10.6 Graphing and Classifying Conics

What you should learn

GOAL 1 Write and graph an equation of a parabola with its vertex at (h, k) and an equation of a circle, ellipse, or hyperbola with its center at (h, k).

GOAL 2 Classify a conic using its equation, as applied in **Example 8**.

Why you should learn it

▼ To model **real-life** situations involving more than one conic, such as the circles that an ice skater uses to practice figure eights in **Ex. 64**.

REAL LIFE

GOAL 1 WRITING AND GRAPHING EQUATIONS OF CONICS

Parabolas, circles, ellipses, and hyperbolas are all curves that are formed by the intersection of a plane and a double-napped cone. Therefore, these shapes are called **conic sections** or simply **conics**.

In previous lessons you studied equations of parabolas with vertices at the origin and equations of circles, ellipses, and hyperbolas with centers at the origin. In this lesson you will study equations of conics that have been translated in the coordinate plane.

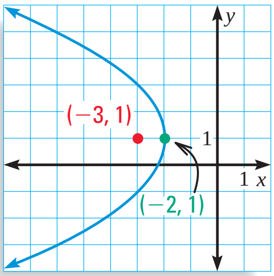
STANDARD FORM OF EQUATIONS OF TRANSLATED CONICS

In the following equations the point (h, k) is the *vertex* of the parabola and the *center* of the other conics.

	Horizontal axis	Vertical axis
CIRCLE	$(x-h)^2 + (y-k)^2 = r^2$	
PARABOLA	$(y-k)^2 = 4p(x-h)$	$(x-h)^2 = 4p(y-k)$
ELLIPSE	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
HYPERBOLA	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

EXAMPLE 1 *Writing an Equation of a Translated Parabola*

Write an equation of the parabola whose vertex is at $(-2, 1)$ and whose focus is at $(-3, 1)$.



SOLUTION

Choose form: Begin by sketching the parabola, as shown. Because the parabola opens to the left, it has the form

$$(y-k)^2 = 4p(x-h)$$

where $p < 0$.

Find h and k: The vertex is at $(-2, 1)$, so $h = -2$ and $k = 1$.

Find p: The distance between the vertex $(-2, 1)$ and the focus $(-3, 1)$ is

$$|p| = \sqrt{(-3-(-2))^2 + (1-1)^2} = 1$$

so $p = 1$ or $p = -1$. Since $p < 0$, $p = -1$.

▶ The standard form of the equation is $(y-1)^2 = -4(x+2)$.

EXAMPLE 2 *Graphing the Equation of a Translated Circle*

Graph $(x - 3)^2 + (y + 2)^2 = 16$.

SOLUTION

Compare the given equation to the standard form of the equation of a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

You can see that the graph is a circle with center at $(h, k) = (3, -2)$ and radius $r = 4$.

Plot the center.

Plot several points that are each **4** units from the center:

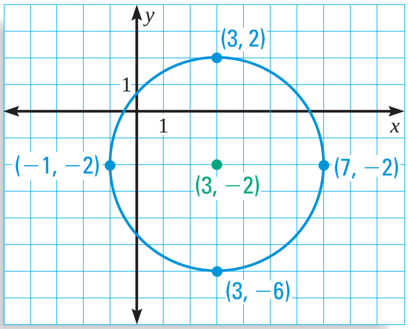
$(3 + 4, -2) = (7, -2)$

$(3 - 4, -2) = (-1, -2)$

$(3, -2 + 4) = (3, 2)$

$(3, -2 - 4) = (3, -6)$

Draw a circle through the points.



EXAMPLE 3 *Writing an Equation of a Translated Ellipse*

STUDENT HELP

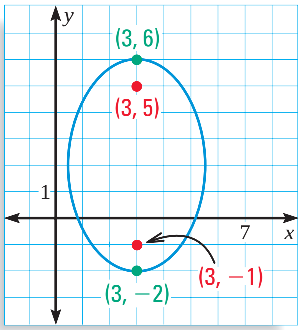
HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

Write an equation of the ellipse with foci at (3, 5) and (3, −1) and vertices at (3, 6) and (3, −2).

