12-1

Arithmetic Sequences and Series

OBJECTIVES
- Find the nth term and arithmetic means of an arithmetic sequence.
- Find the sum of n terms of an arithmetic series.

Real World Application

REAL ESTATE Ofelia Gonzales sells houses in a new development. She makes a commission of \$3750 on the sale of her first house. To encourage aggressive selling, Ms. Gonzales' employer promises a \$500 increase in commission for each additional house sold. Thus, on the sale of her next house, she will earn \$4250 commission. How many houses will Ms. Gonzales have to sell for her total commission in one year to be at least \$65,000? *This problem will be solved in Example 6.*

The set of numbers representing the amount of money earned for each house sold is an example of a **sequence.** A sequence is a function whose domain is the set of natural numbers. The **terms** of a sequence are the range elements of the function. The first term of a sequence is denoted a_1, the second term is a_2, and so on up to the nth term a_n.

Look Back
Refer to Lesson 4-1 for more about natural numbers.

Symbol	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
Term	3	$2\frac{1}{2}$	2	$1\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-1\frac{1}{2}$

The sequence given in the table above is an example of an **arithmetic sequence.** The difference between successive terms of an arithmetic sequence is a constant called the **common difference,** denoted d. In the example above, $d = \frac{1}{2}$.

Arithmetic Sequence

An arithmetic sequence is a sequence in which each term after the first, a_1, is equal to the sum of the preceding term and the common difference, d. The terms of the sequence can be represented as follows.

$$a_1,\ a_1 + d,\ a_1 + 2d,\ \ldots$$

To find the next term in an arithmetic sequence, first find the common difference by subtracting any term from its succeeding term. Then add the common difference to the last term to find the next term in the sequence.

Example 1 **Find the next four terms in the arithmetic sequence $-5, -2, 1, \ldots$**

First, find the common difference.

$a_2 - a_1 = -2 - (-5)$ or 3
$a_3 - a_2 = 1 - (-2)$ or 3

Find the difference between pairs of consecutive terms to verify the common difference.

The common difference is 3.

Add 3 to the third term to get the fourth term, and so on.

$a_4 = 1 + 3$ or 4 $\quad a_5 = 4 + 3$ or 7 $\quad a_6 = 7 + 3$ or 10 $\quad a_7 = 10 + 3$ or 13

The next four terms are 4, 7, 10, and 13.

By definition, the nth term is also equal to $a_{n-1} + d$, where a_{n-1} is the $(n-1)$th term. That is, $a_n = a_{n-1} + d$. This type of formula is called a **recursive formula.** This means that each succeeding term is formulated from one or more previous terms.

The nth term of an arithmetic sequence can also be found when only the first term and the common difference are known. Consider an arithmetic sequence in which $a = -3.7$ and $d = 2.9$. Notice the pattern in the way the terms are formed.

first term	a_1	a	-3.7
second term	a_2	$a + d$	$-3.7 + 1(2.9) = -0.8$
third term	a_3	$a + 2d$	$-3.7 + 2(2.9) = 2.1$
fourth term	a_4	$a + 3d$	$-3.7 + 3(2.9) = 5.0$
fifth term	a_5	$a + 4d$	$-3.7 + 4(2.9) = 7.9$
⋮	⋮	⋮	⋮
nth term	a_n	$a + (n-1)d$	$-3.7 + (n-1)2.9$

The *n*th Term of an Arithmetic Sequence

The nth term of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n-1)d$.

Notice that the preceding formula has four variables: a_n, a_1, n, and d. If any three of these are known, the fourth can be found.

Examples

2 **Find the 47th term in the arithmetic sequence −4, −1, 2, 5, ...**

First, find the common difference.

$a_2 - a_1 = -1 - (-4)$ or 3 $\quad a_3 - a_2 = 2 - (-1)$ or 3 $\quad a_4 - a_3 = 5 - 2$ or 3

The common difference is 3.

Then use the formula for the nth term of an arithmetic sequence.

$$a_n = a_1 + (n-1)d$$
$$a_{47} = -4 + (47 - 1)3 \quad n = 47,\ a_1 = -4, \text{ and } d = 3$$
$$a_{47} = 134$$

3 **Find the first term in the arithmetic sequence for which $a_{19} = 42$ and $d = -\frac{2}{3}$.**

$$a_n = a_1 + (n-1)d$$
$$a_{19} = a_1 + (19 - 1)\left(-\frac{2}{3}\right) \quad n = 19 \text{ and } d = -\frac{2}{3}$$
$$42 = a_1 + (-12) \quad a_{19} = 42$$
$$a_1 = 54$$

Sometimes you may know two terms of an arithmetic sequence that are not in consecutive order. The terms between any two nonconsecutive terms of an arithmetic sequence are called **arithmetic means.** In the sequence below, 38 and 49 are the arithmetic means between 27 and 60.

