14 Dimensional Analysis and Similitude

Introduction

In the process of constructing a mathematical model, we have seen that the variables influencing the behavior must be identified and classified. We must then determine appropriate relationships among those variables retained for consideration. In the case of a single dependent variable this procedure gives rise to some unknown function:

$$y = f(x_1, x_2, \ldots, x_n)$$

where the x_i measure the various factors influencing the phenomenon under investigation. In some situations the discovery of the nature of the function f for the chosen factors comes about by making some reasonable assumption based on a law of nature or previous experience and construction of a mathematical model. We were able to use this methodology in constructing our model on vehicular stopping distance (see Section 2.2). On the other hand, especially for those models designed to predict some physical phenomenon, we may find it difficult or impossible to construct a solvable or tractable explicative model because of the inherent complexity of the problem. In certain instances we might conduct a series of experiments to determine how the dependent variable y is related to various values of the independent variable(s). In such cases we usually prepare a figure or table and apply an appropriate curve-fitting or interpolation method that can be used to predict the value of y for suitable ranges of the independent variable(s). We employed this technique in modeling the elapsed time of a tape recorder in Sections 4.2 and 4.3.

Dimensional analysis is a method for helping determine how the selected variables are related and for reducing significantly the amount of experimental data that must be collected. It is based on the premise that physical quantities have dimensions and that physical laws are not altered by changing the units measuring dimensions. Thus, the phenomenon under investigation can be described by a dimensionally correct equation among the variables. A dimensional analysis provides qualitative information about the model. It is especially important when it is necessary to conduct experiments in the modeling process because the method is helpful in testing the validity of including or neglecting a particular factor, in reducing the number of experiments to be conducted to make predictions, and in improving the usefulness of the results by providing alternatives for the parameters employed to present them. Dimensional analysis has proved useful in physics and engineering for many years and even now plays a role in the study of the life sciences, economics, and operations research. Let's consider an example illustrating how dimensional analysis can be used in the modeling process to increase the efficiency of an experimental design.

Introductory Example: A Simple Pendulum

Consider the situation of a simple pendulum as suggested in Figure 14.1. Let r denote the length of the pendulum, m its mass, and θ the initial angle of displacement from the vertical. One characteristic that is vital in understanding the behavior of the pendulum is the **period**, which is the time required for the pendulum bob to swing through one complete cycle and return to its original position (as at the beginning of the cycle). We represent the period of the pendulum by the dependent variable t.

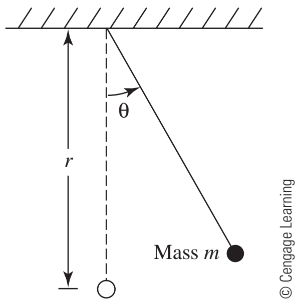


Figure 14.1
A simple pendulum

Problem Identification *For a given pendulum system, determine its period.*

Assumptions First, we list the factors that influence the period. Some of these factors are the length r, the mass m, the initial angle of displacement θ, the acceleration due to gravity g, and frictional forces such as the friction at the hinge and the drag on the pendulum. Assume initially that the hinge is frictionless, that the mass of the pendulum is concentrated at one end of the pendulum, and that the drag force is negligible. Other assumptions about the frictional forces will be examined in Section 14.3. Thus, the problem is to determine or approximate the function

$$t = f(r, m, \theta, g)$$

and test its worthiness as a predictor.

Experimental Determination of the Model Because gravity is essentially constant under the assumptions, the period t is a function of the three variables: length r, mass m, and initial angle of displacement θ. At this point we could systematically conduct experiments to determine how t varies with these three variables. We would want to choose enough values of the independent variables to feel confident in predicting the period t over that range. How many experiments will be necessary?

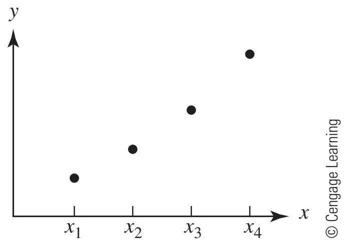
For the sake of illustration, consider a function of one independent variable $y = f(x)$, and assume that four points have been deemed necessary to predict y over a suitable domain for x. The situation is depicted in Figure 14.2. An appropriate curve-fitting or interpolation method could be used to predict y within the domain for x.

Next consider what happens when a second independent variable affects the situation under investigation. We then have a function

$$y = f(x, z)$$

Figure 14.2

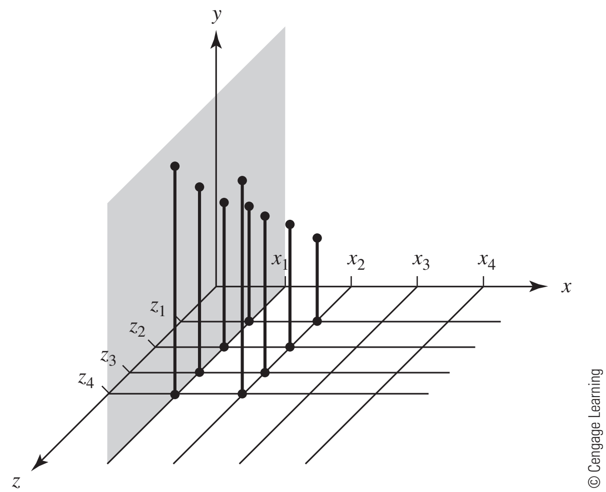
Four points have been deemed necessary to predict y for this function of one variable x.



For each data value of x in Figure 14.2, experiments must be conducted to obtain y for four values of z. Thus, 16 (that is, 4^2) experiments are required. These observations are illustrated in Figure 14.3. Likewise, a function of three variables requires 64 (that is, 4^3) experiments. In general, 4^n experiments are required to predict y when n is the number of arguments of the function, assuming four points for the domain of each argument. Thus, a procedure that reduces the number of arguments of the function f will dramatically reduce the total number of required experiments. Dimensional analysis is one such procedure.

Figure 14.3

Sixteen points are necessary to predict y for this function of the two variables x and z.



The power of dimensional analysis is also apparent when we examine the interpolation curves that would be determined after collecting the data represented in Figures 14.2 and 14.3. Let's assume it is decided to pass a cubic polynomial through the four points shown in Figure 14.2. That is, the four points are used to determine the four constants C_1–C_4 in the interpolating curve:

$$y = C_1x^3 + C_2x^2 + C_3x + C_4$$

Now consider interpolating from Figure 14.3. If for a fixed value of x, say $x = x_1$, we decide to connect our points using a cubic polynomial in z, the equation of the interpolating surface is

$$\begin{aligned} y = {} & D_1x^3 + D_2x^2 + D_3x + D_4 + (D_5x^3 + D_6x^2 + D_7x + D_8)z \\ & + (D_9x^3 + D_{10}x^2 + D_{11}x + D_{12})z^2 + (D_{13}x^3 + D_{14}x^2 + D_{15}x + D_{16})z^3 \end{aligned}$$

Note from the equation that there are 16 constants—$D_1, D_2, \ldots, D_{16}$—to determine, rather than 4 as in the two-dimensional case. This procedure again illustrates the dramatic reduction in effort required when we reduce the number of arguments of the function we will finally investigate.

At this point we make the important observation that the experimental effort required depends more heavily on the number of arguments of the function to be investigated than on the true number of independent variables the modeler originally selected. For example, consider a function of two arguments, say $y = f(x, z)$. The discussion concerning the number of experiments necessary would not be altered if x were some particular combination of several variables. That is, x could be uv/w, where u, v, and w are the variables originally selected in the model.

Consider now the following preview of dimensional analysis, which describes how it reduces our experimental effort. Beginning with a function of n variables (hence, n arguments), the number of arguments is reduced (ordinarily by three) by combining the original variables into products. These resulting $(n-3)$ products are called **dimensionless products** of the original variables. After applying dimensional analysis, we still need to conduct experiments to make our predictions, but the amount of experimental effort that is required will have been reduced exponentially.

In Chapter 2 we discussed the trade-offs of considering additional variables for increased precision versus neglecting variables for simplification. In constructing models based on experimental data, the preceding discussion suggests that the cost of each additional variable is an exponential increase in the number of experimental trials that must be conducted. In the next two sections, we present the main ideas underlying the dimensional analysis process. You may find that some of these ideas are slightly more difficult than some that we have already investigated, but the methodology is powerful when modeling physical behavior.

14.1 Dimensions as Products

The study of physics is based on abstract concepts such as mass, length, time, velocity, acceleration, force, energy, work, and pressure. To each such concept a unit of measurement is assigned. A physical law such as $F = ma$ is true, provided that the units of measurement are consistent. Thus, if mass is measured in kilograms and acceleration in meters per second squared, then the force must be measured in newtons. These units of measurement belong to the MKS (meter–kilogram–second) mass system. It would be inconsistent with the equation $F = ma$ to measure mass in slugs, acceleration in feet per second squared, and force in newtons. In this illustration, force must be measured in pounds, giving the American Engineering System of measurement. There are other systems of measurement, but all are prescribed by international standards so as to be consistent with the laws of physics.

The three primary physical quantities we consider in this chapter are mass, length, and time. We associate with these quantities the dimensions M, L, and T, respectively. The dimensions are symbols that reveal how the numerical value of a quantity changes when the units of measurement change in certain ways. The dimensions of other quantities follow from definitions or from physical laws and are expressed in terms of M, L, and

T. For example, velocity v is defined as the ratio of distance s (dimension L) traveled to time t (dimension T) of travel—that is, $v = st^{-1}$, so the dimension of velocity is LT^{-1}. Similarly, because area is fundamentally a product of two lengths, its dimension is L^2. These dimension expressions hold true regardless of the particular system of measurement, and they show, for example, that velocity may be expressed in meters per second, feet per second, miles per hour, and so forth. Likewise, area can be measured in terms of square meters, square feet, square miles, and so on.

There are still other entities in physics that are more complex in the sense that they are not usually defined directly in terms of mass, length, and time alone; instead, their definitions include other quantities, such as velocity. We associate dimensions with these more complex quantities in accordance with the algebraic operations involved in the definitions. For example, because momentum is the product of mass with velocity, its dimension is $M(LT^{-1})$ or simply MLT^{-1}.

The basic definition of a quantity may also involve dimensionless constants; these are ignored in finding dimensions. Thus, the dimension of kinetic energy, which is one-half (a dimensionless constant) the product of mass with velocity squared, is $M(LT^{-1})^2$ or simply ML^2T^{-2}. As you will see in Example 2, some constants (dimensional constants), such as gravity g, do have an associated dimension, and these must be considered in a dimensional analysis.

These examples illustrate the following important concepts regarding dimensions of physical quantities.

1. We have based the concept of dimension on three physical quantities: mass m, length s, and time t. These quantities are measured in some appropriate system of units whose choice does not affect the assignment of dimensions. (This underlying system must be linear. A dimensional analysis will not work if the scale is logarithmic, for example.)
2. There are other physical quantities, such as area and velocity, that are defined as simple products involving only mass, length, or time. Here we use the term **product** to include any quotient because we may indicate division by negative exponents.
3. There are still other, more complex physical entities, such as momentum and kinetic energy, whose definitions involve quantities other than mass, length, and time. Because the simpler quantities from (1) and (2) are products, these more complex quantities can also be expressed as products involving mass, length, and time by algebraic simplification. We use the term *product* to refer to any physical quantity from item (1), (2), or (3); a product from (1) is trivial because it has only one factor.
4. To each product, there is assigned a **dimension**—that is, an expression of the form

$$M^n L^p T^q \tag{14.1}$$

where n, p, and q are real numbers that may be positive, negative, or zero.

When a basic dimension is missing from a product, the corresponding exponent is understood to be zero. Thus, the dimension $M^2L^0T^{-1}$ may also appear as M^2T^{-1}. When n, p, and q are all zero in an expression of the form (14.1), so that the dimension reduces to

$$M^0L^0T^0 \tag{14.2}$$

the quantity, or product, is said to be *dimensionless*.

