11. The Series RLC Resonance Circuit

Introduction

Thus far we have studied a circuit involving a (1) series resistor R and capacitor C circuit as well as a (2) series resistor R and inductor L circuit. In both cases, it was simpler for the actual experiment to replace the battery and switch with a signal generator producing a square wave. The current through and voltage across the resistor and capacitor, and inductor in the circuit were calculated and measured.

This lab involves a resistor R, capacitor C, and inductor L all in series with a signal generator and this time is experimentally simpler to use a sine wave that a square wave. Also we will introduce the generalized resistance to AC signals called "impedance" for capacitors and inductors. The mathematical techniques will use simple properties of complex numbers which have real and imaginary parts. This will allow you to avoid solving differential equations resulting from the Kirchoff loop rule and instead you will be able to solve problems using a generalized Ohm's law. This is a significant improvement since Ohm's law is an algebraic equation which is much easier to solve than differential equation. Also we will find a new phenomena called "resonance" in the series RLC circuit.

Kirchoff's Loop Rule for a RLC Circuit

The voltage, V_L across an inductor, L is given by

$$V_L = L \frac{d}{dt} i[t] \tag{1}$$

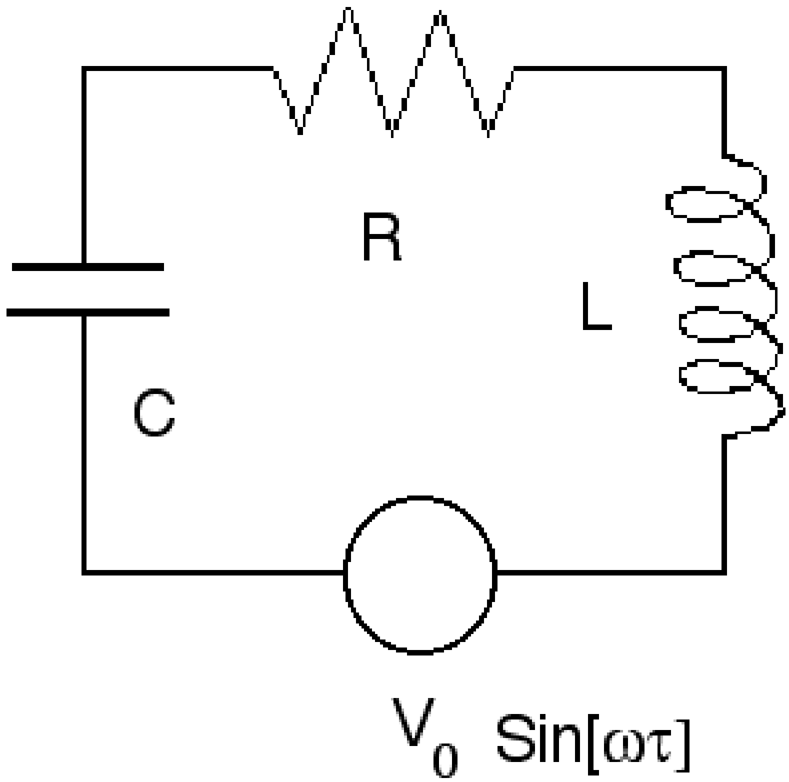
where i[t] is the current which depends upon time, t. The voltage across the capacitor C is

$$V_C = \frac{Q[t]}{C} \tag{2}$$

where the charge Q[t] depends upon time. Finally the voltage across the resistor is

$$V_R = i[t]\, R \tag{3}$$

The voltage produced by the signal generator is a function of time and at first we write the voltage of the signal generator as $V_0\,\mathrm{Sin}[\omega t]$ where $V_0$is the amplitude of the signal generator voltage and ω is the frequency of the signal generator voltage. What we actually have control over is the signal generator voltage frequency f measured in Hz and $\omega=2\pi f$ is the relationship between the two frequencies.



Combining equations (1) through (3) above together with the time varying signal generator we get Kirchoff's loop equation for a series RLC circuit.

$$L \frac{d}{dt} i[t] + \frac{Q[t]}{C} + i[t] R = V_0 \, Sin[\omega t] \qquad (4)$$

You can now take the time derivative of equation (4) and use the definition of current i[t]=dQ[t]/dt to get a linear, second order Inhomogeneous differential equation for the current i[t]

$$L \frac{d^2}{dt^2} i[t] + \frac{i[t]}{C} + R \frac{d}{dt} i[t] = V_0 \, \omega \, Cos[\omega t] \qquad (5)$$

