13-1

Permutations and Combinations

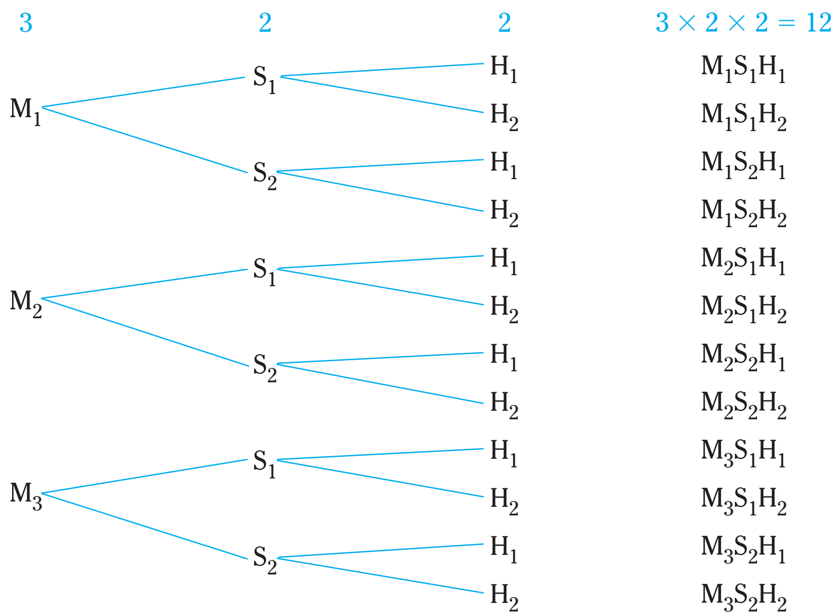
OBJECTIVES

- Solve problems related to the Basic Counting Principle.
- Distinguish between dependent and independent events.
- Solve problems involving permutations or combinations.

Real World Application

EDUCATION Ivette is a freshman at the University of Miami. She is planning her fall schedule for next year. She has a choice of three mathematics courses, two science courses, and two humanities courses. She can only select one course from each area. How many course schedules are possible?

Let M_1, M_2, and M_3 represent the three math courses, S_1 and S_2 the science courses, and H_1 and H_2 the humanities courses. Once Ivette makes a selection from the three mathematics courses she has two choices for her science course. Then, she has two choices for humanities. A **tree diagram** is often used to show all the choices.



Ivette has 12 possible schedules from which to choose.

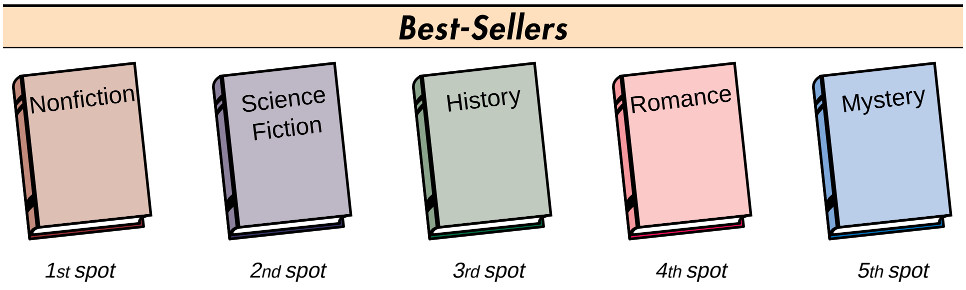
The choice of selecting a mathematics course does *not* affect the choice of ways to select a science or humanities course. Thus, these three choices are called **independent events.** Events that do affect each other are called **dependent events.** An example of dependent events would be the order in which runners finish a race. The first place runner affects the possibilities for second place, the second place runner affects the possibilities for third place, and so on.

The branch of mathematics that studies different possibilities for the arrangement of objects is called **combinatorics.** The example of choosing possible course schedules illustrates a rule of combinatorics known as the **Basic Counting Principle.**

Basic Counting Principle	Suppose one event can be chosen in p different ways, and another independent event can be chosen in q different ways. Then the two events can be chosen successively in $p \cdot q$ ways.

This principle can be extended to any number of independent events. For example, in the previous application, the events are chosen in $p \times q \times r$ or $3 \times 2 \times 2$ different ways.

