101 PROBLEMS IN ALGEBRA

FROM THE TRAINING OF THE USA IMO TEAM

T ANDREESCU & Z FENG

AMT PUBLISHING

101 PROBLEMS IN ALGEBRA

FROM THE TRAINING OF THE USA IMO TEAM

T ANDREESCU & Z FENG

Published by

AMT PUBLISHING

Australian Mathematics Trust
University of Canberra ACT 2601
AUSTRALIA

Telephone: +61 2 6201 5137

AMTT Limited ACN 083 950 341

National Library of Australia Card Number and ISSN
Australian Mathematics Trust Enrichment Series ISSN 1326-0170
101 Problems in Algebra ISBN 1 876420 12 X

THE AUSTRALIAN MATHEMATICS TRUST

ENRICHMENT SERIES

The books in this series are selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solutions. Typically, the problems have occurred in either national or international contests at the secondary school level.

They are intended to be sufficiently detailed at an elementary level for the mathematically inclined or interested to understand but, at the same time, be interesting and sometimes challenging to the undergraduate and the more advanced mathematician. It is believed that these mathematics competition problems are a positive influence on the learning and enrichment of mathematics.

THE AUSTRALIAN MATHEMATICS TRUST

ENRICHMENT SERIES

BOOKS IN THE SERIES

1 AUSTRALIAN MATHEMATICS COMPETITION BOOK 1 1978–1984
JD Edwards, DJ King & PJ O'Halloran

2 MATHEMATICAL TOOLCHEST
AW Plank & NH Williams

3 TOURNAMENT OF TOWNS QUESTIONS AND SOLUTIONS 1984–1989
PJ Taylor

4 AUSTRALIAN MATHEMATICS COMPETITION BOOK 2 1985–1991
PJ O'Halloran, G Pollard & PJ Taylor

5 PROBLEM SOLVING VIA THE AMC
W Atkins

6 TOURNAMENT OF TOWNS QUESTIONS AND SOLUTIONS 1980–1984
PJ Taylor

7 TOURNAMENT OF TOWNS QUESTIONS AND SOLUTIONS 1989–1993
PJ Taylor

8 ASIAN PACIFIC MATHEMATICS OLYMPIADS 1989-2000
H Lausch & C Bosch Giral

9 METHODS OF PROBLEM SOLVING BOOK 1
JB Tabov & PJ Taylor

10 CHALLENGE! 1991-1995
JB Henry, J Dowsey, AR Edwards, LJ Mottershead,
A Nakos & G Vardaro

11 USSR MATHEMATICAL OLYMPIADS 1989-1992
AM Slinko

12 AUSTRALIAN MATHEMATICAL OLYMPIADS 1979-1995
H Lausch & PJ Taylor

13 CHINESE MATHEMATICS COMPETITIONS AND OLYMPIADS 1981-1993
A Liu

14 POLISH & AUSTRIAN MATHEMATICAL OLYMPIADS 1981– 1995
ME Kuczma & E Windischbacher

15 TOURNAMENT OF TOWNS QUESTIONS AND SOLUTIONS 1993–1997
PJ Taylor & AM Storozhev

16 AUSTRALIAN MATHEMATICS COMPETITION BOOK 3 1992–1998
WJ Atkins, JE Munro & PJ Taylor

17 SEEKING SOLUTIONS
JC Burns

18 101 PROBLEMS IN ALGEBRA
T Andreescu & Z Feng

PREFACE

This book contains one hundred highly rated problems used in the training and testing of the USA International Mathematical Olympiad (IMO) team. It is not a collection of one hundred very difficult, impenetrable questions. Instead, the book gradually builds students' algebraic skills and techniques. This work aims to broaden students' view of mathematics and better prepare them for possible participation in various mathematical competitions. It provides in-depth enrichment in important areas of algebra by reorganizing and enhancing students' problem-solving tactics and strategies. The book further stimulates students' interest for future study of mathematics.

INTRODUCTION

In the United States of America, the selection process leading to participation in the International Mathematical Olympiad (IMO) consists of a series of national contests called the American Mathematics Contest 10 (AMC 10), the American Mathematics Contest 12 (AMC 12), the American Invitational Mathematics Examination(AIME), and the United States of America Mathematical Olympiad (USAMO). Participation in the AIME and the USAMO is by invitation only, based on performance in the preceding exams of the sequence. The Mathematical Olympiad Summer Program (MOSP) is a four-week, intense training of 24-30 very promising students who have risen to the top of the American Mathematics Competitions. The six students representing the United States of America in the IMO are selected on the basis of their USAMO scores and further IMO-type testing that takes place during MOSP. Throughout MOSP, full days of classes and extensive problem sets give students thorough preparation in several important areas of mathematics. These topics include combinatorial arguments and identities, generating functions, graph theory, recursive relations, telescoping sums and products, probability, number theory, polynomials, theory of equations, complex numbers in geometry, algorithmic proofs, combinatorial and advanced geometry, functional equations and classical inequalities.

Olympiad-style exams consist of several challenging essay problems. Correct solutions often require deep analysis and careful argument. Olympiad questions can seem impenetrable to the novice, yet most can be solved with elementary high school mathematics techniques, cleverly applied.

Here is some advice for students who attempt the problems that follow.

- Take your time! Very few contestants can solve all the given problems.

- Try to make connections between problems. A very important theme of this work is: all important techniques and ideas featured in the book appear more than once!

- Olympiad problems don't "crack" immediately. Be patient. Try different approaches. Experiment with simple cases. In some cases, working backward from the desired result is helpful.

- Even if you can solve a problem, do read the solutions. They may contain some ideas that did not occur in your solutions, and they may discuss strategic and tactical approaches that can be used elsewhere. The formal solutions are also models of elegant presentation that you should emulate, but they often obscure the torturous process of investigation, false starts, inspiration and attention to detail that led to them. When you read the solutions, try to reconstruct the thinking that went into them. Ask yourself, "What were the key ideas?" "How can I apply these ideas further?"

- Go back to the original problem later, and see if you can solve it in a different way. Many of the problems have multiple solutions, but not all are outlined here.
- All terms in boldface are defined in the *Glossary*. Use the glossary and the reading list to further your mathematical education.
- Meaningful problem solving takes practice. Don't get discouraged if you have trouble at first. For additional practice, use the books on the reading list.

ACKNOWLEDGEMENTS

Thanks to Tiankai Liu who helped in proof reading and preparing solutions.

Many problems are either inspired by or fixed from mathematical contests in different countries and from the following journals:

High-School Mathematics, China
Revista Matematică Timişoara, Romania
Kvant, Russia

We did our best to cite all the original sources of the problems in the solution part. We express our deepest appreciation to the original proposers of the problems.

ABBREVIATIONS AND NOTATIONS

Abbreviations

AHSME	American High School Mathematics Examination
AIME	American Invitational Mathematics Examination
AMC10	American Mathematics Contest 10
AMC12	American Mathematics Contest 12, which replaces AHSME
ARML	American Regional Mathematics League
IMO	International Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad
MOSP	Mathematical Olympiad Summer Program
Putnam	The William Lowell Putnam Mathematical Competition
St. Petersburg	St. Petersburg (Leningrad) Mathematical Olympiad

Notations for Numerical Sets and Fields

$\mathbb{Z}$	the set of integers
$\mathbb{Z}_n$	the set of integers modulo n
$\mathbb{N}$	the set of positive integers
$\mathbb{N}_0$	the set of nonnegative integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers
$\mathbb{Q}^0$	the set of nonnegative rational numbers
$\mathbb{Q}^n$	the set of n-tuples of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{R}^0$	the set of nonnegative real numbers
$\mathbb{R}^n$	the set of n-tuples of real numbers
$\mathbb{C}$	the set of complex numbers

CONTENTS

INTRODUCTORY PROBLEMS

1. INTRODUCTORY PROBLEMS

Problem 1

Let a, b, and c be real and positive parameters. Solve the equation

$$\sqrt{a+bx}+\sqrt{b+cx}+\sqrt{c+ax}=\sqrt{b-ax}+\sqrt{c-bx}+\sqrt{a-cx}.$$

Problem 2

Find the general term of the sequence defined by $x_0 = 3$, $x_1 = 4$ and

$$x_{n+1} = x_{n-1}^2 - nx_n$$

for all $n \in \mathbb{N}$.

Problem 3

Let $x_1, x_2, \ldots, x_n$ be a sequence of integers such that

(i) $-1 \leq x_i \leq 2$, for $i = 1, 2, \ldots, n$;

(ii) $x_1 + x_2 + \cdots + x_n = 19$;

(iii) $x_1^2 + x_2^2 + \cdots + x_n^2 = 99$.

Determine the minimum and maximum possible values of

$$x_1^3 + x_2^3 + \cdots + x_n^3.$$

Problem 4

The function f, defined by

$$f(x) = \frac{ax+b}{cx+d},$$

where a, b, c, and d are nonzero real numbers, has the properties

$$f(19) = 19, \quad f(97) = 97, \quad \text{and} \quad f(f(x)) = x,$$

for all values of x, except $-\dfrac{d}{c}$.

Find the range of f.

Problem 5

Prove that
$$\frac{(a-b)^2}{8a} \le \frac{a+b}{2} - \sqrt{ab} \le \frac{(a-b)^2}{8b}$$
for all $a \ge b > 0$.

Problem 6

Several (at least two) nonzero numbers are written on a board. One may erase any two numbers, say a and b, and then write the numbers $a + \frac{b}{2}$ and $b - \frac{a}{2}$ instead.

Prove that the set of numbers on the board, after any number of the preceding operations, cannot coincide with the initial set.

Problem 7

The polynomial
$$1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$$
may be written in the form
$$a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17},$$
where $y = x + 1$ and a_is are constants.

Find a_2.

Problem 8

Let a, b, and c be distinct nonzero real numbers such that
$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$
Prove that $|abc| = 1$.

Problem 9

Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,
$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \le x \le 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

Problem 10

Find all real numbers x for which

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}.$$

Problem 11

Find the least positive integer m such that

$$\binom{2n}{n}^{\frac{1}{n}} < m$$

for all positive integers n.

Problem 12

Let a, b, c, d, and e be positive integers such that

$$abcde = a + b + c + d + e.$$

Find the maximum possible value of $\max\{a, b, c, d, e\}$.

Problem 13

Evaluate

$$\frac{3}{1! + 2! + 3!} + \frac{4}{2! + 3! + 4!} + \cdots + \frac{2001}{1999! + 2000! + 2001!}.$$

Problem 14

Let $x = \sqrt{a^2 + a + 1} - \sqrt{a^2 - a + 1}$, $a \in \mathbb{R}$.
Find all possible values of x.

Problem 15

Find all real numbers x for which

$$10^x + 11^x + 12^x = 13^x + 14^x.$$

Problem 16

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function such that $f(1,1) = 2$,

$$f(m+1,n) = f(m,n) + m \text{ and } f(m,n+1) = f(m,n) - n$$

for all $m, n \in \mathbb{N}$.
Find all pairs (p,q) such that $f(p,q) = 2001$.

Problem 17

Let f be a function defined on $[0,1]$ such that

$$f(0) = f(1) = 1 \text{ and } |f(a) - f(b)| < |a - b|,$$

for all $a \neq b$ in the interval $[0,1]$.
Prove that

$$|f(a) - f(b)| < \frac{1}{2}.$$

Problem 18

Find all pairs of integers (x, y) such that

$$x^3 + y^3 = (x+y)^2.$$

Problem 19

Let $f(x) = \dfrac{2}{4^x + 2}$ for real numbers x.
Evaluate

$$f\left(\frac{1}{2001}\right) + f\left(\frac{2}{2001}\right) + \cdots + f\left(\frac{2000}{2001}\right).$$

Problem 20

Prove that for $n \geq 6$ the equation

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = 1$$

has integer solutions.

Problem 21

Find all pairs of integers (a, b) such that the polynomial $ax^{17} + bx^{16} + 1$ is divisible by $x^2 - x - 1$.

Problem 22

Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999).$$

What is the largest prime factor of S?

Problem 23

Let x_n be a sequence of nonzero real numbers such that

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}$$

for $n = 3, 4, \ldots$.

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

Problem 24

Solve the equation

$$x^3 - 3x = \sqrt{x+2}.$$

Problem 25

For any sequence of real numbers $A = \{a_1, a_2, a_3, \cdots\}$, define ΔA to be the sequence $\{a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots\}$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19} = a_{92} = 0$.

Find a_1.

Problem 26

Find all real numbers x satisfying the equation

$$2^x + 3^x - 4^x + 6^x - 9^x = 1.$$

Problem 27

Prove that

$$16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17.$$

Problem 28

Determine the number of ordered pairs of integers (m, n) for which $mn \geq 0$ and

$$m^3 + n^3 + 99mn = 33^3.$$

Problem 29

Let a, b, and c be positive real numbers such that $a + b + c \leq 4$ and $ab + bc + ca \geq 4$.

Prove that at least two of the inequalities

$$|a - b| \leq 2, \quad |b - c| \leq 2, \quad |c - a| \leq 2$$

are true.

Problem 30

Evaluate

$$\sum_{k=0}^{n} \frac{1}{(n-k)!(n+k)!}.$$

Problem 31

Let $0 < a < 1$. Solve

$$x^{a^x} = a^{x^a}$$

for positive numbers x.

Problem 32

What is the coefficient of x^2 when

$$(1 + x)(1 + 2x)(1 + 4x) \cdots (1 + 2^n x)$$

is expanded?

Problem 33

Let m and n be distinct positive integers.

Find the maximum value of $|x^m - x^n|$, where x is a real number in the interval $(0, 1)$.

Problem 34

Prove that the polynomial

$$(x - a_1)(x - a_2) \cdots (x - a_n) - 1,$$

where $a_1, a_2, \cdots, a_n$ are distinct integers, cannot be written as the product of two non-constant polynomials with integer coefficients, i.e., it is irreducible.

Problem 35

Find all ordered pairs of real numbers (x, y) for which:

$$\begin{aligned} (1+x)(1+x^2)(1+x^4) &= 1+y^7 \\ \text{and} \quad (1+y)(1+y^2)(1+y^4) &= 1+x^7. \end{aligned}$$

Problem 36

Solve the equation

$$2(2^x - 1)x^2 + (2^{x^2} - 2)x = 2^{x+1} - 2$$

for real numbers x.

Problem 37

Let a be an irrational number and let n be an integer greater than 1. Prove that

$$\left(a + \sqrt{a^2 - 1}\right)^{\frac{1}{n}} + \left(a - \sqrt{a^2 - 1}\right)^{\frac{1}{n}}$$

is an irrational number.

Problem 38

Solve the system of equations

$$\begin{aligned} (x_1 - x_2 + x_3)^2 &= x_2(x_4 + x_5 - x_2) \\ (x_2 - x_3 + x_4)^2 &= x_3(x_5 + x_1 - x_3) \\ (x_3 - x_4 + x_5)^2 &= x_4(x_1 + x_2 - x_4) \\ (x_4 - x_5 + x_1)^2 &= x_5(x_2 + x_3 - x_5) \\ (x_5 - x_1 + x_2)^2 &= x_1(x_3 + x_4 - x_1) \end{aligned}$$

for real numbers x_1, x_2, x_3, x_4, x_5.

Problem 39

Let x, y, and z be complex numbers such that

$$x + y + z = 2,$$

$$x^2 + y^2 + z^2 = 3$$

and

$$xyz = 4.$$

Evaluate

$$\frac{1}{xy + z - 1} + \frac{1}{yz + x - 1} + \frac{1}{zx + y - 1}.$$

Problem 40

Mr. Fat is going to pick three non-zero real numbers and Mr. Taf is going to arrange the three numbers as the coefficients of a quadratic equation

$$\_x^2 + \_x + \_ = 0.$$

Mr. Fat wins the game if and only if the resulting equation has two distinct rational solutions.

Who has a winning strategy?

Problem 41

Given that the real numbers a, b, c, d, and e satisfy simultaneously the relations

$$a + b + c + d + e = 8 \text{ and } a^2 + b^2 + c^2 + d^2 + e^2 = 16,$$

determine the maximum and the minimum value of a.

Problem 42

Find the real zeros of the polynomial

$$P_a(x) = (x^2 + 1)(x - 1)^2 - ax^2,$$

where a is a given real number.

Problem 43

Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n.

Problem 44

Let

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

be a nonzero polynomial with integer coefficients such that $P(r) = P(s) = 0$ for some integers r and s, with $0 < r < s$.

Prove that $a_k \le -s$ for some k.

Problem 45

Let m be a given real number.

Find all complex numbers x such that

$$\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = m^2 + m.$$

Problem 46

The sequence given by $x_0 = a$, $x_1 = b$, and

$$x_{n+1} = \frac{1}{2}\left(x_{n-1} + \frac{1}{x_n}\right).$$

is periodic.

Prove that $ab = 1$.

Problem 47

Let a, b, c, and d be real numbers such that

$$(a^2 + b^2 - 1)(c^2 + d^2 - 1) > (ac + bd - 1)^2.$$

Prove that

$$a^2 + b^2 > 1 \text{ and } c^2 + d^2 > 1.$$

Problem 48

Find all complex numbers z such that

$$(3z + 1)(4z + 1)(6z + 1)(12z + 1) = 2.$$

Problem 49

Let $x_1, x_2, \cdots, x_{n-1}$, be the zeros different from 1 of the polynomial $P(x) = x^n - 1$, $n \geq 2$.

Prove that

$$\frac{1}{1 - x_1} + \frac{1}{1 - x_2} + \cdots + \frac{1}{1 - x_{n-1}} = \frac{n-1}{2}.$$

Problem 50

Let a and b be given real numbers. Solve the system of equations

$$\begin{aligned} \frac{x - y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} &= a, \\ \frac{y - x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} &= b \end{aligned}$$

for real numbers x and y.

ADVANCED PROBLEMS

2. ADVANCED PROBLEMS

Problem 51

Evaluate

$$\binom{2000}{2}+\binom{2000}{5}+\binom{2000}{8}+\cdots+\binom{2000}{2000}.$$

Problem 52

Let x, y, z be positive real numbers such that $x^4+y^4+z^4=1$. Determine with proof the minimum value of

$$\frac{x^3}{1-x^8}+\frac{y^3}{1-y^8}+\frac{z^3}{1-z^8}.$$

Problem 53

Find all real solutions to the equation

$$2^x+3^x+6^x=x^2.$$

Problem 54

Let $\{a_n\}_{n\geq 1}$ be a sequence such that $a_1=2$ and

$$a_{n+1}=\frac{a_n}{2}+\frac{1}{a_n}$$

for all $n\in\mathbb{N}$.

Find an explicit formula for a_n.

Problem 55

Let x, y, and z be positive real numbers. Prove that

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}$$

$$+\frac{z}{z+\sqrt{(z+x)(z+y)}}\quad\leq\quad 1.$$

Problem 56

Find, with proof, all nonzero polynomials $f(z)$ such that

$$f(z^2) + f(z)f(z+1) = 0.$$

Problem 57

Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(n+1) > f(n)$ and

$$f(f(n)) = 3n$$

for all n.

Evaluate $f(2001)$.

Problem 58

Let F be the set of all polynomials $f(x)$ with integers coefficients such that $f(x) = 1$ has at least one integer root.

For each integer $k > 1$, find m_k, the least integer greater than 1 for which there exists $f \in F$ such that the equation $f(x) = m_k$ has exactly k distinct integer roots.

Problem 59

Let $x_1 = 2$ and

$$x_{n+1} = x_n^2 - x_n + 1,$$

for $n \geq 1$.

Prove that

$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 1 - \frac{1}{2^{2^n}}.$$

Problem 60

Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing function such that for all $x, y \in \mathbb{R}^+$,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

Prove that $f(f(x)) = x$.

Problem 61

Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathbb{Q}$.

Problem 62

Let $\frac{3}{4} < a < 1$.

Prove that the equation

$$x^3(x+1) = (x+a)(2x+a)$$

has four distinct real solutions and find these solutions in explicit form.

Problem 63

Let a, b, and c be positive real numbers such that $abc = 1$.

Prove that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

Problem 64

Find all functions f, defined on the set of ordered pairs of positive integers, satisfying the following properties:

$$f(x,x) = x,\ f(x,y) = f(y,x),\ (x+y)f(x,y) = yf(x,x+y).$$

Problem 65

Consider n complex numbers z_k, such that $|z_k| \leq 1$, $k = 1, 2, \ldots, n$. Prove that there exist $e_1, e_2, \ldots, e_n \in \{-1, 1\}$ such that, for any $m \leq n$,

$$|e_1 z_1 + e_2 z_2 + \cdots + e_m z_m| \leq 2.$$

Problem 66

Find a triple of rational numbers (a, b, c) such that

$$\sqrt[3]{\sqrt[3]{2} - 1} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

Problem 67

Find the minimum of

$$\log_{x_1}\left(x_2-\frac{1}{4}\right)+\log_{x_2}\left(x_3-\frac{1}{4}\right)+\cdots+\log_{x_n}\left(x_1-\frac{1}{4}\right)$$

where $x_1, x_2, \ldots, x_n$ are real numbers in the interval $(\frac{1}{4}, 1)$.

Problem 68

Determine $x^2+y^2+z^2+w^2$ if

$$\frac{x^2}{2^2-1^2}+\frac{y^2}{2^2-3^2}+\frac{z^2}{2^2-5^2}+\frac{w^2}{2^2-7^2}=1,$$

$$\frac{x^2}{4^2-1^2}+\frac{y^2}{4^2-3^2}+\frac{z^2}{4^2-5^2}+\frac{w^2}{4^2-7^2}=1,$$

$$\frac{x^2}{6^2-1^2}+\frac{y^2}{6^2-3^2}+\frac{z^2}{6^2-5^2}+\frac{w^2}{6^2-7^2}=1,$$

$$\frac{x^2}{8^2-1^2}+\frac{y^2}{8^2-3^2}+\frac{z^2}{8^2-5^2}+\frac{w^2}{8^2-7^2}=1.$$

Problem 69

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x)+f(y))=(f(x))^2+y$$

for all $x, y \in \mathbb{R}$.

Problem 70

The numbers $1000, 1001, \cdots, 2999$ have been written on a board.

Each time, one is allowed to erase two numbers, say, a and b, and replace them by the number $\frac{1}{2}\min(a,b)$.

After 1999 such operations, one obtains exactly one number c on the board. Prove that $c < 1$.

Problem 71

Let $a_1, a_2, \ldots, a_n$ be real numbers, not all zero.

Prove that the equation

$$\sqrt{1+a_1x}+\sqrt{1+a_2x}+\cdots+\sqrt{1+a_nx}=n$$

has at most one nonzero real root.

Problem 72

Let $\{a_n\}$ be the sequence of real numbers defined by $a_1 = t$ and

$$a_{n+1} = 4a_n(1 - a_n)$$

for $n \geq 1$.

For how many distinct values of t do we have $a_{1998} = 0$?

Problem 73

(a) Do there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \qquad \text{and} \qquad g(f(x)) = x^3$$

for all $x \in \mathbb{R}$?

(b) Do there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \qquad \text{and} \qquad g(f(x)) = x^4$$

for all $x \in \mathbb{R}$?

Problem 74

Let $0 < a_1 \leq a_2 \cdots \leq a_n$, $0 < b_1 \leq b_2 \cdots \leq b_n$ be real numbers such that

$$\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} b_i.$$

Suppose that there exists $1 \leq k \leq n$ such that $b_i \leq a_i$ for $1 \leq i \leq k$ and $b_i \geq a_i$ for $i > k$.

Prove that

$$a_1 a_2 \cdots a_n \geq b_1 b_2 \cdots b_n.$$

Problem 75

Given eight non-zero real numbers $a_1, a_2, \cdots, a_8$, prove that at least one of the following six numbers: $a_1a_3 + a_2a_4$, $a_1a_5 + a_2a_6$, $a_1a_7 + a_2a_8$, $a_3a_5 + a_4a_6$, $a_3a_7 + a_4a_8$, $a_5a_7 + a_6a_8$ is non-negative.

Problem 76

Let a, b and c be positive real numbers such that $abc = 1$.

Prove that

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1.$$

Problem 77

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

holds for all pairs of real numbers (x, y).

Problem 78

Solve the system of equations:

$$x + \frac{3x - y}{x^2 + y^2} = 3$$
$$y - \frac{x + 3y}{x^2 + y^2} = 0.$$

Problem 79

Mr. Fat and Mr. Taf play a game with a polynomial of degree at least 4:

$$x^{2n} + \_x^{2n-1} + \_x^{2n-2} + \cdots + \_x + 1.$$

They fill in real numbers to empty spaces in turn. If the resulting polynomial has no real root, Mr. Fat wins; otherwise, Mr. Taf wins.

If Mr. Fat goes first, who has a winning strategy?

Problem 80

Find all positive integers k for which the following statement is true: if $F(x)$ is a polynomial with integer coefficients satisfying the condition

$$0 \leq F(\mathrm{c}) \leq k \quad \text{for} \quad \mathrm{c} = 0, 1, \ldots, k + 1,$$

then $F(0) = F(1) = \cdots = F(k + 1)$.

Problem 81

The Fibonacci sequence F_n is given by

$$F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n \quad (n \in \mathbb{N}).$$

Prove that

$$F_{2n} = \frac{F_{2n+2}^3 + F_{2n-2}^3}{9} - 2F_{2n}^3$$

for all $n \geq 2$.

Problem 82

Find all functions $u : \mathbb{R} \to \mathbb{R}$ for which there exists a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x)u(y) + f(y)$$

for all $x, y \in \mathbb{R}$.

Problem 83

Let $z_1, z_2, \dots, z_n$ be complex numbers such that

$$|z_1| + |z_2| + \cdots + |z_n| = 1.$$

Prove that there exists a subset S of $\{z_1, z_2, \dots, z_n\}$ such that

$$\left| \sum_{z \in S} z \right| \geq \frac{1}{6}.$$

Problem 84

A polynomial $P(x)$ of degree $n \geq 5$ with integer coefficients and n distinct integer roots is given.

Find all integer roots of $P(P(x))$ given that 0 is a root of $P(x)$.

Problem 85

Two real sequences $x_1, x_2, \dots,$ and $y_1, y_2, \dots,$ are defined in the following way:

$$x_1 = y_1 = \sqrt{3}, \quad x_{n+1} = x_n + \sqrt{1 + x_n^2},$$

and

$$y_{n+1} = \frac{y_n}{1 + \sqrt{1 + y_n^2}}$$

for all $n \geq 1$. Prove that $2 < x_n y_n < 3$ for all $n > 1$.

Problem 86

For a polynomial $P(x)$, define the *difference* of $P(x)$ on the interval $[a, b]$ ($[a, b)$, (a, b), $(a, b]$) as $P(b) - P(a)$.

Prove that it is possible to dissect the interval $[0, 1]$ into a finite number of intervals and color them red and blue alternately such that, for every quadratic polynomial $P(x)$, the total difference of $P(x)$ on red intervals is equal to that of $P(x)$ on blue intervals.

What about cubic polynomials?

Problem 87

Given a cubic equation

$$x^3 + \_x^2 + \_x + \_ = 0,$$

Mr. Fat and Mr. Taf are playing the following game. In one move, Mr. Fat chooses a real number and Mr. Taf puts it in one of the empty spaces. After three moves the game is over. Mr. Fat wins the game if the final equation has three distinct integer roots.

Who has a winning strategy?

Problem 88

Let $n > 2$ be an integer and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for any regular n-gon $A_1A_2\ldots A_n$,

$$f(A_1) + f(A_2) + \cdots + f(A_n) = 0.$$

Prove that f is the zero function.

Problem 89

Let p be a prime number and let $f(x)$ be a polynomial of degree d with integer coefficients such that:

(i) $f(0) = 0, f(1) = 1$;

(ii) for every positive integer n, the remainder upon division of $f(n)$ by p is either 0 or 1.

Prove that $d \geq p - 1$.

Problem 90

Let n be a given positive integer.

Consider the sequence $a_0, a_1, \cdots, a_n$ with $a_0 = \dfrac{1}{2}$ and

$$a_k = a_{k-1} + \frac{a_{k-1}^2}{n},$$

for $k = 1, 2, \cdots, n$.

Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

Problem 91

Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers, not all zero.

(a) Prove that $x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n = 0$ has precisely one positive real root R.

(b) Let $A = \sum_{j=1}^{n} a_j$ and $B = \sum_{j=1}^{n} ja_j$.

Prove that $A^A \leq R^B$.

Problem 92

Prove that there exists a polynomial $P(x, y)$ with real coefficients such that $P(x, y) \geq 0$ for all real numbers x and y, which cannot be written as the sum of squares of polynomials with real coefficients.