Special care must be taken in forming sums of products because just as we cannot add apples and oranges, in an equation we cannot add products that have unlike dimensions. For example, if F denotes force, m mass, and v velocity, we know immediately that the equation

$$F = mv + v^2$$

cannot be correct because mv has dimension MLT^{-1}, whereas v^2 has dimension L^2T^{-2}. These dimensions are unlike; hence, the products mv and v^2 cannot be added. An equation such as this—that is, one that contains among its terms two products having unlike dimensions—is said to be *dimensionally incompatible*. Equations that involve only sums of products having the same dimension are *dimensionally compatible*.

The concept of dimensional compatibility is related to another important concept called **dimensional homogeneity**. In general, an equation that is true regardless of the system of units in which the variables are measured is said to be dimensionally homogeneous. For example, $t = \sqrt{2s/g}$ giving the time a body falls a distance s under gravity (neglecting air resistance) is dimensionally homogeneous (true in all systems), whereas the equation $t = \sqrt{s/16.1}$ is not dimensionally homogeneous (because it depends on a particular system). In particular, if an equation involves only sums of dimensionless products (i.e., it is a dimensionless equation), then the equation is dimensionally homogeneous. Because the products are dimensionless, the factors used for conversion from one system of units to another would simply cancel.

The application of dimensional analysis to a real-world problem is based on the assumption that the solution to the problem is given by a dimensionally homogeneous equation in terms of the appropriate variables. Thus, the task is to determine the form of the desired equation by finding an appropriate dimensionless equation and then solving for the dependent variable. To accomplish this task, we must decide which variables enter into the physical problem under investigation and determine all the dimensionless products among them. In general, there may be infinitely many such products, so they will have to be described rather than actually written out. Certain subsets of these dimensionless products are then used to construct dimensionally homogeneous equations. In Section 14.2 we investigate how the dimensionless products are used to find all dimensionally homogeneous equations. The following example illustrates how the dimensionless products may be found.

EXAMPLE 1 *A Simple Pendulum Revisited*

Consider again the simple pendulum discussed in the introduction. Analyzing the dimensions of the variables for the pendulum problem, we have

Variable	m	g	t	r	θ
Dimension	M	LT^{-2}	T	L	$M^0L^0T^0$

Next we find all the dimensionless products among the variables. Any product of these variables must be of the form

$$m^a g^b t^c r^d \theta^e \tag{14.3}$$

and hence must have dimension

$$(M)^a (LT^{-2})^b (T)^c (L)^d (M^0 L^0 T^0)^e$$

Therefore, a product of the form (14.3) is dimensionless if and only if

$$M^a L^{b+d} T^{c-2b} = M^0 L^0 T^0 \tag{14.4}$$

Equating the exponents on both sides of this last equation leads to the system of linear equations

$$\left.\begin{array}{lllll} a & & & + 0e & = 0 \\ & b & + d & + 0e & = 0 \\ & -2b + c & & + 0e & = 0 \end{array}\right\} \tag{14.5}$$

Solution of the system (14.5) gives $a = 0, c = 2b, d = -b$, where b is arbitrary. Thus, there are infinitely many solutions. Here are some general rules for selecting arbitrary variables: (1) Choose the dependent variable so it will appear only once, (2) select any variable that expedites the solution of the other equations (i.e., a variable that appears in all equations), and (3) choose a variable that always has a zero coefficient, if possible. Notice that the exponent e does not really appear in (14.4) (because it has a zero coefficient in each equation) so it is also arbitrary. One dimensionless product is obtained by setting $b = 0$ and $e = 1$, yielding $a = c = d = 0$. A second, independent dimensionless product is obtained when $b = 1$ and $e = 0$, yielding $a = 0$, $c = 2$, and $d = -1$. These solutions give the dimensionless products

$$\prod_1 = m^0 g^0 t^0 r^0 \theta^1 = \theta$$

$$\prod_2 = m^0 g^1 t^2 r^{-1} \theta^0 = \frac{gt^2}{r}$$

In Section 14.2, we will learn a methodology for relating these products to carry the modeling process to completion. For now, we will develop a relationship in an intuitive manner.

Assuming $t = f(r, m, g, \theta)$, to determine more about the function f, we observe that if the units in which we measure mass are made smaller by some factor (e.g., 10), then the measure of the period t will not change because it is measured in units (T) of time. Because m is the only factor whose dimension contains M, it cannot appear in the model. Similarly, if the scale of the units (L) for measuring length is altered, it cannot change the measure of the period. For this to happen, the factors r and g must appear in the model as r/g, g/r, or, more generally, $(g/r)^k$. This ensures that any linear change in the way length is measured will be canceled. Finally, if we make the units (T) that measure time smaller by a factor of 10, for example, the measure of the period will directly increase by this same factor 10. Thus, to have the dimension of T on the right side of the equation $t = f(r, m, g, \theta)$, g and r must appear as $\sqrt{r/g}$ because T appears to the power -2 in the dimension of g. Note that none of the preceding conditions places any restrictions on the angle θ. Thus, the equation of the period should be of the form

$$t = \sqrt{\frac{r}{g}}\, h(\theta)$$

where the function h must be determined or approximated by experimentation.

We note two things in this analysis that are characteristic of a dimensional analysis. First, in the MLT system, three conditions are placed on the model, so we should generally expect to reduce the number of arguments of the function relating the variables by three. In the pendulum problem we reduced the number of arguments from four to one. Second, all arguments of the function present at the end of a dimensional analysis (in this case, θ) are dimensionless products.

In the problem of the undamped pendulum we assumed that friction and drag were negligible. Before proceeding with experiments (which might be costly), we would like to know whether that assumption is reasonable. Consider the model obtained so far:

$$t = \sqrt{\frac{r}{g}}\, h(\theta)$$

Keeping θ constant while allowing r to vary, form the ratio

$$\frac{t_1}{t_2} = \frac{\sqrt{r_1/g}\, h(\theta_0)}{\sqrt{r_2/g}\, h(\theta_0)} = \sqrt{\frac{r_1}{r_2}}$$

Hence the model predicts that t will vary as $\sqrt{r}$ for constant θ. Thus, if we plot t versus r with fixed θ for some observations, we will expect to get a straight line (Figure 14.4). If we do not obtain a reasonably straight line, then we need to reexamine the assumptions. Note that our judgment here is qualitative. The final measure of the adequacy of any model is always how well it predicts or explains the phenomenon under investigation. Nevertheless, this initial test is useful for eliminating obviously bad assumptions and for choosing among competing sets of assumptions.

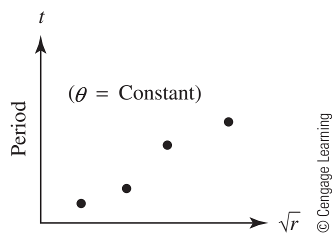


Figure 14.4
Testing the assumptions of the simple pendulum model by plotting the period t versus the square root of the length r for constant displacement θ

Dimensional analysis has helped construct a model $t = f(r, m, g, \theta)$ for the undamped pendulum as $t = \sqrt{r/g}\, h(\theta)$. If we are interested in predicting the behavior of the pendulum, we could isolate the effect of h by holding r constant and varying θ. This provides the ratio

$$\frac{t_1}{t_2} = \frac{\sqrt{r_0/g}\, h(\theta_1)}{\sqrt{r_0/g}\, h(\theta_2)} = \frac{h(\theta_1)}{h(\theta_2)}$$

Hence a plot of t versus θ for several observations would reveal the nature of h. This plot is illustrated in Figure 14.5. We may never discover the true function h relating the variables. In such cases, an empirical model might be constructed from the experimental data, as discussed in Chapter 4. When we are interested in using our model to predict t, based on

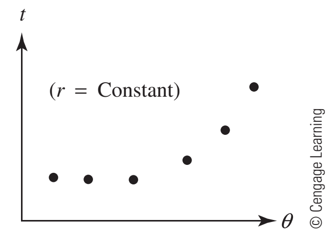


Figure 14.5
Determining the unknown function *h*

experimental results, it is convenient to use the equation $t\sqrt{g/r} = h(\theta)$ and to plot $t\sqrt{g/r}$ versus θ, as in Figure 14.6. Then, for a given value of θ, we would determine $t\sqrt{g/r}$, multiply it by $\sqrt{r/g}$ for a specific r, and finally determine t.

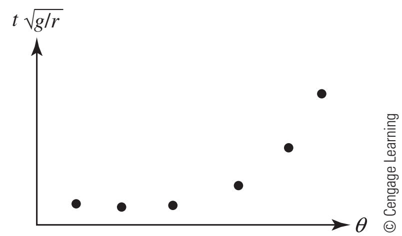


Figure 14.6
Presenting the results for the simple pendulum

EXAMPLE 2 *Wind Force on a Van*

Suppose you are driving a van down a highway with gusty winds. How does the speed of your vehicle affect the wind force you are experiencing?

The force F of the wind on the van is certainly affected by the speed v of the van and the surface area A of the van directly exposed to the wind's direction. Thus, we might hypothesize that the force is proportional to some power of the speed times some power of the surface area; that is,

$$F = kv^aA^b \tag{14.6}$$

for some (dimensionless) constant k. Analyzing the dimensions of the variables gives

Variable	F	k	v	A
Dimension	MLT^{-2}	$M^0L^0T^0$	LT^{-1}	L^2

Hence, dimensionally, Equation (14.6) becomes

$$MLT^{-2} = (M^0L^0T^0)(LT^{-1})^a(L^2)^b$$

This last equation cannot be correct because the dimension M for mass does not enter into the right-hand side with nonzero exponent.

So consider again Equation (14.6). What is missing in our assumption concerning the wind force? Wouldn't the strength of the wind be affected by its density? After some reflection we would probably agree that density does have an effect. If we include the density ρ as a factor, then our refined model becomes

$$F = kv^a A^b \rho^c \tag{14.7}$$

Because density is mass per unit volume, the dimension of density is ML^{-3}. Therefore, dimensionally, Equation (14.7) becomes

$$MLT^{-2} = (M^0L^0T^0)(LT^{-1})^a(L^2)^b(ML^{-3})^c$$

Equating the exponents on both sides of this last equation leads to the system of linear equations:

$$\left.\begin{aligned} c &= 1 \\ a + 2b - 3c &= 1 \\ -a \quad &= -2 \end{aligned}\right\} \tag{14.8}$$

Solution of the system (14.8) gives $a = 2$, $b = 1$, and $c = 1$. When substituted into Equation (14.7), these values give the model

$$F = kv^2A\rho$$

At this point we make an important observation. When it was assumed that $F = kv^aA^b$, the constant was assumed to be dimensionless. Subsequently, our analysis revealed that for a particular medium (so ρ is constant)

$$F \propto Av^2$$

giving $F = k_1Av^2$. However, k_1 does have a dimension associated with it and is called a **dimensional constant**. In particular, the dimension of k_1 is

$$\frac{MLT^{-2}}{L^2(L^2T^{-2})} = ML^{-3}$$

Dimensional constants contain important information and must be considered when performing a dimensional analysis. We consider dimensional constants again in Section 14.3 when we investigate a damped pendulum.

If we assume the density ρ is constant, our model shows that the force of the wind is proportional to the square of the speed of the van times its surface area directly exposed to the wind. We can test the model by collecting data and plotting the wind force F versus v^2A to determine whether the graph approximates a straight line through the origin. This example illustrates one of the ways in which dimensional analysis can be used to test our assumptions and check whether we have a faulty list of variables identifying the problem. Table 14.1 gives a summary of the dimensions of some common physical entities.

Table 14.1 Dimensions of physical entities in the MLT system

Mass	M	Momentum	MLT^{-1}
Length	L	Work	ML^2T^{-2}
Time	T	Density	ML^{-3}
Velocity	LT^{-1}	Viscosity	$ML^{-1}T^{-1}$
Acceleration	LT^{-2}	Pressure	$ML^{-1}T^{-2}$
Specific weight	$ML^{-2}T^{-2}$	Surface tension	MT^{-2}
Force	MLT^{-2}	Power	ML^2T^{-3}
Frequency	T^{-1}	Rotational inertia	ML^2
Angular velocity	T^{-1}	Torque	ML^2T^{-2}
Angular acceleration	T^{-2}	Entropy	ML^2T^{-2}
Angular momentum	ML^2T^{-1}	Heat	ML^2T^{-2}
Energy	ML^2T^{-2}		

14.1 PROBLEMS

1. Determine whether the equation

$$s = s_0 + v_0 t - 0.5gt^2$$

is dimensionally compatible, if s is the position (measured vertically from a fixed reference point) of a body at time t, s_0 is the position at $t = 0$, v_0 is the initial velocity, and g is the acceleration caused by gravity.

