12-2

Geometric Sequences and Series

OBJECTIVES
- Find the *n*th term and geometric means of a geometric sequence.
- Find the sum of *n* terms of a geometric series.

Real World Application

ACCOUNTING Bertha Blackwell is an accountant for a small company. On January 1, 1996, the company purchased $50,000 worth of office copiers. Since this equipment is a company asset, Ms. Blackwell needs to determine how much the copiers are presently worth. She estimates that copiers depreciate at a rate of 45% per year. What value should Ms. Blackwell assign the copiers on her 2001 year-end accounting report? *This problem will be solved in Example 3.*

The following sequence is an example of a **geometric sequence.**

10, 2, 0.4, 0.08, 0.016, … *Can you find the next term?*

The ratio of successive terms in a geometric sequence is a constant called the **common ratio,** denoted r.

Geometric Sequence

A geometric sequence is a sequence in which each term after the first, a_1, is the product of the preceding term and the common ratio, r. The terms of the sequence can be represented as follows, where a_1 is nonzero and r is not equal to 1 or 0.

$$a_1,\ a_1r,\ a_1r^2,\ \ldots.$$

You can find the next term in a geometric sequence as follows.

- First divide any term by the preceding term to find the common ratio.
- Then multiply the last term by the common ratio to find the next term in the sequence.

Example 1 **Determine the common ratio and find the next three terms in each sequence.**

a. $1, -\frac{1}{2}, \frac{1}{4}, \ldots$

First, find the common ratio.

$a_2 \div a_1 = -\frac{1}{2} \div 1$ or $-\frac{1}{2}$ $\qquad$ $a_3 \div a_2 = \frac{1}{4} \div \left(-\frac{1}{2}\right)$ or $-\frac{1}{2}$

The common ratio is $-\frac{1}{2}$.

Multiply the third term by $-\frac{1}{2}$ to get the fourth term, and so on.

$a_4 = \frac{1}{4} \cdot \left(-\frac{1}{2}\right)$ or $-\frac{1}{8}$ $\qquad$ $a_5 = -\frac{1}{8} \cdot \left(-\frac{1}{2}\right)$ or $\frac{1}{16}$ $\qquad$ $a_6 = \frac{1}{16} \cdot \left(-\frac{1}{2}\right)$ or $-\frac{1}{32}$

The next three terms are $-\frac{1}{8}, \frac{1}{16}$, and $-\frac{1}{32}$.

b. $\mathbf{r - 1, -3r + 3, 9r - 9, \ldots}$

First, find the common ratio.

$a_2 \div a_1 = \dfrac{-3r + 3}{r - 1}$

$a_2 \div a_1 = \dfrac{-3(r - 1)}{r - 1}$ *Factor.*

$a_2 \div a_1 = -3$ *Simplify.*

$a_3 \div a_2 = \dfrac{9r - 9}{-3r + 3}$

$a_3 \div a_2 = \dfrac{9(r - 1)}{-3(r - 1)}$ *Factor.*

$a_3 \div a_2 = -3$ *Simplify.*

The common ratio is -3.

Multiply the third term by -3 to get the fourth term, and so on.

$a_4 = -3(9r - 9)$ or $-27r + 27$

$a_5 = -3(-27r + 27)$ or $81r - 81$

$a_6 = -3(81r - 81)$ or $-243r + 243$

The next three terms are $-27r + 27$, $81r - 81$, and $-243r + 243$.

As with arithmetic sequences, geometric sequences can also be defined recursively. By definition, the nth term is also equal to $a_{n-1}r$, where a_{n-1} is the $(n - 1)$th term. That is, $a_n = a_{n-1}r$.

Since successive terms of a geometric sequence can be expressed as the product of the common ratio and the previous term, it follows that each term can be expressed as the product of a_1 and a power of r. The terms of a geometric sequence for which $a_1 = -5$ and $r = 7$ can be represented as follows.

first term	a_1	a_1	-5
second term	a_2	a_1r	$-5 \cdot 7^1 = -35$
third term	a_3	a_1r^2	$-5 \cdot 7^2 = -245$
fourth term	a_4	a_1r^3	$-5 \cdot 7^3 = -1715$
fifth term	a_5	a_1r^4	$-5 \cdot 7^4 = -12{,}005$
⋮	⋮	⋮	⋮
nth term	a_n	ar^{n-1}	$-5 \cdot 7^{n-1}$

The nth Term of a Geometric Sequence

The nth term of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1r^{n-1}$.