Problem 93

For each positive integer n, show that there exists a positive integer k such that

$$k = f(x)(x+1)^{2n} + g(x)(x^{2n}+1)$$

for some polynomials f, g with integer coefficients, and find the smallest such k as a function of n.

Problem 94

Let x be a positive real number.

(a) Prove that

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(x+1)\cdots(x+n)} = \frac{1}{x}.$$

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{n(x+1)\ldots(x+n)} = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2}.$$

Problem 95

Let $n \geq 3$ be an integer, and let

$$X \subseteq S = \{1, 2, \ldots, n^3\}$$

be a set of $3n^2$ elements.

Prove that one can find nine distinct numbers a_i, b_i, c_i $(i = 1, 2, 3)$ in X such that the system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

has a solution (x_0, y_0, z_0) in nonzero integers.

Problem 96

Let $n \geq 3$ be an integer and let $x_1, x_2, \cdots, x_n$ be positive real numbers.

Suppose that $\displaystyle\sum_{j=1}^{n} \frac{1}{1 + x_j} = 1.$

Prove that

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \geq (n-1)\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right).$$

Problem 97

Let $x_1, x_2, \ldots, x_n$ be distinct real numbers. Define the polynomials

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

and

$$Q(x) = P(x)\left(\frac{1}{x - x_1} + \frac{1}{x - x_2} + \cdots + \frac{1}{x - x_n}\right).$$

Let $y_1, y_2, \ldots, y_{n-1}$ be the roots of Q. Show that

$$\min_{i \neq j} |x_i - x_j| < \min_{i \neq j} |y_i - y_j|.$$

Problem 98

Show that for any positive integer n, the polynomial

$$f(x) = (x^2 + x)^{2^n} + 1$$

cannot be written as the product of two non-constant polynomials with integer coefficients.

Problem 99

Let $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ be functions such that

$$a_1 f_1 + a_2 f_2 + a_3 f_3$$

is monotonic for all $a_1, a_2, a_3 \in \mathbb{R}$.

Prove that there exist $c_1, c_2, c_3 \in \mathbb{R}$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

for all $x \in \mathbb{R}$.

Problem 100

Let $x_1, x_2, \ldots, x_n$ be variables, and let $y_1, y_2, \ldots, y_{2^n-1}$ be the sums of nonempty subsets of x_i.

Let $p_k(x_1, \ldots, x_n)$ be the k^{th} **elementary symmetric polynomial** in the y_i (the sum of every product of k distinct y_is).

For which k and n is every coefficient of p_k (as a polynomial in $x_1, \ldots, x_n$) even?

For example, if $n = 2$, then y_1, y_2, y_3 are $x_1, x_2, x_1 + x_2$ and

$$\begin{aligned}
p_1 &= y_1 + y_2 + y_3 = 2x_1 + 2x_2, \\
p_2 &= y_1y_2 + y_2y_3 + y_3y_1 = x_1^2 + x_2^2 + 3x_1x_2, \\
p_3 &= y_1y_2y_3 = x_1^2x_2 + x_1x_2^2.
\end{aligned}$$

Problem 101

Prove that there exist 10 distinct real numbers $a_1, a_2, \ldots, a_{10}$ such that the equation

$$(x - a_1)(x - a_2)\cdots(x - a_{10}) = (x + a_1)(x + a_2)\cdots(x + a_{10})$$

has exactly 5 different real roots.

SOLUTIONS TO INTRODUCTORY PROBLEMS

3. SOLUTIONS TO INTRODUCTORY PROBLEMS

Problem 1 [Romania 1974]

Let a, b, and c be real and positive parameters.

Solve the equation

$$\sqrt{a+bx}+\sqrt{b+cx}+\sqrt{c+ax}=\sqrt{b-ax}+\sqrt{c-bx}+\sqrt{a-cx}.$$

Solution 1

It is easy to see that $x=0$ is a solution. Since the right hand side is a decreasing function of x and the left hand side is an increasing function of x, there is at most one solution.

Thus $x=0$ is the only solution to the equation.

Problem 2

Find the general term of the sequence defined by $x_0=3$, $x_1=4$ and

$$x_{n+1}=x_{n-1}^2-nx_n$$

for all $n \in \mathbb{N}$.

Solution 2

We shall prove by induction that $x_n=n+3$. The claim is evident for $n=0,1$.

For $k \geq 1$, if $x_{k-1}=k+2$ and $x_k=k+3$, then

$$x_{k+1}=x_{k-1}^2-kx_k=(k+2)^2-k(k+3)=k+4,$$

as desired.

This completes the induction.

Problem 3 [AHSME 1999]

Let $x_1, x_2, \ldots, x_n$ be a sequence of integers such that

(i) $-1 \leq x_i \leq 2$, for $i = 1, 2, \ldots, n$;

(ii) $x_1 + x_2 + \cdots + x_n = 19$;

(iii) $x_1^2 + x_2^2 + \cdots + x_n^2 = 99$.

Determine the minimum and maximum possible values of

$$x_1^3 + x_2^3 + \cdots + x_n^3.$$

Solution 3

Let a, b, and c denote the number of -1s, 1s, and 2s in the sequence, respectively. We need not consider the zeros. Then a, b, c are nonnegative integers satisfying

$$-a + b + 2c = 19 \text{ and } a + b + 4c = 99.$$

It follows that $a = 40 - c$ and $b = 59 - 3c$, where $0 \leq c \leq 19$ (since $b \geq 0$), so

$$x_1^3 + x_2^3 + \cdots + x_n^3 = -a + b + 8c = 19 + 6c.$$

When $c = 0$ ($a = 40, b = 59$), the lower bound (19) is achieved.

When $c = 19$ ($a = 21, b = 2$), the upper bound (133) is achieved.

Problem 4 [AIME 1997]

The function f, defined by

$$f(x) = \frac{ax + b}{cx + d},$$

where a, b, c, and d are nonzero real numbers, has the properties

$$f(19) = 19, \quad f(97) = 97, \quad \text{and} \quad f(f(x)) = x,$$

for all values of x, except $-\dfrac{d}{c}$.

Find the range of f.

Solution 4, Alternative 1

For all x, $f(f(x)) = x$, i.e.,

$$\frac{a\left(\dfrac{ax+b}{cx+d}\right) + b}{c\left(\dfrac{ax+b}{cx+d}\right) + d} = x,$$

i.e.

$$\frac{(a^2+bc)x+b(a+d)}{c(a+d)x+bc+d^2}=x,$$

i.e.

$$c(a+d)x^2+(d^2-a^2)x-b(a+d)=0,$$

which implies that $c(a+d)=0$. Since $c\neq 0$, we must have $a=-d$. The conditions $f(19)=19$ and $f(97)=97$ lead to the equations

$$19^2c=2\cdot 19a+b \qquad \text{and} \qquad 97^2c=2\cdot 97a+b.$$

Hence

$$(97^2-19^2)c=2(97-19)a.$$

It follows that $a=58c$, which in turn leads to $b=-1843c$. Therefore

$$f(x)=\frac{58x-1843}{x-58}=58+\frac{1521}{x-58},$$

which never has the value 58.

Thus the range of f is $\mathbb{R}-\{58\}$.

Solution 4, Alternative 2

The statement implies that f is its own inverse. The inverse may be found by solving the equation

$$x=\frac{ay+b}{cy+d}$$

for y. This yields

$$f^{-1}(x)=\frac{dx-b}{-cx+a}.$$

The nonzero numbers a, b, c, and d must therefore be proportional to d, $-b$, $-c$, and a, respectively; it follows that $a=-d$, and the rest is the same as in the first solution.

Problem 5

Prove that

$$\frac{(a-b)^2}{8a}\leq\frac{a+b}{2}-\sqrt{ab}\leq\frac{(a-b)^2}{8b}$$

for all $a\geq b>0$.

Solution 5, Alternative 1

Note that

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2\sqrt{a}}\right)^2\leq 1\leq\left(\frac{\sqrt{a}+\sqrt{b}}{2\sqrt{b}}\right)^2,$$

i.e.

$$\frac{(\sqrt{a}+\sqrt{b})^2(\sqrt{a}-\sqrt{b})^2}{4a} \le (\sqrt{a}-\sqrt{b})^2 \le \frac{(\sqrt{a}+\sqrt{b})^2(\sqrt{a}-\sqrt{b})^2}{4b},$$

i.e.

$$\frac{(a-b)^2}{8a} \le \frac{a-2\sqrt{ab}+b}{2} \le \frac{(a-b)^2}{8b},$$

from which the result follows.

Solution 5, Alternative 2

Note that

$$\frac{a+b}{2} - \sqrt{ab} = \frac{\left(\frac{a+b}{2}\right)^2 - ab}{\frac{a+b}{2}+\sqrt{ab}} = \frac{(a-b)^2}{2(a+b)+4\sqrt{ab}}.$$

Thus the desired inequality is equivalent to

$$4a \ge a+b+2\sqrt{ab} \ge 4b,$$

which is evident as $a \ge b > 0$ (which implies $a \ge \sqrt{ab} \ge b$).

Problem 6 [St. Petersburg 1989]

Several (at least two) nonzero numbers are written on a board. One may erase any two numbers, say a and b, and then write the numbers $a + \frac{b}{2}$ and $b - \frac{a}{2}$ instead.

Prove that the set of numbers on the board, after any number of the preceding operations, cannot coincide with the initial set.

Solution 6

Let S be the sum of the squares of the numbers on the board. Note that S increases in the first operation and does not decrease in any successive operation, as

$$\left(a+\frac{b}{2}\right)^2 + \left(b-\frac{a}{2}\right)^2 = \frac{5}{4}(a^2+b^2) \ge a^2+b^2$$

with equality only if $a = b = 0$.

This completes the proof.

Problem 7 [AIME 1986]

The polynomial

$$1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$$

may be written in the form

$$a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17},$$

where $y = x + 1$ and a_is are constants. Find a_2.

Solution 7, Alternative 1

Let $f(x)$ denote the given expression. Then

$$xf(x) = x - x^2 + x^3 - \cdots - x^{18}$$

and

$$(1 + x)f(x) = 1 - x^{18}.$$

Hence

$$f(x) = f(y-1) = \frac{1 - (y-1)^{18}}{1 + (y-1)} = \frac{1 - (y-1)^{18}}{y}.$$

Therefore a_2 is equal to the coefficient of y^3 in the expansion of

$$1 - (y-1)^{18},$$

i.e.,

$$a_2 = \binom{18}{3} = 816.$$

Solution 7, Alternative 2

Let $f(x)$ denote the given expression. Then

$$\begin{aligned} f(x) = f(y-1) &= 1 - (y-1) + (y-1)^2 - \cdots - (y-1)^{17} \\ &= 1 + (1-y) + (1-y)^2 + \cdots + (1-y)^{17}. \end{aligned}$$

Thus

$$a_2 = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{17}{2} = \binom{18}{3}.$$

Here we used the formula

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

and the fact that

$$\binom{2}{2} = \binom{3}{3} = 1.$$

Problem 8

Let a, b, and c be distinct nonzero real numbers such that

$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}.$$

Prove that $|abc|=1$.

Solution 8

From the given conditions it follows that

$$a-b=\frac{b-c}{bc},\quad b-c=\frac{c-a}{ca},\quad \text{and } c-a=\frac{a-b}{ab}.$$

Multiplying the above equations gives $(abc)^2=1$, from which the desired result follows.

Problem 9 [Putnam 1999]

Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)|-|g(x)|+h(x)=\begin{cases} -1 & \text{if } x<-1 \\ 3x+2 & \text{if } -1\le x\le 0 \\ -2x+2 & \text{if } x>0. \end{cases}$$

Solution 9, Alternative 1

Since $x=-1$ and $x=0$ are the two critical values of the absolute functions, one can suppose that

$$\begin{aligned} F(x) &= a|x+1|+b|x|+cx+d \\ &= \begin{cases} (c-a-b)x+d-a & \text{if } x<-1 \\ (a+c-b)x+a+d & \text{if } -1\le x\le 0 \\ (a+b+c)x+a+d & \text{if } x>0, \end{cases} \end{aligned}$$

which implies that $a=3/2$, $b=-5/2$, $c=-1$, and $d=1/2$. Hence $f(x)=(3x+3)/2$, $g(x)=5x/2$, and $h(x)=-x+\frac{1}{2}$.

Solution 9, Alternative 2

Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r,s)=\frac{r+s+|r-s|}{2}.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x-3,0\}-\max\{5x,0\}+3x+2 \\ &= (-3x-3+|3x+3|)/2-(5x+|5x|)/2+3x+2 \\ &= |(3x+3)/2|-|5x/2|-x+\frac{1}{2}. \end{aligned}$$

Problem 10

Find all real numbers x for which

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}.$$

Solution 10

By setting $2^x = a$ and $3^x = b$, the equation becomes

$$\frac{a^3 + b^3}{a^2b + b^2a} = \frac{7}{6},$$

i.e.

$$\frac{a^2 - ab + b^2}{ab} = \frac{7}{6},$$

i.e.

$$6a^2 - 13ab + 6b^2 = 0,$$

i.e.

$$(2a - 3b)(3a - 2b) = 0.$$

Therefore $2^{x+1} = 3^{x+1}$ or $2^{x-1} = 3^{x-1}$, which implies that $x = -1$ and $x = 1$.

It is easy to check that both $x = -1$ and $x = 1$ satisfy the given equation.

Problem 11 [Romania 1990]

Find the least positive integer m such that

$$\binom{2n}{n}^{\frac{1}{n}} < m$$

for all positive integers n.

Solution 11

Note that

$$\binom{2n}{n} < \binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n} = (1+1)^{2n} = 4^n$$

and for $n = 5$,

$$\binom{10}{5} = 252 > 3^5.$$

Thus $m = 4$.

Problem 12

Let a, b, c, d, and e be positive integers such that

$$abcde = a + b + c + d + e.$$

Find the maximum possible value of $\max\{a, b, c, d, e\}$.

Solution 12, Alternative 1

Suppose that $a \le b \le c \le d \le e$. We need to find the maximum value of e. Since

$$e < a + b + c + d + e \le 5e,$$

then $e < abcde \le 5e$, i.e. $1 < abcd \le 5$.

Hence $(a, b, c, d) = (1, 1, 1, 2)$, $(1, 1, 1, 3)$, $(1, 1, 1, 4)$, $(1, 1, 2, 2)$, or $(1, 1, 1, 5)$, which leads to $\max\{e\} = 5$.

Solution 12, Alternative 2

As before, suppose that $a \le b \le c \le d \le e$. Note that

$$\begin{aligned} 1 &= \frac{1}{bcde} + \frac{1}{cdea} + \frac{1}{deab} + \frac{1}{eabc} + \frac{1}{abcd} \\ &\le \frac{1}{de} + \frac{1}{de} + \frac{1}{de} + \frac{1}{e} + \frac{1}{d} = \frac{3 + d + e}{de}. \end{aligned}$$

Therefore, $de \le 3 + d + e$ or $(d - 1)(e - 1) \le 4$.

If $d = 1$, then $a = b = c = 1$ and $4 + e = e$, which is impossible.

Thus $d - 1 \ge 1$ and $e - 1 \le 4$ or $e \le 5$.

It is easy to see that $(1, 1, 1, 2, 5)$ is a solution.

Therefore $\max\{e\} = 5$.

Comment: The second solution can be used to determine the maximum value of $\{x_1, x_2, \ldots, x_n\}$, when $x_1, x_2, \ldots, x_n$ are positive integers such that

$$x_1 x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n.$$

Problem 13

Evaluate

$$\frac{3}{1! + 2! + 3!} + \frac{4}{2! + 3! + 4!} + \cdots + \frac{2001}{1999! + 2000! + 2001!}.$$

Solution 13

Note that

$$\begin{aligned}\frac{k+2}{k!+(k+1)!+(k+2)!} &= \frac{k+2}{k![1+k+1+(k+1)(k+2)]}\\ &= \frac{1}{k!(k+2)}\\ &= \frac{k+1}{(k+2)!}\\ &= \frac{(k+2)-1}{(k+2)!}\\ &= \frac{1}{(k+1)!}-\frac{1}{(k+2)!}.\end{aligned}$$

By telescoping sum, the desired value is equal to

$$\frac{1}{2}-\frac{1}{2001!}.$$

Problem 14

Let $x=\sqrt{a^2+a+1}-\sqrt{a^2-a+1}$, $a\in\mathbb{R}$.

Find all possible values of x.

Solution 14, Alternative 1

Since

$$\sqrt{a^2+|a|+1}>|a|$$

and

$$x=\frac{2a}{\sqrt{a^2+a+1}+\sqrt{a^2-a+1}},$$

we have

$$|x|<|2a/a|=2.$$

Squaring both sides of

$$x+\sqrt{a^2-a+1}=\sqrt{a^2+a+1}$$

yields

$$2x\sqrt{a^2-a+1}=2a-x^2.$$

Squaring both sides of the above equation gives

$$4(x^2-1)a^2 = x^2(x^2-4) \text{ or } a^2 = \frac{x^2(x^2-4)}{4(x^2-1)}.$$

Since $a^2 \geq 0$, we must have

$$x^2(x^2-4)(x^2-1) \geq 0,$$

Since $|x| < 2$, $x^2-4 < 0$ which forces $x^2-1 < 0$. Therefore, $-1 < x < 1$. Conversely, for every $x \in (-1,1)$ there exists a real number a such that

$$x = \sqrt{a^2+a+1} - \sqrt{a^2-a+1}.$$

Solution 14, Alternative 2

Let $A = (-1/2, \sqrt{3}/2)$, $B = (1/2, \sqrt{3}/2)$, and $P = (a, 0)$. Then P is a point on the x-axis and we are looking for all possible values of $d = PA - PB$.

By the **Triangle Inequality**, $|PA - PB| < |AB| = 1$. And it is clear that all the values $-1 < d < 1$ are indeed obtainable. In fact, for such a d, a half hyperbola of all points Q such that $QA - QB = d$ is well defined. (Points A and B are foci of the hyperbola.)

Since line AB is parallel to the x-axis, this half hyperbola intersects the x- axis, i.e., P is well defined.

Problem 15

Find all real numbers x for which

$$10^x + 11^x + 12^x = 13^x + 14^x.$$

Solution 15

It is easy to check that $x = 2$ is a solution. We claim that it is the only one. In fact, dividing by 13^x on both sides gives

$$\left(\frac{10}{13}\right)^x + \left(\frac{11}{13}\right)^x + \left(\frac{12}{13}\right)^x = 1 + \left(\frac{14}{13}\right)^x.$$

The left hand side is a decreasing function of x and the right hand side is an increasing function of x.

Therefore their graphs can have at most one point of intersection.

Comment: More generally,

$$\begin{aligned} &a^2 + (a+1)^2 + \cdots + (a+k)^2 \\ &\quad = (a+k+1)^2 + (a+k+2)^2 + \cdots + (a+2k)^2 \end{aligned}$$

for $a = k(2k+1)$, $k \in \mathbb{N}$.

Problem 16 [Korean Mathematics Competition 2001]

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function such that $f(1,1) = 2$,

$$f(m+1, n) = f(m,n) + m \text{ and } f(m, n+1) = f(m,n) - n$$

for all $m, n \in \mathbb{N}$.

Find all pairs (p, q) such that $f(p,q) = 2001$.

Solution 16

We have

$$\begin{aligned} f(p,q) &= f(p-1,q) + p - 1 \\ &= f(p-2,q) + (p-2) + (p-1) \\ &= \cdots \\ &= f(1,q) + \frac{p(p-1)}{2} \\ &= f(1,q-1) - (q-1) + \frac{p(p-1)}{2} \\ &= \cdots \\ &= f(1,1) - \frac{q(q-1)}{2} + \frac{p(p-1)}{2} \\ &= 2001. \end{aligned}$$

Therefore

$$\frac{p(p-1)}{2} - \frac{q(q-1)}{2} = 1999,$$

i.e.

$$(p-q)(p+q-1) = 2 \cdot 1999.$$

Note that 1999 is a prime number and that $p - q < p + q - 1$ for $p, q \in \mathbb{N}$. We have the following two cases:

1. $p - q = 1$ and $p + q - 1 = 3998$. Hence $p = 2000$ and $q = 1999$.

2. $p - q = 2$ and $p + q - 1 = 1999$. Hence $p = 1001$ and $q = 999$.

Therefore $(p, q) = (2000, 1999)$ or $(1001, 999)$.

Problem 17 [China 1983]
Let f be a function defined on $[0, 1]$ such that

$$f(0) = f(1) = 1 \text{ and } |f(a) - f(b)| < |a - b|,$$

for all $a \neq b$ in the interval $[0, 1]$.
Prove that

$$|f(a) - f(b)| < \frac{1}{2}.$$

Solution 17
We consider the following cases.

1. $|a - b| \leq 1/2$. Then $|f(a) - f(b)| < |a - b| \leq \frac{1}{2}$, as desired.

2. $|a - b| > 1/2$. By symmetry, we may assume that $a > b$. Then

$$\begin{aligned} |f(a) - f(b)| &= |f(a) - f(1) + f(0) - f(b)| \\ &\leq |f(a) - f(1)| + |f(0) - f(b)| \\ &< |a - 1| + |0 - b| \\ &= 1 - a + b - 0 \\ &= 1 - (a - b) \\ &< \frac{1}{2}, \end{aligned}$$

as desired.

Problem 18
Find all pairs of integers (x, y) such that

$$x^3 + y^3 = (x + y)^2.$$

Solution 18
Since $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$, all pairs of integers $(n, -n)$, $n \in \mathbb{Z}$, are solutions.
Suppose that $x + y \neq 0$. Then the equation becomes

$$x^2 - xy + y^2 = x + y,$$

i.e.

$$x^2 - (y + 1)x + y^2 - y = 0.$$

Treated as a quadratic equation in x, we calculate the discriminant

$$\Delta = y^2 + 2y + 1 - 4y^2 + 4y = -3y^2 + 6y + 1.$$

Solving for $\Delta \geq 0$ yields

$$\frac{3 - 2\sqrt{3}}{3} \leq y \leq \frac{3 + 2\sqrt{3}}{3}.$$

Thus the possible values for y are 0, 1, and 2, which lead to the solutions $(1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 1)$, and $(2, 2)$.

Therefore, the integer solutions of the equation are $(x, y) = (1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, and $(n, -n)$, for all $n \in \mathbb{Z}$.

Problem 19 [Korean Mathematics Competition 2001]

Let

$$f(x) = \frac{2}{4^x + 2}$$

for real numbers x. Evaluate

$$f\left(\frac{1}{2001}\right) + f\left(\frac{2}{2001}\right) + \cdots + f\left(\frac{2000}{2001}\right).$$

Solution 19

Note that f has a half-turn symmetry about point $(1/2, 1/2)$. Indeed,

$$f(1 - x) = \frac{2}{4^{1-x} + 2} = \frac{2 \cdot 4^x}{4 + 2 \cdot 4^x} = \frac{4^x}{4^x + 2},$$

from which it follows that $f(x) + f(1 - x) = 1$.

Thus the desired sum is equal to 1000.

Problem 20

Prove that for $n \geq 6$ the equation

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = 1$$

has integer solutions.

Solution 20

Note that

$$\frac{1}{a^2} = \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2},$$

from which it follows that if $(x_1, x_2, \cdots, x_n) = (a_1, a_2, \cdots, a_n)$ is an integer solution to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = 1,$$

then

$$\begin{aligned}&(x_1, x_2, \cdots, x_{n-1}, x_n, x_{n+1}, x_{n+2}, x_{n+3})\\ &= (a_1, a_2, \cdots, a_{n-1}, 2a_n, 2a_n, 2a_n, 2a_n,)\end{aligned}$$

is an integer solution to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_{n+3}^2} = 1.$$

Therefore we can construct the solutions inductively if there are solutions for $n = 6, 7$, and 8.

Since $x_1 = 1$ is a solution for $n = 1$, $(2, 2, 2, 2)$ is a solution for $n = 4$, and $(2, 2, 2, 4, 4, 4, 4)$ is a solution for $n = 7$.

It is easy to check that $(2, 2, 2, 3, 3, 6)$ and $(2, 2, 2, 3, 4, 4, 12, 12)$ are solutions for $n = 6$ and $n = 8$, respectively. This completes the proof.

Problem 21 [AIME 1988]

Find all pairs of integers (a, b) such that the polynomial

$$ax^{17} + bx^{16} + 1$$

is divisible by $x^2 - x - 1$.

Solution 21, Alternative 1

Let p and q be the roots of $x^2 - x - 1 = 0$. By **Vieta's theorem**, $p + q = 1$ and $pq = -1$. Note that p and q must also be the roots of $ax^{17} + bx^{16} + 1 = 0$. Thus

$$ap^{17} + bp^{16} = -1 \text{ and } aq^{17} + bq^{16} = -1.$$

Multiplying the first of these equations by q^{16}, the second one by p^{16}, and using the fact that $pq = -1$, we find

$$ap + b = -q^{16} \text{ and } aq + b = -p^{16}. \tag{1}$$

Thus

$$a = \frac{p^{16} - q^{16}}{p - q} = (p^8 + q^8)(p^4 + q^4)(p^2 + q^2)(p + q).$$

Since

$$\begin{aligned}
p+q &= 1,\\
p^2+q^2 &= (p+q)^2-2pq = 1+2 = 3,\\
p^4+q^4 &= (p^2+q^2)^2-2p^2q^2 = 9-2 = 7,\\
p^8+q^8 &= (p^4+q^4)^2-2p^4q^4 = 49-2 = 47,
\end{aligned}$$

it follows that $a = 1\cdot 3\cdot 7\cdot 47 = 987$.

Likewise, eliminating a in (1) gives

$$\begin{aligned}
-b &= \frac{p^{17}-q^{17}}{p-q}\\
&= p^{16}+p^{15}q+p^{14}q^2+\cdots+q^{16}\\
&= (p^{16}+q^{16})+pq(p^{14}+q^{14})+p^2q^2(p^{12}+q^{12})\\
&\quad +\cdots+p^7q^7(p^2+q^2)+p^8q^8\\
&= (p^{16}+q^{16})-(p^{14}+q^{14})+\cdots-(p^2+q^2)+1.
\end{aligned}$$

For $n \geq 1$, let $k_{2n} = p^{2n}+q^{2n}$. Then $k_2 = 3$ and $k_4 = 7$, and

$$\begin{aligned}
k_{2n+4} &= p^{2n+4}+q^{2n+4}\\
&= (p^{2n+2}+q^{2n+2})(p^2+q^2)-p^2q^2(p^{2n}+q^{2n})\\
&= 3k_{2n+2}-k_{2n}
\end{aligned}$$

for $n \geq 3$. Then $k_6 = 18$, $k_8 = 47$, $k_{10} = 123$, $k_{12} = 322$, $k_{14} = 843$, $k_{16} = 2207$.

Hence

$$-b = 2207-843+322-123+47-18+7-3+1 = 1597$$

or

$$(a,b) = (987,-1597).$$

Solution 21, Alternative 2

The other factor is of degree 15 and we write

$$(c_{15}x^{15}-c_{14}x^{14}+\cdots+c_1x-c_0)(x^2-x-1) = ax^{17}+bx^{16}+1.$$

Comparing coefficients:

$$\begin{aligned}
& x^0: && c_0 = 1,\\
& x^1: && c_0-c_1 = 0, c_1 = 1\\
& x^2: && -c_0-c_1+c_2 = 0, c_2 = 2,\\
\text{and for } 3 \leq k \leq 15, \quad & x^k: && -c_{k-2}-c_{k-1}+c_k = 0.
\end{aligned}$$

It follows that for $k \leq 15$, $c_k = F_{k+1}$ (**the Fibonacci number**).
Thus $a = c_{15} = F_{16} = 987$ and $b = -c_{14} - c_{15} = -F_{17} = -1597$ or $(a, b) = (987, -1597)$.

Comment: Combining the two methods, we obtain some interesting facts about sequences k_{2n} and F_{2n-1}. Since

$$3F_{2n+3} - F_{2n+5} = 2F_{2n+3} - F_{2n+4} = F_{2n+3} - F_{2n+2} = F_{2n+1},$$

it follows that F_{2n-1} and k_{2n} satisfy the same recursive relation. It is easy to check that $k_2 = F_1 + F_3$ and $k_4 = F_3 + F_5$.
Therefore $k_{2n} = F_{2n-1} + F_{2n+1}$ and

$$F_{2n+1} = k_{2n} - k_{2n-2} + k_{2n-4} - \cdots + (-1)^{n-1}k_2 + (-1)^n.$$

Problem 22 [AIME 1994]

Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999).$$

What is the largest prime factor of S?