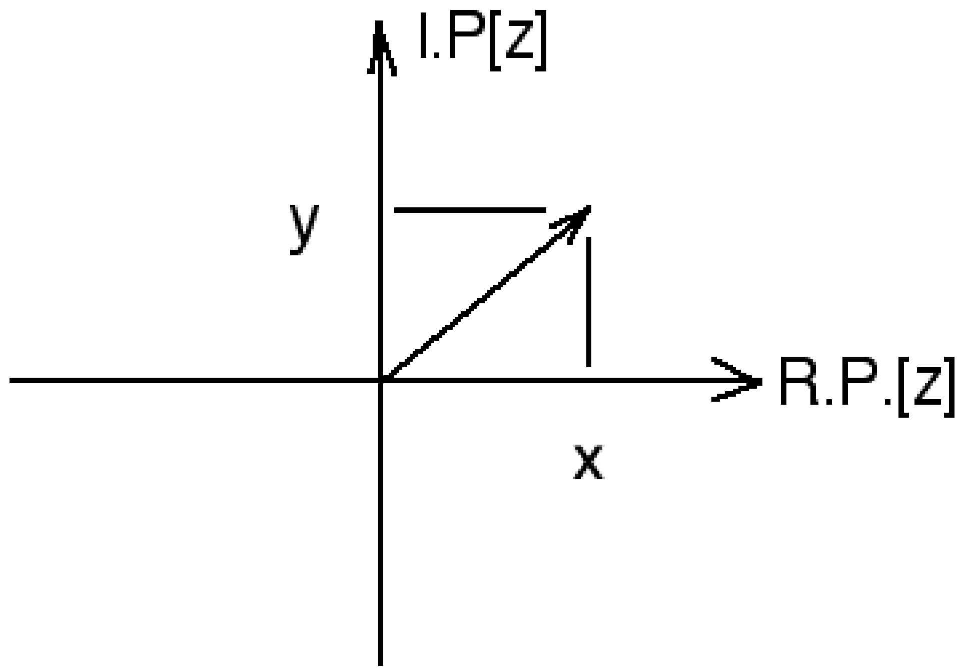
You can solve the differential equation (5) for the current using the techniques in previous labs (in fact equation (5) has the same for as the driven, damped harmonic oscillator). Equation (5) is a linear, second order, Inhomogeneous ordinary differential equation and it is a little complicated to solve. However it is simpler to solve electronics problems if you introduce a generalized resistance or "impedance" and this we do. When introduce complex numbers, the solution to circuits like the series RLC circuit become only slightly more complicated than solving Ohm's law. But first we must review some properties of complex numbers. This will take a little time but it is more than worth it.

Simple Properties of Complex Numbers

The complex number z can be written

$$z = x + i y \tag{6}$$

Note that the i in equation (6) is the imaginary number $i=\sqrt{-1}$ and e=2.7... is the natural number. Hopefully you can distinguish between the imaginary number i and the current i in the equations below. It might be helpful to think of complex numbers as vectors in a two dimensional vector space such that the horizontal component is the real part of the vector and the imaginary part of the vector is the varietal component.



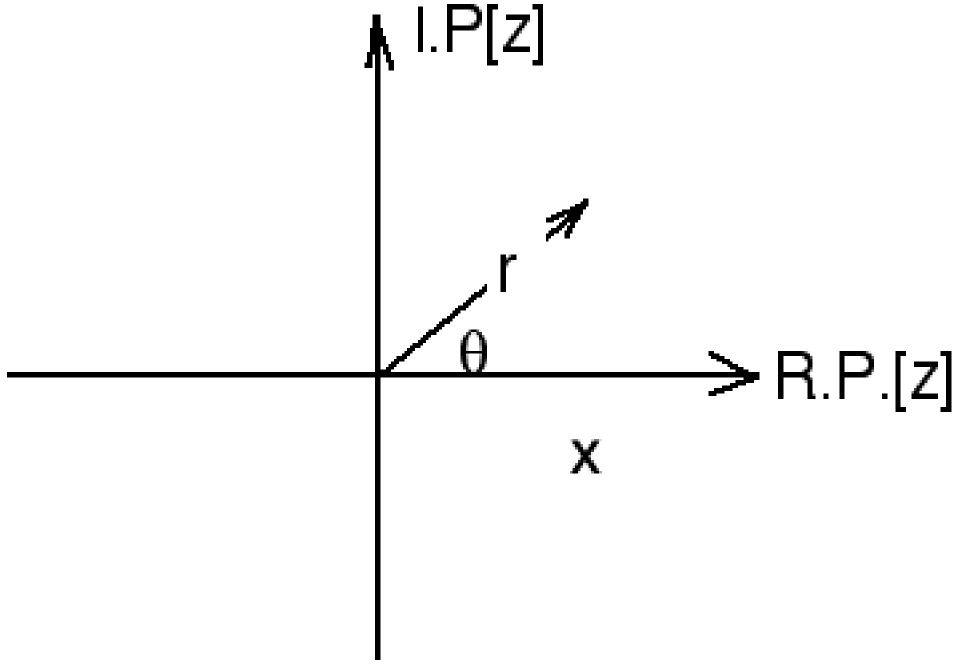
Sometimes we will write x=R.P.[z] by which we mean take the Real Part of the complex number z and we will also write y=I.P.[z]] by which we mean take the Imaginary Part of the complex number z. It might make complex numbers a little less mysterious by thinking of z as a vector in a two dimensional vector space.

The complex conjugate z^* of a complex number z is defined

$$z^* = x - i y \tag{7}$$

so z^* is the mirror image of z. Operationally if you have a complex number z you can construct the complex conjugate z^* by changing the sign of the imaginary part of z.

Sometimes it is convenient to write a complex number in a polar form having a radius component r and an angular position θ



The relationship between the rectangular components x and y and the polar coordinates r and θ is imply

$$x = r\,\mathrm{Cos}[\theta] \quad \text{and} \quad y = r\mathrm{Sin}[\theta] \tag{8}$$

that is, given r and θ you can calculated x and y using equations (3). Note from the Pythagorean theorem

$$r^2 = x^2 + y^2 \quad \text{or} \quad r = \sqrt{x^2 + y^2} \tag{9}$$

and

$$\mathrm{Tan}[\theta] = x / y \quad \text{or} \quad \theta = \mathrm{ArcTan}[x / y]\ . \tag{10}$$

The Euler Relationship

The Euler relation allows you to write $e^{i\phi}$ is a simple an useful form

$$e^{i\phi} = \mathrm{Cos}[\theta] + i\mathrm{Sin}[\theta] \tag{11}$$

At first this formula appears mysterious but it is easily proved using the Taylor series of $e^{i\theta}$ which is

$$e^{i\phi} = 1 + i\phi + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^3}{3!} + \frac{(i\phi)^4}{4!} + \frac{(i\phi)^5}{5!} + \frac{(i\phi)^6}{6!} + \ldots \tag{12}$$

and note that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, ... so tha pattern repeats every four terms. The expansion on the right hand side of equation (12) has odd power terms which are real and even power terms that are imaginary. Grouping the real terms together and the imaginary terms together you get

$$e^{i\phi} = \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + ...\right) + i\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + ...\right) \quad (13)$$

The group of terms in the first set of parenthesis on the right hand side equation (13) is the Taylor series expansion of Cos[ϕ] and the group of terms in the second set of parenthesis on the right hand side of equation (13) is the Taylor series expansion of Sin[ϕ]. Thus equation (11) is proved.