Example 2 **Find an approximation for the 23rd term in the sequence 256, −179.2, 125.44, …**

First, find the common ratio.

$a_2 \div a_1 = -179.2 \div 256$ or -0.7

$a_3 \div a_2 = 125.44 \div (-179.2)$ or -0.7

The common ratio is -0.7.

Then, use the formula for the nth term of a geometric sequence.

$a_n = a_1r^{n-1}$

$a_{23} = 256(-0.7)^{23-1}$ *$n = 23$, $a_1 = 256$, $r = -0.7$*

$a_{23} \approx 0.1000914188$ *Use a calculator.*

The 23rd term is about 0.1.

Geometric sequences can represent growth or decay.

- For a common ratio greater than 1, a sequence may model growth. Applications include compound interest, appreciation of property, and population growth.
- For a positive common ratio less than 1, a sequence may model decay. Applications include some radioactive behavior and depreciation.

Example 3 Real World Application

ACCOUNTING **Refer to the application at the beginning of the lesson. Compute the value of the copiers at the end of the year 2001.**

Since the copiers were purchased at the beginning of the first year, the original purchase price of the copiers represents a_1. If the copiers depreciate at a rate of 45% per year, then they retain 100 − 45 or 55% of their value each year.

Use the formula for the nth term of a geometric sequence to find the value of the copiers six years later or a_7.

$a_n = a_1 r^{n-1}$

$a_7 = 50{,}000 \cdot (0.55)^{7-1}$ *$a_1 = 50{,}000$, $r = 0.55$, $n = 7$*

$a_7 \approx 1384.032031$ *Use a calculator.*

Ms. Blackwell should list the value of the copiers on her report as $1384.03.

The terms between any two nonconsecutive terms of a geometric sequence are called **geometric means.**

Example 4 **Write a sequence that has two geometric means between 48 and −750.**

This sequence will have the form 48, __?__, __?__, −750.

First, find the common ratio.

$a_n = a_1 r^{n-1}$

$a_4 = a_1 r^3$ *Since there will be four terms in the sequence, $n = 4$.*

$-750 = 48r^3$ *$a_4 = -750$ and $a_1 = 48$*

$\frac{-125}{8} = r^3$ *Divide each side by 48 and simplify.*

$\sqrt[3]{-\frac{125}{8}} = r$ *Take the cube root of each side.*

$-2.5 = r$

Then, determine the geometric sequence.

$a_2 = 48(-2.5)$ or -120 $\quad a_3 = -120(-2.5)$ or 300

The sequence is 48, −120, 300, −750.

A **geometric series** is the indicated sum of the terms of a geometric sequence. The lists below show some examples of geometric sequences and their corresponding series.

Geometric Sequence	Geometric Series
3, 9, 27, 81, 243	3 + 9 + 27 + 81 + 243
$16, 4, 1, \frac{1}{4}, \frac{1}{16}$	$16 + 4 + 1 + \frac{1}{4} + \frac{1}{16}$
$a_1, a_2, a_3, a_4, \ldots, a_n$	$a_1 + a_2 + a_3 + a_4 + \cdots + a_n$

To develop a formula for the sum of a finite geometric sequence, S_n, write an expression for S_n and for rS_n, as shown below. Then subtract rS_n from S_n and solve for S_n.

$$S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}$$

$$rS_n = a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} + a_1r^n$$

$$S_n - rS_n = a_1 - a_1r^n$$ *Subtract.*

$$S_n(1 - r) = a_1 - a_1r^n$$ *Factor.*

$$S_n = \frac{a_1 - a_1r^n}{1 - r}$$ *Divide each side by $1 - r$, $r \neq 1$.*

Sum of a Finite Geometric Series

The sum of the first n terms of a finite geometric series is given by $S_n = \frac{a_1 - a_1r^n}{1 - r}$.