Solution 22

Consider each positive integer less than 1000 to be a three-digit number by prefixing 0s to numbers with fewer than three digits. The sum of the products of the digits of all such positive numbers is

$$\begin{aligned}&(0 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 1 + \cdots + 9 \cdot 9 \cdot 9) - 0 \cdot 0 \cdot 0\\ &= (0 + 1 + \cdots + 9)^3 - 0.\end{aligned}$$

However, $p(n)$ is the product of non-zero digits of n. The sum of these products can be found by replacing 0 by 1 in the above expression, since ignoring 0's is equivalent to thinking of them as 1's in the products. (Note that the final 0 in the above expression becomes a 1 and compensates for the contribution of 000 after it is changed to 111.)
Hence

$$S = 46^3 - 1 = (46 - 1)(46^2 + 46 + 1) = 3^3 \cdot 5 \cdot 7 \cdot 103,$$

and the largest prime factor is 103.

Problem 23 [Putnam 1979]

Let x_n be a sequence of nonzero real numbers such that

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}$$

for $n = 3, 4, \ldots$.

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

Solution 23, Alternative 1

We have

$$\frac{1}{x_n} = \frac{2x_{n-2} - x_{n-1}}{x_{n-2}x_{n-1}} = \frac{2}{x_{n-1}} - \frac{1}{x_{n-2}}.$$

Let $y_n = 1/x_n$. Then $y_n - y_{n-1} = y_{n-1} - y_{n-2}$, i.e., y_n is an arithmetic sequence. If x_n is a nonzero integer when n is in an infinite set S, the y_n's for $n \in S$ satisfy $-1 \leq y_n \leq 1$.

Since an arithmetic sequence is unbounded unless the common difference is 0, $y_n - y_{n-1} = 0$ for all n, which in turn implies that $x_1 = x_2 = m$, a nonzero integer.

Clearly, this condition is also sufficient.

Solution 23, Alternative 2

An easy induction shows that

$$x_n = \frac{x_1x_2}{(n-1)x_1 - (n-2)x_2} = \frac{x_1x_2}{(x_1 - x_2)n + (2x_2 - x_1)},$$

for $n = 3, 4, \ldots$.

In this form we see that x_n will be an integer for infinitely many values of n if and only if $x_1 = x_2 = m$ for some nonzero integer m.

Problem 24

Solve the equation

$$x^3 - 3x = \sqrt{x+2}.$$

Solution 24, Alternative 1

It is clear that $x \geq -2$. We consider the following cases.

1. $-2 \leq x \leq 2$. Setting $x = 2\cos a$, $0 \leq a \leq \pi$, the equation becomes

$$8\cos^3 a - 6\cos a = \sqrt{2(\cos a + 1)}.$$

or

$$2\cos 3a = \sqrt{4\cos^2 \frac{a}{2}},$$

from which it follows that $\cos 3a = \cos \frac{a}{2}$.

Then $3a - \frac{a}{2} = 2m\pi$, $m \in \mathbb{Z}$, or $3a + \frac{a}{2} = 2n\pi$, $n \in \mathbb{Z}$.

Since $0 \le a \le \pi$, the solution in this case is

$$x = 2\cos 0 = 2, \ x = 2\cos\frac{4\pi}{5}, \text{ and } x = 2\cos\frac{4\pi}{7}.$$

2. $x > 2$. Then $x^3 - 4x = x(x^2 - 4) > 0$ and

$$x^2 - x - 2 = (x-2)(x+1) > 0$$

or

$$x > \sqrt{x+2}.$$

It follows that

$$x^3 - 3x > x > \sqrt{x+2}.$$

Hence there are no solutions in this case.

Therefore, $x = 2$, $x = 2\cos 4\pi/5$, and $x = 2\cos 4\pi/7$.

Solution 24, Alternative 2

For $x > 2$, there is a real number $t > 1$ such that

$$x = t^2 + \frac{1}{t^2}.$$

The equation becomes

$$\left(t^2 + \frac{1}{t^2}\right)^3 - 3\left(t^2 + \frac{1}{t^2}\right) = \sqrt{t^2 + \frac{1}{t^2} + 2},$$

i.e.

$$t^6 + \frac{1}{t^6} = t + \frac{1}{t},$$

i.e.

$$(t^7 - 1)(t^5 - 1) = 0,$$

which has no solutions for $t > 1$.

Hence there are no solutions for $x > 2$.

For $-2 \le x \le 2$, please see the first solution.

Problem 25 [AIME 1992]

For any sequence of real numbers $A = \{a_1, a_2, a_3, \cdots\}$, define ΔA to be the sequence $\{a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots\}$.

Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19} = a_{92} = 0$.

Find a_1.

Solution 25

Suppose that the first term of the sequence ΔA is d.

Then

$$\Delta A = \{d, d+1, d+2, \ldots\}$$

with the n^{th} term given by $d + (n-1)$.

Hence

$$A = \{a_1, a_1 + d, a_1 + d + (d+1), a_1 + d + (d+1) + (d+2), \ldots\}$$

with the n^{th} term given by

$$a_n = a_1 + (n-1)d + \frac{1}{2}(n-1)(n-2).$$

This shows that a_n is a quadratic polynomial in n with leading coefficient 1/2.

Since $a_{19} = a_{92} = 0$, we must have

$$a_n = \frac{1}{2}(n-19)(n-92),$$

so $a_1 = (1-19)(1-92)/2 = 819$.

Problem 26 [Korean Mathematics Competition 2000]

Find all real numbers x satisfying the equation

$$2^x + 3^x - 4^x + 6^x - 9^x = 1.$$

Solution 26

Setting $2^x = a$ and $3^x = b$, the equation becomes

$$1 + a^2 + b^2 - a - b - ab = 0.$$

Multiplying both sides of the last equation by 2 and completing the squares gives

$$(1-a)^2 + (a-b)^2 + (b-1)^2 = 0.$$

Therefore $1 = 2^x = 3^x$, and $x = 0$ is the only solution.

Problem 27 [China 1992]

Prove that

$$16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17.$$

Solution 27

Note that

$$2\left(\sqrt{k+1} - \sqrt{k}\right) = \frac{2}{\sqrt{k+1} + \sqrt{k}} < \frac{1}{\sqrt{k}}.$$

Therefore

$$\sum_{k=1}^{80} \frac{1}{\sqrt{k}} > 2\sum_{k=1}^{80} \left(\sqrt{k+1} - \sqrt{k}\right) = 16,$$

which proves the lower bound.

On the other hand,

$$2\left(\sqrt{k} - \sqrt{k-1}\right) = \frac{2}{\sqrt{k} + \sqrt{k-1}} > \frac{1}{\sqrt{k}}.$$

Therefore

$$\sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 1 + 2\sum_{k=2}^{80} \left(\sqrt{k} - \sqrt{k-1}\right) = 2\sqrt{80} - 1 < 17,$$

which proves the upper bound. Our proof is complete.

Problem 28 [AHSME 1999]

Determine the number of ordered pairs of integers (m, n) for which $mn \geq 0$ and

$$m^3 + n^3 + 99mn = 33^3.$$

Solution 28

Note that $(m+n)^3 = m^3 + n^3 + 3mn(m+n)$. If $m + n = 33$, then

$$33^3 = (m+n)^3 = m^3 + n^3 + 3mn(m+n) = m^3 + n^3 + 99mn.$$

Hence $m + n - 33$ is a factor of $m^3 + n^3 + 99mn - 33^3$. We have

$$\begin{aligned} &m^3 + n^3 + 99mn - 33^3 \\ &= (m + n - 33)(m^2 + n^2 - mn + 33m + 33n + 33^2) \\ &= \frac{1}{2}(m + n - 33)[(m-n)^2 + (m+33)^2 + (n+33)^2]. \end{aligned}$$

Hence there are 35 solutions altogether: $(0,33)$, $(1,32)$, $\cdots$, $(33,0)$, and $(-33,-33)$.

Comment: More generally, we have

$$\begin{aligned}&a^3+b^3+c^3-3abc\\&=\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2].\end{aligned}$$

Problem 29 [Korean Mathematics Competition 2001]

Let a, b, and c be positive real numbers such that $a+b+c \le 4$ and $ab+bc+ca \ge 4$.

Prove that at least two of the inequalities

$$|a-b| \le 2, \quad |b-c| \le 2, \quad |c-a| \le 2$$

are true.

Solution 29

We have

$$(a+b+c)^2 \le 16,$$

i.e.

$$a^2+b^2+c^2+2(ab+bc+ca) \le 16,$$

i.e.

$$a^2+b^2+c^2 \le 8,$$

i.e.

$$a^2+b^2+c^2-(ab+bc+ca) \le 4,$$

i.e.

$$(a-b)^2+(b-c)^2+(c-a)^2 \le 8,$$

and the desired result follows.

Problem 30

Evaluate

$$\sum_{k=0}^{n} \frac{1}{(n-k)!(n+k)!}.$$

Solution 30

Let S_n denote the desired sum. Then

$$\begin{aligned} S_n &= \frac{1}{(2n)!}\sum_{k=0}^{n}\frac{(2n)!}{(n-k)!(n+k)!} \\ &= \frac{1}{(2n)!}\sum_{k=0}^{n}\binom{2n}{n-k} \\ &= \frac{1}{(2n)!}\sum_{k=0}^{n}\binom{2n}{k} \\ &= \frac{1}{(2n)!}\cdot\frac{1}{2}\left[\sum_{k=0}^{2n}\binom{2n}{k}+\binom{2n}{n}\right] \\ &= \frac{1}{(2n)!}\cdot\frac{1}{2}\left[2^{2n}+\binom{2n}{n}\right] \\ &= \frac{2^{2n-1}}{(2n)!}+\frac{1}{2(n!)^2}. \end{aligned}$$

Problem 31 [Romania 1983]

Let $0 < a < 1$. Solve

$$x^{a^x} = a^{x^a}$$

for positive numbers x.

Solution 31

Taking $\log_a$ yields

$$a^x \log_a x = x^a.$$

Consider functions from $\mathbb{R}^+ \to \mathbb{R}$,

$$f(x) = a^x, \quad g(x) = \log_a x, \quad h(x) = x^a.$$

Then both f and g are decreasing and h is increasing. It follows that $f(x)g(x) = h(x)$ has unique solution $x = a$.

Problem 32

What is the coefficient of x^2 when

$$(1+x)(1+2x)(1+4x)\cdots(1+2^n x)$$

is expanded?

Solution 32

Let

$$f_n(x) = a_{n,0} + a_{n,1}x + \cdots + a_{n,n}x^n = (1+x)(1+2x)\cdots(1+2^n x).$$

It is easy to see that $a_{n,0} = 1$ and

$$a_{n,1} = 1 + 2 + \cdots + 2^n = 2^{n+1} - 1.$$

Since

$$\begin{aligned} f_n(x) &= f_{n-1}(x)(1+2^n x) \\ &= \left(1 + (2^n - 1)x + a_{n-1,2}x^2 + \cdots\right)(1+2^n x) \\ &= 1 + \left(2^{n+1} - 1\right)x + \left(a_{n-1,2} + 2^{2n} - 2^n\right)x^2 + \cdots, \end{aligned}$$

we have

$$\begin{aligned} a_{n,2} &= a_{n-1,2} + 2^{2n} - 2^n \\ &= a_{n-2,2} + 2^{2n-2} - 2^{n-1} + 2^{2n} - 2^n \\ &= \cdots \\ &= a_{1,2} + \left(2^4 + 2^6 + \cdots + 2^{2n}\right) - \left(2^2 + 2^3 + \cdots + 2^n\right) \\ &= 2 + \frac{2^4(2^{2n-2} - 1)}{3} - 4(2^{n-1} - 1) \\ &= \frac{2^{2n+2} - 3\cdot 2^{n+1} + 2}{3} = \frac{\left(2^{n+1} - 1\right)\left(2^{n+1} - 2\right)}{3}. \end{aligned}$$

Problem 33

Let m and n be distinct positive integers.

Find the maximum value of $|x^m - x^n|$, where x is a real number in the interval $(0,1)$.

Solution 33

By symmetry, we can assume that $m > n$. Let $y = x^{m-n}$.

Since $0 < x < 1$, $x^m < x^n$ and $0 < y < 1$. Thus

$$|x^m - x^n| = x^n - x^m = x^n(1 - x^{m-n}) = \left(y^n(1-y)^{m-n}\right)^{\frac{1}{m-n}}.$$

Applying **the AM-GM inequality** yields

$$\begin{aligned} y^n(1-y)^{m-n} &= \left(\frac{n}{m-n}\right)^n \left(\frac{(m-n)y}{n}\right)^n (1-y)^{m-n} \\ &\le \left(\frac{n}{m-n}\right)^n \left(\frac{n\cdot\frac{(m-n)y}{n} + (m-n)(1-y)}{n+m-n}\right)^{n+m-n} \\ &= \frac{n^n(m-n)^{m-n}}{m^m}. \end{aligned}$$

Therefore

$$|x^m - x^n| \le \left(\frac{n^n(m-n)^{m-n}}{m^m}\right)^{\frac{1}{m-n}} = (m-n)\left(\frac{n^n}{m^m}\right)^{\frac{1}{m-n}}.$$

Equality holds if and only if

$$\frac{(m-n)y}{n} = 1 - y$$

or

$$x = \left(\frac{n}{m}\right)^{\frac{1}{m-n}}.$$

Comment: For $m = n + 1$, we have

$$x^n - x^{n+1} \le \frac{n^n}{(n+1)^{n+1}}$$

for real numbers $0 < x < 1$. Equality holds if and only if $x = n/(n+1)$.

Problem 34

Prove that the polynomial

$$(x - a_1)(x - a_2)\cdots(x - a_n) - 1,$$

where $a_1, a_2, \cdots, a_n$ are distinct integers, cannot be written as the product of two non-constant polynomials with integer coefficients, i.e., it is irreducible.

Solution 34

For the sake of contradiction, suppose that

$$f(x) = (x - a_1)(x - a_2)\cdots(x - a_n) - 1$$

is not irreducible. Let $f(x) = p(x)q(x)$ such that $p(x)$ and $q(x)$ are two polynomials with integral coefficients having degree less than n. Then

$$g(x) = p(x) + q(x)$$

is a polynomial with integral coefficients having degree less than n. Since

$$p(a_i)q(a_i) = f(a_i) = -1$$

and both $p(a_i)$ and $q(a_i)$ are integers,

$$|p(a_i)| = |q(a_i)| = 1$$

and

$$p(a_i) + q(a_i) = 0.$$

Thus $g(x)$ has at least n roots. But $\deg g < n$, so $g(x) = 0$. Then

$$p(x) = -q(x) \qquad \text{and} \qquad f(x) = -p(x)^2,$$

which implies that the leading coefficient of $f(x)$ must be a negative integer, which is impossible, since the leading coefficient of $f(x)$ is 1.

Problem 35

Find all ordered pairs of real numbers (x, y) for which:

$$\begin{aligned} (1+x)(1+x^2)(1+x^4) &= 1+y^7 \\ \text{and} \quad (1+y)(1+y^2)(1+y^4) &= 1+x^7. \end{aligned}$$

Solution 35

We consider the following cases.

1. $xy = 0$. Then it is clear that $x = y = 0$ and $(x, y) = (0, 0)$ is a solution.

2. $xy < 0$. By the symmetry, we can assume that $x > 0 > y$. Then $(1+x)(1+x^2)(1+x^4) > 1$ and $1 + y^7 < 1$. There are no solutions in this case.

3. $x, y > 0$ and $x \neq y$. By the symmetry, we can assume that $x > y > 0$. Then

   $$(1+x)(1+x^2)(1+x^4) > 1 + x^7 > 1 + y^7,$$

   showing that there are no solutions in this case.

4. $x, y < 0$ and $x \neq y$. By the symmetry, we can assume that $x < y < 0$. Multiplying by $1 - x$ and $1 - y$ the first and the second equation, respectively, the system now reads

   $$\begin{aligned} 1 - x^8 &= (1+y^7)(1-x) = 1 - x + y^7 - xy^7 \\ 1 - y^8 &= (1+x^7)(1-y) = 1 - y + x^7 - x^7y. \end{aligned}$$

   Subtracting the first equation from the second yields

   $$x^8 - y^8 = (x - y) + (x^7 - y^7) - xy(x^6 - y^6). \qquad (1)$$

   Since $x < y < 0$, $x^8 - y^8 > 0$, $x - y < 0$, $x^7 - y^7 < 0$, $-xy < 0$, and $x^6 - y^6 > 0$. Therefore, the left-hand side of (1) is positive while the right-hand side of (1) is negative.

   Thus there are no solutions in this case.

5. $x = y$. Then solving

$$1 - x^8 = 1 - x + y^7 - xy^7 = 1 - x + x^7 - x^8$$

leads to $x = 0, 1, -1$, which implies that $(x, y) = (0, 0)$ or $(-1, -1)$.

Therefore, $(x, y) = (0, 0)$ and $(-1, -1)$ are the only solutions to the system.

Problem 36

Solve the equation

$$2(2^x - 1)x^2 + (2^{x^2} - 2)x = 2^{x+1} - 2$$

for real numbers x.

Solution 36

Rearranging terms by powers of 2 yields

$$2^{x^2}x + 2^{x+1}(x^2 - 1) - 2(x^2 + x - 1) = 0. \tag{1}$$

Setting $y = x^2 - 1$ and dividing by 2 on the both sides, (1) becomes

$$2^y x + 2^x y - (x + y) = 0$$

or

$$x(2^y - 1) + y(2^x - 1) = 0. \tag{2}$$

Since $f(x) = 2^x - 1$ and x always have the same sign,

$$x(2^y - 1) \cdot y(2^x - 1) \geq 0.$$

Hence if the terms on the left-hand side of (2) are nonzero, they must have the same sign, which in turn implies that their sum is not equal to 0.

Therefore (2) is true if and only if $x = 0$ or $y = 0$, which leads to solutions $x = -1, 0$, and 1.

Problem 37

Let a be an irrational number and let n be an integer greater than 1. Prove that

$$\left(a + \sqrt{a^2 - 1}\right)^{\frac{1}{n}} + \left(a - \sqrt{a^2 - 1}\right)^{\frac{1}{n}}$$

is an irrational number.

Solution 37

Let

$$N = \left(a + \sqrt{a^2 - 1}\right)^{\frac{1}{n}} + \left(a - \sqrt{a^2 - 1}\right)^{\frac{1}{n}},$$

and let

$$b = (a + \sqrt{a^2 - 1})^{\frac{1}{n}}.$$

Then $N = b + 1/b$. For the sake of contradiction, assume that N is rational. Then by using the identity

$$b^{m+1} + \frac{1}{b^{m+1}} = \left(b + \frac{1}{b}\right)\left(b^m + \frac{1}{b^m}\right) - \left(b^{m-1} + \frac{1}{b^{m-1}}\right)$$

repeatedly for $m = 1, 2, \ldots$, we obtain that $b^m + 1/b^m$ is rational for all $m \in N$.

In particular,

$$b^n + \frac{1}{b^n} = a + \sqrt{a^2 - 1} + a - \sqrt{a^2 - 1} = 2a$$

is rational, in contradiction with the hypothesis.

Therefore our assumption is wrong and N is irrational.

Problem 38

Solve the system of equations

$$\begin{aligned}
(x_1 - x_2 + x_3)^2 &= x_2(x_4 + x_5 - x_2) \\
(x_2 - x_3 + x_4)^2 &= x_3(x_5 + x_1 - x_3) \\
(x_3 - x_4 + x_5)^2 &= x_4(x_1 + x_2 - x_4) \\
(x_4 - x_5 + x_1)^2 &= x_5(x_2 + x_3 - x_5) \\
(x_5 - x_1 + x_2)^2 &= x_1(x_3 + x_4 - x_1)
\end{aligned}$$

for real numbers x_1, x_2, x_3, x_4, x_5.

Solution 38

Let $x_{k+5} = x_k$. Adding the five equations gives

$$\sum_{k=1}^{5}(3x_k^2 - 4x_kx_{k+1} + 2x_kx_{k+2}) = \sum_{k=1}^{5}(-x_k^2 + 2x_kx_{k+2}).$$

It follows that

$$\sum_{k=1}^{5}(x_k^2 - x_kx_{k+1}) = 0.$$

Multiplying both sides by 2 and completing the squares yields

$$\sum_{k=1}^{5}(x_k - x_{k+1})^2 = 0,$$

from which $x_1 = x_2 = x_3 = x_4 = x_5$. Therefore the solutions to the system are

$$(x_1, x_2, x_3, x_4, x_5) = (a, a, a, a, a)$$

for $a \in \mathbb{R}$.

Problem 39

Let x, y, and z be complex numbers such that $x+y+z = 2$, $x^2+y^2+z^2 = 3$, and $xyz = 4$.

Evaluate

$$\frac{1}{xy+z-1} + \frac{1}{yz+x-1} + \frac{1}{zx+y-1}.$$

Solution 39

Let S be the desired value. Note that

$$xy + z - 1 = xy + 1 - x - y = (x-1)(y-1).$$

Likewise,

$$yz + x - 1 = (y-1)(x-1)$$

and

$$zx + y - 1 = (z-1)(x-1).$$

Hence

$$\begin{aligned} S &= \frac{1}{(x-1)(y-1)} + \frac{1}{(y-1)(z-1)} + \frac{1}{(z-1)(x-1)} \\ &= \frac{x+y+z-3}{(x-1)(y-1)(z-1)} = \frac{-1}{(x-1)(y-1)(z-1)} \\ &= \frac{-1}{xyz - (xy+yz+zx) + x + y + z - 1} \\ &= \frac{-1}{5-(xy+yz+zx)}. \end{aligned}$$

But

$$2(xy + yz + zx) = (x+y+z)^2 - (x^2+y^2+z^2) = 1.$$

Therefore $S = -2/9$.

Problem 40 [USSR 1990]

Mr. Fat is going to pick three non-zero real numbers and Mr. Taf is going to arrange the three numbers as the coefficients of a quadratic equation

$$\_x^2 + \_x + \_ = 0.$$

Mr. Fat wins the game if and only if the resulting equation has two distinct rational solutions.

Who has a winning strategy?

Solution 40

Mr. Fat has the winning strategy. A set of three distinct rational nonzero numbers a, b, and c, such that $a + b + c = 0$, will do the trick. Let A, B, and C be any arrangement of a, b, and c, and let $f(x) = Ax^2 + Bx + C$. Then

$$f(1) = A + B + C = a + b + c = 0,$$

which implies that 1 is a solution.

Since the product of the two solutions is C/A, the other solution is C/A, and it is different from 1.

Problem 41 [USAMO 1978]

Given that the real numbers a, b, c, d, and e satisfy simultaneously the relations

$$a + b + c + d + e = 8 \text{ and } a^2 + b^2 + c^2 + d^2 + e^2 = 16,$$

determine the maximum and the minimum value of a.

Solution 41, Alternative 1

Since the total of b, c, d, and e is $8 - a$, their average is $x = (8 - a)/4$. Let

$$b = x + b_1, \quad c = x + c_1, \quad d = x + d_1, \quad e = x + e_1.$$

Then $b_1 + c_1 + d_1 + e_1 = 0$ and

$$16 = a^2 + 4x^2 + b_1^2 + c_1^2 + d_1^2 + e_1^2 \geq a^2 + 4x^2 = a^2 + \frac{(8-a)^2}{4} \tag{1}$$

or

$$0 \geq 5a^2 - 16a = a(5a - 16).$$

Therefore $0 \leq a \leq 16/5$, where $a = 0$ if and only if $b = c = d = e = 2$ and $a = 16/5$ if and only if $b = c = d = e = 6/5$.

Solution 41, Alternative 2

By **the RMS-AM inequality**, (1) follows from

$$b^2 + c^2 + d^2 + e^2 \geq \frac{(b+c+d+e)^2}{4} = \frac{(8-a)^2}{4},$$

and the rest of the solution is the same.

Problem 42

Find the real zeros of the polynomial

$$P_a(x) = (x^2+1)(x-1)^2 - ax^2,$$

where a is a given real number.

Solution 42

We have

$$(x^2+1)(x^2-2x+1) - ax^2 = 0.$$

Dividing by x^2 yields

$$\left(x + \frac{1}{x}\right)\left(x - 2 + \frac{1}{x}\right) - a = 0.$$

By setting $y = x + 1/x$, the last equation becomes

$$y^2 - 2y - a = 0.$$

It follows that

$$x + \frac{1}{x} = 1 \pm \sqrt{1+a},$$

which in turn implies that, if $a \geq 0$, then the polynomial $P_a(x)$ has the real zeros

$$x_{1,2} = \frac{1 + \sqrt{1+a} \pm \sqrt{a + 2\sqrt{1+a} - 2}}{2}.$$

In addition, if $a \geq 8$, then $P_a(x)$ also has the real zeros

$$x_{3,4} = \frac{1 - \sqrt{1+a} \pm \sqrt{a - 2\sqrt{1+a} - 2}}{2}.$$

Problem 43

Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n.

Solution 43

We prove a stronger statement:

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \le \frac{1}{\sqrt{3n+1}}.$$

We use induction.

For $n = 1$, the result is evident.

Suppose the statement is true for some positive integer k, i.e.,

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} < \frac{1}{\sqrt{3k+1}}.$$

Then

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2}.$$

In order for the induction step to pass it suffices to prove that

$$\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+4}}.$$

This reduces to

$$\left(\frac{2k+1}{2k+2}\right)^2 < \frac{3k+1}{3k+4},$$

i.e.

$$(4k^2 + 4k + 1)(3k + 4) < (4k^2 + 8k + 4)(3k + 1),$$

i.e.

$$0 < k,$$

which is evident. Our proof is complete.

Comment: By using Stirling numbers, the upper bound can be improved to $1/\sqrt{\pi n}$ for sufficiently large n.

Problem 44 [USAMO Proposal, Gerald Heuer]

Let

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

be a nonzero polynomial with integer coefficients such that

$$P(r) = P(s) = 0$$

for some integers r and s, with $0 < r < s$.

Prove that $a_k \le -s$ for some k.

Solution 44

Write $P(x) = (x - s)x^c Q(x)$ and

$$Q(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_m,$$

where $b_m \neq 0$. Since Q has a positive root, by Descartes' rule of signs, either there must exist some k for which $b_k > 0 \geq b_{k+1}$, or $b_m > 0$.

If there exists a k for which $b_k > 0 \geq b_{k+1}$, then

$$a_{k+1} = -sb_k + b_{k+1} \leq -s.$$

If $b_m > 0$, then $a_m = -sb_m \leq -s$.

In either case, there is a k such that $a_k \leq -s$, as desired.

Problem 45

Let m be a given real number. Find all complex numbers x such that

$$\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = m^2 + m.$$

Solution 45

Completing the square gives

$$\left(\frac{x}{x+1} + \frac{x}{x-1}\right)^2 = \frac{2x^2}{x^2-1} + m^2 + m,$$

i.e.

$$\left(\frac{2x^2}{x^2-1}\right)^2 = \frac{2x^2}{x^2-1} + m^2 + m.$$

Setting $y = 2x^2/(x^2 - 1)$, the above equation becomes

$$y^2 - y - (m^2 + m) = 0,$$

i.e.

$$(y - m - 1)(y + m) = 0.$$

Thus

$$\frac{2x^2}{x^2-1} = -m \text{ or } \frac{2x^2}{x^2-1} = m + 1,$$

which leads to solutions

$$x = \pm\sqrt{\frac{m}{m+2}} \text{ if } m \neq -2 \text{ and } x = \pm\sqrt{\frac{m+1}{m-1}} \text{ if } m \neq 1.$$

Problem 46

The sequence given by $x_0 = a$, $x_1 = b$, and

$$x_{n+1} = \frac{1}{2}\left(x_{n-1} + \frac{1}{x_n}\right).$$

is periodic.

Prove that $ab = 1$.