5, 16, 27, 38, 49, 60

Example 4 **Write an arithmetic sequence that has five arithmetic means between 4.9 and 2.5.**

The sequence will have the form 4.9, __?__, __?__, __?__, __?__, __?__, 2.5. Note that 2.5 is the 7th term of the sequence or a_7.

First, find the common difference, using $n = 7$, $a_7 = 2.5$, and $a_1 = 4.9$

$$a_n = a_1 + (n - 1)d$$
$$2.5 = 4.9 + (7 - 1)d$$
$$2.5 = 4.9 + 6d$$
$$d = -0.4$$

Then determine the arithmetic means.

$$a_2 = 4.9 + (-0.4) \text{ or } 4.5$$
$$a_3 = 4.5 + (-0.4) \text{ or } 4.1$$
$$a_4 = 4.1 + (-0.4) \text{ or } 3.7$$
$$a_5 = 3.7 + (-0.4) \text{ or } 3.3$$
$$a_6 = 3.3 + (-0.4) \text{ or } 2.9$$

The sequence is 4.9, 4.5, 4.1, 3.7, 3.3, 2.9, 2.5.

An indicated sum is 1 + 2 + 3 + 4. The sum 1 + 2 + 3 + 4 is 10.

An **arithmetic series** is the indicated sum of the terms of an arithmetic sequence. The lists below show some examples of arithmetic sequences and their corresponding arithmetic series.

Arithmetic Sequence	Arithmetic Series
$-9, -3, 3, 9, 15$	$-9 + (-3) + 3 + 9 + 15$
$3, \frac{5}{2}, 2, \frac{3}{2}, 1, \frac{1}{2}$	$3 + \frac{5}{2} + 2 + \frac{3}{2} + 1 + \frac{1}{2}$
$a_1, a_2, a_3, a_4, \ldots, a_n$	$a_1 + a_2 + a_3 + a_4 + \cdots + a_n$

The symbol S_n, called the ***n*th partial sum**, is used to represent the sum of the first n terms of a series. To develop a formula for S_n for a finite arithmetic series, a series can be written in two ways and added term by term, as shown below. The second equation for S_n given below is obtained by reversing the order of the terms in the series.

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n \\ +\ S_n &= a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1 \\ \hline 2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) \\ 2S_n &= n(a_1 + a_n) \end{aligned}$$

There are n terms in the series, all of which are $(a_1 + a_n)$.

Therefore, $S_n = \frac{n}{2}(a_1 + a_n)$.

Sum of a Finite Arithmetic Series

The sum of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}(a_1 + a_n)$.

Example 5 **Find the sum of the first 60 terms in the arithmetic series 9 + 14 + 19 + ⋯ + 304.**

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{60} = \frac{60}{2}(9 + 304) \qquad n = 60,\ a_1 = 9,\ a_{60} = 304$$
$$= 9390$$

When the value of the last term, a_n, is not known, you can still determine the sum of the series. Using the formula for the nth term of an arithmetic sequence, you can derive another formula for the sum of a finite arithmetic series.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{n}{2}[a_1 + (a_1 + (n-1)d)] \quad a_n = a_1 + (n-1)d$$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

Example 6 Real World Application

REAL ESTATE **Refer to the application at the beginning of the lesson. How many houses will Ms. Gonzales have to sell for her total commission in one year to be at least $65,000?**

Let S_n = the amount of her desired commission, $65,000.
Let a_1 = the first commission, $3750.
In this example, $d = 500$.

We want to find n, the number of houses that Ms. Gonzales has to sell to have a total commission greater than or equal to $65,000.

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$65{,}000 = \frac{n}{2}[2(3750) + (n-1)(500)] \quad S_n = 65{,}000,\ a_1 = 3750$$

$$130{,}000 = n(7500 + 500n - 500) \quad \textit{Multiply each side by 2.}$$

$$130{,}000 = 7000n + 500n^2 \quad \textit{Simplify.}$$

$$0 = 500n^2 + 7000n - 130{,}000$$

$$0 = 5n^2 + 70n - 1300 \quad \textit{Divide each side by 100.}$$

$$n = \frac{-70 \pm \sqrt{70^2 - 4(5)(-1300)}}{2(5)} \quad \textit{Use the Quadratic Formula.}$$

$$n = \frac{-70 \pm \sqrt{30{,}900}}{10}$$

$$n \approx 10.58 \text{ and } -24.58 \quad \textit{-24.58 is not a possible answer.}$$

Ms. Gonzales must sell 11 or more houses for her total commission to be at least $65,000.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** the first five terms of the sequence defined by $a_n = 6 - 4n$. Is this an arithmetic sequence? Explain.
2. Consider the arithmetic sequence defined by $a_n = \frac{5 - 2n}{2}$.
   a. **Graph** the first five terms of the sequence. Let n be the x-coordinate and a_n be the y-coordinate, and connect the points.
   b. **Describe** the graph found in part **a**.
   c. **Find** the common difference of the sequence and determine its relationship to the graph found in part **a**.

