14.6 – Directional derivative and gradient
University of Massachusetts Amherst
Math 233 – Fall 2013

Recall that if $z = f(x, y)$, the partial derivatives f_x and f_y represent the rates of change of f in the x- and y-directions. That is, the partial derivatives give the rates of change in the directions of the unit vectors $\vec{i}$ and $\vec{j}$.

However, it is often convenient to compute the rates of change of f in an arbitrary direction.

Definition 1. The *directional derivative* of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

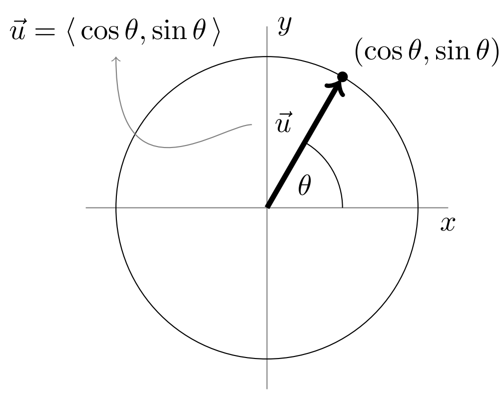
if this limit exists.

This definition is consistent with our previous notion of partial derivatives. Namely, $D_{\vec{i}} f(x, y) = f_x(x, y)$ and $D_{\vec{j}} f(x, y) = f_y(x, y)$.

Theorem 1. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ and*

$$D_{\vec{u}} f(x, y) = a f_x(x, y) + b f_y(x, y).$$

Note. Each unit vector corresponds to a point on the unit circle, and therefore can be represented by an angle.



Example 1. Find the directional derivative $D_{\vec{u}} f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\vec{u}$ is the unit vector given by the angle $\theta = \pi/6$. What is $D_{\vec{u}} f(1, 2)$?

The partial derivatives of $f(x,y)$ define an important vector.

Definition 2. If $f(x,y)$ is a function of two variables, then the *gradient* of f is the vector function ∇f defined by

$$\nabla f(x,y) = \langle\, f_x(x,y), f_y(x,y)\,\rangle = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}.$$

Example 2. Let $f(x,y) = \sin(x) + e^{xy}$. Compute $\nabla f(x,y)$ and $\nabla f(0,1)$.

The directional derivative given in Theorem 1 can also be expressed in terms of the gradient.

Corollary 2. *The directional derivative of a function $f(x,y)$ in the direction of a unit vector $\vec{u}$ is*

$$D_{\vec{u}} f(x,y) = \nabla f(x,y)\cdot\vec{u}.$$

Example 3. Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector $\vec{v} = \langle\, 2, 5\,\rangle$.

Definition 3. The gradient of a function of three variables $f(x,y,z)$ is

$$\begin{aligned}\nabla f(x,y,z) &= \langle\, f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\,\rangle \\ &= f_x(x,y,z)\,\vec{i} + f_y(x,y,z)\,\vec{j} + f_z(x,y,z)\,\vec{k}.\end{aligned}$$

Moreover, the directional derivative along a unit vector $\vec{u} = \langle\, a, b, c\,\rangle$ is

$$\begin{aligned}D_{\vec{u}} f(x,y,z) &= \nabla f(x,y,z)\cdot\vec{u} \\ &= af_x(x,y,z)\,\vec{i} + bf_y(x,y,z)\,\vec{j} + cf_z(x,y,z)\,\vec{k}.\end{aligned}$$

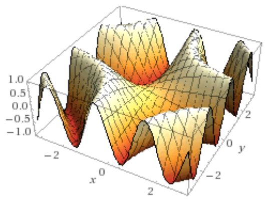
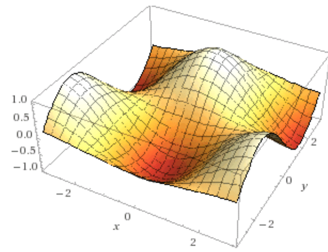
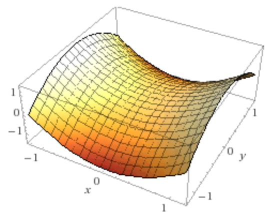
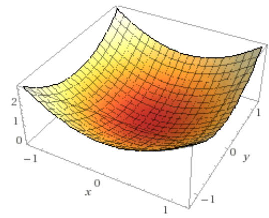
Example 4. If $f(x,y,z) = x\sin(yz)$, find the ∇f and the directional derivative of f at $(1,3,0)$ in the direction of $\vec{v} = \langle\, 1, 2, -1\,\rangle$.