2. Find a dimensionless product relating the torque τ (ML^2T^{-2}) produced by an automobile engine, the engine's rotation rate ψ (T^{-1}), the volume V of air displaced by the engine, and the air density ρ.

3. The various constants of physics often have physical dimensions (dimensional constants) because their values depend on the system in which they are expressed. For example, Newton's law of gravitation asserts that the attractive force between two bodies is proportional to the product of their masses divided by the square of the distance between them, or, symbolically,

$$F = \frac{Gm_1m_2}{r^2}$$

where G is the gravitational constant. Find the dimension of G so that Newton's law is dimensionally compatible.

4. Certain stars, whose light and radial velocities undergo periodic vibrations, are thought to be pulsating. It is hypothesized that the period t of pulsation depends on the star's radius r, its mass m, and the gravitational constant G. (See Problem 3 for the dimension of G.) Express t as a product of m, r, and G, so the equation

$$t = m^a r^b G^c$$

is dimensionally compatible.

5. In checking the dimensions of an equation, you should note that derivatives also possess dimensions. For example, the dimension of ds/dt is LT^{-1} and the dimension of d^2s/dt^2 is LT^{-2}, where s denotes distance and t denotes time. Determine whether the equation

$$\frac{dE}{dt} = \left[mr^2\left(\frac{d^2\theta}{dt^2}\right) mgr\sin\theta\right]\frac{d\theta}{dt}$$

for the time rate of change of total energy E in a pendulum system with damping force is dimensionally compatible.

6. For a body moving along a straight-line path, if the mass of the body is changing over time, then an equation governing its motion is given by

$$m\frac{dv}{dt} = F + u\frac{dm}{dt}$$

where m is the mass of the body, v is the velocity of the body, F is the total force acting on the body, dm is the mass joining or leaving the body in the time interval dt, and u is the velocity of dm at the moment it joins or leaves the body (relative to an observer stationed on the body). Show that the preceding equation is dimensionally compatible.

7. In humans, the hydrostatic pressure of blood contributes to the total blood pressure. The hydrostatic pressure P is a product of blood density ρ, height h of the blood column between the heart and some lower point in the body, and gravity g. Determine

$$P = k\rho^a h^b g^c$$

where k is a dimensionless constant.

8. Assume the force F opposing the fall of a raindrop through air is a product of viscosity μ, velocity v, and the diameter r of the drop. Assume that density is neglected. Find

$$F = k\mu^a v^b r^c$$

where k is a dimensionless constant.

14.1 PROJECT

1. Complete the requirements of "Keeping Dimension Straight," by George E. Strecker, UMAP 564. This module is a very basic introduction to the distinction between dimensions and units. It also provides the student with some practice in using dimensional arguments to properly set up solutions to elementary problems and to recognize errors.

14.2 The Process of Dimensional Analysis

In the preceding section we learned how to determine all dimensionless products among the variables selected in the problem under investigation. Now we investigate how to use the dimensionless products to find all possible dimensionally homogeneous equations among the variables. The key result is Buckingham's theorem, which summarizes the entire theory of dimensional analysis.

Example 1 in the preceding section shows that in general, many dimensionless products may be formed from the variables of a given system. In that example we determined every dimensionless product to be of the form

$$g^b t^{2b} r^{-b} \theta^e \tag{14.9}$$

where b and e are arbitrary real numbers. Each one of these products corresponds to a solution of the homogeneous system of linear algebraic equations given by Equation (14.5). The two products

$$\prod_1 = \theta \quad \text{and} \quad \prod_2 = \frac{gt^2}{r}$$

obtained when $b = 0$, $e = 1$, and $b = 1$, $e = 0$, respectively, are special in the sense that any of the dimensionless products (14.9) can be given as a product of some power of $\prod_1$ times some power of $\prod_2$. Thus, for instance,

$$g^3 t^6 r^{-3} \theta^{1/2} = \prod_1^{1/2} \prod_2^3$$

This observation follows from the fact that $b = 0$, $e = 1$ and $b = 1$, $e = 0$ represent, in some sense, independent solutions of the system (14.5). Let's explore these ideas further.

Consider the following system of m linear algebraic equations in the n unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned} \tag{14.10}$$

The symbols a_{ij} and b_i denote real numbers for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The numbers a_{ij} are called the **coefficients** of the system, and the b_i are referred to as the **constants**. The subscript i in the symbol a_{ij} refers to the ith equation of the system (14.10) and the subscript j refers to the jth unknown x_j to which a_{ij} belongs. Thus, the subscripts serve to locate a_{ij}. It is customary to read a_{13} as "a, one, three" and a_{42} as "a, four, two," for example, rather than "a, thirteen" and "a, forty-two."

A **solution** to the system (14.10) is a sequence of numbers $s_1, s_2, \ldots, s_n$ for which $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ solves each equation in the system. If $b_1 = b_2 = \cdots = b_m = 0$, the system (14.10) is said to be **homogeneous**. The solution $s_1 = s_2 = \cdots = s_n = 0$ always solves the homogeneous system and is called the **trivial solution**. For a homogeneous system there are two solution possibilities: either the trivial solution is the only solution or there are infinitely many solutions.

Whenever $s_1, s_2, \ldots, s_n$ and $s'_1, s'_2, \ldots, s'_n$ are solutions to the homogeneous system, the sequences $s_1 + s'_1, s_2 + s'_2, \ldots, s_n + s'_n$, and $cs_1, cs_2, \ldots, cs_n$ are also solutions for any constant c. These solutions are called the **sum** and **scalar multiple** of the original solutions, respectively. If S and S' refer to the original solutions, then we use the notations $S + S'$ to refer to their sum and cS to refer to a scalar multiple of the first solution. If $S_1, S_2, \ldots, S_k$ is a collection of k solutions to the homogeneous system, then the solution

$$c_1S_1 + c_2S_2 + \cdots + c_kS_k$$

is called a **linear combination** of the k solutions, where $c_1, c_2, \ldots, c_k$ are arbitrary real numbers. It is an easy exercise to show that any linear combination of solutions to the homogeneous system is still another solution to the system.

A set of solutions to a homogeneous system is said to be **independent** if no solution in the set is a linear combination of the remaining solutions in the set. A set of solutions is **complete** if it is independent and every solution is expressible as a linear combination of solutions in the set. For a specific homogeneous system, we seek some complete set of solutions because all other solutions are produced from them using linear combinations. For example, the two solutions corresponding to the two choices $b = 0$, $e = 1$ and $b = 1$, $e = 0$ form a complete set of solutions to the homogeneous system (14.5).

It is not our intent to present the theory of linear algebraic equations. Such a study is appropriate for a course in linear algebra. We do point out that there is an elementary algorithm known as Gaussian elimination for producing a complete set of solutions to a given system of linear equations. Moreover, Gaussian elimination is readily implemented on computers and handheld programmable calculators. The systems of equations we will encounter in this book are simple enough to be solved by the elimination methods learned in intermediate algebra.

How does our discussion relate to dimensional analysis? Our basic goal thus far has been to find all possible dimensionless products among the variables that influence the physical phenomenon under investigation. We developed a homogeneous system of linear algebraic equations to help us determine these dimensionless products. This system of equations usually has infinitely many solutions. Each solution gives the values of the exponents that result in a dimensionless product among the variables. If we sum two solutions, we produce another solution that yields the same dimensionless product as does multiplication of the dimensionless products corresponding to the original two solutions. For example, the sum of the solutions corresponding to $b = 0$, $e = 1$ and $b = 1$, $e = 0$ for Equation (14.5) yields the solution corresponding to $b = 1, e = 1$ with the corresponding dimensionless product from Equation (14.9) given by

$$gt^2r^{-1}\theta = \prod_1 \prod_2$$

The reason for this result is that the unknowns in the system of equations are the exponents in the dimensionless products, and addition of exponents algebraically corresponds to multiplication of numbers having the same base: $x^{m+n} = x^m x^n$. Moreover, multiplication of a solution by a constant produces a solution that yields the same dimensionless product as does raising the product corresponding to the original solution to the power of the constant. For example, -1 times the solution corresponding to $b = 1$, $e = 0$ yields the solution corresponding to $b = -1$, $e = 0$ with the corresponding dimensionless product

$$g^{-1}t^{-2}r = \prod\nolimits_2^{-1}$$

The reason for this last result is that algebraic multiplication of an exponent by a constant corresponds to raising a power to a power, $x^{mn} = (x^m)^n$.

In summary, addition of solutions to the homogeneous system of equations results in multiplication of their corresponding dimensionless products, and multiplication of a solution by a constant results in raising the corresponding product to the power given by that constant. Thus, if S_1 and S_2 are two solutions corresponding to the dimensionless products $\prod_1$ and $\prod_2$, respectively, then the linear combination $aS_1 + bS_2$ corresponds to the dimensionless product

$$\prod\nolimits_1^a \prod\nolimits_2^b$$

It follows from our preceding discussion that a complete set of solutions to the homogeneous system of equations produces all possible solutions through linear combination. The dimensionless products corresponding to a complete set of solutions are therefore called a *complete set of dimensionless products*. All dimensionless products can be obtained by forming powers and products of the members of a complete set.

Next, let's investigate how these dimensionless products can be used to produce all possible dimensionally homogeneous equations among the variables. In Section 14.1 we defined an equation to be dimensionally homogeneous if it remains true regardless of the system of units in which the variables are measured. The fundamental result in dimensional analysis that provides for the construction of all dimensionally homogeneous equations from complete sets of dimensionless products is the following theorem.

Theorem 1

Buckingham's Theorem An equation is dimensionally homogeneous if and only if it can be put into the form

$$f\left(\prod\nolimits_1, \prod\nolimits_2, \ldots, \prod\nolimits_n\right) = 0 \tag{14.11}$$

where f is some function of n arguments and $\{\prod_1, \prod_2, \ldots, \prod_n\}$ is a complete set of dimensionless products.

Let's apply Buckingham's theorem to the simple pendulum discussed in the preceding sections. The two dimensionless products

$$\prod\nolimits_1 = \theta \quad \text{and} \quad \prod\nolimits_2 = \frac{gt^2}{r}$$

form a complete set for the pendulum problem. Thus, according to Buckingham's theorem, there is a function f such that

$$f\left(\theta, \frac{gt^2}{r}\right) = 0$$

Assuming we can solve this last equation for gt^2/r as a function of θ, it follows that

$$t = \sqrt{\frac{r}{g}} h(\theta) \tag{14.12}$$

where h is some function of the single variable θ. Notice that this last result agrees with our intuitive formulation for the simple pendulum presented in Section 14.1. Observe that Equation (14.12) represents only a general form for the relationship among the variables m, g, t, r, and θ. However, it can be concluded from this expression that t does not depend on the mass m and is related to $r^{1/2}$ and $g^{-1/2}$ by some function of the initial angle of displacement θ. Knowing this much, we can determine the nature of the function h experimentally or approximate it, as discussed in Section 14.1.

Consider Equation (14.11) in Buckingham's theorem. For the case in which a complete set consists of a single dimensionless product, for example, $\prod_1$, the equation reduces to the form

$$f\left(\prod_1\right) = 0$$

In this case we assume that the function f has one real root at k (to assume otherwise has little physical meaning). Hence, the solution $\prod_1 = k$ is obtained.

Using Buckingham's theorem, let's reconsider the example from Section 14.1 of the wind force on a van driving down a highway. Because the four variables F, v, A, and ρ were selected and all three equations in (14.8) are independent, a complete set of dimensionless products consists of a single product:

$$\prod_1 = \frac{F}{v^2 A\rho}$$

Application of Buckingham's theorem gives

$$f\left(\prod_1\right) = 0$$

which implies from the preceding discussion that $\prod_1 = k$, or

$$F = kv^2 A\rho$$

where k is a dimensionless constant as before. Thus, when a complete set consists of a *single dimensionless product*, as is generally the case when we begin with four variables, the application of Buckingham's theorem yields the desired relationship *up to a constant of proportionality*. Of course, the predicted proportionality must be tested to determine the adequacy of our list of variables. If the list does prove to be adequate, then the constant of proportionality can be determined by experimentation, thereby completely defining the relationship.

