12 VECTOR GEOMETRY

12.1 Vectors in the Plane

Preliminary Questions

1. Answer true or false. Every nonzero vector is:

(a) equivalent to a vector based at the origin.

(b) equivalent to a unit vector based at the origin.

(c) parallel to a vector based at the origin.

(d) parallel to a unit vector based at the origin.

SOLUTION

(a) This statement is true. Translating the vector so that it is based on the origin, we get an equivalent vector based at the origin.

(b) Equivalent vectors have equal lengths, hence vectors that are not unit vectors, are not equivalent to a unit vector.

(c) This statement is true. A vector based at the origin such that the line through this vector is parallel to the line through the given vector, is parallel to the given vector.

(d) Since parallel vectors do not necessarily have equal lengths, the statement is true by the same reasoning as in (c).

2. What is the length of $-3\mathbf{a}$ if $\|\mathbf{a}\| = 5$?

SOLUTION Using properties of the length we get

$$\|-3\mathbf{a}\| = |-3|\|\mathbf{a}\| = 3\|\mathbf{a}\| = 3 \cdot 5 = 15$$

3. Suppose that $\mathbf{v}$ has components $\langle 3, 1 \rangle$. How, if at all, do the components change if you translate $\mathbf{v}$ horizontally 2 units to the left?

SOLUTION Translating $\mathbf{v} = \langle 3, 1 \rangle$ yields an equivalent vector, hence the components are not changed.

4. What are the components of the zero vector based at $P = (3, 5)$?

SOLUTION The components of the zero vector are always $\langle 0, 0 \rangle$, no matter where it is based.

5. True or false?

(a) The vectors $\mathbf{v}$ and $-2\mathbf{v}$ are parallel.

(b) The vectors $\mathbf{v}$ and $-2\mathbf{v}$ point in the same direction.

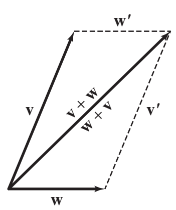
SOLUTION

(a) The lines through $\mathbf{v}$ and $-2\mathbf{v}$ are parallel, therefore these vectors are parallel.

(b) The vector $-2\mathbf{v}$ is a scalar multiple of $\mathbf{v}$, where the scalar is negative. Therefore $-2\mathbf{v}$ points in the opposite direction as $\mathbf{v}$.

6. Explain the commutativity of vector addition in terms of the Parallelogram Law.

SOLUTION To determine the vector $\mathbf{v} + \mathbf{w}$, we translate $\mathbf{w}$ to the equivalent vector $\mathbf{w}'$ whose tail coincides with the head of $\mathbf{v}$. The vector $\mathbf{v} + \mathbf{w}$ is the vector pointing from the tail of $\mathbf{v}$ to the head of $\mathbf{w}'$.



To determine the vector $\mathbf{w} + \mathbf{v}$, we translate $\mathbf{v}$ to the equivalent vector $\mathbf{v}'$ whose tail coincides with the head of $\mathbf{w}$. Then $\mathbf{w} + \mathbf{v}$ is the vector pointing from the tail of $\mathbf{w}$ to the head of $\mathbf{v}'$. In either case, the resulting vector is the vector with the tail at the basepoint of $\mathbf{v}$ and $\mathbf{w}$, and head at the opposite vertex of the parallelogram. Therefore $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

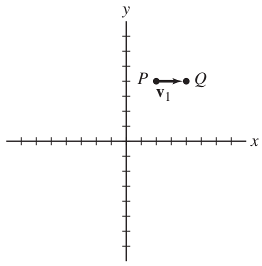
Exercises

1. Sketch the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ with tail P and head Q, and compute their lengths. Are any two of these vectors equivalent?

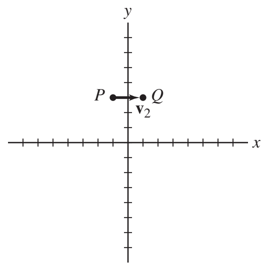
	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
P	$(2, 4)$	$(-1, 3)$	$(-1, 3)$	$(4, 1)$
Q	$(4, 4)$	$(1, 3)$	$(2, 4)$	$(6, 3)$

SOLUTION Using the definitions we obtain the following answers:

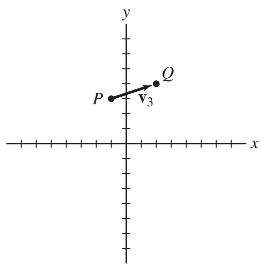
$$\mathbf{v}_1 = \overrightarrow{PQ} = \langle 4-2, 4-4 \rangle = \langle 2, 0 \rangle$$
$$\|\mathbf{v}_1\| = \sqrt{2^2 + 0^2} = 2$$



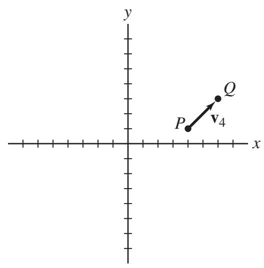
$$\mathbf{v}_2 = \langle 1-(-1), 3-3 \rangle = \langle 2, 0 \rangle$$
$$\|\mathbf{v}_2\| = \sqrt{2^2 + 0^2} = 2$$



$$\mathbf{v}_3 = \langle 2-(-1), 4-3 \rangle = \langle 3, 1 \rangle$$
$$\|\mathbf{v}_3\| = \sqrt{3^2 + 1^2} = \sqrt{10}$$



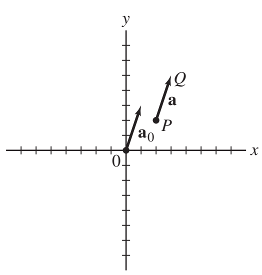
$$\mathbf{v}_4 = \langle 6-4, 3-1 \rangle = \langle 2, 2 \rangle$$
$$\|\mathbf{v}_4\| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$



$\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel and have the same length, hence they are equivalent.

3. What is the terminal point of the vector $\mathbf{a} = \langle 1, 3 \rangle$ based at $P = (2, 2)$? Sketch $\mathbf{a}$ and the vector $\mathbf{a}_0$ based at the origin and equivalent to $\mathbf{a}$.

SOLUTION The terminal point Q of the vector $\mathbf{a}$ is located 1 unit to the right and 3 units up from $P = (2, 2)$. Therefore, $Q = (2+1, 2+3) = (3, 5)$. The vector $\mathbf{a}_0$ equivalent to $\mathbf{a}$ based at the origin is shown in the figure, along with the vector $\mathbf{a}$.



In Exercises 5–8, refer to Figure 21.

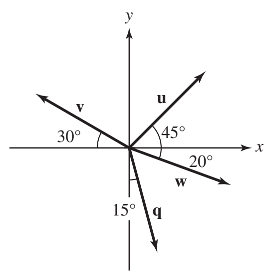


FIGURE 21

5. Find the components of **u**.

SOLUTION Since **u** makes an angle of 45° with the positive x-axis, its components are

$$\|\mathbf{u}\| \langle \cos 45^\circ, \sin 45^\circ \rangle = \|\mathbf{u}\| \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle.$$

7. Find the components of **w**.

SOLUTION Since **w** makes an angle of -20° with the positive x-axis, its components are

$$\|\mathbf{w}\| \langle \cos(-20^\circ), \sin(-20^\circ) \rangle = \|\mathbf{w}\| \left\langle \cos 20^\circ, -\sin 20^\circ \right\rangle.$$

In Exercises 9–12, find the components of $\overrightarrow{PQ}$.

9. $P = (3, 2), \quad Q = (2, 7)$

SOLUTION Using the definition of the components of a vector we have $\overrightarrow{PQ} = \langle 2 - 3, 7 - 2 \rangle = \langle -1, 5 \rangle$.

11. $P = (3, 5), \quad Q = (1, -4)$

SOLUTION By the definition of the components of a vector, we obtain $\overrightarrow{PQ} = \langle 1 - 3, -4 - 5 \rangle = \langle -2, -9 \rangle$.

In Exercises 13–18, calculate.

13. $\langle 2, 1 \rangle + \langle 3, 4 \rangle$

SOLUTION Using vector algebra we have $\langle 2, 1 \rangle + \langle 3, 4 \rangle = \langle 2 + 3, 1 + 4 \rangle = \langle 5, 5 \rangle$.

15. $5 \langle 6, 2 \rangle$

SOLUTION $5\langle 6, 2 \rangle = \langle 5 \cdot 6, 5 \cdot 2 \rangle = \langle 30, 10 \rangle$

17. $\left\langle -\frac{1}{2}, \frac{5}{3} \right\rangle + \left\langle 3, \frac{10}{3} \right\rangle$

SOLUTION The vector sum is $\left\langle -\frac{1}{2}, \frac{5}{3} \right\rangle + \left\langle 3, \frac{10}{3} \right\rangle = \left\langle -\frac{1}{2} + 3, \frac{5}{3} + \frac{10}{3} \right\rangle = \left\langle \frac{5}{2}, 5 \right\rangle.$

19. Which of the vectors (A)–(C) in Figure 22 is equivalent to $\mathbf{v} - \mathbf{w}$?

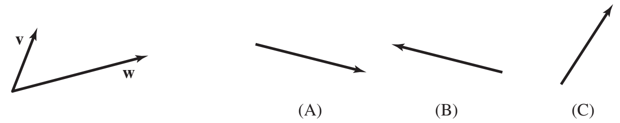
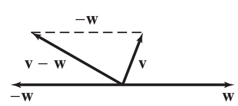


FIGURE 22

SOLUTION The vector $-\mathbf{w}$ has the same length as **w** but points in the opposite direction. The sum $\mathbf{v} + (-\mathbf{w})$, which is the difference $\mathbf{v} - \mathbf{w}$, is obtained by the parallelogram law. This vector is the vector shown in (b).



21. Sketch $2\mathbf{v}$, $-\mathbf{w}$, $\mathbf{v}+\mathbf{w}$, and $2\mathbf{v}-\mathbf{w}$ for the vectors in Figure 24.

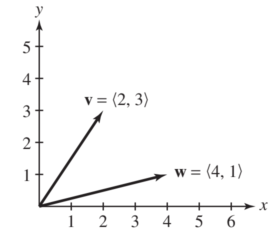
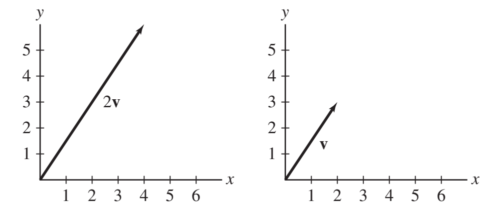
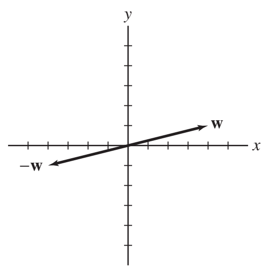


FIGURE 24

SOLUTION The scalar multiple $2\mathbf{v}$ points in the same direction as $\mathbf{v}$ and its length is twice the length of $\mathbf{v}$. It is the vector $2\mathbf{v} = \langle 4, 6\rangle$.



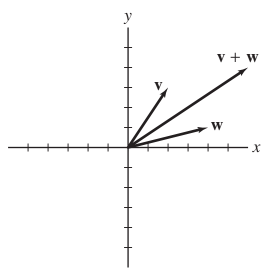
$-\mathbf{w}$ has the same length as $\mathbf{w}$ but points to the opposite direction. It is the vector $-\mathbf{w} = \langle -4, -1\rangle$.



The vector sum $\mathbf{v}+\mathbf{w}$ is the vector:

$$\mathbf{v}+\mathbf{w} = \langle 2, 3\rangle + \langle 4, 1\rangle = \langle 6, 4\rangle.$$

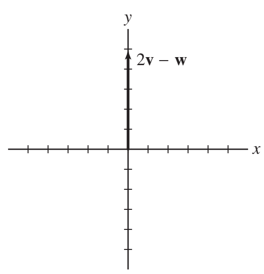
This vector is shown in the following figure:



The vector $2\mathbf{v}-\mathbf{w}$ is

$$2\mathbf{v}-\mathbf{w} = 2\langle 2, 3\rangle - \langle 4, 1\rangle = \langle 4, 6\rangle - \langle 4, 1\rangle = \langle 0, 5\rangle$$

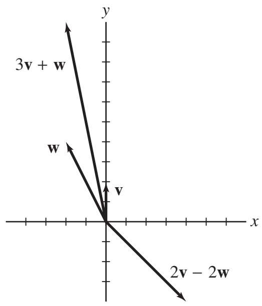
It is shown next:



23. Sketch $\mathbf{v} = \langle 0, 2\rangle$, $\mathbf{w} = \langle -2, 4\rangle$, $3\mathbf{v} + \mathbf{w}$, $2\mathbf{v} - 2\mathbf{w}$.

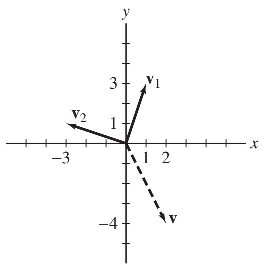
SOLUTION We compute the vectors and then sketch them:

$$3\mathbf{v} + \mathbf{w} = 3\langle 0, 2\rangle + \langle -2, 4\rangle = \langle 0, 6\rangle + \langle -2, 4\rangle = \langle -2, 10\rangle$$
$$2\mathbf{v} - 2\mathbf{w} = 2\langle 0, 2\rangle - 2\langle -2, 4\rangle = \langle 0, 4\rangle - \langle -4, 8\rangle = \langle 4, -4\rangle$$



25. Sketch the vector $\mathbf{v}$ such that $\mathbf{v} + \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ for $\mathbf{v}_1$ and $\mathbf{v}_2$ in Figure 25(A).

SOLUTION Since $\mathbf{v} + \mathbf{v}_1 + \mathbf{v}_2 = 0$, we have that $\mathbf{v} = -\mathbf{v}_1 - \mathbf{v}_2$, and since $\mathbf{v}_1 = \langle 1, 3\rangle$ and $\mathbf{v}_2 = \langle -3, 1\rangle$, then $\mathbf{v} = -\mathbf{v}_1 - \mathbf{v}_2 = \langle 2, -4\rangle$, as seen in this picture.



27. Let $\mathbf{v} = \overrightarrow{PQ}$, where $P = (-2, 5)$, $Q = (1, -2)$. Which of the following vectors with the given tails and heads are equivalent to $\mathbf{v}$?

(a) $(-3, 3)$, $(0, 4)$ **(b)** $(0, 0)$, $(3, -7)$
(c) $(-1, 2)$, $(2, -5)$ **(d)** $(4, -5)$, $(1, 4)$

SOLUTION Two vectors are equivalent if they have the same components. We thus compute the vectors and check whether this condition is satisfied.

$$\mathbf{v} = \overrightarrow{PQ} = \langle 1 - (-2), -2 - 5\rangle = \langle 3, -7\rangle$$

(a) $\langle 0 - (-3), 4 - 3\rangle = \langle 3, 1\rangle$ **(b)** $\langle 3 - 0, -7 - 0\rangle = \langle 3, -7\rangle$
(c) $\langle 2 - (-1), -5 - 2\rangle = \langle 3, -7\rangle$ **(d)** $\langle 1 - 4, 4 - (-5)\rangle = \langle -3, 9\rangle$

We see that the vectors in (b) and (c) are equivalent to $\mathbf{v}$.

In Exercises 29–32, sketch the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$, and determine whether they are equivalent.

29. $A = (1, 1)$, $B = (3, 7)$, $P = (4, -1)$, $Q = (6, 5)$

SOLUTION We compute the vectors and check whether they have the same components:

$$\begin{aligned}\overrightarrow{AB} &= \langle 3 - 1, 7 - 1\rangle = \langle 2, 6\rangle\\ \overrightarrow{PQ} &= \langle 6 - 4, 5 - (-1)\rangle = \langle 2, 6\rangle\end{aligned} \quad\Rightarrow\quad \text{The vectors are equivalent.}$$

31. $A = (-3, 2)$, $B = (0, 0)$, $P = (0, 0)$, $Q = (3, -2)$

SOLUTION We compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$:

$$\begin{aligned}\overrightarrow{AB} &= \langle 0 - (-3), 0 - 2\rangle = \langle 3, -2\rangle\\ \overrightarrow{PQ} &= \langle 3 - 0, -2 - 0\rangle = \langle 3, -2\rangle\end{aligned} \quad\Rightarrow\quad \text{The vectors are equivalent.}$$

In Exercises 33–36, are $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ parallel? And if so, do they point in the same direction?

33. $A = (1, 1), \quad B = (3, 4), \quad P = (1, 1), \quad Q = (7, 10)$

SOLUTION We compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$:

$$\overrightarrow{AB} = \langle 3-1, 4-1 \rangle = \langle 2, 3 \rangle$$
$$\overrightarrow{PQ} = \langle 7-1, 10-1 \rangle = \langle 6, 9 \rangle$$

Since $\overrightarrow{AB} = \frac{1}{3}\langle 6, 9 \rangle$, the vectors are parallel and point in the same direction.

35. $A = (2, 2), \quad B = (-6, 3), \quad P = (9, 5), \quad Q = (17, 4)$

SOLUTION We compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$:

$$\overrightarrow{AB} = \langle -6-2, 3-2 \rangle = \langle -8, 1 \rangle$$
$$\overrightarrow{PQ} = \langle 17-9, 4-5 \rangle = \langle 8, -1 \rangle$$

Since $\overrightarrow{AB} = -\overrightarrow{PQ}$, the vectors are parallel and point in opposite directions.

In Exercises 37–40, let $R = (-2, 7)$. Calculate the following:

37. The length of $\overrightarrow{OR}$

SOLUTION Since $\overrightarrow{OR} = \langle -2, 7 \rangle$, the length of the vector is $\|\overrightarrow{OR}\| = \sqrt{(-2)^2 + 7^2} = \sqrt{53}$.

39. The point P such that $\overrightarrow{PR}$ has components $\langle -2, 7 \rangle$

SOLUTION Denoting $P = (x_0, y_0)$ we have:

$$\overrightarrow{PR} = \langle -2 - x_0, 7 - y_0 \rangle = \langle -2, 7 \rangle$$

Equating corresponding components yields:

$$\begin{array}{c} -2 - x_0 = -2 \\ 7 - y_0 = 7 \end{array} \quad \Rightarrow \quad x_0 = 0, \ y_0 = 0 \quad \Rightarrow \quad P = (0, 0)$$

In Exercises 41–48, find the given vector.

41. Unit vector $\mathbf{e_v}$ where $\mathbf{v} = \langle 3, 4 \rangle$

SOLUTION The unit vector $\mathbf{e_v}$ is the following vector:

$$\mathbf{e_v} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

We find the length of $\mathbf{v} = \langle 3, 4 \rangle$:

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Thus

$$\mathbf{e_v} = \frac{1}{5}\langle 3, 4 \rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

43. Vector of length 4 in the direction of $\mathbf{u} = \langle -1, -1 \rangle$

SOLUTION We first find the unit vector in the direction of $\mathbf{u}$:

$$\mathbf{e_u} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{(-1)^2 + (-1)^2}}\langle -1, -1 \rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle.$$

We now multiply $\mathbf{e_u}$ by 4 to obtain the desired vector:

$$4\mathbf{e_u} = 4\left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \left\langle -\frac{4}{\sqrt{2}}, -\frac{4}{\sqrt{2}} \right\rangle = \left\langle -2\sqrt{2}, -2\sqrt{2} \right\rangle$$

45. Vector of length 2 in the direction opposite to $\mathbf{v} = \mathbf{i} - \mathbf{j}$

SOLUTION We first find the unit vector in the direction of $\mathbf{v}$:

$$\mathbf{e}_{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{1^2 + (-1)^2}}\langle 1, -1\rangle = \frac{1}{\sqrt{2}}\langle 1, -1\rangle = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle.$$

Now multiply by -2 to obtain the desired vector:

$$-2\mathbf{e}_{\mathbf{v}} = -2\left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle = \langle -\sqrt{2}, \sqrt{2}\rangle.$$

47. Unit vector $\mathbf{e}$ making an angle of $\frac{4\pi}{7}$ with the x-axis

SOLUTION The unit vector $\mathbf{e}$ is the following vector:

$$\mathbf{e} = \left\langle \cos\frac{4\pi}{7}, \sin\frac{4\pi}{7} \right\rangle = \langle -0.22, 0.97\rangle.$$

49. Find all scalars λ such that $\lambda\,\langle 2, 3\rangle$ has length 1.

SOLUTION We have:

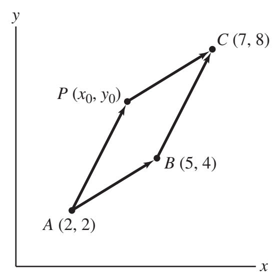
$$\|\lambda\langle 2, 3\rangle\| = |\lambda|\|\langle 2, 3\rangle\| = |\lambda|\sqrt{2^2 + 3^2} = |\lambda|\sqrt{13}$$

The scalar λ must satisfy

$$\begin{aligned} |\lambda|\sqrt{13} &= 1 \\ |\lambda| &= \frac{1}{\sqrt{13}} \end{aligned} \quad\Rightarrow\quad \lambda_1 = \frac{1}{\sqrt{13}},\ \lambda_2 = -\frac{1}{\sqrt{13}}$$

51. What are the coordinates of the point P in the parallelogram in Figure 26(A)?

SOLUTION We denote by A, B, C the points in the figure.



Let $P = (x_0, y_0)$. We compute the following vectors:

$$\begin{aligned} \overrightarrow{PC} &= \langle 7 - x_0, 8 - y_0\rangle \\ \overrightarrow{AB} &= \langle 5 - 2, 4 - 2\rangle = \langle 3, 2\rangle \end{aligned}$$

The vectors $\overrightarrow{PC}$ and $\overrightarrow{AB}$ are equivalent, hence they have the same components. That is:

$$\begin{aligned} 7 - x_0 &= 3 \\ 8 - y_0 &= 2 \end{aligned} \quad\Rightarrow\quad x_0 = 4,\ y_0 = 6 \quad\Rightarrow\quad P = (4, 6)$$

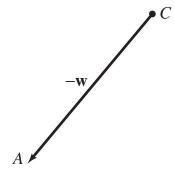
53. Let $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{AC}$, where A, B, C are three distinct points in the plane. Match (a)–(d) with (i)–(iv). (*Hint:* Draw a picture.)

(a) $-\mathbf{w}$ **(b)** $-\mathbf{v}$ **(c)** $\mathbf{w} - \mathbf{v}$ **(d)** $\mathbf{v} - \mathbf{w}$

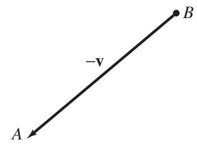
(i) $\overrightarrow{CB}$ **(ii)** $\overrightarrow{CA}$ **(iii)** $\overrightarrow{BC}$ **(iv)** $\overrightarrow{BA}$

SOLUTION

(a) $-\mathbf{w}$ has the same length as $\mathbf{w}$ and points in the opposite direction. Hence: $-\mathbf{w} = \overrightarrow{CA}$.



(b) $-\mathbf{v}$ has the same length as $\mathbf{v}$ and points in the opposite direction. Hence: $-\mathbf{v} = \overrightarrow{BA}$.

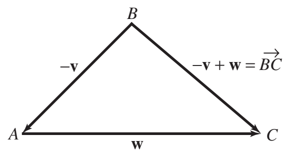


(c) By the parallelogram law we have:

$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\mathbf{v} + \mathbf{w} = \mathbf{w} - \mathbf{v}$$

That is,

$$\mathbf{w} - \mathbf{v} = \overrightarrow{BC}$$

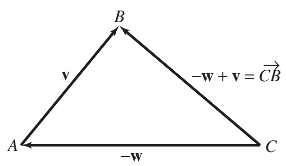


(d) By the parallelogram law we have:

$$\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\mathbf{w} + \mathbf{v} = \mathbf{v} - \mathbf{w}$$

That is,

$$\mathbf{v} - \mathbf{w} = \overrightarrow{CB}.$$



In Exercises 55–58, calculate the linear combination.

55. $3\mathbf{j} + (9\mathbf{i} + 4\mathbf{j})$

SOLUTION We have:

$$3\mathbf{j} + (9\mathbf{i} + 4\mathbf{j}) = 3\langle 0, 1\rangle + 9\langle 1, 0\rangle + 4\langle 0, 1\rangle = \langle 9, 7\rangle$$

57. $(3\mathbf{i} + \mathbf{j}) - 6\mathbf{j} + 2(\mathbf{j} - 4\mathbf{i})$

SOLUTION We have:

$$(3\mathbf{i} + \mathbf{j}) - 6\mathbf{j} + 2(\mathbf{j} - 4\mathbf{i}) = (\langle 3, 0\rangle + \langle 0, 1\rangle) - \langle 0, 6\rangle + 2(\langle 0, 1\rangle - \langle 4, 0\rangle) = \langle -5, -3\rangle$$

59. For each of the position vectors $\mathbf{u}$ with endpoints A, B, and C in Figure 27, indicate with a diagram the multiples $r\mathbf{v}$ and $s\mathbf{w}$ such that $\mathbf{u} = r\mathbf{v} + s\mathbf{w}$. A sample is shown for $\mathbf{u} = \overrightarrow{OQ}$.

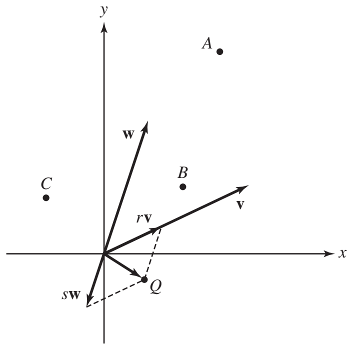
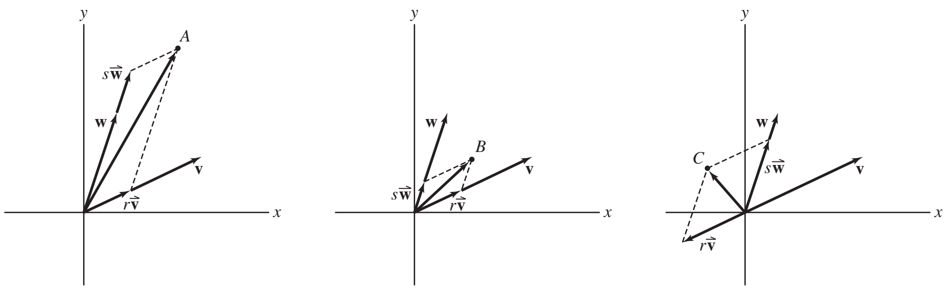


FIGURE 27

SOLUTION See the following three figures:



In Exercises 61 and 62, express $\mathbf{u}$ *as a linear combination* $\mathbf{u} = r\mathbf{v} + s\mathbf{w}$. *Then sketch* $\mathbf{u}, \mathbf{v}, \mathbf{w}$, *and the parallelogram formed by* $r\mathbf{v}$ *and* $s\mathbf{w}$.

61. $\mathbf{u} = \langle 3, -1 \rangle$; $\quad \mathbf{v} = \langle 2, 1 \rangle$, $\mathbf{w} = \langle 1, 3 \rangle$

SOLUTION We have

$$\mathbf{u} = \langle 3, -1 \rangle = r\mathbf{v} + s\mathbf{w} = r\langle 2, 1 \rangle + s\langle 1, 3 \rangle$$

which becomes the two equations

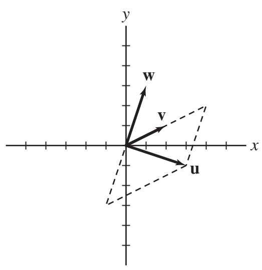
$$3 = 2r + s$$

$$-1 = r + 3s$$

Solving the second equation for r gives $r = -1 - 3s$, and substituting that into the first equation gives $3 = 2(-1 - 3s) + s = -2 - 6s + s$, so $5 = -5s$, so $s = -1$, and thus $r = 2$. In other words,

$$\mathbf{u} = \langle 3, -1 \rangle = 2\langle 2, 1 \rangle - 1\langle 1, 3 \rangle$$

as seen in this sketch:



63. Calculate the magnitude of the force on cables 1 and 2 in Figure 28.

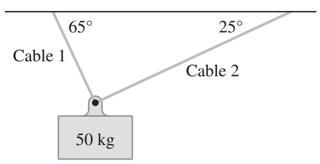
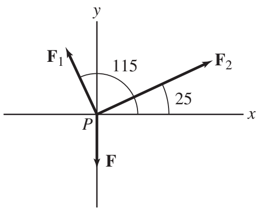


FIGURE 28

SOLUTION The three forces acting on the point P are:

- The force $\mathbf{F}$ of magnitude 50 lb that acts vertically downward.
- The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ that act through cables 1 and 2 respectively.



Since the point P is not in motion we have

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F} = 0 \tag{1}$$

We compute the forces. Letting $\|\mathbf{F}_1\| = f_1$ and $\|\mathbf{F}_2\| = f_2$ we have:

$$\begin{aligned}
\mathbf{F}_1 &= f_1\langle \cos 115^\circ, \sin 115^\circ\rangle = f_1\langle -0.423, 0.906\rangle \\
\mathbf{F}_2 &= f_2\langle \cos 25^\circ, \sin 25^\circ\rangle = f_2\langle 0.906, 0.423\rangle \\
\mathbf{F} &= \langle 0, -50\rangle
\end{aligned}$$

Substituting the forces in (1) gives

$$\begin{aligned}
f_1\langle -0.423, 0.906\rangle + f_2\langle 0.906, 0.423\rangle + \langle 0, -50\rangle &= \langle 0, 0\rangle \\
\langle -0.423 f_1 + 0.906 f_2, 0.906 f_1 + 0.423 f_2 - 50\rangle &= \langle 0, 0\rangle
\end{aligned}$$

We equate corresponding components and get

$$\begin{aligned}
-0.423 f_1 + 0.906 f_2 &= 0 \\
0.906 f_1 + 0.423 f_2 - 50 &= 0
\end{aligned}$$

By the first equation, $f_2 = 0.467 f_1$. Substituting in the second equation and solving for f_1 yields

$$\begin{aligned}
0.906 f_1 + 0.423 \cdot 0.467 f_1 - 50 &= 0 \\
1.104 f_1 = 50 \quad \Rightarrow \quad f_1 = 45.29,\ \ f_2 &= 0.467 f_1 = 21.15
\end{aligned}$$

We conclude that the magnitude of the force on cable 1 is $f_1 = 45.29$ lb and the magnitude of the force on cable 2 is $f_2 = 21.15$ lb.

65. A plane flying due east at 200 km/h encounters a 40-km/h wind blowing in the northeast direction. The resultant velocity of the plane is the vector sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is the velocity vector of the plane and $\mathbf{v}_2$ is the velocity vector of the wind (Figure 30). The angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\frac{\pi}{4}$. Determine the resultant *speed* of the plane (the length of the vector $\mathbf{v}$).

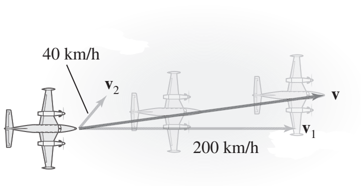
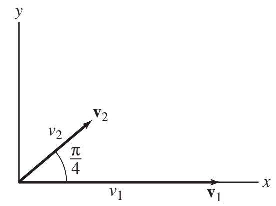


FIGURE 30

SOLUTION The resultant speed of the plane is the length of the sum vector $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. We place the xy-coordinate system as shown in the figure, and compute the components of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. This gives

$$\begin{aligned}
\mathbf{v}_1 &= \langle v_1, 0\rangle \\
\mathbf{v}_2 &= \left\langle v_2 \cos\frac{\pi}{4}, v_2 \sin\frac{\pi}{4}\right\rangle = \left\langle v_2 \cdot \frac{\sqrt{2}}{2}, v_2 \cdot \frac{\sqrt{2}}{2}\right\rangle
\end{aligned}$$



We now compute the sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$:

$$\mathbf{v} = \langle v_1, 0\rangle + \left\langle \frac{\sqrt{2}v_2}{2}, \frac{\sqrt{2}v_2}{2} \right\rangle = \left\langle \frac{\sqrt{2}}{2}v_2 + v_1, \frac{\sqrt{2}}{2}v_2 \right\rangle$$

The resultant speed is the length of $\mathbf{v}$, that is,

$$v = \|\mathbf{v}\| = \sqrt{\left(\frac{\sqrt{2}v_2}{2}\right)^2 + \left(v_1 + \frac{\sqrt{2}v_2}{2}\right)^2} = \sqrt{\frac{v_2^2}{2} + v_1^2 + 2\cdot\frac{\sqrt{2}}{2}v_2v_1 + \frac{v_2^2}{2}} = \sqrt{v_1^2 + v_2^2 + \sqrt{2}v_1v_2}$$

Finally, we substitute the given information $v_1 = 200$ and $v_2 = 40$ in the equation above, to obtain

$$v = \sqrt{200^2 + 40^2 + \sqrt{2}\cdot 200\cdot 40} \approx 230 \text{ km/hr}$$

Further Insights and Challenges

In Exercises 66–68, refer to Figure 31, which shows a robotic arm consisting of two segments of lengths L_1 and L_2.

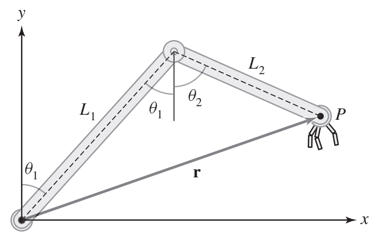


FIGURE 31

67. Let $L_1 = 5$ and $L_2 = 3$. Find $\mathbf{r}$ for $\theta_1 = \frac{\pi}{3}$, $\theta_2 = \frac{\pi}{4}$.

SOLUTION In Exercise 66 we showed that

$$\mathbf{r} = \langle L_1 \sin\theta_1 + L_2 \sin\theta_2, L_1\cos\theta_1 - L_2\cos\theta_2\rangle$$

Substituting the given information we obtain

$$\mathbf{r} = \left\langle 5\sin\frac{\pi}{3} + 3\sin\frac{\pi}{4}, 5\cos\frac{\pi}{3} - 3\cos\frac{\pi}{4}\right\rangle = \left\langle \frac{5\sqrt{3}}{2} + \frac{3\sqrt{2}}{2}, \frac{5}{2} - \frac{3\sqrt{2}}{2}\right\rangle \approx \langle 6.45, 0.38\rangle$$

69. Use vectors to prove that the diagonals $\overline{AC}$ and $\overline{BD}$ of a parallelogram bisect each other (Figure 32). *Hint:* Observe that the midpoint of $\overline{BD}$ is the terminal point of $\mathbf{w} + \frac{1}{2}(\mathbf{v} - \mathbf{w})$.

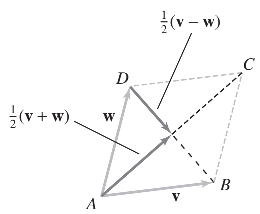
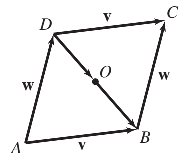


FIGURE 32

SOLUTION We denote by O the midpoint of $\overline{BD}$. Hence,

$$\overrightarrow{DO} = \frac{1}{2}\overrightarrow{DB}$$



Using the Parallelogram Law we have

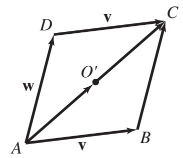
$$\overrightarrow{AO} = \overrightarrow{AD} + \overrightarrow{DO} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{DB}$$

Since $\overrightarrow{AD} = \mathbf{w}$ and $\overrightarrow{DB} = \mathbf{v} - \mathbf{w}$ we get

$$\overrightarrow{AO} = \mathbf{w} + \frac{1}{2}(\mathbf{v} - \mathbf{w}) = \frac{\mathbf{w} + \mathbf{v}}{2} \tag{1}$$

On the other hand, $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \mathbf{w} + \mathbf{v}$, hence the midpoint O' of the diagonal $\overline{AC}$ is the terminal point of $\frac{\mathbf{w}+\mathbf{v}}{2}$. That is,

$$\overrightarrow{AO'} = \frac{\mathbf{w} + \mathbf{v}}{2} \tag{2}$$



We combine (1) and (2) to conclude that O and O' are the same point. That is, the diagonal $\overline{AC}$ and $\overline{BD}$ bisect each other.

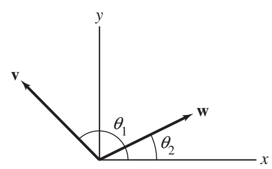
71. Prove that two vectors $\mathbf{v} = \langle a, b\rangle$ and $\mathbf{w} = \langle c, d\rangle$ are perpendicular if and only if

$$ac + bd = 0$$

SOLUTION Suppose that the vectors $\mathbf{v}$ and $\mathbf{w}$ make angles θ_1 and θ_2, which are not $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, respectively, with the positive x-axis. Then their components satisfy

$$\begin{aligned} a &= \|\mathbf{v}\|\cos\theta_1 \\ b &= \|\mathbf{v}\|\sin\theta_1 \end{aligned} \quad\Rightarrow\quad \frac{b}{a} = \frac{\sin\theta_1}{\cos\theta_1} = \tan\theta_1$$

$$\begin{aligned} c &= \|\mathbf{w}\|\cos\theta_2 \\ d &= \|\mathbf{w}\|\sin\theta_2 \end{aligned} \quad\Rightarrow\quad \frac{d}{c} = \frac{\sin\theta_2}{\cos\theta_2} = \tan\theta_2$$



That is, the vectors $\mathbf{v}$ and $\mathbf{w}$ are on the lines with slopes $\frac{b}{a}$ and $\frac{d}{c}$, respectively. The lines are perpendicular if and only if their slopes satisfy

$$\frac{b}{a}\cdot\frac{d}{c} = -1 \quad\Rightarrow\quad bd = -ac \quad\Rightarrow\quad ac + bd = 0$$

We now consider the case where one of the vectors, say $\mathbf{v}$, is perpendicular to the x-axis. In this case $a = 0$, and the vectors are perpendicular if and only if $\mathbf{w}$ is parallel to the x-axis, that is, $d = 0$. So $ac + bd = 0 \cdot c + b \cdot 0 = 0$.

12.2 Vectors in Three Dimensions

Preliminary Questions

1. What is the terminal point of the vector $\mathbf{v} = \langle 3, 2, 1\rangle$ based at the point $P = (1, 1, 1)$?

SOLUTION We denote the terminal point by $Q = (a, b, c)$. Then by the definition of components of a vector, we have

$$\langle 3, 2, 1\rangle = \langle a - 1, b - 1, c - 1\rangle$$

Equivalent vectors have equal components respectively, thus,

$$\begin{aligned} 3 &= a - 1 & & a = 4 \\ 2 &= b - 1 & \Rightarrow \quad & b = 3 \\ 1 &= c - 1 & & c = 2 \end{aligned}$$

The terminal point of $\mathbf{v}$ is thus $Q = (4, 3, 2)$.

2. What are the components of the vector $\mathbf{v} = \langle 3, 2, 1\rangle$ based at the point $P = (1, 1, 1)$?

SOLUTION The component of $\mathbf{v} = \langle 3, 2, 1\rangle$ are $\langle 3, 2, 1\rangle$ regardless of the base point. The component of $\mathbf{v}$ and the base point $P = (1, 1, 1)$ determine the head $Q = (a, b, c)$ of the vector, as found in the previous exercise.

3. If $\mathbf{v} = -3\mathbf{w}$, then (choose the correct answer):
(a) $\mathbf{v}$ and $\mathbf{w}$ are parallel.
(b) $\mathbf{v}$ and $\mathbf{w}$ point in the same direction.

SOLUTION The vectors $\mathbf{v}$ and $\mathbf{w}$ lie on parallel lines, hence these vectors are parallel. Since $\mathbf{v}$ is a scalar multiple of $\mathbf{w}$ by a negative scalar, $\mathbf{v}$ and $\mathbf{w}$ point in opposite directions. Thus, (a) is correct and (b) is not.

4. Which of the following is a direction vector for the line through $P = (3, 2, 1)$ and $Q = (1, 1, 1)$?
(a) $\langle 3, 2, 1\rangle$ **(b)** $\langle 1, 1, 1\rangle$ **(c)** $\langle 2, 1, 0\rangle$

SOLUTION Any vector that is parallel to the vector $\overrightarrow{PQ}$ is a direction vector for the line through P and Q. We compute the vector $\overrightarrow{PQ}$:

$$\overrightarrow{PQ} = \langle 1 - 3, 1 - 2, 1 - 1\rangle = \langle -2, -1, 0\rangle.$$

The vectors $\langle 3, 2, 1\rangle$ and $\langle 1, 1, 1\rangle$ are not constant multiples of $\overrightarrow{PQ}$, hence they are not parallel to $\overrightarrow{PQ}$. However $\langle 2, 1, 0\rangle = -1\langle -2, -1, 0\rangle = -\overrightarrow{PQ}$, hence the vector $\langle 2, 1, 0\rangle$ is parallel to $\overrightarrow{PQ}$. Therefore, the vector $\langle 2, 1, 0\rangle$ is a direction vector for the line through P and Q.

5. How many different direction vectors does a line have?

SOLUTION All the vectors that are parallel to a line are also direction vectors for that line. Therefore, there are infinitely many direction vectors for a line.