Example 1 **Vickie works for a bookstore. Her manager asked her to arrange a set of five best-sellers for a display. The display is to be set up as shown below. The display set is made up of one book from each of 5 categories. There are 4 nonfiction, 4 science fiction, 3 history, 3 romance, and 3 mystery books from which to choose.**



a. Are the choices for each book independent or dependent events?

Since the choice of one type of book does not affect the choice of another type of book, the events are independent.

b. How many different ways can Vickie choose the books for the display?

Vickie has four choices for the first spot in the display, four choices for the second spot, and three choices for each of the next three spots.

1st spot		2nd spot		3rd spot		4th spot		5th spot
4	·	4	·	3	·	3	·	3

This can be represented as $4 \cdot 4 \cdot 3 \cdot 3 \cdot 3$ or 432 different arrangements.

There are 432 possible ways for Vickie to choose books for the display.

The arrangement of objects in a certain order is called a **permutation.** In a permutation, the order of the objects is very important. The symbol $P(n, n)$ denotes the number of permutations of n objects taken all at once. The symbol $P(n, r)$ denotes the number of permutations of n objects taken r at a time.

Permutations $P(n, n)$ and $P(n, r)$	The number of permutations of n objects, taken n at a time is defined as $P(n, n) = n!$. The number of permutations of n objects, taken r at a time is defined as $P(n, r) = \frac{n!}{(n - r)!}$.

Recall that n! is read "n factorial" and, $n! = n(n - 1)(n - 2)\ldots(1)$.

Example 2 **During a judging of a horse show at the Fairfield County Fair, there are three favorite horses: Rye Man, Oiler, and Sea of Gus.**

a. Are the selection of first, second and third place from the three horses independent or dependent events?

b. Assuming there are no ties and the three favorites finish in the top three places, how many ways can the horses win first, second and third places?

a. The choice of a horse for first place does affect the choice for second and third places. For example, if Rye Man is first, then is impossible for it to finish second or third. Therefore, the events are dependent.

b. Since order is important, this situation is a permutation.

Method 1: Tree diagram

There are three possibilities for first place, two for second, and one for third as shown in the tree diagram below. If Rye Man finishes first, then either Oiler or Sea of Gus will finish second. If Oiler finishes second, then Sea of Gus must finish third. Likewise, if Sea of Gus finishes second, then Oiler finishes third.

first	Rye Man		Oiler		Sea of Gus	
second	Oiler	Sea of Gus	Rye Man	Sea of Gus	Rye Man	Oiler
third	Sea of Gus	Oiler	Sea of Gus	Rye Man	Oiler	Rye Man

There are 6 possible ways the horses can win.

Method 2: Permutation formula

This situation depicts three objects taken three at a time and can be represented as $P(3, 3)$.

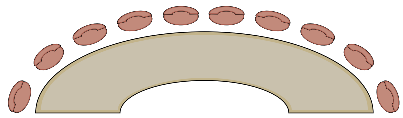
$$P(3, 3) = 3!$$
$$= 3 \cdot 2 \cdot 1 \text{ or } 6$$

Thus, there are 6 ways the horses can win first, second, and third place.

Example 3 **The board of directors of B.E.L.A. Technology Consultants is composed of 10 members.**

a. How many different ways can all the members sit at the conference table as shown?

b. In how many ways can they elect a chairperson, vice-chairperson, treasurer, and secretary, assuming that one person cannot hold more than one office?



a. Since order is important, this situation is a permutation. Also, the 10 members are being taken all at once so the situation can be represented as $P(10, 10)$.

$$P(10, 10) = 10!$$
$$= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \text{ or } 3{,}628{,}800$$

There are 3,628,800 ways that the 10 board members can sit at the table.

b. This is a permutation of 10 people being chosen 4 at a time.

$$P(10, 4) = \frac{10!}{(10-4)!}$$
$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!}$$
$$= 5040$$

There are 5040 ways in which the offices can be filled.

Suppose that in the situation presented in Example 1, Vickie needs to select three types of books from the five types available. There are $P(5, 3)$ or 60 possible arrangements. She can arrange them as shown in the table below.