SOLUTION

Plot the given points and make a rough sketch. The ellipse has a vertical major axis, so its equation is of this form:

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$



Find the center: The center is halfway between the vertices.

$$(h, k) = \left(\frac{3 + 3}{2}, \frac{6 + (-2)}{2}\right) = (3, 2)$$

***Find a*:** The value of a is the distance between the vertex and the center.

$$a = \sqrt{(3 - 3)^2 + (6 - 2)^2} = \sqrt{0 + 4^2} = 4$$

***Find c*:** The value of c is the distance between the focus and the center.

$$c = \sqrt{(3 - 3)^2 + (5 - 2)^2} = \sqrt{0 + 3^2} = 3$$

***Find b*:** Substitute the values of a and c into the equation $b^2 = a^2 - c^2$.

$b^2 = 4^2 - 3^2$

$b^2 = 7$

$b = \sqrt{7}$

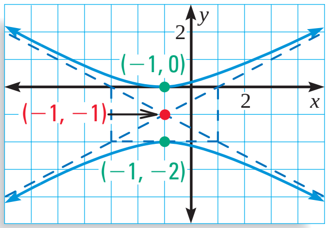
▶ The standard form of the equation is $\frac{(x - 3)^2}{7} + \frac{(y - 2)^2}{16} = 1$.

EXAMPLE 4 *Graphing the Equation of a Translated Hyperbola*

Graph $(y + 1)^2 - \frac{(x + 1)^2}{4} = 1$.

SOLUTION

The y^2-term is positive, so the transverse axis is vertical. Since $a^2 = 1$ and $b^2 = 4$, you know that $a = 1$ and $b = 2$.



Plot the center at $(h, k) = (-1, -1)$. Plot the vertices 1 unit above and below the center at $(-1, 0)$ and $(-1, -2)$.

Draw a rectangle that is centered at $(-1, -1)$ and is $2a = 2$ units high and $2b = 4$ units wide.

Draw the asymptotes through the corners of the rectangle.

Draw the hyperbola so that it passes through the vertices and approaches the asymptotes.

EXAMPLE 5 *Using Circular Models*

COMMUNICATIONS A cellular phone transmission tower located 10 miles west and 5 miles north of your house has a range of 20 miles. A second tower, 5 miles east and 10 miles south of your house, has a range of 15 miles.

a. Write an inequality that describes each tower's range.

b. Do the two regions covered by the towers overlap?

SOLUTION

a. Let the origin represent your house. The first tower is at $(-10, 5)$ and the boundary of its range is a circle with radius 20. Substitute -10 for h, 5 for k, and 20 for r into the standard form of the equation of a circle.

$(x - h)^2 + (y - k)^2 = r^2$ **Standard form of a circle**

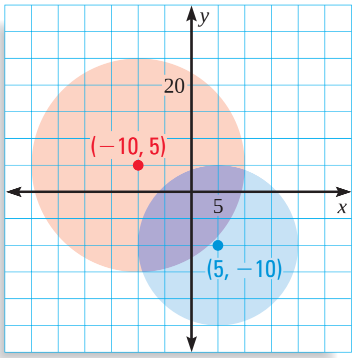
$(x + 10)^2 + (y - 5)^2 < 400$ **Region inside the circle**

The second tower is at $(5, -10)$. The boundary of its range is a circle with radius 15.

$(x - h)^2 + (y - k)^2 = r^2$ **Standard form of a circle**

$(x - 5)^2 + (y + 10)^2 < 225$ **Region inside the circle**

b. One way to tell if the regions overlap is to graph the inequalities. You can see that the regions do overlap.