Example 5 **Find the sum of the first ten terms of the geometric series $16 - 48 + 144 - 432 + \cdots$.**

The formula for the sum of a geometric series can also be written as $S_n = a_1 \frac{1 - r^n}{1 - r}$.

First, find the common ratio.

$a_2 \div a_1 = -48 \div 16$ or -3 $\quad$ $a_4 \div a_3 = -432 \div 144$ or -3

The common ratio is -3.

$$S_n = \frac{a_1 - a_1r^n}{1 - r}$$

$$S_{10} = \frac{16 - 16(-3)^{10}}{1 - (-3)}$$ *$n = 10$, $a_1 = 16$, $r = -3$.*

$$S_{10} = -236{,}192$$ *Use a calculator.*

The sum of the first ten terms of the series is $-236{,}192$.

Banks and other financial institutions use compound interest to determine earnings in accounts or how much to charge for loans. The formula for compound interest is $A = P\left(1 + \frac{r}{n}\right)^{tn}$, where

A = the account balance,
P = the initial deposit or amount of money borrowed,
r = the annual percentage rate (APR),
n = the number of compounding periods per year, and
t = the time in years.

Suppose at the beginning of each quarter you deposit $25 in a savings account that pays an APR of 2% compounded quarterly. Most banks post the interest for each quarter on the last day of the quarter. The chart below lists the additions to the account balance as a result of each successive deposit through the rest of the year. Note that $1 + \frac{r}{n} = 1 + \frac{0.02}{4}$ or 1.005.

Date of Deposit	$A = P\left(1 + \frac{r}{n}\right)^{tn}$	1st Year Additions (to the nearest penny)
January 1	$25 $(1.005)^4$	$25.50
April 1	$25 $(1.005)^3$	$25.38
July 1	$25 $(1.005)^2$	$25.25
October 1	$25 $(1.005)^1$	$25.13
Account balance at the end of one year		$101.26

The chart shows that the first deposit will gain interest through all four compounding periods while the second will earn interest through only three compounding periods. The third and last deposits will earn interest through two and one compounding periods, respectively. The sum of these amounts, $101.26, is the balance of the account at the end of one year. This sum also represents a finite geometric series where $a_1 = 25.13$, $r = 1.005$, and $n = 4$.

$$25.13 + 25.13(1.005) + 25.13(1.005)^2 + 25.13(1.005)^3$$

Example 6 Real World Application

INVESTMENTS Hiroshi wants to begin saving money for college. He decides to deposit $500 at the beginning of each quarter (January 1, April 1, July 1, and October 1) in a savings account that pays an APR of 6% compounded quarterly. The interest for each quarter is posted on the last day of the quarter. Determine Hiroshi's account balance at the end of one year.

The interest is compounded each quarter. So $n = 4$ and the interest rate per period is 6% ÷ 4 or 1.5%. The common ratio r for the geometric series is then 1 + 0.015, or 1.015.

The first term a_1 in this series is the account balance at the end of the first quarter. Thus, $a_1 = 500(1.015)$ or 507.5.

Apply the formula for the sum of a geometric series.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

$$S_4 = \frac{507.5 - 507.5(1.015)^4}{1 - 1.015} \quad n = 4,\ r = 1.015$$

$$S_4 \approx 2076.13$$

Hiroshi's account balance at the end of one year is $2076.13.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** arithmetic and geometric sequences.
2. **Show** that the sequence defined by $a_n = (-3)^{n+1}$ is a geometric sequence.
3. **Explain** why the first term in a geometric sequence must be nonzero.
4. **Find a counterexample** for the statement "The sum of a geometric series cannot be less than its first term."
5. **Determine** whether the given terms form a finite geometric sequence. Write *yes* or *no* and then explain your reasoning.
   a. 3, 6, 18 b. $\sqrt{3}, 3, \sqrt{27}$ c. $x^{-2}, x^{-1}, 1; x \neq 1$
6. Refer to Example 3.
   a. **Make a table** to represent the situation. In the first row, put the number of years, and in the second row, put the value of the computers.
   b. **Graph** the numbers in the table. Let years be the *x*-coordinate, let value be the *y*-coordinate, and connect the points.
   c. **Describe** the graph found in part **b**.