As a first use of the Euler relationship write

$$z = r e^{i\theta} \quad (14)$$

which becomes after using the Euler relation (11)

$$z = r\,(\mathrm{Cos}[\theta] + i\,\mathrm{Sin}[\theta]) \quad (15)$$

and thus after rearrangement

$$z = r\,\mathrm{Cos}[\theta] + i\,r\,\mathrm{Sin}[\theta]$$

Comparison of this equation and equation (6) yields

$$x = r\,\mathrm{Cos}[\theta] \quad \text{and} \quad y = r\,\mathrm{Sin}[\theta]$$

which we knew as equation (8). This should give you a little more confidence in the Euler relationship. These equations can also be used to write

$$\frac{y}{x} = \mathrm{Tan}[\theta] \quad \text{and thus} \quad \theta = \mathrm{ArcTan}[\theta] \quad (16)$$

r is sometimes called the "magnitude" of the complex number z and θ is called the "phase angle". Recall that the complex conjugate z^* of the complex number z is $z^* = x - i y$ and using equations (8)

$$z^* = r\,\mathrm{Cos}[\theta] - i\,r\,\mathrm{Sin}[\theta] \quad (17)$$

Furthermore since the Cos[θ] is an even function of θ we write Cos[θ]=Cos[-θ] and since Sin[θ] is an odd function of θ we may write Sin[θ] = -Sin[-θ] and equation (17) may be written

$$z^* = r\,\mathrm{Cos}[-\theta] + i\,r\,\mathrm{Sin}[-\theta] \quad (18)$$

and if you look at equation (11) or equation (14) it is clear equation (18) may also be written

$$z^* = r\,e^{-i\theta} \quad (19)$$

Thus the complex conjugate of z written in polar form is obtained by keeping r as it is and changing the sign in the exponent of equation (11). These are just about all the properties of complex numbers we need.

Calculations using Complex Numbers

We will need to add two complex numbers $z_1 = x_1 + i y_1$ and $z_2 = x_2 + i y_2$

$$z = z_1 + z_2 \tag{20}$$

but to do this you just as the real parts to get $x = x_1 + x_2$ and the imaginary parts to get $y = y_1 + y_2$. It should be obvious how you subtract one complex number from another.

Multiplication of two complex numbers is obtained easily as well

$$z_1\, z_2 = (x_1 + i\, y_1)\, (x_2 + i\, y_2) \tag{21}$$

The binomial on the right hand side of equation (21) when multiplied out results in four terms two of which are real and two of which are imaginary

$$z_1\, z_2 = (x_1\, x_2 - y_1\, y_2) + i\, (x_1\, y_2 + x_2\, y_1) \tag{22}$$

where we also used $i^2 = -1$. Note in particular if the two numbers are $z = x + i y$ and its complex conjugate $z^* = x - i y$ the imaginary part of the product $z\, z^*$ and we get a real number answer for the product

$$z\, z^* = \left(x^2 + y^2\right) = r^2 \quad \text{and} \quad r = \sqrt{x^2 + y^2} \tag{23}$$

where the last equality follows from equation (9). r obtained by taking the square root of equation (9) is sometime called the magnitude of the complex number or just "magnitude". A complex number can be also written

$$z = r\, e^{i \phi} \tag{24}$$

The multiplication of two numbers is much simpler in polar form (11). Let the two complex numbers be $z_1 = r_1\, e^{i \theta_1}$ and $z_2 = r_2\, e^{i \theta_2}$ so the product is

$$z_1\, z_2 = r_1\, e^{i \theta_1}\, r_2\, e^{i \theta_2} \tag{25}$$

and thus after rearrangement and using the property of multiplication of exponentials

$$z_1\, z_2 = r_1\, r_2\, e^{i\, (\theta_1 + \theta_2)} \tag{26}$$

You can also divide one complex number z_1 by another z_2. (Note that complex numbers are a little different from a two dimensional vector space since you cannot divide one vector by another but you can divide one complex number by another.) Division is most easily done in polar coordinates

$$\frac{z_1}{z_2} = \frac{r_1\, e^{i \theta_1}}{r_2\, e^{i \theta_2}} \tag{27}$$

The right side of equation (26) may be written

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \tag{28}$$

since a property of exponential allows you to write

$$\frac{1}{e^{i\theta}} = e^{-i\theta} \tag{29}$$

If you divide one complex number by another in rectangular coordinates then

$$\frac{z_1}{z_2} = \frac{(x_1 + i y_1)}{(x_2 + i y_2)} \tag{30}$$

The answer we want for the quotient is a real plus and imaginary number. We know from equation (23) we know that multiplying a number by its complex conjugate yields a real number. So it makes sence to multiply the denominator to equation (29) by its complex conjugate and if we do the same to the numerator we have not changed anything because this is just multiplying by one

$$\frac{z_1}{z_2} = \frac{(x_1 + i y_1)}{(x_2 + i y_2)} \frac{(x_2 - i y_2)}{(x_2 - i y_2)} = \frac{(x_1 x_2 + y_1 y_2) + i (y_1 x_2 - x_1 y_2)}{(x_2^2 + y_2^2)} \tag{31}$$