For the case $n = 2$, Equation (14.11) in Buckingham's theorem takes the form

$$f\left(\prod_1, \prod_2\right) = 0 \tag{14.13}$$

If we choose the products in the complete set $\{\prod_1, \prod_2\}$ so that the dependent variable appears in only one of them, for example, $\prod_2$, we can proceed under the assumption that Equation (14.13) can be solved for that chosen product $\prod_2$ in terms of the remaining product $\prod_1$. Such a solution takes the form

$$\prod_2 = H\left(\prod_1\right)$$

and then this latter equation can be solved for the dependent variable. Note that when a complete set consists of more than one dimensionless product, the application of Buckingham's theorem determines the desired relationship *up to an arbitrary function*. After verifying the adequacy of the list of variables, we may be lucky enough to recognize the underlying functional relationship. However, in general we can expect to construct an empirical model, although the task has been eased considerably.

For the general case of n dimensionless products in the complete set for Buckingham's theorem, we again choose the products in the complete set $\{\prod_1, \prod_2, \ldots, \prod_n\}$ so that the dependent variable appears in only one of them, say $\prod_n$ for definiteness. Assuming we can solve Equation (14.11) for that product $\prod_n$ in terms of the remaining ones, we have the form

$$\prod\nolimits_n = H\left(\prod\nolimits_1, \prod\nolimits_2, \ldots, \prod\nolimits_{n-1}\right)$$

We then solve this last equation for the dependent variable.

Summary of Dimensional Analysis Methodology

STEP 1 Decide which variables enter the problem under investigation.

STEP 2 Determine a complete set of dimensionless products $\{\prod_1, \prod_2, \ldots, \prod_n\}$ among the variables. Make sure the dependent variable of the problem appears in only one of the dimensionless products.

STEP 3 Check to ensure that the products found in the previous step are dimensionless and independent. Otherwise you have an algebra error.

STEP 4 Apply Buckingham's theorem to produce all possible dimensionally homogeneous equations among the variables. This procedure yields an equation of the form (14.11).

STEP 5 Solve the equation in Step 4 for the dependent variable.

STEP 6 Test to ensure that the assumptions made in Step 1 are reasonable. Otherwise the list of variables is faulty.

STEP 7 Conduct the necessary experiments and present the results in a useful format.

Let's illustrate the first five steps of this procedure.

EXAMPLE 1 *Terminal Velocity of a Raindrop*

Consider the problem of determining the terminal velocity v of a raindrop falling from a motionless cloud. We examined this problem from a very simplistic point of view in Chapter 2, but let's take another look using dimensional analysis.

What are the variables influencing the behavior of the raindrop? Certainly the terminal velocity will depend on the size of the raindrop given by, say, its radius r. The density ρ of the air and the viscosity μ of the air will also affect the behavior. (Viscosity measures resistance to motion—a sort of internal molecular friction. In gases this resistance is caused by collisions between fast-moving molecules.) The acceleration due to gravity g is another variable to consider. Although the surface tension of the raindrop is a factor that does influence the behavior of the fall, we will ignore this factor. If necessary, surface tension can be taken into account in a later, refined model. These considerations give the following table relating the selected variables to their dimensions:

Variable	v	r	g	ρ	μ
Dimension	LT^{-1}	L	LT^{-2}	ML^{-3}	$ML^{-1}T^{-1}$

Next we find all the dimensionless products among the variables. Any such product must be of the form

$$v^a r^b g^c \rho^d \mu^e \tag{14.14}$$

and hence must have dimension

$$(LT^{-1})^a (L)^b (LT^{-2})^c (ML^{-3})^d (ML^{-1}T^{-1})^e$$

Therefore, a product of the form (14.14) is dimensionless if and only if the following system of equations in the exponents is satisfied:

$$\left.\begin{array}{rrrrrl} & & & d & +e & =0 \\ a & +b & +\ c & -3d & -e & =0 \\ -a & & -2c & & -e & =0 \end{array}\right\} \tag{14.15}$$

Solution of the system (14.15) gives $b = (3/2)d - (1/2)a$, $c = (1/2)d - (1/2)a$, and $e = -d$, where a and d are arbitrary. One dimensionless product $\prod_1$ is obtained by setting $a = 1$, $d = 0$; another, independent dimensionless product $\prod_2$ is obtained when $a = 0$, $d = 1$. These solutions give

$$\prod\nolimits_1 = vr^{-1/2}g^{-1/2} \quad \text{and} \quad \prod\nolimits_2 = r^{3/2}g^{1/2}\rho\mu^{-1}$$

Next, we check the results to ensure that the products are indeed dimensionless:

$$\frac{LT^{-1}}{L^{1/2}(LT^{-2})^{1/2}} = M^0L^0T^0$$

and

$$\frac{L^{3/2}(LT^{-2})^{1/2}(ML^{-3})}{ML^{-1}T^{-1}} = M^0L^0T^0$$

Thus, according to Buckingham's theorem, there is a function f such that

$$f\left(vr^{-1/2}g^{-1/2}, \frac{r^{3/2}g^{1/2}\rho}{\mu}\right) = 0$$

Assuming we can solve this last equation for $vr^{-1/2}g^{-1/2}$ as a function of the second product $\prod_2$, it follows that

$$v = \sqrt{rg}\, h\left(\frac{r^{3/2}g^{1/2}\rho}{\mu}\right)$$

where h is some function of the single product $\prod_2$.

The preceding example illustrates a characteristic feature of dimensional analysis. Normally the modeler studying a given physical system has an intuitive idea of the variables involved and has a working knowledge of general principles and laws (such as Newton's second law) but lacks the precise laws governing the interaction of the variables. Of course, the modeler can always experiment with each independent variable separately, holding the others constant and measuring the effect on the system. Often, however, the efficiency of the experimental work can be improved through an application of dimensional analysis. Although we did not illustrate Steps 6 and 7 of the dimensional analysis process for the preceding example, these steps will be illustrated in Section 14.3.

We now make some observations concerning the dimensional analysis process. Suppose n variables have been identified in the physical problem under investigation. When determining a complete set of dimensionless products, we form a system of three linear algebraic equations by equating the exponents for M, L, and T to zero. That is, we obtain a system of three equations in n unknowns (the exponents). If the three equations are independent, we can solve the system for three of the unknowns in terms of the remaining $n-3$ unknowns (declared to be arbitrary). In this case, we find $n-3$ independent dimensionless products that make up the complete set we seek. For instance, in the preceding example there are five unknowns, a, b, c, d, e, and we determined three of them (b, c, and e) in terms of the remaining $(5-3)$ two arbitrary ones (a and d). Thus, we obtained a complete set of two dimensionless products. When choosing the $n-3$ dimensionless products, we must be sure that the dependent variable appears in only one of them. We can then solve Equation (14.11) guaranteed by Buckingham's theorem for the dependent variable, at least under suitable assumptions on the function f in that equation. (The full story telling when such a solution is possible is the content of an important result in advanced calculus known as the implicit function theorem.)

We acknowledge that we have been rather sketchy in our presentation for solving the system of linear algebraic equations that results in the process of determining all dimensionless products. Recall how to solve simple linear systems by the method of elimination of variables. We conclude this section with another example.

EXAMPLE 2 *Automobile Gas Mileage Revisited*

Consider again the automobile gasoline mileage problem presented in Chapter 2. One of our submodels in that problem was for the force of propulsion F_p. The variables we identified that affect the propulsion force are C_r, the amount of fuel burned per unit time, the amount K of energy contained in each gallon of gasoline, and the speed v. Let's perform a dimensional analysis. The following table relates the various variables to their dimensions:

Variable	F_p	C_r	K	v
Dimension	MLT^{-2}	L^3T^{-1}	$ML^{-1}T^{-2}$	LT^{-1}

Thus, the product

$$F_p^a C_r^b K^c v^d \tag{14.16}$$

must have the dimension

$$(MLT^{-2})^a (L^3T^{-1})^b (ML^{-1}T^{-2})^c (LT^{-1})^d$$

The requirement for a dimensionless product leads to the system

$$\left.\begin{aligned} a \quad\quad + c \quad\quad &= 0 \\ a + 3b - c + d &= 0 \\ -2a - b - 2c - d &= 0 \end{aligned}\right\} \tag{14.17}$$

Solution of the system (14.17) gives $b = -a$, $c = -a$, and $d = a$, where a is arbitrary. Choosing $a = 1$, we obtain the dimensionless product

$$\prod_1 = F_p C_r^{-1} K^{-1} v$$

From Buckingham's theorem there is a function f with $f(\prod_1) = 0$, so $\prod_1$ equals a constant. Therefore,

$$F_p \propto \frac{C_r K}{v}$$

in agreement with the conclusion reached in Chapter 2.

14.2 PROBLEMS

1. Predict the time of revolution for two bodies of mass m_1 and m_2 in empty space revolving about each other under their mutual gravitational attraction.
2. A projectile is fired with initial velocity v at an angle θ with the horizon. Predict the range R.
3. Consider an object that is falling under the influence of gravity. Assume that air resistance is negligible. Using dimensional analysis, find the speed v of the object after it has fallen a distance s. Let $v = f(m, g, s)$, where m is the mass of the object and g is the acceleration due to gravity. Does you answer agree with your knowledge of the physical situation?
4. Using dimensional analysis, find a proportionality relationship for the centrifugal force F of a particle in terms of its mass m, velocity v, and radius r of the curvature of its path.
5. One would like to know the nature of the drag forces experienced by a sphere as it passes through a fluid. It is assumed that the sphere has a low speed. Therefore, the drag force is highly dependent on the viscosity of the fluid. The fluid density is to be neglected. Use the dimensional analysis process to develop a model for drag force F as a function of the radius r and velocity m of the sphere and the viscosity μ of the fluid.
6. The volume flow rate q for laminar flow in a pipe depends on the pipe radius r, the viscosity μ of the fluid, and the pressure drop per unit length dp/dz. Develop a model for the flow rate q as a function of r, μ, and dp/dz.

7. In fluid mechanics, the Reynolds number is a dimensionless number involving the fluid velocity v, density ρ, viscosity μ, and a characteristic length r. Use dimensional analysis to find the Reynolds number.

8. The power P delivered to a pump depends on the specific weight w of the fluid pumped, the height h to which the fluid is pumped, and the fluid flow rate q in cubic feet per second. Use dimensional analysis to determine an equation for power.

9. Find the volume flow rate dV/dt of blood flowing in an artery as a function of the pressure drop per unit length of artery, the radius r of the artery, the blood density ρ, and the blood viscosity μ.

10. The speed of sound in a gas depends on the pressure and the density. Use dimensional analysis to find the speed of sound in terms of pressure and density.

11. The lift force F on a missile depends on its length r, velocity v, diameter δ, and initial angle θ with the horizon; it also depends on the density ρ, viscosity μ, gravity g, and speed of sound s of the air. Show that

$$F = \rho v^2 r^2 h\left(\frac{\delta}{r}, \theta, \frac{\mu}{\rho v r}, \frac{s}{v}, \frac{rg}{v^2}\right)$$

12. The height h that a fluid will rise in a capillary tube decreases as the diameter D of the tube increases. Use dimensional analysis to determine how h varies with D and the specific weight w and surface tension σ of the liquid.

