10.5 Angle Relationships in Circles

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS
G.5.A
G.12.A

Essential Question When a chord intersects a tangent line or another chord, what relationships exist among the angles and arcs formed?

EXPLORATION 1 Angles Formed by a Chord and Tangent Line

Work with a partner. Use dynamic geometry software.

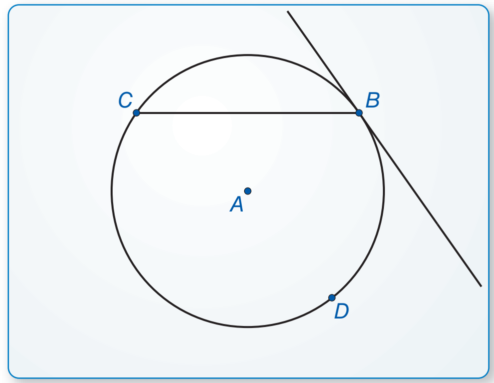
a. Construct a chord in a circle. At one of the endpoints of the chord, construct a tangent line to the circle.

b. Find the measures of the two angles formed by the chord and the tangent line.

c. Find the measures of the two circular arcs determined by the chord.

d. Repeat parts (a)–(c) several times. Record your results in a table. Then write a conjecture that summarizes the data.

Sample



EXPLORATION 2 Angles Formed by Intersecting Chords

Work with a partner. Use dynamic geometry software.

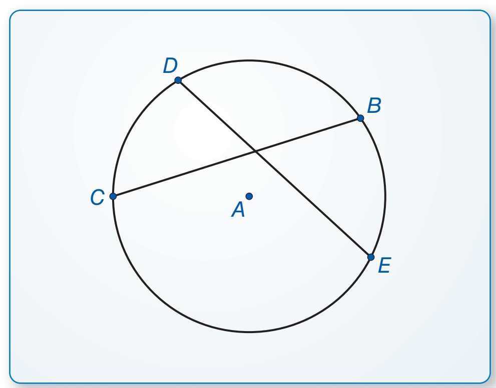
a. Construct two chords that intersect inside a circle.

b. Find the measure of one of the angles formed by the intersecting chords.

c. Find the measures of the arcs intercepted by the angle in part (b) and its vertical angle. What do you observe?

d. Repeat parts (a)–(c) several times. Record your results in a table. Then write a conjecture that summarizes the data.

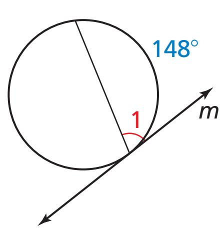
Sample



MAKING MATHEMATICAL ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results.

Communicate Your Answer



3. When a chord intersects a tangent line or another chord, what relationships exist among the angles and arcs formed?

4. Line m is tangent to the circle in the figure at the left. Find the measure of $\angle 1$.

5. Two chords intersect inside a circle to form a pair of vertical angles with measures of 55°. Find the sum of the measures of the arcs intercepted by the two angles.

10.5 Lesson

Core Vocabulary

circumscribed angle, *p. 568*

Previous
tangent
chord
secant

What You Will Learn

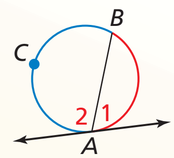
- Find angle and arc measures.
- Use circumscribed angles.

Finding Angle and Arc Measures

Theorem

Theorem 10.14 Tangent and Intersected Chord Theorem

If a tangent and a chord intersect at a point on a circle, then the measure of each angle formed is one-half the measure of its intercepted arc.



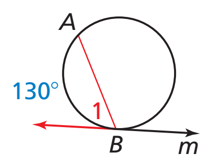
Proof Ex. 33, p. 572

$m\angle 1 = \frac{1}{2}m\overset{\frown}{AB}$ $m\angle 2 = \frac{1}{2}m\overset{\frown}{BCA}$

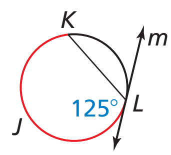
EXAMPLE 1 Finding Angle and Arc Measures

Line m is tangent to the circle. Find the measure of the red angle or arc.

a.



b.



SOLUTION

a. $m\angle 1 = \frac{1}{2}(130°) = 65°$

b. $m\overset{\frown}{KJL} = 2(125°) = 250°$

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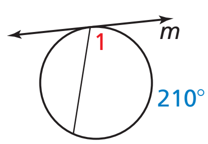
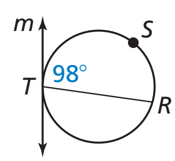
Line *m* is tangent to the circle. Find the indicated measure.