3. Refer to Example 6.

   a. **Explain** why -24.58 is *not* a possible answer.

   b. **Determine** how much money Ms. Gonzales will make if she sells 10 houses.

4. **Describe** the common difference for an arithmetic sequence in which the terms are decreasing.

5. **You Decide** Ms. Brooks defined two sequences, $a_n = (-1)^n$ and $b_n = (-2)^n$, for her class. She asked the class to determine if they were arithmetic sequences. Latonya said the second was an arithmetic sequence and that the first was not. Diana thought the reverse was true. Who is correct? Explain.

Guided Practice

Find the next four terms in each arithmetic sequence.

6. 6, 11, 16, … 7. $-15, -7, 1, \ldots$ 8. $a - 6, a - 2, a + 2, \ldots$

For Exercises 9-15, assume that each sequence or series is arithmetic.

9. Find the 17th term in the sequence for which $a_1 = 10$ and $d = -3$.
10. Find n for the sequence for which $a_n = 37$, $a_1 = -13$, and $d = 5$.
11. What is the first term in the sequence for which $d = -2$ and $a_7 = 3$?
12. Find d for the sequence for which $a_1 = 100$ and $a_{12} = 34$.
13. Write a sequence that has two arithmetic means between 9 and 24.
14. What is the sum of the first 35 terms in the series $7 + 9 + 11 + \cdots$?
15. Find n for a series for which $a_1 = 30$, $d = -4$, and $S_n = -210$.

16. **Theater Design** The right side of the orchestra section of the Nederlander Theater in New York City has 19 rows, and the last row has 27 seats. The numbers of seats in each row increase by 1 as you move toward the back of the section. How many seats are in this section of the theater?

EXERCISES

Practice

Find the next four terms in each arithmetic sequence.

17. $5, -1, -7, \ldots$ 18. $-18, -7, 4, \ldots$ 19. 3, 4.5, 6, …
20. 5.6, 3.8, 2, … 21. $b, b + 4, b + 8, \ldots$ 22. $-x, 0, x, \ldots$
23. $5n, -n, -7n, \ldots$ 24. $5 + k, 5, 5 - k, \ldots$ 25. $2a - 5, 2a + 2, 2a + 9, \ldots$

26. Determine the common difference and find the next three terms of the arithmetic sequence $3 + \sqrt{7}, 5, 7 - \sqrt{7}, \ldots$.

For Exercises 27-34, assume that each sequence or series is arithmetic.

27. Find the 25th term in the sequence for which $a_1 = 8$ and $d = 3$.
28. Find the 18th term in the sequence for which $a_1 = 1.4$ and $d = 0.5$.
29. Find n for the sequence for which $a_n = -41$, $a_1 = 19$, and $d = -5$.
30. Find n for the sequence for which $a_n = 138$, $a_1 = -2$, and $d = 7$.
31. What is the first term in the sequence for which $d = -3$, and $a_{15} = 38$?
32. What is the first term in the sequence for which $d = \frac{1}{3}$ and $a_7 = 10\frac{2}{3}$?
33. Find d for the sequence in which $a_1 = 6$ and $a_{14} = 58$.
34. Find d for the sequence in which $a_1 = 8$ and $a_{11} = 26$.

For Exercises 35-49, assume that each sequence or series is arithmetic.

35. What is the eighth term in the sequence $-4 + \sqrt{5}, -1 + \sqrt{5}, 2 + \sqrt{5}, \ldots$?

36. What is the twelfth term in the sequence $5 - \boldsymbol{i}, 6, 7 + \boldsymbol{i}, \ldots$?

37. Find the 33rd term in the sequence 12.2, 10.5, 8.8, ….

38. Find the 79th term in the sequence $-7, -4, -1, \ldots$.

39. Write a sequence that has one arithmetic mean between 12 and 21.

40. Write a sequence that has two arithmetic means between -5 and 4.

41. Write a sequence that has two arithmetic means between $\sqrt{3}$ and 12.

42. Write a sequence that has three arithmetic means between 2 and 5.

43. Find the sum of the first 11 terms in the series $\frac{3}{2} + 1 + \frac{1}{2} + \ldots$.

44. Find the sum of the first 100 terms in the series $-5 - 4.8 - 4.6 - \ldots$.

45. Find the sum of the first 26 terms in the series $-19 - 13 - 7 - \ldots$.

46. Find n for a series for which $a_1 = -7$, $d = 1.5$, and $S_n = -14$.

47. Find n for a series for which $a_1 = -3$, $d = 2.5$, and $S_n = 31.5$.

48. Write an expression for the nth term of the sequence 5, 7, 9, ….

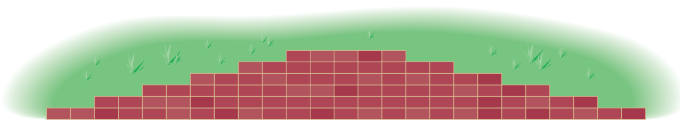
49. Write an expression for the nth term of the sequence $6, -2, -10, \ldots$.