Solution 46

Multiplying by $2x_n$ on both sides of the given recursive relation yields

$$2x_n x_{n+1} = x_{n-1}x_n + 1$$

or

$$2(x_n x_{n+1} - 1) = x_{n-1}x_n - 1.$$

Let $y_n = x_{n-1}x_n - 1$ for $n \in \mathbb{N}$. Since $y_{n+1} = y_n/2$, $\{y_n\}$ is a geometric sequence. If x_n is periodic, then so is y_n, which implies that $y_n = 0$ for all $n \in \mathbb{N}$. Therefore

$$ab = x_0 x_1 = y_1 + 1 = 1.$$

Problem 47

Let a, b, c, and d be real numbers such that

$$(a^2 + b^2 - 1)(c^2 + d^2 - 1) > (ac + bd - 1)^2.$$

Prove that

$$a^2 + b^2 > 1 \text{ and } c^2 + d^2 > 1.$$

Solution 47

For the sake of the contradiction, suppose that one of $a^2 + b^2$ or $c^2 + d^2$ is less than or equal to 1. Since $(ac + bd - 1)^2 \geq 0$, $a^2 + b^2 - 1$ and $c^2 + d^2 - 1$ must have the same sign. Thus both $a^2 + b^2$ and $c^2 + d^2$ are less than 1. Let

$$x = 1 - a^2 - b^2 \text{ and } y = 1 - c^2 - d^2.$$

Then $0 < x, y \leq 1$. Multiplying by 4 on both sides of the given inequality gives

$$\begin{aligned}
4xy &> (2ac + 2bd - 2)^2 = (2 - 2ac - 2bd)^2 \\
&= (a^2 + b^2 + x + c^2 + d^2 + y - 2ac - 2bd)^2 \\
&= [(a - c)^2 + (b - d)^2 + x + y]^2 \\
&\geq (x + y)^2 = x^2 + 2xy + y^2,
\end{aligned}$$

or $0 > x^2 - 2xy + y^2 = (x - y)^2$, which is impossible.

Thus our assumption is wrong and both $a^2 + b^2$ and $c^2 + d^2$ are greater than 1.

Problem 48

Find all complex numbers z such that

$$(3z + 1)(4z + 1)(6z + 1)(12z + 1) = 2.$$

Solution 48

Note that

$$8(3z + 1)6(4z + 1)4(6z + 1)2(12z + 1) = 768,$$

i.e.

$$(24z + 8)(24z + 6)(24z + 4)(24z + 2) = 768.$$

Setting $u = 24z + 5$ and $w = u^2$ yields

$$(u + 3)(u + 1)(u - 1)(u - 3) = 768,$$

i.e.

$$(u^2 - 1)(u^2 - 9) = 768,$$

i.e.

$$w^2 - 10w - 759 = 0,$$

i.e.

$$(w - 33)(w + 23) = 0.$$

Therefore the solutions to the given equation are

$$z = \frac{\pm\sqrt{33} - 5}{24} \text{ and } z = \frac{\pm\sqrt{23}\imath - 5}{24}.$$

Problem 49

Let $x_1, x_2, \cdots, x_{n-1}$, be the zeros different from 1 of the polynomial $P(x) = x^n - 1$, $n \geq 2$.

Prove that

$$\frac{1}{1 - x_1} + \frac{1}{1 - x_2} + \cdots + \frac{1}{1 - x_{n-1}} = \frac{n - 1}{2}.$$

Solution 49, Alternative 1

For $i = 1, 2, \ldots, n$, let $a_i = 1 - x_i$. Let

$$Q(x) = \frac{P(1-x)}{x} = \frac{(1-x)^n - 1}{x}.$$

Then

$$Q(x) = (-1)^n x^{n-1} + (-1)^{n-1}\binom{n}{1}x^{n-2} + \cdots + \binom{n}{2}x - \binom{n}{1}$$

and a_is are the nonzero roots of the polynomial $Q(x)$, as

$$Q(a_i) = \frac{(1-a_i)^n - 1}{a_i} = \frac{x_i^n - 1}{1 - x_i} = 0.$$

Thus the desired sum is the sum of .the reciprocals of the roots of polynomial $Q(x)$, that is,

$$\begin{aligned}
&\frac{1}{1-x_1} + \frac{1}{1-x_2} + \cdots + \frac{1}{1-x_{n-1}} \\
&= \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} \\
&= \frac{a_2a_3\cdots a_n + a_1a_3\cdots a_n + \cdots + a_1a_2\cdots a_{n-1}}{a_1a_2\cdots a_n}
\end{aligned}$$

By **the Vieta's Theorem**, the ratio between

$$S = a_2\cdots a_n + a_1a_3\cdots a_n + \cdots + a_1a_2\cdots a_{n-1}$$

and

$$P = a_1\cdots a_n$$

is equal to the additive inverse of the ratio between the coefficient of x and the constant term in $Q(x)$, i.e., the desired value is equal to

$$\frac{S}{P} = -\frac{\binom{n}{2}}{-\binom{n}{1}} = \frac{n-1}{2},$$

as desired.

Solution 49, Alternative 2

For any polynomial $R(x)$ of degree $n-1$, whose zeros are $x_1, x_2, \ldots, x_{n-1}$, the following identity holds:

$$\frac{1}{x-x_1} + \frac{1}{x-x_2} + \cdots + \frac{1}{x-x_{n-1}} = \frac{R'(x)}{R(x)}.$$

For
$$R(x) = \frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \ldots + x + 1,$$
$R(1) = n$ and
$$R'(1) = (n-1) + (n-2) + \cdot\ \cdot + 1 = \frac{n(n-1)}{2}.$$
It follows that
$$\frac{1}{1-x_1} + \frac{1}{1-x_2} + \cdots + \frac{1}{1-x_{n-1}} = \frac{R'(1)}{R(1)} = \frac{n-1}{2}.$$

Problem 50

Let a and b be given real numbers.
Solve the system of equations
$$\begin{aligned} \frac{x - y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} &= a, \\ \frac{y - x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} &= b \end{aligned}$$
for real numbers x and y.

Solution 50

Let $u = x + y$ and $v = x - y$. Then
$$0 < x^2 - y^2 = uv < 1,\ x = \frac{u+v}{2},\ \text{and } y = \frac{u-v}{2}.$$
Adding the two equations and subtracting the two equations in the original system yields the new system
$$\begin{aligned} u - u\sqrt{uv} &= (a+b)\sqrt{1 - uv} \\ v + v\sqrt{uv} &= (a-b)\sqrt{1 - uv}. \end{aligned}$$
Multiplying the above two equations yields
$$uv(1 - uv) = (a^2 - b^2)(1 - uv),$$
hence $uv = a^2 - b^2$. It follows that
$$u = \frac{(a+b)\sqrt{1 - a^2 + b^2}}{1 - \sqrt{a^2 - b^2}} \text{ and } v = \frac{(a-b)\sqrt{1 - a^2 + b^2}}{1 + \sqrt{a^2 - b^2}},$$
which in turn implies that
$$(x, y) = \left(\frac{a + b\sqrt{a^2 - b^2}}{\sqrt{1 - a^2 + b^2}}, \frac{b + a\sqrt{a^2 - b^2}}{\sqrt{1 - a^2 + b^2}} \right),$$
whenever $0 < a^2 - b^2 < 1$.

SOLUTIONS TO ADVANCED PROBLEMS

4. SOLUTIONS TO ADVANCED PROBLEMS

Problem 51

Evaluate

$$\binom{2000}{2} + \binom{2000}{5} + \binom{2000}{8} + \cdots + \binom{2000}{2000}.$$

Solution 51

Let

$$f(x) = (1+x)^{2000} = \sum_{k=0}^{2000} \binom{2000}{k} x^k.$$

Let $\omega = (-1+\sqrt{3}i)/2$. Then $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$. Hence

$$\begin{aligned}
3&\left(\binom{2000}{2} + \binom{2000}{5} + \cdots + \binom{2000}{2000}\right) \\
&= f(1) + \omega f(\omega) + \omega^2 f(\omega^2) \\
&= 2^{2000} + \omega(1+\omega)^{2000} + \omega^2(1+\omega^2)^{2000} \\
&= 2^{2000} + \omega(-\omega^2)^{2000} + \omega^2(-\omega)^{2000} \\
&= 2^{2000} + \omega^2 + \omega = 2^{2000} - 1.
\end{aligned}$$

Thus the desired value is

$$\frac{2^{2000}-1}{3}.$$

Problem 52

Let x, y, z be positive real numbers such that $x^4 + y^4 + z^4 = 1$. Determine with proof the minimum value of

$$\frac{x^3}{1-x^8} + \frac{y^3}{1-y^8} + \frac{z^3}{1-z^8}.$$

Solution 52

For $0 < u < 1$, let $f(u) = u(1-u^8)$. Let A be a positive real number. By the **AM-GM inequality**,

$$A(f(u))^8 = Au^8(1-u^8)\cdots(1-u^8) \le \left[\frac{Au^8 + 8(1-u^8)}{9}\right]^9.$$

Setting $A = 8$ in the above inequality yields

$$8(f(u))^8 \leq \left(\frac{8}{9}\right)^9$$

or

$$f(u) \leq \frac{8}{\sqrt[4]{3^9}}.$$

It follows that

$$\begin{aligned} \frac{x^3}{1-x^8} + \frac{y^3}{1-y^8} + \frac{z^3}{1-z^8} &= \frac{x^4}{x(1-x^8)} + \frac{y^4}{y(1-y^8)} + \frac{z^4}{z(1-z^8)} \\ &\geq \frac{(x^4+y^4+z^4)\sqrt[4]{3^9}}{8} \\ &= \frac{9\sqrt[4]{3}}{8}, \end{aligned}$$

with equality if and only if

$$x = y = z = \frac{1}{\sqrt[4]{3}}.$$

Comment: This is a simple application of the result of problem 33 in the previous chapter.

Problem 53 [Romania 1990]

Find all real solutions to the equation

$$2^x + 3^x + 6^x = x^2.$$

Solution 53

For $x < 0$, the function $f(x) = 2^x + 3^x + 6^x - x^2$ is increasing, so the equation $f(x) = 0$ has the unique solution $x = -1$.

Assume that there is a solution $s \geq 0$. Then

$$s^2 = 2^s + 3^s + 6^s \geq 3,$$

so $s \geq \sqrt{3}$, and hence $\lfloor s \rfloor \geq 1$.

But then $s \geq \lfloor s \rfloor$ yields

$$2^s \geq 2^{\lfloor s \rfloor} = (1+1)^{\lfloor s \rfloor} \geq 1 + \lfloor s \rfloor \geq s,$$

which in turn implies that

$$6^s > 4^s = (2^s)^2 \geq s^2.$$

So $2^s + 3^s + 6^s > s^2$, a contradiction.

Therefore $x = -1$ is the only solution to the equation.

Problem 54

Let $\{a_n\}_{n\geq 1}$ be a sequence such that $a_1 = 2$ and

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$$

for all $n \in \mathbb{N}$.

Find an explicit formula for a_n.

Solution 54

Solving the equation

$$x = \frac{x}{2} + \frac{1}{x}$$

leads to $x = \pm\sqrt{2}$. Note that

$$\frac{a_{n+1} + \sqrt{2}}{a_{n+1} - \sqrt{2}} = \frac{a_n^2 + 2\sqrt{2}a_n + 2}{a_n^2 - 2\sqrt{2}a_n + 2} = \left(\frac{a_n + \sqrt{2}}{a_n - \sqrt{2}}\right)^2.$$

Therefore,

$$\frac{a_n + \sqrt{2}}{a_n - \sqrt{2}} = \left(\frac{a_1 + \sqrt{2}}{a_1 - \sqrt{2}}\right)^{2^{n-1}} = \left(\sqrt{2} + 1\right)^{2^n}$$

and

$$a_n = \frac{\sqrt{2}\left[\left(\sqrt{2}+1\right)^{2^n} + 1\right]}{\left(\sqrt{2}+1\right)^{2^n} - 1}.$$

Problem 55

Let x, y, and z be positive real numbers. Prove that

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(y+z)(y+x)}} + \frac{z}{z + \sqrt{(z+x)(z+y)}} \leq 1.$$

Solution 55

Note that

$$\sqrt{(x+y)(x+z)} \geq \sqrt{xy} + \sqrt{xz}.$$

In fact, squaring both sides of the above inequality yields

$$x^2 + yz \geq 2x\sqrt{yz},$$

which is evident by the **AM-GM inequality**. Thus

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} \leq \frac{x}{x + \sqrt{xy} + \sqrt{xz}} = \frac{\sqrt{x}}{\sqrt{x} + \sqrt{y} + \sqrt{z}}.$$

Likewise,

$$\frac{y}{y + \sqrt{(y+z)(y+x)}} \leq \frac{\sqrt{y}}{\sqrt{x} + \sqrt{y} + \sqrt{z}},$$

and

$$\frac{z}{z + \sqrt{(z+x)(z+y)}} \leq \frac{\sqrt{z}}{\sqrt{x} + \sqrt{y} + \sqrt{z}}.$$

Adding the last three inequalities leads to the desired result.

Problem 56

Find, with proof, all nonzero polynomials $f(z)$ such that

$$f(z^2) + f(z)f(z+1) = 0.$$

Solution 56

Let $f(z) = az^m(z-1)^n g(z)$, where m and n are non-negative integers and

$$g(z) = (z - z_1)(z - z_2) \cdots (z - z_k),$$

$z_i \neq 0$ and $z_i \neq 1$, for $i = 1, 2, \ldots, k$. The given condition becomes

$$\begin{aligned}
&az^{2m}(z-1)^n(z+1)^n(z^2 - z_1)(z^2 - z_2)\cdots(z^2 - z_k) \\
&= -a^2 z^{m+n}(z+1)^m(z-1)^n(z - z_1)(z - z_2)\cdots(z - z_k) \\
&\quad \cdot(z + 1 - z_1)(z + 1 - z_2)\cdots(z + 1 - z_k).
\end{aligned}$$

Thus $a = -a^2$, and f is nonzero, so $a = -1$. Since $z_i \neq 1$, $1 - z_i \neq 0$. Then $z^{2m} = z^{m+n}$, that is, $m = n$.

Thus f is of the form

$$-z^m(z-1)^m g(z).$$

Dividing by $z^{2m}(z-1)^n(z+1)^n$, the last equation becomes

$$g(z^2) = g(z)g(z+1).$$

We claim that $g(z) \equiv 1$. Suppose not; then clearly g must have at least one complex root $r \neq 0$. Now

$$\begin{aligned} g(r^2) &= g(r)g(r+1) = 0, \\ g(r^4) &= 0, \\ g(r^8) &= 0, \end{aligned}$$

and so on.

Since g cannot have infinitely many roots, all its roots must have absolute value 1.

Now,

$$g((r-1)^2) = g(r-1)g(r) = 0,$$

so $|(r-1)^2| = 1$.

Clearly, if

$$|r| = |(r-1)^2| = 1,$$

then

$$r \in \left\{ \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2} \right\}.$$

But r^2 is also a root of g, so the same should be true of r^2:

$$r^2 \in \left\{ \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2} \right\}.$$

This is absurd. Hence, g cannot have any roots, and $g(z) \equiv 1$.

Therefore, the $f(z)$ are all the polynomials of the form $-z^m(z-1)^m$ for $m \in \mathbb{N}$.

Problem 57

Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all n.

Evaluate $f(2001)$.

Solution 57, Alternative 1

We prove the following lemma.

Lemma For $n = 0, 1, 2, \ldots$,

1. $f(3^n) = 2 \cdot 3^n$; and
2. $f(2 \cdot 3^n) = 3^{n+1}$.

Proof. We use induction.

For $n = 0$, note that $f(1) \neq 1$, otherwise $3 = f(f(1)) = f(1) = 1$, which is impossible. Since $f : \mathbb{N} \to \mathbb{N}$, $f(1) > 1$. Since $f(n+1) > f(n)$, f is increasing. Thus $1 < f(1) < f(f(1)) = 3$ or $f(1) = 2$. Hence $f(2) = f(f(1)) = 3$.

Suppose that for some positive integer $n \geq 1$,

$$f\left(3^n\right) = 2 \cdot 3^n \text{ and } f\left(2 \cdot 3^n\right) = 3^{n+1}.$$

Then,

$$f(3^{n+1}) = f\left(f(2 \cdot 3^n)\right) = 2 \cdot 3^{n+1},$$

and

$$f(2 \cdot 3^{n+1}) = f\left(f(3^{n+1})\right) = 3^{n+2},$$

as desired. This completes the induction. $\square$

There are $3^n - 1$ integers m such that $3^n < m < 2 \cdot 3^n$ and there are $3^n - 1$ integers m' such that

$$f\left(3^n\right) = 2 \cdot 3^n < m' < 3^{n+1} = f\left(2 \cdot 3^n\right).$$

Since f is an increasing function,

$$f\left(3^n + m\right) = 2 \cdot 3^n + m,$$

for $0 \leq m \leq 3^n$. Therefore

$$f\left(2 \cdot 3^n + m\right) = f\left(f\left(3^n + m\right)\right) = 3\left(3^n + m\right)$$

for $0 \leq m \leq 3^n$. Hence

$$f(2001) = f\left(2 \cdot 3^6 + 543\right) = 3\left(3^6 + 543\right) = 3816.$$

Solution 57, Alternative 2

For integer n, let $n_{(3)} = a_1 a_2 \cdots a_\ell$ denote the base 3 representation of n.

Using similar inductions as in the first solution, we can prove that

$$f(n)_{(3)} = \begin{cases} 2a_2 \cdots a_\ell & \text{if } a_1 = 1, \\ 1a_2 \cdots a_\ell 0 & \text{if } a_1 = 2. \end{cases}$$

Since $2001_{(3)} = 2202010$, $f(2001)_{(3)} = 12020100$ or

$$f(2001) = 1 \cdot 3^2 + 2 \cdot 3^4 + 2 \cdot 3^6 + 1 \cdot 3^7 = 3816.$$

Problem 58 [China 1999]

Let F be the set of all polynomials $f(x)$ with integers coefficients such that $f(x) = 1$ has at least one integer root.

For each integer $k > 1$, find m_k, the least integer greater than 1 for which there exists an $f \in F$ such that $f(x) = m_k$ has exactly k distinct integer roots.

Solution 58

Suppose that $f_k \in F$ satisfies the condition that $f_k(x) = m_k$ has exactly k distinct integer roots, and let a be an integer such that $f_k(a) = 1$. Let g_k be the polynomial in F such that

$$g_k(x) = f_k(x + a)$$

for all x.

Now $g_k(0) = f_k(a) = 1$, so the constant term of g_k is 1. Now $g_k(x) = m_k$ has exactly k distinct integer roots $r_1, r_2, \ldots, r_k$, so we can write

$$g_k(x) - m_k = (x - r_1)(x - r_2)\ldots(x - r_k)q_k(x),$$

where $q_k(x)$ is an integer polynomial.

Note that $r_1 r_2 \cdots r_k$ divides the constant term of $g_k(x) - m_k$, which equals $1 - m_k$.

Since $m_k > 1$, $1 - m_k$ cannot be 0,

$$|1 - m_k| \geq |r_1 r_2 \cdots r_k|.$$

Now $r_1, r_2, \cdots r_k$ are distinct integers, and none of them is 0, so

$$|r_1 r_2 \cdots r_k| \geq \left|1 \cdot (-1) \cdot 2 \cdot (-2) \cdot 3 \cdots (-1)^{k+1} \lceil k/2 \rceil\right|,$$

hence

$$m_k \geq \lfloor k/2 \rfloor! \cdot \lceil k/2 \rceil! + 1.$$

This value of m_k is attained by

$$\begin{aligned} g_k(x) &= (-1)^{\binom{k-1}{2}}(x-1)(x+1)(x-2)(x+2) \\ &\quad \cdots \left(x + (-1)^k \lceil k/2 \rceil\right) + \lfloor k/2 \rfloor! \cdot \lceil k/2 \rceil! + 1. \end{aligned}$$

Thus,

$$m_k = \lfloor k/2 \rfloor! \cdot \lceil k/2 \rceil! + 1.$$

Problem 59

Let $x_1 = 2$ and

$$x_{n+1} = x_n^2 - x_n + 1,$$

for $n \geq 1$.

Prove that

$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 1 - \frac{1}{2^{2^n}}.$$

Solution 59

Since $x_1 = 2$ and

$$x_{n+1} - 1 = x_n(x_n - 1),$$

x_n is increasing.

Then $x_n - 1 \neq 0$.

Hence

$$\frac{1}{x_{n+1} - 1} = \frac{1}{x_n(x_n - 1)} = \frac{1}{x_n - 1} - \frac{1}{x_n}$$

or

$$\frac{1}{x_n} = \frac{1}{x_n - 1} - \frac{1}{x_{n+1} - 1},$$

which implies that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1 - \frac{1}{x_{n+1} - 1}.$$

Thus it suffices to prove that, for $n \in \mathbb{N}$,

$$1 - \frac{1}{2^{2^{n-1}}} < 1 - \frac{1}{x_{n+1} - 1} < 1 - \frac{1}{2^{2^n}}.$$

or

$$2^{2^{n-1}} < x_{n+1} - 1 < 2^{2^n}. \tag{1}$$

We use induction to prove (1).

For $n = 1$, $x_2 = x_1^2 - x_1 + 1 = 3$ and (1) becomes $2 < 3 < 4$, which is true.

Now suppose that (1) is true for some positive integer $n = k$, i.e.,

$$2^{2^{k-1}} < x_{k+1} - 1 < 2^{2^k}. \tag{2}$$

Then for $n = k + 1$, the lower bound of (1) follows from

$$x_{k+2} - 1 = x_{k+1}(x_{k+1} - 1) > 2^{2^{k-1}} \cdot 2^{2^{k-1}} = 2^{2^k}.$$

Since x_{k+1} is an integer, the lower bound of (2) implies that

$$x_{k+1} \leq 2^{2^k} \text{ and } x_{k+1} - 1 \leq 2^{2^k} - 1,$$

from which it follows that

$$x_{k+2} - 1 = x_{k+1}(x_{k+1} - 1) \leq 2^{2^k} \cdot \left(2^{2^k} - 1\right) < 2^{2^{k+1}},$$

as desired.

This finishes the induction and we are done.

Problem 60 [Iran 1997]

Suppose that $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a decreasing function such that for all $x, y \in \mathbb{R}^+$,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

Prove that $f(f(x)) = x$.

Solution 60

Setting $y = x$ gives

$$f(2x) + f(2f(x)) = f(2f(x + f(x))).$$

Replacing x with $f(x)$ yields

$$f(2f(x)) + f(2f(f(x))) = f(2f(f(x) + f(f(x)))).$$

Subtracting these two equations gives

$$f(2f(f(x))) - f(2x) = f(2f(f(x) + f(f(x)))) - f(2f(x + f(x))).$$

If $f(f(x)) > x$, the left hand side of this equation is negative, so

$$f(f(x) + f(f(x)) > f(x + f(x))$$

and

$$f(x) + f(f(x)) < x + f(x),$$

a contradiction. A similar contradiction occurs if $f(f(x)) < x$.

Thus $f(f(x)) = x$ as desired.

Comment: In the original formulation f was meant to be a continous function. The solution above shows that this condition is not necessary.

Problem 61 [Nordic Contest 1998]

Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathbb{Q}$.

Solution 61

The only such functions are $f(x) = kx^2$ for rational k. Any such function works, since

$$\begin{aligned}
&f(x+y) + f(x-y) = k(x+y)^2 + k(x-y)^2 \\
&\quad = kx^2 + 2kxy + ky^2 + kx^2 - 2kxy + ky^2 \\
&\quad = 2kx^2 + 2ky^2 \\
&\quad = 2f(x) + 2f(y).
\end{aligned}$$

Now suppose f is any function satisfying

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

Then letting $x = y = 0$ gives $2f(0) = 4f(0)$, so $f(0) = 0$.

We will prove by induction that $f(nz) = n^2 f(z)$ for any positive integer n and any rational number z.

The claim holds for $n = 0$ and $n = 1$; let $n \geq 2$ and suppose the claim holds for $n-1$ and $n-2$.

Then letting $x = (n-1)z$, $y = z$ in the given equation we obtain

$$\begin{aligned}
&f(nz) + f((n-2)z) = f((n-1)z + z) + f((n-1)z - z) \\
&\quad = 2f((n-1)z) + 2f(z)
\end{aligned}$$

so

$$\begin{aligned}
&f(nz) = 2f((n-1)z) + 2f(z) - f((n-2)z) \\
&\quad = 2(n-1)^2 f(z) + 2f(z) - (n-2)^2 f(z) \\
&\quad = (2n^2 - 4n + 2 + 2 - n^2 + 4n - 4) f(z) \\
&\quad = n^2 f(z)
\end{aligned}$$

and the claim holds by induction.

Letting $x = 0$ in the given equation gives

$$f(y) + f(-y) = 2f(0) + 2f(y) = 2f(y),$$

so $f(-y) = f(y)$ for all rational y; thus $f(nz) = n^2 f(z)$ for all integers n.

Now let $k = f(1)$; then for any rational number $x = p/q$,

$$q^2 f(x) = f(qx) = f(p) = p^2 f(1) = kp^2$$

so

$$f(x) = kp^2/q^2 = kx^2.$$

Thus the functions $f(x) = kx^2$, $k \in \mathbb{Q}$, are the only solutions.

Problem 62 [Korean Mathematics Competition 2000]
Let $\frac{3}{4} < a < 1$.
Prove that the equation

$$x^3(x+1) = (x+a)(2x+a)$$

has four distinct real solutions and find these solutions in explicit form.

Solution 62
Look at the given equation as a quadratic equation in a:

$$a^2 + 3xa + 2x^2 - x^3 - x^4 = 0.$$

The discriminant of this equation is

$$9x^2 - 8x^2 + 4x^3 + 4x^4 = (x + 2x^2)^2.$$

Thus

$$a = \frac{-3x \pm (x + 2x^2)}{2}.$$

The first choice $a = -x + x^2$ yields the quadratic equation $x^2 - x - a = 0$, whose solutions are

$$x = \frac{(1 \pm \sqrt{1+4a})}{2}.$$

The second choice $a = -2x - x^2$ yields the quadratic equation

$$x^2 + 2x + a = 0,$$

whose solutions are

$$-1 \pm \sqrt{1-a}.$$

The inequalities

$$-1 - \sqrt{1-a} < -1 + \sqrt{1-a} < \frac{1 - \sqrt{1+4a}}{2} < \frac{1 + \sqrt{1+4a}}{2}$$

show that the four solutions are distinct.

Indeed

$$-1+\sqrt{1-a}<\frac{1-\sqrt{1+4a}}{2}$$

reduces to

$$2\sqrt{1-a}<3-\sqrt{1+4a}$$

which is equivalent to

$$6\sqrt{1+4a}<6+8a,$$

or $3a<4a^2$, which is evident.

Problem 63 [Tournament of Towns 1997]

Let a, b, and c be positive real numbers such that $abc=1$.

Prove that

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\leq 1.$$

Solution 63, Alternative 1

Setting $x=a+b, y=b+c$ and $z=c+a$, the inequality becomes

$$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\leq 1,$$

i.e.

$$\frac{1}{y+1}+\frac{1}{z+1}\leq\frac{x}{x+1},$$

i.e.

$$\frac{y+z+2}{(y+1)(z+1)}\leq\frac{x}{x+1},$$

i.e.

$$xy+xz+2x+y+z+2\leq xyz+xy+xz+x,$$

i.e.

$$x+y+z+2\leq xyz,$$

i.e.

$$2(a+b+c)+2\leq(a+b)(b+c)(c+a),$$

i.e.

$$2(a+b+c)\leq a^2b+ab^2+b^2c+bc^2+c^2a+ca^2.$$

By the **AM-GM inequality**,

$$(a^2b+a^2c+1)\geq 3\sqrt[3]{a^4bc}=3a.$$

Likewise,

$$(b^2c + b^2a + 1) \geq 3b$$

and

$$(c^2a + c^2b + 1) \geq 3c.$$

Therefore we only need to prove that

$$2(a + b + c) + 3 \leq 3(a + b + c),$$

i.e.

$$3 \leq a + b + c,$$

which is evident from **AM-GM inequality** and $abc = 1$.

Solution 63, Alternative 2

Let $a = a_1^3, b = b_1^3, c = c_1^3$. Then $a_1b_1c_1 = 1$. Note that

$$a_1^3 + b_1^3 - a_1^2b_1 - a_1b_1^2 = (a_1 - b_1)(a_1^2 - b_1^2) \geq 0,$$

which implies that

$$a_1^3 + b_1^3 \geq a_1b_1(a_1 + b_1).$$

Therefore,

$$\begin{aligned}
\frac{1}{a + b + 1} &= \frac{1}{a_1^3 + b_1^3 + a_1b_1c_1} \\
&\leq \frac{1}{a_1b_1(a_1 + b_1) + a_1b_1c_1} \\
&= \frac{a_1b_1c_1}{a_1b_1(a_1 + b_1 + c_1)} \\
&= \frac{c_1}{a_1 + b_1 + c_1}.
\end{aligned}$$

Likewise,

$$\frac{1}{b + c + 1} \leq \frac{a_1}{a_1 + b_1 + c_1}$$

and

$$\frac{1}{c + a + 1} \leq \frac{b_1}{a_1 + b_1 + c_1}.$$

Adding the three inequalities yields the desired result.