14.3 A Damped Pendulum

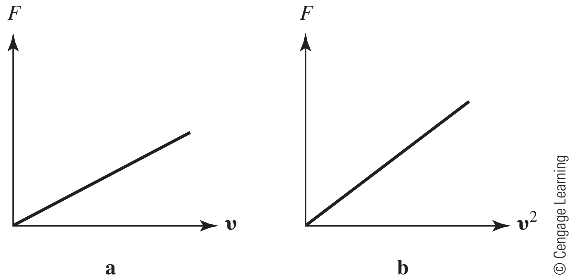
In Section 14.1 we investigated the pendulum problem under the assumptions that the hinge is frictionless, the mass is concentrated at one end of the pendulum, and the drag force is negligible. Suppose we are not satisfied with the results predicted by the constructed model. Then we can refine the model by incorporating drag forces. If F represents the total drag force, the problem now is to determine the function

$$t = f(r, m, g, \theta, F)$$

Let's consider a submodel for the drag force. As we have seen in previous examples, the modeler is usually faced with a trade-off between simplicity and accuracy. For the pendulum it might seem reasonable to expect the drag force to be proportional to some positive power of the velocity. To keep our model simple, we assume that F is proportional to either v or v^2, as depicted in Figure 14.7.

Now we can experiment to determine directly the nature of the drag force. However, we will first perform a dimensional analysis because we expect it to reduce our experimental effort. Assume F is proportional to v so that $F = kv$. For convenience we choose to work with the dimensional constant $k = F/v$, which has dimension MLT^{-2}/LT^{-1}, or simply MT^{-1}. Notice that the dimensional constant captures the assumption about the drag force.

Figure 14.7
Possible submodels for the drag force



Thus, we apply dimensional analysis to the model

$$t = f(r, m, g, \theta, k)$$

An analysis of the dimensions of the variables gives

Variable	t	r	m	g	θ	k
Dimension	T	L	M	LT^{-2}	$M^0L^0T^0$	MT^{-1}

Any product of the variables must be of the form

$$t^a r^b m^c g^d \theta^e k^f \tag{14.18}$$

and hence must have dimension

$$(T)^a (L)^b (M)^c (LT^{-2})^d (M^0L^0T^0)^e (MT^{-1})^f$$

Therefore, a product of the form (14.18) is dimensionless if and only if

$$\left.\begin{array}{rrrrl} & c & & +f & = 0 \\ b & & +\ d & & = 0 \\ a & & -2d & -f & = 0 \end{array}\right\} \tag{14.19}$$

The equations in the system (14.19) are independent, so we know we can solve for three of the variables in terms of the remaining (6 − 3) three variables. We would like to choose the solutions in such a way that t appears in only one of the dimensionless products. Thus, we choose a, e, and f as the arbitrary variables with

$$c = -f,\ b = -d = \frac{-a}{2} + \frac{f}{2},\ d = \frac{a}{2} - \frac{f}{2}$$

Setting $a = 1$, $e = 0$, and $f = 0$, we obtain $c = 0$, $b = -1/2$, and $d = 1/2$ with the corresponding dimensionless product $t\sqrt{g/r}$. Similarly, choosing $a = 0$, $e = 1$, and $f = 0$, we get $c = 0$, $b = 0$, and $d = 0$, corresponding to the dimensionless product θ. Finally, choosing $a = 0$, $e = 0$, and $f = 1$, we obtain $c = -1$, $b = 1/2$, and $d = -1/2$, corresponding to the dimensionless product $k\sqrt{r}/m\sqrt{g}$. Notice that t appears in only the first of these products. From Buckingham's theorem, there is a function h with

$$h\left(t\sqrt{g/r}, \theta, \frac{k\sqrt{r}}{m\sqrt{g}}\right) = 0$$

Assuming we can solve this last equation for $t\sqrt{g/r}$, we obtain

$$t = \sqrt{r/g}\, H\left(\theta, \frac{k\sqrt{r}}{m\sqrt{g}}\right)$$

for some function H of two arguments.

Testing the Model (Step 6)

Given $t = \sqrt{r/g}\, H(\theta, k\sqrt{r}/m\sqrt{g})$, our model predicts that $t_1/t_2 = \sqrt{r_1/r_2}$ if the parameters of the function H (namely, θ and $k\sqrt{r}/m\sqrt{g}$) could be held constant. Now there is no difficulty with keeping θ and k constant. However, varying r while simultaneously keeping $k\sqrt{r}/m\sqrt{g}$ constant is more complicated. Because g is constant, we could try to vary r and m in such a manner that $\sqrt{r}/m$ remains constant. This might be done using a pendulum with a hollow mass to vary m without altering the drag characteristics. Under these conditions we would expect the plot in Figure 14.8.

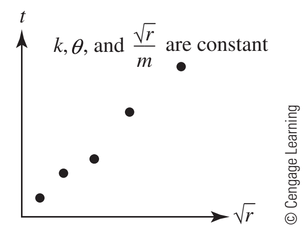


Figure 14.8

A plot of t versus $\sqrt{r}$ keeping the variables k, θ, and $\sqrt{r}/m$ constant

Presenting the Results (Step 7)

As was suggested in predicting the period of the undamped pendulum, we can plot $t\sqrt{g/r} = H(\theta, k\sqrt{r}/m\sqrt{g})$. However, because H is here a function of two arguments, this would yield a three-dimensional figure that is not easy to use. An alternative technique is to plot $t\sqrt{g/r}$ versus $k\sqrt{r}/m\sqrt{g}$ for various values of θ. This is illustrated in Figure 14.9.

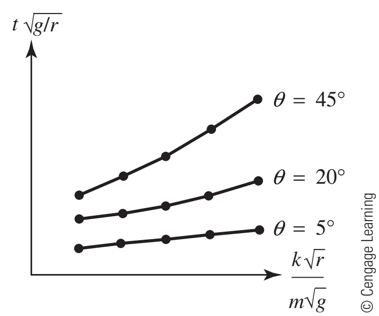


Figure 14.9

Presenting the results

To be safe in predicting t over the range of interest for representative values of θ, it would be necessary to conduct sufficient experiments at various values of $k\sqrt{r}/m\sqrt{g}$. Note that once data are collected, various empirical models could be constructed using an appropriate interpolating scheme for each value of θ.

Choosing Among Competing Models

Because dimensional analysis involves only algebra, it is tempting to develop several models under different assumptions before proceeding with costly experimentation. In the case of the pendulum, under different assumptions, we can develop the following three models (see Problem 1 in the Section 14.3 problem set):

A: $t = \sqrt{r/g}\, h(\theta)$ No drag forces

B: $t = \sqrt{r/g}\, h\left(\theta, \dfrac{k\sqrt{r}}{m\sqrt{g}}\right)$ Drag forces proportional to v: $F = kv$

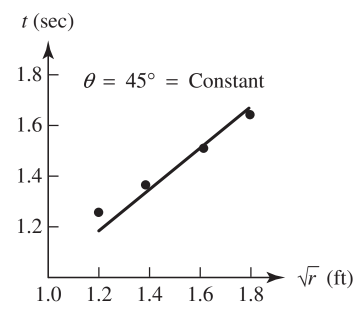
C: $t = \sqrt{r/g}\, h\left(\theta, \dfrac{k_1 r}{m}\right)$ Drag forces proportional to v^2: $F = k_1 v^2$

Because all the preceding models are approximations, it is reasonable to ask which, if any, is suitable in a particular situation. We now describe the experimentation necessary to distinguish among these models, and we present some experimental results.

Model A predicts that when the angle of displacement θ is held constant, the period t is proportional to $\sqrt{r}$. Model B predicts that when θ and $\sqrt{r}/m$ are both held constant, while maintaining the same drag characteristics k, t is proportional to $\sqrt{r}$. Finally, Model C predicts that if θ, r/m, and k_1 are held constant, then t is proportional to $\sqrt{r}$.

The following discussion describes our experimental results for the pendulum.[1] Various types of balls were suspended from a string in such a manner as to minimize the friction at the hinge. The kinds of balls included tennis balls and various types and sizes of plastic balls. A hole was made in each ball to permit variations in the mass without altering appreciably the aerodynamic characteristics of the ball or the location of the center of mass. The models were then compared with one another. In the case of the tennis ball, Model A proved to be superior. The period was independent of the mass, and a plot of t versus $\sqrt{r}$ for constant θ is shown in Figure 14.10.

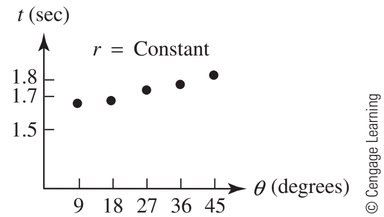
Figure 14.10
Model A for a tennis ball



[1] Data collected by Michael Jaye.

Having decided that $t = \sqrt{r/g}h(\theta)$ is the best of the models for the tennis ball, we isolated the effect of θ by holding r constant to gain insight into the nature of the function h. A plot of t versus θ for constant r is shown in Figure 14.11.

Figure 14.11
Isolating the effect of θ



Note from Figure 14.11 that for small angles of initial displacement θ, the period is virtually independent of θ. However, the displacement effect becomes more noticeable as θ is increased. Thus, for small angles we might hypothesize that $t = c\sqrt{r/g}$ for some constant c. If one plots t versus $\sqrt{r}$ for small angles, the slope of the resulting straight line should be constant.

For larger angles, the experiment demonstrates that the effect of θ needs to be considered. In such cases, one may desire to estimate the period for various angles. For example, if $\theta = 45°$ and we know a particular value of $\sqrt{r}$, we can estimate t from Figure 14.10. Although not shown, plots for several different angles can be graphed in the same figure.

Dimensional Analysis in the Model-Building Process

Let's summarize how dimensional analysis assists in the model-building process. In the determination of a model we must first decide which factors to neglect and which to include. A dimensional analysis provides additional information on how the included factors are related. Moreover, in large problems, we often determine one or more submodels before dealing with the larger problem. For example, in the pendulum problem we had to develop a submodel for drag forces. A dimensional analysis helps us choose among the various submodels.

A dimensional analysis is also useful for obtaining an initial test of the assumptions in the model. For example, suppose we hypothesize that the dependent variable y is some function of five variables, $y = f(x_1, x_2, x_3, x_4, x_5)$. A dimensional analysis in the MLT system in general yields $\prod_1 = h(\prod_2, \prod_3)$, where each $\prod_i$ is a dimensionless product. The model predicts that $\prod_1$ will remain constant if $\prod_2$ and $\prod_3$ are held constant, even though the components of $\prod_2$ and $\prod_3$ may vary. Because there are, in general, an infinite number of ways of choosing $\prod_i$, we should choose those that can be controlled in laboratory experiments. Having determined that $\prod_1 = h(\prod_2, \prod_3)$, we can isolate the effect of $\prod_2$ by holding $\prod_3$ constant and vice versa. This can help explain the functional relationship among the variables. For instance, we say in our example that the period of the pendulum did not depend on the initial displacement for small displacements.

Perhaps the greatest contribution of dimensional analysis is that it reduces the number of experiments required to predict the behavior. If we wanted to conduct experiments to predict values of y for the assumed relationship $y = f(x_1, x_2, x_3, x_4, x_5)$ and it was decided that 5 data points would be necessary over the range of each variable, 5^5, or 3125, experiments would be necessary. Because a two-dimensional chart is required to interpolate conveniently,

y might be plotted against x_1 for five values of x_1, holding x_2, x_3, x_4, x_5 constant. Because x_2, x_3, x_4, and x_5 must vary as well, 5^4, or 625, charts would be necessary. However, after a dimensional analysis yields $\prod_1 = h(\prod_2, \prod_3)$, only 25 data points are required. Moreover, $\prod_1$ can be plotted versus $\prod_2$, for various values of $\prod_3$, on a single chart. Ultimately, the task is far easier after dimensional analysis.

Finally, dimensional analysis helps in presenting the results. It is usually best to present experimental results using those $\prod_i$ that are classical representations within the field of study. For instance, in the field of fluid mechanics there are eight factors that might be significant in a particular situation: velocity v, length r, mass density ρ, viscosity μ, acceleration of gravity g, speed of sound c, surface tension σ, and pressure p. Thus, a dimensional analysis could require as many as five independent dimensionless products. The five generally used are the Reynolds number, Froude number, Mach number, Weber number, and pressure coefficient. These numbers, which are discussed in Section 14.5, are defined as follows:

Reynolds number	$\dfrac{vr\rho}{\mu}$
Froude number	$\dfrac{v^2}{rg}$
Mach number	$\dfrac{v}{c}$
Weber number	$\dfrac{\rho v^2 r}{\sigma}$
Pressure coefficient	$\dfrac{p}{\rho v^2}$

Thus, the application of dimensional analysis becomes quite easy. Depending on which of the eight variables are considered in a particular problem, the following steps are performed.

