$

10. $\frac{(x+3)^2}{144} - \frac{(y-2)^2}{25} = 1$
11. $\frac{(y+6)^2}{1/9} - \frac{(x-2)^2}{1/4} = 1$
12. $\frac{(y-1)^2}{1/4} - \frac{(x+3)^2}{1/16} = 1$
13. $9x^2 - y^2 - 36x - 6y + 18 = 0$
14. $x^2 - 9y^2 + 36y - 72 = 0$
15. $x^2 - 9y^2 + 2x - 54y - 80 = 0$
16. $16y^2 - x^2 + 2x + 64y + 63 = 0$



In Exercises 17–20, find the center, vertices, foci, and the equations of the asymptotes of the hyperbola. Use a graphing utility to graph the hyperbola and its asymptotes.

17. $2x^2 - 3y^2 = 6$
18. $6y^2 - 3x^2 = 18$
19. $9y^2 - x^2 + 2x + 54y + 62 = 0$
20. $9x^2 - y^2 + 54x + 10y + 55 = 0$

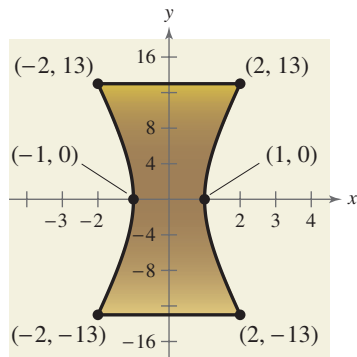
In Exercises 21–26, find the standard form of the equation of the hyperbola with the given characteristics and center at the origin.

21. Vertices: $(0, \pm 2)$; foci: $(0, \pm 4)$
22. Vertices: $(\pm 4, 0)$; foci: $(\pm 6, 0)$
23. Vertices: $(\pm 1, 0)$; asymptotes: $y = \pm 5x$
24. Vertices: $(0, \pm 3)$; asymptotes: $y = \pm 3x$
25. Foci: $(0, \pm 8)$; asymptotes: $y = \pm 4x$
26. Foci: $(\pm 10, 0)$; asymptotes: $y = \pm \frac{3}{4}x$

In Exercises 27–38, find the standard form of the equation of the hyperbola with the given characteristics.

27. Vertices: $(2, 0)$, $(6, 0)$; foci: $(0, 0)$, $(8, 0)$
28. Vertices: $(2, 3)$, $(2, -3)$; foci: $(2, 6)$, $(2, -6)$

29. Vertices: (4, 1), (4, 9); foci: (4, 0), (4, 10)
30. Vertices: (-2, 1), (2, 1); foci: (-3, 1), (3, 1)
31. Vertices: (2, 3), (2, -3);
passes through the point (0, 5)
32. Vertices: (-2, 1), (2, 1);
passes through the point (5, 4)
33. Vertices: (0, 4), (0, 0);
passes through the point $(\sqrt{5}, -1)$
34. Vertices: (1, 2), (1, -2);
passes through the point $(0, \sqrt{5})$
35. Vertices: (1, 2), (3, 2);
asymptotes: $y = x$, $y = 4 - x$
36. Vertices: (3, 0), (3, 6);
asymptotes: $y = 6 - x$, $y = x$
37. Vertices: (0, 2), (6, 2);
asymptotes: $y = \frac{2}{3}x$, $y = 4 - \frac{2}{3}x$
38. Vertices: (3, 0), (3, 4);
asymptotes: $y = \frac{2}{3}x$, $y = 4 - \frac{2}{3}x$
39. **Art** A sculpture has a hyperbolic cross section (see figure).

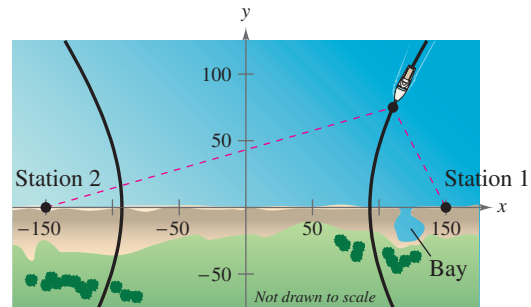


- (a) Write an equation that models the curved sides of the sculpture.
- (b) Each unit in the coordinate plane represents 1 foot. Find the width of the sculpture at a height of 5 feet.
40. **Sound Location** You and a friend live 4 miles apart (on the same “east-west” street) and are talking on the phone. You hear a clap of thunder from lightning in a storm, and 18 seconds later your friend hears the thunder. Find an equation that gives the possible places where the lightning could have occurred. (Assume that the coordinate system is measured in feet and that sound travels at 1100 feet per second.)

