CHAPTER 10

Conic Sections

10.1 The Parabola and the Circle

10.2 The Ellipse and the Hyperbola

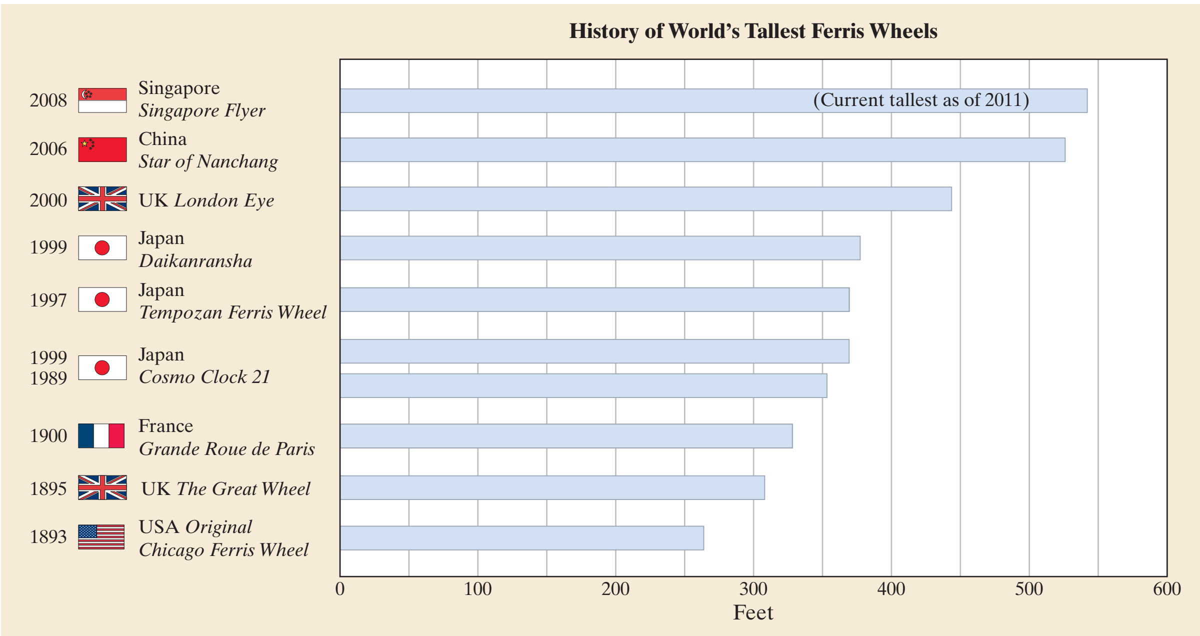
Integrated Review—Graphing Conic Sections

10.3 Solving Nonlinear Systems of Equations

10.4 Nonlinear Inequalities and Systems of Inequalities

In Chapter 8, we analyzed some of the important connections between a parabola and its equation. Parabolas are interesting in their own right but are more interesting still because they are part of a collection of curves known as conic sections. This chapter is devoted to quadratic equations in two variables and their conic section graphs: the parabola, circle, ellipse, and hyperbola.

The original Ferris wheel was named after its designer, George Washington Gale Ferris, Jr., a trained engineer who produced the first Ferris wheel for the 1893 World's Columbian Exposition in Chicago. This very first wheel was 264 feet high and was the Columbian Exposition's most noticeable attraction. Since then, Ferris wheels have gotten ever taller, have been built with ever greater capacities, and have changed their designations from Ferris wheels to giant observation wheels because of their closed capsules. In Exercise 92 of Section 10.1, you will explore the dimensions of the Singapore Flyer, the current record-breaking giant observation wheel.



10.1 The Parabola and the Circle

OBJECTIVES

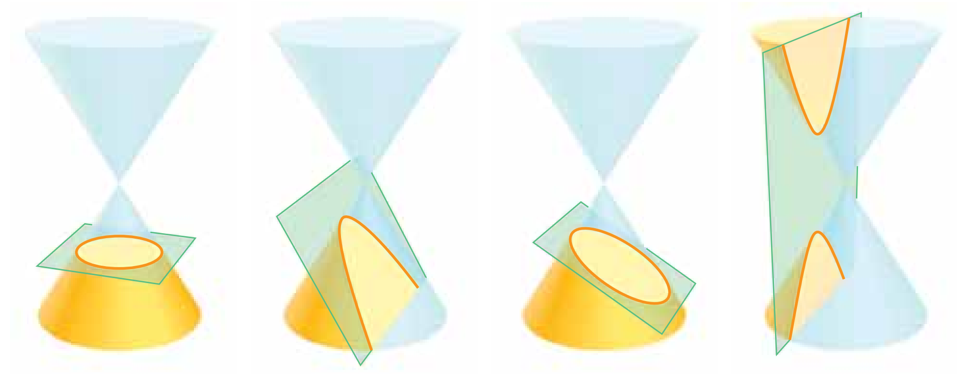
1 Graph Parabolas of the Form $x = a(y - k)^2 + h$ and $y = a(x - h)^2 + k$.