Guided Practice

Determine the common ratio and find the next three terms of each geometric sequence.

7. $\frac{2}{3}, 4, 24, \ldots$ 8. $2, 3, \frac{9}{2}, \ldots$ 9. $1.8, -7.2, 28.8, \ldots$

For Exercises 10–14, assume that each sequence or series is geometric.

10. Find the seventh term of the sequence 7, 2.1, 0.63, ….
11. If $r = 2$ and $a_5 = 24$, find the first term of the sequence.
12. Find the first three terms of the sequence for which $a_4 = 2.5$ and $r = 2$.
13. Write a sequence that has two geometric means between 1 and 27.
14. Find the sum of the first nine terms of the series $0.5 - 1 + 2 - \cdots$.

15. **Investment** Mika Rockwell invests in classic cars. He recently bought a 1978 convertible valued at $20,000. The value of the car is predicted to appreciate at a rate of 3.5% per year. Find the value of the car after 10, 20, and 40 years, assuming that the rate of appreciation remains constant.

EXERCISES

Practice

Determine the common ratio and find the next three terms of each geometric sequence.

16. 10, 2, 0.4, … 17. 8, −20, 50, … 18. $\frac{2}{9}, \frac{2}{3}, 2, \ldots$
19. $\frac{3}{4}, \frac{3}{10}, \frac{3}{25}, \ldots$ 20. −7, 3.5, −1.75, … 21. $3\sqrt{2}, 6, 6\sqrt{2}, \ldots$
22. $9, 3\sqrt{3}, 3, \ldots$ 23. $\boldsymbol{i}, -1, -\boldsymbol{i}, \ldots$ 24. $t^8, t^5, t^2, \ldots$

25. The first term of a geometric sequence is $\frac{a}{b^2}$, and the common ratio is $\frac{b}{a^2}$. Find the next five terms of the geometric sequence.

For Exercises 26–40, assume that each sequence or series is geometric.

26. Find the fifth term of a sequence whose first term is 8 and common ratio is $\frac{3}{2}$.

27. Find the sixth term of the sequence $\frac{1}{2}, -\frac{3}{8}, \frac{9}{32}, \ldots$.

28. Find the seventh term of the sequence 40, 0.4, 0.004, ...

29. Find the ninth term of the sequence $\sqrt{5}, \sqrt{10}, 2\sqrt{5}, \ldots$.

30. If $r = 4$ and $a_6 = 192$, what is the first term of the sequence?

31. If $r = -\sqrt{2}$ and $a_5 = 32\sqrt{2}$, what is the first term of the sequence?

32. Find the first three terms of the sequence for which $a_5 = -6$ and $r = -\frac{1}{3}$.

33. Find the first three terms of the sequence for which $a_5 = 0.32$ and $r = 0.2$.

34. Write a sequence that has three geometric means between 256 and 81.

35. Write a sequence that has two geometric means between -2 and 54.

36. Write a sequence that has one geometric mean between $\frac{4}{7}$ and 7.

37. What is the sum of the first five terms of the series $\frac{5}{3} + 5 + 15 + \cdots$?

38. What is the sum of the first six terms of the series $65 + 13 + 2.6 + \cdots$?

39. Find the sum of the first ten terms of the series $1 - \frac{3}{2} + \frac{9}{4} - \cdots$.

40. Find the sum of the first eight terms of the series $2 + 2\sqrt{3} + 6 + \cdots$.

Applications and Problem Solving

Real World Application

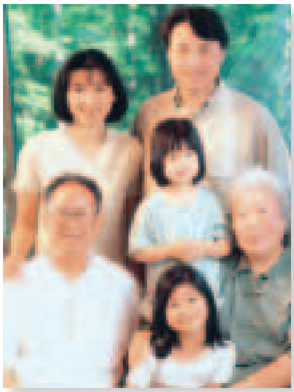
41. **Biology** A cholera bacterium divides every half-hour to produce two complete cholera bacteria.
   a. If an initial colony contains a population of b_0 bacteria, write an equation that will determine the number of bacteria present after t hours.
   b. Suppose a petri dish contains 30 cholera bacteria. Use the equation from part **a** to determine the number of bacteria present 5 hours later.
   c. What assumptions are made in using the formula found in part **a**?