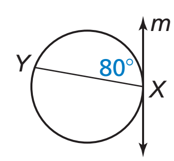
1. $m\angle 1$



2. $m\overset{\frown}{RST}$



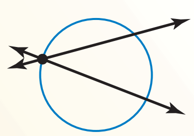
3. $m\overset{\frown}{XY}$



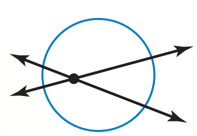
Core Concept

Intersecting Lines and Circles

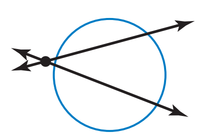
If two nonparallel lines intersect a circle, there are three places where the lines can intersect.



on the circle



inside the circle

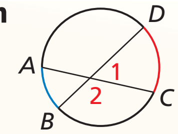


outside the circle

Theorems

Theorem 10.15 Angles Inside the Circle Theorem

If two chords intersect *inside* a circle, then the measure of each angle is one-half the *sum* of the measures of the arcs intercepted by the angle and its vertical angle.



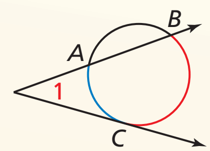
$m\angle 1 = \frac{1}{2}(m\widehat{DC} + m\widehat{AB})$,

$m\angle 2 = \frac{1}{2}(m\widehat{AD} + m\widehat{BC})$

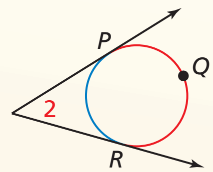
Proof Ex. 35, p. 572

Theorem 10.16 Angles Outside the Circle Theorem

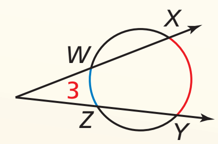
If a tangent and a secant, two tangents, or two secants intersect *outside* a circle, then the measure of the angle formed is one-half the *difference* of the measures of the intercepted arcs.



$m\angle 1 = \frac{1}{2}(m\widehat{BC} - m\widehat{AC})$



$m\angle 2 = \frac{1}{2}(m\widehat{PQR} - m\widehat{PR})$



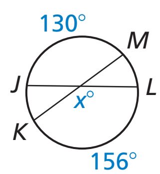
$m\angle 3 = \frac{1}{2}(m\widehat{XY} - m\widehat{WZ})$

Proof Ex. 37, p. 572

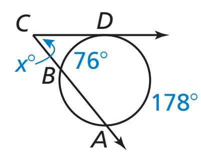
EXAMPLE 2 Finding an Angle Measure

Find the value of x.

a.



b.



SOLUTION

a. The chords $\overline{JL}$ and $\overline{KM}$ intersect inside the circle. Use the Angles Inside the Circle Theorem.

$$x^\circ = \frac{1}{2}(m\widehat{JM} + m\widehat{LK})$$

$$x^\circ = \frac{1}{2}(130^\circ + 156^\circ)$$

$$x = 143$$

▶ So, the value of x is 143.

b. The tangent $\overrightarrow{CD}$ and the secant $\overrightarrow{CB}$ intersect outside the circle. Use the Angles Outside the Circle Theorem.

$$m\angle BCD = \frac{1}{2}(m\widehat{AD} - m\widehat{BD})$$

$$x^\circ = \frac{1}{2}(178^\circ - 76^\circ)$$

$$x = 51$$

▶ So, the value of x is 51.

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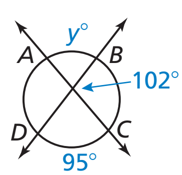
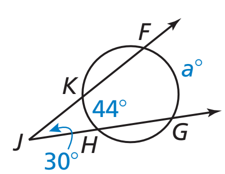
Find the value of the variable.

4.



5.

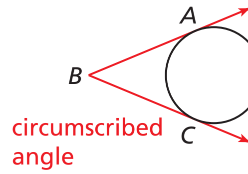


Using Circumscribed Angles

Core Concept

Circumscribed Angle

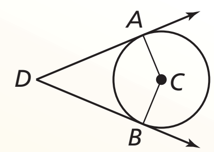
A **circumscribed angle** is an angle whose sides are tangent to a circle.



Theorem

Theorem 10.17 Circumscribed Angle Theorem

The measure of a circumscribed angle is equal to 180° minus the measure of the central angle that intercepts the same arc.