Problem 64 [AIME 1988]

Find all functions f, defined on the set of ordered pairs of positive integers, satisfying the following properties:

$$f(x,x)=x,\ \ f(x,y)=f(y,x),\ \ (x+y)f(x,y)=yf(x,x+y).$$

Solution 64

We claim that $f(x,y)=\operatorname{lcm}(x,y)$, the least common multiple of x and y. It is clear that

$$\operatorname{lcm}(x,x)=x$$

and

$$\operatorname{lcm}(x,y)=\operatorname{lcm}(y,x).$$

Note that

$$\operatorname{lcm}(x,y)=\frac{xy}{\gcd(x,y)}$$

and

$$\gcd(x,y)=\gcd(x,x+y),$$

where $\gcd(u,v)$ denotes the greatest common divisor of u and v. Then

$$\begin{aligned}(x+y)\operatorname{lcm}(x,y) &= (x+y)\cdot\frac{xy}{\gcd(x,y)}\\ &= y\cdot\frac{x(x+y)}{\gcd(x,x+y)}\\ &= y\operatorname{lcm}(x,x+y).\end{aligned}$$

Now we prove that there is only one function satisfying the conditions of the problem.

For the sake of contradiction, assume that there is another function $g(x,y)$ also satisfying the given conditions.

Let S be the set of all pairs of positive integers (x,y) such that $f(x,y)\neq g(x,y)$, and let (m,n) be such a pair with minimal sum $m+n$. It is clear that $m\neq n$, otherwise

$$f(m,n)=f(m,m)=m=g(m,m)=g(m,n).$$

By symmetry ($f(x,y)=f(y,x)$), we can assume that $n-m>0$.

Note that

$$\begin{aligned}nf(m,n-m) &= [m+(n-m)]f(m,n-m)\\ &= (n-m)f(m,m+(n-m))\\ &= (n-m)f(m,n)\end{aligned}$$

or

$$f(m, n - m) = \frac{n - m}{n} \cdot f(m, n).$$

Likewise,

$$g(m, n - m) = \frac{n - m}{n} \cdot g(m, n).$$

Since $f(m,n) \neq g(m,n)$, $f(m, n-m) \neq g(m, n-m)$.

Thus $(m, n-m) \in S$.

But $(m, n-m)$ has a smaller sum $m + (n-m) = n$, a contradiction.

Therefore our assumption is wrong and $f(x,y) = \operatorname{lcm}(x,y)$ is the only solution.

Problem 65 [Romania 1990]

Consider n complex numbers z_k, such that $|z_k| \leq 1$, $k = 1, 2, \ldots, n$.

Prove that there exist $e_1, e_2, \ldots, e_n \in \{-1, 1\}$ such that, for any $m \leq n$,

$$|e_1 z_1 + e_2 z_2 + \cdots + e_m z_m| \leq 2.$$

Solution 65

Call a finite sequence of complex numbers each with absolute value not exceeding 1 a *green sequence.*

Call a green sequence $\{z_k\}_{k=1}^n$ *happy* if it has a *friend* sequence $\{e_k\}_{k=1}^n$ of 1s and -1s, satisfying the condition of the problem.

We will prove by induction on n that all green sequences are happy.

For $n = 2$, this claim is obviously true.

Suppose this claim is true when n equals some number m. For the case of $n = m + 1$, think of the z_k as points in the complex plane.

For each k, let ℓ_k be the line through the origin and the point corresponding to z_k. Among the lines ℓ_1, ℓ_2, ℓ_3, some two are within 60° of each other; suppose they are ℓ_α and ℓ_β, with the leftover one being ℓ_γ.

The fact that ℓ_α and ℓ_β are within 60° of each other implies that there exists some number $e_\beta \in \{-1, 1\}$ such that $z' = z_\alpha + e_\beta z_\beta$ has absolute value at most 1.

Now the sequence $z', z_\gamma, z_4, z_5, \ldots, z_{k+1}$ is a k-term green sequence, so, by the induction hypothesis, it must be happy; let $e', e_\gamma, e_4, e_5, \ldots, e_{k+1}$ be its friend.

Let $e_\alpha = 1$.

Then the sequence $\{e_i\}_{i=1}^{k+1}$ is the friend of $\{z_i\}_{i=1}^{k+1}$. Induction is now complete.

Problem 66 [ARML 1997]

Find a triple of rational numbers (a, b, c) such that

$$\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

Solution 66

Let $x = \sqrt[3]{\sqrt[3]{2}-1}$ and $y = \sqrt[3]{2}$. Then $y^3 = 2$ and $x = \sqrt[3]{y-1}$. Note that

$$1 = y^3 - 1 = (y-1)(y^2+y+1),$$

and

$$y^2+y+1 = \frac{3y^2+3y+3}{3} = \frac{y^3+3y^2+3y+1}{3} = \frac{(y+1)^3}{3},$$

which implies that

$$x^3 = y - 1 = \frac{1}{y^2+y+1} = \frac{3}{(y+1)^3}$$

or

$$x = \frac{\sqrt[3]{3}}{y+1}. \tag{1}$$

On the other hand,

$$3 = y^3 + 1 = (y+1)(y^2-y+1)$$

from which it follows that

$$\frac{1}{y+1} = \frac{y^2-y+1}{3}. \tag{2}$$

Combining (1) and (2), we obtain

$$x = \sqrt[3]{\frac{1}{9}}\left(\sqrt[3]{4} - \sqrt[3]{2} + 1\right).$$

Consequently,

$$(a, b, c) = \left(\frac{4}{9}, -\frac{2}{9}, \frac{1}{9}\right)$$

is a desired triple.

Problem 67 [Romania 1984]

Find the minimum of

$$\log_{x_1}\left(x_2-\frac{1}{4}\right)+\log_{x_2}\left(x_3-\frac{1}{4}\right)+\cdots+\log_{x_n}\left(x_1-\frac{1}{4}\right)$$

where $x_1, x_2, \ldots, x_n$ are real numbers in the interval $(\frac{1}{4}, 1)$.

Solution 67

Since $\log_a x$ is a decreasing function of x when $0 < a < 1$ and, since $(x-1/2)^2 \geq 0$ implies $x^2 \geq x - 1/4$, we have

$$\log_{x_k}\left(x_{k+1}-\frac{1}{4}\right) \geq \log_{x_k} x_{k+1}^2 = 2\log_{x_k} x_{k+1} = 2\frac{\log x_{k+1}}{\log_{x_k}}.$$

It follows that

$$\log_{x_1}\left(x_2-\frac{1}{4}\right)+\log_{x_2}\left(x_3-\frac{1}{4}\right)+\cdots+\log_{x_n}\left(x_1-\frac{1}{4}\right)$$

$$\geq \quad 2\left(\frac{\log x_2}{\log x_1}+\frac{\log x_3}{\log x_2}+\cdots+\frac{\log x_n}{\log x_{n-1}}+\frac{\log x_1}{\log x_n}\right)$$

$$\geq \quad 2n$$

by the **AM-GM inequality**.

Equalities hold if and only if

$$x_1 = x_2 = \cdots = x_n = 1/2.$$

Problem 68 [AIME 1984]

Determine $x^2+y^2+z^2+w^2$ if

$$\frac{x^2}{2^2-1^2}+\frac{y^2}{2^2-3^2}+\frac{z^2}{2^2-5^2}+\frac{w^2}{2^2-7^2}=1,$$

$$\frac{x^2}{4^2-1^2}+\frac{y^2}{4^2-3^2}+\frac{z^2}{4^2-5^2}+\frac{w^2}{4^2-7^2}=1,$$

$$\frac{x^2}{6^2-1^2}+\frac{y^2}{6^2-3^2}+\frac{z^2}{6^2-5^2}+\frac{w^2}{6^2-7^2}=1,$$

$$\frac{x^2}{8^2-1^2}+\frac{y^2}{8^2-3^2}+\frac{z^2}{8^2-5^2}+\frac{w^2}{8^2-7^2}=1.$$

Solution 68

The claim that the given system of equations is satisfied by x^2, y^2, z^2, and w^2 is equivalent to claiming that

$$\frac{x^2}{t-1^2}+\frac{y^2}{t-3^2}+\frac{z^2}{t-5^2}+\frac{w^2}{t-7^2}=1 \tag{1}$$

is satisfied by $t = 4, 16, 36$, and 64.

Multiplying to clear fractions, we find that for all values of t for which it is defined (i.e., $t \neq 1, 9, 25$, and 49), (1) is equivalent to the polynomial equation

$$P(t) = 0,$$

where

$$\begin{aligned} P(t) = {} & (t-1)(t-9)(t-25)(t-49) \\ & -x^2(t-9)(t-25)(t-49) - y^2(t-1)(t-25)(t-49) \\ & -z^2(t-1)(t-9)(t-49) - w^2(t-1)(t-9)(t-25). \end{aligned}$$

Since $\deg P(t) = 4$, $P(t) = 0$ has exactly four zeros $t = 4, 16, 36$, and 64, i.e.,

$$P(t) = (t-4)(t-16)(t-36)(t-64).$$

Comparing the coefficients of t^3 in the two expressions of $P(t)$ yields

$$1+9+25+49+x^2+y^2+z^2+w^2 = 4+16+36+64,$$

from which it follows that

$$x^2+y^2+z^2+w^2 = 36.$$

Problem 69 [Balkan 1997]

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x)+f(y)) = (f(x))^2 + y$$

for all $x, y \in \mathbb{R}$.

Solution 69

Let $f(0) = a$. Setting $x = 0$ in the given condition yields

$$f(f(y)) = a^2 + y,$$

for all $y \in \mathbb{R}$.

Since the range of a^2+y consists of all real numbers, f must be surjective.

Thus there exists $b \in \mathbb{R}$ such that $f(b) = 0$.

Setting $x = b$ in the given condition yields

$$f(f(y)) = f(bf(b) + f(y)) = (f(b))^2 + y = y,$$

for all $y \in \mathbb{R}$. It follows that, for all $x, y \in \mathbb{R}$,

$$\begin{aligned}(f(x))^2 + y &= f(xf(x) + f(y)) \\ &= f[f(f(x))f(x) + f(y)] = f[f(x)f(f(x)) + y] \\ &= f(f(x))^2 + y = x^2 + y,\end{aligned}$$

that is,

$$(f(x))^2 = x^2. \tag{1}$$

It is clear that $f(x) = x$ is a function satisfying the given condition.

Suppose that $f(x) \neq x$. Then there exists some nonzero real number c such that $f(c) = -c$. Setting $x = cf(c) + f(y)$ in (1) yields

$$[f(cf(c) + f(y))]^2 = [cf(c) + f(y)]^2 = [-c^2 + f(y)]^2,$$

for all $y \in \mathbb{R}$, and, setting $x = c$ in the given condition yields

$$f(cf(c) + f(y)) = (f(c))^2 + y = c^2 + y,$$

for all $y \in \mathbb{R}$.

Note that $(f(y))^2 = y^2$.

It follows that

$$[-c^2 + f(y)]^2 = (c^2 + y)^2,$$

or

$$f(y) = -y,$$

for all $y \in \mathbb{R}$, a function which satisfies the given condition.

Therefore the only functions to satisfy the given condition are $f(x) = x$ or $f(x) = -x$, for $x \in \mathbb{R}$.

Problem 70

The numbers $1000, 1001, \cdots, 2999$ have been written on a board.

Each time, one is allowed to erase two numbers, say, a and b, and replace them by the number $\frac{1}{2}\min(a, b)$.

After 1999 such operations, one obtains exactly one number c on the board.

Prove that $c < 1$.

Solution 70

By symmetry, we may assume $a \le b$. Then

$$\frac{1}{2}\min(a,b) = \frac{a}{2}.$$

We have

$$\frac{1}{a} + \frac{1}{b} \le \frac{1}{\left(\frac{a}{2}\right)},$$

from which it follows that the sum of the reciprocals of all the numbers on the board is nondecreasing (i.e., the sum is a monovariant).

At the beginning this sum is

$$S = \frac{1}{1000} + \frac{1}{1001} + \cdots + \frac{1}{2999} \le \frac{1}{\mathrm{c}},$$

where $1/\mathrm{c}$ is the sum at the end. Note that, for $1 \le k \le 999$,

$$\frac{1}{2000-k} + \frac{1}{2000+k} = \frac{4000}{2000^2 - k^2} > \frac{4000}{2000^2} = \frac{1}{1000}.$$

Rearranging terms in S yields

$$\begin{aligned}\frac{1}{\mathrm{c}} \ge \frac{1}{1000} &+ \left(\frac{1}{1001} + \frac{1}{2999}\right) + \left(\frac{1}{1002} + \frac{1}{2998}\right) + \\ &\cdots + \left(\frac{1}{1999} + \frac{1}{2001}\right) + \frac{1}{2000} \\ &> \frac{1}{1000} \times 1000 + \frac{1}{2000} > 1,\end{aligned}$$

or $\mathrm{c} < 1$, as desired.

Problem 71 [Bulgaria 1998]

Let $a_1, a_2, \ldots, a_n$ be real numbers, not all zero.

Prove that the equation

$$\sqrt{1+a_1x} + \sqrt{1+a_2x} + \cdots + \sqrt{1+a_nx} = n$$

has at most one nonzero real root.

Solution 71

Notice that $f_i(x) = \sqrt{1+a_ix}$ is concave. Hence

$$f(x) = \sqrt{1+a_1x} + \cdots + \sqrt{1+a_nx}$$

is concave.

Since $f'(x)$ exists, there can be at most one point on the curve $y = f(x)$ with derivative 0.

Suppose there is more than one nonzero root.

Since $x = 0$ is also a root, we have three real roots $x_1 < x_2 < x_3$. Applying the Mean-Value theorem to $f(x)$ on intervals $[x_1, x_2]$ and $[x_2, x_3]$, we can find two distinct points on the curve with derivative 0, a contradiction.

Therefore, our assumption is wrong and there can be at most one nonzero real root for the equation $f(x) = n$.

Problem 72 [Turkey 1998]

Let $\{a_n\}$ be the sequence of real numbers defined by $a_1 = t$ and

$$a_{n+1} = 4a_n(1 - a_n)$$

for $n \geq 1$.

For how many distinct values of t do we have $a_{1998} = 0$?

Solution 72, Alternative 1

Let $f(x) = 4x(1-x)$. Observe that

$$f^{-1}(0) = \{0, 1\}, \quad f^{-1}(1) = \{1/2\}, \quad f^{-1}([0,1]) = [0,1],$$

and $|\{y : f(y) = x\}| = 2$ for all $x \in [0, 1)$.

Let $A_n = \{x \in \mathbb{R} : f^n(x) = 0\}$; then

$$\begin{aligned} A_{n+1} &= \{x \in \mathbb{R} : f^{n+1}(x) = 0\} \\ &= \{x \in \mathbb{R} : f^n(f(x)) = 0\} = \{x \in \mathbb{R} : f(x) \in A_n\}. \end{aligned}$$

We claim that for all $n \geq 1$, $A_n \subset [0, 1]$, $1 \in A_n$, and

$$|A_n| = 2^{n-1} + 1.$$

For $n = 1$, we have

$$A_1 = \{x \in \mathbb{R} \mid f(x) = 0\} = \{0, 1\},$$

and the claims hold.

Now suppose $n \geq 1$ and $A_n \subset [0,1]$, $1 \in A_n$, and $|A_n| = 2^{n-1} + 1$. Then

$$x \in A_{n+1} \Rightarrow f(x) \in A_n \subset [0,1] \Rightarrow x \in [0, 1],$$

so $A_{n+1} \subset [0, 1]$.

Since $f(0) = f(1) = 0$, we have $f^{n+1}(1) = 0$ for all $n \geq 1$, so $1 \in A_{n+1}$. Now we have

$$\begin{aligned}
|A_{n+1}| &= |\{x : f(x) \in A_n\}| \\
&= \sum_{a \in A_n} |\{x : f(x) = a\}| \\
&= |\{x : f(x) = 1\}| + \sum_{\substack{a \in A_n \\ a \in [0,1)}} |\{x : f(x) = a\}| \\
&= 1 + \sum_{\substack{a \in A_n \\ a \in [0,1)}} 2 \\
&= 1 + 2(|A_n| - 1) \\
&= 1 + 2(2^{n-1} + 1 - 1) \\
&= 2^n + 1.
\end{aligned}$$

Thus the claim holds by induction.

Finally, $a_{1998} = 0$ if and only if $f^{1997}(t) = 0$, so there are $2^{1996} + 1$ such values of t.

Solution 72, Alternative 2

As in the previous solution, observe that if $f(x) \in [0,1]$ then $x \in [0,1]$, so if $a_{1998} = 0$ we must have $t \in [0,1]$.

Now choose $\theta \in [0, \pi/2]$ such that $\sin\theta = \sqrt{t}$.

Observe that for any $\phi \in \mathbb{R}$,

$$f(\sin^2\phi) = 4\sin^2\phi\,(1 - \sin^2\phi) = 4\sin^2\phi\cos^2\phi = \sin^2 2\phi;$$

since $a_1 = \sin^2\theta$, it follows that

$$a_2 = \sin^2 2\theta,\ a_3 = \sin^2 4\theta, \ldots, a_{1998} = \sin^2 2^{1997}\theta.$$

Therefore

$$a_{1998} = 0 \iff \sin 2^{1997}\theta = 0 \iff \theta = \frac{k\pi}{2^{1997}}$$

for some $k \in \mathbb{Z}$.

Thus the values of t which give $a_{1998} = 0$ are

$$\sin^2(k\pi/2^{1997}),$$

$k \in \mathbb{Z}$, giving $2^{1996} + 1$ such values of t.

Problem 73 [IMO 1997 short list]

(a) Do there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \qquad \text{and} \qquad g(f(x)) = x^3$$

for all $x \in \mathbb{R}$?

(b) Do there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \qquad \text{and} \qquad g(f(x)) = x^4$$

for all $x \in \mathbb{R}$?

Solution 73

(a) The conditions imply that $f(x^3) = f(g(f(x))) = [f(x)]^2$, whence

$$x \in \{-1, 0, 1\} \Longrightarrow x^3 = x \Longrightarrow f(x) = [f(x)]^2 \Longrightarrow f(x) \in \{0, 1\}.$$

Thus, there exist different $a, b \in \{-1, 0, 1\}$ such that $f(a) = f(b)$.

But then $a^3 = g(f(a)) = g(f(b)) = b^3$, a contradiction.

Therefore, the desired functions f and g do not exist.

(b) Let

$$g(x) = \begin{cases} |x|^{\ln|x|} & \text{if } |x| \geq 1 \\ |x|^{-\ln|x|} & \text{if } 0 < |x| < 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that g is even and $|a| = |b|$ whenever $g(a) = g(b)$; thus, we are allowed to define f as an even function such that

$$f(x) = y^2, \text{ where } y \text{ is such that } g(\pm y) = x.$$

We claim that the functions f, g described above satisfy the conditions of the problem.

It is clear from the definition of f that $f(g(x)) = x^2$.

Now let $y = \sqrt{f(x)}$.

Then $g(y) = x$ and

$$\begin{aligned} g(f(x)) &= g(y^2) \\ &= \begin{cases} (y^2)^{\ln(y^2)} = y^{4\ln y} = \left(y^{\ln y}\right)^4 & \text{if } y \geq 1 \\ (y^2)^{-\ln(y^2)} = \left(y^{-\ln y}\right)^4 & \text{if } 0 < y < 1 \\ 0 & \text{if } y = 0 \end{cases} \\ &= [g(y)]^4 \\ &= x^4. \end{aligned}$$

Problem 74 [Weichao Wu]

Let $0 < a_1 \le a_2 \cdots \le a_n$, $0 < b_1 \le b_2 \cdots \le b_n$ be real numbers such that

$$\sum_{i=1}^{n} a_i \ge \sum_{i=1}^{n} b_i.$$

Suppose that there exists $1 \le k \le n$ such that $b_i \le a_i$ for $1 \le i \le k$ and $b_i \ge a_i$ for $i > k$.

Prove that

$$a_1 a_2 \cdots a_n \ge b_1 b_2 \cdots b_n.$$

Solution 74, Alternative 1

We define two new sequences. For $i = 1, 2, \ldots, n$, let

$$a_i' = a_k \text{ and } b_i' = \frac{b_i a_k}{a_i}.$$

Then

$$a_i' - b_i' = a_k - \frac{b_i a_k}{a_i} = \frac{a_k}{a_i}(a_i - b_i)$$

or

$$(a_i' - b_i') - (a_i - b_i) = \frac{(a_k - a_i)(a_i - b_i)}{a_i} \ge 0.$$

Therefore

$$n a_k = a_1' + a_2' + \cdots + a_n' \ge b_1' + b_2' + \cdots + b_n'.$$

Applying the **AM-GM inequality** yields

$$\left(\frac{b_1 b_2 \cdots b_n a_k^n}{a_1 a_2 \cdots a_n}\right)^{\frac{1}{n}} = (b_1' b_2' \cdots b_n')^{\frac{1}{n}} \le \frac{b_1' + b_2' + \cdots + b_n'}{n} \le a_k$$

from which the desired result follows.

Solution 74, Alternative 2

We define two new sequences. For $i = 1, 2, \ldots, n$, let

$$a_i' = a_k \text{ and } b_i' = b_i + a_k - a_i > 0.$$

Then

$$b_1' + b_2' + \cdots + b_n' \le n a_k. \tag{1}$$

Note that, for $cy(x - y)(y + c) \ge 0$,

$$\frac{x}{y} \ge \frac{x + c}{y + c}, \; x \ge y \text{ and } c \ge 0;$$

Setting $x = a_i$, $y = b_i$, and $\mathrm{c} = a_k - a_i$, the above inequality implies that $a_i/b_i \geq a'_i/b'_i$, for $i = 1, 2, \ldots, n$. Thus,

$$\frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n} \geq \frac{a'_1 a'_2 \cdots a'_n}{b'_1 b'_2 \cdots b'_n} \tag{2}$$

Using (1) and the **AM-GM inequality** yields

$$(a'_1 a'_2 \cdots a'_n)^{\frac{1}{n}} = a_k \geq \frac{b'_1 + b'_2 + \cdots + b'_n}{n} \geq (b'_1 b'_2 \cdots b'_n)^{\frac{1}{n}}$$

or

$$a'_1 a'_2 \cdots a'_n \geq b'_1 b'_2 \cdots b'_n. \tag{3}$$

It is clear that the desired result follows from (2) and (3).

Problem 75

Given eight non-zero real numbers $a_1, a_2, \cdots, a_8$, prove that at least one of the following six numbers: $a_1a_3 + a_2a_4$, $a_1a_5 + a_2a_6$, $a_1a_7 + a_2a_8$, $a_3a_5 + a_4a_6$, $a_3a_7 + a_4a_8$, $a_5a_7 + a_6a_8$ is non-negative.

Solution 75 [Moscow Olympiad 1978]

First, we introduce some basic knowledge of vector operations.

Let $\mathbf{u} = [\mathbf{a}, \mathbf{b}]$ and $\mathbf{v} = [\mathbf{m}, \mathbf{n}]$ be two vectors.

Define their *dot product* $\mathbf{u} \cdot \mathbf{v} = \mathbf{am} + \mathbf{bn}$.

It is easy to check that

(i) $\mathbf{v} \cdot \mathbf{v} = \mathbf{m^2} + \mathbf{n^2} = |\mathbf{v}|^2$, that is, the dot product of vector with itself is the square of the magnitude of $\mathbf{v}$ and $\mathbf{v} \cdot \mathbf{v} \geq \mathbf{0}$ with equality if and only if $\mathbf{v} = [\mathbf{0}, \mathbf{0}]$;

(ii) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;

(iii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, where $\mathbf{w}$ is a vector;

(iv) $(\mathrm{c}\mathbf{u}) \cdot \mathbf{v} = \mathrm{c}(\mathbf{u} \cdot \mathbf{v})$, where c is a scalar.

When vectors $\mathbf{u}$ and $\mathbf{v}$ are placed tail-by-tail at the origin O, let A and B be the tips of $\mathbf{u}$ and $\mathbf{v}$, respectively. Then $\overrightarrow{AB} = \mathbf{v} - \mathbf{u}$.

Let $\angle AOB = \theta$.

Applying the **law of cosines** to triangle AOB yields

$$\begin{aligned} |\mathbf{v} - \mathbf{u}|^2 &= AB^2 \\ &= OA^2 + OB^2 - 2OA \cdot OB \cos\theta \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta. \end{aligned}$$

It follows that

$$(\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) = \mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}-2|\mathbf{u}||\mathbf{v}|\cos\theta,$$

or

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Consequently, if $0 \le \theta \le 90°$, $\mathbf{u}\cdot\mathbf{v} \ge \mathbf{0}$.

Consider vectors $\mathbf{v}_1 = [a_1, a_2], \mathbf{v}_2 = [a_3, a_4], \mathbf{v}_3 = [a_5, a_6]$, and $\mathbf{v}_4 = [a_7, a_8]$.

Note that the numbers $a_1a_3+a_2a_4$, $a_1a_5+a_2a_6$, $a_1a_7+a_2a_8$, $a_3a_5+a_4a_6$, $a_3a_7+a_4a_8$, $a_5a_7+a_6a_8$ are all the dot products of distinct vectors $\mathbf{v}_i$ and $\mathbf{v}_j$.

Since there are four vectors, when placed tail-by-tail at the origin, at least two of them form a non-obtuse angle, which in turn implies the desired result.

Problem 76 [IMO 1996 short list]

Let a, b and c be positive real numbers such that $abc = 1$.

Prove that

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca} \le 1.$$

Solution 76

We have

$$a^5+b^5 \ge a^2b^2(a+b),$$

because

$$(a^3-b^3)(a^2-b^2) \ge 0,$$

with equality if and only if $a = b$. Hence

$$\begin{aligned}\frac{ab}{a^5+b^5+ab} &\le \frac{ab}{a^2b^2(a+b)+ab}\\ &= \frac{1}{ab(a+b)+1}\\ &= \frac{abc}{ab(a+b+c)}\\ &= \frac{c}{a+b+c}.\end{aligned}$$

Likewise,

$$\frac{bc}{b^5+c^5+bc} \le \frac{a}{a+b+c}$$

and

$$\frac{ca}{c^5 + a^5 + ca} \le \frac{b}{a+b+c}.$$

Adding the last three inequalities leads to the desired result.

Equality holds if and only if $a = b = c = 1$.

Comment: Please compare the solution to this problem with the second solution of problem 13 in this chapter.

Problem 77 [Czech-Slovak match 1997]

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

holds for all pairs of real numbers (x, y).

Solution 77

Clearly, $f(x) = x^2$ satisfies the functional equation.

Now assume that there is a nonzero value a such that $f(a) \neq a^2$.

Let $y = \dfrac{x^2 - f(x)}{2}$ in the functional equation to find that

$$f\left(\frac{f(x) + x^2}{2}\right) = f\left(\frac{f(x) + x^2}{2}\right) + 2f(x)(x^2 - f(x))$$

or $0 = 2f(x)(x^2 - f(x))$. Thus, for each x, either $f(x) = 0$ or $f(x) = x^2$. In both cases, $f(0) = 0$.

Setting $x = a$, it follows from above that either $f(a) = 0$ or $f(a) = 0$ or $f(a) = a^2$.

The latter is false, so $f(a) = 0$.

Now, let $x = 0$ and then $x = a$ in the functional equation to find that

$$f(y) = f(-y), \quad f(y) = f(a^2 - y)$$

and so

$$f(y) = f(-y) = f(a^2 + y);$$

that is, the function is periodic with nonzero period a^2.

Let $y = a^2$ in the original functional equation to obtain

$$f(f(x)) = f(f(x) + a^2) = f(x^2 - a^2) + 4a^2 f(x) = f(x^2) + 4a^2 f(x).$$

However, putting $y = 0$ in the functional equation gives $f(f(x)) = f(x^2)$ for all x.

Thus, $4a^2 f(x) = 0$ for all x. Since a is nonzero, $f(x) = 0$ for all x. Therefore, either $f(x) = x^2$ or $f(x) = 0$.

Problem 78 [*Kvant*]

Solve the system of equations:

$$x + \frac{3x - y}{x^2 + y^2} = 3$$
$$y - \frac{x + 3y}{x^2 + y^2} = 0.$$

Solution 78, Alternative 1

Multiplying the second equation by i and adding it to the first equation yields

$$x + yi + \frac{(3x - y) - (x + 3y)i}{x^2 + y^2} = 3,$$

or

$$x + yi + \frac{3(x - yi)}{x^2 + y^2} - \frac{i(x - yi)}{x^2 + y^2} = 3.$$

Let $z = x + yi$. Then

$$\frac{1}{z} = \frac{x - yi}{x^2 + y^2}.$$

Thus the last equation becomes

$$z + \frac{3 - i}{z} = 3,$$

or

$$z^2 - 3z + (3 - i) = 0.$$

Hence

$$z = \frac{3 \pm \sqrt{-3 + 4i}}{2} = \frac{3 \pm (1 + 2i)}{2},$$

that is, $(x, y) = (2, 1)$ or $(x, y) = (1, -1)$.