42. **Critical Thinking** Consider the geometric sequence with $a_4 = 4$ and $a_7 = 12$.
   a. Find the common ratio and the first term of the sequence.
   b. Find the 28th term of the sequence.

43. **Consumerism** High Tech Electronics advertises a weekly installment plan for the purchase of a popular brand of big screen TV. The buyer pays \$5 at the end of the first week, \$5.50 at the end of the second week, \$6.05 at the end of the third week, and so on for one year.
   a. What will the payments be at the end of the 10th, 20th, and 40th weeks?
   b. Find the total cost of the TV.
   c. Why is the cost found in part **b** not entirely accurate?

44. **Statistics** A number x is said to be the *harmonic mean* of y and z if $\frac{1}{x}$ is the average of $\frac{1}{y}$ and $\frac{1}{z}$.
   a. Find the harmonic mean of 5 and 8.
   b. 8 is the harmonic mean of 20 and another number. What is the number?

45. **Critical Thinking** In a geometric sequence, $a_1 = -2$ and every subsequent term is defined by $a_n = -3a_{n-1}$, where $n > 1$. Find the nth term in the sequence in terms of n.



46. **Genealogy** Wei-Ling discovers through a research of her Chinese ancestry that one of her fifteenth-generation ancestors was a famous military leader. How many descendants does this ancestor have in the fifteenth-generation, assuming each descendent had an average of 2.5 children?

47. **Personal Finance** Tonisha is about to begin her junior year in high school and is looking ahead to her college career. She estimates that by the time she is ready to enter a university she will need at least \$750 to purchase a computer system that will meet her needs. To avoid purchasing the computer on credit, she opens a savings account earning an APR of 2.4%, compounded monthly, and deposits \$25 at the beginning of each month.
   a. Find the balance of the account at the end of the first month.
   b. If Tonisha continues this deposit schedule for the next two years, will she have enough money in her account to buy the computer system? Explain.
   c. Find the least amount of money Tonisha can deposit each month and still have enough money to purchase the computer.

48. **Critical Thinking** Use algebraic methods to determine which term 6561 is of the geometric sequence $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots$.

Mixed Review

49. **Banking** Gloria Castaneda has \$650 in her checking account. She is closing out the account by writing one check for cash against it each week. The first check is for \$20, the second is for \$25, and so on. Each check exceeds the previous one by \$5. In how many weeks will the balance in Ms. Castaneda account be \$0 if there is no service charge? *(Lesson 12-1)*

50. Find the value of $\log_{11} 265$ using the change of base formula. *(Lesson 11-5)*

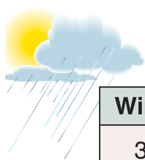
51. Graph the system $xy \geq 2$ and $x - 3y = 2$. *(Lesson 10-8)*

52. Write $3x - 5y + 5 = 0$ in polar form. *(Lesson 9-4)*

53. Write parametric equations of the line $3x + 4y = 5$. *(Lesson 8-6)*

54. If $\csc \theta = 3$ and $0° \leq \theta \leq 90°$, find $\sin \theta$. *(Lesson 7-1)*

55. **Weather** The maximum normal daily temperatures in each season for Lincoln, Nebraska, are given below. Write a sinusoidal function that models the temperatures, using $t = 1$ to represent winter. *(Lesson 6-6)*



Normal Daily Temperatures for Lincoln, Nebraska

Winter	Spring	Summer	Fall
36°	61°	86°	65°

Source: Rand McNally & Company

56. Given $A = 43°$, $b = 20$, and $a = 11$, do these measurements determine one triangle, two triangles, or no triangle? *(Lesson 5-7)*

57. **SAT Practice** **Grid-In** If n and m are integers, and $-(n^2) \leq -\sqrt{49}$ and $m = n + 1$, what is the least possible value of mn?

Extra Practice See p. A48.