2 Graph Circles of the Form $(x - h)^2 + (y - k)^2 = r^2$.

3 Find the Center and the Radius of a Circle, Given Its Equation.

4 Write an Equation of a Circle, Given Its Center and Radius.

Conic sections are named so because each conic section is the intersection of a right circular cone and a plane. The circle, parabola, ellipse, and hyperbola are the conic sections.



Circle Parabola Ellipse Hyperbola

OBJECTIVE

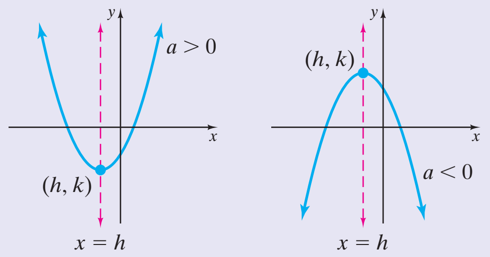
1 Graphing Parabolas

Thus far, we have seen that $f(x)$ or $y = a(x - h)^2 + k$ is the equation of a parabola that opens upward if $a > 0$ or downward if $a < 0$. Parabolas can also open left or right or even on a slant. Equations of these parabolas are not functions of x, of course, since a parabola opening any way other than upward or downward fails the vertical line test. In this section, we introduce parabolas that open to the left and to the right. Parabolas opening on a slant will not be developed in this book.

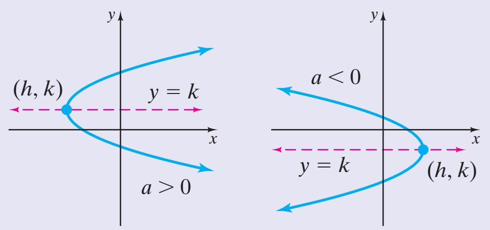
Just as $y = a(x - h)^2 + k$ is the equation of a parabola that opens upward or downward, $x = a(y - k)^2 + h$ is the equation of a parabola that opens to the right or to the left. The parabola opens to the right if $a > 0$ and to the left if $a < 0$. The parabola has vertex (h, k), and its axis of symmetry is the line $y = k$.

Parabolas

$$y = a(x - h)^2 + k$$



$$x = a(y - k)^2 + h$$



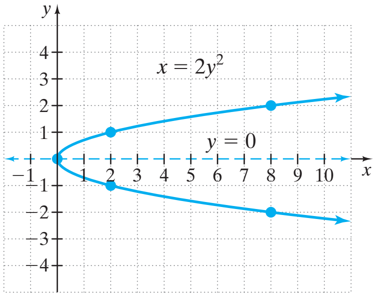
The equations $y = a(x - h)^2 + k$ and $x = a(y - k)^2 + h$ are called **standard forms.**

✓CONCEPT CHECK
Does the graph of the parabola given by the equation $x = -3y^2$ open to the left, to the right, upward, or downward?

EXAMPLE 1 Graph the parabola $x = 2y^2$.

Solution Written in standard form, the equation $x = 2y^2$ is $x = 2(y - 0)^2 + 0$ with $a = 2$, $k = 0$, and $h = 0$. Its graph is a parabola with vertex $(0, 0)$, and its axis of symmetry is the line $y = 0$. Since $a > 0$, this parabola opens to the right. The table shows a few more ordered pair solutions of $x = 2y^2$. Its graph is also shown.

x	y
8	−2
2	−1
0	0
2	1
8	2



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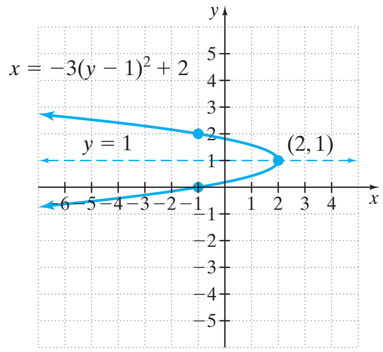
PRACTICE 1 Graph the parabola $x = \frac{1}{2}y^2$.

EXAMPLE 2 Graph the parabola $x = -3(y - 1)^2 + 2$.