Arrangement	Type		
1	nonfiction	science fiction	history
2	nonfiction	history	science fiction
3	nonfiction	romance	mystery
4	nonfiction	mystery	romance
5	science fiction	nonfiction	history
6	science fiction	history	nonfiction
7	science fiction	romance	mystery
8	science fiction	mystery	romance
9	history	nonfiction	science fiction
10	history	science fiction	nonfiction
⋮	⋮	⋮	⋮
60	romance	mystery	nonfiction

Note that arrangements 1, 2, 5, 6, 9 and 10 contain the same three types of books. In each group of three books, there are 3! or 6 ways they can be arranged. Thus, if order is disregarded, there are $\frac{60}{3!}$ or 10 different groups of three types of books that can be selected from the five types. In this situation, called a **combination,** the order in which the books are selected is *not* a consideration.

A combination of n objects taken r at a time is calculated by dividing the number of permutations by the number of arrangements containing the same elements and is denoted by $C(n, r)$.

Combination
***C*(*n*, *r*)**

The number of combinations of n objects taken r at a time is defined as

$$C(n, r) = \frac{n!}{(n-r)!\,r!}.$$

The main difference between a permutation and a combination is whether order is considered (as in permutation) or not (as in combination). For example, for objects E, F, G, and H taken two at a time, the permutations and combinations are listed below.

Permutations				**Combinations**	
EF	FE	GE	HE	EF	FG
EG	FG	GF	HF	EG	FH
EH	FH	GH	HG	EH	GH

Note that in permutations, EF is different from FE. But in combinations, EF is the same as FE.

Example 4 Real World Application

ART **In 1999, The National Art Gallery in Washington, D.C., opened an exhibition of the works of John Singer Sargent (1856–1925). The gallery's curator wanted to select four paintings out of twenty on display to showcase the work of the artist. How many groups of four paintings can be chosen?**

Since order is not important, the selection is a combination of 20 objects taken 4 at a time. It can be represented as $C(20, 4)$.

$$\begin{aligned} C(20, 4) &= \frac{20!}{(20-4)!\,4!} \\ &= \frac{20!}{16!\,4!} \\ &= \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16!}{16!\,4!} \\ &= \frac{20 \cdot 19 \cdot 18 \cdot 17}{4 \cdot 3 \cdot 2 \cdot 1} \\ &= 4845 \end{aligned}$$

Express 20! as $20 \cdot 19 \cdot 18 \cdot 17 \cdot 16!$ since $\frac{16!}{16!} = 1$.

There are 4845 possible groups of paintings.

"Oyster Gatherers of Cancale," 1878

Example 5 **At Grant Senior High School, there are 15 names on the ballot for junior class officers. Five will be selected to form a class committee.**

a. How many different committees of 5 can be formed?

b. In how many ways can a committee of 5 be formed if each student has a different responsibility?

c. If there are 8 girls and 7 boys on the ballot, how many committees of 2 boys and 3 girls can be formed?

a. Order is not important in this situation, so the selection is a combination of 15 people chosen 5 at a time.

$$C(15, 5) = \frac{15!}{(15-5)!\,5!}$$
$$= \frac{15!}{10!\,5!}$$
$$= \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10!}{10!\,5!} \text{ or } 3003$$

There are 3003 different ways to form the committees of 5.

b. Order has to be considered in this situation because each committee member has a different responsibility.

$$P(15, 5) = \frac{15!}{(15-5)!}$$
$$= \frac{15!}{10!} \text{ or } 360{,}360$$

There are 360,360 possible committees.

c. Order is not important. There are three questions to consider.
How many ways can 2 boys be chosen from 7?
How many ways can 3 girls be chosen from 8?
Then, how many ways can 2 boys and 3 girls be chosen together?

Since the events are independent, the answer is the product of the combinations $C(7, 2)$ and $C(8, 3)$.

$$C(7, 2) \cdot C(8, 3) = \frac{7!}{(7-2)!\,2!} \cdot \frac{8!}{(8-3)!\,3!}$$
$$= \frac{7!}{5!\,2!} \cdot \frac{8!}{5!\,3!}$$
$$= 21 \cdot 56 \text{ or } 1176$$

There are 1176 possible committees.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** permutations and combinations.
2. **Write** an expression for the number of ways, out of a standard 52-card deck, that 5-card hands can have 2 jacks and 3 queens.

