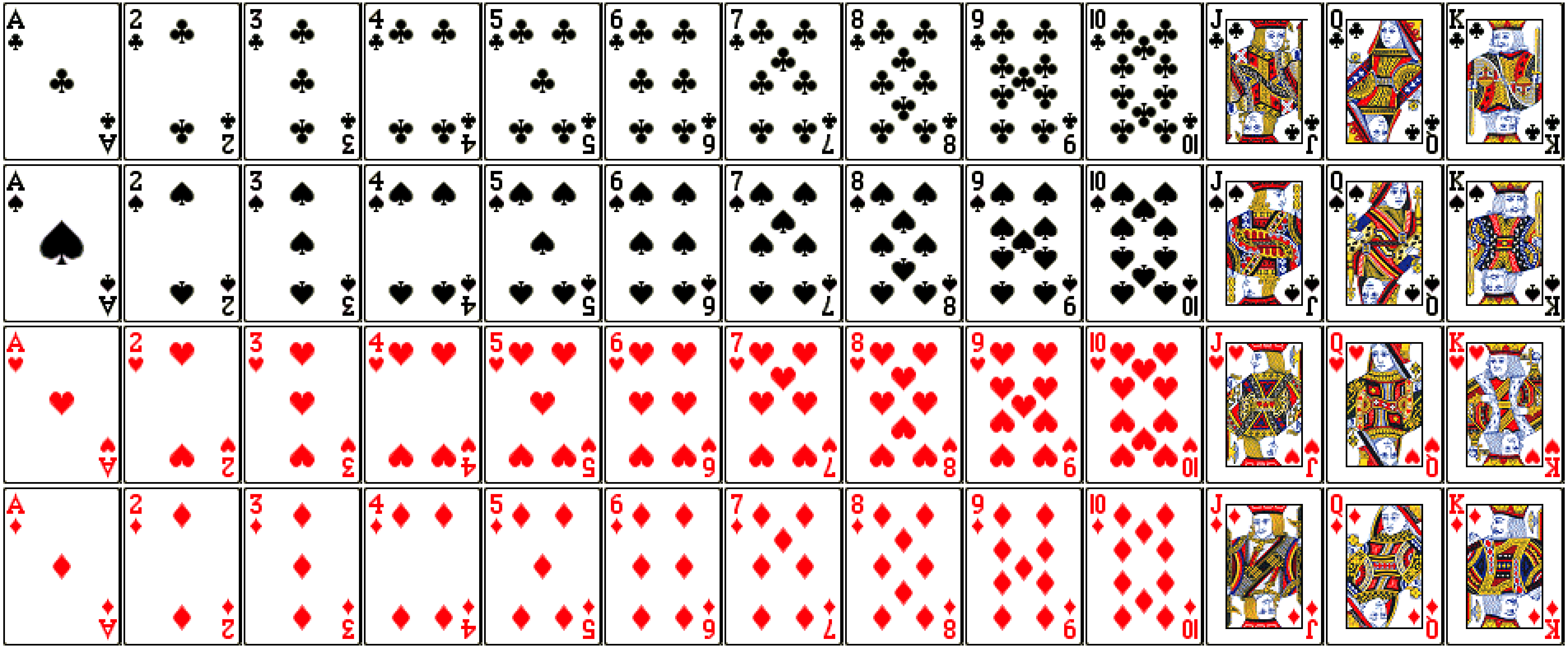
13.3 Permutations and Combinations



There are 6 people who want to use an elevator. There is only room for 4 people. How many ways can 6 people try to fill this elevator (one at a time)?

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$\underline{6} \times \underline{5} \times \underline{4} \times \underline{3}$

$= 360$

Permutations

DEFINITION A **permutation** is an ordering of distinct objects in a straight line. If we select r *different* objects from a set of n objects and arrange them in a straight line, this is called a *permutation of n objects taken r at a time.* The number of permutations of n objects taken r at a time is denoted by $P(n, r)$.

$$P(n, r)$$

P reminds you of the word *permutation.*

r is the number of objects that you are selecting.

n is the number of objects from which you may select.

There are 6 people who want to use an elevator. There is only room for 4 people. How many ways can 6 people try to fill this elevator (one at a time)?