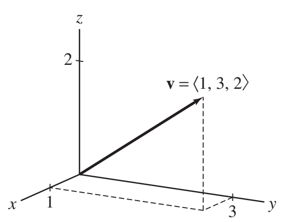
6. True or false? If $\mathbf{v}$ is a direction vector for a line $\mathcal{L}$, then $-\mathbf{v}$ is also a direction vector for $\mathcal{L}$.

SOLUTION True. Every vector that is parallel to $\mathbf{v}$ is a direction vector for the line L. Since $-\mathbf{v}$ is parallel to $\mathbf{v}$, it is also a direction vector for L.

Exercises

1. Sketch the vector $\mathbf{v} = \langle 1, 3, 2\rangle$ and compute its length.

SOLUTION The vector $\mathbf{v} = \langle 1, 3, 2\rangle$ is shown in the following figure:

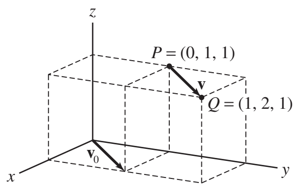


The length of $\mathbf{v}$ is

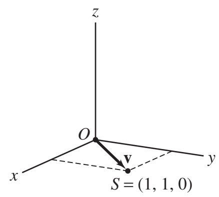
$$\|\mathbf{v}\| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

3. Sketch the vector $\mathbf{v} = \langle 1, 1, 0\rangle$ based at $P = (0, 1, 1)$. Describe this vector in the form $\overrightarrow{PQ}$ for some point Q, and sketch the vector $\mathbf{v}_0$ based at the origin equivalent to $\mathbf{v}$.

SOLUTION The vector $\mathbf{v} = \langle 1, 1, 0\rangle$ based at $P = (0, 1, 1)$ is shown in the figure:



The head Q of the vector $\mathbf{v} = \overrightarrow{PQ}$ is at the point $Q = (0 + 1, 1 + 1, 1 + 0) = (1, 2, 1)$.



The vector $\mathbf{v}_0$ based at the origin and equivalent to $\mathbf{v}$ is

$$v_0 = \langle 1, 1, 0\rangle = \overrightarrow{OS}, \text{ where } S = (1, 1, 0).$$

In Exercises 5–8, find the components of the vector $\overrightarrow{PQ}$.

5. $P = (1, 0, 1), \quad Q = (2, 1, 0)$

SOLUTION By the definition of the vector components we have

$$\overrightarrow{PQ} = \langle 2 - 1, 1 - 0, 0 - 1\rangle = \langle 1, 1, -1\rangle$$

7. $P = (4, 6, 0), \quad Q = \left(-\frac{1}{2}, \frac{9}{2}, 1\right)$

SOLUTION Using the definition of vector components we have

$$\overrightarrow{PQ} = \left\langle -\frac{1}{2} - 4, \frac{9}{2} - 6, 1 - 0\right\rangle = \left\langle -\frac{9}{2}, -\frac{3}{2}, 1\right\rangle$$

In Exercises 9–12, let $R = (1, 4, 3)$.

9. Calculate the length of $\overrightarrow{OR}$.

SOLUTION The length of $\overrightarrow{OR}$ is the distance from $R = (1, 4, 3)$ to the origin. That is,

$$\|\overrightarrow{OR}\| = \sqrt{(1 - 0)^2 + (4 - 0)^2 + (3 - 0)^2} = \sqrt{26} \approx 5.1.$$

11. Find the point P such that $\mathbf{w} = \overrightarrow{PR}$ has components $\langle 3, -2, 3\rangle$, and sketch $\mathbf{w}$.

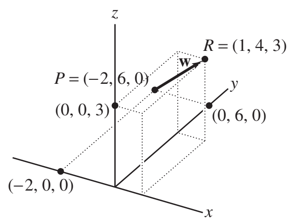
SOLUTION Denoting $P = (x_0, y_0, z_0)$ we get

$$\overrightarrow{PR} = \langle 1 - x_0, 4 - y_0, 3 - z_0\rangle = \langle 3, -2, 3\rangle$$

Equating corresponding components gives

$$\begin{aligned} 1 - x_0 &= 3 \\ 4 - y_0 &= -2 \quad \Rightarrow \quad x_0 = -2,\ y_0 = 6,\ z_0 = 0 \\ 3 - z_0 &= 3 \end{aligned}$$

The point P is, thus, $P = (-2, 6, 0)$.



13. Let $\mathbf{v} = \langle 4, 8, 12\rangle$. Which of the following vectors is parallel to $\mathbf{v}$? Which point in the same direction?

(a) $\langle 2, 4, 6\rangle$ **(b)** $\langle -1, -2, 3\rangle$

(c) $\langle -7, -14, -21\rangle$ **(d)** $\langle 6, 10, 14\rangle$

SOLUTION A vector is parallel to $\mathbf{v}$ if it is a scalar multiple of $\mathbf{v}$. It points in the same direction if the multiplying scalar is positive. Using these properties we obtain the following answer:

(a) $\langle 2, 4, 6\rangle = \frac{1}{2}\mathbf{v} \Rightarrow$ The vectors are parallel and point in the same direction.

(b) $\langle -1, -2, 3\rangle$ is not a scalar multiple of $\mathbf{v}$, hence these vectors are not parallel.

(c) $\langle -7, -14, -21\rangle = -\frac{7}{4}\mathbf{v} \Rightarrow$ The vectors are parallel but point in opposite directions.

(d) $\langle 6, 10, 14\rangle$ is not a constant multiple of $\mathbf{v}$, hence these vectors are not parallel.

In Exercises 14–17, determine whether $\overrightarrow{AB}$ is equivalent to $\overrightarrow{PQ}$.

15. $A = (1, 4, 1) \quad B = (-2, 2, 0)$
$P = (2, 5, 7) \quad Q = (-3, 2, 1)$

SOLUTION We compute the two vectors:

$$\overrightarrow{AB} = \langle -2-1, 2-4, 0-1\rangle = \langle -3, -2, -1\rangle$$
$$\overrightarrow{PQ} = \langle -3-2, 2-5, 1-7\rangle = \langle -5, -3, -6\rangle$$

The components of $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are not equal, hence they are not a translate of each other, that is, the vectors are not equivalent.

17. $A = (1, 1, 0) \quad B = (3, 3, 5)$
$P = (2, -9, 7) \quad Q = (4, -7, 13)$

SOLUTION The vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are the following vectors:

$$\overrightarrow{AB} = \langle 3-1, 3-1, 5-0\rangle = \langle 2, 2, 5\rangle$$
$$\overrightarrow{PQ} = \langle 4-2, -7-(-9), 13-7\rangle = \langle 2, 2, 6\rangle$$

The z-coordinates of the vectors are not equal, hence the vectors are not equivalent.

In Exercises 18–23, calculate the linear combinations.

19. $-2\langle 8, 11, 3\rangle + 4\langle 2, 1, 1\rangle$

SOLUTION Using the operations of vector addition and scalar multiplication we have

$$-2\langle 8, 11, 3\rangle + 4\langle 2, 1, 1\rangle = \langle -16, -22, -6\rangle + \langle 8, 4, 4\rangle = \langle -8, -18, -2\rangle$$

21. $\frac{1}{2}\langle 4, -2, 8\rangle - \frac{1}{3}\langle 12, 3, 3\rangle$

SOLUTION Using the operations on vectors we have

$$\frac{1}{2}\langle 4, -2, 8\rangle - \frac{1}{3}\langle 12, 3, 3\rangle = \langle 2, -1, 4\rangle - \langle 4, 1, 1\rangle = \langle -2, -2, 3\rangle.$$

23. $4\langle 6, -1, 1\rangle - 2\langle 1, 0, -1\rangle + 3\langle -2, 1, 1\rangle$

SOLUTION Using the operations of vector addition and scalar multiplication we have

$$\begin{aligned} 4\langle 6, -1, 1\rangle - 2\langle 1, 0, -1\rangle + 3\langle -2, 1, 1\rangle &= \langle 24, -4, 4\rangle + \langle -2, 0, 2\rangle + \langle -6, 3, 3\rangle \\ &= \langle 16, -1, 9\rangle. \end{aligned}$$

In Exercises 24–27, determine whether or not the two vectors are parallel.

25. $\mathbf{u} = \langle 4, 2, -6 \rangle$, $\mathbf{v} = \langle 2, -1, 3 \rangle$

SOLUTION Since the first component of $\mathbf{u}$ is twice the first component of $\mathbf{v}$, if the two vectors are to be parallel, the second component of $\mathbf{u}$ must be twice the second component of $\mathbf{v}$. But it is not; it is -2 times the second component of $\mathbf{v}$. Thus the two vectors are not parallel.

27. $\mathbf{u} = \langle -3, 1, 4 \rangle$, $\mathbf{v} = \langle 6, -2, 8 \rangle$

SOLUTION Since the first component of $\mathbf{v}$ is -2 times the first component of $\mathbf{u}$, if the two vectors are to be parallel, the third component of $\mathbf{v}$ must be -2 times the third component of $\mathbf{u}$. But it is not; it is 2 times the third component of $\mathbf{u}$. Thus the two vectors are not parallel.

In Exercises 28–31, find the given vector.

29. $\mathbf{e_w}$, where $\mathbf{w} = \langle 4, -2, -1 \rangle$

SOLUTION We first find the length of $\mathbf{w}$:

$$\|\mathbf{w}\| = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$$

Hence,

$$\mathbf{e_w} = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \left\langle \frac{4}{\sqrt{21}}, \frac{-2}{\sqrt{21}}, \frac{-1}{\sqrt{21}} \right\rangle$$

31. Unit vector in the direction opposite to $\mathbf{v} = \langle -4, 4, 2 \rangle$

SOLUTION A unit vector in the direction opposite to $\mathbf{v} = \langle -4, 4, 2 \rangle$ is the following vector:

$$-\mathbf{e_v} = -\frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

We compute the length of $\mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{(-4)^2 + 4^2 + 2^2} = 6$$

The desired vector is, thus,

$$-\mathbf{e_v} = -\frac{1}{6}\langle -4, 4, 2 \rangle = \left\langle \frac{-4}{-6}, \frac{4}{-6}, \frac{2}{-6} \right\rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, -\frac{1}{3} \right\rangle$$

In Exercises 33–40, find a vector parametrization for the line with the given description.

33. Passes through $P = (1, 2, -8)$, direction vector $\mathbf{v} = \langle 2, 1, 3 \rangle$

SOLUTION The vector parametrization for the line is

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v}$$

Inserting the given data we get

$$\mathbf{r}(t) = \langle 1, 2, -8 \rangle + t\langle 2, 1, 3 \rangle = \langle 1 + 2t, 2 + t, -8 + 3t \rangle$$

35. Passes through $P = (4, 0, 8)$, direction vector $\mathbf{v} = 7\mathbf{i} + 4\mathbf{k}$

SOLUTION Since $\mathbf{v} = 7\mathbf{i} + 4\mathbf{k} = \langle 7, 0, 4 \rangle$ we obtain the following parametrization:

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v} = \langle 4, 0, 8 \rangle + t\langle 7, 0, 4 \rangle = \langle 4 + 7t, 0, 8 + 4t \rangle$$

37. Passes through $(1, 1, 1)$ and $(3, -5, 2)$

SOLUTION We use the equation of the line through two points P and Q:

$$\mathbf{r}(t) = (1 - t)\overrightarrow{OP} + t\overrightarrow{OQ}$$

Since $\overrightarrow{OP} = \langle 1, 1, 1 \rangle$ and $\overrightarrow{OQ} = \langle 3, -5, 2 \rangle$ we obtain

$$\mathbf{r}(t) = (1 - t)\langle 1, 1, 1 \rangle + t\langle 3, -5, 2 \rangle = \langle 1 - t, 1 - t, 1 - t \rangle + \langle 3t, -5t, 2t \rangle = \langle 1 + 2t, 1 - 6t, 1 + t \rangle$$

39. Passes through O and (4, 1, 1)

SOLUTION By the equation of the line through two points we get

$$\mathbf{r}(t) = (1-t)\langle 0,0,0\rangle + t\langle 4,1,1\rangle = \langle 0,0,0\rangle + \langle 4t, t, t\rangle = \langle 4t, t, t\rangle$$

In Exercises 41–44, find parametric equations for the lines with the given description.

41. Perpendicular to the xy-plane, passes through the origin

SOLUTION A direction vector for the line is a vector parallel to the z-axis, for instance, we may choose $\mathbf{v} = \langle 0,0,1\rangle$. The line passes through the origin (0, 0, 0), hence we obtain the following parametrization:

$$\mathbf{r}(t) = \langle 0,0,0\rangle + t\langle 0,0,1\rangle = \langle 0,0,t\rangle$$

or $x = 0$, $y = 0$, $z = t$.

43. Parallel to the line through (1, 1, 0) and (0, −1, −2), passes through (0, 0, 4)

SOLUTION The direction vector is $\mathbf{v} = \langle 0-1, -1-1, -2-0\rangle = \langle -1,-2,-2\rangle$. Hence, using the equation of a line we obtain

$$\mathbf{r}(t) = \langle 0,0,4\rangle + t\langle -1,-2,-2\rangle = \langle -t, -2t, 4-2t\rangle$$

45. Which of the following is a parametrization of the line through $P = (4, 9, 8)$ perpendicular to the xz-plane (Figure 17)?

(a) $\mathbf{r}(t) = \langle 4,9,8\rangle + t\,\langle 1,0,1\rangle$
(b) $\mathbf{r}(t) = \langle 4,9,8\rangle + t\,\langle 0,0,1\rangle$
(c) $\mathbf{r}(t) = \langle 4,9,8\rangle + t\,\langle 0,1,0\rangle$
(d) $\mathbf{r}(t) = \langle 4,9,8\rangle + t\,\langle 1,1,0\rangle$

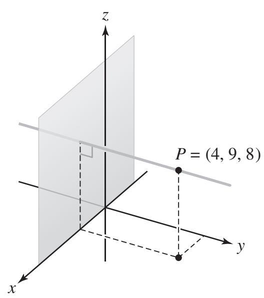


FIGURE 17

SOLUTION Since the line is perpendicular to the xz-plane, all of its points have x-coordinate equal to 4 and z-coordinate equal to 8 (see diagram). Thus only the y-coordinate varies, and the correct answer is **(c)**, $\langle 4,9,8\rangle + t\,\langle 0,1,0\rangle$.

47. Show that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ define the same line, where

$$\mathbf{r}_1(t) = \langle 3,-1,4\rangle + t\,\langle 8,12,-6\rangle$$
$$\mathbf{r}_2(t) = \langle 11,11,-2\rangle + t\,\langle 4,6,-3\rangle$$

Hint: Show that $\mathbf{r}_2(t)$ passes through (3, −1, 4) and that the direction vectors for $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are parallel.

SOLUTION We observe first that the direction vectors of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are multiples of each other:

$$\langle 8,12,-6\rangle = 2\,\langle 4,6,-3\rangle$$

Therefore $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are parallel. To show they coincide, it suffices to prove that they share a point in common, so we verify that $\mathbf{r}_1(0) = \langle 3,-1,4\rangle$ lies on $\mathbf{r}_2(t)$ by solving for t:

$$\begin{aligned}\langle 3,-1,4\rangle &= \langle 11,11,-2\rangle + t\,\langle 4,6,-3\rangle\\ \langle 3,-1,4\rangle - \langle 11,11,-2\rangle &= t\,\langle 4,6,-3\rangle\\ \langle -8,-12,6\rangle &= t\,\langle 4,6,-3\rangle\end{aligned}$$

This equation is satisfied for $t = -2$, so $\mathbf{r}_1$ and $\mathbf{r}_2$ coincide.

49. Find two different vector parametrizations of the line through $P = (5, 5, 2)$ with direction vector $\mathbf{v} = \langle 0, -2, 1\rangle$.

SOLUTION Two different parameterizations are

$$\mathbf{r}_1(t) = \langle 5, 5, 2\rangle + t\,\langle 0, -2, 1\rangle$$
$$\mathbf{r}_2(t) = \langle 5, 5, 2\rangle + t\,\langle 0, -20, 10\rangle$$

51. Show that the lines $\mathbf{r}_1(t) = \langle -1, 2, 2\rangle + t\,\langle 4, -2, 1\rangle$ and $\mathbf{r}_2(t) = \langle 0, 1, 1\rangle + t\,\langle 2, 0, 1\rangle$ do not intersect.

SOLUTION The two lines intersect if there exist parameter values t_1 and t_2 such that

$$\langle -1, 2, 2\rangle + t_1\langle 4, -2, 1\rangle = \langle 0, 1, 1\rangle + t_2\langle 2, 0, 1\rangle$$
$$\langle -1 + 4t_1, 2 - 2t_1, 2 + t_1\rangle = \langle 2t_2, 1, 1 + t_2\rangle$$

Equating corresponding components yields

$$-1 + 4t_1 = 2t_2$$
$$2 - 2t_1 = 1$$
$$2 + t_1 = 1 + t_2$$

The second equation implies $t_1 = \frac{1}{2}$. Substituting into the first and third equations we get

$$-1 + 4\cdot\frac{1}{2} = 2t_2 \quad\Rightarrow\quad t_2 = \frac{1}{2}$$
$$2 + \frac{1}{2} = 1 + t_2 \quad\Rightarrow\quad t_2 = \frac{3}{2}$$

We conclude that the equations do not have solutions, which means that the two lines do not intersect.

53. Determine whether the lines $\mathbf{r}_1(t) = \langle 0, 1, 1\rangle + t\,\langle 1, 1, 2\rangle$ and $\mathbf{r}_2(s) = \langle 2, 0, 3\rangle + s\,\langle 1, 4, 4\rangle$ intersect, and if so, find the point of intersection.

SOLUTION The lines intersect if there exist parameter values t and s such that

$$\langle 0, 1, 1\rangle + t\langle 1, 1, 2\rangle = \langle 2, 0, 3\rangle + s\langle 1, 4, 4\rangle$$
$$\langle t, 1 + t, 1 + 2t\rangle = \langle 2 + s, 4s, 3 + 4s\rangle \tag{1}$$

Equating corresponding components we get

$$t = 2 + s$$
$$1 + t = 4s$$
$$1 + 2t = 3 + 4s$$

Substituting t from the first equation into the second equation we get

$$\begin{aligned} 1 + 2 + s &= 4s \\ 3s &= 3 \end{aligned} \quad\Rightarrow\quad s = 1,\ t = 2 + s = 3$$

We now check whether $s = 1$, $t = 3$ satisfy the third equation:

$$1 + 2\cdot 3 = 3 + 4\cdot 1$$
$$7 = 7$$

We conclude that $s = 1$, $t = 3$ is the solution of (1), hence the two lines intersect. To find the point of intersection we substitute $s = 1$ in the right-hand side of (1) to obtain

$$\langle 2 + 1, 4\cdot 1, 3 + 4\cdot 1\rangle = \langle 3, 4, 7\rangle$$

The point of intersection is the terminal point of this vector, that is, $(3, 4, 7)$.

55. A meteor follows a trajectory $\mathbf{r}(t) = \langle 2, 1, 4\rangle + t\,\langle 3, 2, -1\rangle$ km. with t in minutes, near the surface of the earth, which is represented by the xy-plane. Determine at what time the meteor hits the ground.

SOLUTION Since the surface of the earth is the xy-plane, the z direction is the height above the ground. The meteor hits the ground when the z component of $\mathbf{r}(t)$ is zero; this happens when $4 - t = 0$, or $t = 4$. The meteor hits the ground at $t = 4$ minutes.

57. Find the components of the vector $\mathbf{v}$ whose tail and head are the midpoints of segments $\overline{AC}$ and $\overline{BC}$ in Figure 18. [Note that the midpoint of (a_1, a_2, a_3) and (b_1, b_2, b_3) is $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$.]

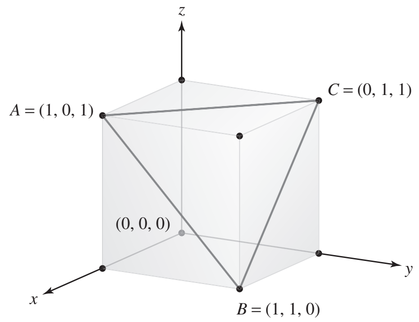
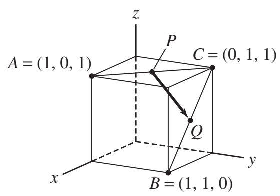


FIGURE 18

SOLUTION We denote by P and Q the midpoints of the segments $\overline{AC}$ and $\overline{BC}$ respectively. Thus,

$$\mathbf{v} = \overrightarrow{PQ} \tag{1}$$



We use the formula for the midpoint of a segment to find the coordinates of the points P and Q. This gives

$$P = \left(\frac{1+0}{2}, \frac{0+1}{2}, \frac{1+1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}, 1\right)$$

$$Q = \left(\frac{1+0}{2}, \frac{1+1}{2}, \frac{0+1}{2}\right) = \left(\frac{1}{2}, 1, \frac{1}{2}\right)$$

Substituting in (1) yields the following vector:

$$\mathbf{v} = \overrightarrow{PQ} = \left\langle \frac{1}{2} - \frac{1}{2}, 1 - \frac{1}{2}, \frac{1}{2} - 1 \right\rangle = \left\langle 0, \frac{1}{2}, -\frac{1}{2} \right\rangle.$$

59. A box that weighs 1000 kg is hanging from a crane at the dock. The crane has a square 20 m by 20 m framework as in Figure 19, with four cables, each of the same length, supporting the box. The box hangs 10 m below the level of the framework. Find the magnitude of the force acting on each cable.

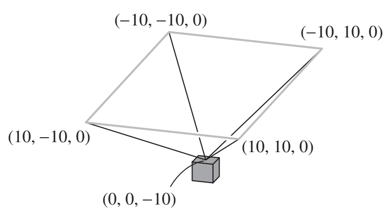


FIGURE 19

SOLUTION By symmetry, the magnitude of the force acting on all four cables is the same; denote this magnitude by f. The directions of the four forces exerted by the cables are

$$\langle 10, -10, 0 \rangle - \langle 0, 0, -10 \rangle = \langle 10, -10, 10 \rangle$$

$$\langle -10, -10, 0 \rangle - \langle 0, 0, -10 \rangle = \langle -10, -10, 10 \rangle$$

$$\langle -10, 10, 0\rangle - \langle 0, 0, -10\rangle = \langle -10, 10, 10\rangle$$
$$\langle 10, 10, 0\rangle - \langle 0, 0, -10\rangle = \langle 10, 10, 10\rangle .$$

The sum of the forces in these four directions must balance the downward force, which is $\langle 0, 0, -1000\rangle$. Thus we get

$$f\,\langle 10, -10, 10\rangle + f\,\langle -10, -10, 10\rangle + f\,\langle -10, 10, 10\rangle + f\,\langle 10, 10, 10\rangle = \langle 0, 0, -1000\rangle ,$$

so that

$$f\,\langle 0, 0, 40\rangle = \langle 0, 0, -1000\rangle ,$$

which gives $f = -25$. The force has a magnitude of 25 newtons.

Further Insights and Challenges

In Exercises 60–66, we consider the equations of a line in ***symmetric form****, when $a \neq 0$, $b \neq 0$, $c \neq 0$.*

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad \boxed{10}$$

61. Find the symmetric equations of the line through $P_0 = (-2, 3, 3)$ with direction vector $\mathbf{v} = \langle 2, 4, 3\rangle$.

SOLUTION Using $(x_0, y_0, z_0) = (-2, 3, 3)$ and $\langle a, b, c\rangle = \langle 2, 4, 3\rangle$ in Equation (10) gives

$$\frac{x + 2}{2} = \frac{y - 3}{4} = \frac{z - 3}{3}$$

63. Find the symmetric equations of the line

$$x = 3 + 2t, \quad y = 4 - 9t, \quad z = 12t$$

SOLUTION If we solve each equation fot t, we get:

$$t = \frac{x - 3}{2}, \quad t = \frac{4 - y}{9}, \quad t = \frac{z}{12}$$

When we set these equations equal to each other, we get:

$$\frac{x - 3}{2} = \frac{4 - y}{9} = \frac{z}{12}$$

65. Find a vector parametrization for the line $\frac{x}{2} = \frac{y}{7} = \frac{z}{8}$.

SOLUTION If we let t equal these three terms, as follows:

$$t = \frac{x}{2} = \frac{y}{7} = \frac{z}{8}$$

then we can break it up into three equations:

$$t = \frac{x}{2}, \quad t = \frac{y}{7}, \quad t = \frac{z}{8}$$

and solving for x, y, and z gives us:

$$x = 2t, \quad y = 7t, \quad z = 8t$$

and writing this in vector form gives us

$$\mathbf{r}(t) = t\,\langle 2, 7, 8\rangle$$

67. A median of a triangle is a segment joining a vertex to the midpoint of the opposite side. Referring to Figure 20(A), prove that three medians of triangle ABC intersect at the terminal point P of the vector $\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$. The point P is the *centroid* of the triangle. *Hint:* Show, by parametrizing the segment $\overline{AA'}$, that P lies two-thirds of the way from A to A'. It will follow similarly that P lies on the other two medians.

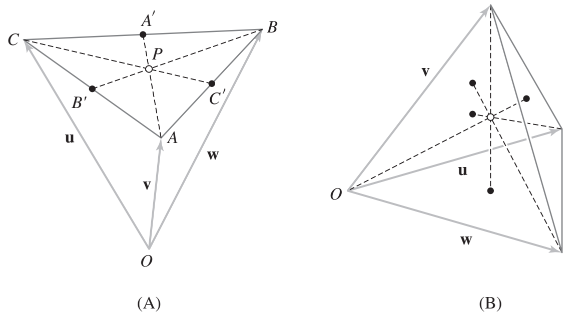


FIGURE 20

SOLUTION From Figure 20(A),

$$\overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{AC'} = \mathbf{v} + \frac{1}{2}(\mathbf{w}-\mathbf{v}) = \frac{1}{2}(\mathbf{v}+\mathbf{w})$$

$$\overrightarrow{OD} = \mathbf{u}$$

The line through the points C and C' has the parametrization

$$t\mathbf{u} + (1-t)\frac{\mathbf{v}+\mathbf{w}}{2} \tag{1}$$

Similarly, the line through B and B' has the parametrization

$$t\mathbf{w} + (1-t)\frac{\mathbf{v}+\mathbf{u}}{2} \tag{2}$$

And the line through A and A' has the parametrization

$$t\mathbf{v} + (1-t)\frac{\mathbf{u}+\mathbf{w}}{2} \tag{3}$$

Now, setting $t = \frac{1}{3}$ in (1), (2) and (3) yields $\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$. We conclude that the terminal point of this vector lies on each one of the lines, hence it is their point of intersection.

12.3 Dot Product and the Angle Between Two Vectors

Preliminary Questions

1. Is the dot product of two vectors a scalar or a vector?

SOLUTION The dot product of two vectors is the sum of products of scalars, hence it is a scalar.

2. What can you say about the angle between **a** and **b** if $\mathbf{a}\cdot\mathbf{b} < 0$?

SOLUTION Since the cosine of the angle between **a** and **b** satisfies $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$, also $\cos\theta < 0$. By definition $0 \le \theta \le \pi$, but since $\cos\theta < 0$ then θ is in $[\pi/2, \pi]$. In other words, the angle between **a** and **b** is obtuse.

3. Which property of dot products allows us to conclude that if **v** is orthogonal to both **u** and **w**, then **v** is orthogonal to $\mathbf{u}+\mathbf{w}$?

SOLUTION One property is that two vectors are orthogonal if and only if the dot product of the two vectors is zero. The second property is the Distributive Law. Since **v** is orthogonal to **u** and **w**, we have $\mathbf{v}\cdot\mathbf{u} = 0$ and $\mathbf{v}\cdot\mathbf{w} = 0$. Therefore,

$$\mathbf{v}\cdot(\mathbf{u}+\mathbf{w}) = \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{w} = 0 + 0 = 0$$

We conclude that **v** is orthogonal to $\mathbf{u}+\mathbf{w}$.

4. Which is the projection of $\mathbf{v}$ along $\mathbf{v}$: (a) $\mathbf{v}$ or (b) $\mathbf{e_v}$?

SOLUTION The projection of $\mathbf{v}$ along itself is $\mathbf{v}$, since

$$\mathbf{v}_{||\mathbf{v}} = \left(\frac{\mathbf{v}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \mathbf{v}.$$

5. Let $\mathbf{u}_{||\mathbf{v}}$ be the projection of $\mathbf{u}$ along $\mathbf{v}$. Which of the following is the projection $\mathbf{u}$ along the vector $2\mathbf{v}$ and which is the projection of $2\mathbf{u}$ along $\mathbf{v}$?

(a) $\frac{1}{2}\mathbf{u}_{||\mathbf{v}}$ **(b)** $\mathbf{u}_{||\mathbf{v}}$ **(c)** $2\mathbf{u}_{||\mathbf{v}}$

SOLUTION Since $\mathbf{u}_{\|\mathbf{v}}$ is the projection of $\mathbf{u}$ along $\mathbf{v}$, we have,

$$\mathbf{u}_{\|\mathbf{v}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$

The projection of $\mathbf{u}$ along the vector $2\mathbf{v}$ is

$$\mathbf{u}_{\|2\mathbf{v}} = \left(\frac{\mathbf{u}\cdot(2\mathbf{v})}{(2\mathbf{v})\cdot(2\mathbf{v})}\right)2\mathbf{v} = \left(\frac{2(\mathbf{u}\cdot\mathbf{v})}{4(\mathbf{v}\cdot\mathbf{v})}\right)2\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \mathbf{u}_{\|\mathbf{v}}.$$

That is, $\mathbf{u}_{\|\mathbf{v}}$ is the projection of $\mathbf{u}$ along $2\mathbf{v}$. Notice that the projection of $\mathbf{u}$ along $\mathbf{v}$ is the projection of $\mathbf{u}$ along the unit vector $\mathbf{e_v}$, hence it depends on the direction of $\mathbf{v}$ rather than on the length of $\mathbf{v}$. Therefore, the projection of $\mathbf{u}$ along $\mathbf{v}$ and along $2\mathbf{v}$ is the same vector.

For the second question,

$$(2\mathbf{u})_{\|\mathbf{v}} = \left(\frac{(2\mathbf{u})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = 2\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = 2\mathbf{u}_{\|\mathbf{v}}.$$

That is, the projection of $2\mathbf{u}$ along $\mathbf{v}$ is twice the projection of $\mathbf{u}$ along $\mathbf{v}$.

6. Which of the following is equal to $\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$?

(a) $\mathbf{u}\cdot\mathbf{v}$ **(b)** $\mathbf{u}\cdot\mathbf{e_v}$ **(c)** $\mathbf{e_u}\cdot\mathbf{e_v}$

SOLUTION By the Theorems on the Dot Product and the Angle Between Vectors, we have

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u}}{\|\mathbf{u}\|}\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|} = \mathbf{e_u}\cdot\mathbf{e_v}$$

The correct answer is (c).

Exercises

In Exercises 1–12, compute the dot product.

1. $\langle 1, 2, 1\rangle \cdot \langle 4, 3, 5\rangle$

SOLUTION Using the definition of the dot product we obtain

$$\langle 1, 2, 1\rangle \cdot \langle 4, 3, 5\rangle = 1\cdot 4 + 2\cdot 3 + 1\cdot 5 = 15$$

3. $\langle 0, 1, 0\rangle \cdot \langle 7, 41, -3\rangle$

SOLUTION The dot product is

$$\langle 0, 1, 0\rangle \cdot \langle 7, 41, -3\rangle = 0\cdot 7 + 1\cdot 41 + 0\cdot(-3) = 41$$

5. $\langle 3, 1\rangle \cdot \langle 4, -7\rangle$

SOLUTION The dot product of the two vectors is the following scalar:

$$\langle 3, 1\rangle \cdot \langle 4, -7\rangle = 3\cdot 4 + 1\cdot(-7) = 5$$

7. $\mathbf{k}\cdot\mathbf{j}$

SOLUTION By the orthogonality of $\mathbf{j}$ and $\mathbf{k}$, we have $\mathbf{k}\cdot\mathbf{j} = 0$

9. $(\mathbf{i}+\mathbf{j})\cdot(\mathbf{j}+\mathbf{k})$

SOLUTION By the distributive law and the orthogonality of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ we have

$$(\mathbf{i}+\mathbf{j})\cdot(\mathbf{j}+\mathbf{k}) = \mathbf{i}\cdot\mathbf{j} + \mathbf{i}\cdot\mathbf{k} + \mathbf{j}\cdot\mathbf{j} + \mathbf{j}\cdot\mathbf{k} = 0+0+1+0 = 1$$

11. $(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}-5\mathbf{k})$

SOLUTION We use properties of the dot product to obtain

$$\begin{aligned}(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}-5\mathbf{k}) &= 3\mathbf{i}\cdot\mathbf{i}+2\mathbf{i}\cdot\mathbf{j}-5\mathbf{i}\cdot\mathbf{k}+3\mathbf{j}\cdot\mathbf{i}+2\mathbf{j}\cdot\mathbf{j}-5\mathbf{j}\cdot\mathbf{k}+3\mathbf{k}\cdot\mathbf{i}+2\mathbf{k}\cdot\mathbf{j}-5\mathbf{k}\cdot\mathbf{k}\\ &= 3\|\mathbf{i}\|^2+2\|\mathbf{j}\|^2-5\|\mathbf{k}\|^2 = 3\cdot 1+2\cdot 1-5\cdot 1 = 0\end{aligned}$$

In Exercises 13–18, determine whether the two vectors are orthogonal and, if not, whether the angle between them is acute or obtuse.

13. $\langle 1, 1, 1\rangle, \quad \langle 1, -2, -2\rangle$

SOLUTION We compute the dot product of the two vectors:

$$\langle 1, 1, 1\rangle\cdot\langle 1, -2, -2\rangle = 1\cdot 1+1\cdot(-2)+1\cdot(-2) = -3$$

Since the dot product is negative, the angle between the vectors is obtuse.

15. $\langle 1, 2, 1\rangle, \quad \langle 7, -3, -1\rangle$

SOLUTION We compute the dot product:

$$\langle 1, 1, 1\rangle\cdot\langle 3, -2, -1\rangle = 1\cdot 3+1\cdot(-2)+1\cdot(-1) = 0$$

The dot product is zero, hence the vectors are orthogonal.

17. $\left\langle \frac{12}{5}, -\frac{4}{5}\right\rangle, \quad \left\langle \frac{1}{2}, -\frac{7}{4}\right\rangle$

SOLUTION We find the dot product of the two vectors:

$$\left\langle \frac{12}{5}, -\frac{4}{5}\right\rangle\cdot\left\langle \frac{1}{2}, -\frac{7}{4}\right\rangle = \frac{12}{5}\cdot\frac{1}{2}+\left(-\frac{4}{5}\right)\cdot\left(-\frac{7}{4}\right) = \frac{12}{10}+\frac{28}{20} = \frac{13}{5}$$

The dot product is positive, hence the angle between the vectors is acute.

In Exercises 19–22, find the cosine of the angle between the vectors.

19. $\langle 0, 3, 1\rangle, \quad \langle 4, 0, 0\rangle$

SOLUTION Since $\langle 0, 3, 1\rangle\cdot\langle 4, 0, 0\rangle = 0\cdot 4+3\cdot 0+1\cdot 0 = 0$, the vectors are orthogonal, that is, the angle between them is $\theta = 90^\circ$ and $\cos\theta = 0$.

21. $\mathbf{i}+\mathbf{j}, \quad \mathbf{j}+2\mathbf{k}$

SOLUTION We use the formula for the cosine of the angle between two vectors. Let $\mathbf{v} = \mathbf{i}+\mathbf{j}$ and $\mathbf{w} = \mathbf{j}+2\mathbf{k}$. We compute the following values:

$$\begin{aligned}\|\mathbf{v}\| &= \|\mathbf{i}+\mathbf{j}\| = \sqrt{1^2+1^2} = \sqrt{2}\\ \|\mathbf{w}\| &= \|\mathbf{j}+2\mathbf{k}\| = \sqrt{1^2+2^2} = \sqrt{5}\\ \mathbf{v}\cdot\mathbf{w} &= (\mathbf{i}+\mathbf{j})\cdot(\mathbf{j}+2\mathbf{k}) = \mathbf{i}\cdot\mathbf{j}+2\mathbf{i}\cdot\mathbf{k}+\mathbf{j}\cdot\mathbf{j}+2\mathbf{j}\cdot\mathbf{k} = \|\mathbf{j}\|^2 = 1\end{aligned}$$

Hence,

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{1}{\sqrt{2}\sqrt{5}} = \frac{1}{\sqrt{10}}.$$

In Exercises 23–28, find the angle between the vectors. Use a calculator if necessary.

23. $\langle 2, \sqrt{2}\rangle, \quad \langle 1+\sqrt{2}, 1-\sqrt{2}\rangle$

SOLUTION We write $\mathbf{v} = \left\langle 2, \sqrt{2}\right\rangle$ and $\mathbf{w} = \left\langle 2, \sqrt{2}\right\rangle$. To use the formula for the cosine of the angle θ between two vectors we need to compute the following values:

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{4+2} = \sqrt{6}\\ \|\mathbf{w}\| &= \sqrt{(1+\sqrt{2})^2+(1-\sqrt{2})^2} = \sqrt{6}\\ \mathbf{v}\cdot\mathbf{w} &= 2+2\sqrt{2}+\sqrt{2}-2 = 3\sqrt{2}\end{aligned}$$

Hence,

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{3\sqrt{2}}{\sqrt{6}\sqrt{6}} = \frac{\sqrt{2}}{2}$$

and so,

$$\theta = \cos^{-1}\frac{\sqrt{2}}{2} = \pi/4$$

25. $\langle 1, 1, 1\rangle, \quad \langle 1, 0, 1\rangle$

SOLUTION We denote $\mathbf{v} = \langle 1, 1, 1\rangle$ and $\mathbf{w} = \langle 1, 0, 1\rangle$. To use the formula for the cosine of the angle θ between two vectors we need to compute the following values:

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}\\ \|\mathbf{w}\| &= \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}\\ \mathbf{v}\cdot\mathbf{w} &= 1 + 0 + 1 = 2\end{aligned}$$

Hence,

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{2}{\sqrt{3}\sqrt{2}} = \frac{\sqrt{6}}{3}$$

and so,

$$\theta = \cos^{-1}\frac{\sqrt{6}}{3} \approx 0.615$$

27. $\langle 0, 1, 1\rangle, \quad \langle 1, -1, 0\rangle$

SOLUTION We denote $\mathbf{v} = \langle 0, 1, 1\rangle$ and $\mathbf{w} = \langle 1, -1, 0\rangle$. To use the formula for the cosine of the angle θ between two vectors we need to compute the following values:

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}\\ \|\mathbf{w}\| &= \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}\\ \mathbf{v}\cdot\mathbf{w} &= 0 + (-1) + 0 = -1\end{aligned}$$

Hence,

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-1}{\sqrt{2}\sqrt{2}} = -\frac{1}{2}$$

and so,

$$\theta = \cos^{-1} -\frac{1}{2} = \frac{2\pi}{3}$$

29. Find all values of b for which the vectors are orthogonal.

(a) $\langle b, 3, 2\rangle, \quad \langle 1, b, 1\rangle$ **(b)** $\langle 4, -2, 7\rangle, \quad \left\langle b^2, b, 0\right\rangle$

SOLUTION

(a) The vectors are orthogonal if and only if the scalar product is zero. That is,

$$\begin{aligned}\langle b, 3, 2\rangle \cdot \langle 1, b, 1\rangle &= 0\\ b\cdot 1 + 3\cdot b + 2\cdot 1 &= 0\\ 4b + 2 &= 0 \quad \Rightarrow \quad b = -\frac{1}{2}\end{aligned}$$

(b) We set the scalar product of the two vectors equal to zero and solve for b. This gives

$$\begin{aligned}\langle 4, -2, 7\rangle \cdot \langle b^2, b, 0\rangle &= 0\\ 4b^2 - 2b + 7\cdot 0 &= 0\\ 2b(2b-1) &= 0 \quad \Rightarrow \quad b = 0 \text{ or } b = \frac{1}{2}\end{aligned}$$

31. Find two vectors that are not multiples of each other and are both orthogonal to $\langle 2, 0, -3\rangle$.

SOLUTION We denote by $\langle a, b, c\rangle$, a vector orthogonal to $\langle 2, 0, -3\rangle$. Hence,

$$\begin{aligned}\langle a, b, c\rangle \cdot \langle 2, 0, -3\rangle &= 0\\ 2a + 0 - 3c &= 0\\ 2a - 3c &= 0 \quad \Rightarrow \quad a = \frac{3}{2}c\end{aligned}$$

Thus, the vectors orthogonal to $\langle 2, 0, -3\rangle$ are of the form

$$\left\langle \frac{3}{2}c, b, c \right\rangle.$$

We may find two such vectors by setting $c = 0$, $b = 1$ and $c = 2$, $b = 2$. We obtain

$$\mathbf{v}_1 = \langle 0, 1, 0\rangle, \quad \mathbf{v}_2 = \langle 3, 2, 2\rangle.$$

33. Find $\mathbf{v} \cdot \mathbf{e}$ where $\|\mathbf{v}\| = 3$, $\mathbf{e}$ is a unit vector, and the angle between $\mathbf{e}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$.