41. **Sound Location** Three listening stations located at (3300, 0), (3300, 1100), and (-3300, 0) monitor an explosion. The last two stations detect the explosion 1 second and 4 seconds after the first, respectively. Determine the coordinates of the explosion. (Assume that the coordinate system is measured in feet and that sound travels at 100 feet per second.)

Model It

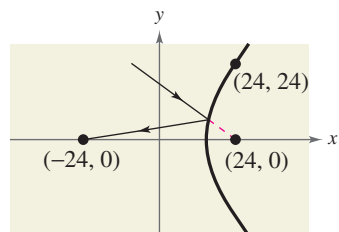
42. **LORAN** Long distance radio navigation for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations, 300 miles apart, are positioned on the rectangular coordinate system at points with coordinates (-150, 0) and (150, 0), and that a ship is traveling on a hyperbolic path with coordinates (x, 75) (see figure).



- (a) Find the x -coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).
- (b) Determine the distance between the ship and station 1 when the ship reaches the shore.
- (c) The ship wants to enter a bay located between the two stations. The bay is 30 miles from station 1. What should the time difference be between the pulses?
- (d) The ship is 60 miles offshore when the time difference in part (c) is obtained. What is the position of the ship?

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- 43. Hyperbolic Mirror** A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at a focus will be reflected to the other focus. The focus of a hyperbolic mirror (see figure) has coordinates $(24, 0)$. Find the vertex of the mirror if the mount at the top edge of the mirror has coordinates $(24, 24)$.



- 44. Running Path** Let $(0, 0)$ represent a water fountain located in a city park. Each day you run through the park along a path given by

$$x^2 + y^2 - 200x - 52,500 = 0$$

where x and y are measured in meters.

- What type of conic is your path? Explain your reasoning.
- Write the equation of the path in standard form. Sketch a graph of the equation.
- After you run, you walk to the water fountain. If you stop running at $(-100, 150)$, how far must you walk for a drink of water?

In Exercises 45–60, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

- $x^2 + y^2 - 6x + 4y + 9 = 0$
- $x^2 + 4y^2 - 6x + 16y + 21 = 0$
- $4x^2 - y^2 - 4x - 3 = 0$
- $y^2 - 6y - 4x + 21 = 0$
- $y^2 - 4x^2 + 4x - 2y - 4 = 0$
- $x^2 + y^2 - 4x + 6y - 3 = 0$
- $x^2 - 4x - 8y + 2 = 0$
- $4x^2 + y^2 - 8x + 3 = 0$
- $4x^2 + 3y^2 + 8x - 24y + 51 = 0$
- $4y^2 - 2x^2 - 4y - 8x - 15 = 0$
- $25x^2 - 10x - 200y - 119 = 0$
- $4y^2 + 4x^2 - 24x + 35 = 0$
- $4x^2 + 16y^2 - 4x - 32y + 1 = 0$
- $2y^2 + 2x + 2y + 1 = 0$
- $100x^2 + 100y^2 - 100x + 400y + 409 = 0$
- $4x^2 - y^2 + 4x + 2y - 1 = 0$

Synthesis

True or False? In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.

- In the standard form of the equation of a hyperbola, the larger the ratio of b to a , the larger the eccentricity of the hyperbola.
- In the standard form of the equation of a hyperbola, the trivial solution of two intersecting lines occurs when $b = 0$.
- Consider a hyperbola centered at the origin with a horizontal transverse axis. Use the definition of a hyperbola to derive its standard form.
- Writing** Explain how the central rectangle of a hyperbola can be used to sketch its asymptotes.
- Think About It** Change the equation of the hyperbola so that its graph is the bottom half of the hyperbola.

$$9x^2 - 54x - 4y^2 + 8y + 41 = 0$$

- 66. Exploration** A circle and a parabola can have 0, 1, 2, 3, or 4 points of intersection. Sketch the circle given by $x^2 + y^2 = 4$. Discuss how this circle could intersect a parabola with an equation of the form $y = x^2 + C$. Then find the values of C for each of the five cases described below. Use a graphing utility to verify your results.

- No points of intersection
- One point of intersection
- Two points of intersection
- Three points of intersection
- Four points of intersection

Skills Review

In Exercises 67–72, factor the polynomial completely.

- $x^3 - 16x$
- $x^2 + 14x + 49$
- $2x^3 - 24x^2 + 72x$
- $6x^3 - 11x^2 - 10x$
- $16x^3 + 54$
- $4 - x + 4x^2 - x^3$

In Exercises 73–76, sketch a graph of the function. Include two full periods.

- $y = 2 \cos x + 1$
- $y = \sin \pi x$
- $y = \tan 2x$
- $y = -\frac{1}{2} \sec x$