3. You Decide Ms. Sloan asked her students how many ways 5 patients in a hospital could be assigned to 7 identical private rooms. Anita said that the problem dealt with computing $C(7, 5)$. Sam disagreed, saying that $P(7, 5)$ was the correct way to answer the question. Who is correct? Why?

4. Draw a tree diagram to illustrate all of the possible T-shirts available that come in sizes small, medium, large, and extra large and in the colors blue, green and gray.

Guided Practice

5. A restaurant offers the choice of an entrée, a vegetable, a dessert, and a drink for a lunch special. If there are 4 entrees, 3 vegetables, 5 desserts and 5 drinks available to choose from, how many different lunches are available?

6. Are choosing a movie to see and choosing a snack to buy dependent or independent events?

Find each value.

7. $P(6, 6)$
8. $P(5, 3)$
9. $\dfrac{P(12, 8)}{P(6, 4)}$
10. $C(7, 4)$
11. $C(20, 15)$
12. $C(4, 3) \cdot C(5, 2)$

13. If a group of 10 students sits in the same row in an auditorium, how many possible ways can they be arranged?

14. How many baseball lineups of 9 players can be formed from a team that has 15 members if all players can play any position?

15. **Postal Service** The U.S. Postal Service uses 5-digit ZIP codes to route letters and packages to their destinations.
   a. How many ZIP codes are possible if the numbers 0 through 9 are used for each of the 5 digits?
   b. Suppose that when the first digit is 0, the second, third, and fourth digits cannot be 0. How many 5-digit ZIP codes are possible if the first digit is 0?
   c. In 1983, the U.S. Postal Service introduced the ZIP +4, which added 4 more digits to the existing 5-digit ZIP codes. Using the numbers 0 through 9, how many additional ZIP codes were possible?

EXERCISES

Practice

16. If you toss a coin, then roll a die, and then spin a 4-colored spinner with equal sections, how many outcomes are possible?

17. How many ways can 7 classes be scheduled, if each class is offered in each of 7 periods?

18. Find the number of different 7-digit telephone numbers where:
   a. the first digit cannot be zero.
   b. only even digits are used.
   c. the complete telephone numbers are multiples of 10.
   d. the first three digits are 593 in that order.

State whether the events are *independent* or *dependent*.

19. selecting members for a team

20. tossing a penny, rolling a die, then tossing a dime

21. deciding the order in which to complete your homework assignments

Find each value.

22. $P(8, 8)$ **23.** $P(6, 4)$ **24.** $P(5, 3)$

25. $P(7, 4)$ **26.** $P(9, 5)$ **27.** $P(10, 7)$

28. $\frac{P(6, 3)}{P(4, 2)}$ **29.** $\frac{P(6, 4)}{P(5, 3)}$ **30.** $\frac{P(6, 3) \cdot P(7, 5)}{P(9, 6)}$

31. $C(5, 3)$ **32.** $C(10, 5)$ **33.** $C(4, 2)$

34. $C(12, 4)$ **35.** $C(9, 9)$ **36.** $C(14, 7)$

37. $C(3, 2) \cdot C(8, 3)$ **38.** $C(7, 3) \cdot C(8, 5)$ **39.** $C(5, 1) \cdot C(4, 2) \cdot C(8, 2)$

40. A pizza shop has 14 different toppings from which to choose. How many different 4-topping pizzas can be made?

41. If you make a fruit salad using 5 different fruits and you have 14 different varieties from which to choose, how many different fruit salads can you make?

42. How many different 12-member juries can be formed from a group of 18 people?

43. A bag contains 3 red, 5 yellow, and 8 blue marbles. How many ways can 2 red, 1 yellow, and 2 blue marbles be chosen?

44. How many different ways can 11 paintings be displayed on a wall?

45. From a standard 52-card deck, find how many 5-card hands are possible that have:

a. 3 hearts and 2 clubs.

b. 1 ace, 2 jacks, and 2 kings.

c. all face cards.