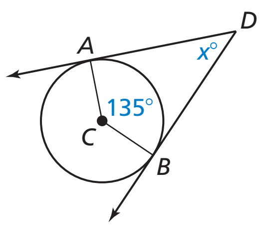
$m\angle ADB = 180° - m\angle ACB$

Proof Ex. 38, p. 572

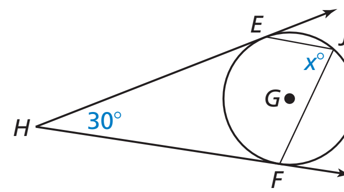
EXAMPLE 3 Finding Angle Measures

Find the value of x.

a.



b.



SOLUTION

a. Use the Circumscribed Angle Theorem to find $m\angle ADB$.

$m\angle ADB = 180° - m\angle ACB$	Circumscribed Angle Theorem
$x° = 180° - 135°$	Substitute.
$x = 45$	Subtract.

So, the value of x is 45.

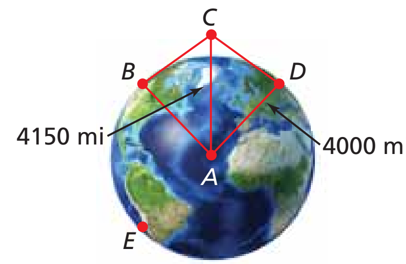
b. Use the Measure of an Inscribed Angle Theorem (Theorem 10.10) and the Circumscribed Angle Theorem to find $m\angle EJF$.

$m\angle EJF = \frac{1}{2}m\widehat{EF}$	Measure of an Inscribed Angle Theorem
$m\angle EJF = \frac{1}{2}m\angle EGF$	Definition of minor arc
$m\angle EJF = \frac{1}{2}(180° - m\angle EHF)$	Circumscribed Angle Theorem
$m\angle EJF = \frac{1}{2}(180° - 30°)$	Substitute.
$x = \frac{1}{2}(180 - 30)$	Substitute.
$x = 75$	Simplify.

So, the value of x is 75.

EXAMPLE 4 Modeling with Mathematics

The northern lights are bright flashes of colored light between 50 and 200 miles above Earth. A flash occurs 150 miles above Earth at point C. What is the measure of $\overparen{BD}$, the portion of Earth from which the flash is visible? (Earth's radius is approximately 4000 miles.)



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SOLUTION

1. **Understand the Problem** You are given the approximate radius of Earth and the distance above Earth that the flash occurs. You need to find the measure of the arc that represents the portion of Earth from which the flash is visible.

2. **Make a Plan** Use properties of tangents, triangle congruence, and angles outside a circle to find the arc measure.

3. **Solve the Problem** Because $\overline{CB}$ and $\overline{CD}$ are tangents, $\overline{CB} \perp \overline{AB}$ and $\overline{CD} \perp \overline{AD}$ by the Tangent Line to Circle Theorem (Theorem 10.1). Also, $\overline{BC} \cong \overline{DC}$ by the External Tangent Congruence Theorem (Theorem 10.2), and $\overline{CA} \cong \overline{CA}$ by the Reflexive Property of Congruence (Theorem 2.1). So, $\triangle ABC \cong \triangle ADC$ by the Hypotenuse-Leg Congruence Theorem (Theorem 5.9). Because corresponding parts of congruent triangles are congruent, $\angle BCA \cong \angle DCA$. Solve right $\triangle CBA$ to find that $m\angle BCA \approx 74.5°$. So, $m\angle BCD \approx 2(74.5°) = 149°$.

$m\angle BCD = 180° - m\angle BAD$	Circumscribed Angle Theorem
$m\angle BCD = 180° - m\overparen{BD}$	Definition of minor arc
$149° \approx 180° - m\overparen{BD}$	Substitute.
$31° \approx m\overparen{BD}$	Solve for $m\overparen{BD}$.

▶ The measure of the arc from which the flash is visible is about 31°.

4. **Look Back** You can use inverse trigonometric ratios to find $m\angle BAC$ and $m\angle DAC$.

$$m\angle BAC = \cos^{-1}\left(\frac{4000}{4150}\right) \approx 15.5°$$

$$m\angle DAC = \cos^{-1}\left(\frac{4000}{4150}\right) \approx 15.5°$$

So, $m\angle BAD \approx 15.5° + 15.5° = 31°$, and therefore $m\overparen{BD} \approx 31°$.

COMMON ERROR

Because the value for $m\angle BCD$ is an approximation, use the symbol $\approx$ instead of $=$.