So we have achieved are goal of writing the quotient as a real number plus and imaginary number specifically

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2)}{(x_2^2 + y_2^2)} + i \frac{(y_1 x_2 - x_1 y_2)}{(x_2^2 + y_2^2)} \tag{32}$$

Solving the Series RLC Circuit with Complex Numbers

Suppose the signal generator voltage is a Sine function $V_s[t] = V_0 \text{Cos}[\omega t]$ where the amplitude V_0 is a real number. Using the Euler formula we also know that signal generator voltage can be written

$$V_s[t] = \text{R.P.}\left[V_0 e^{i\omega t}\right] \tag{33}$$

We want to solve for the current i[t] in the series RLC circuit and this current is the same everywhere in the circuit by conservation of charge. Also we expect the current to be a Sine function or Cosine function since the signal generator voltage is a Cosine function of time. Thus we guess

$$i[t] = i_0 e^{i\omega t} \tag{34}$$

where i_0 is the amplitude of the current and it is independent of time and i_0 possibly a complex number. At the end of the calculation we will take the real part of the current as our answer since we took the real part in equation (32).

Substitution of equation (33) into equation (1) which give the voltage across the inductor yields

$$V_L = L \frac{d}{dt} i[t] = i\omega L i_0 e^{i\omega t} \tag{35}$$

Equation (3) for the voltage across the resistor is easy to write with equation (33) for the current

$$V_R = R\, i[t] = R\, i_0\, e^{i\omega t} \tag{36}$$

Equation (2) for the voltage across the capacitor is a little more complicated since the current i[t] does not appear directly. But recall that

$$Q[t] = \int i[t]\, dt \tag{37}$$

so from equation (33) for the current we get

$$Q[t] = i_0 \int e^{i\omega t}\, dt = \frac{i_0}{i\omega} e^{i\omega t} \tag{38}$$

which can now be used in equation (2) to obtain

$$V_C = \frac{i_0\, e^{i\omega t}}{i\omega C} \tag{39}$$

Now we use equations (32), (33), (34) and (38) in the Kirchoff loop rule $V_R + V_L + V_C = V_s$ and obtain

$$R\, i_0\, e^{i\omega t} + i\omega L i_0\, e^{i\omega t} + \frac{i_0\, e^{i\omega t}}{i\omega C} = V_0\, e^{i\omega t} \tag{40}$$

which looks complicated but after simplifying by cancelling the exponential we get

$$\left(R + i\omega L + \frac{1}{i\omega C}\right) i_0 = V_0 \tag{41}$$

Notice this is just Ohm's law if we take R for the resistance of the resistor, $i\omega L$ as the generalized resistance of the inductor, $1 / i\omega C$ as the generalized resistance of the capacitor C. The generalized resistance of the inductor is called inductive reactance X_L and the generalized resistance of the capacitor is called capacitive reactance X_C. Also, the generalized resistance is called impedance Z. So we will write

$$Z_R = R \tag{42}$$

$$Z_L = i\omega L \tag{43}$$

$$Z_C = \frac{1}{i\omega C} \tag{44}$$

and Ohm's Law obtained from equation (40) and is just

$$(Z_R + Z_L + Z_C)\, i_0 = V \tag{45}$$

The total impedance Z_T is just

$$Z_T = Z_R + Z_L + Z_C = \left(R + i\omega L + \frac{1}{i\omega C}\right)$$

Note that

$$\frac{1}{i} = \frac{1}{i}\frac{i}{i} = \frac{i}{-1} = -i$$

so the total impedance can also be written

$$Z_T = Z_R + Z_L + Z_C = R + i\left(\omega L - \frac{1}{\omega C}\right) \tag{46}$$

$Z_T = |Z_T|\, e^{i\phi}$ where is the magnitude $|Z_T|$ and phase ϕ of the impedance and these are easily obtained from equation (46)

$$|Z_T| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \quad \text{and} \quad \phi = \text{ArcTan}\left[\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}\right] \tag{47}$$

We usually want use Ohm's law to find the current so solving (45) yields

$$i_0 = \frac{V_0}{|Z_T|\, e^{i\phi}} = |i_0|\, e^{-i\phi} \tag{48}$$

where the magnitude of the current $|i_0|$ is given by

$$|i_0| = \frac{V_0}{|Z_T|\, e^{i\phi}} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{49}$$

and equation (47) gives the phase ϕ. Remember the current has to be the same everywhere in the circuit due to conservation of charge. Equation (48) tells that the current is NOT in phase with the voltage of the signal generation V_s since this voltage has zero phase. Equation (48) tells that the current is in phase with the voltage across the resistor $V_R = i_0 R$ since

$$V_R = \frac{R V_0\, e^{-i\phi}}{|Z_T|} \tag{50}$$

The magnitude of the voltage across the resistor is $R V_0 / |Z_T|$. The voltage across the resistor either "lags" or "leads" the voltage of the signal generator depending on the sign of ϕ.