SOLUTION Since $\mathbf{v} \cdot \mathbf{e} = \|\mathbf{v}\|\|\mathbf{e}\| \cos 2\pi/3$, and $\|\mathbf{v}\| = 3$ and $\|\mathbf{e}\| = 1$, we have $\mathbf{v} \cdot \mathbf{e} = 3 \cdot 1 \cdot (-1/2) = -3/2$.

In Exercises 35–38, simplify the expression.

35. $(\mathbf{v} - \mathbf{w}) \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w}$

SOLUTION By properties of the dot product we obtain

$$(\mathbf{v} - \mathbf{w}) \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|^2 - \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|^2$$

37. $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} - (\mathbf{v} + \mathbf{w}) \cdot \mathbf{w}$

SOLUTION We use properties of the dot product to write

$$\begin{aligned}(\mathbf{v} + \mathbf{w}) \cdot \mathbf{v} - (\mathbf{v} + \mathbf{w}) \cdot \mathbf{w} &= \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{w}\\ &= \|\mathbf{v}\|^2 + \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} - \|\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2\end{aligned}$$

In Exercises 39–42, use the properties of the dot product to evaluate the expression, assuming that $\mathbf{u} \cdot \mathbf{v} = 2$, $\|\mathbf{u}\| = 1$, *and* $\|\mathbf{v}\| = 3$.

39. $\mathbf{u} \cdot (4\mathbf{v})$

SOLUTION Using properties of the dot product we get

$$\mathbf{u} \cdot (4\mathbf{v}) = 4(\mathbf{u} \cdot \mathbf{v}) = 4 \cdot 2 = 8.$$

41. $2\mathbf{u} \cdot (3\mathbf{u} - \mathbf{v})$

SOLUTION By properties of the dot product we obtain

$$\begin{aligned}2\mathbf{u} \cdot (3\mathbf{u} - \mathbf{v}) &= (2\mathbf{u}) \cdot (3\mathbf{u}) - (2\mathbf{u}) \cdot \mathbf{v} = 6(\mathbf{u} \cdot \mathbf{u}) - 2(\mathbf{u} \cdot \mathbf{v})\\ &= 6\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) = 6 \cdot 1^2 - 2 \cdot 2 = 2\end{aligned}$$

43. Find the angle between $\mathbf{v}$ and $\mathbf{w}$ if $\mathbf{v} \cdot \mathbf{w} = -\|\mathbf{v}\|\ \|\mathbf{w}\|$.

SOLUTION Using the formula for dot product, and the given equation $\mathbf{v} \cdot \mathbf{w} = -\|\mathbf{v}\|\ \|\mathbf{w}\|$, we get:

$$\|\mathbf{v}\|\ \|\mathbf{w}\| \cos\theta = -\|\mathbf{v}\|\ \|\mathbf{w}\|,$$

which implies $\cos\theta = -1$, and so the angle between the two vectors is $\theta = \pi$.

45. Assume that $\|\mathbf{v}\| = 3$, $\|\mathbf{w}\| = 5$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\theta = \frac{\pi}{3}$.

(a) Use the relation $\|\mathbf{v} + \mathbf{w}\|^2 = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w})$ to show that $\|\mathbf{v} + \mathbf{w}\|^2 = 3^2 + 5^2 + 2\mathbf{v} \cdot \mathbf{w}$.

(b) Find $\|\mathbf{v} + \mathbf{w}\|$.

SOLUTION For part (a), we use the distributive property to get:

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2 &= (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w})\\ &= \mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}+\mathbf{w}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w}\\ &= \|\mathbf{v}\|^2+2\mathbf{v}\cdot\mathbf{w}+\|\mathbf{w}\|^2\\ &= 3^2+5^2+2\mathbf{v}\cdot\mathbf{w}\end{aligned}$$

For part (b), we use the definition of dot product on the previous equation to get:

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2 &= 3^2+5^2+2\mathbf{v}\cdot\mathbf{w}\\ &= 34+2\cdot 3\cdot 5\cdot\cos\pi/3\\ &= 34+15=49\end{aligned}$$

Thus, $\|\mathbf{v}+\mathbf{w}\| = \sqrt{49} = 7$.

47. Show that if $\mathbf{e}$ and $\mathbf{f}$ are unit vectors such that $\|\mathbf{e}+\mathbf{f}\| = \frac{3}{2}$, then $\|\mathbf{e}-\mathbf{f}\| = \frac{\sqrt{7}}{2}$. *Hint:* Show that $\mathbf{e}\cdot\mathbf{f} = \frac{1}{8}$.

SOLUTION We use the relation of the dot product with length and properties of the dot product to write

$$\begin{aligned}9/4 = \|\mathbf{e}+\mathbf{f}\|^2 &= (\mathbf{e}+\mathbf{f})\cdot(\mathbf{e}+\mathbf{f}) = \mathbf{e}\cdot\mathbf{e}+\mathbf{e}\cdot\mathbf{f}+\mathbf{f}\cdot\mathbf{e}+\mathbf{f}\cdot\mathbf{f}\\ &= \|\mathbf{e}\|^2+2\mathbf{e}\cdot\mathbf{f}+\|\mathbf{f}\|^2 = 1^2+2\mathbf{e}\cdot\mathbf{f}+1^2 = 2+2\mathbf{e}\cdot\mathbf{f}\end{aligned}$$

We now find $\mathbf{e}\cdot\mathbf{f}$:

$$9/4 = 2+2\mathbf{e}\cdot\mathbf{f} \quad\Rightarrow\quad \mathbf{e}\cdot\mathbf{f} = 1/8$$

Hence, using the same method as above, we have:

$$\begin{aligned}\|\mathbf{e}-\mathbf{f}\|^2 &= (\mathbf{e}-\mathbf{f})\cdot(\mathbf{e}-\mathbf{f}) = \mathbf{e}\cdot\mathbf{e}-\mathbf{e}\cdot\mathbf{f}-\mathbf{f}\cdot\mathbf{e}+\mathbf{f}\cdot\mathbf{f}\\ &= \|\mathbf{e}\|^2-2\mathbf{e}\cdot\mathbf{f}+\|\mathbf{f}\|^2 = 1^2-2\mathbf{e}\cdot\mathbf{f}+1^2 = 2-2\mathbf{e}\cdot\mathbf{f} = 2-2/8 = 7/4.\end{aligned}$$

Taking square roots, we get:

$$\|\mathbf{e}-\mathbf{f}\| = \frac{\sqrt{7}}{2}$$

49. Find the angle θ in the triangle in Figure 12.

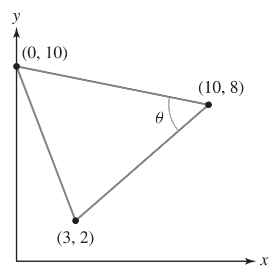
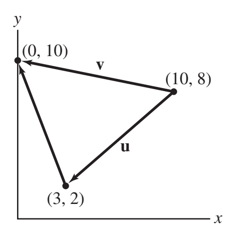


FIGURE 12

SOLUTION We denote by $\mathbf{u}$ and $\mathbf{v}$ the vectors in the figure.



Hence,

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|\,\|\mathbf{u}\|} \tag{1}$$

We find the vectors $\mathbf{v}$ and $\mathbf{u}$, and then compute their length and the dot product $\mathbf{v}\cdot\mathbf{u}$. This gives

$$\begin{aligned}
\mathbf{v} &= \langle 0-10, 10-8\rangle = \langle -10, 2\rangle\\
\mathbf{u} &= \langle 3-10, 2-8\rangle = \langle -7, -6\rangle\\
\|\mathbf{v}\| &= \sqrt{(-10)^2+2^2} = \sqrt{104}\\
\|\mathbf{u}\| &= \sqrt{(-7)^2+(-6)^2} = \sqrt{85}\\
\mathbf{v}\cdot\mathbf{u} &= \langle -10, 2\rangle\cdot\langle -7, -6\rangle = (-10)\cdot(-7) + 2\cdot(-6) = 58
\end{aligned}$$

Substituting these values in (1) yields

$$\cos\theta = \frac{58}{\sqrt{104}\sqrt{85}} \approx 0.617$$

Hence the angle of the triangle is 51.91°.

51. (a) Draw $\mathbf{u}_{||\mathbf{v}}$ and $\mathbf{v}_{||\mathbf{u}}$ for the vectors appearing as in Figure 14.
(b) Which of $\mathbf{u}_{||\mathbf{v}}$ and $\mathbf{v}_{||\mathbf{u}}$ has the greater magnitude?

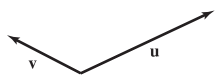
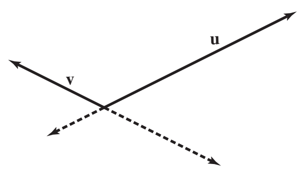


FIGURE 14

SOLUTION
(a)



(b) The component of $\mathbf{u}$ parallel to $\mathbf{v}$, $\mathbf{u}_{||\mathbf{v}}$, has the greater magnitude.

In Exercises 53–60, find the projection of $\mathbf{u}$ *along* $\mathbf{v}$.

53. $\mathbf{u} = \langle 2, 5\rangle, \quad \mathbf{v} = \langle 1, 1\rangle$

SOLUTION We first compute the following dot products:

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= \langle 2, 5\rangle\cdot\langle 1, 1\rangle = 7\\
\mathbf{v}\cdot\mathbf{v} &= \|\mathbf{v}\|^2 = 1^2 + 1^2 = 2
\end{aligned}$$

The projection of $\mathbf{u}$ along $\mathbf{v}$ is the following vector:

$$\mathbf{u}_{||\mathbf{v}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{7}{2}\mathbf{v} = \left\langle \frac{7}{2}, \frac{7}{2}\right\rangle$$

55. $\mathbf{u} = \langle -1, 2, 0\rangle, \quad \mathbf{v} = \langle 2, 0, 1\rangle$

SOLUTION The projection of $\mathbf{u}$ along $\mathbf{v}$ is the following vector:

$$\mathbf{u}_{\|\mathbf{v}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$

We compute the values in this expression:

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= \langle -1, 2, 0\rangle\cdot\langle 2, 0, 1\rangle = -1\cdot 2 + 2\cdot 0 + 0\cdot 1 = -2\\
\mathbf{v}\cdot\mathbf{v} &= \|\mathbf{v}\|^2 = 2^2 + 0^2 + 1^2 = 5
\end{aligned}$$

Hence,

$$\mathbf{u}_{\|\mathbf{v}} = -\frac{2}{5}\langle 2, 0, 1\rangle = \left\langle -\frac{4}{5}, 0, -\frac{2}{5}\right\rangle.$$

57. $\mathbf{u} = 5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}, \quad \mathbf{v} = \mathbf{k}$

SOLUTION The projection of $\mathbf{u}$ along $\mathbf{v}$ is the following vector:

$$\mathbf{u}_{\|\mathbf{v}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$

We compute the dot products:

$$\mathbf{u}\cdot\mathbf{v} = (5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k})\cdot\mathbf{k} = -4\mathbf{k}\cdot\mathbf{k} = -4$$
$$\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2 = \|\mathbf{k}\|^2 = 1$$

Hence,

$$\mathbf{u}_{\|\mathbf{v}} = \frac{-4}{1}\mathbf{k} = -4\mathbf{k}$$

59. $\mathbf{u} = \langle a, b, c\rangle, \quad \mathbf{v} = \mathbf{i}$

SOLUTION The component of $\mathbf{u}$ along $\mathbf{v}$ is a, since

$$\mathbf{u}\cdot\mathbf{e_v} = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\cdot\mathbf{i} = a$$

Therefore, the projection of $\mathbf{u}$ along $\mathbf{v}$ is the vector

$$\mathbf{u}_{\|\mathbf{v}} = (\mathbf{u}\cdot\mathbf{e_v})\mathbf{e_v} = a\mathbf{i}$$

In Exercises 61 and 62, compute the component of $\mathbf{u}$ *along* $\mathbf{v}$.

61. $\mathbf{u} = \langle 3, 2, 1\rangle, \quad \mathbf{v} = \langle 1, 0, 1\rangle$

SOLUTION We first compute the following dot products:

$$\mathbf{u}\cdot\mathbf{v} = \langle 3, 2, 1\rangle\cdot\langle 1, 0, 1\rangle = 4$$
$$\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2 = 1^2 + 1^2 = 2$$

The component of $\mathbf{u}$ along $\mathbf{v}$ is the length of the projection of $\mathbf{u}$ along $\mathbf{v}$

$$\left\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}\right\| = \frac{4}{2}\|\mathbf{v}\| = 2\|\mathbf{v}\| = 2\sqrt{2}$$

63. Find the length of $\overline{OP}$ in Figure 15.

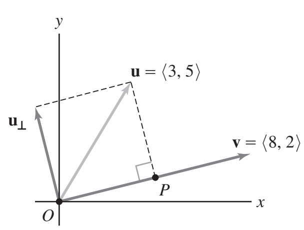


FIGURE 15

SOLUTION This is just the component of $\mathbf{u} = \langle 3, 5\rangle$ along $\mathbf{v} = \langle 8, 2\rangle$. We first compute the following dot products:

$$\mathbf{u}\cdot\mathbf{v} = \langle 3, 5\rangle\cdot\langle 8, 2\rangle = 34$$
$$\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2 = 8^2 + 2^2 = 68$$

The component of $\mathbf{u}$ along $\mathbf{v}$ is the length of the projection of $\mathbf{u}$ along $\mathbf{v}$

$$\left\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}\right\| = \frac{34}{68}\|\mathbf{v}\| = \frac{34}{68}\sqrt{68}$$

In Exercises 65–70, find the decomposition $\mathbf{a} = \mathbf{a}_{\|\mathbf{b}} + \mathbf{a}_{\perp\mathbf{b}}$ *with respect to* $\mathbf{b}$.

65. $\mathbf{a} = \langle 1, 0\rangle, \quad \mathbf{b} = \langle 1, 1\rangle$

SOLUTION

Step 1. We compute $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{b}$

$$\mathbf{a}\cdot\mathbf{b} = \langle 1, 0\rangle \cdot \langle 1, 1\rangle = 1\cdot 1 + 0\cdot 1 = 1$$
$$\mathbf{b}\cdot\mathbf{b} = \|\mathbf{b}\|^2 = 1^2 + 1^2 = 2$$

Step 2. We find the projection of $\mathbf{a}$ along $\mathbf{b}$:

$$\mathbf{a}_{\|\mathbf{b}} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \frac{1}{2}\langle 1, 1\rangle = \left\langle \frac{1}{2}, \frac{1}{2}\right\rangle$$

Step 3. We find the orthogonal part as the difference:

$$\mathbf{a}_{\perp\mathbf{b}} = \mathbf{a} - \mathbf{a}_{\|\mathbf{b}} = \langle 1, 0\rangle - \left\langle \frac{1}{2}, \frac{1}{2}\right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$$

Hence,

$$\mathbf{a} = \mathbf{a}_{\|\mathbf{b}} + \mathbf{a}_{\perp\mathbf{b}} = \left\langle \frac{1}{2}, \frac{1}{2}\right\rangle + \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle.$$

67. $\mathbf{a} = \langle 4, -1, 0\rangle, \quad \mathbf{b} = \langle 0, 1, 1\rangle$

SOLUTION We first compute $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{b}$ to find the projection of $\mathbf{a}$ along $\mathbf{b}$:

$$\mathbf{a}\cdot\mathbf{b} = \langle 4, -1, 0\rangle \cdot \langle 0, 1, 1\rangle = 4\cdot 0 + (-1)\cdot 1 + 0\cdot 1 = -1$$
$$\mathbf{b}\cdot\mathbf{b} = \|\mathbf{b}\|^2 = 0^2 + 1^2 + 1^2 = 2$$

Hence,

$$\mathbf{a}_{\|\mathbf{b}} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \frac{-1}{2}\langle 0, 1, 1\rangle = \left\langle 0, -\frac{1}{2}, -\frac{1}{2}\right\rangle$$

We now find the vector $\mathbf{a}_{\perp\mathbf{b}}$ orthogonal to $\mathbf{b}$ by computing the difference:

$$\mathbf{a} - \mathbf{a}_{\|\mathbf{b}} = \langle 4, -1, 0\rangle - \left\langle 0, -\frac{1}{2}, -\frac{1}{2}\right\rangle = \left\langle 4, -\frac{1}{2}, \frac{1}{2}\right\rangle$$

Thus, we have

$$\mathbf{a} = \mathbf{a}_{\|\mathbf{b}} + \mathbf{a}_{\perp\mathbf{b}} = \left\langle 0, -\frac{1}{2}, -\frac{1}{2}\right\rangle + \left\langle 4, -\frac{1}{2}, \frac{1}{2}\right\rangle.$$

69. $\mathbf{a} = \langle x, y\rangle, \quad \mathbf{b} = \langle 1, -1\rangle$

SOLUTION We first compute $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{b}$ to find the projection of $\mathbf{a}$ along $\mathbf{b}$:

$$\mathbf{a}\cdot\mathbf{b} = \langle x, y\rangle \cdot \langle 1, -1\rangle = x - y$$
$$\mathbf{b}\cdot\mathbf{b} = \|\mathbf{b}\|^2 = 1^2 + (-1)^2 = 2$$

Hence,

$$\mathbf{a}_{||} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \frac{x-y}{2}\langle 1, -1\rangle = \left\langle \frac{x-y}{2}, \frac{y-x}{2}\right\rangle$$

We now find the vector $\mathbf{a}_{\perp}$ orthogonal to $\mathbf{b}$ by computing the difference:

$$\mathbf{a} - \mathbf{a}_{||} = \langle x, y\rangle - \left\langle \frac{x-y}{2}, \frac{y-x}{2}\right\rangle = \left\langle \frac{x+y}{2}, \frac{x+y}{2}\right\rangle$$

Thus, we have

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_{\perp} = \left\langle \frac{x-y}{2}, \frac{y-x}{2}\right\rangle + \left\langle \frac{x+y}{2}, \frac{x+y}{2}\right\rangle$$

71. Let $\mathbf{e}_\theta = \langle \cos\theta, \sin\theta \rangle$. Show that $\mathbf{e}_\theta \cdot \mathbf{e}_\psi = \cos(\theta - \psi)$ for any two angles θ and ψ.

SOLUTION First, $\mathbf{e}_\theta$ is a unit vector since by a trigonometric identity we have

$$\|\mathbf{e}_\theta\| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

The cosine of the angle α between $\mathbf{e}_\theta$ and the vector $\mathbf{i}$ in the direction of the positive x-axis is

$$\cos\alpha = \frac{\mathbf{e}_\theta \cdot \mathbf{i}}{\|\mathbf{e}_\theta\| \cdot \|\mathbf{i}\|} = \mathbf{e}_\theta \cdot \mathbf{i} = ((\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}) \cdot \mathbf{i} = \cos\theta$$

The solution of $\cos\alpha = \cos\theta$ for angles between 0 and π is $\alpha = \theta$. That is, the vector $\mathbf{e}_\theta$ makes an angle θ with the x-axis. We now use the trigonometric identity

$$\cos\theta\cos\psi + \sin\theta\sin\psi = \cos(\theta - \psi)$$

to obtain the following equality:

$$\mathbf{e}_\theta \cdot \mathbf{e}_\psi = \langle \cos\theta, \sin\theta \rangle \cdot \langle \cos\psi, \sin\psi \rangle = \cos\theta\cos\psi + \sin\theta\sin\psi = \cos(\theta - \psi)$$

In Exercises 73–76, refer to Figure 16.

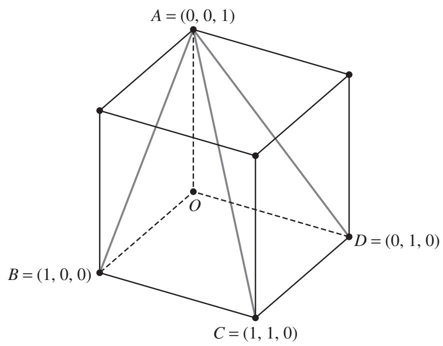
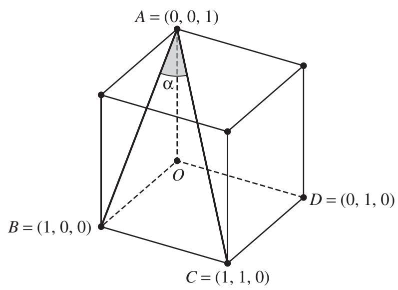


FIGURE 16 Unit cube in $\mathbf{R}^3$.

73. Find the angle between $\overline{AB}$ and $\overline{AC}$.

SOLUTION The cosine of the angle α between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is

$$\cos\alpha = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|} \tag{1}$$



We compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and then calculate their dot product and lengths. We get

$$\overrightarrow{AB} = \langle 1-0, 0-0, 0-1 \rangle = \langle 1, 0, -1 \rangle$$
$$\overrightarrow{AC} = \langle 1-0, 1-0, 0-1 \rangle = \langle 1, 1, -1 \rangle$$
$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \langle 1, 0, -1 \rangle \cdot \langle 1, 1, -1 \rangle = 1 \cdot 1 + 0 \cdot 1 + (-1) \cdot (-1) = 2$$
$$\|\overrightarrow{AB}\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$
$$\|\overrightarrow{AC}\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

Substituting in (1) and solving for $0 \le \alpha \le 90^\circ$ gives

$$\cos\alpha = \frac{2}{\sqrt{2}\cdot\sqrt{3}} \approx 0.816 \quad \Rightarrow \quad \alpha \approx 35.31^\circ.$$

75. Calculate the projection of $\overrightarrow{AC}$ along $\overrightarrow{AD}$.

SOLUTION $\overline{DC}$ is perpendicular to the face OAD of the cube. Hence, it is orthogonal to the segment $\overline{AD}$ on this face. Therefore, the projection of the vector $\overrightarrow{AC}$ along $\overrightarrow{AD}$ is the vector $\overrightarrow{AD}$ itself.

77. The methane molecule CH_4 consists of a carbon molecule bonded to four hydrogen molecules that are spaced as far apart from each other as possible. The hydrogen atoms then sit at the vertices of a tetrahedron, with the carbon atom at its center, as in Figure 17. We can model this with the carbon atom at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the hydrogen atoms at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$. Use the dot product to find the bond angle α formed between any two of the line segments from the carbon atom to the hydrogen atoms.

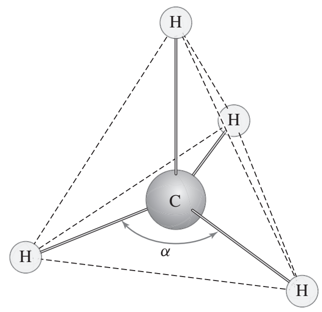


FIGURE 17 A methane molecule.

SOLUTION Use the atoms at $(0, 0, 0)$ and $(1, 1, 0)$. The vectors from C to these atoms are

$$\mathbf{u} = \langle 0, 0, 0\rangle - \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle = \left\langle -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle$$
$$\mathbf{v} = \langle 1, 1, 0\rangle - \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle = \left\langle \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle.$$

The angle α between these two vectors is given by

$$\cos\alpha = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-\frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\left(-\frac{1}{2}\right)}{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}} = \frac{-1/4}{3/4} = -\frac{1}{3}.$$

It follows that

$$\alpha = \cos^{-1}\left(-\frac{1}{3}\right) \approx 1.9106 \approx 109.471^\circ.$$

79. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors and set $\mathbf{u} = \mathbf{e_v} + \mathbf{e_w}$. Use the dot product to show that the angle between $\mathbf{u}$ and $\mathbf{v}$ is equal to the angle between $\mathbf{u}$ and $\mathbf{w}$. Explain this result geometrically with a diagram.

SOLUTION We denote by α the angle between $\mathbf{u}$ and $\mathbf{v}$ and by β the angle between $\mathbf{u}$ and $\mathbf{w}$. Since $\mathbf{e_v}$ and $\mathbf{e_w}$ are vectors in the directions of $\mathbf{v}$ and $\mathbf{w}$ respectively, α is the angle between $\mathbf{u}$ and $\mathbf{e_v}$ and β is the angle between $\mathbf{u}$ and $\mathbf{e_w}$. The cosines of these angles are thus

$$\cos\alpha = \frac{\mathbf{u}\cdot\mathbf{e_v}}{\|\mathbf{u}\|\|\mathbf{e_v}\|} = \frac{\mathbf{u}\cdot\mathbf{e_v}}{\|\mathbf{u}\|}; \quad \cos\beta = \frac{\mathbf{u}\cdot\mathbf{e_w}}{\|\mathbf{u}\|\|\mathbf{e_w}\|} = \frac{\mathbf{u}\cdot\mathbf{e_w}}{\|\mathbf{u}\|}$$

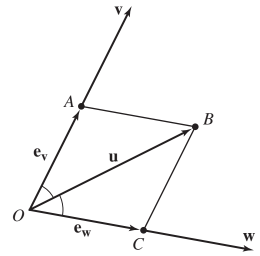
To show that $\cos\alpha = \cos\beta$ (which implies that $\alpha = \beta$) we must show that

$$\mathbf{u}\cdot\mathbf{e_v} = \mathbf{u}\cdot\mathbf{e_w}.$$

We compute the two dot products:

$$\mathbf{u}\cdot\mathbf{e_v} = (\mathbf{e_v} + \mathbf{e_w})\cdot\mathbf{e_v} = \mathbf{e_v}\cdot\mathbf{e_v} + \mathbf{e_w}\cdot\mathbf{e_v} = 1 + \mathbf{e_w}\cdot\mathbf{e_v}$$
$$\mathbf{u}\cdot\mathbf{e_w} = (\mathbf{e_v} + \mathbf{e_w})\cdot\mathbf{e_w} = \mathbf{e_v}\cdot\mathbf{e_w} + \mathbf{e_w}\cdot\mathbf{e_w} = \mathbf{e_v}\cdot\mathbf{e_w} + 1$$

We see that $\mathbf{u} \cdot \mathbf{e_v} = \mathbf{u} \cdot \mathbf{e_w}$. We conclude that $\cos\alpha = \cos\beta$, hence $\alpha = \beta$. Geometrically, $\mathbf{u}$ is a diagonal in the rhombus $OABC$ (see figure), hence it bisects the angle $\sphericalangle AOC$ of the rhombus.



81. Calculate the force (in newtons) required to push a 40-kg wagon up a 10° incline (Figure 19).

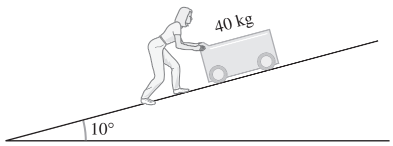
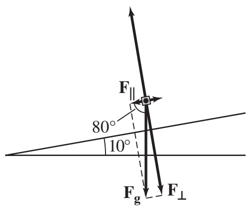


FIGURE 19

SOLUTION Gravity exerts a force $\mathbf{F}_g$ of magnitude $40g$ newtons where $g = 9.8$. The magnitude of the force required to push the wagon equals the component of the force $\mathbf{F}_g$ along the ramp. Resolving $\mathbf{F}_g$ into a sum $\mathbf{F}_g = \mathbf{F}_\parallel + \mathbf{F}_\perp$, where $\mathbf{F}_\parallel$ is the force along the ramp and $\mathbf{F}_\perp$ is the force orthogonal to the ramp, we need to find the magnitude of $\mathbf{F}_\parallel$. The angle between $\mathbf{F}_g$ and the ramp is $90^\circ - 10^\circ = 80^\circ$. Hence,

$$\mathbf{F}_\parallel = \|\mathbf{F}_g\| \cos 80^\circ = 40 \cdot 9.8 \cdot \cos 80^\circ \approx 68.07N.$$



Therefore the minimum force required to push the wagon is 68.07 N. (Actually, this is the force required to keep the wagon from sliding down the hill; any slight amount greater than this force will serve to push it up the hill.)

83. A light beam travels along the ray determined by a unit vector $\mathbf{L}$, strikes a flat surface at point P, and is reflected along the ray determined by a unit vector $\mathbf{R}$, where $\theta_1 = \theta_2$ (Figure 21). Show that if $\mathbf{N}$ is the unit vector orthogonal to the surface, then

$$\mathbf{R} = 2(\mathbf{L} \cdot \mathbf{N})\mathbf{N} - \mathbf{L}$$

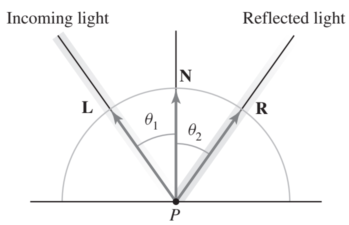
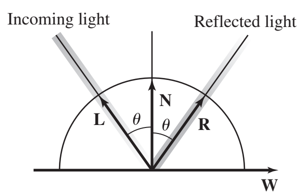


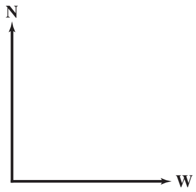
FIGURE 21

SOLUTION We denote by $\mathbf{W}$ a unit vector orthogonal to $\mathbf{N}$ in the direction shown in the figure, and let $\theta_1 = \theta_2 = \theta$.



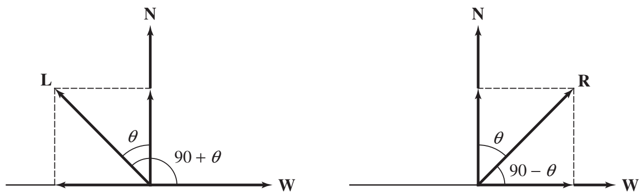
We resolve the unit vectors **R** and **L** into a sum of forces along **N** and **W**. This gives

$$\mathbf{R} = \cos(90-\theta)\mathbf{W} + \cos\theta\mathbf{N} = \sin\theta\mathbf{W} + \cos\theta\mathbf{N}$$
$$\mathbf{L} = -\cos(90-\theta)\mathbf{W} + \cos\theta\mathbf{N} = -\sin\theta\mathbf{W} + \cos\theta\mathbf{N} \quad (1)$$



Now, since

$$\mathbf{L}\cdot\mathbf{N} = \|\mathbf{L}\|\|\mathbf{N}\|\cos\theta = 1\cdot 1\cos\theta = \cos\theta$$



we have by (1):

$$2(\mathbf{L}\cdot\mathbf{N})\mathbf{N} - \mathbf{L} = (2\cos\theta)\mathbf{N} - \mathbf{L} = (2\cos\theta)\mathbf{N} - ((-\sin\theta)\mathbf{W} + (\cos\theta)\mathbf{N})$$
$$= (2\cos\theta)\mathbf{N} + (\sin\theta)\mathbf{W} - (\cos\theta)\mathbf{N} = (\sin\theta)\mathbf{W} + (\cos\theta)\mathbf{N} = \mathbf{R}$$

85. Prove that $\|\mathbf{v}+\mathbf{w}\|^2 - \|\mathbf{v}-\mathbf{w}\|^2 = 4\mathbf{v}\cdot\mathbf{w}$.

SOLUTION We compute the following values:

$$\|\mathbf{v}+\mathbf{w}\|^2 = (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 + 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2$$
$$\|\mathbf{v}-\mathbf{w}\|^2 = (\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 - 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2$$

Hence,

$$\|\mathbf{v}+\mathbf{w}\|^2 - \|\mathbf{v}-\mathbf{w}\|^2 = (\|\mathbf{v}\|^2 + 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2) - (\|\mathbf{v}\|^2 - 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2) = 4\mathbf{v}\cdot\mathbf{w}$$

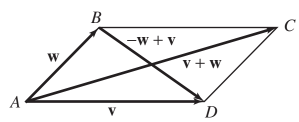
87. Show that the two diagonals of a parallelogram are perpendicular if and only if its sides have equal length. *Hint:* Use Exercise 86 to show that $\mathbf{v}-\mathbf{w}$ and $\mathbf{v}+\mathbf{w}$ are orthogonal if and only if $\|\mathbf{v}\| = \|\mathbf{w}\|$.

SOLUTION We denote the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ by

$$\mathbf{w} = \overrightarrow{AB}, \quad \mathbf{v} = \overrightarrow{AD}.$$

Then,

$$\overrightarrow{AC} = \mathbf{w}+\mathbf{v}, \quad \overrightarrow{BD} = -\mathbf{w}+\mathbf{v}.$$



The diagonals are perpendicular if and only if the vectors $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are orthogonal. By Exercise 86 these vectors are orthogonal if and only if the norms of the sum $(\mathbf{v}+\mathbf{w}) + (\mathbf{v}-\mathbf{w}) = 2\mathbf{v}$ and the difference $(\mathbf{v}+\mathbf{w}) - (\mathbf{v}-\mathbf{w}) = 2\mathbf{w}$ are equal, that is,

$$\|2\mathbf{v}\| = \|2\mathbf{w}\|$$
$$2\|\mathbf{v}\| = 2\|\mathbf{w}\| \quad\Rightarrow\quad \|\mathbf{v}\| = \|\mathbf{w}\|$$

89. Verify that $(\lambda\mathbf{v})\cdot\mathbf{w} = \lambda(\mathbf{v}\cdot\mathbf{w})$ for any scalar λ.

SOLUTION We denote the components of the vectors $\mathbf{v}$ and $\mathbf{w}$ by

$$\mathbf{v} = \langle a_1, a_2, a_3\rangle \quad \mathbf{w} = \langle b_1, b_2, b_3\rangle$$

Thus,

$$\begin{aligned}(\lambda\mathbf{v})\cdot\mathbf{w} &= (\lambda\langle a_1, a_2, a_3\rangle)\cdot\langle b_1, b_2, b_3\rangle = \langle \lambda a_1, \lambda a_2, \lambda a_3\rangle\cdot\langle b_1, b_2, b_3\rangle\\ &= \lambda a_1 b_1 + \lambda a_2 b_2 + \lambda a_3 b_3\end{aligned}$$

Recalling that λ, a_i, and b_i are scalars and using the definitions of scalar multiples of vectors and the dot product, we get

$$(\lambda\mathbf{v})\cdot\mathbf{w} = \lambda(a_1b_1 + a_2b_2 + a_3b_3) = \lambda\left(\langle a_1, a_2, a_3\rangle\cdot\langle b_1, b_2, b_3\rangle\right) = \lambda(\mathbf{v}\cdot\mathbf{w})$$

Further Insights and Challenges

91. In this exercise, we prove the Cauchy–Schwarz inequality: If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then

$$|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\| \qquad \boxed{6}$$

(a) Let $f(x) = \|x\mathbf{v} + \mathbf{w}\|^2$ for x a scalar. Show that $f(x) = ax^2 + bx + c$, where $a = \|\mathbf{v}\|^2$, $b = 2\mathbf{v}\cdot\mathbf{w}$, and $c = \|\mathbf{w}\|^2$.

(b) Conclude that $b^2 - 4ac \le 0$. *Hint:* Observe that $f(x) \ge 0$ for all x.

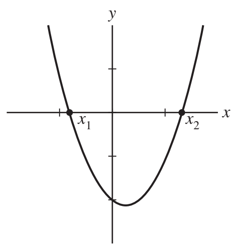
SOLUTION

(a) We express the norm as a dot product and compute it:

$$\begin{aligned}f(x) &= \|x\mathbf{v} + \mathbf{w}\|^2 = (x\mathbf{v} + \mathbf{w})\cdot(x\mathbf{v} + \mathbf{w})\\ &= x^2\mathbf{v}\cdot\mathbf{v} + x\mathbf{v}\cdot\mathbf{w} + x\mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2x^2 + 2(\mathbf{v}\cdot\mathbf{w})x + \|\mathbf{w}\|^2\end{aligned}$$

Hence, $f(x) = ax^2 + bx + c$, where $a = \|\mathbf{v}\|^2$, $b = 2\mathbf{v}\cdot\mathbf{w}$, and $c = \|\mathbf{w}\|^2$.

(b) If f has distinct real roots x_1 and x_2, then $f(x)$ is negative for x between x_1 and x_2, but this is impossible since f is the square of a length.



$f(x) = ax^2 + bx + c,\ a > 0$

Using properties of quadratic functions, it follows that f has a nonpositive discriminant. That is, $b^2 - 4ac \le 0$. Substituting the values for a, b, and c, we get

$$\begin{aligned}4(\mathbf{v}\cdot\mathbf{w})^2 - 4\|\mathbf{v}\|^2\|\mathbf{w}\|^2 &\le 0\\ (\mathbf{v}\cdot\mathbf{w})^2 &\le \|\mathbf{v}\|^2\|\mathbf{w}\|^2\end{aligned}$$

Taking the square root of both sides we obtain

$$|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\|\mathbf{w}\|$$

93. This exercise gives another proof of the relation between the dot product and the angle θ between two vectors $\mathbf{v} = \langle a_1, b_1\rangle$ and $\mathbf{w} = \langle a_2, b_2\rangle$ in the plane. Observe that $\mathbf{v} = \|\mathbf{v}\|\langle\cos\theta_1, \sin\theta_1\rangle$ and $\mathbf{w} = \|\mathbf{w}\|\langle\cos\theta_2, \sin\theta_2\rangle$, with θ_1 and θ_2 as in Figure 24. Then use the addition formula for the cosine to show that

$$\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta$$

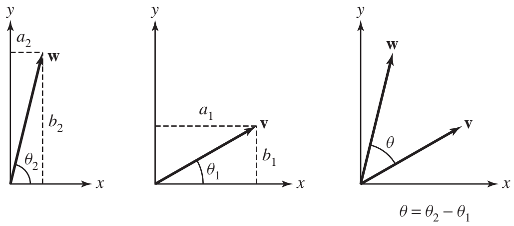


FIGURE 24

SOLUTION Using the trigonometric function for angles in right triangles, we have

$$a_2 = \|\mathbf{v}\| \sin\theta_1, \qquad a_1 = \|\mathbf{v}\| \cos\theta_1$$
$$b_2 = \|\mathbf{w}\| \sin\theta_2, \qquad b_1 = \|\mathbf{w}\| \cos\theta_2$$

Hence, using the given identity we obtain

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= \langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = a_1b_1 + a_2b_2 = \|\mathbf{v}\|\cos\theta_1\|\mathbf{w}\|\cos\theta_2 + \|\mathbf{v}\|\sin\theta_1\|\mathbf{w}\|\sin\theta_2\\ &= \|\mathbf{v}\|\|\mathbf{w}\|(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta_1 - \theta_2)\end{aligned}$$

That is,

$$\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$$

95. Let **v** be a nonzero vector. The angles α, β, γ between **v** and the unit vectors **i**, **j**, **k** are called the direction angles of **v** (Figure 25). The cosines of these angles are called the **direction cosines** of **v**. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

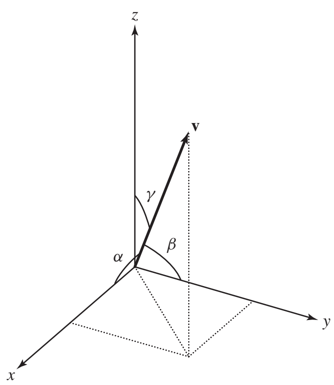


FIGURE 25 Direction angles of **v**.

SOLUTION We use the relation between the dot product and the angle between two vectors to write

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|} = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|}\\ \cos\beta &= \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\|\mathbf{j}\|} = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|}\\ \cos\gamma &= \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\|\mathbf{k}\|} = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|}\end{aligned} \tag{1}$$

We compute the values involved in (1). Letting $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ we get

$$\mathbf{v}\cdot\mathbf{i} = \langle v_1, v_2, v_3\rangle \cdot \langle 1, 0, 0\rangle = v_1$$
$$\mathbf{v}\cdot\mathbf{j} = \langle v_1, v_2, v_3\rangle \cdot \langle 0, 1, 0\rangle = v_2$$
$$\mathbf{v}\cdot\mathbf{k} = \langle v_1, v_2, v_3\rangle \cdot \langle 0, 0, 1\rangle = v_3$$
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{2}$$

We now substitute (2) into (1) to obtain

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

Finally, we compute the sum of squares of the direction cosines:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{v_1}{\|\mathbf{v}\|}\right)^2 + \left(\frac{v_2}{\|\mathbf{v}\|}\right)^2 + \left(\frac{v_3}{\|\mathbf{v}\|}\right)^2 = \frac{1}{\|\mathbf{v}\|^2}(v_1^2 + v_2^2 + v_3^2) = \frac{1}{\|\mathbf{v}\|^2} \cdot \|\mathbf{v}\|^2 = 1$$

97. The set of all points $X = (x, y, z)$ equidistant from two points P, Q in $\mathbf{R}^3$ is a plane (Figure 26). Show that X lies on this plane if

$$\overrightarrow{PQ} \cdot \overrightarrow{OX} = \frac{1}{2}\left(\|\overrightarrow{OQ}\|^2 - \|\overrightarrow{OP}\|^2\right) \qquad \boxed{7}$$

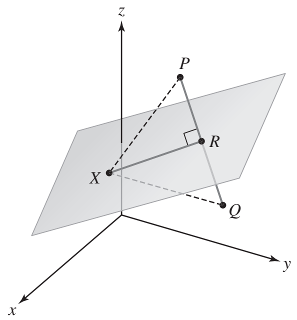


FIGURE 26

Hint: If R is the midpoint of $\overline{PQ}$, then X is equidistant from P and Q if and only if $\overrightarrow{XR}$ is orthogonal to $\overrightarrow{PQ}$.