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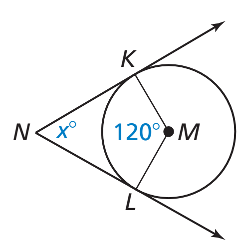
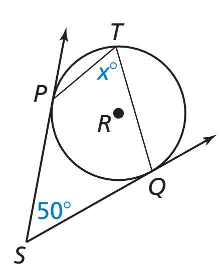
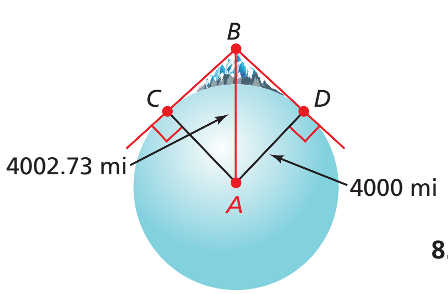
Find the value of x.

6.



7.





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8. You are on top of Mount Rainier on a clear day. You are about 2.73 miles above sea level at point B. Find $m\overparen{CD}$, which represents the part of Earth that you can see.

10.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

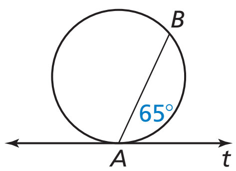
1. **COMPLETE THE SENTENCE** Points A, B, C, and D are on a circle, and $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ at point P. If $m\angle APC = \frac{1}{2}(m\overset{\frown}{BD} - m\overset{\frown}{AC})$, then point P is ______ the circle.

2. **WRITING** Explain how to find the measure of a circumscribed angle.

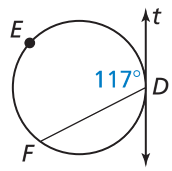
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, line t is tangent to the circle. Find the indicated measure. *(See Example 1.)*

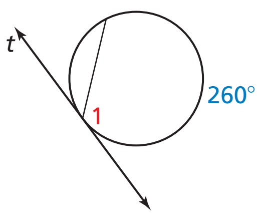
3. $m\overset{\frown}{AB}$



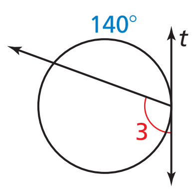
4. $m\overset{\frown}{DEF}$



5. $m\angle 1$

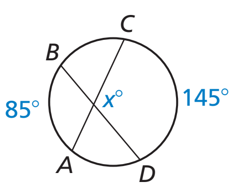


6. $m\angle 3$



In Exercises 7–14, find the value of x. *(See Examples 2 and 3.)*

7.



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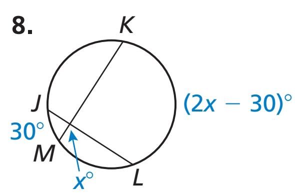

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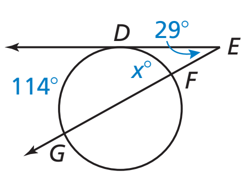

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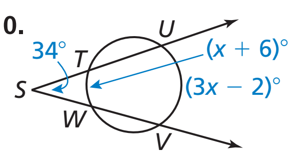

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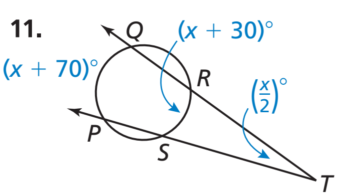

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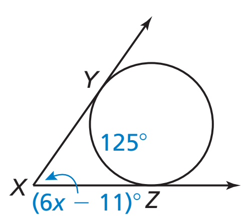

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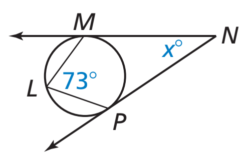
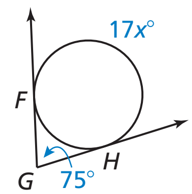
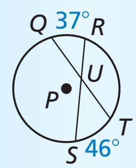


14.



ERROR ANALYSIS In Exercises 15 and 16, describe and correct the error in finding the angle measure.

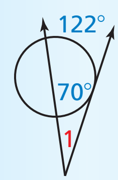
15.



$m\angle SUT = m\overset{\frown}{ST} = 46°$

So, $m\angle SUT = 46°$.

16.

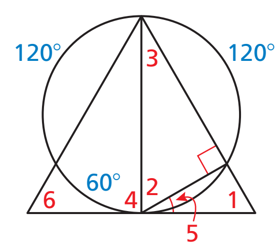


$m\angle 1 = 122° - 70°$

$= 52°$

So, $m\angle 1 = 52°$.