The voltage across the inductor L is given by equation (35) with (48) and neglecting the $e^{i\omega t}$ factor since it is unimportant here

$$V_L = i\omega L\, |i_0|\, e^{-i\phi} = \omega L\, |i_0|\, e^{-i(\phi - \pi/2)} \tag{51}$$

since the imaginary number $i = e^{i\pi/2}$ by the Euler formula. The voltage across the inductor has a phase of - $\pi/2$ or -90° relative the current in the inductor. The voltage across the capacitor is given by (49) with (48) and neglecting the $e^{i\omega t}$ factor since it is unimportant here right now

$$V_C = \frac{|i_0| e^{-i\phi}}{i\,\omega\,C} = -i\,\frac{|i_0| e^{-i\phi}}{\omega\,C} = \frac{|i_0| e^{-i(\phi+\pi/2)}}{\omega\,C} \tag{52}$$

since minus the imaginary number is also $-i = e^{-i\pi/2}$ by the Euler formula. The voltage across the capacitor has a phase of $+\pi/2$ or +90° relative the current in the capacitor.

The Resonance Phenomena for the Series RLC Circuit

The magnitude of the voltage across the resistor can be written using equation (49) for the current

$$|V_R| = \frac{V_0\,R}{\sqrt{R^2 + \left(\omega\,L - \frac{1}{\omega C}\right)^2}} \tag{53}$$

Suppose R=10kΩ=10000Ω, L= 6 mH=0.006 H, and C=25. $\mu\mu$F = $25 \times 10^{-12}\,F$ and assume the amplitude of the signal generator voltage is V_0=12 volts. (You values for R, L, and C as well as V_0 will be different in your experiment. Make sure you resonance freqency is accessible to both you signal generator and oscilloscope. Also, try to pick R, L, and C so that your resonance curve is "narrow".) Use equation (53) to graph the voltage across the resistor versus the signal generator frequency:

```
Clear[V0, R, L, C0];
R = 10 000.;
L = 0.006;
C0 = 25. * 10^-12;
V0 = 12.;
```

The graph of V_R versus ω has a peak when (ω L-$\frac{1}{\omega C}$) = 0 since under this condition the denominator of equation (53) is as small as possible and $V_R = V_0/R$. Solving (ω L-$\frac{1}{\omega C}$) = 0 for ω yields the resonance frequency ω_0 is given by

```
ω0 = 1/Sqrt[L * C0]
```

2.58199×10^6

The corresponding frequency f of Hz of the signal generator is

```
f = ω0 / (2 π)
```

410 936.

which is about 400 kHz. The period T of the oscilloscope must be in the region

```
T = 1 / f
```

2.43347×10^{-6}

or T=2.4 μsec. Equation (53) is input into *Mathematica* with

$$V[\omega\_] := \frac{V0 * R}{\sqrt{R^2 + \left(\omega * L - \frac{1}{\omega * C0}\right)^2}}$$

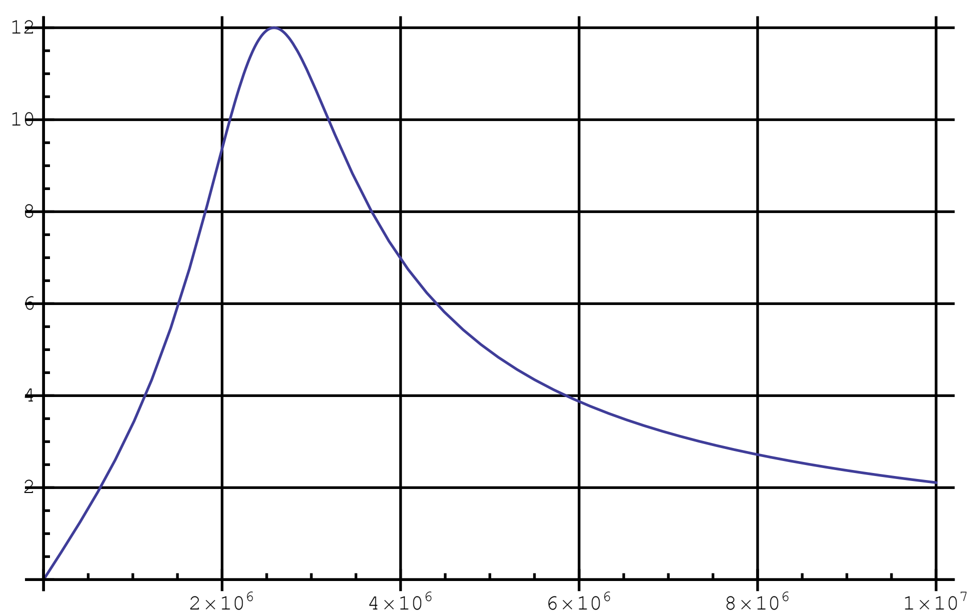
The voltage at the resonance frequency ω_0 is just about 12 volts as predicted

$V[2.5 * 10^6]$

11.9404

and the graph V[ω] versus ω is produced with

```
Plot[V[ω], {ω, 0, 10. * 10^6}, GridLines → Automatic]
```



which is a fairly narrow or sharply peaked graph. Notice the location of the peak is about 2.6×10^6 Rad/sec as predicted. This is a so-called "Resonance Curve" and note it is not symmetric. One experiment you will do is to take data of the voltage V_R across the resistor versus the frequency f of the signal generator and from this data you will construct a graph as above.

When taking data on the resonance curve above, change the frequency f or ω and then measure the voltage across the resistor. IMPORTANT: Each time you change the frequency ω of the signal generator, make sure you adjust the output amplitude of the signal generator so that the amplitude is the same (say 12. Volts) for all the frequency measurements. The reason for this is that the total impedance of the RLC circuit changes with frequency ω. The most efficient transfer of power from the signal generator to the RLC circuit occurs when the impedance of the RLC circuit equals the output impedance of the signal generator. The impedance of the RLC circuit changes with ω and so the "load" seen by the signal generator changes with frequency and the current changes as well. This phenomena is called "loading" of the signal generator.