SOLUTION Let R be the midpoint of the segment $\overline{PQ}$. The points $X = (x, y, z)$ that are equidistant from P and Q are the points for which the vector $\overrightarrow{XR}$ is orthogonal to $\overrightarrow{PQ}$. That is,

$$\overrightarrow{XR} \cdot \overrightarrow{PQ} = 0 \qquad (1)$$

Since $\overrightarrow{XR} = \overrightarrow{XO} + \overrightarrow{OR}$ we have by (1):

$$O = \left(\overrightarrow{XO} + \overrightarrow{OR}\right) \cdot \overrightarrow{PQ} = \overrightarrow{XO} \cdot \overrightarrow{PQ} + \overrightarrow{OR} \cdot \overrightarrow{PQ} = -\overrightarrow{OX} \cdot \overrightarrow{PQ} + \overrightarrow{OR} \cdot \overrightarrow{PQ}$$

Transferring sides we get

$$\overrightarrow{OX} \cdot \overrightarrow{PQ} = \overrightarrow{OR} \cdot \overrightarrow{PQ} \qquad (2)$$

We now write $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$ on the right-hand-side of (2), and $\overrightarrow{OR} = \dfrac{\overrightarrow{OP} + \overrightarrow{OQ}}{2}$. We get

$$\begin{aligned} \overrightarrow{OX} \cdot \overrightarrow{PQ} &= \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OQ}\right) \cdot \left(\overrightarrow{PO} + \overrightarrow{OQ}\right) = \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OQ}\right) \cdot \left(\overrightarrow{OQ} - \overrightarrow{OP}\right) \\ &= \frac{1}{2}\left(\overrightarrow{OP} \cdot \overrightarrow{OQ} - \overrightarrow{OP} \cdot \overrightarrow{OP} + \overrightarrow{OQ} \cdot \overrightarrow{OQ} - \overrightarrow{OQ} \cdot \overrightarrow{OP}\right) = \frac{1}{2}\left(\|\overrightarrow{OQ}\|^2 - \|\overrightarrow{OP}\|^2\right) \end{aligned}$$

Thus, we showed that the vector equation of the plane is

$$\overrightarrow{OX} \cdot \overrightarrow{PQ} = \frac{1}{2}\left(\|\overrightarrow{OQ}\|^2 - \|\overrightarrow{OP}\|^2\right).$$

99. Use Eq. (7) to find the equation of the plane consisting of all points $X = (x, y, z)$ equidistant from $P = (2, 1, 1)$ and $Q = (1, 0, 2)$.

SOLUTION Using Eq. (7) with $X = (x, y, z)$, $P = (2, 1, 1)$, and $Q = (1, 0, 2)$ gives

$$\langle x, y, z\rangle \cdot \langle -1, -1, 1\rangle = \frac{1}{2}\left((\sqrt{5})^2 - (\sqrt{6})^2\right) = -\frac{1}{2}$$

This gives us $-1x - 1y + 1z = -\frac{1}{2}$, which leads to $2x + 2y - 2z = 1$.

12.4 The Cross Product

Preliminary Questions

1. What is the (1, 3) minor of the matrix $\begin{vmatrix} 3 & 4 & 2 \\ -5 & -1 & 1 \\ 4 & 0 & 3 \end{vmatrix}$?

SOLUTION The (1, 3) minor is obtained by crossing out the first row and third column of the matrix. That is,

$$\begin{vmatrix} 3 & 4 & 2 \\ -5 & -1 & 1 \\ 4 & 0 & 3 \end{vmatrix} \quad \Rightarrow \quad \begin{vmatrix} -5 & -1 \\ 4 & 0 \end{vmatrix}$$

2. The angle between two unit vectors $\mathbf{e}$ and $\mathbf{f}$ is $\frac{\pi}{6}$. What is the length of $\mathbf{e} \times \mathbf{f}$?

SOLUTION We use the Formula for the Length of the Cross Product:

$$\|\mathbf{e} \times \mathbf{f}\| = \|\mathbf{e}\| \|\mathbf{f}\| \sin\theta$$

Since $\mathbf{e}$ and $\mathbf{f}$ are unit vectors, $\|\mathbf{e}\| = \|\mathbf{f}\| = 1$. Also $\theta = \frac{\pi}{6}$, therefore,

$$\|\mathbf{e} \times \mathbf{f}\| = 1 \cdot 1 \cdot \sin\frac{\pi}{6} = \frac{1}{2}$$

The length of $\mathbf{e} \times \mathbf{f}$ is $\frac{1}{2}$.

3. What is $\mathbf{u} \times \mathbf{w}$, assuming that $\mathbf{w} \times \mathbf{u} = \langle 2, 2, 1 \rangle$?

SOLUTION By anti-commutativity of the cross product, we have

$$\mathbf{u} \times \mathbf{w} = -\mathbf{w} \times \mathbf{u} = -\langle 2, 2, 1 \rangle = \langle -2, -2, -1 \rangle$$

4. Find the cross product without using the formula:

(a) $\langle 4, 8, 2 \rangle \times \langle 4, 8, 2 \rangle$

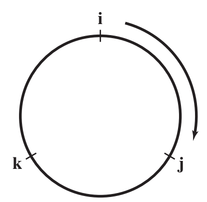
(b) $\langle 4, 8, 2 \rangle \times \langle 2, 4, 1 \rangle$

SOLUTION By properties of the cross product, the cross product of parallel vectors is the zero vector. In particular, the cross product of a vector with itself is the zero vector. Since $\langle 4, 8, 2 \rangle = 2\langle 2, 4, 1 \rangle$, the vectors $\langle 4, 8, 2 \rangle$ and $\langle 2, 4, 1 \rangle$ are parallel. We conclude that

$$\langle 4, 8, 2 \rangle \times \langle 4, 8, 2 \rangle = \mathbf{0} \quad \text{and} \quad \langle 4, 8, 2 \rangle \times \langle 2, 4, 1 \rangle = \mathbf{0}.$$

5. What are $\mathbf{i} \times \mathbf{j}$ and $\mathbf{i} \times \mathbf{k}$?

SOLUTION The cross product $\mathbf{i} \times \mathbf{j}$ and $\mathbf{i} \times \mathbf{k}$ are determined by the right-hand rule. We can also use the following figure to determine these cross-products:



We get

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \text{ and } \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

6. When is the cross product $\mathbf{v} \times \mathbf{w}$ equal to zero?

SOLUTION The cross product $\mathbf{v} \times \mathbf{w}$ is equal to zero if one of the vectors $\mathbf{v}$ or $\mathbf{w}$ (or both) is the zero vector, or if $\mathbf{v}$ and $\mathbf{w}$ are parallel vectors.

7. Which of the following are meaningful and which are not? Explain.

(a) $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$

(b) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

(c) $\|\mathbf{w}\|(\mathbf{u} \cdot \mathbf{v})$

(d) $\|\mathbf{w}\|(\mathbf{u} \times \mathbf{v})$

SOLUTION

(a) Since $\mathbf{u} \cdot \mathbf{v}$ is a scalar, this product does not make sense: taking the cross product of a scalar with a vector is not defined.

(b) $\mathbf{u} \times \mathbf{v}$ is a vector, so the result of this expression is the dot product of two vectors, which is defined and is a scalar.

(c) Both $\|\mathbf{w}\|$ and $\mathbf{u} \cdot \mathbf{v}$ are scalars, so this expression is defined and is just the product of two real numbers.

(d) Since $\|\mathbf{w}\|$ is a scalar and $\mathbf{u} \times \mathbf{v}$ is a vector, this expression is defined. It is a multiple of the vector $\mathbf{u} \times \mathbf{v}$.

8. Which of the following vectors are equal to $\mathbf{j} \times \mathbf{i}$?

(a) $\mathbf{i} \times \mathbf{k}$

(b) $-\mathbf{k}$

(c) $\mathbf{i} \times \mathbf{j}$

SOLUTION Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ (see Exercise 5 above), we see that $\mathbf{j} \times \mathbf{i} = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}$. Thus **(b)** is equal to $\mathbf{j} \times \mathbf{i}$, but **(c)** is not. Finally, since $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, **(a)** is not equal to $\mathbf{j} \times \mathbf{i}$ either.

Exercises

In Exercises 1–4, calculate the 2×2 *determinant.*

1. $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$

SOLUTION Using the definition of 2×2 determinant we get

$$\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 1 \cdot 3 - 2 \cdot 4 = -5$$

3. $\begin{vmatrix} -6 & 9 \\ 1 & 1 \end{vmatrix}$

SOLUTION We evaluate the determinant to obtain

$$\begin{vmatrix} -6 & 9 \\ 1 & 1 \end{vmatrix} = -6 \cdot 1 - 9 \cdot 1 = -15$$

In Exercises 5–8, calculate the 3×3 *determinant.*

5. $\begin{vmatrix} 1 & 2 & 1 \\ 4 & -3 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

SOLUTION Using the definition of 3×3 determinant we obtain

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 1 \\ 4 & -3 & 0 \\ 1 & 0 & 1 \end{vmatrix} &= 1\begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & -3 \\ 1 & 0 \end{vmatrix} \\ &= 1 \cdot (-3 \cdot 1 - 0 \cdot 0) - 2 \cdot (4 \cdot 1 - 0 \cdot 1) + 1 \cdot (4 \cdot 0 - (-3) \cdot 1) \\ &= -3 - 8 + 3 = -8\end{aligned}$$

7. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -4 & 2 \end{vmatrix}$

SOLUTION We have

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -4 & 2 \end{vmatrix} &= 1\begin{vmatrix} 4 & 6 \\ -4 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 6 \\ -3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ -3 & -4 \end{vmatrix} \\ &= 1(4\cdot 2 - 6\cdot(-4)) - 2(2\cdot 2 - 6\cdot(-3)) + 3(2\cdot(-4) - 4\cdot(-3)) \\ &= 32 - 44 + 12 = 0 \end{aligned}$$

In Exercises 9–12, calculate $\mathbf{v}\times\mathbf{w}$.

9. $\mathbf{v} = \langle 1, 2, 1\rangle, \quad \mathbf{w} = \langle 3, 1, 1\rangle$

SOLUTION Using the definition of the cross product we get

$$\begin{aligned} \mathbf{v}\times\mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{k} \\ &= (2-1)\mathbf{i} - (1-3)\mathbf{j} + (1-6)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - 5\mathbf{k} \end{aligned}$$

11. $\mathbf{v} = \left\langle \frac{2}{3}, 1, \frac{1}{2}\right\rangle, \quad \mathbf{w} = \langle 4, -6, 3\rangle$

SOLUTION We have

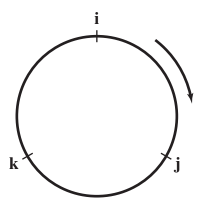
$$\begin{aligned} \mathbf{v}\times\mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & 1 & \frac{1}{2} \\ 4 & -6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} \\ -6 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{2}{3} & \frac{1}{2} \\ 4 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{2}{3} & 1 \\ 4 & -6 \end{vmatrix}\mathbf{k} \\ &= (3+3)\,\mathbf{i} - (2-2)\,\mathbf{j} + (-4-4)\,\mathbf{k} = 6\mathbf{i} - 8\mathbf{k} \end{aligned}$$

In Exercises 13–16, use the relations in Eq. (5) to calculate the cross product.

13. $(\mathbf{i}+\mathbf{j})\times\mathbf{k}$

SOLUTION We use basic properties of the cross product to obtain

$$(\mathbf{i}+\mathbf{j})\times\mathbf{k} = \mathbf{i}\times\mathbf{k} + \mathbf{j}\times\mathbf{k} = -\mathbf{j} + \mathbf{i}$$



$$\mathbf{i}\times\mathbf{k} = -\mathbf{j}$$
$$\mathbf{j}\times\mathbf{k} = \mathbf{i}$$

15. $(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})\times(\mathbf{j} - \mathbf{k})$

SOLUTION Using the distributive law we obtain

$$\begin{aligned} (\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})\times(\mathbf{j} - \mathbf{k}) &= (\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})\times\mathbf{j} - (\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})\times(\mathbf{k}) \\ &= \mathbf{i}\times\mathbf{j} + 2\mathbf{k}\times\mathbf{j} - \mathbf{i}\times\mathbf{k} - (-3\mathbf{j})\times\mathbf{k} \\ &= \mathbf{i} + \mathbf{j} + \mathbf{k} \end{aligned}$$

In Exercises 17–22, calculate the cross product assuming that

$$\mathbf{u}\times\mathbf{v}=\langle 1,1,0\rangle,\quad \mathbf{u}\times\mathbf{w}=\langle 0,3,1\rangle,\quad \mathbf{v}\times\mathbf{w}=\langle 2,-1,1\rangle$$

17. $\mathbf{v}\times\mathbf{u}$

SOLUTION Using the properties of the cross product we obtain

$$\mathbf{v}\times\mathbf{u}=-\mathbf{u}\times\mathbf{v}=\langle -1,-1,0\rangle$$

19. $\mathbf{w}\times(\mathbf{u}+\mathbf{v})$

SOLUTION Using the properties of the cross product we obtain

$$\mathbf{w}\times(\mathbf{u}+\mathbf{v})=\mathbf{w}\times\mathbf{u}+\mathbf{w}\times\mathbf{v}=-\mathbf{u}\times\mathbf{w}-\mathbf{v}\times\mathbf{w}=\langle -2,-2,-2\rangle.$$

21. $(\mathbf{u}-2\mathbf{v})\times(\mathbf{u}+2\mathbf{v})$

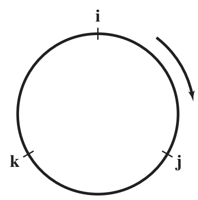
SOLUTION Using the properties of the cross product we obtain

$$\begin{aligned}(\mathbf{u}-2\mathbf{v})\times(\mathbf{u}+2\mathbf{v})&=(\mathbf{u}-2\mathbf{v})\times\mathbf{u}+(\mathbf{u}-2\mathbf{v})\times 2\mathbf{v}=\mathbf{u}\times\mathbf{u}-2\mathbf{v}\times\mathbf{u}+\mathbf{u}\times 2\mathbf{v}-4\mathbf{v}\times\mathbf{v}\\&=0+2\mathbf{u}\times\mathbf{v}+2\mathbf{u}\times\mathbf{v}-0=0+4\mathbf{u}\times\mathbf{v}=\langle 4,4,0\rangle\end{aligned}$$

23. Let $\mathbf{v}=\langle a,b,c\rangle$. Calculate $\mathbf{v}\times\mathbf{i}$, $\mathbf{v}\times\mathbf{j}$, and $\mathbf{v}\times\mathbf{k}$.

SOLUTION We write $\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ and use the distributive law:

$$\begin{aligned}\mathbf{v}\times\mathbf{i}&=(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\times\mathbf{i}=a\mathbf{i}\times\mathbf{i}+b\mathbf{j}\times\mathbf{i}+c\mathbf{k}\times\mathbf{i}=a\cdot\mathbf{0}-b\mathbf{k}+c\mathbf{j}=-b\mathbf{k}+c\mathbf{j}=\langle 0,c,-b\rangle\\\mathbf{v}\times\mathbf{j}&=(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\times\mathbf{j}=a\mathbf{i}\times\mathbf{j}+b\mathbf{j}\times\mathbf{j}+c\mathbf{k}\times\mathbf{j}=a\mathbf{k}+b\mathbf{0}-c\mathbf{i}=a\mathbf{k}-c\mathbf{i}=\langle -c,0,a\rangle\\\mathbf{v}\times\mathbf{k}&=(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\times\mathbf{k}=a\mathbf{i}\times\mathbf{k}+b\mathbf{j}\times\mathbf{k}+c\mathbf{k}\times\mathbf{k}=-a\mathbf{j}+b\mathbf{i}+c\mathbf{0}=-a\mathbf{j}+b\mathbf{i}=\langle b,-a,0\rangle\end{aligned}$$



In Exercises 25 and 26, refer to Figure 17.

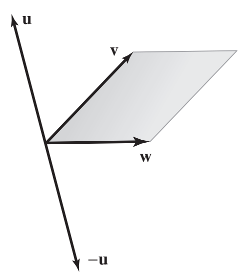


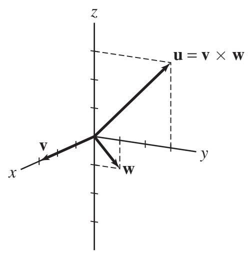
FIGURE 17

25. Which of $\mathbf{u}$ and $-\mathbf{u}$ is equal to $\mathbf{v}\times\mathbf{w}$?

SOLUTION The direction of $\mathbf{v}\times\mathbf{w}$ is determined by the right-hand rule, that is, our thumb points in the direction of $\mathbf{v}\times\mathbf{w}$ when the fingers of our right hand curl from $\mathbf{v}$ to $\mathbf{w}$. Therefore $\mathbf{v}\times\mathbf{w}$ equals $-\mathbf{u}$ rather than $\mathbf{u}$.

27. Let $\mathbf{v}=\langle 3,0,0\rangle$ and $\mathbf{w}=\langle 0,1,-1\rangle$. Determine $\mathbf{u}=\mathbf{v}\times\mathbf{w}$ using the geometric properties of the cross product rather than the formula.

SOLUTION The cross product $\mathbf{u}=\mathbf{v}\times\mathbf{w}$ is orthogonal to $\mathbf{v}$.



Since $\mathbf{v}$ lies along the x-axis, $\mathbf{u}$ lies in the yz-plane, therefore $\mathbf{u} = \langle 0, b, c\rangle$. $\mathbf{u}$ is also orthogonal to $\mathbf{w}$, so $\mathbf{u} \cdot \mathbf{w} = 0$. This gives $\mathbf{u} \cdot \mathbf{w} = \langle 0, b, c\rangle \cdot \langle 0, 1, -1\rangle = b - c = 0 \Rightarrow b = c$. Thus, $\mathbf{u} = \langle 0, b, b\rangle$. By the right-hand rule, $\mathbf{u}$ points to the positive z-direction so $b > 0$. We compute the length of $\mathbf{u}$. Since $\mathbf{v} \cdot \mathbf{w} = \langle 3, 0, 0\rangle \cdot \langle 0, 1, -1\rangle = 0$, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. Hence,

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin \frac{\pi}{2} = \|\mathbf{v}\|\|\mathbf{w}\| = 3 \cdot \sqrt{2}.$$

Also since $b > 0$, we have

$$\|\mathbf{u}\| = \|\langle 0, b, b\rangle\| = \sqrt{2b^2} = b\sqrt{2}$$

Equating the lengths gives

$$b\sqrt{2} = 3\sqrt{2} \quad \Rightarrow \quad b = 3.$$

We conclude that $\mathbf{u} = \mathbf{v} \times \mathbf{w} = \langle 0, 3, 3\rangle$.

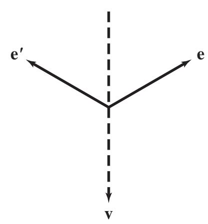
29. Show that if $\mathbf{v}$ and $\mathbf{w}$ lie in the yz-plane, then $\mathbf{v} \times \mathbf{w}$ is a multiple of $\mathbf{i}$.

SOLUTION $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$. Since $\mathbf{v}$ and $\mathbf{w}$ lie in the yz-plane, $\mathbf{v} \times \mathbf{w}$ must lie along the x axis which is perpendicular to yz-plane. That is, $\mathbf{v} \times \mathbf{w}$ is a scalar multiple of the unit vector $\mathbf{i}$.

31. Let $\mathbf{e}$ and $\mathbf{e}'$ be unit vectors in $\mathbf{R}^3$ such that $\mathbf{e} \perp \mathbf{e}'$. Use the geometric properties of the cross product to compute $\mathbf{e} \times (\mathbf{e}' \times \mathbf{e})$.

SOLUTION Let $\mathbf{u} = \mathbf{e} \times \left(\mathbf{e}' \times \mathbf{e}\right)$ and $\mathbf{v} = \mathbf{e}' \times \mathbf{e}$. The vector $\mathbf{v}$ is orthogonal to $\mathbf{e}'$ and $\mathbf{e}$, hence $\mathbf{v}$ is orthogonal to the plane π defined by $\mathbf{e}'$ and $\mathbf{e}$. Now $\mathbf{u}$ is orthogonal to $\mathbf{v}$, hence $\mathbf{u}$ lies in the plane π orthogonal to $\mathbf{v}$. $\mathbf{u}$ is orthogonal to $\mathbf{e}$, which is in this plane, hence $\mathbf{u}$ is a multiple of $\mathbf{e}'$:

$$\mathbf{u} = \lambda \mathbf{e}' \tag{1}$$



The right-hand rule implies that $\mathbf{u}$ is in the direction of $\mathbf{e}'$, hence $\lambda > 0$. To find λ, we compute the length of $\mathbf{u}$:

$$\|\mathbf{v}\| = \|\mathbf{e}' \times \mathbf{e}\| = \|\mathbf{e}'\|\|\mathbf{e}\| \sin \frac{\pi}{2} = 1 \cdot 1 \cdot 1 = 1$$

$$\|\mathbf{u}\| = \|\mathbf{e} \times \mathbf{v}\| = \|\mathbf{e}\|\|\mathbf{v}\| \sin \frac{\pi}{2} = 1 \cdot 1 \cdot 1 = 1 \tag{2}$$

Combining (1), (2), and $\lambda > 0$ we conclude that

$$\mathbf{u} = \mathbf{e} \times \left(\mathbf{e}' \times \mathbf{e}\right) = \mathbf{e}'.$$

33. An electron moving with velocity $\mathbf{v}$ in the plane experiences a force $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$, where q is the charge on the electron and $\mathbf{B}$ is a uniform magnetic field pointing directly out of the page. Which of the two vectors $\mathbf{F}_1$ or $\mathbf{F}_2$ in Figure 18 represents the force on the electron? Remember that q is negative.

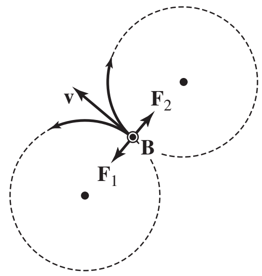
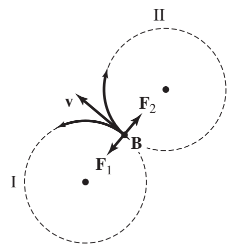


FIGURE 18 The magnetic field vector **B** points directly out of the page.

SOLUTION Since the magnetic field **B** points directly out of the page (toward us), the right-hand rule implies that the cross product $\mathbf{v} \times \mathbf{B}$ is in the direction of $\mathbf{F}_2$ (see figure).



Since $\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$ and $q < 0$, the force **F** on the electron is represented by the opposite vector $\mathbf{F}_1$.

35. Verify identity (12) for vectors $\mathbf{v} = \langle 3, -2, 2 \rangle$ and $\mathbf{w} = \langle 4, -1, 2 \rangle$.

SOLUTION We compute the cross product $\mathbf{v} \times \mathbf{w}$:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 2 \\ 4 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -2 \\ 4 & -1 \end{vmatrix} \mathbf{k}$$
$$= (-4+2)\mathbf{i} - (6-8)\mathbf{j} + (-3+8)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} = \langle -2, 2, 5 \rangle$$

We now find the dot product $\mathbf{v} \cdot \mathbf{w}$:

$$\mathbf{v} \cdot \mathbf{w} = \langle 3, -2, 2 \rangle \cdot \langle 4, -1, 2 \rangle = 3 \cdot 4 + (-2) \cdot (-1) + 2 \cdot 2 = 18$$

Finally we compute the squares of the lengths of **v**, **w** and $\mathbf{v} \times \mathbf{w}$:

$$\|\mathbf{v}\|^2 = 3^2 + (-2)^2 + 2^2 = 17$$
$$\|\mathbf{w}\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$
$$\|\mathbf{v} \times \mathbf{w}\|^2 = (-2)^2 + 2^2 + 5^2 = 33$$

We now verify the equality:

$$\|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2 = 17 \cdot 21 - 18^2 = 33 = \|\mathbf{v} \times \mathbf{w}\|^2$$

37. Find the area of the parallelogram spanned by **v** and **w** in Figure 19.

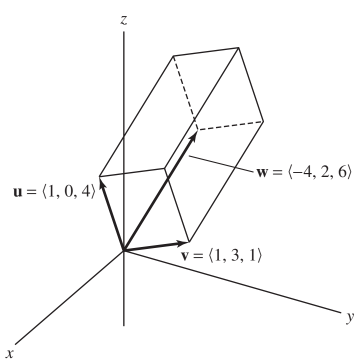


FIGURE 19

SOLUTION The area of the parallelogram equals the length of the cross product of the two vectors $\mathbf{v} = \langle 1, 3, 1\rangle$ and $\mathbf{w} = \langle -4, 2, 6\rangle$. We calculate the cross product as follows:

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ -4 & 2 & 6 \end{vmatrix} = (18-2)\mathbf{i} - (6+4)\mathbf{j} + (2+12)\mathbf{k} = 16\mathbf{i} - 10\mathbf{j} + 14\mathbf{k}$$

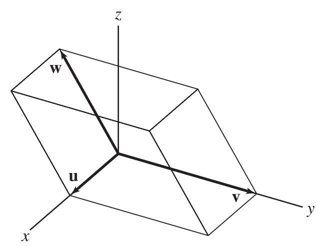
The length of this vector $16\mathbf{i} - 10\mathbf{j} + 14\mathbf{k}$ is $\sqrt{16^2+10^2+14^2} = 2\sqrt{138}$. Thus, the area of the parallelogram is $2\sqrt{138}$.

39. Sketch and compute the volume of the parallelepiped spanned by

$$\mathbf{u} = \langle 1, 0, 0\rangle, \qquad \mathbf{v} = \langle 0, 2, 0\rangle, \qquad \mathbf{w} = \langle 1, 1, 2\rangle$$

SOLUTION Using $\mathbf{u} = \langle 1, 0, 0\rangle$, $\mathbf{v} = \langle 0, 2, 0\rangle$, and $\mathbf{w} = \langle 1, 1, 2\rangle$, the volume is given by the following scalar triple product:

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(4-0) - 0 + 0 = 4.$$



41. Calculate the area of the parallelogram spanned by $\mathbf{u} = \langle 1, 0, 3\rangle$ and $\mathbf{v} = \langle 2, 1, 1\rangle$.

SOLUTION The area of the parallelogram is the length of the vector $\mathbf{u}\times\mathbf{v}$. We first compute this vector:

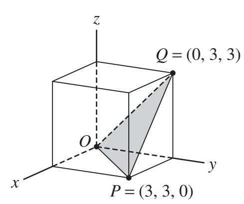
$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}\mathbf{k} = -3\mathbf{i} - (1-6)\mathbf{j} + \mathbf{k} = -3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$$

The area A is the length

$$A = \|\mathbf{u}\times\mathbf{v}\| = \sqrt{(-3)^2+5^2+1^2} = \sqrt{35} \approx 5.92.$$

43. Sketch the triangle with vertices at the origin O, $P = (3, 3, 0)$, and $Q = (0, 3, 3)$, and compute its area using cross products.

SOLUTION The triangle OPQ is shown in the following figure.



The area S of the triangle is half of the area of the parallelogram determined by the vectors $\overrightarrow{OP} = \langle 3, 3, 0\rangle$ and $\overrightarrow{OQ} = \langle 0, 3, 3\rangle$. Thus,

$$S = \frac{1}{2}\|\overrightarrow{OP}\times\overrightarrow{OQ}\| \tag{1}$$

We compute the cross product:

$$\overrightarrow{OP}\times\overrightarrow{OQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 0 \\ 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 3 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ 0 & 3 \end{vmatrix}\mathbf{k}$$

$$= 9\mathbf{i} - 9\mathbf{j} + 9\mathbf{k} = 9\langle 1, -1, 1\rangle$$

Substituting into (1) gives

$$S = \frac{1}{2}\|9\langle 1, -1, 1\rangle\| = \frac{9}{2}\|\langle 1, -1, 1\rangle\| = \frac{9}{2}\sqrt{1^2 + (-1)^2 + 1^2} = \frac{9\sqrt{3}}{2} \approx 7.8$$

The area of the triangle is $S = \frac{9\sqrt{3}}{2} \approx 7.8$.

45. Use cross products to find the area of the triangle in the xy-plane defined by $(1, 2)$, $(3, 4)$, and $(-2, 2)$.

SOLUTION Think of the triangle as lying in the xy-plane in a three-dimensional coordinate system, and let $P = (1, 2, 0)$, $Q = (3, 4, 0)$, and $R = (-2, 2, 0)$. Then the area T of the triangle is given by equation (7) in the text:

$$T = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$$

We compute the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$:

$$\overrightarrow{PQ} = \langle 3 - 1, 4 - 2, 0 - 0\rangle = \langle 2, 2, 0\rangle$$
$$\overrightarrow{PR} = \langle -2 - 1, 2 - 2, 0 - 0\rangle = \langle -3, 0, 0\rangle$$

We now find the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$ by computing the following determinant:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ -3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 0 \\ -3 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 2 \\ -3 & 0 \end{vmatrix}\mathbf{k} = 6\mathbf{k}$$

Thus, we get

$$T = \frac{1}{2}\|6\mathbf{k}\| = \frac{1}{2} \cdot 6 = 3.$$

47. Check that the four points $P(2, 4, 4)$, $Q(3, 1, 6)$, $R(2, 8, 0)$, and $S(7, 2, 1)$ all lie in a plane. Then use vectors to find the area of the quadrilateral they define.

SOLUTION The points P, Q, and R determine a plane with normal vector $\mathbf{n}$; S lies in that plane if $\overrightarrow{PS}$ is perpendicular to $\mathbf{n}$. To find $\mathbf{n}$, we compute

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1, -3, 2\rangle \times \langle 0, 4, -4\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 0 & 4 & -4 \end{vmatrix} \\ &= \begin{vmatrix} -3 & 2 \\ 4 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -3 \\ 0 & 4 \end{vmatrix}\mathbf{k} \\ &= 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}. \end{aligned}$$

Since

$$\langle 4, 4, 4\rangle \cdot \overrightarrow{PS} = \langle 4, 4, 4\rangle \cdot \langle 5, -2, -3\rangle = 0,$$

the normal vector is also orthogonal to $\overrightarrow{PS}$, so that the vector $\overrightarrow{PS}$, and therefore the point S, also lies in the plane. So all four points lie in a plane.

To find the area of the quadrilateral of which they are the vertices, divide the quadrilateral into the two triangles $\triangle PQR$ and $\triangle SQR$. The area of each of these triangles is given by equation (7) in the text. First we must compute various vectors:

$$\overrightarrow{PQ} = \langle 3 - 2, 1 - 4, 6 - 4\rangle = \langle 1, -3, 2\rangle$$
$$\overrightarrow{PR} = \langle 2 - 2, 8 - 4, 0 - 4\rangle = \langle 0, 4, -4\rangle$$
$$\overrightarrow{SQ} = \langle 3 - 7, 1 - 2, 6 - 1\rangle = \langle -4, -1, 5\rangle$$
$$\overrightarrow{SR} = \langle 2 - 7, 8 - 2, 0 - 1\rangle = \langle -5, 6, -1\rangle.$$

To find the area of $\triangle PQR$, we must compute $\overrightarrow{PQ} \times \overrightarrow{PR}$; to find the area of $\triangle SQR$ we must compute $\overrightarrow{SQ} \times \overrightarrow{SR}$:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 0 & 4 & -4 \end{vmatrix} = \begin{vmatrix} -3 & 2 \\ 4 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -3 \\ 0 & 4 \end{vmatrix}\mathbf{k} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\overrightarrow{SQ} \times \overrightarrow{SR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -1 & 5 \\ -5 & 6 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 5 \\ 6 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & 5 \\ -5 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & -1 \\ -5 & 6 \end{vmatrix}\mathbf{k} = -29\mathbf{i} - 29\mathbf{j} - 29\mathbf{k}.$$

The area of the quadrilateral, $\mathcal{S}$, is given by

$$\mathcal{S} = A(\triangle PQS) + A(\triangle RQS) = \frac{1}{2}\|4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}\| + \frac{1}{2}\|-29\mathbf{i} - 29\mathbf{j} - 29\mathbf{k}\| = \frac{1}{2}(4\sqrt{3} + 29\sqrt{3}) = \frac{33}{2}\sqrt{3}.$$

In Exercises 49–51, verify the identity using the formula for the cross product.

49. $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$

SOLUTION Let $\mathbf{v} = \langle a, b, c\rangle$ and $\mathbf{w} = \langle d, e, f\rangle$. By the definition of the cross product we have

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} b & c \\ e & f \end{vmatrix}\mathbf{i} - \begin{vmatrix} a & c \\ d & f \end{vmatrix}\mathbf{j} + \begin{vmatrix} a & b \\ d & e \end{vmatrix}\mathbf{k} = (bf - ec)\mathbf{i} - (af - dc)\mathbf{j} + (ae - db)\mathbf{k}$$

We also have

$$-\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -d & -e & -f \\ a & b & c \end{vmatrix} = (-ec + bf)\mathbf{i} - (-dc + af)\mathbf{j} + (-db + ea)\mathbf{k}$$

Thus, $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$, as desired.

51. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

SOLUTION We let $\mathbf{u} = \langle a_1, a_2, a_3\rangle$, $\mathbf{v} = \langle b_1, b_2, b_3\rangle$ and $\mathbf{w} = \langle c_1, c_2, c_3\rangle$. Computing the left-hand side gives

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle \times \langle c_1, c_2, c_3\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 + b_2 & a_3 + b_3 \\ c_2 & c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 + b_1 & a_3 + b_3 \\ c_1 & c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 + b_1 & a_2 + b_2 \\ c_1 & c_2 \end{vmatrix}\mathbf{k} \\ &= (c_3(a_2 + b_2) - c_2(a_3 + b_3))\,\mathbf{i} - (c_3\,(a_1 + b_1) - c_1\,(a_3 + b_3))\,\mathbf{j} \\ &\quad + (c_2(a_1 + b_1) - c_1(a_2 + b_2))\,\mathbf{k}\end{aligned}$$

We now compute the right-hand-side of the equality:

$$\begin{aligned}\mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}\mathbf{k} + \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\mathbf{i} \\ &\quad - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\mathbf{k} \\ &= (a_2c_3 - a_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k} \\ &\quad + (b_2c_3 - b_3c_2)\mathbf{i} - (b_1c_3 - b_3c_1)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k} \\ &= (a_2c_3 - a_3c_2 + b_2c_3 - b_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1 + b_1c_3 - b_3c_1)\mathbf{j} \\ &\quad + (a_1c_2 - a_2c_1 + b_1c_2 - b_2c_1)\mathbf{k} \\ &= (c_3(a_2 + b_2) - c_2(a_3 + b_3))\,\mathbf{i} - (c_3(a_1 + b_1) - c_1(a_3 + b_3))\,\mathbf{j} \\ &\quad + (c_2(a_1 + b_1) - c_1(a_2 + b_2))\,\mathbf{k}\end{aligned}$$

The results are the same. Hence,

$$(\mathbf{u}+\mathbf{v})\times\mathbf{w}=\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}.$$

53. Verify the relations (5).

SOLUTION We must verify the following relations:

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\quad \mathbf{j}\times\mathbf{k}=\mathbf{i},\quad \mathbf{k}\times\mathbf{i}=\mathbf{j},\quad \mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$$

We compute the cross products using the definition of the cross product. This gives

$$\mathbf{i}\times\mathbf{j}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\0&1&0\end{vmatrix}=\begin{vmatrix}0&0\\1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\0&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\0&1\end{vmatrix}\mathbf{k}=\mathbf{k}$$

$$\mathbf{j}\times\mathbf{k}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\0&0&1\end{vmatrix}=\begin{vmatrix}1&0\\0&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&0\\0&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\0&0\end{vmatrix}\mathbf{k}=\mathbf{i}$$

$$\mathbf{k}\times\mathbf{i}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\1&0&0\end{vmatrix}=\begin{vmatrix}0&1\\0&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&1\\1&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&0\\1&0\end{vmatrix}\mathbf{k}=\mathbf{j}$$

$$\mathbf{i}\times\mathbf{i}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\1&0&0\end{vmatrix}=\begin{vmatrix}0&0\\0&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\1&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\1&0\end{vmatrix}\mathbf{k}=\mathbf{0}$$

$$\mathbf{j}\times\mathbf{j}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\0&1&0\end{vmatrix}=\begin{vmatrix}1&0\\1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&0\\0&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\0&1\end{vmatrix}\mathbf{k}=\mathbf{0}$$

$$\mathbf{k}\times\mathbf{k}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\0&0&1\end{vmatrix}=\begin{vmatrix}0&1\\0&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&1\\0&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&0\\0&0\end{vmatrix}\mathbf{k}=\mathbf{0}$$

55. The components of the cross product have a geometric interpretation. Show that the absolute value of the $\mathbf{k}$-component of $\mathbf{v}\times\mathbf{w}$ is equal to the area of the parallelogram spanned by the projections $\mathbf{v}_0$ and $\mathbf{w}_0$ onto the xy-plane (Figure 21).

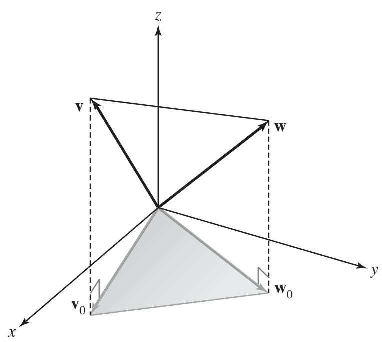


FIGURE 21

SOLUTION Let $\mathbf{v}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{w}=\langle b_1,b_2,b_3\rangle$, hence, $\mathbf{v}_0=\langle a_1,a_2,0\rangle$ and $\mathbf{w}_0=\langle b_1,b_2,0\rangle$. The area S of the parallelogram spanned by $\mathbf{v}_0$ and $\mathbf{w}_0$ is the following value:

$$S=\|\mathbf{v}_0\times\mathbf{w}_0\| \tag{1}$$

We compute the cross product:

$$\mathbf{v}_0\times\mathbf{w}_0=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&0\\b_1&b_2&0\end{vmatrix}=\begin{vmatrix}a_2&0\\b_2&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}a_1&0\\b_1&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}\mathbf{k}$$

$$=0\mathbf{i}-0\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}=\langle 0,0,a_1b_2-a_2b_1\rangle$$

Using (1) we have

$$S = \sqrt{0^2 + 0^2 + (a_1b_2 - a_2b_1)^2} = |a_1b_2 - a_2b_1| \tag{2}$$

We now compute $\mathbf{v} \times \mathbf{w}$:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

The $\mathbf{k}$-component of $\mathbf{v} \times \mathbf{w}$ is, thus,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \tag{3}$$

By (2) and (3) we obtain the desired result.

57. Show that three points P, Q, R are collinear (lie on a line) if and only if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$.

SOLUTION The points P, Q, and R lie on one line if and only if the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel. By basic properties of the cross product this is equivalent to $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$.

59. Solve the equation $\langle 1, 1, 1\rangle \times \mathbf{X} = \langle 1, -1, 0\rangle$, where $\mathbf{X} = \langle x, y, z\rangle$. *Note:* There are infinitely many solutions.

SOLUTION Let $\mathbf{X} = \langle a, b, c\rangle$. We compute the cross product:

$$\langle 1, 1, 1\rangle \times \langle a, b, c\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ a & c \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} \mathbf{k}$$
$$= (c-b)\mathbf{i} - (c-a)\mathbf{j} + (b-a)\mathbf{k} = \langle c-b, a-c, b-a\rangle$$

The equation for $\mathbf{X}$ is, thus,

$$\langle c-b, a-c, b-a\rangle = \langle 1, -1, 0\rangle$$

Equating corresponding components we get

$$\begin{aligned} c - b &= 1 \\ a - c &= -1 \\ b - a &= 0 \end{aligned}$$

The third equation implies $a = b$. Substituting in the first and second equations gives

$$\begin{aligned} c - a &= 1 \\ a - c &= -1 \end{aligned} \quad \Rightarrow \quad c = a + 1$$

The solution is thus, $b = a$, $c = a + 1$. The corresponding solutions $\mathbf{X}$ are

$$\mathbf{X} = \langle a, b, c\rangle = \langle a, a, a+1\rangle$$

Therefore any vector of the form $\langle a, a, a+1\rangle$ where a is an arbitrary constant is a solution. For example, setting $a = 0$ gives $\langle 0, 0, 1\rangle$.

61. Let $\mathbf{X} = \langle x, y, z \rangle$. Show that $\mathbf{i} \times \mathbf{X} = \mathbf{v}$ has a solution if and only if $\mathbf{v}$ is contained in the yz-plane (the $\mathbf{i}$-component is zero).

SOLUTION Let $\mathbf{X} = \langle a, b, c \rangle$. We compute the cross product:

$$\langle 1, 1, 1 \rangle \times \langle a, b, c \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ a & c \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} \mathbf{k}$$

$$= (c - b)\mathbf{i} - (c - a)\mathbf{j} + (b - a)\mathbf{k} = \langle c - b, a - c, b - a \rangle$$

The equation for $\mathbf{X}$ is, thus,

$$\langle c - b, a - c, b - a \rangle = \langle 1, -1, 0 \rangle$$

Equating corresponding components we get

$$\begin{aligned} c - b &= 1 \\ a - c &= -1 \\ b - a &= 0 \end{aligned}$$

The third equation implies $a = b$. Substituting in the first and second equations gives

$$\begin{aligned} c - a &= 1 \\ a - c &= -1 \end{aligned} \quad \Rightarrow \quad c = a + 1$$

The solution is thus, $b = a$, $c = a + 1$. The corresponding solutions $\mathbf{X}$ are

$$\mathbf{X} = \langle a, b, c \rangle = \langle a, a, a + 1 \rangle$$

One possible solution is obtained for $a = 0$, that is, $\mathbf{X} = \langle 0, 0, 1 \rangle$.