P(6, 4) =

$\underline{6} \times \underline{5} \times \underline{4} \times \underline{3}$

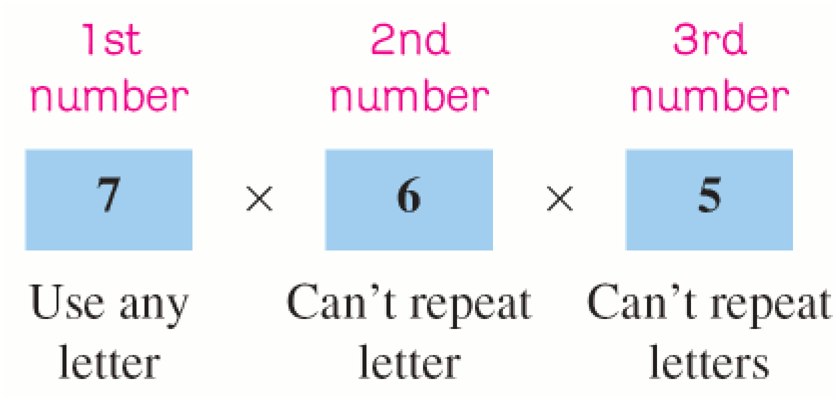
= 360

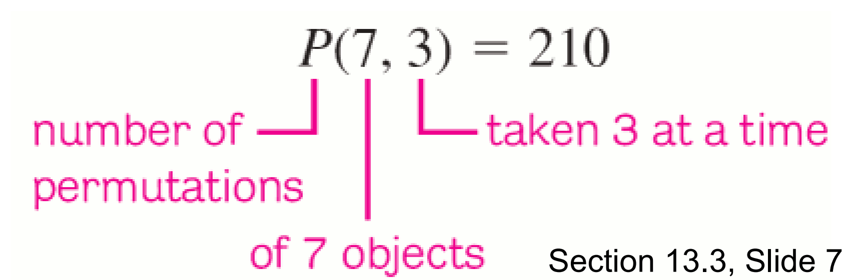
Permutations

- Example: How many permutations are there of the letters a, b, c, d, e, f, and g if we take the letters three at a time? Write the answer using $P(n, r)$ notation.

Permutations

- Example: How many permutations are there of the letters *a*, *b*, *c*, *d*, *e*, *f*, and *g* if we take the letters three at a time? Write the answer using *P*(*n*, *r*) notation.

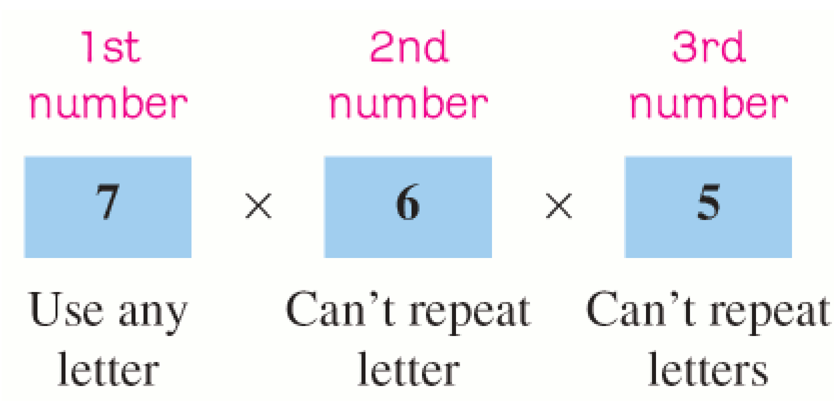


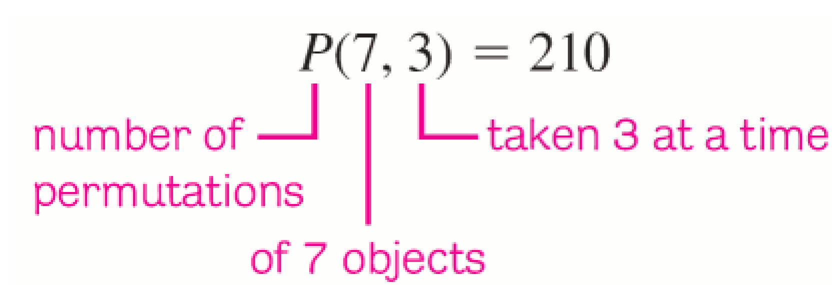


P(n,r) describes a slot diagram.

n = number in first slot
r = number of slots

n	(n-1)	(n-2)	(n-3)	…	(last #)
1^{st}	2^{nd}	3^{rd}	4^{th}		r^{th}





How many ways are there to arrange 5 books on a bookshelf?

How many ways are there to arrange 5 books on a bookshelf?

P(5,5) =

$\underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$

= 120

Shortcut/Defintion

DEFINITION If n is a counting number, the symbol $n!$, called *n factorial*, stands for the product $n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \cdots \cdot 2 \cdot 1$. We define $0! = 1$.

Example: 5! = 5x4x3x2x1

Example:

Compute $(5 - 2)!$

Example:

Compute (5 - 2)!