The Effect of Changing C in the Resonance Frequency and on the Width of the Resonance Curve

The location of the peak ω_0 should decrease if we make C larger. For example, if 100 time larger than before that is C= 2500 $\mu\mu$F=2.5 * 10^{-9}F then the graph appears

```
Clear[V0, R, L, C0];
R = 10 000.;
L = 0.006;
C0 = 2500. * 10^-12;
V0 = 12.;
ω0 = 1/Sqrt[L * C0]
```

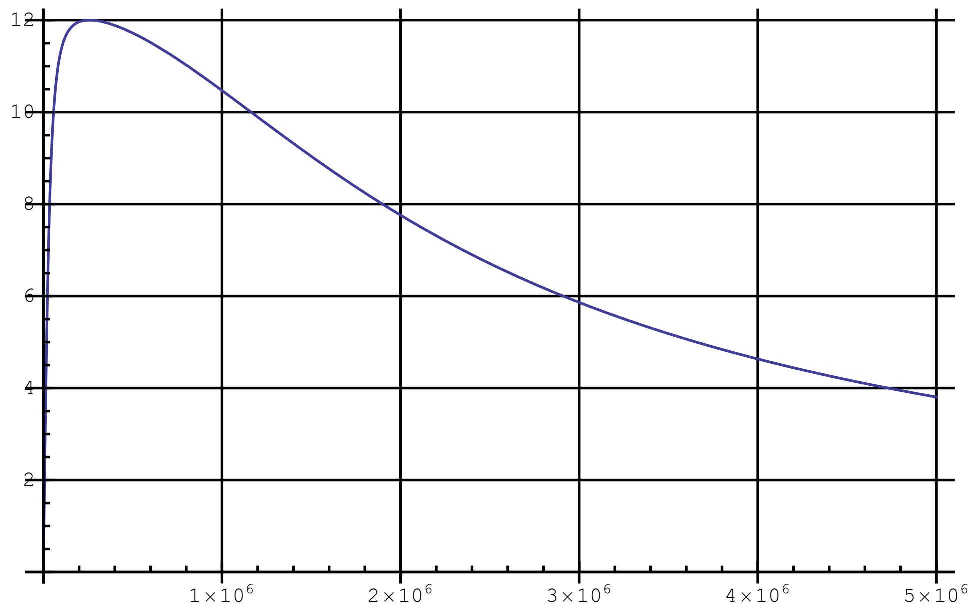
$$\omega 0 = \frac{1}{\sqrt{L * C0}}$$

```
258 199.
```

so the resonance frequency is about ω0=260,000=0.26×10^6 which is about 10 times smaller than before. This is born about by the new resonance curve:

$$V[\omega\_] := \frac{V0 * R}{\sqrt{R^2 + \left(\omega * L - \frac{1}{\omega * C0}\right)^2}}$$

```
Plot[V[ω], {ω, 0, 5.0 * 10^6}, GridLines → Automatic]
```



Notice location of the peak $\omega 0$ is at a frequency 1/10 the one before. Also notice that the "width" of the curve has increased as C0 was increased. We should try to understand this behavior.

The Location of the Peak of the Resonance Curve and the Width of the Resonance Curve

The peak of the resonance curve is obtained by taking the derivative of V_R=$V_R[\omega]$ (that is equation (53) with respect to ω and set the derivative to zero to get the location of the peak ω_0. You can use *Mathematica* to do this for you.

```
Clear[R, V0, C0, L, ω];
∂ω V[ω]
```

$$-\frac{R\,V0\left(L+\frac{1}{C0\,\omega^2}\right)\left(-\frac{1}{C0\,\omega}+L\,\omega\right)}{\left(R^2+\left(-\frac{1}{C0\,\omega}+L\,\omega\right)^2\right)^{3/2}}$$

Setting the derivative to zero, that is $(dV_R/d\omega) = 0$ and solving for the $\omega 0$ at the peak is done via

$$\text{Solve}\left[-\frac{R\,V0\left(L+\frac{1}{C0\,\omega^2}\right)\left(-\frac{1}{C0\,\omega}+L\,\omega\right)}{\left(R^2+\left(-\frac{1}{C0\,\omega}+L\,\omega\right)^2\right)^{3/2}}==0,\ \{\omega\}\right]$$

$$\left\{\left\{\omega\to-\frac{1}{\sqrt{C0}\,\sqrt{L}}\right\},\ \left\{\omega\to-\frac{i}{\sqrt{C0}\,\sqrt{L}}\right\},\ \left\{\omega\to\frac{i}{\sqrt{C0}\,\sqrt{L}}\right\},\ \left\{\omega\to\frac{1}{\sqrt{C0}\,\sqrt{L}}\right\}\right\}$$

There are actually four roots to the equation but one root is negative and two roots are imaginary so these roots are not physically attainable. The remaining root is $\omega=1/\sqrt{LC}$ and this is the same as what we got before for $\omega 0$ from more simplistic reasoning.