In Exercises 63–66, the ***torque*** *about the origin O due to a force $\mathbf{F}$ acting on an object with position vector $\mathbf{r}$ is the vector quantity $\tau = \mathbf{r} \times \mathbf{F}$. If several forces $\mathbf{F}_j$ act at positions $\mathbf{r}_j$, then the net torque (units: N-m or lb-ft) is the sum*

$$\tau = \sum \mathbf{r}_j \times \mathbf{F}_j$$

Torque measures how much the force causes the object to rotate. By Newton's Laws, τ is equal to the rate of change of angular momentum.

63. Calculate the torque τ about O acting at the point P on the mechanical arm in Figure 22(A), assuming that a 25-newton force acts as indicated. Ignore the weight of the arm itself.

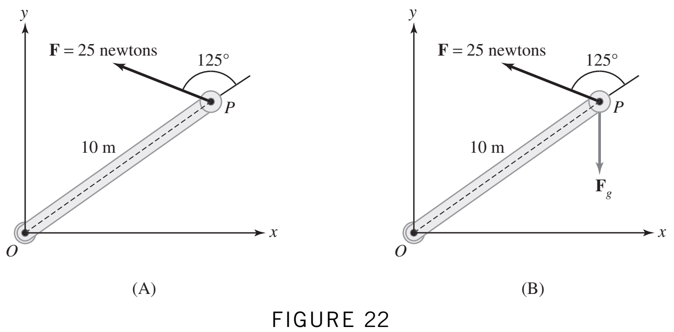


FIGURE 22

SOLUTION We denote by O and P the points shown in the figure and compute the position vector $\mathbf{r} = \overrightarrow{OP}$ and the force vector $\mathbf{F}$.

Denoting by θ the angle between the arm and the x-axis we have

$$\mathbf{r} = \overrightarrow{OP} = 10\,(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

The angle between the force vector $\mathbf{F}$ and the x-axis is $(\theta + 125°)$, hence,

$$\mathbf{F} = 25\left(\cos\left(\theta + 125°\right)\mathbf{i} + \sin\left(\theta + 125°\right)\mathbf{j}\right)$$

The torque τ about O acting at the point P is the cross product $\tau = \mathbf{r} \times \mathbf{F}$. We compute it using the cross products of the unit vectors $\mathbf{i}$ and $\mathbf{j}$:

$$\begin{aligned}\tau = \mathbf{r} \times \mathbf{F} &= 10\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \times 25\left(\cos\left(\theta + 125^\circ\right)\mathbf{i} + \sin\left(\theta + 125^\circ\right)\mathbf{j}\right)\\ &= 250\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \times \left(\cos\left(\theta + 125^\circ\right)\mathbf{i} + \sin\left(\theta + 125^\circ\right)\mathbf{j}\right)\\ &= 250\left(\cos\theta\sin\left(\theta + 125^\circ\right)\mathbf{k} + \sin\theta\cos\left(\theta + 125^\circ\right)(-\mathbf{k})\right)\\ &= 250\left(\sin\left(\theta + 125^\circ\right)\cos\theta - \sin\theta\cos\left(\theta + 125^\circ\right)\right)\mathbf{k}\end{aligned}$$

We now use the identity $\sin\alpha\cos\beta - \sin\beta\cos\alpha = \sin(\alpha - \beta)$ to obtain

$$\tau = 250\sin\left(\theta + 125^\circ - \theta\right)\mathbf{k} = 250\sin 125^\circ\mathbf{k} \approx 204.79\mathbf{k}$$

65. Let τ be the net torque about O acting on the robotic arm of Figure 23. Assume that the two segments of the arms have mass m_1 and m_2 (in kilograms) and that a weight of m_3 kg is located at the endpoint P. In calculating the torque, we may assume that the entire mass of each arm segment lies at the midpoint of the arm (its center of mass). Show that the position vectors of the masses m_1, m_2, and m_3 are

$$\begin{aligned}\mathbf{r}_1 &= \frac{1}{2}L_1(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j})\\ \mathbf{r}_2 &= L_1(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) + \frac{1}{2}L_2(\sin\theta_2\mathbf{i} - \cos\theta_2\mathbf{j})\\ \mathbf{r}_3 &= L_1(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}) + L_2(\sin\theta_2\mathbf{i} - \cos\theta_2\mathbf{j})\end{aligned}$$

Then show that

$$\tau = -g\left(L_1\left(\frac{1}{2}m_1 + m_2 + m_3\right)\sin\theta_1 + L_2\left(\frac{1}{2}m_2 + m_3\right)\sin\theta_2\right)\mathbf{k}$$

where $g = 9.8\ \text{m/s}^2$. To simplify the computation, note that all three gravitational forces act in the $-\mathbf{j}$ direction, so the $\mathbf{j}$-components of the position vectors $\mathbf{r}_i$ do not contribute to the torque.

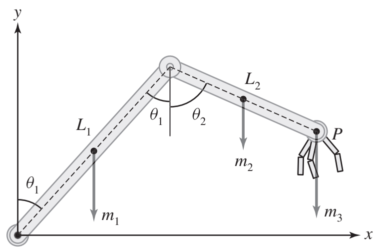
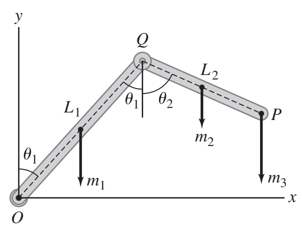


FIGURE 23

SOLUTION We denote by O, P, and Q the points shown in the figure.



The coordinates of O and Q are

$$O = (0, 0), \qquad Q = (L_1\sin\theta_1, L_1\cos\theta_1)$$

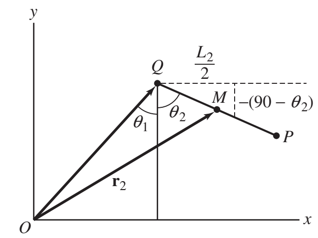
The midpoint of the segment OQ is, thus,

$$\left(\frac{0 + L_1\sin\theta_1}{2}, \frac{0 + L_1\cos\theta_1}{2}\right) = \left(\frac{L_1\sin\theta_1}{2}, \frac{L_1\cos\theta_1}{2}\right)$$

Since the mass m_1 is assumed to lie at the midpoint of the arm, the position vector of m_1 is

$$\mathbf{r}_1 = \frac{L_1}{2}\left(\sin\theta_1\mathbf{i} + \cos\theta_1\mathbf{j}\right) \tag{1}$$

We now find the position vector $\mathbf{r}_2$ of m_2. We have (see figure)



$$\mathbf{r}_2 = \overrightarrow{OQ} + \overrightarrow{QM} \tag{2}$$

$$\overrightarrow{OQ} = L_1 \sin\theta_1 \mathbf{i} + L_1 \cos\theta_1 \mathbf{j} = L_1 (\sin\theta_1 \mathbf{i} + \cos\theta_1 \mathbf{j}) \tag{3}$$

The vector $\overrightarrow{QM}$ makes an angle of $-(90^\circ - \theta_2)$ with the x axis and has length $\frac{L_2}{2}$, hence,

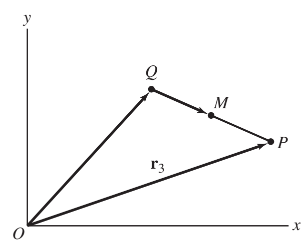
$$\overrightarrow{QM} = \frac{L_2}{2}\left(\cos\left(-\left(90^\circ - \theta_2\right)\right)\mathbf{i} + \sin\left(-\left(90^\circ - \theta_2\right)\right)\mathbf{j}\right) = \frac{L_2}{2}(\sin\theta_2 \mathbf{i} - \cos\theta_2 \mathbf{j}) \tag{4}$$

Combining (2), (3) and (4) we get

$$\mathbf{r}_2 = L_1 (\sin\theta_1 \mathbf{i} + \cos\theta_1 \mathbf{j}) + \frac{L_2}{2}(\sin\theta_2 \mathbf{i} - \cos\theta_2 \mathbf{j}) \tag{5}$$

Finally, we find the position vector $\mathbf{r}_3$:

$$\mathbf{r}_3 = \overrightarrow{OQ} + \overrightarrow{QP} = \overrightarrow{OQ} + 2\overrightarrow{QM}$$



Substituting (3) and (4) we get

$$\mathbf{r}_3 = L_1 (\sin\theta_1 \mathbf{i} + \cos\theta_1 \mathbf{j}) + L_2 (\sin\theta_2 \mathbf{i} - \cos\theta_2 \mathbf{j}) \tag{6}$$

The net torque is the following vector:

$$\tau = \mathbf{r}_1 \times (-gm_1\mathbf{j}) + \mathbf{r}_2 \times (-gm_2\mathbf{j}) + \mathbf{r}_3 \times (-gm_3\mathbf{j})$$

In computing the cross products, the $\mathbf{j}$ components of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ do not contribute to the torque since $\mathbf{j} \times \mathbf{j} = \mathbf{0}$. We thus consider only the $\mathbf{i}$ components of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ in (1), (5) and (6). This gives

$$\begin{aligned}
\tau &= \frac{L_1}{2}\sin\theta_1 \mathbf{i} \times (-gm_1\mathbf{j}) + \left(L_1 \sin\theta_1 + \frac{L_2}{2}\sin\theta_2\right)\mathbf{i} \times (-gm_2\mathbf{j}) + (L_1\sin\theta_1 + L_2\sin\theta_2)\,\mathbf{i} \times (-gm_3\mathbf{j}) \\
&= -\frac{L_1 g m_1 \sin\theta_1}{2}\mathbf{k} - \left(L_1 g m_2 \sin\theta_1 + \frac{L_2 g m_2}{2}\sin\theta_2\right)\mathbf{k} - (L_1 g m_3 \sin\theta_1 + L_2 g m_3 \sin\theta_2)\,\mathbf{k} \\
&= -\left(L_1\left(\frac{1}{2}gm_1 + gm_2 + gm_3\right)\sin\theta_1 + L_2\left(\frac{1}{2}gm_2 + gm_3\right)\sin\theta_2\right)\mathbf{k} \\
&= -g\left(L_1\left(\frac{1}{2}m_1 + m_2 + m_3\right)\sin\theta_1 + L_2\left(\frac{1}{2}m_2 + m_3\right)\sin\theta_2\right)\mathbf{k}
\end{aligned}$$

Further Insights and Challenges

67. Show that 3×3 determinants can be computed using the **diagonal rule**: Repeat the first two columns of the matrix and form the products of the numbers along the six diagonals indicated. Then add the products for the diagonals that slant from left to right and subtract the products for the diagonals that slant from right to left.

$$\det(A) = \left| \begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & \\ - & - & - & + & + & + \end{array} \right.$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

SOLUTION Using the definition of 3×3 determinants given in Eq. (2) we get

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Using the definition of 2×2 determinants given in Eq. (1) we get

$$\begin{aligned} \det(A) &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

69. Prove that $\mathbf{v} \times \mathbf{w} = \mathbf{v} \times \mathbf{u}$ if and only if $\mathbf{u} = \mathbf{w} + \lambda\mathbf{v}$ for some scalar λ. Assume that $\mathbf{v} \neq \mathbf{0}$.

SOLUTION Transferring sides and using the distributive law and the property of parallel vectors, we obtain the following equivalent equalities:

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \mathbf{v} \times \mathbf{u} \\ \mathbf{0} &= \mathbf{v} \times \mathbf{u} - \mathbf{v} \times \mathbf{w} \\ \mathbf{0} &= \mathbf{v} \times (\mathbf{u} - \mathbf{w}) \end{aligned}$$

This holds if and only if there exists a scalar λ such that

$$\begin{aligned} \mathbf{u} - \mathbf{w} &= \lambda\mathbf{v} \\ \mathbf{u} &= \mathbf{w} + \lambda\mathbf{v} \end{aligned}$$

71. Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$, then either (i) $\mathbf{u}$ and $\mathbf{v}$ are parallel, or (ii) $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$.

SOLUTION By the theorem on basic properties of the cross product, part (c), it follows that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$ if and only if

- $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ or
- $\mathbf{w} = \lambda\,(\mathbf{u} \times \mathbf{v})$

We consider the two possibilities.

1. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ is equivalent to $\mathbf{u}$ and $\mathbf{v}$ being parallel vectors or one of them being the zero vector.
2. The cross product $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, hence $\mathbf{w} = \lambda\,(\mathbf{u} \times \mathbf{v})$ implies that $\mathbf{w}$ is also orthogonal to $\mathbf{u}$ and $\mathbf{v}$ (for $\lambda \neq 0$) or $\mathbf{w} = \mathbf{0}$ (for $\lambda = 0$).

Conclusions: $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$ implies that either $\mathbf{u}$ and $\mathbf{v}$ are parallel, or $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, or one of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is the zero vector.

73. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be nonzero vectors. Assume that $\mathbf{b}$ and $\mathbf{c}$ are not parallel, and set

$$\mathbf{v} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \qquad \mathbf{w} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

(a) Prove that:

(i) $\mathbf{v}$ lies in the plane spanned by $\mathbf{b}$ and $\mathbf{c}$.

(ii) $\mathbf{v}$ is orthogonal to $\mathbf{a}$.

(b) Prove that $\mathbf{w}$ also satisfies (i) and (ii). Conclude that $\mathbf{v}$ and $\mathbf{w}$ are parallel.

(c) Show algebraically that $\mathbf{v} = \mathbf{w}$ (Figure 24).

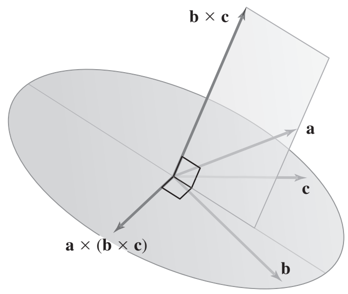


FIGURE 24

SOLUTION

(a) Since $\mathbf{v}$ is the cross product of $\mathbf{a}$ and another vector $(\mathbf{b} \times \mathbf{c})$, then $\mathbf{v}$ is orthogonal to $\mathbf{a}$. Furthermore, $\mathbf{v}$ is orthogonal to $(\mathbf{b} \times \mathbf{c})$, so it is orthogonal to the normal vector to the plane containing $\mathbf{b}$ and $\mathbf{c}$, so $\mathbf{v}$ must be in that plane.

(b) $\mathbf{w} \cdot \mathbf{a} = ((\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}) \cdot \mathbf{a} = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{a}) = 0$ (since $\mathbf{a} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a}$ and $\mathbf{b} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b}$). Thus, $\mathbf{w}$ is orthogonal to $\mathbf{a}$. Also, $\mathbf{w}$ is a multiple of $\mathbf{b}$ and $\mathbf{c}$, so $\mathbf{w}$ must be in the plane containing $\mathbf{b}$ and $\mathbf{c}$.

Now, if $\mathbf{a}$ is perpendicular to the plane spanned by $\mathbf{b}$ and $\mathbf{c}$, then $\mathbf{a}$ is parallel to $\mathbf{b} \times \mathbf{c}$ and so $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 0$, which means $\mathbf{v} = 0$, but also $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = 0$ which means $\mathbf{w} = 0$. Thus, $\mathbf{v}$ and $\mathbf{w}$ are parallel (in fact, equal).

Now, if $\mathbf{a}$ is not perpendicular to the plane spanned by $\mathbf{b}$ and $\mathbf{c}$, then the set of vectors on that plane that are also perpendicular to $\mathbf{a}$ form a line, and thus all such vectors are parallel. We conclude that $\mathbf{v}$ and $\mathbf{w}$, being on that plane and perpendicular to $\mathbf{a}$, are parallel.

(c) On the one hand,

$$\mathbf{v} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \langle a_1, a_2, a_3 \rangle \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - b_3c_2) & (b_3c_1 - b_1c_3) & (b_1c_2 - b_2c_1) \end{vmatrix}$$

$$= \big\langle a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3), a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1),$$
$$a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)\big\rangle$$

but on the other hand,

$$\mathbf{w} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$
$$= (a_1c_1 + a_2c_2 + a_3c_3)\langle b_1, b_2, b_3\rangle - (a_1b_1 + a_2b_2 + a_3b_3)\langle c_1, c_2, c_3\rangle$$
$$= \big\langle a_2c_2b_1 + a_3c_3b_1 - a_2b_2c_1 - a_3b_3c_1, a_1c_1b_2 + a_3c_3b_2 - a_1b_1c_2 - a_3b_3c_2,$$
$$a_1c_1b_3 + a_2c_2b_3 - a_1b_1c_3 - a_2b_2c_3\big\rangle$$
$$= \big\langle a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3), a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1),$$
$$a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)\big\rangle$$

which is the same as $\mathbf{v}$.

75. Show that if $\mathbf{a}$, $\mathbf{b}$ are nonzero vectors such that $\mathbf{a} \perp \mathbf{b}$, then there exists a vector $\mathbf{X}$ such that

$$\mathbf{a} \times \mathbf{X} = \mathbf{b} \qquad \boxed{1}$$

Hint: Show that if $\mathbf{X}$ is orthogonal to $\mathbf{b}$ and is not a multiple of $\mathbf{a}$, then $\mathbf{a} \times \mathbf{X}$ is a multiple of $\mathbf{b}$.

SOLUTION We define the following vectors:

$$\mathbf{X} = \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a}\|^2}, \quad \mathbf{c} = \mathbf{a} \times \mathbf{X} \qquad (1)$$

We show that $\mathbf{c} = \mathbf{b}$. Since $\mathbf{X}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{X}$ is orthogonal to the plane of $\mathbf{a}$ and $\mathbf{b}$. But $\mathbf{c}$ is orthogonal to $\mathbf{X}$, hence $\mathbf{c}$ is contained in the plane of $\mathbf{a}$ and $\mathbf{b}$, that is, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are in the same plane. Now the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are in one plane, and the vectors $\mathbf{c}$ and $\mathbf{b}$ are orthogonal to $\mathbf{a}$. It follows that $\mathbf{c}$ and $\mathbf{b}$ are parallel. We now show that $\|\mathbf{c}\| = \|\mathbf{b}\|$. We use the cross-product identity to obtain

$$\|\mathbf{c}\|^2 = \|\mathbf{a}\times\mathbf{X}\|^2 = \|\mathbf{a}\|^2\|\mathbf{X}\|^2 - (\mathbf{a}\cdot\mathbf{X})^2$$

$\mathbf{X}$ is orthogonal to $\mathbf{a}$, hence $\mathbf{a}\cdot\mathbf{X} = 0$, and we obtain

$$\|\mathbf{c}\|^2 = \|\mathbf{a}\|^2\|\mathbf{X}\|^2 = \|\mathbf{a}\|^2\left\|\frac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\|^2}\right\|^2 = \frac{\|\mathbf{a}\|^2}{\|\mathbf{a}\|^4}\|\mathbf{a}\times\mathbf{b}\|^2 = \frac{1}{\|\mathbf{a}\|^2}\|\mathbf{a}\times\mathbf{b}\|^2.$$

By the given data, $\mathbf{a}$ and $\mathbf{b}$ are orthogonal vectors, so $\|\mathbf{a}\times\mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2$, and then

$$\|\mathbf{c}\|^2 = \frac{1}{\|\mathbf{a}\|^2}\|\mathbf{a}\times\mathbf{b}\|^2 = \frac{1}{\|\mathbf{a}\|^2}\left(\|\mathbf{a}\|^2\|\mathbf{a}\|^2\right) = \|\mathbf{b}\|^2 \Rightarrow \|\mathbf{c}\| = \|\mathbf{b}\|. \tag{2}$$

Since $\mathbf{c}$ and $\mathbf{b}$ are parallel, it follows that $\mathbf{c} = \mathbf{b}$ or $\mathbf{c} = -\mathbf{b}$. We thus proved that the vector $\mathbf{X} = \dfrac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\|^2}$ satisfies $\mathbf{a}\times\mathbf{X} = \mathbf{b}$ or $\mathbf{a}\times\mathbf{X} = -\mathbf{b}$. If $\mathbf{a}\times\mathbf{X} = -\mathbf{b}$, then $\mathbf{a}\times(-\mathbf{X}) = \mathbf{b}$. Hence, there exists a vector $\mathbf{X}$ such that $\mathbf{a}\times\mathbf{X} = \mathbf{b}$.

77. Assume that $\mathbf{v}$ and $\mathbf{w}$ lie in the first quadrant in $\mathbf{R}^2$ as in Figure 25. Use geometry to prove that the area of the parallelogram is equal to $\det\begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix}$.

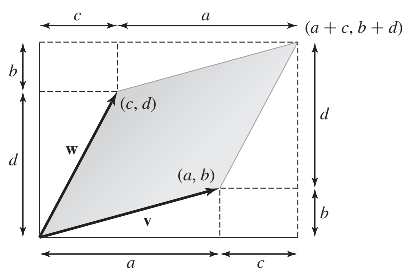
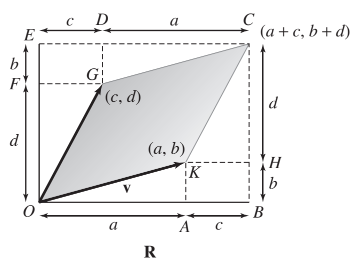


FIGURE 25

SOLUTION We denote the components of $\mathbf{u}$ and $\mathbf{v}$ by

$$\mathbf{u} = \langle c, d\rangle$$
$$\mathbf{v} = \langle a, b\rangle$$

We also denote by $O, A, B, C, D, E, F, G, H, K$ the points shown in the figure.



Since $OGCK$ is a parallelogram, it follows by geometrical properties that the triangles OFG and KHC and also the triangles DGC and AKO are congruent. It also follows that the rectangles $EFDG$ and $ABHK$ have equal areas. We use the following notation:

A: The area of the parallelogram
S: The area of the rectangle $OBCE$
S_1: The area of the rectangle $EFDG$
S_2: The area of the triangle OFG
S_3: The area of the triangle DGC

Hence,

$$A = S - 2(S_1 + S_2 + S_3) \tag{1}$$

Using the formulas for the areas of rectangles and triangles we have (see figure)

$$S = OB \cdot OE = (a + c)(d + b)$$
$$S_1 = bc, \quad S_2 = \frac{cd}{2}, \quad S_3 = \frac{ab}{2}$$

Substituting into (1) we get

$$\begin{aligned} A &= (a + c)(d + b) - 2\left(bc + \frac{cd}{2} + \frac{ab}{2}\right) \\ &= ad + ab + cd + cb - 2bc - cd - ab \\ &= ad - bc \end{aligned} \tag{2}$$

On the other hand,

$$\det\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{3}$$

By (2) and (3) we obtain the desired result.

79. In the notation of Exercise 78, suppose that **a**, **b**, **c** are mutually perpendicular as in Figure 26(B). Let S_F be the area of face F. Prove the following three-dimensional version of the Pythagorean Theorem:

$$S_A^2 + S_B^2 + S_C^2 = S_D^2$$

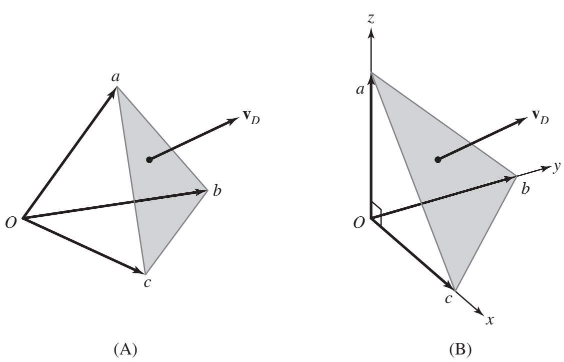


FIGURE 26 The vector $\mathbf{v}_D$ is perpendicular to the face.

SOLUTION Since $\|\mathbf{v}_D\| = S_D$ then using Exercise 78 we obtain

$$\begin{aligned} S_D^2 &= \|\mathbf{v}_D\|^2 = \mathbf{v}_D \cdot \mathbf{v}_D = (\mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C) \cdot (\mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C) \\ &= \mathbf{v}_A \cdot \mathbf{v}_A + \mathbf{v}_A \cdot \mathbf{v}_B + \mathbf{v}_A \cdot \mathbf{v}_C + \mathbf{v}_B \cdot \mathbf{v}_A + \mathbf{v}_B \cdot \mathbf{v}_B + \mathbf{v}_B \cdot \mathbf{v}_C + \mathbf{v}_C \cdot \mathbf{v}_A + \mathbf{v}_C \cdot \mathbf{v}_B + \mathbf{v}_C \cdot \mathbf{v}_C \\ &= \|\mathbf{v}_A\|^2 + \|\mathbf{v}_B\|^2 + \|\mathbf{v}_C\|^2 + 2\left(\mathbf{v}_A \cdot \mathbf{v}_B + \mathbf{v}_A \cdot \mathbf{v}_C + \mathbf{v}_B \cdot \mathbf{v}_C\right) \end{aligned} \tag{1}$$

Now, the normals $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$ to the coordinate planes are mutually orthogonal, hence,

$$\mathbf{v}_A \cdot \mathbf{v}_B = \mathbf{v}_A \cdot \mathbf{v}_C = \mathbf{v}_B \cdot \mathbf{v}_C = 0 \tag{2}$$

Combining (1) and (2) and using the relations $\|\mathbf{v}_F\| = S_F$ we obtain

$$S_D^2 = S_A^2 + S_B^2 + S_C^2$$

12.5 Planes in 3-Space

Preliminary Questions

1. What is the equation of the plane parallel to $3x + 4y - z = 5$ passing through the origin?

SOLUTION The two planes are parallel, therefore the vector $\mathbf{n} = \langle 3, 4, -1\rangle$ that is normal to the given plane is also normal to the plane we need to find. This plane is passing through the origin, hence we may substitute $\langle x_0, y_0, z_0\rangle = \langle 0, 0, 0\rangle$ in the vector form of the equation of the plane. This gives

$$\mathbf{n}\cdot\langle x, y, z\rangle = \mathbf{n}\cdot\langle x_0, y_0, z_0\rangle$$
$$\langle 3, 4, -1\rangle\cdot\langle x, y, z\rangle = \langle 3, 4, -1\rangle\cdot\langle 0, 0, 0\rangle = 0$$

or in scalar form

$$3x + 4y - z = 0$$

2. The vector $\mathbf{k}$ is normal to which of the following planes?

(a) $x = 1$ **(b)** $y = 1$ **(c)** $z = 1$

SOLUTION The planes $x = 1$, $y = 1$, and $z = 1$ are orthogonal to the x, y, and z-axes respectively. Since the plane $z = 1$ is orthogonal to the z-axis, the vector $\mathbf{k}$ is normal to this plane.

3. Which of the following planes is not parallel to the plane $x + y + z = 1$?

(a) $2x + 2y + 2z = 1$ **(b)** $x + y + z = 3$

(c) $x - y + z = 0$

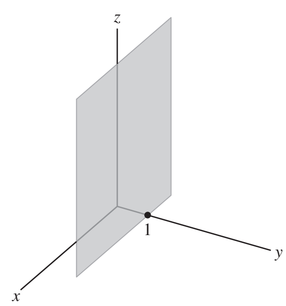
SOLUTION The two planes are parallel if vectors that are normal to the planes are parallel. The vector $\mathbf{n} = \langle 1, 1, 1\rangle$ is normal to the plane $x + y + z = 1$. We identify the following normals:

- $\mathbf{v} = \langle 2, 2, 2\rangle$ is normal to plane (a)
- $\mathbf{u} = \langle 1, 1, 1\rangle$ is normal to plane (b)
- $\mathbf{w} = \langle 1, -1, 1\rangle$ is normal to plane (c)

The vectors $\mathbf{v}$ and $\mathbf{u}$ are parallel to $\mathbf{n}$, whereas $\mathbf{w}$ is not. (These vectors are not constant multiples of each other). Therefore, only plane (c) is not parallel to the plane $x + y + z = 1$.

4. To which coordinate plane is the plane $y = 1$ parallel?

SOLUTION The plane $y = 1$ is parallel to the xz-plane.



5. Which of the following planes contains the z-axis?

(a) $z = 1$ **(b)** $x + y = 1$ **(c)** $x + y = 0$

SOLUTION The points on the z-axis are the points with zero x and y coordinates. A plane contains the z-axis if and only if the points $(0, 0, c)$ satisfy the equation of the plane for all values of c.

(a) Plane (a) does not contain the z-axis, rather it is orthogonal to this axis. Only the point $(0, 0, 1)$ is on the plane.

(b) $x = 0$ and $y = 0$ do not satisfy the equation of the plane, since $0 + 0 \neq 1$. Therefore the plane does not contain the z-axis.

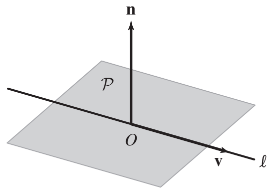
(c) The plane $x + y = 0$ contains the z-axis since $x = 0$ and $y = 0$ satisfy the equation of the plane.

6. Suppose that a plane $\mathcal{P}$ with normal vector **n** and a line $\mathcal{L}$ with direction vector **v** both pass through the origin and that $\mathbf{n} \cdot \mathbf{v} = 0$. Which of the following statements is correct?

(a) $\mathcal{L}$ is contained in $\mathcal{P}$.

(b) $\mathcal{L}$ is orthogonal to $\mathcal{P}$.

SOLUTION The direction vector of the line $\mathcal{L}$ is orthogonal to the vector **n** that is normal to the plane. Therefore, $\mathcal{L}$ is either parallel or contained in the plane. Since the origin is common to $\mathcal{L}$ and $\mathcal{P}$, the line is contained in the plane. That is, statement (a) is correct.



Exercises

In Exercises 1–8, write the equation of the plane with normal vector **n** *passing through the given point in the scalar form $ax + by + cz = d$.*

1. $\mathbf{n} = \langle 1, 3, 2\rangle, \quad (4, -1, 1)$

SOLUTION The vector equation is

$$\langle 1, 3, 2\rangle \cdot \langle x, y, z\rangle = \langle 1, 3, 2\rangle \cdot \langle 4, -1, 1\rangle = 4 - 3 + 2 = 3$$

To obtain the scalar forms we compute the dot product on the left-hand side of the previous equation:

$$x + 3y + 2z = 3$$

or in the other scalar form:

$$(x - 4) + 3(y + 1) + 2(z - 1) + 4 - 3 + 2 = 3$$
$$(x - 4) + 3(y + 1) + 2(z - 1) = 0$$

3. $\mathbf{n} = \langle -1, 2, 1\rangle, \quad (4, 1, 5)$

SOLUTION The vector form is

$$\langle -1, 2, 1\rangle \cdot \langle x, y, z\rangle = \langle -1, 2, 1\rangle \cdot \langle 4, 1, 5\rangle = -4 + 2 + 5 = 3$$

To obtain the scalar form we compute the dot product above:

$$-x + 2y + z = 3$$

or in the other scalar form:

$$-(x - 4) + 2(y - 1) + (z - 5) = 3 + 4 - 2 - 5 = 0$$
$$-(x - 4) + 2(y - 1) + (z - 5) = 0$$

5. $\mathbf{n} = \mathbf{i}, \quad (3, 1, -9)$

SOLUTION We find the vector form of the equation of the plane. We write the vector $\mathbf{n} = \mathbf{i}$ as $\mathbf{n} = \langle 1, 0, 0\rangle$ and obtain

$$\langle 1, 0, 0\rangle \cdot \langle x, y, z\rangle = \langle 1, 0, 0\rangle \cdot \langle 3, 1, -9\rangle = 3 + 0 + 0 = 3$$

Computing the dot product above gives the scalar form:

$$x + 0 + 0 = 3$$
$$x = 3$$

Or in the other scalar form:

$$(x - 3) + 0 \cdot (y - 1) + 0 \cdot (z + 9) = 3 - 3 = 0$$

7. $\mathbf{n} = \mathbf{k}$, $(6, 7, 2)$

SOLUTION We write the normal $\mathbf{n} = \mathbf{k}$ in the form $\mathbf{n} = \langle 0, 0, 1\rangle$ and obtain the following vector form of the equation of the plane:

$$\langle 0, 0, 1\rangle \cdot \langle x, y, z\rangle = \langle 0, 0, 1\rangle \cdot \langle 6, 7, 2\rangle = 0 + 0 + 2 = 2$$

We compute the dot product to obtain the scalar form:

$$0x + 0y + 1z = 2$$
$$z = 2$$

or in the other scalar form:

$$0(x - 6) + 0(y - 7) + 1(z - 2) = 0$$

9. Write down the equation of any plane through the origin.

SOLUTION We can use any equation $ax + by + cz = d$ which contains the point $(x, y, z) = (0, 0, 0)$. One solution (and there are many) is $x + y + z = 0$.

11. Which of the following statements are true of a plane that is parallel to the yz-plane?

(a) $\mathbf{n} = \langle 0, 0, 1\rangle$ is a normal vector.

(b) $\mathbf{n} = \langle 1, 0, 0\rangle$ is a normal vector.

(c) The equation has the form $ay + bz = d$

(d) The equation has the form $x = d$

SOLUTION

(a) For $\mathbf{n} = \langle 0, 0, 1\rangle$ a normal vector, the plane would be parallel to the xy-plane, not the yz-plane. This statement is false.

(b) For $\mathbf{n} = \langle 1, 0, 0\rangle$ a normal vector, the plane would be parallel to the yz-plane. This statement is true.

(c) For the equation $ay + bz = d$, this plane intersects the yz-plane at $y = 0, z = d/b$ or $y = d/a, z = 0$ depending on whether a or b is non-zero, but it is not equal to the yz-plane (which has equation $x = d$) Thus, it is not parallel to the yz-plane This statement is false.

(d) For the equation of the form $x = d$, this has $\langle 1, 0, 0\rangle$ as a normal vector and is parallel to the yz-plane. This statement is true.

In Exercises 13–16, find a vector normal to the plane with the given equation.

13. $9x - 4y - 11z = 2$

SOLUTION Using the scalar form of the equation of the plane, a vector normal to the plane is the coefficients vector:

$$\mathbf{n} = \langle 9, -4, -11\rangle$$

15. $3(x - 4) - 8(y - 1) + 11z = 0$

SOLUTION Using the scalar form of the equation of the plane, $3x - 8y + 11z = 4$ a vector normal to the plane is the coefficients vector:

$$\mathbf{n} = \langle 3, -8, 11\rangle$$

In Exercises 17–20, find an equation of the plane passing through the three points given.

17. $P = (2, -1, 4)$, $Q = (1, 1, 1)$, $R = (3, 1, -2)$

SOLUTION We go through the steps below:

Step 1. Find the normal vector $\mathbf{n}$. The vectors $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$ lie on the plane, hence the cross product $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is normal to the plane. We compute the cross product:

$$\mathbf{a} = \overrightarrow{PQ} = \langle 1 - 2, 1 - (-1), 1 - 4\rangle = \langle -1, 2, -3\rangle$$
$$\mathbf{b} = \overrightarrow{PR} = \langle 3 - 2, 1 - (-1), -2 - 4\rangle = \langle 1, 2, -6\rangle$$

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -3 \\ 1 & 2 & -6 \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 2 & -6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -3 \\ 1 & -6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{k}$$
$$= -6\mathbf{i} - 9\mathbf{j} - 4\mathbf{k} = \langle -6, -9, -4 \rangle$$

Step 2. Choose a point on the plane. We choose any one of the three points on the plane, for instance $Q = (1, 1, 1)$. Using the vector form of the equation of the plane we get

$$\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \langle x_0, y_0, z_0 \rangle$$
$$\langle -6, -9, -4 \rangle \cdot \langle x, y, z \rangle = \langle -6, -9, -4 \rangle \cdot \langle 1, 1, 1 \rangle$$

Computing the dot products we obtain the following equation:

$$-6x - 9y - 4z = -6 - 9 - 4 = -19$$
$$6x + 9y + 4z = 19$$

19. $P = (1, 0, 0), \quad Q = (0, 1, 1), \quad R = (2, 0, 1)$

SOLUTION We use the vector form of the equation of the plane:

$$\mathbf{n} \cdot \langle x, y, z \rangle = d \tag{1}$$

To find the normal vector to the plane, $\mathbf{n}$, we first compute the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ that lie in the plane, and then find the cross product of these vectors. This gives

$$\overrightarrow{PQ} = \langle 0, 1, 1 \rangle - \langle 1, 0, 0 \rangle = \langle -1, 1, 1 \rangle$$
$$\overrightarrow{PR} = \langle 2, 0, 1 \rangle - \langle 1, 0, 0 \rangle = \langle 1, 0, 1 \rangle$$
$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{k}$$
$$= \mathbf{i} + 2\mathbf{j} - \mathbf{k} = \langle 1, 2, -1 \rangle \tag{2}$$

We now choose any one of the three points in the plane, say $P = (1, 0, 0)$, and compute d:

$$d = \mathbf{n} \cdot \overrightarrow{OP} = \langle 1, 2, -1 \rangle \cdot \langle 1, 0, 0 \rangle = 1 \cdot 1 + 2 \cdot 0 + (-1) \cdot 0 = 1 \tag{3}$$

Finally we substitute (2) and (3) into (1) to obtain the following equation of the plane:

$$\langle 1, 2, -1 \rangle \cdot \langle x, y, z \rangle = 1$$
$$x + 2y - z = 1$$

In Exercises 21–28, find the equation of the plane with the given description.

21. Passes through O and is parallel to $4x - 9y + z = 3$

SOLUTION The vector $\mathbf{n} = \langle 4, -9, 1 \rangle$ is normal to the plane $4x - 9y + z = 3$, and so is also normal to the parallel plane. Setting $\mathbf{n} = \langle 4, -9, 1 \rangle$ and $(x_0, y_0, z_0) = (0, 0, 0)$ in the vector equation of the plane yields

$$\langle 4, -9, 1 \rangle \cdot \langle x, y, z \rangle = \langle 4, -9, 1 \rangle \cdot \langle 0, 0, 0 \rangle = 0$$
$$4x - 9y + z = 0$$

23. Passes through $(4, 1, 9)$ and is parallel to $x = 3$

SOLUTION The vector form of the plane $x = 3$ is

$$\langle 1, 0, 0\rangle \cdot \langle x, y, z\rangle = 3$$

Hence, $\mathbf{n} = \langle 1, 0, 0\rangle$ is normal to this plane. This vector is also normal to the parallel plane. Setting $(x_0, y_0, z_0) = (4, 1, 9)$ and $\mathbf{n} = \langle 1, 0, 0\rangle$ in the vector equation of the plane yields

$$\langle 1, 0, 0\rangle \cdot \langle x, y, z\rangle = \langle 1, 0, 0\rangle \cdot \langle 4, 1, 9\rangle = 4 + 0 + 0 = 4$$

or

$$x + 0 + 0 = 4 \quad \Rightarrow \quad x = 4$$

25. Passes through $(-2, -3, 5)$ and has normal vector $\mathbf{i} + \mathbf{k}$

SOLUTION We substitute $\mathbf{n} = \langle 1, 0, 1\rangle$ and $(x_0, y_0, z_0) = (-2, -3, 5)$ in the vector equation of the plane to obtain

$$\langle 1, 0, 1\rangle \cdot \langle x, y, z\rangle = \langle 1, 0, 1\rangle \cdot \langle -2, -3, 5\rangle$$

or

$$\begin{aligned} x + 0 + z &= -2 + 0 + 5 = 3 \\ x + z &= 3 \end{aligned}$$

27. Contains the lines $\mathbf{r}_1(t) = \langle 2, 1, 0\rangle + \langle t, 2t, 3t\rangle$ and $\mathbf{r}_2(t) = \langle 2, 1, 0\rangle + \langle 3t, t, 8t\rangle$

SOLUTION Since the plane contains the lines $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, the direction vectors $\mathbf{v}_1 = \langle 1, 2, 3\rangle$ and $\mathbf{v}_2 = \langle 3, 1, 8\rangle$ of the lines lie in the plane. Therefore the cross product $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ is normal to the plane. We compute the cross product:

$$\begin{aligned} \mathbf{n} = \langle 1, 2, 3\rangle \times \langle 3, 1, 8\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 8 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{k} \\ &= 13\mathbf{i} + \mathbf{j} - 5\mathbf{k} = \langle 13, 1, -5\rangle \end{aligned}$$

We now must choose a point on the plane. Since the line $\mathbf{r}_1(t) = \langle 2 + t, 1 + 2t, 3t\rangle$ is contained in the plane, all of its points are on the plane. We choose the point corresponding to $t = 0$, that is,

$$\langle x_0, y_0, z_0\rangle = \langle 2, 1, 0\rangle$$

We now use the vector equation of the plane to determine the equation of the desired plane:

$$\begin{aligned} \mathbf{n} \cdot \langle x, y, z\rangle &= \mathbf{n} \cdot \langle x_0, y_0, z_0\rangle \\ \langle 13, 1, -5\rangle \cdot \langle x, y, z\rangle &= \langle 13, 1, -5\rangle \cdot \langle 2, 1, 0\rangle \\ 13x + y - 5z &= 26 + 1 + 0 = 27 \\ 13x + y - 5z &= 27 \end{aligned}$$

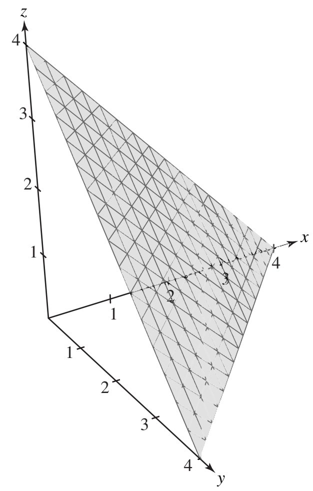
29. Are the planes $\frac{1}{2}x + 2y - z = 5$ and $3x + 12y - 6z = 1$ parallel?