Solution The equation $x = -3(y - 1)^2 + 2$ is in the form $x = a(y - k)^2 + h$ with $a = -3$, $k = 1$, and $h = 2$. Since $a < 0$, the parabola opens to the left. The vertex (h, k) is $(2, 1)$, and the axis of symmetry is the line $y = 1$. When $y = 0$, $x = -1$, so the x-intercept is $(-1, 0)$. Again, we obtain a few ordered pair solutions and then graph the parabola.

x	y
2	1
−1	0
−1	2
−10	3
−10	−1



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PRACTICE 2 Graph the parabola $x = -2(y + 4)^2 - 1$.

EXAMPLE 3 Graph $y = -x^2 - 2x + 15$.

Solution Complete the square on x to write the equation in standard form.

$y - 15 = -x^2 - 2x$ Subtract 15 from both sides.

$y - 15 = -1(x^2 + 2x)$ Factor −1 from the terms $-x^2 - 2x$.

Answer to Concept Check:
to the left

The coefficient of x is 2. Find the square of half of 2.

$$\frac{1}{2}(2) = 1 \quad \text{and} \quad 1^2 = 1$$

$$y - 15 - 1(1) = -1(x^2 + 2x + 1) \quad \text{Add } -1(1) \text{ to both sides.}$$

$$y - 16 = -1(x + 1)^2 \quad \text{Simplify the left side and factor the right side.}$$

$$y = -(x + 1)^2 + 16 \quad \text{Add 16 to both sides.}$$

The equation is now in standard form $y = a(x - h)^2 + k$ with $a = -1$, $h = -1$, and $k = 16$.

The vertex is then (h, k), or $(-1, 16)$.

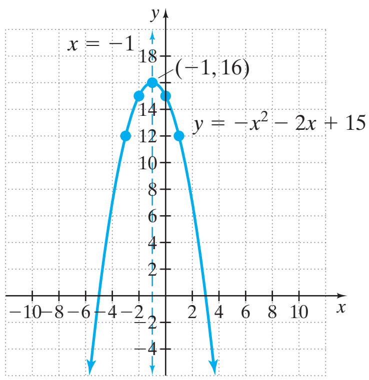
A second method for finding the vertex is by using the formula $\frac{-b}{2a}$.

$$x = \frac{-(-2)}{2(-1)} = \frac{2}{-2} = -1$$

$$y = -(-1)^2 - 2(-1) + 15 = -1 + 2 + 15 = 16$$

Again, we see that the vertex is $(-1, 16)$, and the axis of symmetry is the vertical line $x = -1$. The y-intercept is (0, 15). Now we can use a few more ordered pair solutions to graph the parabola.

x	y
−1	16
0	15
−2	15
1	12
−3	12
3	0
−5	0



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PRACTICE 3 Graph $y = -x^2 + 4x + 6$.

EXAMPLE 4 Graph $x = 2y^2 + 4y + 5$.

Solution Notice that this equation is quadratic in y, so its graph is a parabola that opens to the left or the right. We can complete the square on y, or we can use the formula $\frac{-b}{2a}$ to find the vertex.

Since the equation is quadratic in y, the formula gives us the y-value of the vertex.

$$y = \frac{-4}{2 \cdot 2} = \frac{-4}{4} = -1$$

$$x = 2(-1)^2 + 4(-1) + 5 = 2 \cdot 1 - 4 + 5 = 3$$

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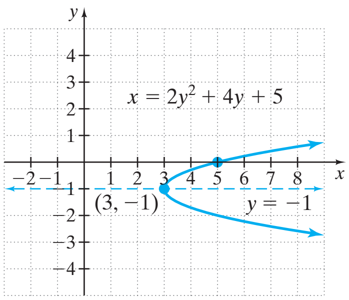
The vertex is $(3, -1)$, and the axis of symmetry is the line $y = -1$. The parabola opens to the right since $a > 0$. The x-intercept is $(5, 0)$.



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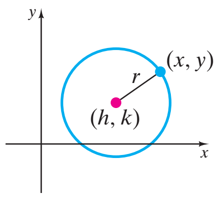
PRACTICE
4 Graph $x = 3y^2 + 6y + 4$.

OBJECTIVE
2 Graphing Circles



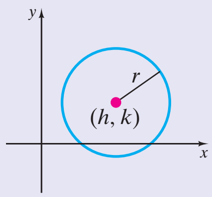
Another conic section is the **circle.** A circle is the set of all points in a plane that are the same distance from a fixed point called the **center.** The distance is called the **radius** of the circle. To find a standard equation for a circle, let (h, k) represent the center of the circle and let (x, y) represent any point on the circle. The distance between (h, k) and (x, y) is defined to be the circle's radius, r units. We can find this distance r by using the distance formula.

$$r = \sqrt{(x - h)^2 + (y - k)^2}$$

$$r^2 = (x - h)^2 + (y - k)^2 \qquad \text{Square both sides.}$$

Circle

The graph of $(x - h)^2 + (y - k)^2 = r^2$ is a circle with center (h, k) and radius r.