The Width of the Resonance Curve: You Might Want to Skip this Section at First

We also need the second derivative $d^2\,V_R[\omega]\big/d\omega^2$ evaluated at $\omega=\omega_0$ below so we do it now:

$$\partial_\omega\left(-\frac{R\,V0\left(L+\frac{1}{C0\,\omega^2}\right)\left(-\frac{1}{C0\,\omega}+L\,\omega\right)}{\left(R^2+\left(-\frac{1}{C0\,\omega}+L\,\omega\right)^2\right)^{3/2}}\right)$$

$$\frac{3\,R\,V0\left(L+\frac{1}{C0\,\omega^2}\right)^2\left(-\frac{1}{C0\,\omega}+L\,\omega\right)^2}{\left(R^2+\left(-\frac{1}{C0\,\omega}+L\,\omega\right)^2\right)^{5/2}}-\frac{R\,V0\left(L+\frac{1}{C0\,\omega^2}\right)^2}{\left(R^2+\left(-\frac{1}{C0\,\omega}+L\,\omega\right)^2\right)^{3/2}}+\frac{2\,R\,V0\left(-\frac{1}{C0\,\omega}+L\,\omega\right)}{C0\,\omega^3\left(R^2+\left(-\frac{1}{C0\,\omega}+L\,\omega\right)^2\right)^{3/2}}$$

which is quite complicated but we evaluate the second derivative at $\omega=\omega_0$ to get

```
% /. ω -> 1/Sqrt[L*C0]
```

$$\frac{12\,L^2\left(\frac{L}{\sqrt{C0\,L}}-\frac{\sqrt{C0\,L}}{C0}\right)^2 R\,V0}{\left(\left(\frac{L}{\sqrt{C0\,L}}-\frac{\sqrt{C0\,L}}{C0}\right)^2+R^2\right)^{5/2}}-\frac{4\,L^2\,R\,V0}{\left(\left(\frac{L}{\sqrt{C0\,L}}-\frac{\sqrt{C0\,L}}{C0}\right)^2+R^2\right)^{3/2}}+\frac{2\,(C0\,L)^{3/2}\left(\frac{L}{\sqrt{C0\,L}}-\frac{\sqrt{C0\,L}}{C0}\right) R\,V0}{C0\left(\left(\frac{L}{\sqrt{C0\,L}}-\frac{\sqrt{C0\,L}}{C0}\right)^2+R^2\right)^{3/2}}$$

which is still a bit of a mess but simplifying with algebra produces a simple result for the second derivative

```
Simplify[%]
```

$$-\frac{4\,L^2\sqrt{R^2}\;V0}{R^3}$$

(The second derivative is negative since the function is expanded about a MAXIMUM.) The reason we need the second derivative is that next we do a Taylor series of the voltage $V_R[\omega]$ about $\omega=\omega_0$ expanded to second order. The *Mathematica* **Series** function does this for us. Actually we write $\omega=\omega_0+\Delta\omega$ and expanded $\Delta\omega$ about $\Delta\omega=0$:

```
Clear[R];
Series[V_R[ω_0 + Δω], {Δω, 0, 2}]
```

$$V_R[\omega_0]+V_R'[\omega_0]\,\Delta\omega+\frac{1}{2}V_R''[\omega_0]\,\Delta\omega^2+O[\Delta\omega]^3$$

and again (1) $V_R[\omega_0] = V_0/R$ (2) $V_R'[\omega_0]=0$ at the peak and (3) we also calculated $V_R''[\omega_0]=-4L^2\,V_0/R^2$ above. So if you solve for $V_R[\omega]-V_R[\omega_0]$ you get

$$V_R[\omega]-V_R[\omega_0] = \frac{1}{2}V_R''[\omega_0]\,\Delta\omega^2 \tag{54}$$

Usually people want to calculate the width $\Delta\omega^2$ for which the voltage is 50% or 0.5 of its maximum value $V_R[\omega_0]=V_0/R$. Using equation (54) we get

$$\frac{1}{2} = \frac{V_R[\omega]-V_R[\omega_0]}{V_R[\omega_0]} = \frac{1}{2}\,\frac{V_R''[\omega_0]}{V_R[\omega_0]}\,\Delta\omega^2 \tag{55}$$

The graph has its peak when $\Delta\omega$=0 and the voltage there is V0/R. The width is controlled by $V_R''[\omega_0]$ that is the coefficient of the $\Delta\omega^2$.

$$\frac{1}{2} = \left(\frac{\frac{4 (L^2\ V0)}{R^2}}{\frac{V0}{R}}\right) \frac{\Delta\omega^2}{2}$$

thus we get

$$\Delta\omega^2 = \frac{R}{4\ L^2} \quad \text{or} \quad \Delta\omega = \frac{\sqrt{R}}{2\ L} \tag{56}$$

Since $\Delta\omega$ is a frequency we want to have some frequency to compare $\Delta\omega$ in order to decide if it is large of small. Thus we compute the relative width ($\Delta\omega$ relative $\omega_0 = 1\big/\sqrt{LC}$ the resonance frequency.

$$\frac{\Delta\omega}{\omega_0} = \frac{\frac{\sqrt{R}}{2\ L}}{\frac{1}{\sqrt{LC}}} = \sqrt{\frac{RC}{4\ L}} \tag{57}$$

Notice the relative width increases as C increases a result we obtained just by graphing the voltage $V_R[\omega]$ using numerical values of R, L, and C.