SOLUTION The planes $\frac{1}{2}x + 2y - z = 5$ and $3x + 12y - 6z = 1$ are parallel if and only if the vectors $\mathbf{n}_1 = \left\langle \frac{1}{2}, 2, -1\right\rangle$ and $\mathbf{n}_2 = \langle 3, 12, -6\rangle$ normal to the planes are parallel. Since $\mathbf{n}_2 = 6\mathbf{n}_1$ the planes are parallel.

In Exercises 31–35, draw the plane given by the equation.

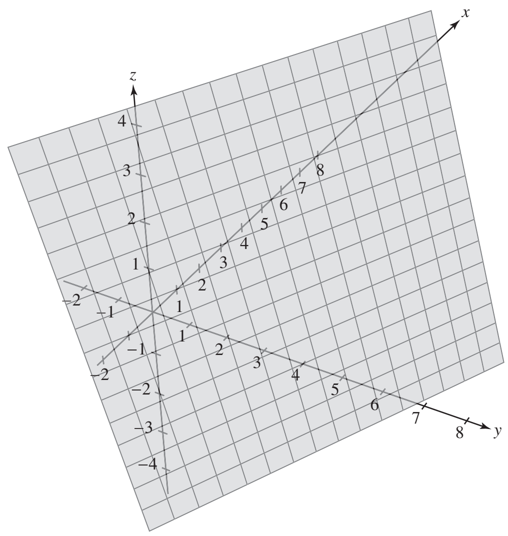
31. $x + y + z = 4$

SOLUTION



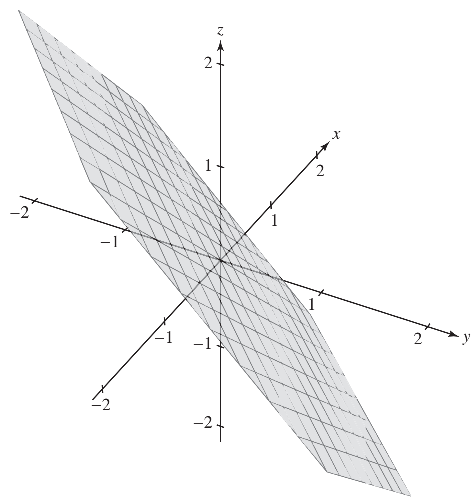
33. $12x - 6y + 4z = 6$

SOLUTION



35. $x + y + z = 0$

SOLUTION



37. Find an equation of the plane $\mathcal{P}$ in Figure 8.

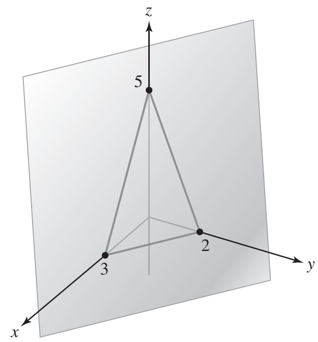


FIGURE 8

SOLUTION We must find the equation of the plane passing though the points $P = (3, 0, 0)$, $Q = (0, 2, 0)$, and $R = (0, 0, 5)$. We use the following steps:

Step 1. Find a normal vector $\mathbf{n}$. The vectors $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$ lie in the plane, hence the cross product $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is normal to the plane. We compute the cross product:

$$\mathbf{a} = \overrightarrow{PQ} = \langle 0-3, 2-0, 0-0 \rangle = \langle -3, 2, 0 \rangle$$

$$\mathbf{b} = \overrightarrow{PR} = \langle 0-3, 0-0, 5-0 \rangle = \langle -3, 0, 5 \rangle$$

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 0 \\ -3 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 0 \\ -3 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 2 \\ -3 & 0 \end{vmatrix} \mathbf{k}$$

$$= 10\mathbf{i} + 15\mathbf{j} + 6\mathbf{k} = \langle 10, 15, 6 \rangle$$

Step 2. Choose a point on the plane. We choose one of the points on the plane, say $P = (3, 0, 0)$. Substituting $\mathbf{n} = \langle 10, 15, 6 \rangle$ and $(x_0, y_0, z_0) = (3, 0, 0)$ in the vector form of the equation of the plane gives

$$\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \langle x_0, y_0, z_0 \rangle$$

$$\langle 10, 15, 6 \rangle \cdot \langle x, y, z \rangle = \langle 10, 15, 6 \rangle \cdot \langle 3, 0, 0 \rangle$$

Computing the dot products we get the following scalar form of the equation of the plane:

$$10x + 15y + 6z = 10 \cdot 3 + 0 + 0 = 30$$

$$10x + 15y + 6z = 30$$

In Exercises 39–42, find the intersection of the line and the plane.

39. $x + y + z = 14, \quad \mathbf{r}(t) = \langle 1, 1, 0 \rangle + t \langle 0, 2, 4 \rangle$

SOLUTION The line has parametric equations

$$x = 1, \quad y = 1 + 2t, \quad z = 4t$$

To find a value of t for which (x, y, z) lies on the plane, we substitute the parametric equations in the equation of the plane and solve for t:

$$x + y + z = 14$$

$$1 + (1 + 2t) + 4t = 14$$

$$6t = 12 \quad \Rightarrow \quad t = 2$$

The point P of intersection has coordinates

$$x = 1, \quad y = 1 + 2 \cdot 2 = 5, \quad z = 4 \cdot 2 = 8$$

That is, $P = (1, 5, 8)$.

41. $z = 12, \quad \mathbf{r}(t) = t\,\langle -6, 9, 36\rangle$

SOLUTION The parametric equations of the line are

$$x = -6t, \quad y = 9t, \quad z = 36t \tag{1}$$

We substitute the parametric equations in the equation of the plane and solve for t:

$$z = 12$$
$$36t = 12 \quad \Rightarrow \quad t = \frac{1}{3}$$

The value of the parameter at the point of intersection is $t = \frac{1}{3}$. Substituting into (1) gives the coordinates of the point P of intersection:

$$x = -6 \cdot \frac{1}{3} = -2, \quad y = 9 \cdot \frac{1}{3} = 3, \quad z = 36 \cdot \frac{1}{3} = 12$$

That is,

$$P = (-2, 3, 12).$$

In Exercises 43–48, find the trace of the plane in the given coordinate plane.

43. $3x - 9y + 4z = 5, \quad yz$

SOLUTION The yz-plane has the equation $x = 0$, hence the intersection of the plane with the yz-plane must satisfy both $x = 0$ and the equation of the plane $3x - 9y + 4z = 5$. That is, this is the set of all points $(0, y, z)$ in the yz-plane such that $-9y + 4z = 5$.

45. $3x + 4z = -2, \quad xy$

SOLUTION The trace of the plane $3x + 4z = -2$ in the xy coordinate plane is the set of all points that satisfy the equation of the plane and the equation $z = 0$ of the xy coordinate plane. Thus, we substitute $z = 0$ in $3x + 4z = -2$ to obtain the line $3x = -2$ or $x = -\frac{2}{3}$ in the xy-plane.

47. $-x + y = 4, \quad xz$

SOLUTION The trace of the plane $-x + y = 4$ on the xz-plane is the set of all points that satisfy both the equation of the given plane and the equation $y = 0$ of the xz-plane. That is, the set of all points $(x, 0, z)$ such that $-x + 0 = 4$, or $x = -4$. This is a vertical line in the xz-plane.

49. Does the plane $x = 5$ have a trace in the yz-plane? Explain.

SOLUTION The yz-plane has the equation $x = 0$, hence the x-coordinates of the points in this plane are zero, whereas the x-coordinates of the points in the plane $x = 5$ are 5. Thus, the two planes have no common points.

51. Give equations for two distinct planes whose trace in the yz-plane has equation $y = 4z$.

SOLUTION The yz-plane has the equation $x = 0$, hence the trace of a plane $ax + by + cz = 0$ in the yz-plane is obtained by substituting $x = 0$ in the equation of the plane. Therefore, the following two planes have trace $y = 4z$ (that is, $y - 4z = 0$) in the yz-plane:

$$x + y - 4z = 0; \quad 2x + y - 4z = 0$$

53. Find all planes in $\mathbf{R}^3$ whose intersection with the xz-plane is the line with equation $3x + 2z = 5$.

SOLUTION The intersection of the plane $ax + by + cz = d$ with the xz-plane is obtained by substituting $y = 0$ in the equation of the plane. This gives the following line in the xz-plane:

$$ax + cz = d$$

This is the equation of the line $3x + 2z = 5$ if and only if for some $\lambda \neq 0$,

$$a = 3\lambda, \quad c = 2\lambda, \quad d = 5\lambda$$

Notice that b can have any value. The planes are thus

$$(3\lambda)x + by + (2\lambda)z = 5\lambda, \quad \lambda \neq 0.$$

In Exercises 55–60, compute the angle between the two planes, defined as the angle θ (between 0 and π) between their normal vectors (Figure 9).

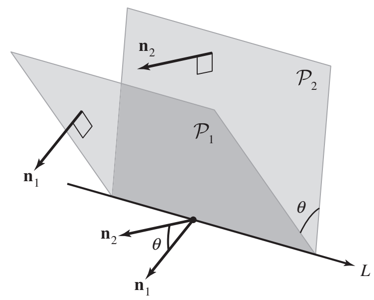


FIGURE 9 By definition, the angle between two planes is the angle between their normal vectors.

55. Planes with normals $\mathbf{n}_1 = \langle 1, 0, 1\rangle$, $\mathbf{n}_2 = \langle -1, 1, 1\rangle$

SOLUTION Using the formula for the angle between two vectors we get

$$\cos\theta = \frac{\mathbf{n}_1\cdot\mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\langle 1,0,1\rangle\cdot\langle -1,1,1\rangle}{\|\langle 1,0,1\rangle\|\|\langle -1,1,1\rangle\|} = \frac{-1+0+1}{\sqrt{1^2+0+1^2}\sqrt{(-1)^2+1^2+1^2}} = 0$$

The solution for $0 \le \theta < \pi$ is $\theta = \frac{\pi}{2}$.

57. $2x + 3y + 7z = 2$ and $4x - 2y + 2z = 4$

SOLUTION The planes $2x + 3y + 7z = 2$ and $4x - 2y + 2z = 4$ have the normals $\mathbf{n}_1 = \langle 2, 3, 7\rangle$ and $\mathbf{n}_2 = \langle 4, -2, 2\rangle$ respectively. The cosine of the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is

$$\cos\theta = \frac{\mathbf{n}_1\cdot\mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\langle 2,3,7\rangle\cdot\langle 4,-2,2\rangle}{\|\langle 2,3,7\rangle\|\|\langle 4,-2,2\rangle\|} = \frac{8-6+14}{\sqrt{2^2+3^2+7^2}\sqrt{4^2+(-2)^2+2^2}} = \frac{16}{\sqrt{62}\sqrt{24}} \approx 0.415$$

The solution for $0 \le \theta < \pi$ is $\theta = 1.143$ rad or $\theta = 65.49^\circ$.

59. $3(x-1) - 5y + 2(z-12) = 0$ and the plane with normal $\mathbf{n} = \langle 1, 0, 1\rangle$

SOLUTION The plane $3(x-1) - 5y + 2(z-12) = 0$ has the normal $\mathbf{n}_1 = \langle 3, -5, 2\rangle$, and our second plane has given normal $\mathbf{n}_2 = \langle 1, 0, 1\rangle$. We use the formula for the angle between two vectors:

$$\cos\theta = \frac{\mathbf{n}_1\cdot\mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\langle 3,-5,2\rangle\cdot\langle 1,0,1\rangle}{\|\langle 3,-5,2\rangle\|\|\langle 1,0,1\rangle\|} = \frac{3+0+2}{\sqrt{3^2+(-5)^2+2^2}\sqrt{1^2+0+1^2}} = \frac{5}{\sqrt{38}\sqrt{2}} \approx 0.5735$$

The solution for $0 \le \theta < \pi$ is $\theta = 0.96$ rad or $\theta = 55^\circ$.

61. Find an equation of a plane making an angle of $\frac{\pi}{2}$ with the plane $3x + y - 4z = 2$.

SOLUTION The angle θ between two planes (chosen so that $0 \le \theta < \pi$) is defined as the angle between their normal vectors. The following vector is normal to the plane $3x + y - 4z = 2$:

$$\mathbf{n}_1 = \langle 3, 1, -4\rangle$$

Let $\mathbf{n}\cdot\langle x, y, z\rangle = d$ denote the equation of a plane making an angle of $\frac{\pi}{2}$ with the given plane, where $\mathbf{n} = \langle a, b, c\rangle$. Since the two planes are perpendicular, the dot product of their normal vectors is zero. That is,

$$\mathbf{n}\cdot\mathbf{n}_1 = \langle a, b, c\rangle\cdot\langle 3, 1, -4\rangle = 3a + b - 4c = 0 \quad\Rightarrow\quad b = -3a + 4c$$

Thus, the required planes (there is more than one plane) have the following normal vector:

$$\mathbf{n} = \langle a, -3a + 4c, c\rangle$$

We obtain the following equation:

$$\begin{aligned} \mathbf{n}\cdot\langle x, y, c\rangle &= d \\ \langle a, -3a+4c, c\rangle\cdot\langle x, y, z\rangle &= d \\ ax + (4c - 3a)y + cz &= d \end{aligned}$$

Every choice of the values of a, c and d yields a plane with the desired property. For example, we set $a = c = d = 1$ to obtain

$$x + y + z = 1$$

63. Find a plane that is perpendicular to the two planes $x + y = 3$ and $x + 2y - z = 4$.

SOLUTION The vector forms of the equations of the planes are $\langle 1, 1, 0\rangle \cdot \langle x, y, z\rangle = 3$ and $\langle 1, 2, -1\rangle \cdot \langle x, y, z\rangle = 4$, hence the vectors $\mathbf{n}_1 = \langle 1, 1, 0\rangle$ and $\mathbf{n}_2 = \langle 1, 2, -1\rangle$ are normal to the planes. We denote the equation of the planes which are perpendicular to the two planes by

$$ax + by + cz = d \tag{1}$$

Then, the normal $\mathbf{n} = \langle a, b, c\rangle$ to the planes is orthogonal to the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ of the given planes. Therefore, $\mathbf{n} \cdot \mathbf{n}_1 = 0$ and $\mathbf{n} \cdot \mathbf{n}_2 = 0$ which gives us

$$\langle a, b, c\rangle \cdot \langle 1, 1, 0\rangle = 0, \quad \langle a, b, c\rangle \cdot \langle 1, 2, -1\rangle = 0$$

We obtain the following equations:

$$\begin{cases} a + b = 0 \\ a + 2b - c = 0 \end{cases}$$

The first equation implies that $b = -a$. Substituting in the second equation we get $a - 2a - c = 0$, or $c = -a$. Substituting $b = -a$ and $c = -a$ in (1) gives (for $a \neq 0$):

$$ax - ay - az = d \quad \Rightarrow \quad x - y - z = \frac{d}{a}$$

$\frac{d}{a}$ is an arbitrary constant which we denote by f. The planes which are perpendicular to the given planes are, therefore,

$$x - y - z = f$$

65. Let $\mathcal{L}$ denote the intersection of the planes $x - y - z = 1$ and $2x + 3y + z = 2$. Find parametric equations for the line $\mathcal{L}$. *Hint:* To find a point on $\mathcal{L}$, substitute an arbitrary value for z (say, $z = 2$) and then solve the resulting pair of equations for x and y.

SOLUTION We use Exercise 62 to find a direction vector for the line of intersection $\mathcal{L}$ of the planes $x - y - z = 1$ and $2x + 3y + z = 2$. We identify the normals $\mathbf{n}_1 = \langle 1, -1, -1\rangle$ and $\mathbf{n}_2 = \langle 2, 3, 1\rangle$ to the two planes respectively. Hence, a direction vector for $\mathcal{L}$ is the cross product $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$. We find it here:

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 3 & 1 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} = \langle 2, -3, 5\rangle$$

We now need to find a point on $\mathcal{L}$. We choose $z = 2$, substitute in the equations of the planes and solve the resulting equations for x and y. This gives

$$\begin{matrix} x - y - 2 = 1 \\ 2x + 3y + 2 = 2 \end{matrix} \quad \text{or} \quad \begin{matrix} x - y = 3 \\ 2x + 3y = 0 \end{matrix}$$

The 1st equation implies that $y = x - 3$. Substituting in the 2nd equation and solving for x gives

$$2x + 3(x - 3) = 0$$
$$5x = 9 \quad \Rightarrow \quad x = \frac{9}{5}, \quad y = \frac{9}{5} - 3 = -\frac{6}{5}$$

We conclude that the point $\left(\frac{9}{5}, -\frac{6}{5}, 2\right)$ is on $\mathcal{L}$. We now use the vector parametrization of a line to obtain the following parametrization for $\mathcal{L}$:

$$\mathbf{r}(t) = \left\langle \frac{9}{5}, -\frac{6}{5}, 2 \right\rangle + t\langle 2, -3, 5\rangle$$

This yields the parametric equations

$$x = \frac{9}{5} + 2t, \quad y = -\frac{6}{5} - 3t, \quad z = 2 + 5t$$

67. Two vectors $\mathbf{v}$ and $\mathbf{w}$, each of length 12, lie in the plane $x + 2y - 2z = 0$. The angle between $\mathbf{v}$ and $\mathbf{w}$ is $\pi/6$. This information determines $\mathbf{v} \times \mathbf{w}$ up to a sign ± 1. What are the two possible values of $\mathbf{v} \times \mathbf{w}$?

SOLUTION The length of $\mathbf{v}\times\mathbf{w}$ is $\|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$, but since both vectors have length 12 and since the angle between them is $\pi/6$, then the length of $\mathbf{v}\times\mathbf{w}$ is $12\cdot 12\cdot 1/2 = 72$. The direction of $\mathbf{v}\times\mathbf{w}$ is perpendicular to the plane containing them, which is the plane $x+2y-2z=0$, which has normal vector $\mathbf{n}=\langle 1,2,-2\rangle$. Since $\mathbf{v}\times\mathbf{w}$ must have length 72 and must be parallel to $\langle 1,2,-2\rangle$, then it must be ± 72 times the unit vector $\langle 1,2,-2\rangle/\sqrt{1^2+2^2+(-2)^2} = \langle 1/3, 2/3, -2/3\rangle$. Thus,

$$\mathbf{v}\times\mathbf{w} = \pm 72\cdot\langle 1/3, 2/3, -2/3\rangle = \pm 24\cdot\langle 1,2,-2\rangle$$

69. In this exercise, we show that the orthogonal distance D from the plane $\mathcal{P}$ with equation $ax+by+cz=d$ to the origin O is equal to (Figure 10)

$$D = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

Let $\mathbf{n}=\langle a,b,c\rangle$, and let P be the point where the line through $\mathbf{n}$ intersects $\mathcal{P}$. By definition, the orthogonal distance from $\mathcal{P}$ to O is the distance from P to O.

(a) Show that P is the terminal point of $\mathbf{v} = \left(\frac{d}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}$.

(b) Show that the distance from P to O is D.

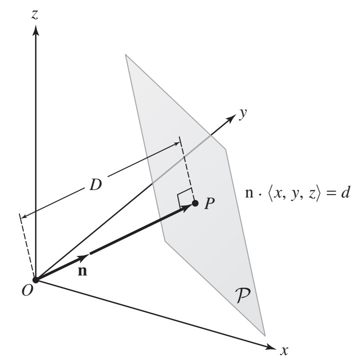


FIGURE 10

SOLUTION Let $\mathbf{v}$ be the vector $\mathbf{v} = \left(\frac{d}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}$. Then $\mathbf{v}$ is parallel to $\mathbf{n}$ and the two vectors are on the same ray.

(a) First we must show that the terminal point of $\mathbf{v}$ lies on the plane $ax+by+cz=d$. Since the terminal point of $\mathbf{v}$ is the point

$$\left(\frac{d}{\mathbf{n}\cdot\mathbf{n}}\right)(a,b,c) = \left(\frac{da}{a^2+b^2+c^2}, \frac{db}{a^2+b^2+c^2}, \frac{dc}{a^2+b^2+c^2}\right)$$

then we need only show that this point satisfies $ax+by+cz=d$. Plugging in, we find:

$$ax+by+cz = a\cdot\frac{da}{a^2+b^2+c^2} + b\cdot\frac{db}{a^2+b^2+c^2} + c\cdot\frac{dc}{a^2+b^2+c^2} = \frac{a^2d+b^2d+c^2d}{a^2+b^2+c^2} = d$$

(b) We now show that the distance from P to O is D. This distance is just the length of the vector $\mathbf{v}$, which is:

$$\|\mathbf{v}\| = \left(\frac{|d|}{\mathbf{n}\cdot\mathbf{n}}\right)\|\mathbf{n}\| = \frac{|d|}{\|\mathbf{n}\|} = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

as desired.

Further Insights and Challenges

In Exercises 71 and 72, let $\mathcal{P}$ be a plane with equation

$$ax+by+cz=d$$

and normal vector $\mathbf{n}=\langle a,b,c\rangle$. For any point Q, there is a unique point P on $\mathcal{P}$ that is closest to Q, and is such that $\overline{PQ}$ is orthogonal to $\mathcal{P}$ (Figure 11).

71. Show that the point P on $\mathcal{P}$ closest to Q is determined by the equation

$$\overrightarrow{OP} = \overrightarrow{OQ} + \left(\frac{d - \overrightarrow{OQ} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n} \qquad \boxed{7}$$

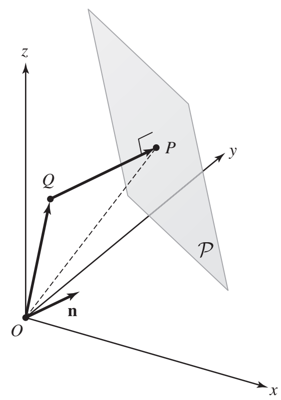
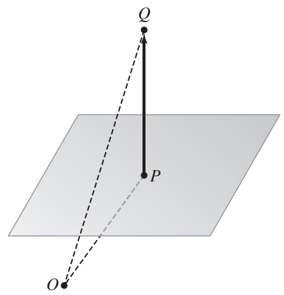


FIGURE 11

SOLUTION Since $\overrightarrow{PQ}$ is orthogonal to the plane $\mathcal{P}$, it is parallel to the vector $\mathbf{n} = \langle a, b, c\rangle$ which is normal to the plane. Hence,

$$\overrightarrow{PQ} = \lambda\mathbf{n} \tag{1}$$



Since $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$, we have $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$. Thus, by (1) we get

$$\overrightarrow{OQ} - \overrightarrow{OP} = \lambda\mathbf{n} \quad\Rightarrow\quad \overrightarrow{OP} = \overrightarrow{OQ} - \lambda\mathbf{n} \tag{2}$$

The point P is on the plane, hence $\overrightarrow{OP}$ satisfies the vector form of the equation of the plane, that is,

$$\mathbf{n} \cdot \overrightarrow{OP} = d \tag{3}$$

Substituting (2) into (3) and solving for λ yields

$$\mathbf{n} \cdot \left(\overrightarrow{OQ} - \lambda\mathbf{n}\right) = d$$

$$\mathbf{n} \cdot \overrightarrow{OQ} - \lambda\mathbf{n} \cdot \mathbf{n} = d$$

$$\lambda\mathbf{n} \cdot \mathbf{n} = \mathbf{n} \cdot \overrightarrow{OQ} - d \quad\Rightarrow\quad \lambda = \frac{\mathbf{n} \cdot \overrightarrow{OQ} - d}{\mathbf{n} \cdot \mathbf{n}} \tag{4}$$

Finally, we combine (2) and (4) to obtain

$$\overrightarrow{OP} = \overrightarrow{OQ} + \left(\frac{d - \mathbf{n} \cdot \overrightarrow{OQ}}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n}$$

73. Use Eq. (7) to find the point P closest to $Q = (2, 1, 2)$ on the plane $x + y + z = 1$.

SOLUTION We identify $\mathbf{n} = \langle 1, 1, 1\rangle$ as a vector normal to the plane. By Eq. (7) the closest point P to Q is determined by

$$\overrightarrow{OP} = \overrightarrow{OQ} + \left(\frac{d - \overrightarrow{OQ} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n}$$

We substitute $\mathbf{n} = \langle 1, 1, 1\rangle$, $\overrightarrow{OQ} = \langle 2, 1, 2\rangle$ and $d = 1$ in this equation to obtain

$$\overrightarrow{OP} = \langle 2, 1, 2\rangle + \frac{1 - \langle 2, 1, 2\rangle \cdot \langle 1, 1, 1\rangle}{\langle 1, 1, 1\rangle \cdot \langle 1, 1, 1\rangle}\langle 1, 1, 1\rangle = \langle 2, 1, 2\rangle + \frac{1-(2+1+2)}{1+1+1}\langle 1, 1, 1\rangle$$
$$= \langle 2, 1, 2\rangle - \frac{4}{3}\langle 1, 1, 1\rangle = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle$$

The terminal point $P = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ of $\overrightarrow{OP}$ is the closest point to $Q = (2, 1, 2)$ on the plane.

75. Use Eq. (8) to find the distance from $Q = (1, 1, 1)$ to the plane $2x + y + 5z = 2$.

SOLUTION By Eq. (8), the distance from $Q = \langle x_1, y_1, z_1\rangle$ to the plane $ax + by + cz = d$ is

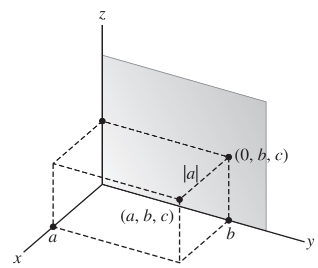
$$\ell = \frac{|ax_1 + by_1 + cz_1 - d|}{\|\mathbf{n}\|} \tag{1}$$

We identify the vector $\mathbf{n} = \langle 2, 1, 5\rangle$ as a normal to the plane $2x + y + 5z = 2$. Also $a = 2$, $b = 1$, $c = 5$, $d = 2$, and $(x_1, y_1, z_1) = (1, 1, 1)$. Substituting in (1) above we get

$$\ell = \frac{|2\cdot 1 + 1\cdot 1 + 5\cdot 1 - 2|}{\|\langle 2, 1, 5\rangle\|} = \frac{6}{\sqrt{2^2+1^2+5^2}} = \frac{6}{\sqrt{30}} \approx 1.095$$

77. What is the distance from $Q = (a, b, c)$ to the plane $x = 0$? Visualize your answer geometrically and explain without computation. Then verify that Eq. (8) yields the same answer.

SOLUTION The plane $x = 0$ is the yz-coordinate plane. The closest point to Q on the plane is the projection of Q on the plane, which is the point $Q' = (0, b, c)$. Therefore the distance from Q to the plane is the length of the vector $\overrightarrow{Q'Q} = \langle a, 0, 0\rangle$ which is $|a|$.



We now verify that Eq. (8) gives the same answer. The plane $x = 0$ has the vector parametrization $\langle 1, 0, 0\rangle \cdot \langle x, y, z\rangle = 0$, hence $\mathbf{n} = \langle 1, 0, 0\rangle$. The coefficients of the plane $x = 0$ are $A = 1$, $B = C = D = 0$. Also $(x_1, y_1, z_1) = (a, b, c)$. Substituting this value in Eq. (8) we get

$$\frac{|Ax_1 + By_1 + Cz_1 - D|}{\|\mathbf{n}\|} = \frac{|1\cdot a + 0 + 0 - 0|}{\|\langle 1, 0, 0\rangle\|} = \frac{|a|}{\sqrt{1^2+0^2+0^2}} = |a|$$

The two answers agree, as expected.

12.6 A Survey of Quadric Surfaces

Preliminary Questions

1. True or false? All traces of an ellipsoid are ellipses.

SOLUTION This statement is true, mostly. All traces of an ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ are ellipses, except for the traces obtained by intersecting the ellipsoid with the planes $x = \pm a$, $y = \pm b$ and $z = \pm c$. These traces reduce to the single points $(\pm a, 0, 0)$, $(0, \pm b, 0)$ and $(0, 0, \pm c)$ respectively.

2. True or false? All traces of a hyperboloid are hyperbolas.

SOLUTION The statement is false. For a hyperbola in the standard orientation, the horizontal traces are ellipses (or perhaps empty for a hyperbola of two sheets), and the vertical traces are hyperbolas.

3. Which quadric surfaces have both hyperbolas and parabolas as traces?

SOLUTION The hyperbolic paraboloid $z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$ has vertical trace curves which are parabolas. If we set $x = x_0$ or $y = y_0$ we get

$$z = \left(\frac{x_0}{a}\right)^2 - \left(\frac{y}{b}\right)^2 \quad \Rightarrow \quad z = -\left(\frac{y}{b}\right)^2 + C$$
$$z = \left(\frac{x}{a}\right)^2 - \left(\frac{y_0}{b}\right)^2 \quad \Rightarrow \quad z = \left(\frac{x}{a}\right)^2 + C$$

The hyperbolic paraboloid has vertical traces which are hyperbolas, since for $z = z_0$, $(z_0 > 0)$, we get

$$z_0 = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

4. Is there any quadric surface whose traces are all parabolas?

SOLUTION There is no quadric surface whose traces are all parabolas.

5. A surface is called **bounded** if there exists $M > 0$ such that every point on the surface lies at a distance of at most M from the origin. Which of the quadric surfaces are bounded?

SOLUTION The only quadric surface that is bounded is the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

All other quadric surfaces are not bounded, since at least one of the coordinates can increase or decrease without bound.

6. What is the definition of a parabolic cylinder?

SOLUTION A parabolic cylinder consists of all vertical lines passing through a parabola $\mathcal{C}$ in the xy-plane.

Exercises

In Exercises 1–6, state whether the given equation defines an ellipsoid or hyperboloid, and if a hyperboloid, whether it is of one or two sheets.

1. $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{5}\right)^2 = 1$

SOLUTION This equation is the equation of an ellipsoid.

3. $x^2 + 3y^2 + 9z^2 = 1$

SOLUTION We rewrite the equation as follows:

$$x^2 + \left(\frac{y}{\frac{1}{\sqrt{3}}}\right)^2 + \left(\frac{z}{\frac{1}{3}}\right)^2 = 1$$

This equation defines an ellipsoid.

5. $x^2 - 3y^2 + 9z^2 = 1$

SOLUTION We rewrite the equation in the form

$$x^2 - \left(\frac{y}{\frac{1}{\sqrt{3}}}\right)^2 + \left(\frac{z}{\frac{1}{3}}\right)^2 = 1$$

This is the equation of a hyperboloid of one sheet.

In Exercises 7–12, state whether the given equation defines an elliptic paraboloid, a hyperbolic paraboloid, or an elliptic cone.

7. $z = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2$

SOLUTION This equation defines an elliptic paraboloid.

9. $z = \left(\frac{x}{9}\right)^2 - \left(\frac{y}{12}\right)^2$

SOLUTION This equation defines a hyperbolic paraboloid.

11. $3x^2 - 7y^2 = z$

SOLUTION Rewriting the equation as

$$z = \left(\frac{x}{\frac{1}{\sqrt{3}}}\right)^2 - \left(\frac{y}{\frac{1}{\sqrt{7}}}\right)^2$$

we identify it as the equation of a hyperbolic paraboloid.

In Exercises 13–20, state the type of the quadric surface and describe the trace obtained by intersecting with the given plane.

13. $x^2 + \left(\frac{y}{4}\right)^2 + z^2 = 1, \quad y = 0$

SOLUTION The equation $x^2 + \left(\frac{y}{4}\right)^2 + z^2 = 1$ defines an ellipsoid. The xz-trace is obtained by substituting $y = 0$ in the equation of the ellipsoid. This gives the equation $x^2 + z^2 = 1$ which defines a circle in the xz-plane.

15. $x^2 + \left(\frac{y}{4}\right)^2 + z^2 = 1, \quad z = \frac{1}{4}$

SOLUTION The quadric surface is an ellipsoid, since its equation has the form $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ for $a = 1$, $b = 4$, $c = 1$. To find the trace obtained by intersecting the ellipsoid with the plane $z = \frac{1}{4}$, we set $z = \frac{1}{4}$ in the equation of the ellipsoid. This gives

$$lx^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = 1$$

$$x^2 + \frac{y^2}{16} = \frac{15}{16}$$

To get the standard form we divide by $\frac{15}{16}$ to obtain

$$\frac{x^2}{\frac{15}{16}} + \frac{y^2}{\frac{16 \cdot 15}{16}} = 1 \quad \Rightarrow \quad \left(\frac{x}{\frac{\sqrt{15}}{4}}\right)^2 + \left(\frac{y}{\sqrt{15}}\right)^2 = 1 \tag{1}$$

We conclude that the trace is an ellipse on the xy-plane, whose equation is given in (1).

17. $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 - 5z^2 = 1, \quad y = 1$

SOLUTION Rewriting the equation in the form

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 - \left(\frac{z}{\frac{1}{\sqrt{5}}}\right)^2 = 1$$

we identify it as the equation of a hyperboloid of one sheet. Substituting $y = 1$ we get

$$\frac{x^2}{9} + \frac{1}{25} - 5z^2 = 1$$

$$\frac{x^2}{9} - 5z^2 = \frac{24}{25}$$

$$\frac{25}{24 \cdot 9}x^2 - \frac{25 \cdot 5}{24}z^2 = 1$$

$$\left(\frac{x}{\frac{6\sqrt{6}}{5}}\right)^2 - \left(\frac{z}{\frac{2}{5}\sqrt{\frac{6}{5}}}\right)^2 = 1$$

Thus, the trace on the plane $y = 1$ is a hyperbola.

19. $y = 3x^2, \quad z = 27$

SOLUTION This equation defines a parabolic cylinder, consisting of all vertical lines passing through the parabola $y = 3x^2$ in the xy-plane. Hence, the trace of the cylinder on the plane $z = 27$ is the parabola $y = 3x^2$ on this plane, that is, the following set:

$$\left\{(x, y, z) : y = 3x^2,\ z = 27\right\}.$$

21. Match each of the ellipsoids in Figure 13 with the correct equation:

(a) $x^2 + 4y^2 + 4z^2 = 16$ **(b)** $4x^2 + y^2 + 4z^2 = 16$

(c) $4x^2 + 4y^2 + z^2 = 16$

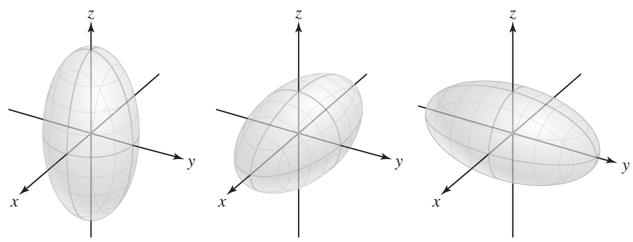


FIGURE 13

SOLUTION

(a) We rewrite the equation in the form

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$$

The ellipsoid intersects the x, y, and z axes at the points $(\pm 4, 0, 0)$, $(0, \pm 2, 0)$, and $(0, 0, \pm 2)$, hence (B) is the corresponding figure.

(b) We rewrite the equation in the form

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$$

The x, y, and z intercepts are $(\pm 2, 0, 0)$, $(0, \pm 4, 0)$, and $(0, 0, \pm 2)$ respectively, hence (C) is the correct figure.

(c) We write the equation in the form

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{4}\right)^2 = 1$$

The x, y, and z intercepts are $(\pm 2, 0, 0)$, $(0, \pm 2, 0)$, and $(0, 0, \pm 4)$ respectively, hence the corresponding figure is (A).

23. What is the equation of the surface obtained when the elliptic paraboloid $z = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2$ is rotated about the x-axis by 90°? Refer to Figure 14.

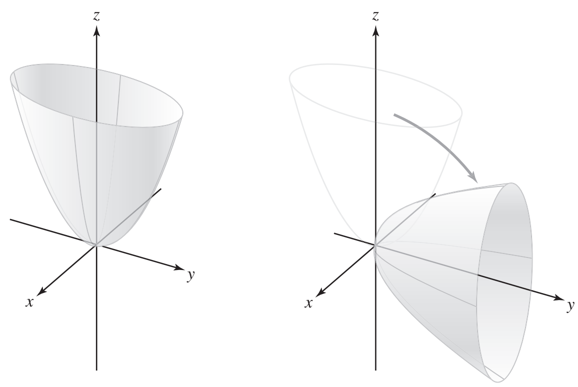


FIGURE 14

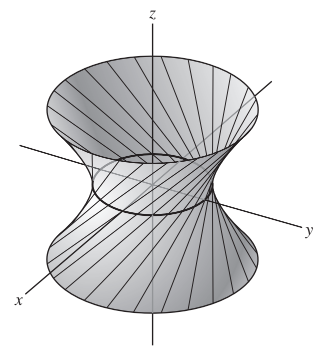
SOLUTION The axis of symmetry of the resulting surface is the y-axis rather than the z-axis. Interchanging y and z in the given equation gives the following equation of the rotated paraboloid:

$$y = \left(\frac{x}{2}\right)^2 + \left(\frac{z}{4}\right)^2$$

In Exercises 25–38, sketch the given surface.