$(5-2)! = 3! = 3 \times 2 \times 1 = 6$

Factorial Notation

- Example: Compute $\dfrac{8!}{5!3!}$.

Factorial Notation

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- Solution:

$$\frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 8 \cdot 7 = 56.$$

Cancel 5!. Cancel 3!, which equals 6.

Factorial Notation

FORMULA FOR COMPUTING $P(n, r)$

$$P(n, r) = \frac{n!}{(n-r)!}$$

There are 6 people who want to use an elevator. There is only room for 4 people. How many ways can 6 people try to fill this elevator (one at a time)?

$P(6, 4) =$

$= 360$

When we care about the order of objects, like books on a bookshelf, we have a permutation.

When we do not care about the order of objects, like 2 people wining a raffle, we have a combination.

Combinations

FORMULA FOR COMPUTING $C(n, r)$ If we choose r objects from a set of n objects, we say that we are forming a **combination** of n objects taken r at a time. The notation $C(n, r)$ denotes the number of such combinations.† Also,

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r! \cdot (n - r)!}.$$

Example: A person would like to run 4 errands, but only has time for 2. How many pairs of errands could be tried?

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Order does not matter = combination.

$$C(4,2) = \frac{4!}{(4-2)!\ 2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$$

Combinations

- Example: How many three-element sets can be chosen from a set of five objects?
- Solution: Order is not important, so it is clear that this is a combination problem.

$$C(5, 3) = \frac{5!}{3! \cdot (5 - 3)!} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot 2 \cdot 1} = \frac{20}{2} = 10$$

Combinations

- Example: How many four-person committees can be formed from a set of 10 people?

Combinations

- Example: How many four-person committees can be formed from a set of 10 people?
- Solution: Order is not important, so it is clear that this is a combination problem.

$$C(10, 4) = \frac{10!}{4! \cdot (10-4)!} = \frac{10 \cdot \overset{3}{\cancel{9}} \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 210$$

Example: At a vation spot there are 7 sites to visit, but you only have time for 5. How many different combinations do you have to choose from?

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Order does not matter = combination.

$C(7,5) = 21$

Combinations

- Example: In the game of poker, five cards are drawn from a standard 52-card deck. How many different poker hands are possible?

Combinations

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- Solution:

$$C(52, 5) = \frac{52!}{5!47!} = \frac{\overset{13}{\cancel{52}} \cdot \overset{17}{\cancel{51}} \cdot \overset{10}{\cancel{50}} \cdot 49 \cdot \overset{24}{\cancel{48}}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 2{,}598{,}960$$

Combinations

- Example: In the game of bridge, a hand consists of 13 cards drawn from a standard 52-card deck. How many different bridge hands are there?

Combinations

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- Solution:

$$C(52, 13) = \frac{52!}{13!39!} = 635{,}013{,}559{,}600$$

Combining counting methods.

Sometimes you will have more than one counting idea to find the total number of possibilities.

Example:

2 men and 2 women from a firm will attend a conference. The firm has 10 men and 9 women to choose from. How many group possibilities are there?

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1^{st} task : choose 2 men from 10

2^{nd} task : choose 2 women from 10

Use a slot diagram $\underset{1^{st}}{\underline{\quad}} \times \underset{2^{nd}}{\underline{\quad}}$

Combining Counting Methods

Stage 1: Select the two women from the nine available.

$$C(9, 2) = \frac{9!}{2!7!} = 36 \text{ ways}$$

Stage 2: Select the two men from the ten available.

$$C(10, 2) = \frac{10!}{2!8!} = 45 \text{ ways}$$

Thus, choosing the women and then choosing the men can be done in $36 \cdot 45 = 1{,}620$ ways.

Example:
How many different outcomes are there for rolling a die and then drawing 2 cards from a deck of cards?

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1^{st} task : roll a die

2^{nd} task : draw 2 cards from 52
(order does not matter)

Use a slot diagram $\underset{1^{st}}{\_\_\_} \times \underset{2^{nd}}{\_\_\_}$

Example:

How many different outcomes are there for rolling a die and then drawing 2 cards from a deck of cards?