Solution 78, Alternative 2

Multiplying the first equation by y, the second by x, and adding up yields

$$2xy + \frac{(3x - y)y - (x + 3y)x}{x^2 + y^2} = 3y,$$

or $2xy - 1 = 3y$. It follows that $y \neq 0$ and

$$x = \frac{3y + 1}{2y}.$$

Substituting this into the second equation of the given system gives

$$y\left[\left(\frac{3y+1}{2y}\right)^2+y^2\right]-\left(\frac{3y+1}{2y}\right)-3y=0,$$

or

$$4y^4-3y^2-1=0.$$

It follows that $y^2=1$ and that the solutions to the system are $(2,1)$ and $(1,-1)$.

Problem 79 [China 1995]

Mr. Fat and Mr. Taf play a game with a polynomial of degree at least 4:

$$x^{2n}+\_x^{2n-1}+\_x^{2n-2}+\cdots+\_x+1.$$

They fill in real numbers to empty spaces in turn.

If the resulting polynomial has no real root, Mr. Fat wins; otherwise, Mr. Taf wins.

If Mr. Fat goes first, who has a winning strategy?

Solution 79

Mr. Taf has a winning strategy.

We say a blank space is *odd* (*even*) if it is the coefficient of an odd (even) power of x.

First Mr. Taf will fill in arbitrary real numbers into one of the remaining even spaces, if there are any.

Since there are only $n-1$ even spaces, there will be at least one odd space left after $2n-3$ plays, that is, the given polynomial becomes

$$p(x)=q(x)+\_x^s+\_x^{2t-1},$$

where s and $2t-1$ are distinct positive integers and $q(x)$ is a fixed polynomial.

We claim that there is a real number a such that

$$p(x)=q(x)+ax^s+\_x^{2t-1}$$

will always have a real root regardless of the coefficient of x^{2t-1}.

Then Mr. Taf can simply fill in a in front of x^s and win the game.

Now we prove our claim. Let b be the coefficient of x^{2t-1} in $p(x)$. Note that

$$\begin{aligned}&\frac{1}{2^{2t-1}}p(2)+p(-1)\\&=\left(\frac{1}{2^{2t-1}}q(2)+2^{s-2t+1}a+b\right)+[q(-1)+(-1)^s a-b]\\&=\left(\frac{1}{2^{2t-1}}q(2)+q(-1)\right)+a[2^{s-2t+1}+(-1)^s].\end{aligned}$$

Since $s\neq 2t-1$, $2^{s-2t+1}+(-1)^s\neq 0$.

Thus

$$a=-\frac{\frac{1}{2^{2t-1}}q(2)+q(-1)}{2^{s-2t+1}+(-1)^s}$$

is well defined such that a is independent of b and

$$\frac{1}{2^{2t-1}}p(2)+p(-1)=0.$$

It follows that either $p(-1)=p(2)=0$ or $p(-1)$ and $p(2)$ have different signs, which implies that there is a real root of $p(x)$ in between -1 and 2.

In either case, $p(x)$ has a real root regardless of the coefficient of x^{2t-1}, as claimed.

Our proof is thus complete.

Problem 80 [IMO 1997 short list]

Find all positive integers k for which the following statement is true: if $F(x)$ is a polynomial with integer coefficients satisfying the condition

$$0\leq F(\mathrm{c})\leq k \quad \text{for} \quad \mathrm{c}=0,1,\ldots,k+1,$$

then $F(0)=F(1)=\cdots=F(k+1)$.

Solution 80

The statement is true if and only if $k\geq 4$.

We start by proving that it does hold for each $k\geq 4$.

Consider any polynomial $F(x)$ with integer coefficients satisfying the inequality $0\leq F(\mathrm{c})\leq k$ for each $\mathrm{c}\in\{0,1,\ldots,k+1\}$.

Note first that $F(k+1)=F(0)$, since $F(k+1)-F(0)$ is a multiple of $k+1$ not exceeding k in absolute value.

Hence

$$F(x)-F(0)=x(x-k-1)G(x),$$

where $G(x)$ is a polynomial with integer coefficients.

Consequently,

$$k \geq |F(c) - F(0)| = c(k+1-c)|G(c)| \tag{1}$$

for each $c \in \{1, 2, \ldots, k\}$.

The equality $c(k+1-c) > k$ holds for each $c \in \{2, 3, \ldots, k-1\}$, as it is equivalent to $(c-1)(k-c) > 0$.

Note that the set $\{2, 3, \ldots, k-1\}$ is not empty if $k \geq 3$, and for any c in this set, (1) implies that $|G(c)| < 1$.

Since $G(c)$ is an integer, $G(c) = 0$.

Thus

$$F(x) - F(0) = x(x-2)(x-3)\cdots(x-k+1)(x-k-1)H(x), \tag{2}$$

where $H(x)$ is a polynomial with integer coefficients.

To complete the proof of our claim, it remains to show that $H(1) = H(k) = 0$.

Note that for $c = 1$ and $c = k$, (2) implies that

$$k \geq |F(c) - F(0)| = (k-2)! \cdot k \cdot |H(c)|.$$

For $k \geq 4$, $(k-2)! > 1$.

Hence $H(c) = 0$.

We established that the statement in the question holds for any $k \geq 4$.

But the proof also provides information for the smaller values of k as well.

More exactly, if $F(x)$ satisfies the given condition then 0 and $k+1$ are roots of $F(x)$ and $F(0)$ for any $k \geq 1$; and if $k \geq 3$ then 2 must also be a root of $F(x) - F(0)$.

Taking this into account, it is not hard to find the following counterexamples:

$$\begin{array}{ll} F(x) = x(2-x) & \text{for } k = 1, \\ F(x) = x(3-x) & \text{for } k = 2, \\ F(x) = x(4-x)(x-2)^2 & \text{for } k = 3. \end{array}$$

Problem 81 [Korean Mathematics Competition 2001]
The Fibonacci sequence F_n is given by

$$F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n \quad (n \in \mathbb{N}).$$

Prove that

$$F_{2n} = \frac{F_{2n+2}^3 + F_{2n-2}^3}{9} - 2F_{2n}^3$$

for all $n \geq 2$.

Solution 81
Note that

$$F_{2n+2} - 3F_{2n} = F_{2n+1} - 2F_{2n} = F_{2n-1} - F_{2n} = -F_{2n-2},$$

whence

$$3F_{2n} - F_{2n+2} - F_{2n-2} = 0 \tag{1}$$

for all $n \geq 2$.
Setting $a = 3F_{2n}, b = -F_{2n+2}$, and $c = -F_{2n-2}$ in the algebraic identity

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

gives

$$27F_{2n}^3 - F_{2n+2}^3 - F_{2n-2}^3 - 9F_{2n+2}F_{2n}F_{2n-2} = 0.$$

Applying (1) twice gives

$$\begin{aligned} F_{2n+2}F_{2n-2} - F_{2n}^2 &= (3F_{2n} - F_{2n-2})F_{2n-2} - F_{2n}^2 \\ &= F_{2n}(3F_{2n-2} - F_{2n}) - F_{2n-2}^2 = F_{2n}F_{2n-4} - F_{2n-2}^2 \\ &= \cdots = F_6F_2 - F_4^2 = -1. \end{aligned}$$

The desired result follows from

$$9F_{2n+2}F_{2n}F_{2n-2} - 9F_{2n}^3 = 9F_{2n}(F_{2n+2}F_{2n-2} - F_{2n}^2) = -9F_{2n}.$$

Problem 82 [Romania 1998]

Find all functions $u : \mathbb{R} \to \mathbb{R}$ for which there exists a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x)u(y) + f(y)$$

for all $x, y \in \mathbb{R}$.

Solution 82

The solutions are $u(x) = a^x$, $a \in \mathbb{R}^+$.

To see that these work, take $f(x) = x$ for $a = 1$.

If $a \neq 1$, take $f(x) = a^x - 1$; then

$$f(x+y) = a^{x+y} - 1 = (a^x - 1)a^y + a^y - 1 = f(x)u(y) + f(y)$$

for all x, $y \in \mathbb{R}$.

Now suppose $u : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ are functions for which f is strictly monotonic and $f(x+y) = f(x)u(y) + f(y)$ for all x, $y \in \mathbb{R}$.

We must show that u is of the form $u(x) = a^x$ for some $a \in \mathbb{R}^+$. First, letting $y = 0$, we obtain $f(x) = f(x)u(0) + f(0)$ for all $x \in \mathbb{R}$.

Thus, $u(0) \neq 1$ would imply $f(x) = f(0)/(1 - u(0))$ for all x, which would contradict the fact that f is strictly monotonic, so we must have $u(0) = 1$ and $f(0) = 0$.

Then $f(x) \neq 0$ for all $x \neq 0$.

Next, we have

$$f(x)u(y) + f(y) = f(x+y) = f(x) + f(y)u(x),$$

or

$$f(x)(u(y) - 1) = f(y)(u(x) - 1)$$

for all $x, y \in \mathbb{R}$. That is,

$$\frac{u(x) - 1}{f(x)} = \frac{u(y) - 1}{f(y)}$$

for all $xy \neq 0$.

It follows that there exists $C \in \mathbb{R}$ such that

$$\frac{u(x) - 1}{f(x)} = C$$

for all $x \neq 0$.

Thus, $u(x) = 1 + Cf(x)$ for $x \neq 0$; since $u(0) = 1$, $f(0) = 0$, this equation also holds for $x = 0$.

If $C = 0$, then $u(x) = 1$ for all x, and we are done.

Otherwise, observe

$$\begin{aligned} u(x+y) &= 1 + Cf(x+y) \\ &= 1 + Cf(x)u(y) + Cf(y) \\ &= u(y) + Cf(\dot{x})u(y) \\ &= u(x)u(y) \end{aligned}$$

for all $x, y \in \mathbb{R}$.

Thus $u(nx) = u(x)^n$ for all $n \in \mathbb{Z}$, $x \in \mathbb{R}$.

Since $u(x) = 1 + Cf(x)$ for all x, u is strictly monotonic, and $u(-x) = 1/u(x)$ for all x, so $u(x) > 0$ for all x as $u(0) = 1$.

Let $a = u(1) > 0$; then $u(n) = a^n$ for all $n \in \mathbb{N}$, and

$$u(p/q) = (u(p))^{1/q} = a^{p/q}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$, so $u(x) = a^x$ for all $x \in \mathbb{Q}$.

Since u is monotonic and the rationals are dense in $\mathbb{R}$, we have $u(x) = a^x$ for all $x \in \mathbb{R}$.

Thus all solutions are of the form $u(x) = a^x$, $a \in \mathbb{R}^+$.

Problem 83 [China 1986]

Let $z_1, z_2, \ldots, z_n$ be complex numbers such that

$$|z_1| + |z_2| + \cdots + |z_n| = 1.$$

Prove that there exists a subset S of $\{z_1, z_2, \ldots, z_n\}$ such that

$$\left| \sum_{z \in S} z \right| \geq \frac{1}{6}.$$

Solution 83, Alternative 1

Let ℓ_1, ℓ_2, and ℓ_3 be three rays from origin that form angles of $60°, 180°$, and $300°$, respectively, with the positive x-axis.

For $i = 1, 2, 3$, let $\mathcal{R}_i$ denote the region between ℓ_i and ℓ_{i+1} (here $\ell_4 = \ell_1$), including the ray ℓ_i. Then

$$1 = \sum_{z_k \in \mathcal{R}_1} |z_k| + \sum_{z_k \in \mathcal{R}_2} |z_k| + \sum_{z_k \in \mathcal{R}_3} |z_k|.$$

By the Pigeonhole Principle, at least one of the above sums is not less than $1/3$.

Say it's $\mathcal{R}_3$ (otherwise, we apply a rotation, which does not effect the magnitude of a complex number). Let $z_k = x_k + iy_k$. Then for $z_k \in \mathcal{R}_3$, $x_k = |x_k| \geq |z_k|/2$.

Consequently,

$$\left|\sum_{z_k \in \mathcal{R}_3} z_k\right| \geq \left|\sum_{z_k \in \mathcal{R}_3} x_k\right| \geq \frac{1}{2} \sum_{z_k \in \mathcal{R}_3} |z_k| \geq \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6},$$

as desired.

Solution 83, Alternative 2

We prove a stronger statement: there is subset S of $\{z_1, z_2, \ldots, z_n\}$ such that

$$\left|\sum_{z \in S} z\right| \geq \frac{1}{4}.$$

For $1 \leq k \leq n$, let $z_k = x_k + iy_k$. Then

$$\begin{aligned}
1 &= |z_1| + |z_2| + \cdots + |z_n| \\
&\leq (|x_1| + |y_1|) + (|x_2| + |y_2|) + \cdots + (|x_n| + |y_n|) \\
&= \sum_{x_k \geq 0} |x_k| + \sum_{x_k < 0} |x_k| + \sum_{y_k \geq 0} |y_k| + \sum_{y_k < 0} |y_k|.
\end{aligned}$$

By the Pigeonhole Principle, at least one of the above sums is not less than $1/4$. By symmetry, we may assume that

$$\frac{1}{4} \leq \sum_{x_k \geq 0} |x_k| = \left|\sum_{x_k \geq 0} x_k\right|.$$

Consequently,

$$\left|\sum_{x_k \geq 0} z_k\right| \geq \left|\sum_{x_k \geq 0} x_k\right| \geq \frac{1}{4}.$$

Comment: Using advanced mathematics, the lower bound can be further improved to $1/\pi$.

Problem 84 [Czech-Slovak Match 1998]

A polynomial $P(x)$ of degree $n \geq 5$ with integer coefficients and n distinct integer roots is given.

Find all integer roots of $P(P(x))$ given that 0 is a root of $P(x)$.

Solution 84

The roots of $P(x)$ are clearly integer roots of $P(P(x))$; we claim there are no other integer roots.

We prove our claim by contradiction. Suppose, on the contrary, that $P(P(k)) = 0$ for some integer k such that $P(k) \neq 0$.

Let

$$P(x) = a(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n),$$

where $a, r_1, r_2, \ldots, r_n$ are integers,

$$r_1 = 0 \leq |r_2| \leq |r_3| \leq \cdots \leq |r_n|.$$

Since $P(k) \neq 0$, we must have $|k - r_i| \geq 1$ for all i.

Since the r_i are all distinct, at most two of $|k - r_2|, |k - r_3|, |k - r_4|$ equal 1, so

$$|a(k - r_2) \cdots (k - r_{n-1})| \geq |a||k - r_2||k - r_3||k - r_4| \geq 2,$$

and $|P(k)| \geq 2|k(k - r_n)|$.

Also note that $P(k) = r_{i_0}$ for some i_0, so $|P(k)| \leq |r_n|$.

Now we consider the following two cases:

1. $|k| \geq |r_n|$. Then $|P(k)| \geq 2|k(k - r_n)| \geq 2|k| > |r_n|$, a contradiction.

2. $|k| < |r_n|$, that is, $1 \leq |k| \leq |r_n| - 1$. Let a, b, c be real numbers, $a \leq b$. For $x \in [a, b]$, the function

   $$f(x) = x(\mathrm{c} - x)$$

   reaches its minimum value at an endpoint $x = a$ or $x = b$, or at both endpoints.

   Thus

   $$|k(k - r_n)| = |k||r_n - k| \geq |k|(|r_n| - |k|) \geq |r_n| - 1.$$

   It follows that

   $$|r_n| \geq |P(k)| \geq 2|k(k - r_n)| \geq 2(|r_n| - 1),$$

   which implies that $|r_n| \leq 2$. Since $n \geq 5$, this is only possible if

   $$P(x) = (x + 2)(x + 1)x(x - 1)(x - 2).$$

   But then it is impossible to have $k \neq r_i$ and $|k| \leq |r_n|$, a contradiction.

Thus our assumption was incorrect, and the integer roots of $P(P(x))$ are exactly all the integer roots of $P(x)$.

Problem 85 [Belarus 1999]

Two real sequences $x_1, x_2, \ldots,$ and $y_1, y_2, \ldots,$ are defined in the following way:

$$x_1 = y_1 = \sqrt{3}, \quad x_{n+1} = x_n + \sqrt{1 + x_n^2},$$

and

$$y_{n+1} = \frac{y_n}{1 + \sqrt{1 + y_n^2}}$$

for all $n \geq 1$. Prove that $2 < x_n y_n < 3$ for all $n > 1$.

Solution 85, Alternative 1

Let $z_n = 1/y_n$ and note that the recursion for y_n is equivalent to

$$z_{n+1} = z_n + \sqrt{1 + z_n^2}.$$

Also note that $z_2 = \sqrt{3} = x_1$; since the x_is and z_is satisfy the same recursion, this means that $z_n = x_{n-1}$ for all $n > 1$.

Thus,

$$x_n y_n = \frac{x_n}{z_n} = \frac{x_n}{x_{n-1}}.$$

Note that

$$\sqrt{1 + x_{n-1}^2} > x_{n-1}.$$

Thus $x_n > 2x_{n-1}$ and $x_n y_n > 2$, which is the lower bound of the desired inequality.

Since x_ns are increasing for $n > 1$, we have

$$x_{n-1}^2 \geq x_1^2 = 3 > \frac{1}{3},$$

which implies that

$$2x_{n-1} > \sqrt{1 + x_{n-1}^2}.$$

Thus $3x_{n-1} > x_n$, which leads to the upper bound of the desired inequality.

Solution 85, Alternative 2

Setting $x_n = \cot\theta_n$ for $0 < \theta_n < 90°$ yields

$$x_{n+1} = \cot\theta_n + \sqrt{1 + \cot^2\theta_n} = \cot\theta_n + \csc\theta_n = \cot\left(\frac{\theta_n}{2}\right).$$

Since $\theta_1 = 30°$, we have in general $\theta_n = \frac{30°}{2^{n-1}}$. Similar calculation shows that

$$y_n = \tan(2\theta_n) = \frac{2\tan\theta_n}{1-\tan^2\theta_n}.$$

It follows that

$$x_n y_n = \frac{2}{1-\tan^2\theta_n}.$$

Since $\tan\theta_n \neq 0$, $\tan^2\theta_n$ is positive and $x_n y_n > 2$.

And since for $n > 1$ we have $\theta_n < 30°$, we also have

$$\tan^2\theta_n < \frac{1}{3}$$

so that $x_n y_n < 3$.

Comment: From the closed forms for x_n and y_n in the second solution, we can see the relationship

$$y_n = \frac{1}{x_{n-1}}$$

used in the first solution.

Problem 86 [China 1995]

For a polynomial $P(x)$, define the *difference* of $P(x)$ on the interval $[a, b]$ ($[a, b)$, (a, b), $(a, b]$) as $P(b) - P(a)$.

Prove that it is possible to dissect the interval $[0, 1]$ into a finite number of intervals and color them red and blue alternately such that, for every quadratic polynomial $P(x)$, the total difference of $P(x)$ on red intervals is equal to that of $P(x)$ on blue intervals.

What about cubic polynomials?

Solution 86

For an interval i, let $\Delta_i P$ denote the difference of polynomial P on i.

For a positive real number c and a set $S \subseteq \mathbb{R}$, let $S + c$ denote the set obtained by shifting S in the positive direction by c.

We prove a more general result.

Lemma

Let ℓ be a positive real number, and let k be a positive integer. It is always possible to dissect interval $I_k = [0, 2^k\ell]$ into a finite number of intervals and color them red and blue alternatively such that, for every polynomial $P(x)$ with $\deg P \leq k$, the total difference of $P(x)$ on the red intervals is equal to that on the blue intervals.

Proof

We induct on k.

For $k = 1$, we can just use intervals $[0, \ell]$ and $(\ell, 2\ell]$. It is easy to see that a linear or constant polynomial has the same difference on the two intervals.

Suppose that the statement is true for $k = n$, where n is a positive integer; that is, there exists a set R_n of red disjoint intervals and a set B_n of blue disjoint intervals such that $R_n \cap B_n = \emptyset$, $R_n \cup B_n = I_n$, and, for any polynomials $P(x)$ with $\deg P \le n$, the total differences of P on R_n is equal to that of P on B_n.

Now consider polynomial $f(x)$ with $\deg f \le n + 1$. Define

$$g(x) = f(x + 2^n \ell) \text{ and } h(x) = f(x) - g(x).$$

Then $\deg h \le n$. By the induction hypothesis,

$$\sum_{b \in B_n} \Delta_b h = \sum_{r \in R_n} \Delta_r h,$$

or

$$\sum_{b \in B_n} \Delta_b f + \sum_{r \in R_n} \Delta_r g = \sum_{r \in R_n} \Delta_r f + \sum_{r \in B_n} \Delta_b g.$$

It follows that

$$\sum_{b \in B'_{n+1}} \Delta_b f = \sum_{r \in R'_{n+1}} \Delta_r f,$$

where

$$\begin{aligned} R'_{n+1} &= R_n \cup (B_n + 2^n \ell), \\ \text{and} \quad B'_{n+1} &= B_n \cup (R_n + 2^n \ell). \end{aligned}$$

(If R'_{n+1} and B'_{n+1} both contain the number $2^n \ell$, that number may be removed from one of them.)

It is clear that B'_{n+1} and R'_{n+1} form a dissection of I_{n+1} and, for any polynomial f with $\deg f \le n + 1$, the total difference of f on B'_{n+1} is equal to that of f on R'_{n+1}.

The only possible trouble left is that the colors in $B'_{n+1} \cup R'_{n+1}$ might not be alternating (which can happen at the end of the I_n and the beginning of $I_n + 2^n \ell$).

But note that if intervals $i_1 = [a_1, b_1]$ and $i_2 = [b_1, c_1]$ are in the same color, then

$$\Delta_{i_1} f + \Delta_{i_2} f = \Delta_{i_3} f,$$

where $i_3 = [a_1, c_1]$.

Thus, in $B'_{n+1} \cup R'_{n+1}$, we can simply put consecutive same color intervals into one bigger interval of the same color.

Thus, there exists a dissection

$$I_{n+1} = B_{n+1} \cup R_{n+1}$$

such that, for every polynomial $f(x)$ with $\deg f \leq n+1$,

$$\sum_{b \in B_{n+1}} \Delta_b f = \sum_{r \in R_{n+1}} \Delta_r f.$$

This completes the induction and the proof of the lemma. □

Setting first $\ell = \frac{1}{4}$ and then $\ell = \frac{1}{8}$ in the lemma, it is clear that the answer to each of the given questions is "yes."

Problem 87 [USSR 1990]

Given a cubic equation

$$x^3 + \_x^2 + \_x + \_ = 0,$$

Mr. Fat and Mr. Taf are playing the following game.

In one move, Mr. Fat chooses a real number and Mr. Taf puts it in one of the empty spaces.

After three moves the game is over.

Mr. Fat wins the game if the final equation has three distinct integer roots.

Who has a winning strategy?

Solution 87

Mr. Fat has a winning strategy.

Let the polynomial be $x^3 + ax^2 + bx + \mathrm{c}$. Mr. Fat can pick 0 first. We consider the following cases:

(a) Mr. Taf chooses $a = 0$, yielding the polynomial equation

$$x^3 + bx + \mathrm{c} = 0.$$

Mr. Fat then picks the number $-(mnp)^2$, where m, n, and p are three positive integers such that

$$m^2 + n^2 = p^2.$$

If Mr. Taf chooses $b = -(mnp)^2$, then Mr. Fat will choose $\mathrm{c} = 0$. The given polynomial becomes

$$x(x - mnp)(x + mnp).$$

If Mr. Taf chooses $c = -(mnp)^2$, then Mr. Fat will choose

$$b = m^2n^2 - n^2p^2 - p^2m^2.$$

The given polynomial becomes

$$(x + m^2)(x + n^2)(x - p^2).$$

(b) Mr. Taf chooses $b = 0$, yielding the equation

$$x^3 + ax^2 + c = 0.$$

Mr. Fat then picks the number

$$m^2(m + 1)^2(m^2 + m + 1)^3,$$

where m is an integer greater than 1.

If Mr. Taf chooses

$$a = m^2(m + 1)^2(m^2 + m + 1)^3,$$

then Mr. Fat can choose

$$c = -m^8(m + 1)^8(m^2 + m + 1)^6.$$

The polynomial becomes

$$(x - mp)[x + (m + 1)p][x + m(m + 1)p],$$

where

$$p = m^2(m + 1)^2(m^2 + m + 1)^2.$$

If Mr. Taf chooses

$$c = m^2(m + 1)^2(m^2 + m + 1)^3,$$

then Mr. Fat can choose

$$a = -(m^2 + m + 1)^2.$$

The polynomial becomes

$$(x + mq)[x - (m + 1)q][x - m(m + 1)q],$$

where

$$q = m^2 + m + 1.$$

(c) Mr. Taf chooses c = 0.

Then the problem reduces to problem 40 of the previous chapter. Mr. Fat needs only to pick two integers a and b such that

$$ab(a-1)(b-1) \neq 0$$

and $a+b=-1$.

The polynomial becomes either $x(x-1)(x-a)$ or $x(x-1)(x-b)$.

Our proof is complete.

Below is an example of what Mr. Fat and Mr. Taf could do:

F	T	F	T	F	Roots
0	a	-3600	b	0	$-60, 0, 60$
"	"	"	c	-481	$-16, -9, 25$
"	b	$4\cdot 9\cdot 7^3$	a	$-2^8\cdot 3^8\cdot 7^6$	$-8\cdot 27\cdot 49$, $-4\cdot 27\cdot 49$, $8\cdot 9\cdot 49$
"	"	"	c	-49	$-14, 21, 42$
"	c	2	a	-3	$-3, 0, 1$
"	"	"	b	-3	$0, 1, 2$

Problem 88 [Romania 1996]

Let $n > 2$ be an integer and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for any regular n-gon $A_1A_2\ldots A_n$,

$$f(A_1) + f(A_2) + \cdots + f(A_n) = 0.$$

Prove that f is the zero function.

Solution 88

We identify $\mathbb{R}^2$ with the complex plane and let $\zeta = e^{2\pi i/n}$.

Then the condition is that for any $z \in \mathbb{C}$ and any positive real t,

$$\sum_{j=1}^{n} f(z + t\zeta^j) = 0.$$

In particular, for each of $k = 1, \ldots, n$, we obtain

$$\sum_{j=1}^{n} f(z - \zeta^k + \zeta^j) = 0.$$

Summing over k, we have

$$\sum_{m=1}^{n}\sum_{k=1}^{n} f(z-(1-\zeta^m)\zeta^k)=0.$$

For $m=n$ the inner sum is $nf(z)$; for other m, the inner sum again runs over a regular polygon, hence is 0.

Thus $f(z)=0$ for all $z\in\mathbb{C}$.

Problem 89 [IMO 1997 short list]

Let p be a prime number and let $f(x)$ be a polynomial of degree d with integer coefficients such that:

(i) $f(0)=0, f(1)=1$;

(ii) for every positive integer n, the remainder upon division of $f(n)$ by p is either 0 or 1.

Prove that $d\geq p-1$.

Solution 89, Alternative 1

For the sake of the contradiction, assume that $d\leq p-2$.

Then by **Lagrange's interpolation formula** the polynomial $f(x)$ is determined by its values at 0, 1, ..., $p-2$; that is,

$$\begin{aligned} f(x) &= \sum_{k=0}^{p-2} f(k)\frac{x\cdots(x-k+1)(x-k-1)\cdots(x-p+2)}{k\cdots 1\cdot(-1)\cdots(k-p+2)} \\ &= \sum_{k=0}^{p-2} f(k)\frac{x\cdots(x-k+1)}{k!(-1)^{p-k}}\,\frac{(x-k-1)\cdots(x-p+2)}{(p-k-2)!}. \end{aligned}$$

Setting $x=p-1$ gives

$$\begin{aligned} f(p-1) &= \sum_{k=0}^{p-2} f(k)\frac{(p-1)(p-2)\cdots(p-k)}{(-1)^{p-k}k!} \\ &\equiv \sum_{k=0}^{p-2} f(k)\frac{(-1)^k k!}{(-1)^{p-k}k!} \\ &\equiv (-1)^p\sum_{k=0}^{p-2} f(k) \pmod p. \end{aligned}$$

It follows that

$$S(f) := f(0)+f(1)+\cdots+f(p-1)\equiv 0 \pmod p. \tag{1}$$

On the other hand, (ii) implies that $S(f) \equiv j \pmod{p}$, where j denotes the number of those $k \in \{0, 1, \dots, p-1\}$ for which $f(k) \equiv 1 \pmod{p}$.

But (i) implies that $1 \leq j \leq p-1$.