The equation $(x - h)^2 + (y - k)^2 = r^2$ is called **standard form.**

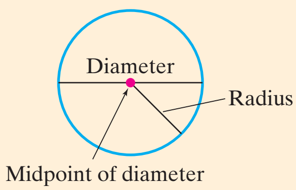
If an equation can be written in the standard form

$$(x - h)^2 + (y - k)^2 = r^2$$

then its graph is a circle, which we can draw by graphing the center (h, k) and using the radius r.

Helpful Hint

Notice that the radius is the *distance* from the center of the circle to any point of the circle. Also notice that the *midpoint* of a diameter of a circle is the center of the circle.

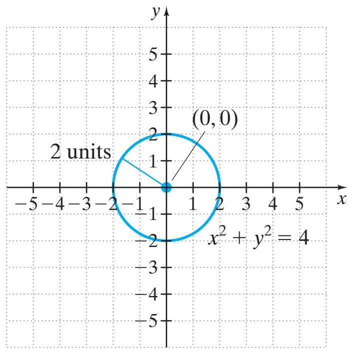


EXAMPLE 5 Graph $x^2 + y^2 = 4$.

Solution The equation can be written in standard form as

$$(x - 0)^2 + (y - 0)^2 = 2^2$$

The center of the circle is (0, 0), and the radius is 2. Its graph is shown.



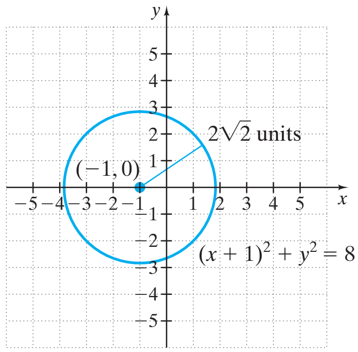
PRACTICE 5 Graph $x^2 + y^2 = 25$.

Helpful Hint

Notice the difference between the equation of a circle and the equation of a parabola. The equation of a circle contains both x^2 and y^2 terms on the same side of the equation with equal coefficients. The equation of a parabola has either an x^2 term or a y^2 term but not both.

EXAMPLE 6 Graph $(x + 1)^2 + y^2 = 8$.

Solution The equation can be written as $(x + 1)^2 + (y - 0)^2 = 8$ with $h = -1$, $k = 0$, and $r = \sqrt{8}$. The center is $(-1, 0)$, and the radius is $\sqrt{8} = 2\sqrt{2} \approx 2.8$.



PRACTICE 6 Graph $(x - 3)^2 + (y + 2)^2 = 4$.

✓CONCEPT CHECK

In the graph of the equation $(x - 3)^2 + (y - 2)^2 = 5$, what is the distance between the center of the circle and any point on the circle?

OBJECTIVE

3 Finding the Center and the Radius of a Circle

To find the center and the radius of a circle from its equation, write the equation in standard form. To write the equation of a circle in standard form, we complete the square on both x and y.

Answer to Concept Check:
$\sqrt{5}$ units

EXAMPLE 7 Graph $x^2 + y^2 + 4x - 8y = 16$.

Solution Since this equation contains x^2 and y^2 terms on the same side of the equation with equal coefficients, its graph is a circle. To write the equation in standard form, group the terms involving x and the terms involving y and then complete the square on each variable.

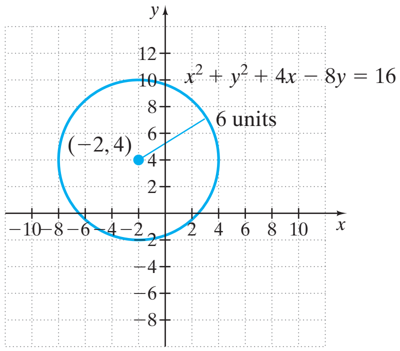
$$(x^2 + 4x) + (y^2 - 8y) = 16$$

Thus, $\frac{1}{2}(4) = 2$ and $2^2 = 4$. Also, $\frac{1}{2}(-8) = -4$ and $(-4)^2 = 16$. Add 4 and then 16 to both sides.

$$(x^2 + 4x + 4) + (y^2 - 8y + 16) = 16 + 4 + 16$$

$$(x + 2)^2 + (y - 4)^2 = 36 \quad \text{Factor.}$$

This circle the center $(-2, 4)$ and radius 6, as shown.