The Effect of Increasing R in the Resonance Frequency and on the Width of the Resonance Curve

The location of the peak ω_0 should not be changed if R is changed but the relative width should increase according to equation (57). For example, if R is 100 times larger than in the first example, that is R=1,000,000 Ω=1.0 MegΩ then the graph appears

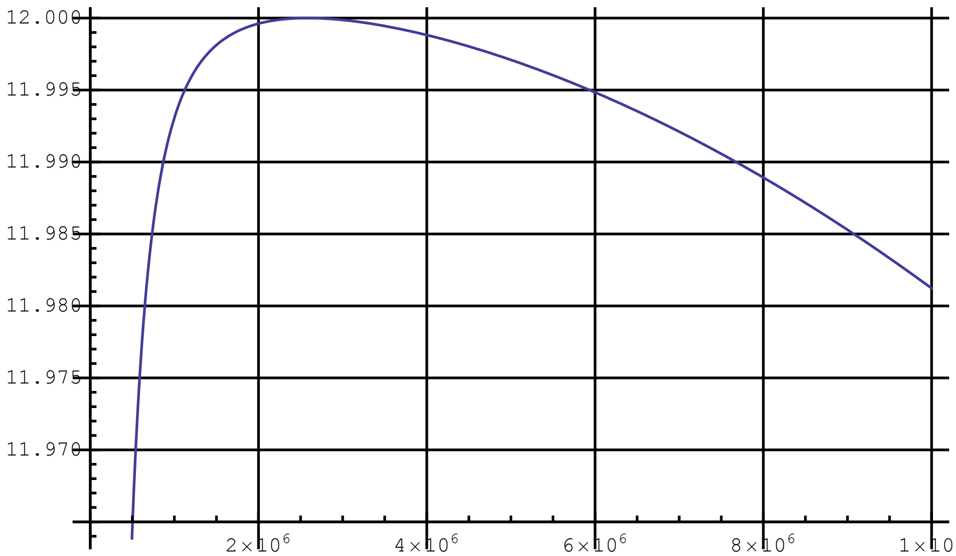
```
Clear[V0, R, L, C0];
R = 1 000 000.;
L = 0.006;
C0 = 25. * 10^-12;
V0 = 12.;
ω0 = 1/Sqrt[L * C0]
```

2.58199×10^6

so the resonance frequency is about ω0=2,600,000=2.6×10^6 which is the same as in the first example. This is born about by the new resonance curve:

$$V[\omega\_] := \frac{V0 * R}{\sqrt{R^2 + \left(\omega * L - \frac{1}{\omega * C0}\right)^2}}$$

```
Plot[V[ω], {ω, 0, 10.0 * 10^6}, GridLines → Automatic]
```



Notice location of the peak $\omega 0$ is as before. Also notice that the "width" of the curve has increased as R was increased. We now understand this behavior.

Laboratory Exercises

PART A: Place a signal generator in series with a resistor R, inductor L, and a capacitor C. Pretty much any output level (the output voltage) of the signal generator will do OK but after you get the oscilloscope working properly make a note of the maximum voltage in your lab notebook. Choose a **Sine wave** and make the frequency f of the signal generator such that $f=\frac{\omega_0}{2\pi}$ with $\omega_0=\frac{1}{\sqrt{LC}}$. With channel 1 of the oscilloscope, measure the voltage across the signal generator and with channel 2 measure the **voltage across the resistor** V_R. **Measure** V_R with the signal generator ω (actually f) at FIVE ω below the resonance frequency and FIVE ω above the resonance frequency. BEFORE you measure V_R MAKE SURE the voltage of the signal generator measured by Channel 1 of the oscilloscope is the same as V_R at the resonance frequency.

Increase R (with L and C fixed) and see how ω_0 and $\Delta\omega$ change. Is the behavior what you expected? Explain.

Increase L (with R and C fixed) and see how ω_0 and $\Delta\omega$ change. Is the behavior what you expected? Explain.

Increase C (with R and L fixed) and see how ω_0 and $\Delta\omega$ change. Is the behavior what you expected? Explain.

PART B: With channel 1 of the oscilloscope, measure the voltage across the signal generator and with channel 2 measure the voltage V_L across the inductor L. Recall that the voltage across the inductor is

$$V_L = \left| i_0 \right| \omega L \, e^{-i\,(\phi-\pi/2)} \tag{58}$$

where ω is the frequency of the signal generator and the amplitude of the current is given by

$$| i_0 | = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{59}$$

and the phase is

$$\phi = \text{ArcTan}\left[\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}\right] \tag{60}$$

Measure the voltage across the inductor with channel 2 and compare with the voltage across the signal generator. Note the voltage across the inductor depends upon the frequency of the signal generator ω. Measure the amplitude of V_L at five frequencies below resonance and at five frequencies above resonance. The remarks about loading the signal generator apply here as well.

Phase Relationship: The voltage across the inductor is out of phase with the voltage across the signal generator by ϕ-90°. At resonance the numerator is zero of the argument of ArcTan in equation (60) so ϕ is zero at resonance ω. **Observe the Voltage across the inductor on channel 2 of the oscilloscope an**d m**ake sure you understand that the voltage across the inductor is -90° out of phase with the voltage across the signal generator at resonance. Sketch two diagrams of what you see.**

PART C: Measure the voltage across the capacitor V_C on channel 2 of the oscilloscope and compare with the signal generator voltage on channel 1. The voltage across the capacitor is given by equation (52)

$$V_C = \frac{\left| i_0 \right| e^{-i\,(\phi+\pi/2)}}{\omega C} \tag{61}$$

where $| i_0 |$ and ϕ are as given in PART B above.

Measure the voltage across the capacitor with channel 2 and compare with the voltage across the signal generator. Note the voltage across the capacitor depends upon the frequency of the signal generator ω. Measure the amplitude of V_C at five frequencies below resonance and at five frequencies above resonance. The remarks about loading the signal generator apply here as well.

Phase Relationship: The voltage across the inductor is out of phase with the voltage across the signal generator by ϕ-+90°. At resonance the numerator is zero of the argument of ArcTan in equation (60) so

ϕ is zero at resonance ω. **Observe the Voltage across the capacitor on channel 2 of the oscilloscope an**d m**ake sure you understand that the voltage across the capacitor is +90° out of phase with the voltage across the signal generator at resonance. Sketch two diagrams of what you see.**