So $S(f) \not\equiv 0 \pmod{p}$, which contradicts (1).

Thus our original assumption was wrong, and our proof is complete.

Solution 89, Alternative 2

Again, we approach the problem indirectly.

Assume that $d \leq p-2$, and let

$$f(x) = a_{p-2}x^{p-2} + \cdots + a_1 x + a_0.$$

Then

$$S(f) = \sum_{k=0}^{p-1} f(k) = \sum_{k=0}^{p-1}\sum_{i=0}^{p-2} a_i k^i = \sum_{i=0}^{p-2} a_i \sum_{k=0}^{p-1} k^i = \sum_{i=0}^{p-2} a_i S_i,$$

where $S_i = \sum_{k=0}^{p-1} k^i$.

We claim that $S_i \equiv 0 \pmod{p}$ for all $i = 0, 1, \dots, p-2$.

We use strong induction on i to prove our claim.

The statement is true for $i = 0$ as $S_0 = p$.

Now suppose that $S_0 \equiv S_1 \equiv \cdots \equiv S_{i-1} \equiv 0 \pmod{p}$ for some $1 \leq i \leq p-2$. Note that

$$\begin{aligned}
0 \equiv p^{i+1} &= \sum_{k=1}^{p} k^{i+1} - \sum_{k=0}^{p-1} k^{i+1} = \sum_{k=0}^{p-1}[(k+1)^{i+1} - k^{i+1}] \\
&= \sum_{k=0}^{p-1}\sum_{j=0}^{i}\binom{i+1}{j}k^j = (i+1)S_i + \sum_{j=0}^{i-1}\binom{i+1}{j}S_j \\
&\equiv (i+1)S_i \pmod{p}
\end{aligned}$$

Since $0 < i+1 < p$, it follows that $S_i \equiv 0 \pmod{p}$. This completes the induction and the proof of the claim. Therefore,

$$S(f) = \sum_{i=0}^{p-2} a_i S_i \equiv 0 \pmod{p}.$$

The rest is the same as in the first solution.

Problem 90

Let n be a given positive integer.

Consider the sequence $a_0, a_1, \cdots, a_n$ with $a_0 = \frac{1}{2}$ and

$$a_k = a_{k-1} + \frac{a_{k-1}^2}{n},$$

for $k = 1, 2, \cdots, n$.

Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

Solution 90, Alternative 1

We prove a stronger statement: For $k = 1, 2, \ldots, n$,

$$\frac{n+1}{2n-k+2} < a_k < \frac{n}{2n-k}. \tag{1}$$

We use induction to prove both inequalities.

We first prove the upper bound. For $k = 1$, it is easy to check that

$$a_1 = \frac{1}{2} + \frac{1}{4n} = \frac{2n+1}{4n} < \frac{n}{2n-1}.$$

Suppose that

$$a_k < \frac{n}{2n-k},$$

for some positive integer $k < n$. Then

$$\begin{aligned} a_{k+1} &= \frac{a_k}{n}(n + a_k) \\ &< \frac{1}{2n-k}\left(n + \frac{n}{2n-k}\right) \\ &= \frac{n(2n-k+1)}{(2n-k)^2} \\ &< \frac{n}{2n-k-1}, \end{aligned}$$

as

$$(2n-k+1)(2n-k-1) = (2n-k)^2 - 1 < (2n-k)^2.$$

Thus our induction step is complete. In particular, for $k = n-1$,

$$a_n = a_{k+1} < \frac{n}{2n-(n-1)-1} = 1,$$

as desired.

Now we prove the upper bound. For $k = 1$, it is easy to check that

$$a_1 = \frac{2n+1}{4n} > \frac{n+1}{2n+1}.$$

Suppose that

$$a_k > \frac{n+1}{2n-k+2},$$

for some positive integer $k < n$. Then

$$a_{k+1} = a_k + \frac{a_k^2}{n} > \frac{n+1}{2n-k+2} + \frac{(n+1)^2}{n(2n-k+2)^2}.$$

It follows that

$$\begin{aligned} a_{k+1} - \frac{n+1}{2n-k+1} &\geq -\frac{n+1}{(2n-k+1)(2n-k+2)} + \frac{(n+1)^2}{n(2n-k+2)^2} \\ &= \frac{n+1}{2n-k+2)^2}\left(\frac{n+1}{n} - \frac{2n-k+2}{2n-k+1}\right) \\ &= \frac{n+1}{2n-k+2)^2}\left(\frac{1}{n} - \frac{1}{2n-k+1}\right) > 0. \end{aligned}$$

This complete the induction step. In particular, for $n = k - 1$, we obtain

$$a_n = a_{k+1} > \frac{n+1}{2n-(n-1)+1} = \frac{n+1}{n+2} = 1 - \frac{1}{n+2} > 1 - \frac{1}{n},$$

as desired.

Solution 90, Alternative 2

Rewriting the given condition as

$$\frac{1}{a_k} = \frac{1}{a_{k-1} + \dfrac{a_{k-1}^2}{n}} = \frac{n}{a_{k-1}(n + a_{k-1})} = \frac{1}{a_{k-1}} - \frac{1}{n + a_{k-1}}$$

yields

$$\frac{1}{a_{k-1}} - \frac{1}{a_k} = \frac{1}{n + a_{k-1}}, \tag{2}$$

for $k = 1, 2, \ldots, n$.

It is clear that a_ks are increasing.

Thus

$$a_n > a_{n-1} > \cdots > a_0 = \frac{1}{2}.$$

Thus (2) implies that
$$\frac{1}{a_{k-1}} - \frac{1}{a_k} < \frac{1}{n}$$
for $k = 1, 2, \ldots, n$.

Telescoping summation gives
$$\frac{1}{a_0} - \frac{1}{a_n} < 1$$
or
$$\frac{1}{a_n} > \frac{1}{a_0} - 1 = 2 - 1 = 1,$$
that is, $a_n < 1$, which gives the desired upper bound.

Since $a_n < 1$, and, since a_ks are increasing, $\frac{1}{2} = a_0 < a_k \le a_n < 1$ for $k = 1, 2, \ldots, n$.

Then (2) implies
$$\frac{1}{a_{k-1}} - \frac{1}{a_k} = \frac{1}{n + a_{k-1}} > \frac{1}{n+1},$$
for $k = 1, 2, \ldots, n$.

Telescoping sum gives
$$\frac{1}{a_0} - \frac{1}{a_n} > \frac{n}{n+1}$$
or
$$\frac{1}{a_n} < \frac{1}{a_0} - \frac{n}{n+1} = \frac{n+2}{n+1},$$
that is,
$$a_n > \frac{n+1}{n+2} = 1 - \frac{1}{n+2} > 1 - \frac{1}{n},$$
which is the desired lower bound.

Problem 91 [IMO 1996 short list]

Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers, not all zero.

(a) Prove that $x^n - a_1 x^{n-1} - \cdots - a_{n-1}x - a_n = 0$ has precisely one positive real root R.

(b) Let $A = \sum_{j=1}^n a_j$ and $B = \sum_{j=1}^n j a_j$.

Prove that $A^A \le R^B$.

Solution 91

(a) Consider the function

$$f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}.$$

Note that f decreases from ∞ to 0 as x increases from 0 to ∞.

Hence there is a unique real number R such that $f(R) = 1$, that is, there exists a unique positive real root R of the given polynomial.

(b) Let $c_j = a_j/A$.

Then c_js are non-negative and $\sum c_j = 1$.

Since $-\ln x$ is a convex function on the interval $(0, \infty)$, by **Jensen's inequality**,

$$\sum_{j=1}^{n} c_j \left(-\ln \frac{A}{R^j}\right) \geq -\ln\left(\sum_{j=1}^{n} c_j \frac{A}{R^j}\right) = -\ln(f(R)) = 0.$$

It follows that

$$\sum_{j=1}^{n} c_j(-\ln A + j \ln R) \geq 0$$

or

$$\sum_{j=1}^{n} c_j \ln A \leq \sum_{j=1}^{n} j c_j \ln R.$$

Substituting $c_j = a_j/A$, we obtain the desired inequality.

Comment: Please compare the solution of (a) with that of the problem 15 in the last chapter.

Problem 92

Prove that there exists a polynomial $P(x, y)$ with real coefficients such that $P(x, y) \geq 0$ for all real numbers x and y, which cannot be written as the sum of squares of polynomials with real coefficients.

Solution 92

We claim that

$$P(x, y) = (x^2 + y^2 - 1)x^2y^2 + \frac{1}{27}$$

is a polynomial satisfying the given conditions.

First we prove that $P(x, y) \geq 0$ for all real numbers x and y.

If $x^2 + y^2 - 1 \geq 0$, then it is clear that $P(x, y) > 0$; if $x^2 + y^2 - 1 < 0$, then applying the **AM-GM inequality** gives

$$(1 - x^2 - y^2)x^2y^2 \leq \left(\frac{1 - x^2 - y^2 + x^2 + y^2}{3}\right)^3 = \frac{1}{27},$$

or

$$(x^2 + y^2 - 1)x^2y^2 \geq -\frac{1}{27}.$$

It follows that $P(x, y) \geq 0$.

We are left to prove that $P(x, y)$ cannot be written as the sum of squares of polynomials with real coefficients.

For the sake of contradiction, assume that

$$P(x, y) = \sum_{i=1}^{n} Q_i(x, y)^2.$$

Since $\deg P = 6$, $\deg Q_i \leq 3$.

Thus

$$\begin{aligned} Q_i(x, y) \quad &= \quad A_ix^3 + B_ix^2y + C_ixy^2 + D_iy^3 \\ &\quad\quad +E_ix^2 + F_ixy + G_iy^2 + H_ix + I_iy + J_i. \end{aligned}$$

Comparing the coefficients, in $P(x, y)$ and $\sum_{i=1}^{n} Q_i(x, y)^2$, of terms x^6 and y^6 gives

$$\sum_{i=1}^{n} A_i^2 = \sum_{i=1}^{n} D_i^2 = 0,$$

or $A_i = D_i = 0$ for all i.

Then, comparing those of x^4 and y^4 gives

$$\sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n} G_i^2 = 0,$$

or $E_i = G_i = 0$ for all i.

Next, comparing those of x^2 and y^2 gives

$$\sum_{i=1}^{n} H_i^2 = \sum_{i=1}^{n} I_i^2 = 0,$$

or $H_i = I_i = 0$ for all i.

Thus,

$$Q_i(x, y) = B_ix^2y + C_ixy^2 + F_ixy + J_i.$$

But, finally, comparing the coefficients of the term x^2y^2, we have

$$\sum_{i=1}^{n} F_i^2 = -1,$$

which is impossible for real numbers F_i.

Thus our assumption is wrong, and our proof is complete.

Problem 93 [IMO 1996 short list]

For each positive integer n, show that there exists a positive integer k such that

$$k = f(x)(x+1)^{2n} + g(x)(x^{2n}+1)$$

for some polynomials f, g with integer coefficients, and find the smallest such k as a function of n.

Solution 93

First we show that such a k exists.

Note that $x+1$ divides $1 - x^{2n}$. Then for some polynomial $a(x)$ with integer coefficients, we have

$$(1+x)a(x) = 1 - x^{2n} = 2 - (1 + x^{2n}),$$

or

$$2 = (1+x)a(x) + (1 + x^{2n}).$$

Raising both sides to the $(2n)^{\text{th}}$ power, we obtain

$$2^{2n} = (1+x)^{2n}(a(x))^{2n} + (1+x^{2n})b(x),$$

where $b(x)$ is a polynomial with integer coefficients.

This shows that a k satisfying the condition of the problem exists. Let k_0 be the minimum such k.

Let $2n = 2^r \cdot q$, where r is a positive integer and q is an odd integer.

We claim that $k_0 = 2^q$.

First we prove that 2^q divides k_0. Let $t = 2^r$. Note that $x^{2n} + 1 = (x^t + 1)Q(x)$, where

$$Q(x) = x^{t(q-1)} - x^{t(q-2)} + \cdots - x^t + 1.$$

The roots of $x^t + 1$ are

$$\omega_m = \cos\left(\frac{(2m-1)\pi}{t}\right) + i\sin\left(\frac{(2m-1)\pi}{t}\right), \quad m = 1, 2, \ldots, t;$$

that is,

$$R(x) = x^t + 1 = (x - \omega_1)(x - \omega_2)\cdots(x - \omega_t).$$

Let $f(x)$ and $g(x)$ be polynomials with integer coefficients such that

$$\begin{aligned} k_0 &= f(x)(x+1)^{2n} + g(x)(x^{2n}+1) \\ &= f(x)(x+1)^{2n} + g(x)Q(x)(x^t+1). \end{aligned}$$

It follows that

$$f(\omega_m)(\omega_m + 1)^n = k_0, \quad 1 \le m \le t. \tag{1}$$

Since r is positive, t is even. So

$$2 = R(-1) = (1+\omega_1)(1+\omega_2)\cdots(1+\omega_t).$$

Since $f(\omega_1)f(\omega_2)\cdots f(\omega_t)$ is a **symmetric polynomial** in $\omega_1, \omega_2, \ldots, \omega_t$ with integer coefficients, it can be expressed as a polynomial with integer coefficients in the **elementary symmetric functions** in $\omega_1, \omega_2, \ldots, \omega_t$ and therefore

$$F = f(\omega_1)f(\omega_2)\cdots f(\omega_t)$$

is an integer.

Taking the product over $m = 1, 2, \ldots, t$, (1) gives $2^{2n}F = k_0^t$ or $2^{2^r \cdot q}F = k_0^{2^r}$. It follows that 2^q divides k_0.

It now suffices to prove that $k_0 \le 2^q$.

Note that $Q(-1) = 1$.

It follows that

$$Q(x) = (x+1)\mathrm{c}(x) + 1,$$

where $\mathrm{c}(x)$ is a polynomial with integer coefficients.

Hence

$$(x+1)^{2n}(\mathrm{c}(x))^{2n} = (Q(x)-1)^{2n} = Q(x)d(x) + 1, \tag{2}$$

for some polynomial $d(x)$ with integer coefficients.

Also observe that, for any fixed m,

$$\{\omega_m^{2j-1} : j = 1, 2, \ldots, t\} = \{\omega_1, \omega_2, \ldots, \omega_t\}.$$

Thus

$$(1+\omega_m)(1+\omega_m^3)\cdots(1+\omega_m^{2t-1}) = R(-1) = 2,$$

and writing

$$1 + \omega_m^{2j-1} = (1+\omega_m)(1 - \omega_m + \omega_m^2 - \cdots + \omega_m^{2j-2}),$$

we find that for some polynomial $h(x)$, independent of m, with integer coefficients such that

$$(1+\omega_m)^t h(\omega_m) = 2.$$

But then $(x+1)h(x)-2$ is divisible by x^t+1 and hence we can write

$$(x+1)h(x) = 2 + (x^t+1)u(x),$$

for some polynomial $u(x)$ with integer coefficients.

Raising both sides to the power q we obtain

$$(x+1)^{2n}(h(x))^q = 2^q + (x^t+1)v(x), \tag{3}$$

where $v(x)$ is a polynomial with integer coefficients.

Using (2) and (3) we obtain

$$\begin{aligned}
&(x+1)^{2n}(\mathrm{c}(x))^{2n}(x^t+1)v(x) \\
&\quad = Q(x)d(x)(x^t+1)v(x) + (x^t+1)v(x) \\
&\quad = Q(x)d(x)(x^t+1)v(x) + (x+1)^{2n}(h(x))^q - 2^q,
\end{aligned}$$

that is,

$$2^q = f_1(x)(x+1)^{2n} + g_1(x)(x^{2n}+1),$$

where $f_1(x)$ and $g_1(x)$ are polynomials with integer coefficients.

Hence $k_0 \le 2^q$, as desired.

Our proof is thus complete.

Problem 94 [USAMO 1998 proposal, Kiran Kedlaya]

Let x be a positive real number.

(a) Prove that

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(x+1)\cdots(x+n)} = \frac{1}{x}.$$

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{n(x+1)\dots(x+n)} = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2}.$$

Solution 94

We use infinite telescoping sums to solve the problem.

(a) Equivalently, we have to show that

$$\sum_{n=1}^{\infty} \frac{n!x}{n(x+1)\cdots(x+n)} = 1.$$

Note that

$$\frac{x}{n(x+n)} = \frac{1}{n} - \frac{1}{x+n}.$$

It follows that

$$\frac{n!x}{n(x+1)\cdots(x+n)} = \frac{(n-1)!}{(x+1)\cdots(x+n-1)} - \frac{n!}{(x+1)\cdots(x+n)},$$

and this telescoping summation yields the desired result.

(b) Let

$$f(x) = \sum_{n=1}^{\infty} \frac{(n-1)!}{n(x+1)\ldots(x+n)}.$$

Then, by (a), $f(x) < \frac{1}{x}$.

In particular, $f(x)$ converges to 0 as x approaches ∞, so we can write f as an infinite telescoping series

$$f(x) = \sum_{k=1}^{\infty} [f(x+k-1) - f(x+k)]. \tag{1}$$

On the other hand, the result in (a) gives

$$\begin{aligned} f(x-1) - f(x) &= \sum_{n=1}^{\infty} \frac{(n-1)!}{n(x+1)\ldots(x+n-1)} \left(\frac{1}{x} - \frac{1}{x+n}\right) \\ &= \frac{1}{x} \sum_{n=1}^{\infty} \frac{(n-1)!}{(x+1)\cdots(x+n)} \\ &= \frac{1}{x^2}. \end{aligned}$$

Substituting the last equation to (1) gives

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2},$$

as desired.

Problem 95 [Romania 1996]

Let $n \geq 3$ be an integer, and let

$$X \subseteq S = \{1, 2, \ldots, n^3\}$$

be a set of $3n^2$ elements.

Prove that one can find nine distinct numbers a_i, b_i, c_i $(i = 1, 2, 3)$ in X such that the system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

has a solution (x_0, y_0, z_0) in nonzero integers.

Solution 95

Label the elements of X in increasing order $x_1 < \cdots < x_{3n^2}$, and put

$$\begin{aligned} X_1 &= \{x_1, \ldots, x_{n^2}\}, \\ X_2 &= \{x_{n^2+1}, \ldots, x_{2n^2}\}, \\ X_3 &= \{x_{2n^2+1}, \ldots, x_{3n^2}\}. \end{aligned}$$

Define the function $f : X_1 \times X_2 \times X_3 \to S \times S$ as follows:

$$f(a, b, c) = (b - a, c - b).$$

The domain of f contains n^6 elements.

The range of f, on the other hand, is contained in the subset of $S \times S$ of pairs whose sum is at most n^3, a set of cardinality

$$\sum_{k=1}^{n^3-1} k = \frac{n^3(n^3-1)}{2} < \frac{n^6}{2}.$$

By the **Pigeonhole Principle**, some three triples (a_i, b_i, c_i) $(i = 1, 2, 3)$ map to the same pair, in which case $x = b_1 - c_1, y = c_1 - a_1, z = a_1 - b_1$ is a solution in nonzero integers.

Note that a_i cannot equal b_j since X_1 and X_2 are disjoint, and that $a_1 = a_2$ implies that the triples (a_1, b_1, c_1) and (a_2, b_2, c_2) are identical, a contradiction.

Hence the nine numbers chosen are indeed distinct.

Problem 96 [Xuanguo Huang]

Let $n \geq 3$ be an integer and let $x_1, x_2, \cdots, x_n$ be positive real numbers. Suppose that

$$\sum_{j=1}^{n} \frac{1}{1+x_j} = 1.$$

Prove that

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \geq (n-1)\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right).$$

Solution 96

By symmetry, we may assume that $x_1 \leq x_2 \leq \cdots \leq x_n$. We have the following lemma.

Lemma For $1 \leq i < j \leq n$,

$$\frac{\sqrt{x_i}}{1+x_i} \geq \frac{\sqrt{x_j}}{1+x_j}.$$

Proof. Since $n \geq 3$, and, since

$$\sum_{i=1}^{n} \frac{1}{1+x_i} = 1,$$

we have

$$1 > \frac{1}{1+x_i} + \frac{1}{1+x_j} = \frac{2+x_i+x_j}{(1+x_i)(1+x_j)}$$

or

$$1 + x_i + x_j + x_i x_j > 2 + x_i + x_j.$$

It follows that $x_i x_j > 1$. Thus

$$\begin{aligned}
\frac{\sqrt{x_i}}{1+x_i} - \frac{\sqrt{x_j}}{1+x_j} &= \frac{\sqrt{x_i}(1+x_j) - \sqrt{x_j}(1+x_i)}{(1+x_i)(1+x_j)} \\
&= \frac{(\sqrt{x_i} - \sqrt{x_j})(1 - \sqrt{x_i x_j})}{(1+x_i)(1+x_j)} \\
&\geq 0,
\end{aligned}$$

as desired. □

By the lemma, we have

$$\frac{\sqrt{x_1}}{1+x_1} \geq \frac{\sqrt{x_2}}{1+x_2} \geq \cdots \geq \frac{\sqrt{x_n}}{1+x_n},$$

and, since

$$\frac{1}{\sqrt{x_1}} \geq \frac{1}{\sqrt{x_2}} \geq \cdots \geq \frac{1}{\sqrt{x_n}},$$

it follows by the **Chebyshev Inequality**

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}}\sum_{i=1}^{n}\frac{\sqrt{x_i}}{1+x_i} \leq \sum_{i=1}^{n}\left(\frac{1}{\sqrt{x_i}}\cdot\frac{\sqrt{x_i}}{1+x_i}\right) = \sum_{i=1}^{n}\frac{1}{1+x_i} = 1. \quad (1)$$

By the **Cauchy-Schwartz Inequality**, we have

$$\sum_{i=1}^{n}\frac{\sqrt{x_i}}{1+x_i}\sum_{i=1}^{n}\frac{1+x_i}{\sqrt{x_i}} \geq n^2,$$

or

$$\sum_{i=1}^{n}\frac{\sqrt{x_i}}{1+x_i}\left(\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}} + \sum_{i=1}^{n}\sqrt{x_i}\right) \geq n^2. \quad (2)$$

Multiplying by

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}}$$

on both sides of (2) and applying (1) gives

$$\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}} + \sum_{i=1}^{n}\sqrt{x_i} \geq n\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}},$$

which in turn implies the desired inequality.

Problem 97

Let $x_1, x_2, \ldots, x_n$ be distinct real numbers.
Define the polynomials

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

and

$$Q(x) = P(x) \left(\frac{1}{x - x_1} + \frac{1}{x - x_2} + \cdots + \frac{1}{x - x_n} \right).$$

Let $y_1, y_2, \ldots, y_{n-1}$ be the roots of Q.
Show that

$$\min_{i \neq j} |x_i - x_j| < \min_{i \neq j} |y_i - y_j|.$$

Solution 97

By symmetry, we may assume that

$$d = \min_{i \neq j} |y_i - y_j| = y_2 - y_1.$$

Let $s_k = y_1 - x_k$, for $k = 1, 2, \ldots, n$.
By symmetry, we may also assume that $s_1 < s_2 < \cdots < s_n$, i.e., $x_1 > x_2 > \cdots > x_n$.
For the sake of contradiction, assume that

$$d \leq \min_{i \neq j} |x_i - x_j| = \min_{i<j} x_i - x_j = \min_{i<j} s_j - s_i. \tag{1}$$

Since P has no double roots, it shares none with Q.
Then

$$P(y_i) \left(\frac{1}{y_i - x_1} + \frac{1}{y_i - x_2} + \cdots + \frac{1}{y_i - x_n} \right) = Q(y_1) = 0,$$

or

$$\frac{1}{y_i - x_1} + \frac{1}{y_i - x_2} + \cdots + \frac{1}{y_i - x_n} = 0.$$

In particular, setting $i = 1$ and $i = 2$ gives

$$\sum_{k=1}^{n} \frac{1}{s_k} = \sum_{k=1}^{n} \frac{1}{s_k + d} = 0. \tag{2}$$

We claim that there is a k such that $s_k(s_k + d) < 0$, otherwise, we have

$$\frac{1}{s_k + d} < \frac{1}{s_k}$$

for all k, which in turn implies that

$$\sum_{k=1}^{n} \frac{1}{s_k + d} < \sum_{k=1}^{n} \frac{1}{s_k},$$

which contradicts (2).

Let j be the number of ks such that $s_k(s_k+d) < 0$, that is, $s_k < 0 < s_k+d$. A simple but critical fact is that $s_k + d$ and s_{k+j} have the same sign. In fact, suppose that

$$s_1 < \cdots < s_i < s_{i+1} < \cdots < s_{i+j} < 0 < s_{i+j+1} < \cdots < s_n;$$

then

$$s_1 + d < \cdots < s_i + d < 0 < s_{i+1} + d < \cdots < s_n + d.$$

Then $s_{k+j} > 0$ if and only if $k > i + 1$, that is $s_k + d > 0$.

From (1), we obtain $s_k + d \le s_{k+j}$, and, since $s_k + d$ and s_{k+j} have the same sign, we obtain

$$\frac{1}{s_k + d} \ge \frac{1}{s_{k+j}},$$

for all $k = 1, 2, \cdots, n - j$. Therefore,

$$\sum_{k=1}^{n-j} \frac{1}{s_{k+j}} \le \sum_{k=1}^{n-j} \frac{1}{s_k + d},$$

or

$$\sum_{k=j+1}^{n} \frac{1}{s_k} \le \sum_{k=1}^{n-j} \frac{1}{s_k + d}. \tag{3}$$

Also note that

$$\sum_{k=1}^{j} \frac{1}{s_k} < 0 < \sum_{k=n-j+1}^{n} \frac{1}{s_k + d}. \tag{4}$$

Adding (3) and (4) yields

$$\sum_{k=1}^{n} \frac{1}{s_k} < \sum_{k=1}^{n} \frac{1}{s_k + d},$$

which contradicts (2).

Thus our assumption is wrong and our proof is complete.

Problem 98 [Romania 1998]

Show that for any positive integer n, the polynomial

$$f(x) = (x^2 + x)^{2^n} + 1$$

cannot be written as the product of two non-constant polynomials with integer coefficients.

Solution 98

Note that $f(x) = g(h(x))$, where $h(x) = x^2 + x$ and $g(y) = y^{2^n} + 1$.
Since

$$g(y+1) = (y+1)^{2^n} + 1 = y^{2^n} + \left(\sum_{k=1}^{2^n-1} \binom{2^n}{k} y^k \right) + 2,$$

and $\binom{2^n}{k}$ is even for $1 \le k \le 2^n - 1$, g is irreducible, by Eisenstein's criterion.

Now let p be a non-constant factor of f, and let r be a root of p.
Then $g(h(r)) = f(r) = 0$, so $s := h(r)$ is a root of g.
Since $s = r^2 + r \in \mathbb{Q}(r)$, we have $\mathbb{Q}(s) \subset \mathbb{Q}(r)$, so

$$\deg p \ge \deg(\mathbb{Q}(r)/\mathbb{Q}) \ge \deg(\mathbb{Q}(s)/\mathbb{Q}) = \deg g = 2^n.$$

Thus every factor of f has degree at least 2^n.
Therefore, if f is reducible, we can write $f(x) = A(x)B(x)$ where A and B have degree 2^n.
Next, observe that

$$\begin{aligned} f(x) &\equiv (x^2 + x)^{2^n} + 1 \\ &\equiv x^{2^{n+1}} + x^{2^n} + 1 \equiv (x^2 + x + 1)^{2^n} \pmod{2}. \end{aligned}$$

Since $x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$, by unique factorization we must have

$$A(x) \equiv B(x) \equiv (x^2 + x + 1)^{2^{n-1}} \equiv x^{2^n} + x^{2^{n-1}} + 1 \pmod{2}.$$

Thus, if we write

$$\begin{aligned} A(x) &= a_{2^n} x^{2^n} + \cdots + a_0, \\ B(x) &= b_{2^n} x^{2^n} + \cdots + b_0, \end{aligned}$$

then a_{2^n}, $a_{2^{n-1}}$, a_0, b_{2^n}, $b_{2^{n-1}}$, b_0 are odd and all the other coefficients are even. Since f is monic, we may assume without loss of generality that $a_{2^n} = b_{2^n} = 1$; also, $a_0 b_0 = f(0) = 1$, but $a_0 > 0$, $b_0 > 0$ as f has no real roots, so $a_0 = b_0 = 1$.