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PRACTICE 7 Graph $x^2 + y^2 + 6x - 2y = 6$.

OBJECTIVE 4

Writing Equations of Circles

Since a circle is determined entirely by its center and radius, this information is all we need to write an equation of a circle.

EXAMPLE 8 Find an equation of the circle with center $(-7, 3)$ and radius 10.

Solution Using the given values $h = -7$, $k = 3$, and $r = 10$, we write the equation

$$(x - h)^2 + (y - k)^2 = r^2$$

or

$$[x - (-7)]^2 + (y - 3)^2 = 10^2 \quad \text{Substitute the given values.}$$

or

$$(x + 7)^2 + (y - 3)^2 = 100$$

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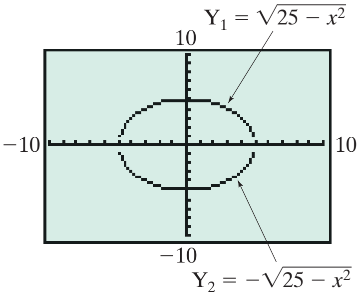
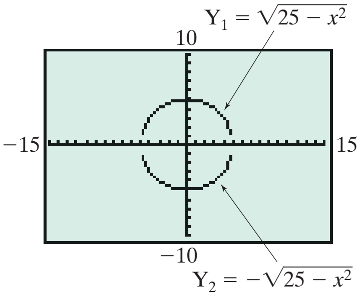
PRACTICE 8 Find an equation of the circle with center $(-2, -5)$ and radius 9.

Graphing Calculator Explorations





To graph an equation such as $x^2 + y^2 = 25$ with a graphing calculator, we first solve the equation for y.

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \pm\sqrt{25 - x^2}$$

The graph of $y = \sqrt{25 - x^2}$ will be the top half of the circle, and the graph of $y = -\sqrt{25 - x^2}$ will be the bottom half of the circle.

To graph, press [Y=] and enter $Y_1 = \sqrt{25 - x^2}$ and $Y_2 = -\sqrt{25 - x^2}$.

Insert parentheses around $25 - x^2$ so that $\sqrt{25 - x^2}$ and not $\sqrt{25} - x^2$ is graphed.

The top graph to the left does not appear to be a circle because we are currently using a standard window and the screen is rectangular. This causes the tick marks on the x-axis to be farther apart than the tick marks on the y-axis and, thus, creates the distorted circle. If we want the graph to appear circular, we must define a square window by using a feature of the graphing calculator or by redefining the window to show the x-axis from −15 to 15 and the y-axis from −10 to 10. Using a square window, the graph appears as shown on the bottom to the left.

Use a graphing calculator to graph each circle.

1. $x^2 + y^2 = 55$
2. $x^2 + y^2 = 20$
3. $5x^2 + 5y^2 = 50$
4. $6x^2 + 6y^2 = 105$
5. $2x^2 + 2y^2 - 34 = 0$
6. $4x^2 + 4y^2 - 48 = 0$
7. $7x^2 + 7y^2 - 89 = 0$
8. $3x^2 + 3y^2 - 35 = 0$

Vocabulary, Readiness & Video Check

Use the choices below to fill in each blank. Some choices may be used more than once.

radius	center	vertex
diameter	circle	conic sections

1. The circle, parabola, ellipse, and hyperbola are called the ______________.

2. For a parabola that opens upward, the lowest point is the ______________.

3. A ______________ is the set of all points in a plane that are the same distance from a fixed point. The fixed point is called the ______________.

4. The midpoint of a diameter of a circle is the ______________.

5. The distance from the center of a circle to any point of the circle is called the ______________.

6. Twice a circle's radius is its ______________.

Martin-Gay Interactive Videos

See Video 10.1

Watch the section lecture video and answer the following questions.

OBJECTIVE 1

7. Based on Example 1 and the lecture before, would you say that parabolas of the form $x = a(y - k)^2 + h$ are functions? Why or why not?

OBJECTIVE 2

8. Based on the lecture before Example 2, what would be the standard form of a circle with its center at the origin? Simplify your answer.

OBJECTIVE 3

9. From Example 3, if you know the center and radius of a circle, how can you write that circle's equation?

OBJECTIVE 4

10. From Example 4, why do we need to complete the square twice when writing this equation of a circle in standard form?

10.1 Exercise Set

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The graph of each equation is a parabola. Determine whether the parabola opens upward, downward, to the left, or to the right. Do not graph. See Examples 1 through 4.