Applications and Problem Solving

50. Keyboarding Antonio has found that he can input statistical data into his computer at the rate of 2 data items faster each half hour he works. One Monday, he starts work at 9:00 A.M., inputting at a rate of 3 data items per minute. At what rate will Antonio be inputting data into the computer by lunchtime (noon)?

51. Critical Thinking Show that if x, y, z, and w are the first four terms of an arithmetic sequence, then $x + w - y = z$.

52. Construction The Arroyos are planning to build a brick patio that approximates the shape of a trapezoid. The shorter base of the trapezoid needs to start with a row of 5 bricks, and each row must increase by 2 bricks on each side until there are 25 rows. How many bricks do the Arroyos need to buy?



53. Critical Thinking The measures of the angles of a convex polygon form an arithmetic sequence. The least measurement in the sequence is 85°. The greatest measurement is 215°. Find the number of sides in this polygon.

54. Geometry The sum of the interior angles of a triangle is 180°.

a. What are the sums of the interior angles of polygons with 4, 5, 6, and 7 sides?

b. Show that these sums (beginning with the triangle) form an arithmetic sequence.

c. Find the sum of the interior angles of a 35-sided polygon.

55. Critical Thinking Consider the sequence of odd natural numbers.

a. What is S_5?

b. What is S_{10}?

c. Make a conjecture as to the pattern that emerges concerning the sum. Write an algebraic proof verifying your conjecture.

56. Sports At the 1998 Winter X-Games held in Crested Butte, Colorado, Jennie Waara, from Sweden, won the women's snowboarding slope-style competition. Suppose that in one of the qualifying races, Ms. Waara traveled 5 feet in the first second, and the distance she traveled increased by 7 feet each subsequent second. If Ms. Waara reached the finish line in 15 seconds, how far did she travel?

57. Entertainment A radio station advertises a contest with ten cash prizes totaling \$5510. There is to be a \$100 difference between each successive prize. Find the amounts of the least and greatest prizes the radio station will award.

58. Critical Thinking Some sequences involve a pattern but are not arithmetic. Find the sum of the first 72 terms in the sequence 6, 8, 2, …, where $a_n = a_{n-1} - a_{n-2}$.

Mixed Review

59. Personal Finance If Parker Hamilton invests \$100 at 7% compounded continuously, how much will he have at the end of 15 years? *(Lesson 11-3)*

60. Find the coordinates of the center, foci, and vertices of the ellipse whose equation is $4x^2 + 25y^2 + 250y + 525 = 0$. Then graph the equation. *(Lesson 10-3)*

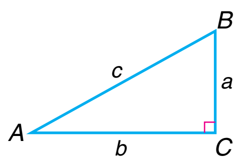
61. Find $6\left(\cos\frac{5\pi}{8} + \boldsymbol{i}\sin\frac{5\pi}{8}\right) \div 12\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)$. Then express the quotient in rectangular form. *(Lesson 9-7)*

62. Find the inner product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if $\vec{\mathbf{u}} = \langle 2, -1, 3\rangle$ and $\vec{\mathbf{v}} = \langle 5, 3, 0\rangle$. *(Lesson 8-4)*

63. Write the standard form of the equation of a line for which the length of the normal is 5 units and the normal makes an angle of 30° with the positive x-axis. *(Lesson 7-6)*

64. Graph $y = \sec 2\theta - 3$. *(Lesson 6-7)*

65. Solve triangle ABC if $B = 19°32'$ and $c = 4.5$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-5)*



66. Find the discriminant of $4p^2 - 3p + 2 = 0$. Describe the nature of its roots. *(Lesson 4-2)*

67. Determine the slant asymptote of $f(x) = \dfrac{x^2 - 4x + 2}{x - 3}$. *(Lesson 3-7)*

68. Triangle ABC is represented by the matrix $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 3 & -4 \end{bmatrix}$. Find the image of the triangle after a rotation of 270° counterclockwise about the origin. *(Lesson 2-4)*

69. SAT/ACT Practice If $a - 4b = 15$ and $4a - b = 15$, then $a - b = ?$

A 3 **B** 4 **C** 6 **D** 15 **E** 30

Extra Practice See p. A48.