1^{st} task : roll a die = 6 ways

2^{nd} task : draw 2 cards from 52 = C(52,2)
(order does not matter)

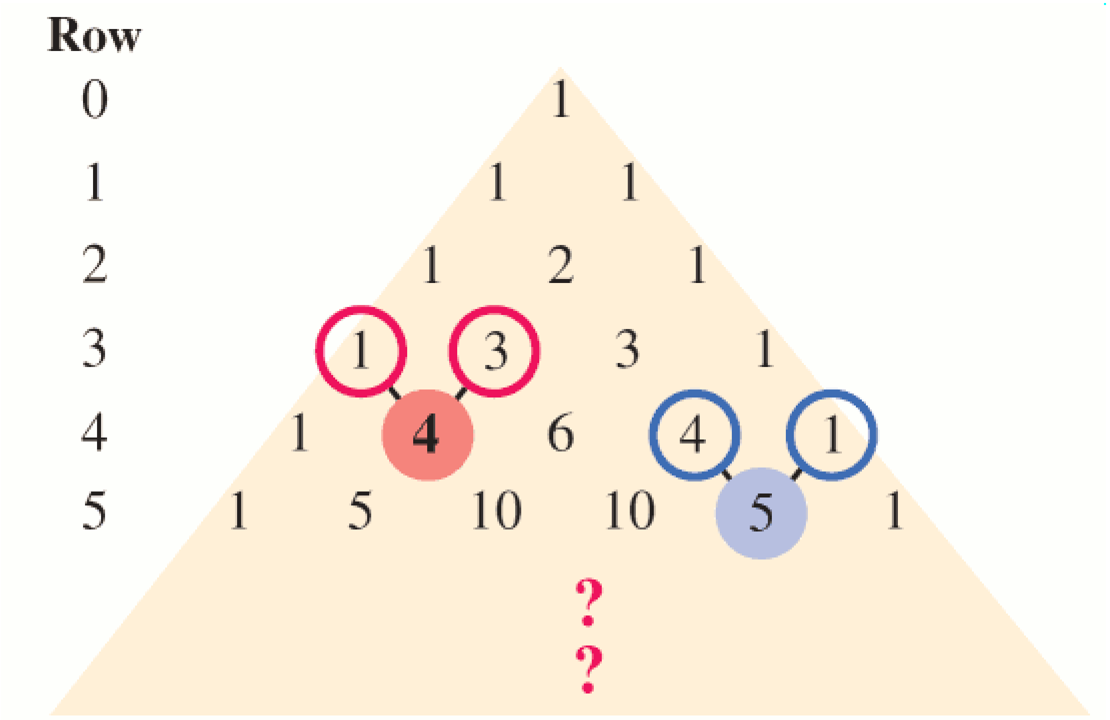
Use a slot diagram $\underset{1^{st}}{\underline{\;6\;}} \times \underset{2^{nd}}{\underline{1326}} = 7956$

Pascal's Triangle

- numbers are written on diagonals
- on the outsides write '1'
- on the inside each number is the sum of the numbers to its upper left and upper right.

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

Combining Counting Methods



Pascal's triangle.

Combining Counting Methods

PASCAL'S TRIANGLE COUNTS THE SUBSETS OF A SET The *n*th row of Pascal's triangle counts the subsets of various sizes of an *n*-element set.

For example, consider the set {1, 2, 3, 4} and the 4th row of Pascal's triangle: 1 4 6 4 1.

$\varnothing$ —— 1 zero-element set

{1}, {2}, {3}, {4} —— 4 one-element sets

{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4} —— 6 two-element sets

{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4} —— 4 three-element sets

{1, 2, 3, 4} —— 1 four-element set

Combining Counting Methods

ENTRIES OF PASCAL'S TRIANGLE AS $C(n, r)$ The rth entry of the nth row of Pascal's triangle is $C(n, r)$.

For example, consider the 4th row of Pascal's triangle: 1 4 6 4 1.

$$C(4, 0) = 1$$

$$C(4, 1) = 4$$

$$C(4, 2) = 6$$

$$C(4, 3) = 4$$

$$C(4, 4) = 1$$

Combining Counting Methods

- Example: Assume that a pharmaceutical company has developed five antibiotics and four immune system stimulators. In how many ways can we choose a treatment program consisting of three antibiotics and two immune system stimulators to treat a disease? Use Pascal's triangle to speed your computations.
- Solution: We will count this in two stages: (a) choosing the antibiotics, (b) choosing the immune system simulators.

(continued on next slide)

1st – choose 3 antibiotics from 5

2nd – choose 2 immune system simulators from 4

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Combining Counting Methods

Stage 1: Choosing 3 antibiotics from 5 can be done in *C*(5, 3) ways.

1	5	10	10	5	1
$C(5, 0)$	$C(5, 1)$	$C(5, 2)$	$C(5, 3)$		

Stage 2: Choosing 2 immune system simulators from 4 can be done in *C*(4, 2) ways.

1	4	6	4	1
$C(4, 0)$	$C(4, 1)$	$C(4, 2)$		

Total: $C(5, 3) \times C(4, 2) = 10 \times 6 = 60$ ways.