You can also check whether the distance between the two towers is less than the sum of the ranges.

$$\sqrt{(-10 - 5)^2 + (5 - (-10))^2} \stackrel{?}{<} 20 + 15$$

$$15\sqrt{2} \stackrel{?}{<} 35$$

$$21.2 < 35 \checkmark$$

▶ The regions do overlap.

FOCUS ON APPLICATIONS

REAL LIFE **CELLULAR PHONES** work only when there is a transmission tower nearby to retrieve the signal. Because of the need for many towers, they are often designed to blend in with the environment.

GOAL 2 CLASSIFYING A CONIC FROM ITS EQUATION

The equation of any conic can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which is called a **general second-degree equation** in x and y. The expression $B^2 - 4AC$ is called the **discriminant** of the equation and can be used to determine which type of conic the equation represents.

CONCEPT SUMMARY — CLASSIFYING CONICS

If the graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is a conic, then the type of conic can be determined as follows.

DISCRIMINANT	TYPE OF CONIC
$B^2 - 4AC < 0$, $B = 0$, and $A = C$	Circle
$B^2 - 4AC < 0$ and either $B \neq 0$ or $A \neq C$	Ellipse
$B^2 - 4AC = 0$	Parabola
$B^2 - 4AC > 0$	Hyperbola

If $B = 0$, each axis of the conic is horizontal or vertical. If $B \neq 0$, the axes are neither horizontal nor vertical.

EXAMPLE 6 *Classifying a Conic*

a. Classify the conic given by $2x^2 + y^2 - 4x - 4 = 0$.

b. Graph the equation in part (a).

SOLUTION

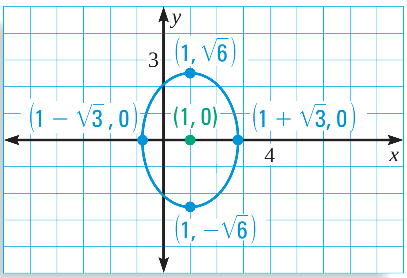
a. Since $A = 2$, $B = 0$, and $C = 1$, the value of the discriminant is as follows:

$$B^2 - 4AC = 0^2 - 4(2)(1) = -8$$

▶ Because $B^2 - 4AC < 0$ and $A \neq C$, the graph is an ellipse.

b. To graph the ellipse, first complete the square as follows.

$$2x^2 + y^2 - 4x - 4 = 0$$
$$(2x^2 - 4x) + y^2 = 4$$
$$2(x^2 - 2x) + y^2 = 4$$
$$2(x^2 - 2x + \underline{?}) + y^2 = 4 + 2(\underline{?})$$
$$2(x^2 - 2x + 1) + y^2 = 4 + 2(1)$$
$$2(x - 1)^2 + y^2 = 6$$
$$\frac{(x-1)^2}{3} + \frac{y^2}{6} = 1$$



By comparing this equation to $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, you can see that $h = 1$, $k = 0$, $a = \sqrt{6}$, and $b = \sqrt{3}$. Use these facts to draw the ellipse.

STUDENT HELP

Look Back
For help with completing the square, see p. 282.

EXAMPLE 7 *Classifying a Conic*

a. Classify the conic given by $4x^2 - 9y^2 + 32x - 144y - 548 = 0$.

b. Graph the equation in part (a).

SOLUTION

a. Since $A = 4$, $B = 0$, and $C = -9$, the value of the discriminant is as follows:

$$B^2 - 4AC = 0^2 - 4(4)(-9) = 144$$

▸ Because $B^2 - 4AC > 0$, the graph is a hyperbola.

b. To graph the hyperbola, first complete the square as follows.

$$4x^2 - 9y^2 + 32x - 144y - 548 = 0$$

$$(4x^2 + 32x) - (9y^2 + 144y) = 548$$

$$4(x^2 + 8x + \underline{\ ?\ }) - 9(y^2 + 16y + \underline{\ ?\ }) = 548 + 4(\underline{\ ?\ }) - 9(\underline{\ ?\ })$$