In Exercises 17–22, find the indicated angle measure. Justify your answer.



17. $m\angle 1$

18. $m\angle 2$

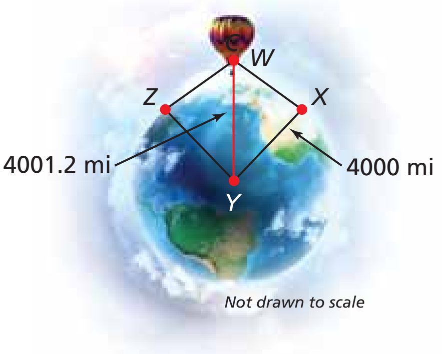
19. $m\angle 3$

20. $m\angle 4$

21. $m\angle 5$

22. $m\angle 6$

23. PROBLEM SOLVING You are flying in a hot air balloon about 1.2 miles above the ground. Find the measure of the arc that represents the part of Earth you can see. The radius of Earth is about 4000 miles. *(See Example 4.)*



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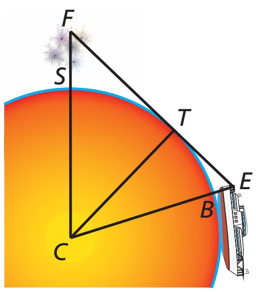
24. PROBLEM SOLVING You are watching fireworks over San Diego Bay S as you sail away in a boat. The highest point the fireworks reach F is about 0.2 mile above the bay. Your eyes E are about 0.01 mile above the water. At point B you can no longer see the fireworks because of the curvature of Earth. The radius of Earth is about 4000 miles, and $\overline{FE}$ is tangent to Earth at point T. Find $m\overset{\frown}{SB}$. Round your answer to the nearest tenth.



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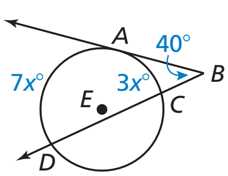
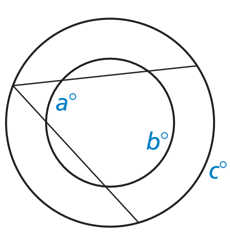
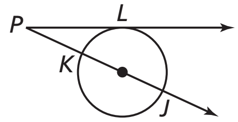
25. MATHEMATICAL CONNECTIONS In the diagram, $\overrightarrow{BA}$ is tangent to $\odot E$. Write an algebraic expression for $m\overset{\frown}{CD}$ in terms of x. Then find $m\overset{\frown}{CD}$.



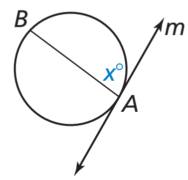
26. MATHEMATICAL CONNECTIONS The circles in the diagram are concentric. Write an algebraic expression for c in terms of a and b.



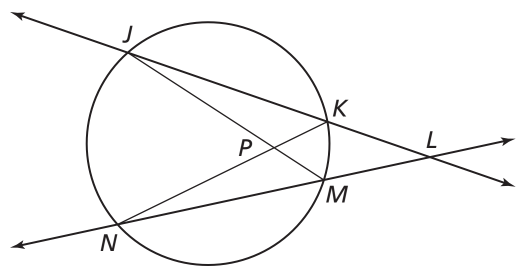
27. ABSTRACT REASONING In the diagram, $\overrightarrow{PL}$ is tangent to the circle, and $\overline{KJ}$ is a diameter. What is the range of possible angle measures of $\angle LPJ$? Explain your reasoning.



28. ABSTRACT REASONING In the diagram, $\overline{AB}$ is any chord that is not a diameter of the circle. Line m is tangent to the circle at point A. What is the range of possible values of x? Explain your reasoning. (The diagram is not drawn to scale.)



29. PROOF In the diagram, $\overleftrightarrow{JL}$ and $\overleftrightarrow{NL}$ are secant lines that intersect at point L. Prove that $m\angle JPN > m\angle JLN$.



30. MAKING AN ARGUMENT Your friend claims that it is possible for a circumscribed angle to have the same measure as its intercepted arc. Is your friend correct? Explain your reasoning.

31. REASONING Points A and B are on a circle, and t is a tangent line containing A and another point C.

a. Draw two diagrams that illustrate this situation.