1. Choose an appropriate subset from the preceding five dimensionless products.
2. Apply Buckingham's theorem.
3. Test the reasonableness of the choice of variables.
4. Conduct the necessary experiments and present the results in a useful format.

We illustrate an application of these steps to a fluid mechanics problem in Section 14.5.

14.3 PROBLEMS

1. For the damped pendulum,
   a. Assume that F is proportional to v^2 and use dimensional analysis to show that $t = \sqrt{r/g}h(\theta, rk_1/m)$.
   b. Assume that F is proportional to v^2 and describe an experiment to test the model $t = \sqrt{r/g}h(\theta, rk_1/m)$.

2. Under appropriate conditions, all three models for the pendulum imply that t is proportional to $\sqrt{r}$. Explain how the conditions distinguish among the three models by considering how m must vary in each case.

3. Use a model employing a differential equation to predict the period of a simple frictionless pendulum for small initial angles of displacement. (*Hint*: Let $\sin\theta = \theta$.) Under these conditions, what should be the constant of proportionality? Compare your results with those predicted by Model A in the text.[2]

14.4 Examples Illustrating Dimensional Analysis

EXAMPLE 1 *Explosion Analysis*[3]

In excavation and mining operations, it is important to be able to predict the size of a crater resulting from a given explosive such as TNT in some particular soil medium. Direct experimentation is often impossible or too costly. Thus, it is desirable to use small laboratory or field tests and then scale these up in some manner to predict the results for explosions far greater in magnitude.

We may wonder how the modeler determines which variables to include in the initial list. Experience is necessary to intelligently determine which variables can be neglected. Even with experience, however, the task is usually difficult in practice, as this example will illustrate. It also illustrates that the modeler must often change the list of variables to get usable results.

Problem Identification *Predict the crater volume V produced by a spherical explosive located at some depth d in a particular soil medium.*

Assumptions and Model Formulation Initially, let's assume that the craters are geometrically similar (see Chapter 2), where the crater size depends on three variables: the radius r of the crater, the density ρ of the soil, and the mass W of the explosive. These variables are composed of only two primary dimensions, length L and mass M, and a dimensional analysis results in only one dimensionless product (see Problem 1a in the Section 14.4 problem set):

$$\prod_r = r\left(\frac{\rho}{W}\right)^{1/3}$$

According to Buckingham's theorem, $\prod_r$ must equal a constant. Thus, the crater dimensions of radius or depth vary with the cube root of the mass of the explosive. Because the crater

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[2]For students who have studied differential equations.

[3]This example is adapted with permission from R. M. Schmidt, "A Centrifuge Cratering Experiment: Development of a Gravity-Scaled Yield Parameter." In *Impact and Explosion Cratering*, edited by D. J. Roddy, R. O. Pepin, and R. B. Merrill (New York: Pergamon, 1977), pp. 1261–1276.

volume is proportional to r^3, it follows that the volume of the crater is proportional to the mass of the explosive for constant soil density. Symbolically, we have

$$V \propto \frac{W}{\rho} \tag{14.20}$$

Experiments have shown that the proportionality (14.20) is satisfactory for small explosions (less than 300 lb of TNT) at zero depth in soils, such as moist alluvium, that have good cohesion. For larger explosions, however, the rule proves unsatisfactory. Other experiments suggest that gravity plays a key role in the explosion process, and because we want to consider extraterrestial craters as well, we need to incorporate gravity as a variable.

If gravity is taken into account, then we assume crater size to be dependent on four variables: crater radius r, density of the soil ρ, gravity g, and charge energy E. Here, the charge energy is the mass W of the explosive times its specific energy. Applying a dimensional analysis to these four variables again leads to a single dimensionless product (see Problem 1b in the 14.4 problem set):

$$\prod_{rg} = r\left(\frac{\rho g}{E}\right)^{1/4}$$

Thus, $\prod_{rg}$ equals a constant and the linear crater dimensions (radius or depth of the crater) vary with the one-fourth root of the energy (or mass) of the explosive for a constant soil density. This leads to the following proportionality known as the quarter-root scaling and is a special case of *gravity scaling*:

$$V \propto \left(\frac{E}{\rho g}\right)^{3/4} \tag{14.21}$$

Experimental evidence indicates that gravity scaling holds for large explosions (more than 100 tons of TNT) where the stresses in the cratering process are much larger than the material strengths of the soil. The proportionality (14.21) predicts that crater volume decreases with increased gravity. The effect of gravity on crater formation is relevant in the study of extraterrestial craters. Gravitational effects can be tested experimentally using a centrifuge to increase gravitational accelerations.[4]

A question of interest to explosion analysts is whether the material properties of the soil do become less important with increased charge size and increased gravity. Let's consider the case in which the soil medium is characterized only by its density ρ. Thus, the crater volume V depends on the explosive, soil density ρ, gravity g, and the depth of burial d of the charge. In addition, the explicit role of material strength or cohesion has been tested and the strength–gravity transition is shown to be a function of charge size and soil strength.

We now describe our explosive in more detail than in previous models. To characterize an explosive, three independent variables are needed: size, energy field, and explosive density δ. The size can be given as charge mass W, as charge energy E, or as the radius α of the spherical explosive. The energy yield can be given as a measure of the specific

[4]See the papers by R. M. Schmidt (1977, 1980) and by Schmidt and Holsapple (1980), cited in Further Reading, which discuss the effects when a centrifuge is used to perform explosive cratering tests under the influence of gravitational acceleration up to 480 G, where 1 G is the terrestrial gravity field strength of 981 cm/sec^2.

energy Q_e or the energy density per unit volume Q_V. The following equations relate the variables:

$$W = \frac{E}{Q_e}$$
$$Q_V = \delta Q_e$$
$$\alpha^3 = \left(\frac{3}{4\pi}\right)\left(\frac{W}{\delta}\right)$$

One choice of these variables leads to the model formulation

$$V = f(W, Q_e, \delta, \rho, g, d)$$

Because there are seven variables under consideration and the MLT system is being used, a dimensional analysis generally will result in four (7 − 3) dimensionless products. The dimensions of the variables are shown in the following table.

Variable	V	W	Q_e	δ	ρ	g	d
Dimension	L^3	M	L^2T^{-2}	ML^{-3}	ML^{-3}	LT^{-2}	L

Any product of the variables must be of the form

$$V^a W^b Q_e^c \delta^e \rho^f g^k d^m \tag{14.22}$$

and hence have dimensions

$$(L^3)^a (M^b)(L^2T^{-2})^c (ML^{-3})^{e+f} (LT^{-2})^k (L)^m$$

Therefore, a product of the form (14.22) is dimensionless if and only if the exponents satisfy the following homogeneous system of equations:

$$\begin{aligned}
M:&\quad b + e + f = 0\\
L:&\quad 3a + 2c - 3e - 3f + k + m = 0\\
T:&\quad -2c - 2k = 0
\end{aligned}$$

Solution to this system produces

$$b = \frac{k-m}{3} - a, \quad c = -k, \quad e = a - f + \frac{k-m}{3}$$

where a, f, k, and m are arbitrary. By setting one of these arbitrary exponents equal to 1 and the other three equal to 0, in succession, we obtain the following set of dimensionless products:

$$\frac{V\delta}{W}, \quad \left(\frac{g}{Q_e}\right)\left(\frac{W}{\delta}\right)^{1/3}, \quad d\left(\frac{\delta}{W}\right)^{1/3}, \quad \frac{\rho}{\delta}$$

(Convince yourself that these are dimensionless.) Because the dimensions of ρ and δ are equal, we can rewrite these dimensionless products as follows:

$$\prod_1 = \frac{V\rho}{W}$$

$$\prod_2 = \left(\frac{g}{Q_e}\right)\left(\frac{W}{\delta}\right)^{1/3}$$

$$\prod_3 = d\left(\frac{\rho}{W}\right)^{1/3}$$

$$\prod_4 = \frac{\rho}{\delta}$$

so $\prod_1$ is consistent with the dimensionless product implied by Equation (14.20). Then, applying Buckingham's theorem, we obtain the model

$$h\left(\prod_1, \prod_2, \prod_3, \prod_4\right) = 0 \qquad (14.23)$$

or

$$V = \frac{W}{\rho} H\left(\frac{gW^{1/3}}{Q_e\delta^{1/3}}, \frac{d\delta^{1/3}}{W^{1/3}}, \frac{\rho}{\delta}\right)$$

Presenting the Results For oil-base clay, the value of ρ is approximately 1.53 g/cm^2; for wet sand, 1.65; and for desert alluvium, 1.60. For TNT, δ has the value 2.23 g/cm^3. Thus, $0.69 < \prod_4 < 0.74$, so for simplicity we can assume that for these soils and TNT, $\prod_4$ is constant. Then, Equation (14.23) becomes

$$h\left(\prod_1, \prod_2, \prod_3\right) = 0 \qquad (14.24)$$

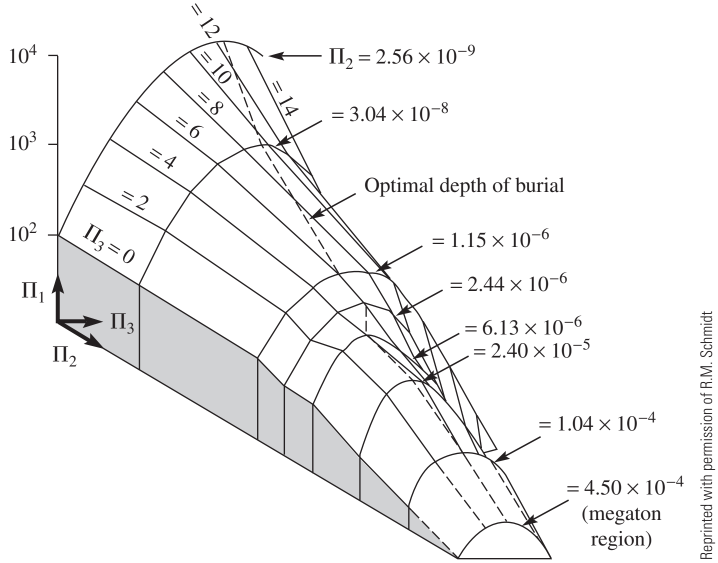

Figure 14.12
A plot of the surface $h(\prod_1, \prod_2, \prod_3) = 0$, showing the crater volume parameter $\prod_1$ as a function of gravity-scaled yield $\prod_2$ and depth of burial parameter $\prod_3$ (reprinted by permission of R. M. Schmidt)

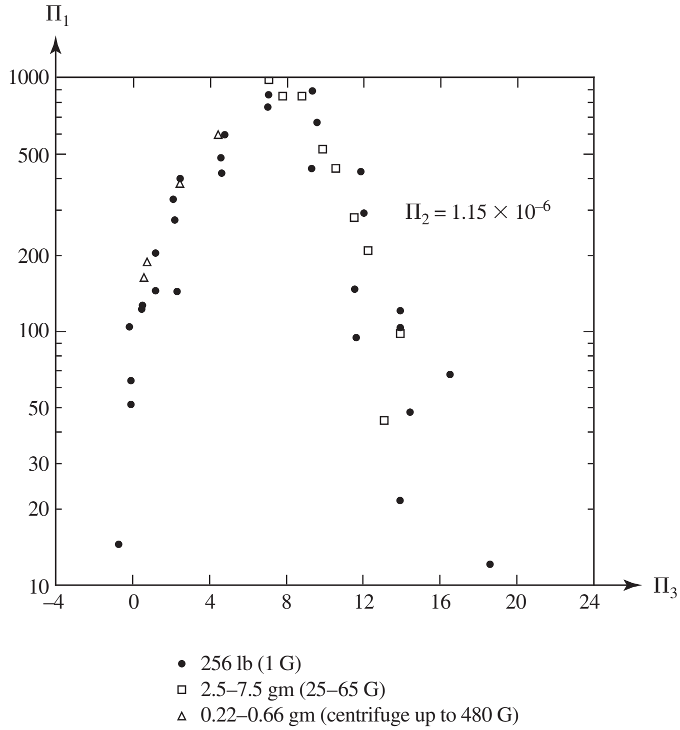

Figure 14.13 Data values for a cross section of the surface depicted in Figure 14.12 (data reprinted by permission from R. M. Schmidt)

R. M. Schmidt gathered experimental data to plot the surface described by Equation (14.24). A plot of the surface is depicted in Figure 14.12, showing the crater and volume parameter $\prod_1$ as a function of the scaled energy charge $\prod_2$ and the depth of the burial parameter $\prod_3$. Cross-sectional data for the surface parallel to the $\prod_1 \prod_3$ plane when $\prod_2 = 1.15 \times 10^{-6}$ are depicted in Figure 14.13.