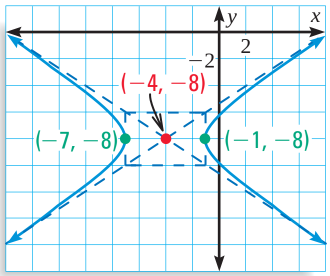
$$4(x^2 + 8x + 16) - 9(y^2 + 16y + 64) = 548 + 4(16) - 9(64)$$

$$4(x + 4)^2 - 9(y + 8)^2 = 36$$

$$\frac{(x + 4)^2}{3^2} - \frac{(y + 8)^2}{2^2} = 1$$

By comparing this equation to $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, you can see that $h = -4$, $k = -8$, $a = 3$, and $b = 2$.

To draw the hyperbola, plot the center at $(h, k) = (-4, -8)$ and the vertices at $(-7, -8)$ and $(-1, -8)$. Draw a rectangle $2a = 6$ units wide and $2b = 4$ units high and centered at $(-4, -8)$. Draw the asymptotes through the corners of the rectangle. Then draw the hyperbola so that it passes through the vertices and approaches the asymptotes.



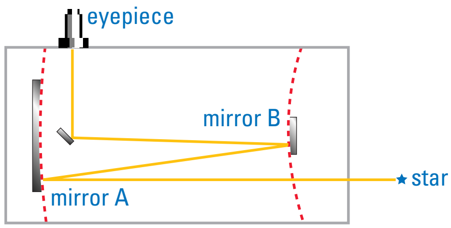
REAL LIFE Astronomy

EXAMPLE 8 *Classifying Conics in Real Life*

The diagram at the right shows the mirrors in a Cassegrain telescope. The equations of the two mirrors are given below. Classify each mirror as parabolic, elliptical, or hyperbolic.

a. Mirror A: $y^2 - 72x - 450 = 0$

b. Mirror B: $88.4x^2 - 49.7y^2 - 4390 = 0$



SOLUTION

EQUATION	$B^2 - 4AC$	TYPE OF MIRROR
a. $y^2 - 72x - 450 = 0$	$0^2 - 4(0)(1) = 0$	Parabolic
b. $88.4x^2 - 49.7y^2 - 4390 = 0$	$0^2 - 4(88.4)(-49.7) > 0$	Hyperbolic

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain why circles, ellipses, parabolas, and hyperbolas are called conic sections.

Concept Check ✓

2. How are the graphs of $x^2 + y^2 = 25$ and $(x - 1)^2 + (y + 2)^2 = 25$ alike? How are they different?

3. How can the discriminant $B^2 - 4AC$ be used to classify the graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$?

Skill Check ✓

Write an equation for the conic section.

4. Circle with center at $(4, -1)$ and radius 7
5. Ellipse with foci at $(2, -4)$ and $(5, -4)$ and vertices at $(-1, -4)$ and $(8, -4)$
6. Parabola with vertex at $(3, -2)$ and focus at $(3, -4)$
7. Hyperbola with foci at $(5, 2)$ and $(5, -6)$ and vertices at $(5, 0)$ and $(5, -4)$

Classify the conic section.

8. $x^2 + 2x - 4y + 4 = 0$
9. $3x^2 - 5y^2 - 6x + y - 2 = 0$
10. $x^2 + y^2 + 7x - 4y - 8 = 0$
11. $-5x^2 - 2y^2 + x - 3y + 1 = 0$
12. **COMMUNICATIONS** Look back at Example 5. Suppose there is a tower 25 miles east and 30 miles north of your house with a range of 25 miles. Does the region covered by this tower overlap the regions covered by the two towers in Example 5? Illustrate your answer with a graph.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 954.