Applications and Problem Solving

Real World Application

46. Home Security A home security company offers a security system that uses the numbers 0 through 9, inclusive, for a 5-digit security code.

a. How many different security codes are possible?

b. If no digits can be repeated, how many security codes are available?

c. Suppose the homeowner does not want to use 0 as one of the digits and wants only two of the digits to be odd. How many codes can be formed if the digits can be repeated? If no repetitions are allowed, how many codes are available?

47. Baseball How many different 9-player teams can be fielded if there are 3 players who can only play catcher, 4 players who can only play first base, 6 players who can only pitch, and 14 players who can play in any of the remaining 6 positions?

48. Transportation In a train yard, there are 12 flatcars, 10 tanker cars, 15 boxcars, and 5 livestock cars.

a. If the cars must be connected according to their final destinations, how many different ways can they be arranged?

b. How many ways can the train be made up if it is to have 30 cars?

c. How many trains can be formed with 3 livestock cars, 6 flatcars, 6 tanker cars, and 5 boxcars?

49. Critical Thinking Prove $P(n, n - 1) = P(n, n)$.

50. Entertainment Three couples have reserved seats for a Broadway musical. Find how many different ways they can sit if:

a. there are no seating restrictions.

b. two members of each couple wish to sit together.

51. **Botany** A researcher with the U.S. Department of Agriculture is conducting an experiment to determine how well certain crops can survive adverse weather conditions. She has gathered 6 corn plants, 3 wheat plants, and 2 bean plants. She needs to select four plants at random for the experiment.

a. In how many ways can this be done?

b. If exactly 2 corn plants must be included, in how many ways can the plants be selected?

52. **Geometry** How many lines are determined by 10 points, no 3 of which are collinear?

53. **Critical Thinking** There are 6 permutations of the digits 1, 6, and 7.

167 176 617 671 716 761

The average of these six numbers is $\frac{3108}{6} = 518$ which is equal to $37(1 + 6 + 7)$. If the digits are 0, 4, and 7, then the average of the six permutations is $\frac{2442}{6} = 407$ or $37(0 + 4 + 7)$.

a. Use this pattern to find the average of the six permutations of 2, 5, and 9.

b. Will this pattern hold for all sets of three digits? If so, prove it.

Mixed Review

54. **Banking** Cynthia has a savings account that has an annual yield of 5.8%. Find the balance of the account after each of the first three years if her initial balance is $2140. *(Lesson 12-8)*

55. Find the sum of the first ten terms of the series $1^3 + 2^3 + 3^3 + \cdots$. *(Lesson 12-5)*

56. Solve $7.1^x = 83.1$ using logarithms. Round to the nearest hundredth. *(Lesson 11-6)*

57. Find the value of x to the nearest tenth such that $x = e^{0.346}$. *(Lesson 11-3)*

58. **Communications** A satellite dish tracks a satellite directly overhead. Suppose the graph of the equation $y = 4x^2$ models the shape of the dish when it is oriented in this position. Later in the day, the dish is observed to have rotated approximately 45°. Find an equation that models the new orientation of the dish. *(Lesson 10-7)*

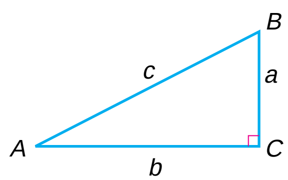
59. Graph the system of polar equations $r = 2$, and $r = 2\cos 2\theta$. Then solve the system and compare the common solutions. *(Lesson 9-2)*

60. Find the initial vertical and horizontal velocities of a rock thrown with an initial velocity of 28 feet per second at an angle of 45° with the horizontal. *(Lesson 8-7)*

61. Solve $\sin 2x + 2\sin x = 0$ for $0° \le x \le 360°$. *(Lesson 7-5)*

62. State the amplitude, period, and phase shift for the function $y = 8\cos(\theta - 30°)$. *(Lesson 6-5)*

63. Given the triangle at the right, solve the triangle if $A = 27°$ and $b = 15.2$. Round angle measures to the nearest degree and side measures to the nearest tenth. *(Lesson 5-5)*



64. **SAT/ACT Practice** What is the number of degrees through which the hour hand of a clock moves in 2 hours 12 minutes?

A 66° **B** 72° **C** 126° **D** 732° **E** 792°

Extra Practice See p. A51.