Experiments have shown that the physical effect of increasing gravity is to reduce crater volume for a given charge yield. This result suggests that increased gravity can be compensated for by increasing the size of the charge to maintain the same cratering efficiency. Note also that Figures 14.12 and 14.13 can be used for prediction once an empirical interpolating model is constructed from the data. Holsapple and Schmidt (1982) extend these methods to impact cratering, and Housen, Schmidt, and Holsapple (1983) extend them to crater ejecta scaling.

EXAMPLE 2 *How Long Should You Roast a Turkey?*

One general rule for roasting a turkey is the following: Set the oven to 400 °F and allow 20 min per pound for cooking. How good is this rule?

Assumptions Let t denote the cooking time for the turkey. Now, on what variables does t depend? Certainly the size of the turkey is a factor that must be considered. Let's assume that the turkeys are geometrically similar and use l to denote some characteristic dimension of the uncooked meat; specifically, we assume that l represents the length of the turkey. Another factor is the difference between the temperature of the raw meat and the oven ΔT_m. (We know from experience that it takes longer to cook a bird that is nearly frozen than it does to cook one that is initially at room temperature.) Because the turkey will have to reach a certain interior temperature before it is considered fully cooked, the difference ΔT_c between the temperature of the cooked meat and the oven is a variable determining the cooking time. Finally, we know that different foods require different cooking times independent of size; it takes only 10 min or so to bake a pan of cookies, whereas a roast beef or turkey requires several hours. A measure of the factor representing the differences between foods is the *coefficient of heat conduction* for a particular uncooked food. Let k denote the coefficient of heat conduction for a turkey. Thus, we have the following model formulation for the cooking time:

$$t = f(\Delta T_m, \Delta T_c, k, l)$$

Dimensional Analysis Consider the dimensions of the independent variables. The temperature variables ΔT_m and ΔT_c measure the energy per volume and therefore have the dimension ML^2T^{-2}/L^3, or simply $ML^{-1}T^{-2}$. Now, what about the heat conduction variable k? **Thermal conductivity** k is defined as the amount of energy crossing one unit cross-sectional area per second divided by the gradient perpendicular to the area. That is,

$$k = \frac{\text{energy}/(\text{area} \times \text{time})}{\text{temperature}/\text{length}}$$

Accordingly, the dimension of k is $(ML^2T^{-2})(L^{-2}T^{-1})/(ML^{-1}T^{-2})(L^{-1})$, or simply L^2T^{-1}. Our analysis gives the following table:

Variable	ΔT_m	ΔT_c	k	l	t
Dimension	$ML^{-1}T^{-2}$	$ML^{-1}T^{-2}$	L^2T^{-1}	L	T

Any product of the variables must be of the form

$$\Delta T_m^a \Delta T_c^b k^c l^d t^e \tag{14.25}$$

and hence have dimension

$$(ML^{-1}T^{-2})^a (ML^{-1}T^{-2})^b (L^2T^{-1})^c (L)^d (T)^e$$

Therefore, a product of the form (14.25) is dimensionless if and only if the exponents satisfy

$$\begin{aligned}
M&: \quad a + b = 0 \\
L&: \quad -a - b + 2c + d = 0 \\
T&: \quad -2a - 2b - c + e = 0
\end{aligned}$$

Solution of this system of equations gives

$$a = -b, \quad c = e, \quad d = -2e$$

where b and e are arbitrary constants. If we set $b = 1, e = 0$, we obtain $a = -1$, $c = 0$, and $d = 0$; likewise, $b = 0$, $e = 1$ produces $a = 0$, $c = 1$, and $d = -2$. These independent solutions yield the complete set of dimensionless products:

$$\prod_1 = \Delta T_m^{-1} \Delta T_c \quad \text{and} \quad \prod_2 = kl^{-2}t$$

From Buckingham's theorem, we obtain

$$h\left(\prod_1, \prod_2\right) = 0$$

or

$$t = \left(\frac{l^2}{k}\right) H\left(\frac{\Delta T_c}{\Delta T_m}\right) \tag{14.26}$$

The rule stated in our opening remarks gives the roasting time for the turkey in terms of its weight w. Let's assume the turkeys are geometrically similar, or $V \propto l^3$. If we assume the turkey is of constant density (which is not quite correct because the bones and flesh differ in density), then, because weight is density times volume and volume is proportional to l^3, we get $w \propto l^3$. Moreover, if we set the oven to a constant baking temperature and specify that the turkey must initially be near room temperature (65 °F), then $\Delta T_c / \Delta T_m$ is a dimensionless constant. Combining these results with Equation (14.26), we get the proportionality

$$t \propto w^{2/3} \tag{14.27}$$

because k is constant for turkeys. Thus, the required cooking time is proportional to weight raised to the two-thirds power. Therefore, if t_1 hours are required to cook a turkey weighing w_1 pounds and t_2 is the time for a weight of w_2 pounds,

$$\frac{t_1}{t_2} = \left(\frac{w_1}{w_2}\right)^{2/3}$$

it follows that a doubling of the weight of a turkey increases the cooking time by the factor $2^{2/3} \approx 1.59$.

How does our result (14.27) compare to the rule stated previously? Assume that ΔT_m, ΔT_c, and k are independent of the length or weight of the turkey, and consider cooking a 23-lb turkey versus an 8-lb bird. According to our rule, the ratio of cooking times is given by

$$\frac{t_1}{t_2} = \left(\frac{20 \cdot 23}{20 \cdot 8}\right) = 2.875$$

On the other hand, from dimensionless analysis and Equation (14.27),

$$\frac{t_1}{t_2} = \left(\frac{23}{8}\right)^{2/3} \approx 2.02$$

Thus, the rule predicts it will take nearly three times as long to cook a 23-lb bird as it will to cook an 8-lb turkey. Dimensional analysis predicts it will take only twice as long. Which rule is correct? Why have so many cooks overcooked a turkey?

Testing the Results Suppose that turkeys of various sizes are cooked in an oven preheated to 325 °F. The initial temperature of the turkeys is 65 °F. All the turkeys are removed from the oven when their internal temperature, measured by a meat thermometer, reaches 195 °F. The (hypothetical) cooking times for the various turkeys are recorded as shown in the following table.

w (lb)	5	10	15	20
t (hr)	2	3.4	4.5	5.4

A plot of t versus $w^{2/3}$ is shown in Figure 14.14. Because the graph approximates a straight line through the origin, we conclude that $t \propto w^{2/3}$, as predicted by our model.

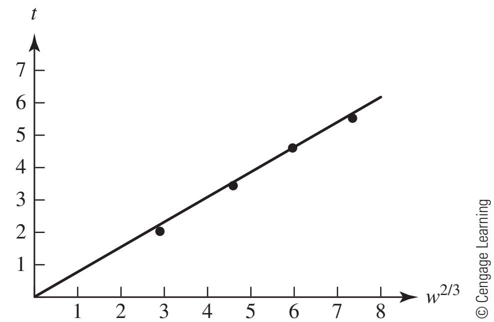

Figure 14.14
Plot of cooking times versus weight to the two-thirds power reveals the predicted proportionality.

14.4 PROBLEMS

1. a. Use dimensional analysis to establish the cube-root law

$$r\left(\frac{\rho}{W}\right)^{1/3} = \text{constant}$$

for scaling of explosions, where r is the radius or depth of the crater, ρ is the density of the soil medium, and W the mass of the explosive.

b. Use dimensional analysis to establish the one-fourth-root law

$$r\left(\frac{\rho g}{E}\right)^{1/4} = \text{constant}$$

for scaling of explosions, where r is the radius or depth of the crater, ρ is the density of the soil medium, g is gravity, and E is the charge energy of the explosive.

2. a. Show that the products $\prod_1, \prod_2, \prod_3, \prod_4$ for the refined explosion model presented in the text are dimensionless products.

b. Assume ρ is essentially constant for the soil being used and restrict the explosive to a specific type, say TNT. Under these conditions, ρ/δ is essentially constant, yielding $\prod_1 = f(\prod_2, \prod_3)$. You have collected the following data with $\prod_2 = 1.5 \times 10^{-6}$.

$\prod_3$	0	2	4	6	8	10	12	14
$\prod_1$	15	150	425	750	825	425	250	90

i. Construct a scatterplot of $\prod_1$ versus $\prod_3$. Does a trend exist?

ii. How accurate do you think the data are? Find an empirical model that captures the *trend* of the data with accuracy commensurate with your appraisal of the accuracy of the data.

iii. Use your empirical model to predict the volume of a crater using TNT in desert alluvium with (CGS system) $W = 1500$ g, $\rho = 1.53$ g/cm^3, and $\prod_3 = 12.5$.

3. Consider a zero-depth burst, spherical explosive in a soil medium. Assume the value of the crater volume V depends on the explosive size, energy yield, and explosive energy, as well as on the strength Y of the soil (considered a resistance to pressure with dimensions $ML^{-1}T^{-2}$), soil density ρ, and gravity g. In this problem, assume

$$V = f(W, Q_e, \delta, Y, \rho, g)$$

and use dimensional analysis to produce the following *mass set* of dimensionless products.

$$\prod_1 = \frac{V\rho}{W} \qquad \prod_2 = \left(\frac{g}{Q_e}\right)\left(\frac{W}{\delta}\right)^{1/3}$$

$$\prod_3 = \frac{Y}{\delta Q_e} \qquad \prod_4 = \frac{\rho}{\delta}$$

4. For the explosion process and material characteristics discussed in Problem 3, consider

$$V = f(E, Q_v, \delta, Y, \rho, g)$$

and use dimensional analysis to produce the following *energy set* of dimensionless products.

$$\overline{\prod}_1 = \frac{VQ_v}{E} \qquad \overline{\prod}_2 = \frac{\rho g E^{1/3}}{Q_v^{4/3}}$$

$$\overline{\prod}_3 = \frac{Y}{Q_v} \qquad \overline{\prod}_4 = \frac{\rho}{\delta}$$

5. Repeat Problem 4 for

$$V = f(E, Q_e, \delta, Y, \rho, g)$$

and use dimensional analysis to produce the following *gravity set* of dimensionless products.

$$\overline{\overline{\prod}}_1 = V\left(\frac{\rho g}{E}\right)^{3/4} \qquad \overline{\overline{\prod}}_2 = \left(\frac{1}{Q_e}\right)\left(\frac{g^3 E}{\delta}\right)^{1/4}$$

$$\overline{\overline{\prod}}_3 = \frac{Y}{\delta Q_e} \qquad \overline{\overline{\prod}}_4 = \frac{\rho}{\delta}$$

6. An experiment consists of dropping spheres into a tank of heavy oil and measuring the times of descent. It is desired that a relationship for the time of descent be determined and verified by experimentation. Assume the time of descent is a function of mass m, gravity g, radius r, viscosity μ, and distance traveled d. Neglect fluid density. That is,

$$t = f(m, g, r, \mu, d)$$

a. Use dimensional analysis to find a relationship for the time of descent.

b. How will the spheres be chosen to verify that the time of descent relationship is independent of fluid density? Assuming you have verified the assumptions on fluid density, describe how you would determine the nature of your function experimentally.

c. Using differential equations techniques, find the velocity of the sphere as a function of time, radius, mass, viscosity, gravity, and fluid density. Using this result and that found in part (a), predict under what conditions fluid density may be neglected. (*Hints:* Use the results of Problem 5 in Section 14.2 as a submodel for drag force. Consider the buoyant force.)[5]

7. A windmill is rotated by air flow to produce power to pump water. It is desired to find the power output P of the windmill. Assume that P is a function of the density of the air ρ, viscosity of the air μ, diameter of the windmill d, wind speed v, and the rotational speed of the windmill ω (measured in radians per second). Thus,

$$P = f(\rho, \mu, d, v, \omega)$$

a. Using dimensional analysis, find a relationship for P. Be sure to check your products to make sure that they are dimensionless.

b. Do your results make common sense? Explain.

c. Discuss how you would design an experiment to determine the nature of your function.