WRITING EQUATIONS **Write an equation for the conic section.**

13. Circle with center at $(9, 3)$ and radius 4
14. Circle with center at $(-4, 2)$ and radius 3
15. Parabola with vertex at $(1, -2)$ and focus at $(1, 1)$
16. Parabola with vertex at $(-3, 1)$ and directrix $x = -8$
17. Ellipse with vertices at $(2, -3)$ and $(2, 6)$ and foci at $(2, 0)$ and $(2, 3)$
18. Ellipse with vertices at $(-2, 2)$ and $(4, 2)$ and co-vertices at $(1, 1)$ and $(1, 3)$
19. Hyperbola with vertices at $(5, -4)$ and $(5, 4)$ and foci at $(5, -6)$ and $(5, 6)$
20. Hyperbola with vertices at $(-4, 2)$ and $(1, 2)$ and foci at $(-7, 2)$ and $(4, 2)$

STUDENT HELP

HOMEWORK HELP
Examples 1, 3: Exs. 13–20
Examples 2, 4: Exs. 21–28
Example 5: Exs. 63, 64
Examples 6, 7: Exs. 29–62
Example 8: Exs. 65–67

GRAPHING **Graph the equation. Identify the important characteristics of the graph, such as the center, vertices, and foci.**

21. $(x - 6)^2 + (y - 2)^2 = 4$
22. $(x + 7)^2 = 12(y - 3)$
23. $\frac{(y - 8)^2}{16} - \frac{(x + 3)^2}{4} = 1$
24. $\frac{(x - 3)^2}{25} + \frac{(y + 6)^2}{49} = 1$
25. $\frac{(x + 1)^2}{16} + \frac{y^2}{9} = 1$
26. $\frac{x^2}{16} - (y + 4)^2 = 1$
27. $(x + 7)^2 + (y - 1)^2 = 1$
28. $(y - 4)^2 = 3(x + 2)$

CLASSIFYING **Classify the conic section.**

29. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

30. $x^2 - 4y^2 + 3x - 26y - 30 = 0$

31. $4x^2 - 9y^2 + 18y + 3x = 0$

32. $x^2 + y^2 - 10x - 2y + 10 = 0$

33. $36x^2 + 16y^2 - 25x + 22y + 2 = 0$

34. $4x^2 + 4y^2 - 16x + 4y - 60 = 0$

35. $9y^2 - x^2 + 2x + 54y + 62 = 0$

36. $16x^2 + 25y^2 - 18x - 20y + 8 = 0$

37. $x^2 - 2x + 8y + 9 = 0$

38. $2y^2 - 8y - 4x + 10 = 0$

39. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

40. $9x^2 - y^2 + 54x + 10y + 55 = 0$

41. $x^2 + y^2 - 4x - 2y - 4 = 0$

42. $16x^2 + 9y^2 + 24x - 36y + 23 = 0$

43. $16y^2 - x^2 + 2x + 64y + 63 = 0$

44. $x^2 - 4x + 16y + 17 = 0$

MATCHING **Match the equation with its graph.**

45. $9x^2 - 4y^2 + 36x - 24y - 36 = 0$

46. $y^2 - 2y - 4x + 9 = 0$

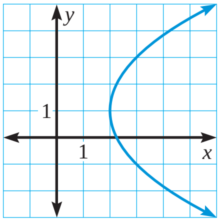
47. $9x^2 + 4y^2 + 36x + 24y + 36 = 0$

48. $y^2 - x^2 + 6y + 4x + 4 = 0$

49. $4x^2 + 9y^2 - 16x + 54y + 61 = 0$

50. $x^2 + y^2 - 4x + 6y + 4 = 0$

A.



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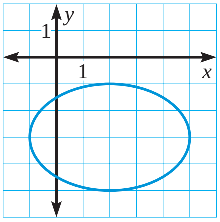
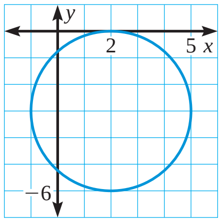


C.