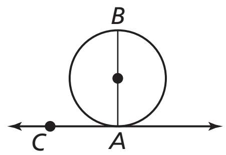
b. Write an equation for $m\overset{\frown}{AB}$ in terms of $m\angle BAC$ for each diagram.

c. For what measure of $\angle BAC$ can you use either equation to find $m\overset{\frown}{AB}$? Explain.

32. REASONING $\triangle XYZ$ is an equilateral triangle inscribed in $\odot P$. $\overline{AB}$ is tangent to $\odot P$ at point X, $\overline{BC}$ is tangent to $\odot P$ at point Y, and $\overline{AC}$ is tangent to $\odot P$ at point Z. Draw a diagram that illustrates this situation. Then classify $\triangle ABC$ by its angles and sides. Justify your answer.

33. PROVING A THEOREM To prove the Tangent and Intersected Chord Theorem (Theorem 10.14), you must prove three cases.

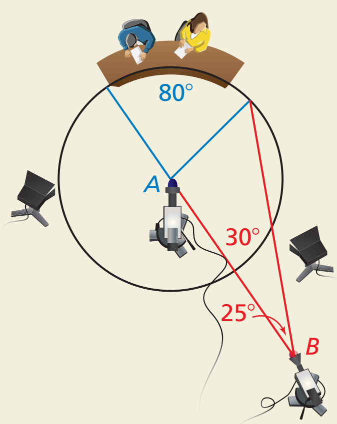
a. The diagram shows the case where $\overline{AB}$ contains the center of the circle. Use the Tangent Line to Circle Theorem (Theorem 10.1) to write a paragraph proof for this case.



b. Draw a diagram and write a proof for the case where the center of the circle is in the interior of $\angle CAB$.

c. Draw a diagram and write a proof for the case where the center of the circle is in the exterior of $\angle CAB$.

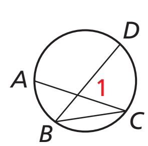
34. HOW DO YOU SEE IT? In the diagram, television cameras are positioned at A and B to record what happens on stage. The stage is an arc of $\odot A$. You would like the camera at B to have a 30° view of the stage. Should you move the camera closer or farther away? Explain your reasoning.



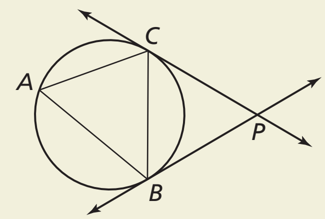
35. PROVING A THEOREM Write a proof of the Angles Inside the Circle Theorem (Theorem 10.15).

Given Chords $\overline{AC}$ and $\overline{BD}$ intersect inside a circle.

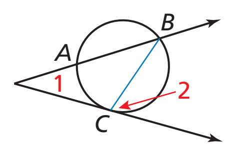
Prove $m\angle 1 = \frac{1}{2}(m\overset{\frown}{DC} + m\overset{\frown}{AB})$



36. THOUGHT PROVOKING In the figure, $\overleftrightarrow{BP}$ and $\overleftrightarrow{CP}$ are tangent to the circle. Point A is any point on the major arc formed by the endpoints of the chord $\overline{BC}$. Label all congruent angles in the figure. Justify your reasoning.



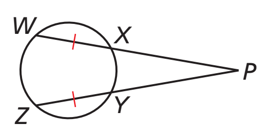
37. PROVING A THEOREM Use the diagram below to prove the Angles Outside the Circle Theorem (Theorem 10.16) for the case of a tangent and a secant. Then copy the diagrams for the other two cases on page 567 and draw appropriate auxiliary segments. Use your diagrams to prove each case.



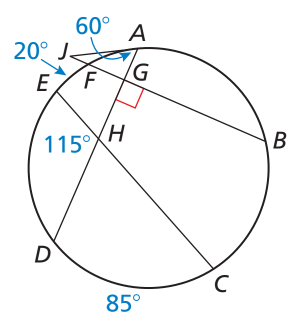
38. PROVING A THEOREM Prove that the Circumscribed Angle Theorem (Theorem 10.17) follows from the Angles Outside the Circle Theorem (Theorem 10.16).

In Exercises 39 and 40, find the indicated measure(s). Justify your answer.

39. Find $m\angle P$ when $m\overset{\frown}{WZY} = 200°$.



40. Find $m\overset{\frown}{AB}$ and $m\overset{\frown}{ED}$.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. *(Skills Review Handbook)*

41. $x^2 + x = 12$

42. $x^2 = 12x + 35$

43. $-3 = x^2 + 4x$