25. $x^2 + y^2 - z^2 = 1$

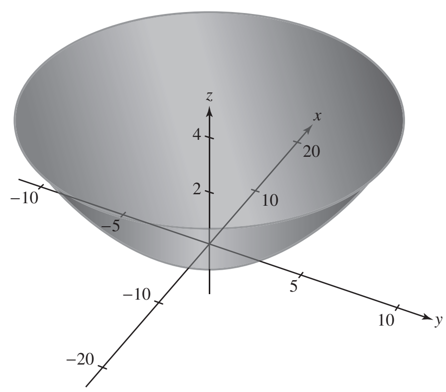
SOLUTION This equation defines a hyperboloid of one sheet. The trace on the plane $z = z_0$ is the circle $x^2 + y^2 = 1 + z_0^2$. The trace on the plane $y = y_0$ is the hyperbola $x^2 - z^2 = 1 - y_0^2$ and the trace on the plane $x = x_0$ is the hyperbola $y^2 - z^2 = 1 - x_0^2$. We obtain the following surface:



Graph of $x^2 + y^2 - z^2 = 1$

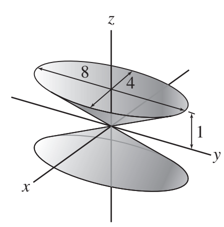
27. $z = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{8}\right)^2$

SOLUTION This equation defines an elliptic paraboloid, as shown in the following figure:



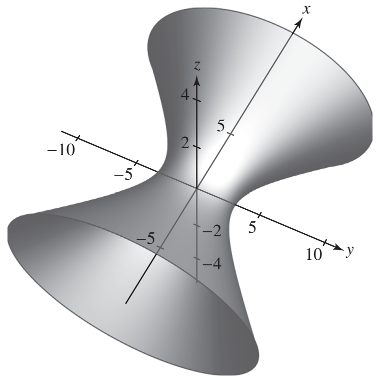
29. $z^2 = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{8}\right)^2$

SOLUTION This equation defines the following elliptic cone:



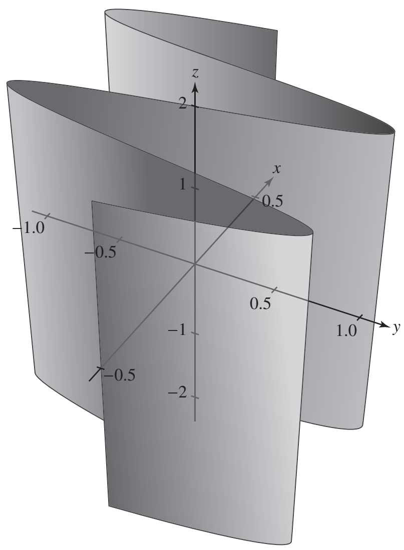
31. $x^2 - y^2 + 9z^2 = 9$

SOLUTION This is the equation of a hyperboloid of one sheet oriented along the y-axis. The graph of the surface is shown below:



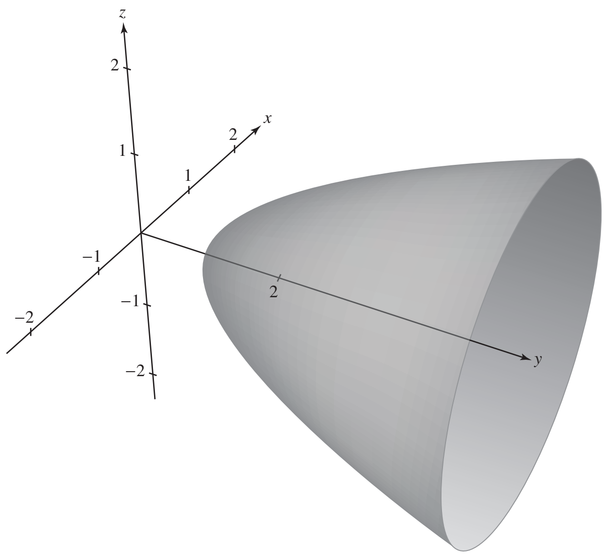
33. $x = \sin y$

SOLUTION This is the equation of a cylindrical surface oriented along the z-axis whose cross-section is the curve $x = \sin y$. The graph of the surface is shown below:



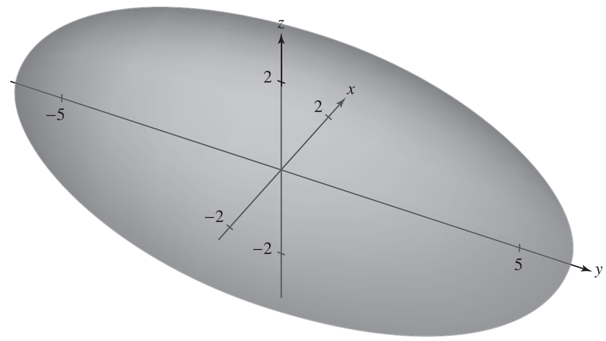
35. $x = 1 + y^2 + z^2$

SOLUTION This is the equation of an elliptic paraboloid oriented along the x-axis. The graph of the surface is shown below:



37. $x^2 + 9y^2 + 4z^2 = 36$

SOLUTION This is the equation of an ellipsoid. The graph of the surface is shown below:



39. Find the equation of the ellipsoid passing through the points marked in Figure 15(A).

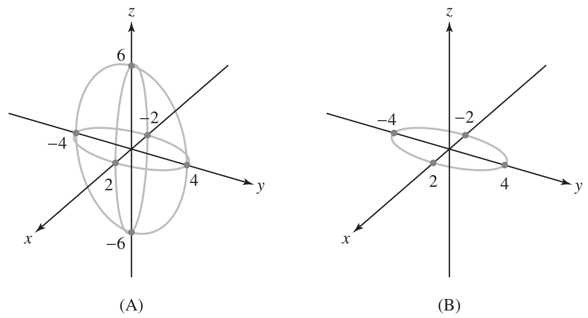


FIGURE 15

SOLUTION The equation of an ellipsoid is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \tag{1}$$

The x, y and z intercepts are $(\pm a, 0, 0)$, $(0, \pm b, 0)$ and $(0, 0, \pm c)$ respectively. The x, y and z intercepts of the desired ellipsoid are $(\pm 2, 0, 0)$, $(0, \pm 4, 0)$ and $(0, 0, \pm 6)$ respectively, hence $a = 2$, $b = 4$ and $c = 6$. Substituting into (1) we get

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{6}\right)^2 = 1.$$

41. Find the equation of the hyperboloid shown in Figure 16(A).

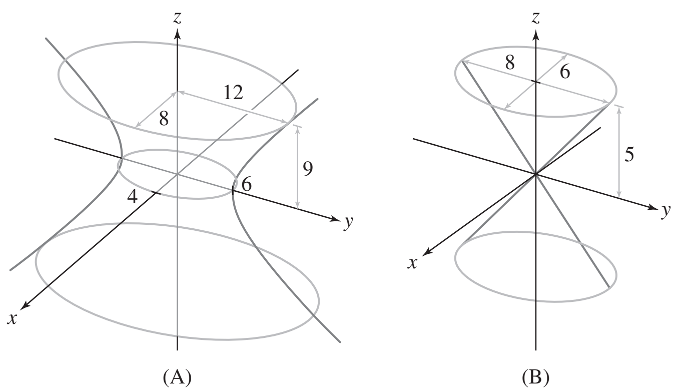


FIGURE 16

SOLUTION The hyperboloid in the figure is of one sheet and the intersections with the planes $z = z_0$ are ellipses. Hence, the equation of the hyperboloid has the form

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1 \tag{1}$$

Substituting $z = 0$ we get

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

By the given information this ellipse has x and y intercepts at the points $(\pm 4, 0)$ and $(0, \pm 6)$ hence $a = 4$, $b = 6$. Substituting in (1) we get

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{6}\right)^2 - \left(\frac{z}{c}\right)^2 = 1 \tag{2}$$

Substituting $z = 9$ we get

$$\frac{x^2}{16} + \frac{y^2}{36} - \frac{9^2}{c^2} = 1$$

$$\frac{x^2}{16} + \frac{y^2}{36} = 1 + \frac{81}{c^2} = \frac{c^2 + 81}{c^2}$$

$$\frac{c^2x^2}{16(81 + c^2)} + \frac{c^2y^2}{36(81 + c^2)} = 1$$

$$\left(\frac{x}{\frac{4}{c}\sqrt{81 + c^2}}\right)^2 + \left(\frac{y}{\frac{6}{c}\sqrt{81 + c^2}}\right)^2 = 1$$

By the given information the following must hold:

$$\begin{array}{l} \frac{4}{c}\sqrt{81 + c^2} = 8 \\ \frac{6}{c}\sqrt{81 + c^2} = 12 \end{array} \quad \Rightarrow \quad \frac{\sqrt{81 + c^2}}{c} = 2 \quad \Rightarrow \quad 81 + c^2 = 4c^2 \quad \Rightarrow \quad 3c^2 = 81$$

Thus, $c = 3\sqrt{3}$, and by substituting in (2) we obtain the following equation:

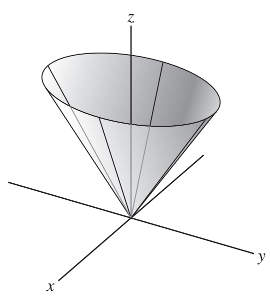
$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{6}\right)^2 - \left(\frac{z}{3\sqrt{3}}\right)^2 = 1$$

43. Determine the vertical traces of elliptic and parabolic cylinders in standard form.

SOLUTION The vertical traces of elliptic or parabolic cylinders are one or two vertical lines, or an empty set.

45. Let $\mathcal{C}$ be an ellipse in a horizonal plane lying above the xy-plane. Which type of quadric surface is made up of all lines passing through the origin and a point on $\mathcal{C}$?

SOLUTION The quadric surface is the upper part of an elliptic cone.



Further Insights and Challenges

47. Let $\mathcal{S}$ be the hyperboloid $x^2 + y^2 = z^2 + 1$ and let $P = (\alpha, \beta, 0)$ be a point on $\mathcal{S}$ in the (x, y)-plane. Show that there are precisely two lines through P entirely contained in $\mathcal{S}$ (Figure 17). *Hint:* Consider the line $\mathbf{r}(t) = \langle \alpha + at, \beta + bt, t\rangle$ through P. Show that $\mathbf{r}(t)$ is contained in $\mathcal{S}$ if (a, b) is one of the two points on the unit circle obtained by rotating (α, β) through $\pm\frac{\pi}{2}$. This proves that a hyperboloid of one sheet is a **doubly ruled surface**, which means that it can be swept out by moving a line in space in two different ways.

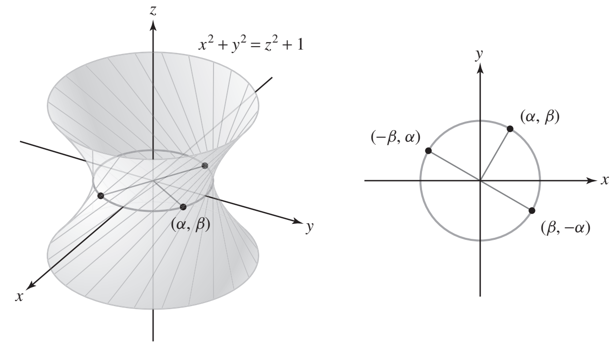


FIGURE 17

SOLUTION The parametric equations of the lines through $P = (\alpha, \beta, 0)$ have the form

$$x = \alpha + ks, \quad y = \beta + \ell s, \quad z = ms$$

Setting the parameter $t = ms$ and replacing $\frac{k}{m}$ and $\frac{\ell}{m}$ by a and b, respectively, we obtain the following (normalized) form

$$x = \alpha + at, \quad y = \beta + bt, \quad z = t$$

The line is entirely contained in S if and only if for all values of the parameter t, the following equality holds:

$$(\alpha + at)^2 + (\beta + bt)^2 = t^2 + 1$$

That is, for all t,

$$\alpha^2 + 2\alpha at + a^2t^2 + \beta^2 + 2\beta bt + b^2t^2 = t^2 + 1$$
$$(a^2 + b^2 - 1)t^2 + 2(\alpha a + \beta b)t + (\alpha^2 + \beta^2 - 1) = 0$$

This equality holds for all t if and only if all the coefficients are zero. That is, if and only if

$$\begin{cases} a^2 + b^2 - 1 = 0 \\ \alpha a + \beta b = 0 \\ \alpha^2 + \beta^2 - 1 = 0 \end{cases}$$

The first and the third equations imply that (a, b) and (α, β) are points on the unit circle $x^2 + y^2 = 1$. The second equation implies that the vector $\mathbf{u} = \langle a, b \rangle$ is orthogonal to the vector $\mathbf{v} = \langle \alpha, \beta \rangle$ (since $\mathbf{u} \cdot \mathbf{v} = a\alpha + b\beta = 0$).

Conclusions: There are precisely two lines through P entirely contained in S. For the direction vectors $(a, b, 1)$ of these lines, (a, b) is obtained by rotating (α, β) through $\pm\frac{\pi}{2}$ about the origin.

In Exercises 48 and 49, let $\mathcal{C}$ be a curve in $\mathbf{R}^3$ not passing through the origin. The cone on $\mathcal{C}$ is the surface consisting of all lines passing through the origin and a point on $\mathcal{C}$ [Figure 18(A)].

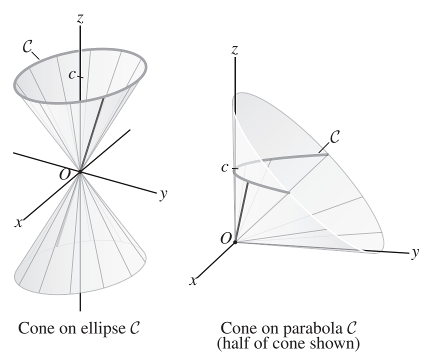


FIGURE 18

49. Let a and c be nonzero constants and let $\mathcal{C}$ be the parabola at height c consisting of all points (x, ax^2, c) [Figure 18(B)]. Let $\mathcal{S}$ be the cone consisting of all lines passing through the origin and a point on $\mathcal{C}$. This exercise shows that $\mathcal{S}$ is also an elliptic cone.

(a) Show that $\mathcal{S}$ has equation $yz = acx^2$.

(b) Show that under the change of variables $y = u + v$ and $z = u - v$, this equation becomes $acx^2 = u^2 - v^2$ or $u^2 = acx^2 + v^2$ (the equation of an elliptic cone in the variables x, v, u).

SOLUTION

(a) A point P on the parabola $\mathcal{C}$ has the form $P = \left(x_0, ax_0^2, c\right)$, hence the parametric equations of the line through the origin and P are

$$x = tx_0, \quad y = tax_0^2, \quad z = tc.$$

Then

$$yz = tax_0^2 ct = ac(tx_0)^2 = acx^2.$$

(b) Define new variables $z = u - v$ and $y = u + v$. The equation in part **(a)** becomes

$$(u+v)(u-v) = acx^2$$
$$u^2 - v^2 = acx^2 \quad \Rightarrow \quad u^2 = acx^2 + v^2$$

This is the equation of an elliptic cone in the variables x, v, u. We, thus, showed that the cone on the parabola $\mathcal{C}$ is transformed to an elliptic cone by the transformation (change of variables) $y = u + v, z = u - v, x = x$.

12.7 Cylindrical and Spherical Coordinates

Preliminary Questions

1. Describe the surfaces $r = R$ in cylindrical coordinates and $\rho = R$ in spherical coordinates.

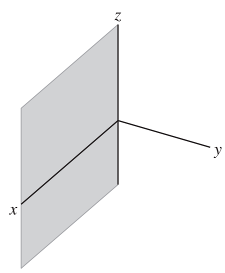
SOLUTION The surface $r = R$ consists of all points located at a distance R from the z-axis. This surface is the cylinder of radius R whose axis is the z-axis. The surface $\rho = R$ consists of all points located at a distance R from the origin. This is the sphere of radius R centered at the origin.

2. Which statement about cylindrical coordinates is correct?

(a) If $\theta = 0$, then P lies on the z-axis.

(b) If $\theta = 0$, then P lies in the xz-plane.

SOLUTION The equation $\theta = 0$ defines the half-plane of all points that project onto the ray $\theta = 0$, that is, onto the nonnegative x-axis. This half plane is part of the (x, z)-plane, therefore if $\theta = 0$, then P lies in the (x, z)-plane.



The half-plane $\theta = 0$

For instance, the point $P = (1, 0, 1)$ satisfies $\theta = 0$, but it does not lie on the z-axis. We conclude that statement (b) is correct and statement (a) is false.

3. Which statement about spherical coordinates is correct?

(a) If $\phi = 0$, then P lies on the z-axis.

(b) If $\phi = 0$, then P lies in the xy-plane.

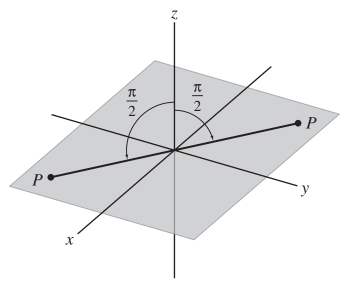
SOLUTION The equation $\phi = 0$ describes the nonnegative z-axis. Therefore, if $\phi = 0$, P lies on the z-axis as stated in (a). Statement (b) is false, since the point $(0, 0, 1)$ satisfies $\phi = 0$, but it does not lie in the (x, y)-plane.

4. The level surface $\phi = \phi_0$ in spherical coordinates, usually a cone, reduces to a half-line for two values of ϕ_0. Which two values?

SOLUTION For $\phi_0 = 0$, the level surface $\phi = 0$ is the upper part of the z-axis. For $\phi_0 = \pi$, the level surface $\phi = \pi$ is the lower part of the z-axis. These are the two values of ϕ_0 where the level surface $\phi = \phi_0$ reduces to a half-line.

5. For which value of ϕ_0 is $\phi = \phi_0$ a plane? Which plane?

SOLUTION For $\phi_0 = \frac{\pi}{2}$, the level surface $\phi = \frac{\pi}{2}$ is the xy-plane.



Exercises

In Exercises 1–4, convert from cylindrical to rectangular coordinates.

1. $(4, \pi, 4)$

SOLUTION By the given data $r = 4$, $\theta = \pi$ and $z = 4$. Hence,

$$\begin{aligned} x &= r\cos\theta = 4\cos\pi = 4\cdot(-1) = -4 \\ y &= r\sin\theta = 4\sin\pi = 4\cdot 0 \\ z &= 4 \end{aligned} \quad\Rightarrow\quad (x, y, z) = (-4, 0, 4)$$

3. $\left(0, \dfrac{\pi}{5}, \dfrac{1}{2}\right)$

SOLUTION We have $r = 0$, $\theta = \frac{\pi}{5}$, $z = \frac{1}{2}$. Thus,

$$\begin{aligned} x &= r\cos\theta = 0\cdot\cos\frac{\pi}{5} = 0 \\ y &= r\sin\theta = 0\cdot\sin\frac{\pi}{5} = 0 \\ z &= \frac{1}{2} \end{aligned} \quad\Rightarrow\quad (x, y, z) = \left(0, 0, \frac{1}{2}\right)$$

In Exercises 5–10, convert from rectangular to cylindrical coordinates.

5. $(1, -1, 1)$

SOLUTION We are given that $x = 1$, $y = -1$, $z = 1$. We find r:

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Next we find θ. The point $(x, y) = (1, -1)$ lies in the fourth quadrant, hence,

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1, \quad \frac{3\pi}{2} \le \theta \le 2\pi \quad\Rightarrow\quad \theta = \frac{7\pi}{4}$$

We conclude that the cylindrical coordinates of the point are

$$(r, \theta, z) = \left(\sqrt{2}, \frac{7\pi}{4}, 1\right).$$

7. $(1, \sqrt{3}, 7)$

SOLUTION We have $x = 1$, $y = \sqrt{3}$, $z = 7$. We first find r:

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

Since the point $(x, y) = \left(1, \sqrt{3}\right)$ lies in the first quadrant, $0 \le \theta \le \frac{\pi}{2}$. Hence,

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}, \quad 0 \le \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

The cylindrical coordinates are thus

$$(r, \theta, z) = \left(2, \frac{\pi}{3}, 7\right).$$

9. $\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}, 2\right)$

SOLUTION We have $x = \frac{5}{\sqrt{2}}$, $y = \frac{5}{\sqrt{2}}$, $z = 2$. We find r:

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{5}{\sqrt{2}}\right)^2} = \sqrt{25} = 5$$

Since the point $(x, y) = \left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$ is in the first quadrant, $0 \le \theta \le \frac{\pi}{2}$, therefore,

$$\tan\theta = \frac{y}{x} = \frac{5/\sqrt{2}}{5/\sqrt{2}} = 1, \quad 0 \le \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

The corresponding cylindrical coordinates are

$$(r, \theta, z) = \left(5, \frac{\pi}{4}, 2\right).$$

In Exercises 11–16, describe the set in cylindrical coordinates.

11. $x^2 + y^2 \le 1$

SOLUTION The inequality describes a solid cylinder of radius 1 centered on the z-axis. Since $x^2 + y^2 = r^2$, this inequality can be written as $r^2 \le 1$.

13. $y^2 + z^2 \le 4, \quad x = 0$

SOLUTION The projection of the points in this set onto the xy-plane are points on the y axis, thus $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. Therefore, $y = r\sin\frac{\pi}{2} = r \cdot 1 = r$ or $y = r\sin\left(\frac{3\pi}{2}\right) = -r$. In both cases, $y^2 = r^2$, thus the inequality $y^2 + z^2 \le 4$ becomes $r^2 + z^2 \le 4$. In cylindrical coordinates, we obtain the following inequality

$$r^2 + z^2 \le 4, \quad \theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{3\pi}{2}$$

15. $x^2 + y^2 \le 9, \quad x \ge y$

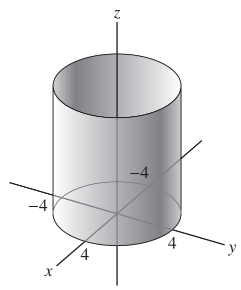
SOLUTION The equation $x^2 + y^2 \le 9$ in cylindrical coordinates becomes $r^2 \le 9$, which becomes $r \le 3$. However, we also have the restriction that $x \ge y$. This means that the projection of our set onto the xy plane is below and to the right of the line $y = x$. In other words, our θ is restricted to $-3\pi/4 \le \theta \le \pi/4$. In conclusion, the answer is:

$$r \le 3, \qquad -3\pi/4 \le \theta \le \pi/4$$

In Exercises 17–26, sketch the set (described in cylindrical coordinates).

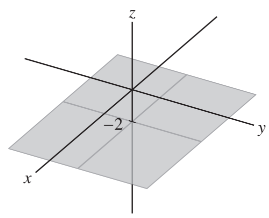
17. $r = 4$

SOLUTION The surface $r = 4$ consists of all points located at a distance 4 from the z-axis. It is a cylinder of radius 4 whose axis is the z-axis. The cylinder is shown in the following figure:



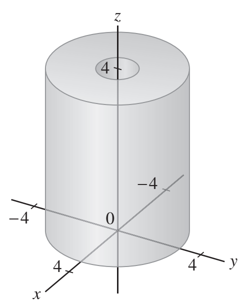
19. $z = -2$

SOLUTION $z = -2$ is the horizontal plane at height -2, shown in the following figure:



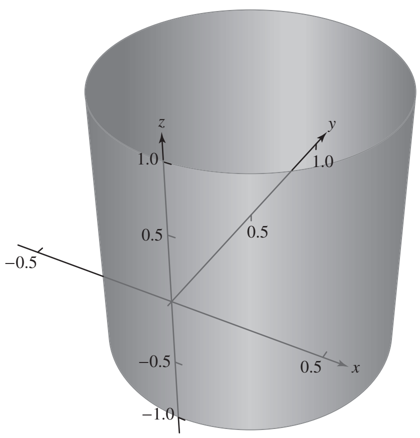
21. $1 \leq r \leq 3, \quad 0 \leq z \leq 4$

SOLUTION The region $1 \leq r \leq 3$, $0 \leq z \leq 4$ is shown in the following figure:



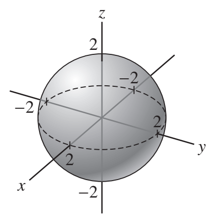
23. $r = \sin\theta$ (*Hint:* Convert to rectangular.)

SOLUTION To convert to rectangular coordinates, multiply both sides by r, giving $r^2 = r\sin\theta = y$, so that $x^2 + y^2 = y$. This simplifies to $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$, which is a cylinder oriented in the z direction whose base in the xy-plane is the circle of radius $\frac{1}{2}$ centered at $\left(0, \frac{1}{2}\right)$. A sketch of the surface is below:



25. $z^2 + r^2 \le 4$

SOLUTION The region $z^2 + r^2 \le 4$ is shown in the following figure:



In rectangular coordinates the inequality is $z^2 + \left(x^2 + y^2\right) \le 4$, or $x^2 + y^2 + z^2 \le 4$, which is a ball of radius 2.

In Exercises 27–32, find an equation of the form $r = f(\theta, z)$ in cylindrical coordinates for the following surfaces.

27. $z = x + y$

SOLUTION We substitute $x = r\cos\theta$, $y = r\sin\theta$ to obtain the following equation in cylindrical coordinates:

$$\begin{aligned} z &= r\cos\theta + r\sin\theta \\ z &= r(\cos\theta + \sin\theta) \end{aligned} \quad\Rightarrow\quad r = \frac{z}{\cos\theta + \sin\theta}.$$

29. $\dfrac{x^2}{yz} = 1$

SOLUTION We rewrite the equation in the form

$$\frac{x}{\frac{y}{x} z} = 1$$

Substituting $x = r\cos\theta$ and $\frac{y}{x} = \tan\theta$ we get

$$\begin{aligned} \frac{r\cos\theta}{(\tan\theta)\, z} &= 1 \\ r &= \frac{z\tan\theta}{\cos\theta} \end{aligned}$$

31. $x^2 + y^2 = 4$

SOLUTION Since $x^2 + y^2 = r^2$, the equation in cylindrical coordinates is, $r^2 = 4$ or $r = 2$.

In Exercises 33–38, convert from spherical to rectangular coordinates.

33. $\left(3, 0, \dfrac{\pi}{2}\right)$

SOLUTION We are given that $\rho = 3$, $\theta = 0$, $\phi = \frac{\pi}{2}$. Using the relations between spherical and rectangular coordinates we have

$$\begin{aligned} x &= \rho\sin\phi\cos\theta = 3\sin\frac{\pi}{2}\cos 0 = 3\cdot 1\cdot 1 = 3 \\ y &= \rho\sin\phi\sin\theta = 3\sin\frac{\pi}{2}\sin 0 = 3\cdot 1\cdot 0 = 0 \quad\Rightarrow\quad (x, y, z) = (3, 0, 0) \\ z &= \rho\cos\phi = 3\cos\frac{\pi}{2} = 3\cdot 0 = 0 \end{aligned}$$

35. $(3, \pi, 0)$

SOLUTION We have $\rho = 3$, $\theta = \pi$, $\phi = 0$. Hence,

$$\begin{aligned} x &= \rho\sin\phi\cos\theta = 3\sin 0\cos\pi = 0 \\ y &= \rho\sin\phi\sin\theta = 3\sin 0\sin\pi = 0 \quad\Rightarrow\quad (x, y, z) = (0, 0, 3) \\ z &= \rho\cos\phi = 3\cos 0 = 3 \end{aligned}$$

37. $\left(6, \dfrac{\pi}{6}, \dfrac{5\pi}{6}\right)$

SOLUTION Since $\rho = 6$, $\theta = \frac{\pi}{6}$, and $\phi = \frac{5\pi}{6}$ we get

$$\begin{aligned} x &= \rho \sin\phi \cos\theta = 6 \sin\frac{5\pi}{6}\cos\frac{\pi}{6} = 6\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \\ y &= \rho\sin\phi\sin\theta = 6\sin\frac{5\pi}{6}\sin\frac{\pi}{6} = 6\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{3}{2} \qquad \Rightarrow \quad (x, y, z) = \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, -3\sqrt{3}\right) \\ z &= \rho\cos\phi = 6\cos\frac{5\pi}{6} = 6\cdot\left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3} \end{aligned}$$

In Exercises 39–44, convert from rectangular to spherical coordinates.

39. $(\sqrt{3}, 0, 1)$

SOLUTION By the given data $x = \sqrt{3}$, $y = 0$, and $z = 1$. We find the radial coordinate:

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(\sqrt{3}\right)^2 + 0^2 + 1^2} = 2$$

The angular coordinate θ satisfies

$$\tan\theta = \frac{y}{x} = \frac{0}{\sqrt{3}} = 0 \quad \Rightarrow \quad \theta = 0 \quad \text{or} \quad \theta = \pi$$

Since the point $(x, y) = \left(\sqrt{3}, 0\right)$ lies in the first quadrant, the correct choice is $\theta = 0$. The angle of declination ϕ satisfies

$$\cos\phi = \frac{z}{\rho} = \frac{1}{2}, \quad 0 \le \phi \le \pi \quad \Rightarrow \quad \phi = \frac{\pi}{3}$$

The spherical coordinates of the given points are thus

$$(\rho, \theta, \phi) = \left(2, 0, \frac{\pi}{3}\right)$$

41. $(1, 1, 1)$

SOLUTION We have $x = y = z = 1$. The radial coordinate is

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

The angular coordinate θ is determined by $\tan\theta = \frac{y}{x} = \frac{1}{1} = 1$ and by the quadrant of the point $(x, y) = (1, 1)$, that is, $\theta = \frac{\pi}{4}$. The angle of declination ϕ satisfies

$$\cos\phi = \frac{z}{\rho} = \frac{1}{\sqrt{3}}, \quad 0 \le \phi \le \pi \quad \Rightarrow \quad \phi = 0.955$$

The spherical coordinates are thus

$$\left(\sqrt{3}, \frac{\pi}{4}, 0.955\right)$$

43. $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}, \sqrt{3}\right)$

SOLUTION We have $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$, and $z = \sqrt{3}$. Thus

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\sqrt{3}\right)^2} = 2$$

The angular coordinate θ satisfies $0 \le \theta \le \frac{\pi}{2}$, since the point $(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is in the first quadrant. Also $\tan\theta = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$, hence the angle is $\theta = \frac{\pi}{3}$. The angle of declination ϕ satisfies

$$\cos\phi = \frac{z}{\rho} = \frac{\sqrt{3}}{2}, \quad 0 \le \phi \le \pi \quad \Rightarrow \quad \phi = \frac{\pi}{6}$$

We conclude that

$$(\rho, \theta, \phi) = \left(2, \frac{\pi}{3}, \frac{\pi}{6}\right)$$

In Exercises 45 and 46, convert from cylindrical to spherical coordinates.

45. $(2, 0, 2)$

SOLUTION We are given that $r = 2, \theta = 0, z = 2$. Using the conversion formulas, we have

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2} = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$
$$\theta = \theta = 0$$
$$\phi = \cos^{-1}(z/\rho) = \cos^{-1}(2/(2\sqrt{2})) = \pi/4$$

In Exercises 47 and 48, convert from spherical to cylindrical coordinates.

47. $\left(4, 0, \frac{\pi}{4}\right)$

SOLUTION We are given that $\rho = 4, \theta = 0$, and $\phi = \pi/4$. To find r, we use the formulas $x = r\cos\theta$ and $x = \rho\cos\theta\sin\phi$ to get $r\cos\theta = \rho\cos\theta\sin\phi$, and so

$$r = \rho\sin\phi = 4\sin\pi/4 = 2\sqrt{2}$$

Clearly $\theta = 0$, and as for z,

$$z = \rho\cos\phi = 4\cos\pi/4 = 2\sqrt{2}$$

So, in cylindrical coordinates, our point is $(2\sqrt{2}, 0, 2\sqrt{2})$

In Exercises 49–54, describe the given set in spherical coordinates.

49. $x^2 + y^2 + z^2 \le 1$

SOLUTION Substituting $\rho^2 = x^2 + y^2 + z^2$ we obtain $\rho^2 \le 1$ or $0 \le \rho \le 1$.

51. $x^2 + y^2 + z^2 = 1, \quad x \ge 0, \quad y \ge 0, \quad z \ge 0$

SOLUTION By $\rho^2 = x^2 + y^2 + z^2$, we get $\rho^2 = 1$ or $\rho = 1$. The inequalities $x \ge 0$, $y \ge 0$ determine the first quadrant, which is also determined by $0 \le \theta \le \frac{\pi}{2}$. Finally, $z \ge 0$ gives $\cos\phi = \frac{z}{\rho} \ge 0$. Also $0 \le \phi \le \pi$, hence $0 \le \phi \le \frac{\pi}{2}$. We obtain the following description:

$$\rho = 1, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \frac{\pi}{2}$$

53. $y^2 + z^2 \le 4, \quad x = 0$

SOLUTION We substitute $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$ in the given inequality. This gives

$$4 \ge \rho^2\sin^2\theta\sin^2\phi + \rho^2\cos^2\phi \tag{1}$$

The equality $x = 0$ determines that $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ (and the origin). In both cases, $\sin^2\theta = 1$. Hence by (1) we get

$$\begin{aligned} \rho^2\sin^2\phi + \rho^2\cos^2\phi &\le 4 \\ \rho^2(1) &\le 4 \\ \rho &\le 2 \end{aligned}$$

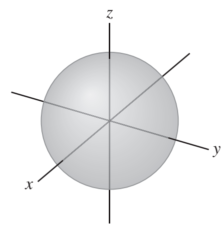
We obtain the following description:

$$\left\{(\rho, \theta, \phi) : 0 \le \rho \le 2, \theta = \frac{\pi}{2} \text{ or } \theta = \frac{3\pi}{2}\right\}$$

In Exercises 55–64, sketch the set of points (described in spherical coordinates).

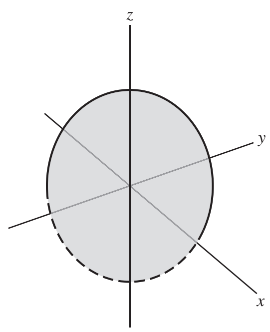
55. $\rho = 4$

SOLUTION $\rho = 4$ describes the sphere of radius 4. This is shown in the following figure:



57. $\rho = 2, \quad \theta = \dfrac{\pi}{4}$

SOLUTION The equation $\rho = 2$ is a sphere of radius 2, and the equation $\theta = \frac{\pi}{4}$ is the vertical plane $y = x$. These two surfaces intersect in a (vertical) circle of radius 2, as seen here.

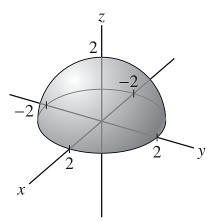


59. $\rho = 2, \quad 0 \le \phi \le \dfrac{\pi}{2}$

SOLUTION The set

$$\rho = 2, \quad 0 \le \phi \le \frac{\pi}{2}$$

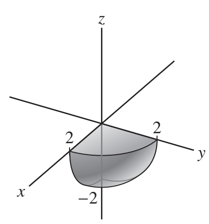
is shown in the following figure:



It is the upper half of the sphere with radius 2.

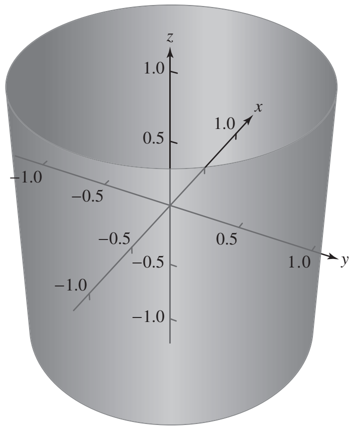
61. $\rho \le 2, \quad 0 \le \theta \le \dfrac{\pi}{2}, \quad \dfrac{\pi}{2} \le \phi \le \pi$

SOLUTION This set is the part of the ball of radius 2 which is below the first quadrant of the xy-plane, as shown in the following figure:



63. $\rho = \csc\phi$

SOLUTION Multiplying both sides by $\sin\phi$ gives $\rho\sin\phi = 1$; since $\rho^2\sin^2\phi = x^2 + y^2$, this is the equation of the surface $x^2 + y^2 = 1$, which is a cylinder oriented parallel to the z-axis having cross-section the circle of radius 1 centered at the origin. This set is shown in the following figure:



In Exercises 65–70, find an equation of the form $\rho = f(\theta, \phi)$ in spherical coordinates for the following surfaces.

65. $z = 2$

SOLUTION Since $z = \rho\cos\phi$, we have $\rho\cos\phi = 2$, or $\rho = \frac{2}{\cos\phi}$.

67. $x = z^2$

SOLUTION Substituting $x = \rho\cos\theta\sin\phi$ and $z = \rho\cos\phi$ we obtain

$$\rho\cos\theta\sin\phi = \rho^2\cos^2\phi$$
$$\cos\theta\sin\phi = \rho\cos^2\phi$$
$$\rho = \frac{\cos\theta\sin\phi}{\cos^2\phi} = \frac{\cos\theta\tan\phi}{\cos\phi}$$

69. $x^2 - y^2 = 4$

SOLUTION We substitute $x = \rho\cos\theta\sin\phi$ and $y = \rho\sin\theta\sin\phi$ to obtain

$$4 = \rho^2\cos^2\theta\sin^2\phi - \rho^2\sin^2\theta\sin^2\phi = \rho^2\sin^2\phi\left(\cos^2\theta - \sin^2\theta\right)$$

Using the identity $\cos^2\theta - \sin^2\theta = \cos 2\theta$ we get

$$4 = \rho^2\sin^2\phi\cos 2\theta$$
$$\rho^2 = \frac{4}{\sin^2\phi\cos 2\theta}$$

We take the square root of both sides. Since $0 < \phi < \pi$ we have $\sin\phi > 0$, hence,

$$\rho = \frac{2}{\sin\phi\sqrt{\cos 2\theta}}$$

71. Which of (a)–(c) is the equation of the cylinder of radius R in spherical coordinates? Refer to Figure 17.

(a) $R\rho = \sin\phi$ **(b)** $\rho\sin\phi = R$ **(c)** $\rho = R\sin\phi$

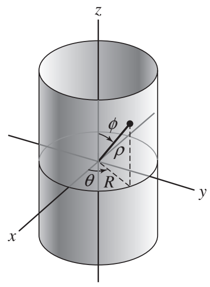


FIGURE 17

SOLUTION The equation of the cylinder of radius R in rectangular coordinates is $x^2 + y^2 = R^2$ (z is unlimited). Substituting the formulas for x and y in terms of ρ, θ and ϕ yields

$$R^2 = \rho^2 \cos^2\theta \sin^2\phi + \rho^2 \sin^2\theta \sin^2\phi = \rho^2 \sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) = \rho^2 \sin^2\phi$$

Hence,

$$R^2 = \rho^2 \sin^2\phi$$

We take the square root of both sides. Since $0 \le \phi \le \pi$, we have $\sin\phi \ge 0$, therefore,

$$R = \rho \sin\phi$$

Equation (b) is the correct answer.

73. Find the spherical angles (θ, ϕ) for Helsinki, Finland (60.1° N, 25.0° E), and São Paulo, Brazil (23.52° S, 46.52° W).

SOLUTION For Helsinki, θ is 25° and ϕ is $90 - 60.1 = 29.9°$.

For São Paulo, θ is $360 - 46.52 = 313.48°$ and ϕ is $90 + 23.52 = 113.52°$.

75. Consider a rectangular coordinate system with its origin at the center of the earth, z-axis through the North Pole, and x-axis through the prime meridian. Find the rectangular coordinates of Sydney, Australia (34° S, 151° E), and Bogotá, Colombia (4° 32′ N, 74° 15′ W). A minute is $1/60°$. Assume that the earth is a sphere of radius $R = 6370$ km.

SOLUTION We first find the angle (θ, ϕ) for the two towns. For Sydney $\theta = 151°$, since its longitude lies to the east of Greenwich, that is, in the positive θ direction. Sydney's latitude is south of the equator, hence $\phi = 90 + 34 = 124°$.

For Bogotá, we have $\theta = 360° - 74°15' = 285°45'$, since $74°15'W$ refers to $74°15'$ in the negative θ direction. The latitude is north of the equator hence $\phi = 90° - 4°32' = 85°28'$.

We now use the formulas of x,y and z in terms of ρ, θ, ϕ to find the rectangular coordinates of the two towns. (Notice that $285°45' = 285.75°$ and $85°28' = 85.47°$).

Sydney:

$$x = \rho \cos\theta \sin\phi = 6370 \cos 151° \sin 124° = -4618.8$$
$$y = \rho \sin\theta \sin\phi = 6370 \sin 151° \sin 124° = 2560$$
$$z = \rho \cos\phi = 6370 \cos 124° = -3562.1$$

Bogotá:

$$x = \rho \cos\theta \sin\phi = 6370 \cos 285.75° \sin 85.47° = 1723.7$$
$$y = \rho \sin\theta \sin\phi = 6370 \sin 285.75° \sin 85.47° = -6111.7$$
$$z = \rho \cos\phi = 6370 \cos 85.47° = 503.1$$

77. Find an equation of the form $z = f(r, \theta)$ in cylindrical coordinates for $z^2 = x^2 - y^2$.

SOLUTION In cylindrical coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. Hence,

$$z^2 = x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta$$

We use the identity $\cos^2\theta - \sin^2\theta = \cos 2\theta$ to obtain

$$z^2 = r^2 \cos 2\theta \quad \Rightarrow \quad z = \pm r\sqrt{\cos 2\theta}$$

79. An apple modeled by taking all the points in and on a sphere of radius 2 in. is cored with a vertical cylinder of radius 1 in. Use inequalities in cylindrical coordinates to describe the set of all points that remain in the apple once the core is removed.

SOLUTION The sphere together with its interior is, in rectangular coordinates, the set of points with $x^2 + y^2 + z^2 \le 4$. In cylindrical coordinates, this is the set of points with $r^2 + z^2 \le 4$. So we can parametrize the sphere and its interior as $-2 \le z \le 2$ and $0 \le r \le \sqrt{4 - z^2}$. The vertical cylinder together with its interior is parametrized by $r \le 1$. The cylinder intersects the sphere when $1^2 + z^2 = 4$, or $z = \pm\sqrt{3}$. When $|z| > \sqrt{3}$, all points in the sphere lie inside the cylinder, so are gone when the cylinder is removed. So removing the cylinder from the sphere gives a set of points parametrized as $-\sqrt{3} < z < \sqrt{3}$, $1 < r \le \sqrt{4 - z^2}$.

81. Explain the following statement: If the equation of a surface in cylindrical or spherical coordinates does not involve the coordinate θ, then the surface is rotationally symmetric with respect to the z-axis.

SOLUTION Suppose the point $P = (\rho_0, \theta_0, \phi_0)$ (in spherical coordinates) or (r_0, θ_0, z_0) (in cylindrical coordinates) lies on the surface. Since the equation of the surface does not involve the coordinate θ, we may substitute any value of θ for θ_0 and still get a point on the surface. But changing θ amounts to rotating P around the z-axis. Therefore all the points obtained by rotating P around the z-axis are on the surface and hence the surface is rotationally symmetric with respect to the z-axis.

83. Find equations $r = g(\theta, z)$ (cylindrical) and $\rho = f(\theta, \phi)$ (spherical) for the hyperboloid $x^2 + y^2 = z^2 + 1$ (Figure 18). Do there exist points on the hyperboloid with $\phi = 0$ or π? Which values of ϕ occur for points on the hyperboloid?

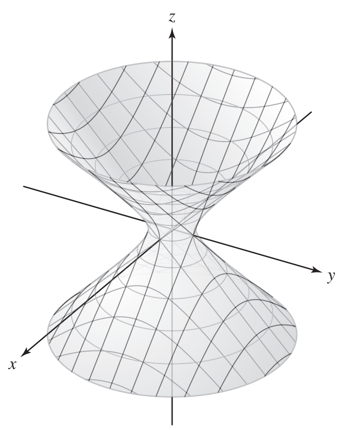


FIGURE 18 The hyperboloid $x^2 + y^2 = z^2 + 1$.