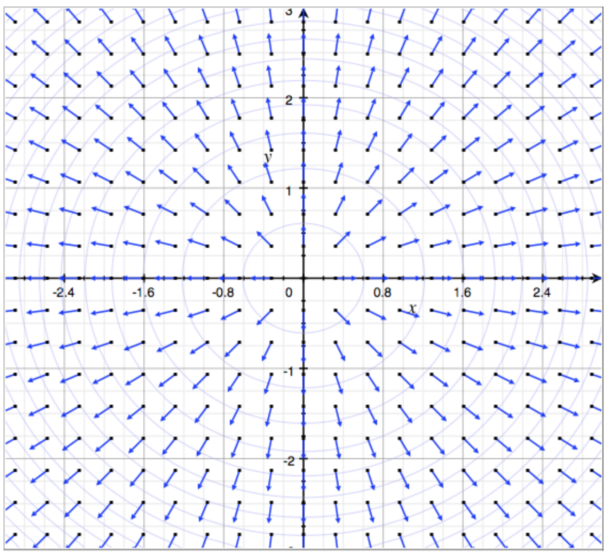
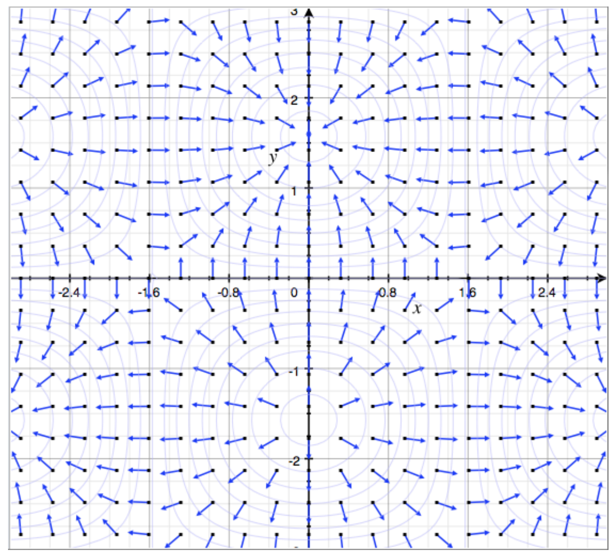
Theorem 3. *Let f be a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\vec{u}}f$ is $|\nabla f|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector ∇f. That is, the rate of change in f is greatest in the direction of the gradient vector ∇f.*

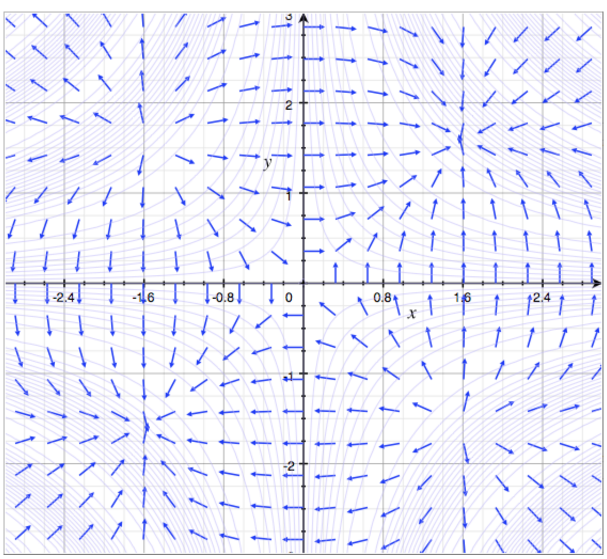
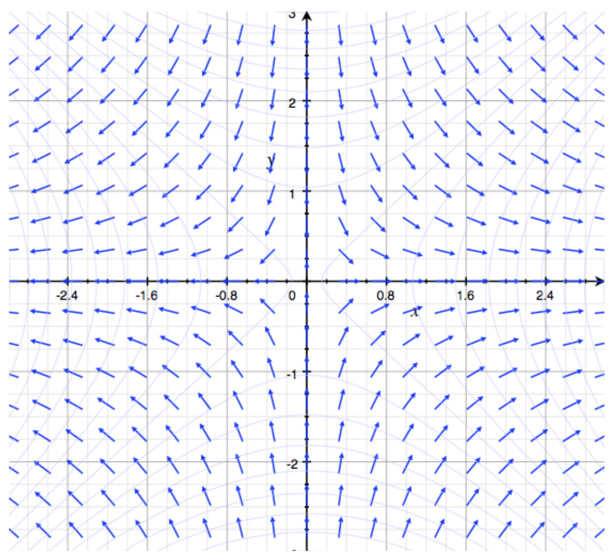
Example 5. Let $f(x,y) = xe^y$.

(1) Find the rate of change of f at the point $P(2,0)$ in the direction of $Q(\frac{1}{2}, 2)$.
(2) In what direction does f have the maximum rate of change?
(3) What is this maximum rate of change?

Example 6. Match the each surfaces with its gradient vector field.

Definition 4. Let $f(x, y, z) = k$ be a level surface of f, and let $P(x_0, y_0, z_0)$ be a point on this level surface. The *tangent plane to the level surface at* P is given by the equation

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Example 7. Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

(The normal line is the line orthogonal to the tangent plane at the point of tangency.)