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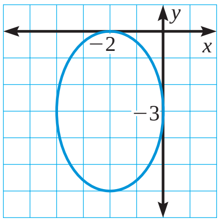
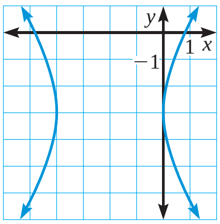
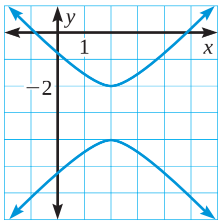


E.



F.



FOCUS ON APPLICATIONS

REAL LIFE **WHISPER DISHES** are two parabolic dishes set up facing directly toward each other. A person listening at the focus of one dish is able to hear even the softest sound made at the focus of the other dish.

CLASSIFYING AND GRAPHING **Classify the conic section and write its equation in standard form. Then graph the equation.**

51. $y^2 - 12y + 4x + 4 = 0$

52. $x^2 + y^2 - 6x - 8y + 24 = 0$

53. $9x^2 - y^2 - 72x + 8y + 119 = 0$

54. $4x^2 + y^2 - 48x - 4y + 48 = 0$

55. $x^2 + 4y^2 - 2x - 8y + 1 = 0$

56. $x^2 + y^2 - 12x - 24y + 36 = 0$

57. $16x^2 - y^2 + 16y - 128 = 0$

58. $x^2 + 9y^2 + 8x + 4y + 7 = 0$

59. $x^2 + y^2 - 12x - 12y + 36 = 0$

60. $y^2 - 2x - 20y + 94 = 0$

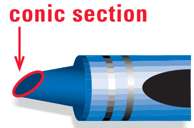
61. $x^2 + 4x - 8y + 12 = 0$

62. $-9x^2 + 4y^2 - 36x - 16y - 164 = 0$

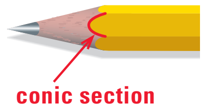
63. **WHISPER DISHES** The whisper dish shown at the left can be seen at the Thronateeska Discovery Center in Albany, Georgia. Two dishes are positioned so that their vertices are 50 feet apart. The focus of each dish is 3 feet from its vertex. Write equations for the cross sections of the dishes so that the vertex of one dish is at the origin and the vertex of the other dish is on the positive x-axis.

64. **FIGURE SKATING** To practice making a figure eight, a figure skater will skate along two circles etched in the ice. Write equations for two externally tangent circles that are each 6 feet in diameter so that the center of one circle is at the origin and the center of the other circle is on the positive y-axis.

65. **VISUAL THINKING** A new crayon has a cone-shaped tip. When it is used for the first time, a flat spot is worn on the tip. The edge of the flat spot is a conic section, as shown. What type(s) of conic could it be?



66. **VISUAL THINKING** When a pencil is sharpened the tip becomes a cone. On a pencil with flat sides, the intersection of the cone with each flat side is a conic section. What type of conic is it?



67. **ASTRONOMY** A Gregorian telescope contains two mirrors whose cross sections can be modeled by the equations $405x^2 + 729y^2 - 295{,}245 = 0$ and $-120y^2 - 1440x = 0$. What types of mirrors are they?

Test Preparation

68. **MULTIPLE CHOICE** Which of the following is an equation of the hyperbola with vertices at (3, 5) and (3, −1) and foci at (3, 7) and (3, −3)?

Ⓐ $\frac{(x-3)^2}{25} - \frac{(y-2)^2}{9} = 1$ Ⓑ $\frac{(y-2)^2}{9} - \frac{(x-3)^2}{25} = 1$

Ⓒ $\frac{(y-2)^2}{9} - \frac{(x-3)^2}{7} = 1$ Ⓓ $\frac{(x-3)^2}{9} - \frac{(y-2)^2}{16} = 1$

Ⓔ $\frac{(y-2)^2}{9} - \frac{(x-3)^2}{16} = 1$

69. **MULTIPLE CHOICE** What conic does $25x^2 + y^2 - 100x - 2y + 76 = 0$ represent?