1. $y = x^2 - 7x + 5$
2. $y = -x^2 + 16$
3. $x = -y^2 - y + 2$
4. $x = 3y^2 + 2y - 5$
5. $y = -x^2 + 2x + 1$
6. $x = -y^2 + 2y - 6$

The graph of each equation is a parabola. Find the vertex of the parabola and then graph it. See Examples 1 through 4.

7. $x = 3y^2$
8. $x = 5y^2$
9. $x = -2y^2$
10. $x = -4y^2$
11. $y = -4x^2$
12. $y = -2x^2$
13. $x = (y - 2)^2 + 3$
14. $x = (y - 4)^2 - 1$
15. $y = -3(x - 1)^2 + 5$
16. $y = -4(x - 2)^2 + 2$
17. $x = y^2 + 6y + 8$
18. $x = y^2 - 6y + 6$
19. $y = x^2 + 10x + 20$
20. $y = x^2 + 4x - 5$
21. $x = -2y^2 + 4y + 6$
22. $x = 3y^2 + 6y + 7$

The graph of each equation is a circle. Find the center and the radius and then graph the circle. See Examples 5 through 7.

23. $x^2 + y^2 = 9$
24. $x^2 + y^2 = 25$
25. $x^2 + (y - 2)^2 = 1$
26. $(x - 3)^2 + y^2 = 9$
27. $(x - 5)^2 + (y + 2)^2 = 1$
28. $(x + 3)^2 + (y + 3)^2 = 4$
29. $x^2 + y^2 + 6y = 0$
30. $x^2 + 10x + y^2 = 0$
31. $x^2 + y^2 + 2x - 4y = 4$
32. $x^2 + y^2 + 6x - 4y = 3$
33. $(x + 2)^2 + (y - 3)^2 = 7$
34. $(x + 1)^2 + (y - 2)^2 = 5$
35. $x^2 + y^2 - 4x - 8y - 2 = 0$
36. $x^2 + y^2 - 2x - 6y - 5 = 0$

Hint: For Exercises 37 through 42, first divide the equation through by the coefficient of x^2 (or y^2).

37. $3x^2 + 3y^2 = 75$
38. $2x^2 + 2y^2 = 18$
39. $6(x - 4)^2 + 6(y - 1)^2 = 24$
40. $7(x - 1)^2 + 7(y - 3)^2 = 63$
41. $4(x + 1)^2 + 4(y - 3)^2 = 12$
42. $5(x - 2)^2 + 5(y + 1) = 50$

Write an equation of the circle with the given center and radius. See Example 8.

43. $(2, 3); 6$
44. $(-7, 6); 2$
45. $(0, 0); \sqrt{3}$
46. $(0, -6); \sqrt{2}$
47. $(-5, 4); 3\sqrt{5}$
48. the origin; $4\sqrt{7}$

MIXED PRACTICE

Sketch the graph of each equation. If the graph is a parabola, find its vertex. If the graph is a circle, find its center and radius.

49. $x = y^2 - 3$
50. $x = y^2 + 2$
51. $y = (x - 2)^2 - 2$
52. $y = (x + 3)^2 + 3$
53. $x^2 + y^2 = 1$
54. $x^2 + y^2 = 49$
55. $x = (y + 3)^2 - 1$
56. $x = (y - 1)^2 + 4$
57. $(x - 2)^2 + (y - 2)^2 = 16$
58. $(x + 3)^2 + (y - 1)^2 = 9$
59. $x = -(y - 1)^2$
60. $x = -2(y + 5)^2$
61. $(x - 4)^2 + y^2 = 7$
62. $x^2 + (y + 5)^2 = 5$
63. $y = 5(x + 5)^2 + 3$
64. $y = 3(x - 4)^2 + 2$
65. $\frac{x^2}{8} + \frac{y^2}{8} = 2$
66. $2x^2 + 2y^2 = \frac{1}{2}$
67. $y = x^2 + 7x + 6$
68. $y = x^2 - 2x - 15$
69. $x^2 + y^2 + 2x + 12y - 12 = 0$
70. $x^2 + y^2 + 6x + 10y - 2 = 0$
71. $x = y^2 + 8y - 4$
72. $x = y^2 + 6y + 2$
73. $x^2 - 10y + y^2 + 4 = 0$
74. $x^2 + y^2 - 8y + 5 = 0$
75. $x = -3y^2 + 30y$
76. $x = -2y^2 - 4y$
77. $5x^2 + 5y^2 = 25$
78. $\frac{x^2}{3} + \frac{y^2}{3} = 2$
79. $y = 5x^2 - 20x + 16$
80. $y = 4x^2 - 40x + 105$

REVIEW AND PREVIEW

Graph each equation. See Section 3.3.