SOLUTION For the cylindrical coordinates (r, θ, z) we have $x^2 + y^2 = r^2$. Substituting into the equation $x^2 + y^2 = z^2 + 1$ gives

$$r^2 = z^2 + 1 \quad \Rightarrow \quad r = \sqrt{z^2 + 1}$$

For the spherical coordinates (ρ, θ, ϕ) we have $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ and $z = \rho\cos\phi$. We substitute into the equation of the hyperboloid $x^2 + y^2 = z^2 + 1$ and simplify to obtain

$$\begin{aligned} \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta &= \rho^2 \cos^2\phi + 1 \\ \rho^2 \sin^2\phi \left(\cos^2\theta + \sin^2\theta\right) &= \rho^2 \cos^2\phi + 1 \\ \rho^2 \left(\sin^2\phi - \cos^2\phi\right) &= 1 \end{aligned}$$

Using the trigonometric identity $\cos 2\phi = \cos^2\phi - \sin^2\phi$ we get

$$\rho^2 \cdot \left(-\cos 2\phi\right) = 1 \quad \Rightarrow \quad \rho = \sqrt{-\frac{1}{\cos 2\phi}}$$

For $\phi = 0$ and $\phi = \pi$ we have $\cos 2 \cdot 0 = 1$ and $\cos 2\pi = 1$. In both cases $-\frac{1}{\cos 2\phi} = -1 < 0$, hence there is no real value of ρ satisfying $\rho = \sqrt{-\frac{1}{\cos 2\phi}}$. We conclude that there are no points on the hyperboloid with $\phi = 0$ or π.

To obtain a real ρ such that $\rho = \sqrt{-\frac{1}{\cos 2\phi}}$, we must have $-\frac{1}{\cos 2\phi} > 0$. That is, $\cos 2\phi < 0$ (and of course $0 \le \phi \le \pi$). The corresponding values of ϕ are

$$\frac{\pi}{2} < 2\phi \le \frac{3\pi}{2} \quad \Rightarrow \quad \frac{\pi}{4} < \phi \le \frac{3\pi}{4}$$

Further Insights and Challenges

In Exercises 84–88, a ***great circle*** *on a sphere S with center O and radius R is a circle obtained by intersecting S with a plane that passes through O (Figure 19). If P and Q are not antipodal (on opposite sides), there is a unique great circle through P and Q on S (intersect S with the plane through O, P, and Q). The geodesic distance from P to Q is defined as the length of the smaller of the two circular arcs of this great circle.*

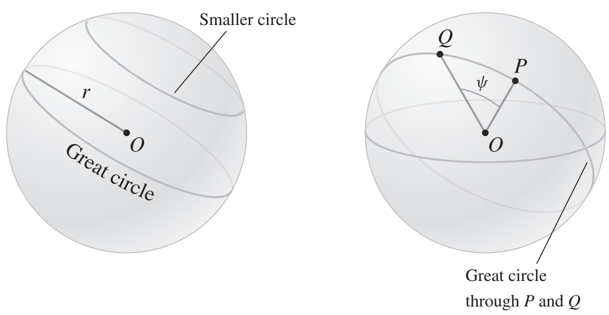


FIGURE 19

85. Show that the geodesic distance from $Q = (a, b, c)$ to the North Pole $P = (0, 0, R)$ is equal to $R\cos^{-1}\left(\frac{c}{R}\right)$.

SOLUTION Let ψ be the central angle between P and Q, that is, the angle between the vectors $\mathbf{v} = \overrightarrow{OP}$ and $\mathbf{u} = \overrightarrow{OQ}$. By Exercise 84 the geodesic distance from P to Q is $R\psi$. We find ψ. By the formula for the cosine of the angle between two vectors, we have

$$\cos\psi = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \tag{1}$$

We compute the values in this quotient:

$$\mathbf{u}\cdot\mathbf{v} = \langle 0, 0, R\rangle \cdot \langle a, b, c\rangle = 0 + 0 + Rc = Rc$$

$$\|\mathbf{v}\| = \|\overrightarrow{OP}\| = R$$

$$\|\mathbf{u}\| = \|\overrightarrow{OQ}\| = \sqrt{a^2 + b^2 + c^2} = R$$

Substituting in (1) we get

$$\cos\psi = \frac{Rc}{R^2} = \frac{c}{R} \quad \Rightarrow \quad \psi = \cos^{-1}\left(\frac{c}{R}\right)$$

The geodesic distance from Q to P is thus

$$R\psi = R\cos^{-1}\left(\frac{c}{R}\right)$$

87. Show that the central angle ψ between points P and Q on a sphere (of any radius) with angular coordinates (θ, ϕ) and (θ', ϕ') is equal to

$$\psi = \cos^{-1}\left(\sin\phi\sin\phi'\cos(\theta - \theta') + \cos\phi\cos\phi'\right)$$

Hint: Compute the dot product of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Check this formula by computing the geodesic distance between the North and South Poles.

SOLUTION We denote the vectors $\mathbf{u} = \overrightarrow{OP}$ and $\mathbf{v} = \overrightarrow{OQ}$. By the formula for the angle between two vectors we have

$$\psi = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$$

Denoting by R the radius of the sphere, we have $\|\mathbf{u}\| = \|\mathbf{v}\| = R$, hence,

$$\psi = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{R^2}\right) \tag{1}$$

The rectangular coordinates of **u** and **v** are

u	v
$x = R\sin\phi\cos\theta$	$x' = R\sin\phi'\cos\theta'$
$y = R\sin\phi\sin\theta$	$y' = R\sin\phi'\sin\theta'$
$z = R\cos\phi$	$z' = R\cos\phi'$

Hence,

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= R^2\sin\phi\cos\theta\sin\phi'\cos\theta' + R^2\sin\phi\sin\theta\sin\phi'\sin\theta' + R^2\cos\phi\cos\phi' \\ &= R^2\left[\sin\phi\sin\phi'\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\right) + \cos\phi\cos\phi'\right]\end{aligned}$$

We use the identity $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ to obtain

$$\mathbf{u}\cdot\mathbf{v} = R^2\left(\sin\phi\sin\phi'\cos\left(\theta-\theta'\right) + \cos\phi\cos\phi'\right)$$

Substituting in (1) we obtain

$$\psi = \cos^{-1}\left(\sin\phi\sin\phi'\cos\left(\theta-\theta'\right) + \cos\phi\cos\phi'\right) \tag{2}$$

We now check this formula in the case where P and Q are the north and south poles respectively. In this case $\theta = \theta' = 0$, $\phi = 0$, $\phi' = \pi$. Substituting in (2) gives

$$\psi = \cos^{-1}\left(\sin 0\sin\pi\cos 0 + \cos 0\cos\pi\right) = \cos^{-1}(-1) = \pi$$

Using Exercise 84, the geodesic distance between the two poles is $R\psi = R\pi$, in accordance with the formula for the length of a semicircle.

CHAPTER REVIEW EXERCISES

In Exercises 1–6, let $\mathbf{v} = \langle -2, 5\rangle$ *and* $\mathbf{w} = \langle 3, -2\rangle$.

1. Calculate $5\mathbf{w} - 3\mathbf{v}$ and $5\mathbf{v} - 3\mathbf{w}$.

SOLUTION We use the definition of basic vector operations to compute the two linear combinations:

$$\begin{aligned}5\mathbf{w} - 3\mathbf{v} &= 5\langle 3, -2\rangle - 3\langle -2, 5\rangle = \langle 15, -10\rangle + \langle 6, -15\rangle = \langle 21, -25\rangle \\ 5\mathbf{v} - 3\mathbf{w} &= 5\langle -2, 5\rangle - 3\langle 3, -2\rangle = \langle -10, 25\rangle + \langle -9, 6\rangle = \langle -19, 31\rangle\end{aligned}$$

3. Find the unit vector in the direction of **v**.

SOLUTION The unit vector in the direction of **v** is

$$\mathbf{e}_{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

We compute the length of **v**:

$$\|\mathbf{v}\| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$$

Hence,

$$\mathbf{e}_{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5\rangle}{\sqrt{29}} = \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right\rangle.$$

5. Express **i** as a linear combination $r\mathbf{v} + s\mathbf{w}$.

SOLUTION We use basic properties of vector algebra to write

$$\mathbf{i} = r\mathbf{v} + s\mathbf{w} \tag{1}$$

$$\langle 1, 0\rangle = r\langle -2, 5\rangle + s\langle 3, -2\rangle = \langle -2r + 3s, 5r - 2s\rangle$$

The vector are equivalent, hence,

$$1 = -2r + 3s$$
$$0 = 5r - 2s$$

The second equation implies that $s = \frac{5}{2}r$. We substitute in the first equation and solve for r:

$$1 = -2r + 3 \cdot \frac{5}{2}r$$
$$1 = \frac{11}{2}r$$
$$r = \frac{2}{11} \quad \Rightarrow \quad s = \frac{5}{2} \cdot \frac{2}{11} = \frac{5}{11}$$

Substituting in (1) we obtain

$$\mathbf{i} = \frac{2}{11}\mathbf{v} + \frac{5}{11}\mathbf{w}.$$

7. If $P = (1, 4)$ and $Q = (-3, 5)$, what are the components of $\overrightarrow{PQ}$? What is the length of $\overrightarrow{PQ}$?

SOLUTION By the Definition of Components of a Vector we have

$$\overrightarrow{PQ} = \langle -3 - 1, 5 - 4\rangle = \langle -4, 1\rangle$$

The length of $\overrightarrow{PQ}$ is

$$\|\overrightarrow{PQ}\| = \sqrt{(-4)^2 + 1^2} = \sqrt{17}.$$

9. Find the vector with length 3 making an angle of $\frac{7\pi}{4}$ with the positive x-axis.

SOLUTION We denote the vector by $\mathbf{v} = \langle a, b\rangle$. $\mathbf{v}$ makes an angle $\theta = \frac{7\pi}{4}$ with the x-axis, and its length is 3, hence,

$$a = \|\mathbf{v}\|\cos\theta = 3\cos\frac{7\pi}{4} = \frac{3}{\sqrt{2}}$$
$$b = \|\mathbf{v}\|\sin\theta = 3\sin\frac{7\pi}{4} = -\frac{3}{\sqrt{2}}$$

That is,

$$\mathbf{v} = \langle a, b\rangle = \left\langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}} \right\rangle.$$

11. Find the value of β for which $\mathbf{w} = \langle -2, \beta\rangle$ is parallel to $\mathbf{v} = \langle 4, -3\rangle$.

SOLUTION If $\mathbf{v} = \langle 4, -3\rangle$ and $\mathbf{w} = \langle -2, \beta\rangle$ are parallel, there exists a scalar λ such that $\mathbf{w} = \lambda\mathbf{v}$. That is,

$$\langle -2, \beta\rangle = \lambda\langle 4, -3\rangle = \langle 4\lambda, -3\lambda\rangle$$

yielding

$$-2 = 4\lambda \quad \text{and} \quad \beta = -3\lambda$$

These equations imply that $\lambda = -\frac{1}{2}$ and $\lambda = -\frac{\beta}{3}$. Equating the two expressions for λ gives

$$-\frac{1}{2} = -\frac{\beta}{3} \quad \text{or} \quad \beta = \frac{3}{2}.$$

13. Let $\mathbf{w} = \langle 2, -2, 1\rangle$ and $\mathbf{v} = \langle 4, 5, -4\rangle$. Solve for $\mathbf{u}$ if $\mathbf{v} + 5\mathbf{u} = 3\mathbf{w} - \mathbf{u}$.

SOLUTION Using vector algebra we have

$$\begin{aligned}\mathbf{v} + 5\mathbf{u} &= 3\mathbf{w} - \mathbf{u}\\ 6\mathbf{u} &= 3\mathbf{w} - \mathbf{v}\\ \mathbf{u} &= \frac{1}{2}\mathbf{w} - \frac{1}{6}\mathbf{v} = \left\langle 1, -1, \frac{1}{2}\right\rangle - \left\langle \frac{4}{6}, \frac{5}{6}, -\frac{4}{6}\right\rangle = \left\langle \frac{1}{3}, -\frac{11}{6}, \frac{7}{6}\right\rangle\end{aligned}$$

15. Find a parametrization $\mathbf{r}_1(t)$ of the line passing through $(1, 4, 5)$ and $(-2, 3, -1)$. Then find a parametrization $\mathbf{r}_2(t)$ of the line parallel to $\mathbf{r}_1$ passing through $(1, 0, 0)$.

SOLUTION Since the points $P = (-2, 3, -1)$ and $Q = (1, 4, 5)$ are on the line l_1, the vector $\overrightarrow{PQ}$ is a direction vector for the line. We find this vector:

$$\overrightarrow{PQ} = \langle 1 - (-2), 4 - 3, 5 - (-1)\rangle = \langle 3, 1, 6\rangle$$

Substituting $\mathbf{v} = \langle 3, 1, 6\rangle$ and $P_0 = \langle 1, 4, 5\rangle$ in the vector parametrization of the line we obtain the following equation for l_1:

$$\begin{aligned}\mathbf{r}_1(t) &= \overrightarrow{OP_0} + t\mathbf{v}\\ \mathbf{r}_1(t) &= \langle 1, 4, 5\rangle + t\langle 3, 1, 6\rangle = \langle 1 + 3t, 4 + t, 5 + 6t\rangle\end{aligned}$$

The line l_2 is parallel to l_1, hence $\overrightarrow{PQ} = \langle 3, 1, 6\rangle$ is also a direction vector for l_2. Substituting $\mathbf{v} = \langle 3, 1, 6\rangle$ and $P_0 = (1, 0, 0)$ in the vector parametrization of the line we obtain the following equation for l_2:

$$\begin{aligned}\mathbf{r}_2(t) &= \overrightarrow{OP_0} + t\mathbf{v}\\ \mathbf{r}_2(t) &= \langle 1, 0, 0\rangle + t\langle 3, 1, 6\rangle = \langle 1 + 3t, t, 6t\rangle\end{aligned}$$

17. Find a and b such that the lines $\mathbf{r}_1 = \langle 1, 2, 1\rangle + t\langle 1, -1, 1\rangle$ and $\mathbf{r}_2 = \langle 3, -1, 1\rangle + t\langle a, b, -2\rangle$ are parallel.

SOLUTION The lines are parallel if and only if the direction vectors $\mathbf{v}_1 = \langle 1, -1, 1\rangle$ and $\mathbf{v}_2 = \langle a, b, -2\rangle$ are parallel. That is, if and only if there exists a scalar λ such that:

$$\begin{aligned}\mathbf{v}_2 &= \lambda\mathbf{v}_1\\ \langle a, b, -2\rangle &= \lambda\langle 1, -1, 1\rangle = \langle \lambda, -\lambda, \lambda\rangle\end{aligned}$$

We obtain the following equations:

$$\begin{aligned}a &= \lambda\\ b &= -\lambda \quad \Rightarrow \quad a = -2, \quad b = 2\\ -2 &= \lambda\end{aligned}$$

19. Sketch the vector sum $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$ for the vectors in Figure 1(A).

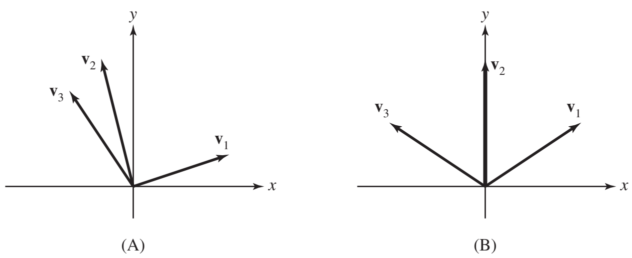
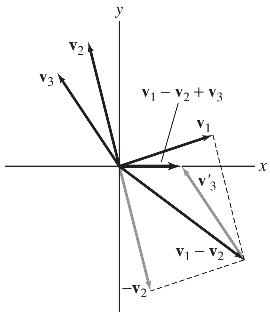


FIGURE 1

SOLUTION Using the Parallelogram Law we obtain the vector sum shown in the figure.



We first add $\mathbf{v}_1$ and $-\mathbf{v}_2$, then we add $\mathbf{v}_3$ to $\mathbf{v}_1 - \mathbf{v}_2$.

In Exercises 21–26, let $\mathbf{v} = \langle 1, 3, -2\rangle$ *and* $\mathbf{w} = \langle 2, -1, 4\rangle$.

21. Compute $\mathbf{v}\cdot\mathbf{w}$.

SOLUTION Using the definition of the dot product we have

$$\mathbf{v}\cdot\mathbf{w} = \langle 1, 3, -2\rangle \cdot \langle 2, -1, 4\rangle = 1\cdot 2 + 3\cdot(-1) + (-2)\cdot 4 = 2 - 3 - 8 = -9$$

23. Compute $\mathbf{v}\times\mathbf{w}$.

SOLUTION We use the definition of the cross product as a "determinant":

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}\mathbf{k}$$
$$= (12-2)\mathbf{i} - (4+4)\mathbf{j} + (-1-6)\mathbf{k} = 10\mathbf{i} - 8\mathbf{j} - 7\mathbf{k} = \langle 10, -8, -7\rangle$$

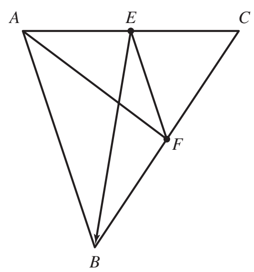
25. Find the volume of the parallelepiped spanned by $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u} = \langle 1, 2, 6\rangle$.

SOLUTION The volume V of the parallelepiped spanned by $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$ is the following determinant:

$$V = \left|\det\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \\ \mathbf{u} \end{pmatrix}\right| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & 2 & 6 \end{vmatrix} = \left| 1\cdot\begin{vmatrix} -1 & 4 \\ 2 & 6 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}\right|$$
$$= |1\cdot(-6-8) - 3(12-4) - 2(4+1)| = 48$$

27. Use vectors to prove that the line connecting the midpoints of two sides of a triangle is parallel to the third side.

SOLUTION Let E and F be the midpoints of sides AC and BC in a triangle ABC (see figure).



We must show that

$$\overrightarrow{EF} \parallel \overrightarrow{AB}$$

Using the Parallelogram Law we have

$$\overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BF} \tag{1}$$

By the definition of the points E and F,

$$\overrightarrow{EA} = \frac{1}{2}\overrightarrow{CA}; \quad \overrightarrow{BF} = \frac{1}{2}\overrightarrow{BC}$$

We substitute (1) to obtain

$$\begin{aligned}\overrightarrow{EF} &= \frac{1}{2}\overrightarrow{CA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{AB} + \frac{1}{2}\left(\overrightarrow{CA} + \overrightarrow{BC}\right)\\ &= \overrightarrow{AB} + \frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{CA}\right) = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BA} = \overrightarrow{AB} - \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AB}\end{aligned}$$

Therefore, $\overrightarrow{EF}$ is a constant multiple of $\overrightarrow{AB}$, which implies that $\overrightarrow{EF}$ and $\overrightarrow{AB}$ are parallel vectors.

29. Calculate the component of $\mathbf{v} = \left\langle -2, \frac{1}{2}, 3\right\rangle$ along $\mathbf{w} = \langle 1, 2, 2\rangle$.

SOLUTION We first compute the following dot products:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= \langle -2, \frac{1}{2}, 3\rangle \cdot \langle 1, 2, 2\rangle = 5\\ \mathbf{w}\cdot\mathbf{w} &= \|\mathbf{w}\|^2 = 1^2 + 2^2 + 2^2 = 9\end{aligned}$$

The component of $\mathbf{v}$ along $\mathbf{w}$ is the following number:

$$\left\|\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w}\right\| = \frac{5}{9}\|\mathbf{w}\| = \frac{5}{9}\cdot 3 = \frac{5}{3}$$

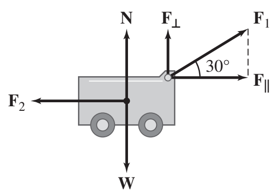
31. A 50-kg wagon is pulled to the right by a force $\mathbf{F}_1$ making an angle of 30° with the ground. At the same time, the wagon is pulled to the left by a horizontal force $\mathbf{F}_2$.

(a) Find the magnitude of $\mathbf{F}_1$ in terms of the magnitude of $\mathbf{F}_2$ if the wagon does not move.

(b) What is the maximal magnitude of $\mathbf{F}_1$ that can be applied to the wagon without lifting it?

SOLUTION

(a) By Newton's Law, at equilibrium, the total force acting on the wagon is zero.



We resolve the force $\mathbf{F}_1$ into its components:

$$\mathbf{F}_1 = \mathbf{F}_{\|} + \mathbf{F}_{\perp}$$

where $\mathbf{F}_{\|}$ is the horizontal component and $\mathbf{F}_{\perp}$ is the vertical component. Since the wagon does not move, the magnitude of $\mathbf{F}_{\|}$ must be equal to the magnitude of $\mathbf{F}_2$. That is,

$$\|\mathbf{F}_{\|}\| = \|\mathbf{F}_1\|\cos 30^\circ = \|\mathbf{F}_2\|$$

The above equation gives:

$$\|\mathbf{F}_1\|\frac{\sqrt{3}}{2} = \|\mathbf{F}_2\| \quad \Rightarrow \quad \|\mathbf{F}_1\| = \frac{2\|\mathbf{F}_2\|}{\sqrt{3}}$$

(b) The maximum magnitude of force $\mathbf{F}_1$ that can be applied to the wagon without lifting the wagon is found by comparing the vertical forces:

$$\begin{aligned}\|\mathbf{F}_1\|\sin 30^\circ &= 9.8\cdot 50\\ \|\mathbf{F}_1\|\cdot\frac{1}{2} &= 9.8\cdot 50 \quad \Rightarrow \quad \|\mathbf{F}_1\| = 9.8\cdot 100 = 980\text{ N}\end{aligned}$$

In Exercises 33–36, let $\mathbf{v} = \langle 1, 2, 4\rangle$, $\mathbf{u} = \langle 6, -1, 2\rangle$, *and* $\mathbf{w} = \langle 1, 0, -3\rangle$. *Calculate the given quantity.*

33. $\mathbf{v} \times \mathbf{w}$

SOLUTION We use the definition of the cross product as a determinant to compute $\mathbf{v} \times \mathbf{w}$:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 4 \\ 1 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 0 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}\mathbf{k}$$
$$= (-6-0)\mathbf{i} - (-3-4)\mathbf{j} + (0-2)\mathbf{k} = -6\mathbf{i} + 7\mathbf{j} - 2\mathbf{k} = \langle -6, 7, -2\rangle$$

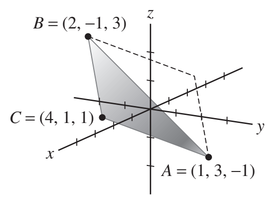
35. $\det\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}$

SOLUTION We compute the determinant:

$$\det\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{vmatrix} 6 & -1 & 2 \\ 1 & 2 & 4 \\ 1 & 0 & -3 \end{vmatrix} = 6 \cdot \begin{vmatrix} 2 & 4 \\ 0 & -3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}$$
$$= 6 \cdot (-6-0) + 1 \cdot (-3-4) + 2 \cdot (0-2) = -47$$

37. Use the cross product to find the area of the triangle whose vertices are $(1, 3, -1)$, $(2, -1, 3)$, and $(4, 1, 1)$.

SOLUTION Let $A = (1, 3, -1)$, $B = (2, -1, 3)$ and $C = (4, 1, 1)$.



The area S of the triangle ABC is half the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Using the Formula for the Area of the Parallelogram, we conclude that the area of the triangle is:

$$S = \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| \tag{1}$$

We first compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\overrightarrow{AB} = \langle 2-1, -1-3, 3-(-1)\rangle = \langle 1, -4, 4\rangle$$
$$\overrightarrow{AC} = \langle 4-1, 1-3, 1-(-1)\rangle = \langle 3, -2, 2\rangle$$

We compute the cross product of the two vectors:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 4 \\ 3 & -2 & 2 \end{vmatrix} = \begin{vmatrix} -4 & 4 \\ -2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -4 \\ 3 & -2 \end{vmatrix}\mathbf{k}$$
$$= (-8-(-8))\mathbf{i} - (2-12)\mathbf{j} + (-2-(-12))\mathbf{k}$$
$$= 10\mathbf{j} + 10\mathbf{k} = \langle 0, 10, 10\rangle = 10\langle 0, 1, 1\rangle$$

The length of $\overrightarrow{AB} \times \overrightarrow{AC}$ is, thus:

$$\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \|10\langle 0, 1, 1\rangle\| = 10\|\langle 0, 1, 1\rangle\| = 10\sqrt{0^2 + 1^2 + 1^2} = 10\sqrt{2}$$

Substituting in (1) gives the following area:

$$S = \frac{1}{2} \cdot 10\sqrt{2} = 5\sqrt{2}.$$

39. Show that if the vectors $\mathbf{v}$, $\mathbf{w}$ are orthogonal, then $\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

SOLUTION The vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, hence:

$$\mathbf{v}\cdot\mathbf{w} = 0 \tag{1}$$

Using the relation of the dot product with length and properties of the dot product we obtain:

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2 &= (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{v}\cdot(\mathbf{v}+\mathbf{w}) + \mathbf{w}\cdot(\mathbf{v}+\mathbf{w})\\ &= \mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 + 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2 \end{aligned} \tag{2}$$

Combining (1) and (2) we get:

$$\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$

41. Find $\|\mathbf{e}-4\mathbf{f}\|$, assuming that $\mathbf{e}$ and $\mathbf{f}$ are unit vectors such that $\|\mathbf{e}+\mathbf{f}\| = \sqrt{3}$.

SOLUTION We use the relation of the dot product with length and properties of the dot product to write

$$\begin{aligned}3 = \|\mathbf{e}+\mathbf{f}\|^2 &= (\mathbf{e}+\mathbf{f})\cdot(\mathbf{e}+\mathbf{f}) = \mathbf{e}\cdot\mathbf{e} + \mathbf{e}\cdot\mathbf{f} + \mathbf{f}\cdot\mathbf{e} + \mathbf{f}\cdot\mathbf{f}\\ &= \|\mathbf{e}\|^2 + 2\mathbf{e}\cdot\mathbf{f} + \|\mathbf{f}\|^2 = 1^2 + 2\mathbf{e}\cdot\mathbf{f} + 1^2 = 2 + 2\mathbf{e}\cdot\mathbf{f}\end{aligned}$$

We now find $\mathbf{e}\cdot\mathbf{f}$:

$$3 = 2 + 2\mathbf{e}\cdot\mathbf{f} \quad\Rightarrow\quad \mathbf{e}\cdot\mathbf{f} = 1/2$$

Hence, using the same method as above, we have:

$$\begin{aligned}\|\mathbf{e}-4\mathbf{f}\|^2 &= (\mathbf{e}-4\mathbf{f})\cdot(\mathbf{e}-4\mathbf{f})\\ &= \|\mathbf{e}\|^2 - 2\cdot\mathbf{e}\cdot 4\mathbf{f} + \|4\mathbf{f}\|^2 = 1^2 - 8\mathbf{e}\cdot\mathbf{f} + 4^2 = 17 - 4 = 13\end{aligned}$$

Taking square roots, we get:

$$\|\mathbf{e}-4\mathbf{f}\| = \sqrt{13}$$

43. Show that the equation $\langle 1, 2, 3\rangle \times \mathbf{v} = \langle -1, 2, a\rangle$ has no solution for $a \neq -1$.

SOLUTION By properties of the cross product, the vector $\langle -1, 2, a\rangle$ is orthogonal to $\langle 1, 2, 3\rangle$, hence the dot product of these vectors is zero. That is:

$$\langle -1, 2, a\rangle \cdot \langle 1, 2, 3\rangle = 0$$

We compute the dot product and solve for a:

$$\begin{aligned}-1 + 4 + 3a &= 0\\ 3a &= -3 \quad\Rightarrow\quad a = -1\end{aligned}$$

We conclude that if the given equation is solvable, then $a = -1$.

45. Use the identity

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$$

to prove that

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) + \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) + \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0}$$

SOLUTION The given identity implies that:

$$\begin{aligned}\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &= (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}\\ \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) &= (\mathbf{v}\cdot\mathbf{u})\,\mathbf{w} - (\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}\\ \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) &= (\mathbf{w}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{w}\cdot\mathbf{u})\,\mathbf{v}\end{aligned}$$

Adding the three equations and using the commutativity of the dot product we find that:

$$\begin{aligned}&\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) + \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) + \mathbf{w}\times(\mathbf{u}\times\mathbf{v})\\ &\quad= (\mathbf{u}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{u})\,\mathbf{v} + (\mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v})\,\mathbf{w} + (\mathbf{w}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w})\,\mathbf{u} = \mathbf{0}\end{aligned}$$

47. Write the equation of the plane $\mathcal{P}$ with vector equation

$$\langle 1, 4, -3\rangle \cdot \langle x, y, z\rangle = 7$$

in the form

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$$

Hint: You must find a point $P = (x_0, y_0, z_0)$ on $\mathcal{P}$.

SOLUTION We identify the vector $\mathbf{n} = \langle a, b, c\rangle = \langle 1, 4, -3\rangle$ that is normal to the plane, hence we may choose,

$$a = 1, \quad b = 4, \quad c = -3.$$

We now must find a point in the plane. The point $(x_0, y_0, z_0) = (0, 1, -1)$, for instance, satisfies the equation of the plane, therefore the equation may be written in the form:

$$1(x - 0) + 4(y - 1) - 3(z - (-1)) = 0$$

or

$$(x - 0) + 4(y - 1) - 3(z + 1) = 0$$

49. Find the plane through $P = (4, -1, 9)$ containing the line $\mathbf{r}(t) = \langle 1, 4, -3\rangle + t\langle 2, 1, 1\rangle$.

SOLUTION Since the plane contains the line, the direction vector of the line, $\mathbf{v} = \langle 2, 1, 1\rangle$, is in the plane. To find another vector in the plane, we use the points $A = (1, 4, -3)$ and $B = (4, -1, 9)$ that lie in the plane, and compute the vector $\mathbf{u} = \overrightarrow{AB}$:

$$\mathbf{u} = \overrightarrow{AB} = \langle 4 - 1, -1 - 4, 9 - (-3)\rangle = \langle 3, -5, 12\rangle$$

We now compute the cross product $\mathbf{n} = \mathbf{v} \times \mathbf{u}$ that is normal to the plane:

$$\mathbf{n} = \mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 3 & -5 & 12 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -5 & 12 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 3 & 12 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 3 & -5 \end{vmatrix}\mathbf{k}$$
$$= (12 + 5)\mathbf{i} - (24 - 3)\mathbf{j} + (-10 - 3)\mathbf{k} = 17\mathbf{i} - 21\mathbf{j} - 13\mathbf{k} = \langle 17, -21, -13\rangle$$

Finally, we use the vector form of the equation of the plane with $\mathbf{n} = \langle 17, -21, -13\rangle$ and $P_0 = (4, -1, 9)$ to obtain the following equation:

$$\mathbf{n} \cdot \langle x, y, z\rangle = \mathbf{n} \cdot \langle x_0, y_0, z_0\rangle$$
$$\langle 17, -21, -13\rangle \cdot \langle x, y, z\rangle = \langle 17, -21, -13\rangle \cdot \langle 4, -1, 9\rangle$$
$$17x - 21y - 13z = 17 \cdot 4 + 21 - 13 \cdot 9 = -28$$

The equation of the plane is, thus,

$$17x - 21y - 13z = -28.$$

51. Find the trace of the plane $3x - 2y + 5z = 4$ in the xy-plane.

SOLUTION The xy-plane has equation $z = 0$, therefore the intersection of the plane $3x - 2y + 5z = 4$ with the xy-plane must satisfy both $z = 0$ and the equation of the plane. Therefore the trace has the following equation:

$$3x - 2y + 5 \cdot 0 = 4 \quad \Rightarrow \quad 3x - 2y = 4$$

We conclude that the trace of the plane in the xy-plane is the line $3x - 2y = 4$ in the xy-plane.

In Exercises 53–58, determine the type of the quadric surface.

53. $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 + 2z^2 = 1$

SOLUTION Writing the equation in the form:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{\frac{1}{\sqrt{2}}}\right)^2 = 1$$

we identify the quadric surface as an ellipsoid.

55. $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 - 2z = 0$

SOLUTION We rewrite this equation as:

$$2z = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2$$

or

$$z = \left(\frac{x}{3\sqrt{2}}\right)^2 + \left(\frac{y}{4\sqrt{2}}\right)^2$$

This is the equation of an elliptic paraboloid.

57. $\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 - 2z^2 = 0$

SOLUTION This equation may be rewritten in the form

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 = \left(\frac{z}{\frac{1}{\sqrt{2}}}\right)^2$$

we identify the quadric surface as an elliptic cone.

59. Determine the type of the quadric surface $ax^2 + by^2 - z^2 = 1$ if:

(a) $a < 0, \quad b < 0$

(b) $a > 0, \quad b > 0$

(c) $a > 0, \quad b < 0$

SOLUTION

(a) If $a < 0, b < 0$ then for all x, y and z we have $ax^2 + by^2 - z^2 < 0$, hence there are no points that satisfy $ax^2 + by^2 - z^2 = 1$. Therefore it is the empty set.

(b) For $a > 0$ and $b > 0$ we rewrite the equation as

$$\left(\frac{x}{\frac{1}{\sqrt{a}}}\right)^2 + \left(\frac{y}{\frac{1}{\sqrt{b}}}\right)^2 - z^2 = 1$$

which is the equation of a hyperboloid of one sheet.

(c) For $a > 0$, $b < 0$ we rewrite the equation in the form

$$\left(\frac{x}{\frac{1}{\sqrt{a}}}\right)^2 - \left(\frac{y}{\frac{1}{\sqrt{|b|}}}\right)^2 - z^2 = 1$$

which is the equation of a hyperboloid of two sheets.

61. Convert $(x, y, z) = (3, 4, -1)$ from rectangular to cylindrical and spherical coordinates.

SOLUTION In cylindrical coordinates (r, θ, z) we have

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$$

Therefore, $r = \sqrt{3^2+4^2} = 5$ and $\tan\theta = \frac{4}{3}$. The projection of the point $(3, 4, -1)$ onto the xy-plane is the point $(3, 4)$, in the first quadrant. Therefore, the corresponding value of θ is $\tan^{-1}\frac{4}{3} \approx 0.93$ rad. The cylindrical coordinates are, thus,

$$(r, \theta, z) = \left(5, \tan^{-1}\frac{4}{3}, -1\right)$$

The spherical coordinates (ρ, θ, ϕ) satisfy

$$\rho = \sqrt{x^2+y^2+z^2}, \quad \tan\theta = \frac{y}{x}, \quad \cos\phi = \frac{z}{\rho}$$

Therefore,

$$\rho = \sqrt{3^2+4^2+(-1)^2} = \sqrt{26}$$
$$\tan\theta = \frac{4}{3}$$
$$\cos\phi = \frac{-1}{\sqrt{26}}$$

The angle θ is the same as in the cylindrical coordinates, that is, $\theta = \tan^{-1}\frac{4}{3}$. The angle ϕ is the solution of $\cos\phi = \frac{-1}{\sqrt{26}}$ that satisfies $0 \le \phi \le \pi$, that is, $\phi = \cos^{1}\left(\frac{-1}{\sqrt{26}}\right) \approx 1.77$ rad. The spherical coordinates are, thus,

$$(\rho, \theta, \phi) = \left(\sqrt{26}, \tan^{-1}\frac{4}{3}, \cos^{-1}\left(\frac{-1}{\sqrt{26}}\right)\right).$$

63. Convert the point $(\rho, \theta, \phi) = \left(3, \frac{\pi}{6}, \frac{\pi}{3}\right)$ from spherical to cylindrical coordinates.

SOLUTION By the given information, $\rho = 3$, $\theta = \frac{\pi}{6}$, and $\phi = \frac{\pi}{3}$. We must determine the cylindrical coordinates (r, θ, z). The angle θ is the same as in spherical coordinates. We find z using the relation $\cos\phi = \frac{z}{\rho}$, or $z = \rho\cos\phi$. We obtain

$$z = \rho\cos\phi = 3\cos\frac{\pi}{3} = 3\cdot\frac{1}{2} = \frac{3}{2}$$

We find r using the relation $\rho^2 = x^2+y^2+z^2 = r^2+z^2$, or $r = \sqrt{\rho^2 - z^2}$, we get

$$r = \sqrt{3^2 - \left(\frac{3}{2}\right)^2} = \sqrt{\frac{27}{4}} = \frac{3\sqrt{3}}{2}$$

Hence, in cylindrical coordinates we obtain the following description:

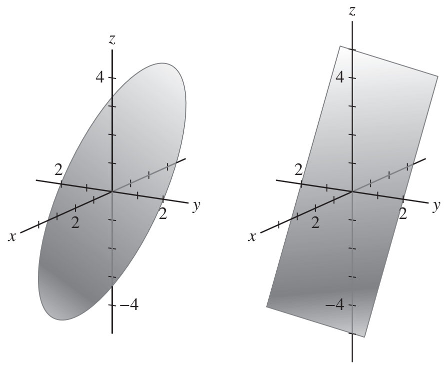
$$(r, \theta, z) = \left(\frac{3\sqrt{3}}{2}, \frac{\pi}{6}, \frac{3}{2}\right).$$

65. Sketch the graph of the cylindrical equation $z = 2r\cos\theta$ and write the equation in rectangular coordinates.

SOLUTION To obtain the equation in rectangular coordinates, we substitute $x = r\cos\theta$ in the equation $z = 2r\cos\theta$:

$$z = 2r\cos\theta = 2x \quad \Rightarrow \quad z = 2x$$

This is the equation of a plane normal to the xz-plane, whose intersection with the xz-plane is the line $z = 2x$. The graph of the plane is shown in the following figure (the same plane drawn twice, using the cylindrical coordinates' equation and using the rectangular coordinates' equation):



67. Show that the cylindrical equation

$$r^2(1 - 2\sin^2\theta) + z^2 = 1$$

is a hyperboloid of one sheet.

SOLUTION We rewrite the equation in the form

$$r^2 - 2(r\sin\theta)^2 + z^2 = 1$$

To write this equation in rectangular coordinates, we substitute $r^2 = x^2 + y^2$ and $r\sin\theta = y$. This gives

$$x^2 + y^2 - 2y^2 + z^2 = 1$$
$$x^2 - y^2 + z^2 = 1$$

We now can identify the surface as a hyperboloid of one sheet.

69. Describe how the surface with spherical equation

$$\rho^2(1 + A\cos^2\phi) = 1$$

depends on the constant A.

SOLUTION To identify the surface we convert the equation to rectangular coordinates. We write

$$\rho^2 + A\rho^2\cos^2\phi = 1$$

To obtain the following equation in terms of x, y, z only, we substitute $\rho^2 = x^2 + y^2 + z^2$ and $\rho\cos\phi = z$:

$$x^2 + y^2 + z^2 + Az^2 = 1$$
$$x^2 + y^2 + (1 + A)z^2 = 1 \qquad (1)$$

Case 1: $A < -1$. Then $A + 1 < 0$ and the equation can be rewritten in the form

$$x^2 + y^2 - \left(\frac{z}{|1 + A|^{-1/2}}\right)^2 = 1$$

The corresponding surface is a hyperboloid of one sheet.

Case 2: $A = -1$. Equation (1) becomes:

$$x^2 + y^2 = 1$$

In R^3, this equation describes a cylinder with the z-axis as its central axis.

Case 3: $A > -1$. Then equation (1) can be rewritten as

$$x^2 + y^2 + \left(\frac{z}{(1 + A)^{-1/2}}\right)^2 = 1$$

Then if $A = 0$ the equation $x^2 + y^2 + z^2 = 1$ describes the unit sphere in R^3. Otherwise, the surface is an ellipsoid.

71. Let c be a scalar, $\mathbf{a}$ and $\mathbf{b}$ be vectors, and $\mathbf{X} = \langle x, y, z\rangle$. Show that the equation $(\mathbf{X} - \mathbf{a}) \cdot (\mathbf{X} - \mathbf{b}) = c^2$ defines a sphere with center $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and radius R, where $R^2 = c^2 + \left\|\frac{1}{2}(\mathbf{a} - \mathbf{b})\right\|^2$.

SOLUTION We evaluate the following length:

$$\begin{aligned}
\|\mathbf{x}-\mathbf{m}\|^2 = \left\|\mathbf{x}-\frac{1}{2}(\mathbf{a}+\mathbf{b})\right\|^2 &= \left((\mathbf{x}-\mathbf{a})+\frac{1}{2}(\mathbf{a}-\mathbf{b})\right)\cdot\left((\mathbf{x}-\mathbf{b})-\frac{1}{2}(\mathbf{a}-\mathbf{b})\right)\\
&= (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b})-\frac{1}{2}(\mathbf{x}-\mathbf{a})\cdot(\mathbf{a}-\mathbf{b})+\frac{1}{2}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{x}-\mathbf{b})-\frac{1}{4}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\
&= (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b})+\frac{1}{2}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{x}-\mathbf{b}-\mathbf{x}+\mathbf{a})-\frac{1}{4}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\
&= (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b})+\frac{1}{2}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})-\frac{1}{4}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\
&= (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b})+\frac{1}{4}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\
&= (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b})+\left\|\frac{1}{2}(\mathbf{a}-\mathbf{b})\right\|^2
\end{aligned}$$

Since $R^2 = c^2 + \|\frac{1}{2}(\mathbf{a}-\mathbf{b})\|^2$ we get

$$\|\mathbf{x}-\mathbf{m}\|^2 = (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b}) + R^2 - c^2$$

We conclude that if $(\mathbf{x}-\mathbf{a})(\mathbf{x}-\mathbf{b}) = c^2$ then $\|\mathbf{x}-\mathbf{m}\|^2 = R^2$. That is, the equation $(\mathbf{x}-\mathbf{a})(\mathbf{x}-\mathbf{b}) = c^2$ defines a sphere with center $\mathbf{m}$ and radius R.