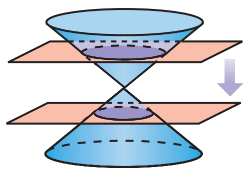
Ⓐ Parabola Ⓑ Circle Ⓒ Ellipse

Ⓓ Hyperbola Ⓔ Not enough information

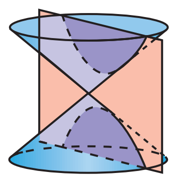
★ **Challenge**

70. **DEGENERATE CONICS** A *degenerate* conic occurs when the intersection of a plane with a double-napped cone is something other than a parabola, circle, ellipse, or hyperbola.

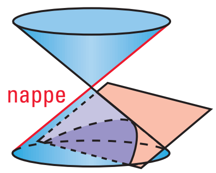
a. Imagine a plane perpendicular to the axis of a double-napped cone. As the plane passes through the cone, the intersection is a circle whose radius decreases and then increases. At what point is the intersection something other than a circle? What is the intersection?



b. Imagine a plane parallel to the axis of a double-napped cone. As the plane passes through the cone, the intersection is a hyperbola whose vertices get closer together and then farther apart. At what point is the intersection something other than a hyperbola? What is the intersection?



c. Imagine a plane parallel to the nappe passing through a double-napped cone. As the plane passes through the cone, the intersection is a parabola that gets narrower and then flips and gets wider. At what point is the intersection something other than a parabola? What is the intersection?



EXTRA CHALLENGE
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MIXED REVIEW

SYSTEMS **Solve the system using any algebraic method.** (Review 3.2 for 10.7)

71. $x - y = 10$
$3x - 2y = 25$

72. $4x + 3y = 1$
$-3x - 6y = 3$

73. $4x + y = 2$
$6x + 3y = 0$

74. $2x - 3y = 0$
$x + 6y = 14$

75. $23x = 68$
$x + 3y = 19$

76. $x = y$
$123x - 18y = 17$

EVALUATING LOGARITHMIC EXPRESSIONS **Evaluate the expression.** (Review 8.4)

77. $\log_7 7^5$

78. $\log_4 64$

79. $\log_5 1$

80. $\log_{1/3} 9$

81. $\log_{25} 625$

82. $\log 0.0001$

SOLVING EQUATIONS **Solve the equation.** (Review 8.8)

83. $\frac{40}{1 + 6e^{-4x}} = 20$

84. $\frac{10}{1 + 9e^{-2x}} = 1$

85. $\frac{8}{1 + 8e^{-x}} = 7$

86. $\frac{15}{1 + 3e^{-6x}} = 3$

87. $\frac{24}{1 + 5e^{-4x}} = 9$

88. $\frac{9}{1 + 2e^{-3x}} = 7$

MATH & History

History of Conic Sections

APPLICATION LINK www.mcdougallittell.com

THEN

IN 200 B.C. conic sections were studied thoroughly for the first time by a Greek mathematician named Apollonius. Six hundred years later, the Egyptian mathematician Hypatia simplified the works of Apollonius, making it accessible to many more people. For centuries, conics were studied and appreciated only for their mathematical beauty rather than for their occurrence in nature or practical use.

NOW

TODAY astronomers know that the paths of celestial objects, such as planets and comets, are conic sections. For example, a comet's path can be parabolic, hyperbolic, or elliptical.

Tell what type of path each comet follows. Which comet(s) will pass by the sun more than once?

1. $3550x^2 + 14{,}200x + 7100y - 13{,}050 = 0$

2. $2200x^2 + 4600y^2 - 13{,}200x - 18{,}400y + 12{,}900 = 0$

3. $5000x^2 - 6500y^2 + 20{,}000x - 52{,}000y - 695{,}000 = 0$