81. $y = 2x + 5$
82. $y = -3x + 3$
83. $y = 3$
84. $x = -2$

Rationalize each denominator and simplify if possible. See Section 7.5.

85. $\frac{1}{\sqrt{3}}$
86. $\frac{\sqrt{5}}{\sqrt{8}}$
87. $\frac{4\sqrt{7}}{\sqrt{6}}$
88. $\frac{10}{\sqrt{5}}$

CONCEPT EXTENSIONS

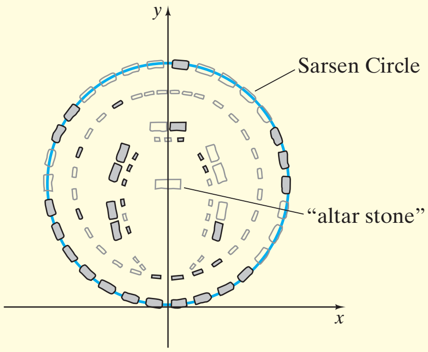
For Exercises 89 and 90, explain the error in each statement.

89. The graph of $x = 5(y + 5)^2 + 1$ is a parabola with vertex $(-5, 1)$ and opening to the right.
90. The graph of $x^2 + (y + 3)^2 = 10$ is a circle with center $(0, -3)$ and radius 5.
91. ***The Sarsen Circle*** The first image that comes to mind when one thinks of Stonehenge is the very large sandstone blocks with sandstone lintels across the top. The Sarsen Circle of Stonehenge is the outer circle of the sandstone blocks, each of which weighs up to 50 tons. There were originally 30 of these monolithic blocks, but only 17 remain upright to this day. The "altar stone" lies at the center of this circle, which has a diameter of 33 meters.
   a. What is the radius of the Sarsen Circle?
   b. What is the circumference of the Sarsen Circle? Round your result to 2 decimal places.

c. Since there were originally 30 Sarsen stones located on the circumference, how far apart would the centers of the stones have been? Round to the nearest tenth of a meter.

d. Using the axes in the drawing, what are the coordinates of the center of the circle?

e. Use parts (a) and (d) to write the equation of the Sarsen Circle.



92. Although there are many larger observation wheels on the horizon, as of this writing the largest observation wheel in the world is the Singapore Flyer. From the Flyer, you can see up to 45 km away. Each of the 28 enclosed capsules holds 28 passengers and completes a full rotation every 32 minutes. Its diameter is 150 meters, and the height of this giant wheel is 165 meters. (*Source:* singaporeflyer.com)

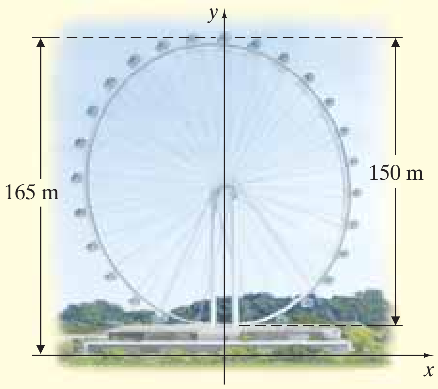
a. What is the radius of the Singapore Flyer?

b. How close is the wheel to the ground?

c. How high is the center of the wheel from the ground?

d. Using the axes in the drawing, what are the coordinates of the center of the wheel?

e. Use parts (a) and (d) to write an equation of the Singapore Flyer.



93. In 1893, Pittsburgh bridge builder George Ferris designed and built a gigantic revolving steel wheel whose height was 264 feet and diameter was 250 feet. This Ferris wheel opened at the 1893 exposition in Chicago. It had 36 wooden cars, each capable of holding 60 passengers. (*Source: The Handy Science Answer Book*)

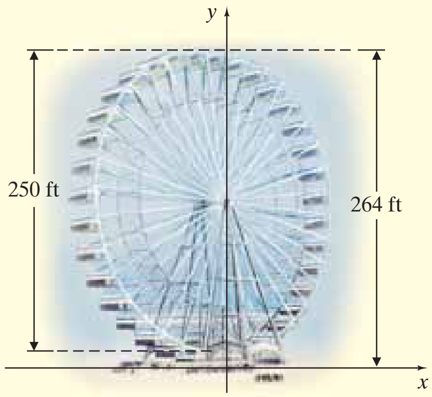
a. What was the radius of this Ferris wheel?

b. How close was the wheel to the ground?

c. How high was the center of the wheel from the ground?

d. Using the axes in the drawing, what are the coordinates of the center of the wheel?

e. Use parts (a) and (d) to write an equation of the wheel.