8. For a sphere traveling through a liquid, assume that the drag force F_D is a function of the fluid density ρ, fluid viscosity μ, radius of the sphere r, and speed of the sphere v. Use dimensional analysis to find a relationship for the drag force

$$F_D = f(\rho, \mu, r, v)$$

Make sure you provide some justification that the given independent variables influence the drag force.

14.4 PROJECT

1. Complete the requirements for the module, "Listening to the Earth: Controlled Source Seismology," by Richard G. Montgomery, UMAP 292-293. This module develops the elementary theory of wave reflection and refraction and applies it to a model of the earth's subsurface. The model shows how information on layer depth and sound velocity may

[5] For students who have studied differential equations.

be obtained to provide data on width, density, and composition of the subsurface. This module is a good introduction to controlled seismic methods and requires no previous knowledge of either physics or geology.

14.4 Further Reading

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14.5 Similitude

Suppose we are interested in the effects of wave action on a large ship at sea, heat loss of a submarine and the drag force it experiences in its underwater environment, or the wind effects on an aircraft wing. Quite often, because it is physically impossible to duplicate the actual phenomenon in the laboratory, we study a scaled-down model in a simulated environment to predict accurately the performance of the physical system. The actual physical system for which the predictions are to be made is called the **prototype**. How do we scale experiments in the laboratory to ensure that the effects observed for the model will be the same effects experienced by the prototype?

Although extreme care must be exercised in using simulations, the dimensional products resulting from dimensional analysis of the problem can provide insight into how the scaling for a model should be done. The idea comes from Buckingham's theorem. If the physical system can be described by a dimensionally homogeneous equation in the variables, then it can be put into the form

$$f\left(\prod_1, \prod_2, \ldots, \prod_n\right) = 0$$

for a complete set of dimensionless products. Assume that the independent variable of the problem appears only in the product $\prod_n$ and that

$$\prod_n = H\left(\prod_1, \prod_2, \ldots, \prod_{n-1}\right)$$

For the solution to the model and the prototype to be the same, it is sufficient that the value of all independent dimensionless products $\prod_1, \prod_2, \ldots, \prod_{n-1}$ be the same for the model and the prototype.

For example, suppose the Reynolds number $v r \rho / \mu$ appears as one of the dimensionless products in a fluid mechanics problem, where v represents fluid velocity, r a characteristic dimension (such as the diameter of a sphere or the length of a ship), ρ the fluid density, and μ the fluid viscosity. These values refer to the prototype. Next, let v_m, r_m, ρ_m, and μ_m denote the corresponding values for the scaled-down model. For the effects on the model and the prototype to be the same, we want the two Reynolds numbers to agree so that

$$\frac{v_m r_m \rho_m}{\mu_m} = \frac{v r \rho}{\mu}$$

The last equation is referred to as a *design condition* to be satisfied by the model. If the length of the prototype is too large for the laboratory experiments so that we have to scale down the length of the model, say $r_m = r/10$, then the same Reynolds number for the model and the prototype can be achieved by using the same fluid ($\rho_m = \rho$ and $\mu_m = \mu$) and varying the velocity, $v_m = 10v$. If it is impractical to scale the velocity by the factor of 10, we can instead scale it by a lesser amount $0 < k < 10$ and use a different fluid so that the equation

$$\frac{k \rho_m}{10 \mu_m} = \frac{\rho}{\mu}$$

is satisfied. We do need to be careful in generalizing the results from the scaled-down model to the prototype. Certain factors (such as surface tension) that may be negligible for the prototype may become significant for the model. Such factors would have to be taken into account before making any predictions for the prototype.

EXAMPLE 1 *Drag Force on a Submarine*

We are interested in the drag forces experienced by a submarine to be used for deep-sea oceanographic explorations. We assume that the variables affecting the drag D are fluid velocity v, characteristic dimension r (here, the length of the submarine), fluid density ρ, the fluid viscosity μ, and the velocity of sound in the fluid c. We wish to predict the drag force by studying a model of the prototype. How will we scale the experiments for the model?

A major stumbling block in our problem is in describing shape factors related to the physical object being modeled—in this case, the submarine. Let's consider submarines that are ellipsoidal in shape. In two dimensions, if a is the length of the major axis and b is the length of the minor axis of an ellipse, we can define $r_1 = a/b$ and assign a characteristic dimension such as r, the length of the submarine. In three dimensions, define also $r_2 = a/b'$, where a is the original major axis and b' is the second minor axis. Then r, r_1, and r_2 describe the shape of the submarine. In a more irregularly shaped object, additional shape factors would be required. The basic idea is that the object can be described using a characteristic dimension and an appropriate collection of shape factors. In the case of our three-dimensional ellipsoidal submarine, the shape factors r_1 and r_2 are needed. These shape factors are dimensionless constants.

Returning to our list of six fluid mechanics variables D, v, r, ρ, μ, and c, notice that we are neglecting surface tension (because it is small) and that gravity is not being considered. Thus, it is expected that a dimensionless analysis will produce three $(6 - 3)$ independent dimensionless products. We can choose the following three products for convenience:

Reynolds number	$R = \dfrac{vr\rho}{\mu}$
Mach number	$M = \dfrac{v}{c}$
Pressure coefficient	$P = \dfrac{p}{\rho v^2}$

The added shape factors are dimensionless so that Buckingham's theorem gives the equation

$$h(P, M, R, r_1, r_2) = 0$$

Assuming that we can solve for P yields

$$P = H(M, R, r_1, r_2)$$

Substituting $P = p/\rho v^2$ and solving for p gives

$$p = \rho v^2 H(R, M, r_1, r_2)$$

Remembering that the total drag force is the pressure (force per unit area) times the area (which is proportional to r^2 for geometrically similar objects) and gives the proportionality $D \propto pr^2$, or

$$D = kpv^2r^2H(R, M, r_1, r_2) \tag{14.28}$$

Now a similar equation must hold to give the proportionality for the model

$$D_m = kp_m v_m^2 r_m^2 H(R_m, M_m, r_{1m}, r_{2m}) \tag{14.29}$$

Because the prototype and model equations refer to the same physical system, both equations are identical in form. Therefore, the design conditions for the model require that

Condition (a)	$R_m = R$
Condition (b)	$M_m = M$
Condition (c)	$r_{1m} = r_1$
Condition (d)	$r_{2m} = r_2$

Note that if conditions (a)–(d) are satisfied, then Equations (14.28) and (14.29) give

$$\frac{D_m}{D} = \frac{\rho_m v_m^2 r_m^2}{\rho v^2 r^2} \tag{14.30}$$

Thus, D can be computed once D_m is measured. Note that the design conditions (c) and (d) imply geometric similarity between the model and the prototype submarine

$$\frac{a_m}{b_m} = \frac{a}{b} \quad \text{and} \quad \frac{a_m}{b'_m} = \frac{a}{b'}$$

If the velocities are small compared to the speed of sound in a fluid, then v/c can be considered constant in accordance with condition (b). If the same fluid is used for both the model and prototype, then condition (a) is satisfied if

$$v_m r_m = vr$$

or

$$\frac{v_m}{v} = \frac{r}{r_m}$$

which states that the velocity of the model must increase inversely as the scaling factor r_m/r. Under these conditions, Equation (14.30) yields

$$\frac{D_m}{D} = \frac{\rho_m v_m^2 r_m^2}{\rho v^2 r^2} = 1$$

If increasing the velocity of the scaled model proves unsatisfactory in the laboratory, then a different fluid may be considered for the scaled model ($\rho_m \neq \rho$ and $\mu_m \neq \mu$). If the ratio v/c is small enough to neglect, then both v_m and r_m can be varied to ensure that

$$\frac{v_m r_m \rho_m}{\mu_m} = \frac{vr\rho}{\mu}$$

in accordance with condition (a). Having chosen values that satisfy design condition (a), and knowing the drag on the scaled model, we can use Equation (14.30) to compute the drag on the prototype. Consider the additional difficulties if the velocities are sufficiently great that we must satisfy condition (b) as well.

A few comments are in order. One distinction between the Reynolds number and the other four numbers in fluid mechanics is that the Reynolds number contains the viscosity of the fluid. Dimensionally, the Reynolds number is proportional to the ratio of the inertia forces of an element of fluid to the viscous force acting on the fluid. In certain problems the numerical value of the Reynolds number may be significant. For example, the flow of a fluid in a pipe is virtually always parallel to the edges of the pipe (giving *laminar flow*) if the Reynolds number is less than 2000. Reynolds numbers in excess of 3000 almost always indicate turbulent flow. Normally, there is a critical Reynolds number between 2000 and 3000 at which the flow becomes turbulent.

The design condition (a) mentioned earlier requires the Reynolds number of the model and the prototype to be the same. This requirement precludes the possibility of laminar flow in the prototype being represented by turbulent flow in the model, and vice versa. The equality of the Reynolds number for a model and prototype is important in all problems in which viscosity plays a significant role.

The Mach number is the ratio of fluid velocity to the speed of sound in the fluid. It is generally important for problems involving objects moving with high speed in fluids, such as projectiles, high-speed aircraft, rockets, and submarines. Physically, if the Mach number is the same in model and prototype, the effect of the compressibility force in the fluid relative to the inertia force will be the same for model and prototype. This is the situation that is required by condition (b) in our example on the submarine.

14.5 PROBLEMS

1. A model of an airplane wing is tested in a wind tunnel. The model wing has an 18-in. chord, and the prototype has a 4-ft chord moving at 250 mph. Assuming the air in the wind tunnel is at atmospheric pressure, at what velocity should wind tunnel tests be conducted so that the Reynolds number of the model is the same as that of the prototype?

2. Two smooth balls of equal weight but different diameters are dropped from an airplane. The ratio of their diameters is 5. Neglecting compressibility (assume constant Mach number), what is the ratio of the terminal velocities of the balls? Are the flows completely similar?

3. Consider predicting the pressure drop Δp between two points along a smooth horizontal pipe under the condition of steady laminar flow. Assume

$$\Delta p = f(s, d, \rho, \mu, v)$$

where s is the control distance between two points in the pipe, d is the diameter of the pipe, ρ is the fluid density, μ is the fluid viscosity, and v is the velocity of the fluid.

   a. Determine the design conditions for a scaled model of the prototype.

   b. Must the model be geometrically similar to the prototype?

   c. May the same fluid be used for model and prototype?

   d. Show that if the same fluid is used for both model and prototype, then the equation is

$$\Delta p = \frac{\Delta p_m}{n^2}$$

   where $n = d/d_m$.

4. It is desired to study the velocity v of a fluid flowing in a smooth open channel. Assume that

$$v = f(r, \rho, \mu, \sigma, g)$$

where r is the characteristic length of the channel cross-sectional area divided by the wetted perimeter, ρ is the fluid density, μ is the fluid viscosity, σ is the surface tension, and g is the acceleration of gravity.

   a. Describe the appropriate pair of shape factors r_1 and r_2.

   b. Show that

$$\frac{v^2}{gr} = H\left(\frac{\rho v r}{\mu}, \frac{\rho v^2 r}{\sigma}, r_1, r_2\right)$$

   Discuss the design conditions required of the model.

c. Will it be practical to use the same fluid in the model and the prototype?

d. Suppose the surface tension σ is ignored and the design conditions are satisfied. If $r_m = r/n$, what is the equation for the velocity of the prototype? When is the equation compatible with the design conditions?

e. What is the equation for the velocity v if gravity is ignored? What if viscosity is ignored? What fluid would you use if you were to ignore viscosity?

14.5 Further Reading

Massey, Bernard S. *Units, Dimensional Analysis and Physical Similarity*. London: Van Nostrand Reinhold, 1971.