Therefore,

$$\begin{aligned}
&([x^{2^n+2^{n-1}}] + [x^{2^{n-1}}])(g(x)h(x)) \\
&\equiv \left(\sum_{i=2^{n-1}}^{2^n} a_i b_{2^n+2^{n-1}-i}\right) + \left(\sum_{i=0}^{2^{n-1}} a_i b_{2^{n-1}-i}\right) \\
&\equiv a_{2^n} b_{2^{n-1}} + a_{2^{n-1}} b_{2^n} + a_0 b_{2^{n-1}} + a_{2^{n-1}} b_0 \\
&\equiv 2(a_{2^{n-1}} + b_{2^{n-1}}) \equiv 0 \pmod 4
\end{aligned}$$

as $a_{2^{n-1}} + b_{2^{n-1}}$ is even.

But

$$([x^{2^n+2^{n-1}}] + [x^{2^{n-1}}])(f(x)) = \binom{2^n}{2^{n-1}} = 2\binom{2^n - 1}{2^{n-1} - 1},$$

and $\binom{2^n-1}{2^{n-1}-1}$ is odd by **Lucas's theorem**, so

$$\left([x^{2^n+2^{n-1}}] + [x^{2^{n-1}}]\right)\left(f(x)\right) \equiv 2 \pmod 4,$$

a contradiction.

Hence f is irreducible.

Problem 99 [Iran 1998]

Let $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ be functions such that

$$a_1 f_1 + a_2 f_2 + a_3 f_3$$

is monotonic for all $a_1, a_2, a_3 \in \mathbb{R}$.

Prove that there exist $c_1, c_2, c_3 \in \mathbb{R}$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

for all $x \in \mathbb{R}$.

Solution 99, Alternative 1

We establish the following lemma.

Lemma: Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions such that f is nonconstant and $af + bg$ is monotonic for all $a, b \in \mathbb{R}$. Then there exists $c \in \mathbb{R}$ such that $g - cf$ is a constant function.

Proof. Let s, t be two real numbers such that $f(s) \neq f(t)$.

Let
$$u = \frac{g(s) - g(t)}{f(s) - f(t)}.$$
Let $h_1 = g - d_1 f$ for some $d_1 \in \mathbb{R}$.

Then h_1 is monotonic. But
$$h_1(s) - h_1(t) = g(s) - g(t) - d_1(f(s) - f(t)) = (f(s) - f(t))(u - d_1).$$
Since $f(s) - f(t) \neq 0$ is fixed, the monotonicity of h_1 depends only on the sign of $u - d_1$.

Since f is nonconstant, there exist $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) \neq f(x_2)$. Let
$$\mathrm{c} = \frac{g(x_1) - g(x_2)}{f(x_1) - f(x_2)}$$
and $h = g - \mathrm{c}f$.

Then $r = h(x_1) = h(x_2)$ and the monotonicity of $h_1 = g - d_1 f$, for each d_1, depends only on the sign of $\mathrm{c} - d_1$.

We claim that $h = g - \mathrm{c}f$ is a constant function.

We prove our claim by contradiction.

Suppose, on the contrary, that there exists $x_3 \in \mathbb{R}$ such that $h(x_3) \neq r$.

Since $f(x_1) \neq f(x_2)$, at least one of $f(x_1) \neq f(x_3)$ and $f(x_2) \neq f(x_3)$ is true.

Without loss of generality, suppose that $f(x_1) \neq f(x_3)$.

Let
$$\mathrm{c}' = \frac{g(x_1) - g(x_3)}{f(x_1) - f(x_3)}.$$
Then the monotonicity of h_1 also depends only on the sign of $\mathrm{c}' - d_1$.

Since $h(x_3) \neq r = h(x_1)$,
$$\mathrm{c} \neq \frac{g(x_1) - g(x_3)}{f(x_1) - f(x_3)} = \mathrm{c}';$$
hence $\mathrm{c} - d_1 \neq \mathrm{c}' - d_1$.

So there exists some d_1 such that h_1 is both strictly increasing and decreasing, which is impossible.

Therefore our assumption is false and h is a constant function. □

Now we prove our main result.

If f_1, f_2, f_3 are all constant functions, the result is trivial.

Without loss of generality, suppose that f_1 is nonconstant.

For $a_3 = 0$, we apply the lemma to f_1 and f_2, so $f_2 = cf_1 + d$; for $a_2 = 0$, we apply the lemma to f_1 and f_3, so $f_3 = c'f_1 + d'$.

Here c, c', d, d' are constant.

We have

$$(c'd - cd')f_1 + d'f_2 - df_3 = (c'd - cd')f_1 + d'(cf_1 + d) - d(c'f_1 + d') = 0.$$

If $(c'd - cd', d', -d) \neq (0,0,0)$, then let

$$(c_1, c_2, c_3) = (c'd - cd', d', -d)$$

and we are done.

Otherwise, $d = d' = 0$ and f_2, f_3 are constant multiples of f_1.

Then the problem is again trivial.

Solution 99, Alternative 2

Define the vector

$$v(x) = (f_1(x), f_2(x), f_3(x))$$

for $x \in \mathbb{R}$.

If the $v(x)$ span a proper subspace of $\mathbb{R}^3$, we can find a vector (c_1, c_2, c_3) orthogonal to that subspace, and then $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all $x \in \mathbb{R}$.

So suppose the $v(x)$ span all of $\mathbb{R}^3$.

Then there exist $x_1 < x_2 < x_3 \in \mathbb{R}$ such that $v(x_1)$, $v(x_2)$, $v(x_3)$ are linearly independent, and so the 3×3 matrix A with $A_{ij} = f_j(x_i)$ has linearly independent rows.

But then A is invertible, and its columns also span $\mathbb{R}^3$.

This means we can find c_1, c_2, c_3 such that

$$\sum_{i=1}^{3} c_i(f_i(x_1), f_i(x_2), f_i(x_3)) = (0,1,0),$$

and the function $c_1 f_1 + c_2 f_2 + c_3 f_3$ is then not monotonic, a contradiction.

Problem 100 [USAMO 1999 proposal, Richard Stong]

Let $x_1, x_2, \ldots, x_n$ be variables, and let $y_1, y_2, \ldots, y_{2^n-1}$ be the sums of nonempty subsets of x_i.

Let $p_k(x_1, \ldots, x_n)$ be the k^{th} **elementary symmetric polynomial** in the y_i (the sum of every product of k distinct y_i's).

For which k and n is every coefficient of p_k (as a polynomial in $x_1, \ldots, x_n$) even?

For example, if $n = 2$, then y_1, y_2, y_3 are $x_1, x_2, x_1 + x_2$ and

$$\begin{aligned}
p_1 &= y_1 + y_2 + y_3 = 2x_1 + 2x_2, \\
p_2 &= y_1y_2 + y_2y_3 + y_3y_1 = x_1^2 + x_2^2 + 3x_1x_2, \\
p_3 &= y_1y_2y_3 = x_1^2x_2 + x_1x_2^2.
\end{aligned}$$

Solution 100

We say a polynomial p_k is *even* if every coefficient of p_k is even.

Otherwise, we say p_k is *not even.*

For any fixed positive integer n, we say a nonnegative integer k is *bad for n* if $k = 2^n - 2^j$ for some nonnegative integer j.

We will show by induction on n that $p_k(x_1, x_2, \cdots, x_n)$ is not even if and only if k is bad for n.

For $n = 1$, $p_1(x_1) = x_1$ is not even and $k = 1$ is bad for $n = 1$ as $k = 1 = 2^1 - 2^0 = 2^n - 2^0$.

Suppose that the claim is true for a certain n.

We now consider $p_k(x_1, x_2, \ldots, x_{n+1})$.

Let $\sigma_k(y_1, y_2, \ldots, y_s)$ be the k^{th} elementary symmetric polynomial.

We have the following useful, but easy to prove, facts:

1. $\sigma_k(y_1, y_2, \cdots, y_s, 0) = \sigma_k(y_1, y_2, \cdots, y_s)$;

2. For all $1 \le r \le s$,
$$\sigma_k(y_1, \ldots, y_s) = \sum_{i+j=k} [\sigma_i(y_1, \cdots, y_r)\sigma_j(y_{r+1}, \cdots, y_s)];$$

3. $$\begin{aligned}
&\sigma_k(x + y_1, x + y_2, \ldots, x + y_s) \\
&= \sum_{i_1<i_2<\cdots<i_k} (x + y_{i_1})(x + y_{i_2}) \cdots (x + y_{i_k}) \\
&= \sum_{i_1<i_2<\cdots<i_k} \sum_{r=0}^{k} \sum_{\substack{s_1<s_2<\cdots<s_r \\ \{s_1,\cdots,s_r\}\subseteq\{i_1,\cdots,i_k\}}} y_{s_1}y_{s_2}\cdots y_{s_r}x^{k-r}
\end{aligned}$$

$$= \sum_{r=0}^{k} \binom{s-r}{k-r} \sigma_r(y_1, \cdots, y_s) x^{k-r}.$$

Hence

$$\begin{aligned}
&p_k(x_1, x_2, \cdots, x_{n+1}) \\
&= \sum_{i+j=k} [p_i(x_1, \cdots, x_n) \cdot \\
&\qquad \sigma_j(x_{n+1}, x_1 + x_{n+1}, \cdots, x_1 + x_2 + \cdots + x_{n+1})] \\
&= \sum_{i+j=k} \sum_{r=0}^{j} \binom{2^n - r}{j - r} p_i(x_1, \cdots, x_n) p_r(x_1, \cdots, x_n) x_{n+1}^{j-r}.
\end{aligned}$$

By the induction hypothesis, every term of $p_r(x_1, x_2 \cdots, x_n)$ is even unless $r = 2^n - 2^t$, for some $0 \le t \le n$.

For such r, note that

$$\binom{2^n - r}{j - r} = \binom{2^t}{j - r}$$

is even unless $j - r = 0$ or $j - r = 2^t$.

Therefore, taking coefficients modulo 2,

$$\begin{aligned}
&p_k(x_1, x_2, \cdots, x_{n+1}) \\
&\equiv \sum_{i+j=k} p_i(x_1, x_2, \cdots, x_n) p_j(x_1, x_2, \cdots x_n) \\
&\quad + \sum_{t=0}^{n} p_{k-2^n}(x_1, x_2, \cdots, x_n) p_{2^n - 2^t}(x_1, x_2, \cdots, x_n) x^{2^t}.
\end{aligned}$$

By the induction hypothesis, the terms in the first sum are even unless $k - 2^n = 2^n - 2^u$ for some $0 \le u \le n$, that is $k = 2^{n+1} - 2^u$.

In the second sum, every term appears twice except the term

$$p_{k/2}(x_1, x_2, \cdots, x_n)^2,$$

for k even.

By the induction hypothesis, this term is even unless $k/2 = 2^n - 2^v$, for some $0 \le v \le n$, that is $k = 2^{n+1} - 2^{v+1}$.

It follows that $p_k(x_1, x_2, \cdots x_{n+1})$ is even unless $k = 2^{n+1} - 2^w$ for some $0 \le w \le n + 1$, i.e., k is bad for $n + 1$:

Furthermore, note that the odd coefficients in

$$p_k(x_1, x_2, \cdots, x_{n+1})$$

occur for different powers of x_{n+1}.

Therefore, the condition that k is bad for $n+1$ is also sufficient for

$$p_k(x_1, x_2, \cdots, x_{n+1})$$

to be odd.

Our induction is complete.

Problem 101 [Russia 2000]

Prove that there exist 10 distinct real numbers $a_1, a_2, \ldots, a_{10}$ such that the equation

$$(x-a_1)(x-a_2)\cdots(x-a_{10}) = (x+a_1)(x+a_2)\cdots(x+a_{10})$$

has exactly 5 different real roots.

Solution 101

We show that $\{a_1, a_2, \ldots, a_{10}\} = \{7, 6, \ldots, -2\}$ is a group of numbers satisfying the conditions given in the problem.

The given equality becomes

$$(x-2)(x-1)x(x+1)(x+2)g(x^2) = 0,$$

where

$$\begin{aligned} g(u) \quad &= \quad 2[((7+6+\cdots+3)u^2 + \\ &\qquad (7\cdot 6\cdot 5 + 7\cdot 6\cdot 4 + \cdots + 5\cdot 4\cdot 3)u + 7\cdot 6\cdots 3]. \end{aligned}$$

If $g(u) =$ has no real solutions, then $g(x^2) = 0$ has no real solutions.

If u_1 and u_2 are real solutions of $g(u) = 0$, then $u_1 + u_2 < 0$ and $u_1 u_2 > 0$, that is, both u_1 and u_2 are negative.

It follows again that $g(x^2) = 0$ has no real solutions.

Our proof is complete.

GLOSSARY

Arithmetic-Geometric Mean Inequality (AM–GM Inequality)

If $a_1, a_2, \ldots, a_n$ are n nonnegative numbers, then

$$\frac{1}{n}(a_1 + a_2 + \cdots + a_n) \geq (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Binomial Coefficient

The coefficient of x^k in the expansion of $(x+1)^n$ is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Cauchy-Schwarz Inequality

For any real numbers $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

with equality if and only if a_i and b_i are proportional, $i = 1, 2, \ldots, n$.

Chebyshev Inequality

1. Let $x_1, x_2 \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be two sequences of real numbers, such that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$. Then

$$\frac{1}{n}(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) \leq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

2. Let $x_1, x_2 \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be two sequences of real numbers, such that $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$. Then

$$\frac{1}{n}(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) \geq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

De Moivre's Formula

For any angle α and for any integer n,

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha.$$

Elementary Symmetric Polynomials (Functions)

Given indeterminates x_1, ..., x_n, the elementary symmetric functions $s_1, \ldots, s_n$ are defined by the relation (in another indeterminate t)

$$(t + x_1)\cdot \cdot (t + x_n) = t^n + s_1 t^{n-1} + \cdots + s_{n-1}t + s_n.$$

That is, s_k is the sum of the products of the x_i taken k at a time. It is a basic result that every symmetric polynomial in $x_1, \ldots, x_n$ can be (uniquely) expressed as a polynomial in the s_i, and vice versa.

Fibonacci Numbers

Sequence defined recursively by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, for all $n \in \mathbb{N}$.

Jensen's Inequality

If f is concave up on an interval $[a, b]$ and λ_1, λ_2, ..., λ_n are nonnegative numbers with sum equal to 1, then

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)$$

for any $x_1, x_2, \ldots, x_n$ in the interval $[a, b]$. If the function is concave down, the inequality is reversed.

Lagrange's Interpolation Formula

Let $x_0, x_1, \ldots, x_n$ be distinct real numbers, and let $y_0, y_1, \ldots, y_n$ be arbitrary real numbers. Then there exists a unique polynomial $P(x)$ of degree at most n such that $P(x_i) = y_i$, $i = 0, 1, \ldots, n$. This is the polynomial given by

$$P(x) = \sum_{i=0}^{n} y_i \frac{(x - x_0)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}.$$

Law of Cosines

Let ABC be a triangle. Then

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC\cos A.$$

Lucas' Theorem

Let p be a prime; let a and b be two positive integers such that

$$a = a_k p^k + a_{k-1} p^{k-1} + \cdots a_1 p + a_0, b = b_k p^k + b_{k-1} p^{k-1} + \cdots b_1 p + b_0,$$

where $0 \le a_i, b_i < p$ are integers for $i = 0, 1, \ldots, k$. Then

$$\binom{a}{b} \equiv \binom{a_k}{b_k} \binom{a_{k-1}}{b_{k-1}} \cdots \binom{a_1}{b_1} \binom{a_0}{b_0} \pmod{p}.$$

Pigeonhole Principle

If n objects are distributed among $k < n$ boxes, some box contains at least two objects.

Root Mean Square-Arithmetic Mean Inequality (RMS–AM Inequality)

For positive numbers $x_1, x_2, \ldots, x_n$,

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_k^2}{n}} \ge \frac{x_1 + x_2 + \cdots + x_k}{n}.$$

More generally, let $a_1, a_2, \ldots, a_n$ be any positive numbers for which $a_1 + a_2 + \cdots + a_n = 1$. For positive numbers $x_1, x_2, \ldots, x_n$ we define

$$\begin{aligned}
M_{-\infty} &= \min\{x_1, x_2, \ldots, x_k\}, \\
M_{\infty} &= \max\{x_1, x_2, \ldots, x_k\}, \\
M_0 &= x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \\
M_t &= (a_1 x_1^t + a_2 x_2^t + \cdots + a_k x_k^t)^{1/t},
\end{aligned}$$

where t is a non-zero real number. Then

$$M_{-\infty} \le M_s \le M_t \le M_{\infty}$$

for $s \le t$.

Triangle Inequality

Let $z = a + bi$ be a complex number. Define the absolute value of z to be

$$|z| = \sqrt{a^2 + b^2}.$$

Let α and β be two complex numbers. The inequality

$$|\alpha + \beta| \leq |\alpha| + |\beta|$$

is called the triangle inequality.

Let $\alpha = \alpha_1 + \alpha_2 i$ and $\beta = \beta_1 + \beta_2 i$, where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers. Then $\alpha + \beta = (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2)i$.

Vectors $\mathbf{u} = [\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}]$, $\mathbf{v} = [\boldsymbol{\beta_1}, \boldsymbol{\beta_2}]$, and $\mathbf{w} = [\boldsymbol{\alpha_1} + \boldsymbol{\beta_1}, \boldsymbol{\alpha_2} + \boldsymbol{\beta_2}]$ form a triangle with sides lengths $|\alpha|, |\beta|$, and $|\alpha + \beta|$.

The triangle inequality restates the fact that the length of any side of a triangle is less than the sum of the lengths of the other two sides.

Vieta's Theorem

Let $x_1, x_2, \ldots, x_n$ be the roots of polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

where $a_n \neq 0$ and $a_0, a_1, \ldots, a_n \in \mathbb{C}$. Let s_k be the sum of the products of the x_i taken k at a time. Then

$$s_k = (-1)^k \frac{a_{n-k}}{a_n},$$

that is,

$$x_1 + x_2 + \cdots + x_n = -\frac{a_{n-1}}{a_n};$$

$$x_1 x_2 + \cdots + x_i x_j + x_{n-1} x_n = \frac{a_{n-2}}{a_n};$$

$$\cdots$$

$$x_1 x_2 \cdots x_n = (-1)^n \frac{a_0}{a_n}.$$

Trigonometric Identities

$$\sin^2 a + \cos^2 a = 1,$$

$$\tan x = \frac{\sin a}{\cos a},$$

$$\cot x = \frac{1}{\tan a}$$

addition and subtraction formulas:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$$
$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b};$$

double-angle formulas:

$$\sin 2a = 2 \sin a \cos a,$$
$$\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a,$$
$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a},$$

triple-angle formulas:

$$\sin 3a = 3 \sin a - 4 \sin^3 a,$$
$$\cos 3a = 4 \cos^3 a - 3 \cos a,$$
$$\tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a};$$

half-angle formulas:

$$\sin a = \frac{2 \tan \frac{a}{2}}{1 + \tan^2 \frac{a}{2}},$$
$$\cos a = \frac{1 - \tan^2 \frac{a}{2}}{1 + \tan^2 \frac{a}{2}},$$
$$\tan a = \frac{2 \tan \frac{a}{2}}{1 - \tan^2 \frac{a}{2}};$$

sum-to-product formulas:

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2},$$
$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2},$$
$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a \cos b};$$

difference-to-product formulas:

$$\sin a - \sin b = 2 \sin \frac{a-b}{2} \cos \frac{a+b}{2},$$

$$\cos a - \cos b = -2\sin\frac{a-b}{2}\sin\frac{a+b}{2},$$
$$\tan a - \tan b = \frac{\sin(a-b)}{\cos a\cos b};$$

product-to-sum formulas:

$$2\sin a\cos b = \sin(a+b) + \sin(a-b),$$
$$2\cos a\cos b = \cos(a+b) + \cos(a-b),$$
$$2\sin a\sin b = -\cos(a+b) + \cos(a-b).$$

FURTHER READING

1. Andreescu, T. Kedlaya, K.; Zeitz, P., *Mathematical Contests 1995-1996: Olympiad Problems from around the World, with Solutions*, American Mathematics Competitions, 1997.

2. Andreescu, T. Kedlaya, K., *Mathematical Contests 1996-1997: Olympiad Problems from around the World, with Solutions*, American Mathematics Competitions, 1998.

3. Andreescu, T. Kedlaya, K., *Mathematical Contests 1997-1998: Olympiad Problems from around the World, with Solutions*, American Mathematics Competitions, 1999.

4. Andreescu, T. Feng, Z., *Mathematical Olympiads: Problems and Solutions from around the World, 1998-1999*, Mathematical Association of America, 2000.

5. Andreescu, T. Gelca, R., *Mathematical Olympiad Challenges*, Birkhäuser, 2000.

6. Barbeau, E., *Polynomials*, Springer-Verlag, 1989.

7. Beckenbach, E. F., Bellman, R., *An Introduction to Inequalities*, New Mathematical Library, Vol. 3, Mathematical Association of America, 1961.

8. Chinn, W. G., Steenrod, N. E., *First Concepts of Topology*, New Mathematical Library, Vol. 27, Random House, 1966.

9. Cofman, J., *What to Solve?*, Oxford Science Publications, 1990.

10. Coxeter, H. S. M., Greitzer, S. L., *Geometry Revisited*, New Mathematical Library, Vol. 19, Mathematical Association of America, 1967.

11. Doob, M., *The Canadian Mathematical Olympiad 1969-1993*, University of Toronto Press, 1993.

12. Engel, A., *Problem-Solving Strategies*, Problem Books in Mathematics, Springer, 1998.

13. Fomin, D., Kirichenko, A., *Leningrad Mathematical Olympiads 1987-1991*, MathPro Press, 1994.

14. Fomin, D., Genkin, S., Itenberg, I., *Mathematical Circles*, American Mathematical Society, 1996.

15. Graham, R. L., Knuth, D. E., Patashnik, O., *Concrete Mathematics*, Addison-Wesley, 1989.

16. Greitzer, S. L., *International Mathematical Olympiads, 1959-1977*, New Mathematical Library, Vol. 27, Mathematical Association of America, 1978.

17. Grossman, I., Magnus, W., *Groups and Their Graphs*, New Mathematical Library, Vol. 14, Mathematical Association of America, 1964.

18. Kazarinoff, N. D., *Geometric Inequalities*, New Mathematical Library, Vol. 4, Random House, 1961.

19. Klamkin, M., *International Mathematical Olympiads, 1978-1985*, New Mathematical Library, Vol. 31, Mathematical Association of America, 1986.

20. Klamkin, M., *USA Mathematical Olympiads, 1972-1986*, New Mathematical Library, Vol. 33, Mathematical Association of America, 1988.

21. Kürschák, J., *Hungarian Problem Book, Volumes I & II*, New Mathematical Library, Vols. 11 & 12, Mathematical Association of America, 1967.

22. Kuczma, M., *144 Problems of the Austrian-Polish Mathematics Competition 1978-1993*, The Academic Distribution Center, 1994.

23. Larson, L. C., *Problem-Solving Through Problems*, Springer-Verlag, 1983.

24. Lausch, H., Bosch Giral, C., *The Asian Pacific Mathematics Olympiad 1989-2000*, AMT Publishing, Canberra, 2001.

25. Liu, A., *Chinese Mathematics Competitions and Olympiads 1981-1993*, AMT Publishing, Canberra, 1998.

26. Lozansky, E., Rousseau, C. *Winning Solutions*, Springer, 1996.

27. Ore, O., *Graphs and their uses, Random House*, 1963.

28. Ore, O., *Invitation to number theory, Random House*, 1967.

29. Sharygin, I. F., *Problems in Plane Geometry*, Mir, Moscow, 1988.

30. Sharygin, I. F., *Problems in Solid Geometry*, Mir, Moscow, 1986.

31. Shklarsky, D. O, Chentzov, N. N; Yaglom, I. M., *The USSR Olympiad Problem Book*, Freeman, 1962.

32. Slinko, A., *USSR Mathematical Olympiads 1989-1992*, AMT Publishing, Canberra, 1997.

33. Soifer, A., *Colorado Mathematical Olympiad: The first ten years*, Center for excellence in mathematics education, 1994.

34. Szekely, G. J., *Contests in Higher Mathematics*, Springer- Verlag, 1996.

35. Stanley, R. P., *Enumerative Combinatorics*, Cambridge University Press, 1997.

36. Taylor, P. J., *Tournament of Towns 1980-1984*, AMT Publishing, Canberra, 1993.

37. Taylor, P. J., *Tournament of Towns 1984-1989*, AMT Publishing, Canberra, 1992.

38. Taylor, P. J., *Tournament of Towns 1989-1993*, AMT Publishing, Canberra, 1994.

39. Taylor, P. J., Storozhev, A., *Tournament of Towns 1993-1997*, AMT Publishing, Canberra, 1998.

40. Tomescu, I., *Problems in Combinatorics and Graph Theory*, Wiley, 1985.

41. Vanden Eynden, C., *Elementary Number Theory*, McGraw-Hill, 1987.

42. Wilf, H. S., *Generatingfunctionology*, Academic Press, 1994.

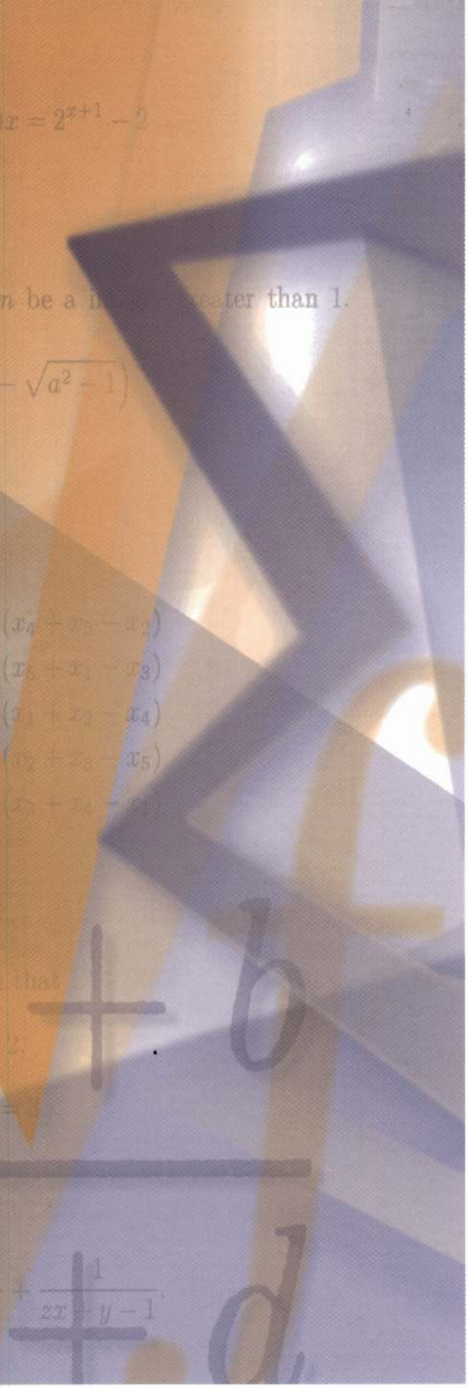
43. Wilson, R., *Introduction to graph theory*, Academic Press, 1972.

44. Yaglom , I. M., *Geometric Transformations*, New Mathematical Library, Vol. 8, Random House, 1962.

45. Yaglom , I. M., *Geometric Transformations II*, New Mathematical Library, Vol. 21, Random House, 1968.

46. Yaglom , I. M., *Geometric Transformations III*, New Mathematical Library, Vol. 24, Random House, 1973.

47. Zeitz, P., *The Art and Craft of Problem Solving*, John Wiley & Sons, 1999.



Titu Andreescu is Director of the American Mathematics Competitions, serves as Head Coach of the USA International Mathematical Olympiad (IMO) Team, is Chair of the USA Mathematical Olympiad Committee, and is Director of the USA Mathematical Olympiad Summer Program. Originally from Romania, Prof. Andreescu received the Distinguished Professor Award from the Romanian Ministry of Education in 1983, then after moving to the USA was awarded the Edyth May Sliffe Award for Distinguished High School Mathematics Teaching from the MAA in 1994. In addition, he received a Certificate of Appreciation presented by the President of the MAA for "his outstanding service as coach of the USA Mathematical Olympiad Program in preparing the USA team for its perfect performance in Hong Kong at the 1994 IMO".

Zuming Feng graduated with a PhD from Johns Hopkins University with an emphasis on Algebraic Number Theory and Elliptic Curves. He teaches at Phillips Exeter Academy. He also serves as a coach of the USA International Mathematical Olympiad (IMO) Team, is a member of the USA Mathematical Olympiad Committee, and is an assistant director of the USA Mathematical Olympiad Summer Program. He received the Edyth May Sliffe Award for Distinguished High School Mathematics Teaching from the MAA in 1996.

THE AUSTRALIAN MATHEMATICS TRUST

ENRICHMENT SERIES

ISBN 1 876420 12 X