94. The world's largest-diameter Ferris wheel currently operating is the Cosmo Clock 21 at Yokohama City, Japan. It has a 60-armed wheel, its diameter is 100 meters, and it has a height of 105 meters. (*Source: The Handy Science Answer Book*)

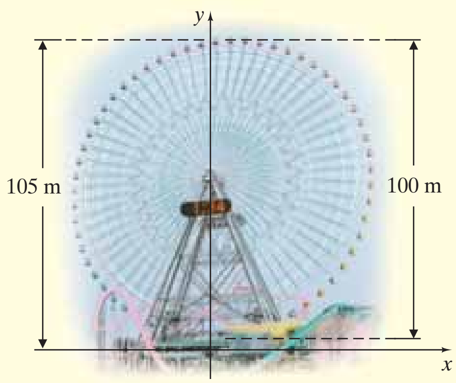
a. What is the radius of this Ferris wheel?

b. How close is the wheel to the ground?

c. How high is the center of the wheel from the ground?

d. Using the axes in the drawing, what are the coordinates of the center of the wheel?

e. Use parts (a) and (d) to write an equation of the wheel.

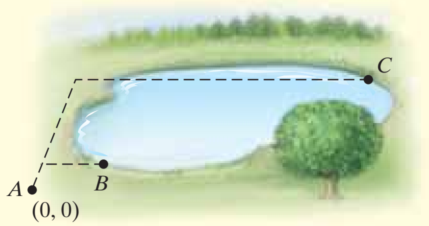


95. If you are given a list of equations of circles and parabolas and none are in standard form, explain how you would determine which is an equation of a circle and which is an equation of a parabola. Explain also how you would distinguish the upward or downward parabolas from the left-opening or right-opening parabolas.

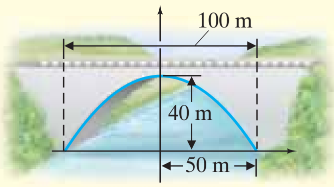
△ **96.** Determine whether the triangle with vertices $(2, 6)$, $(0, -2)$, and $(5, 1)$ is an isosceles triangle.

Solve.

97. Two surveyors need to find the distance across a lake. They place a reference pole at point A in the diagram. Point B is 3 meters east and 1 meter north of the reference point A. Point C is 19 meters east and 13 meters north of point A. Find the distance across the lake, from B to C.



98. A bridge constructed over a bayou has a supporting arch in the shape of a parabola. Find an equation of the parabolic arch if the length of the road over the arch is 100 meters and the maximum height of the arch is 40 meters.



Use a graphing calculator to verify each exercise. Use a square viewing window.

99. Exercise 77. **100.** Exercise 78.

101. Exercise 79. **102.** Exercise 80.

10.2 The Ellipse and the Hyperbola

OBJECTIVES

1 Define and Graph an Ellipse.

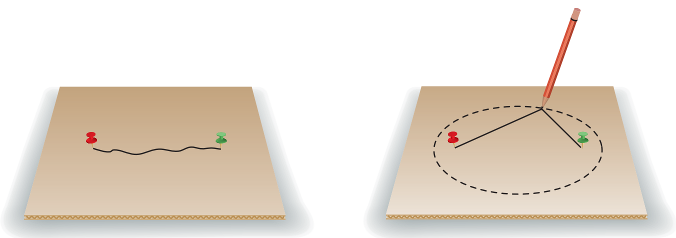
2 Define and Graph a Hyperbola.

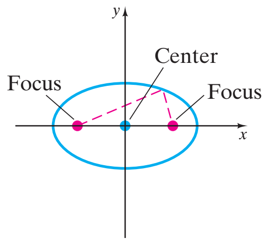
OBJECTIVE 1

Graphing Ellipses

An **ellipse** can be thought of as the set of points in a plane such that the sum of the distances of those points from two fixed points is constant. Each of the two fixed points is called a **focus.** (The plural of focus is **foci.**) The point midway between the foci is called the **center.**

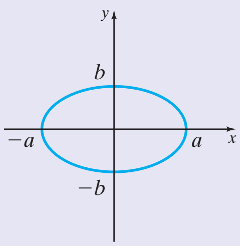
An ellipse may be drawn by hand by using two thumbtacks, a piece of string, and a pencil. Secure the two thumbtacks in a piece of cardboard, for example, and tie each end of the string to a tack. Use your pencil to pull the string tight and draw the ellipse. The two thumbtacks are the foci of the drawn ellipse.





Ellipse with Center (0, 0)

The graph of an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse with center $(0, 0)$. The x-intercepts are $(a, 0)$ and $(-a, 0)$, and the y-intercepts are $(0, b)$, and $(0, -b)$.



The **standard form** of an ellipse with center $(0, 0)$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.