14. MODELING OF THIN-WALLED SHELLS AND PLATES. INTRODUCTION TO THE THEORY OF SHELL FINITE ELEMENT MODELS

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14.1 Plate and shell theories

Plane structures are called plates if the thickness of structure is significantly less than the other dimensions, moreover if the structure is loaded perpendicularly to its plane. The plate can be bounded along its sides by an optional geometrical object; the kinematic boundary conditions can be various (point-supported, rigidly or elastically supported along the sides, simply supported, etc.) [1]. The plate can be considered as the extension of a beam in two dimensions, because both implies the dominance of the bending load and most commonly the load is introduced transversely. Nevertheless, there are significant differences too, since e.g. the flexure of the beam can be either straight or curved, on the other hand the midplane of a plate is always flat. If the midplane of the plate is curved then it is no longer plate but a shell [2]. In the sequel we overview the most important details of the theory of plates and shells.

14.2 The basic equations of Kirchhoff plate theory

The Kirchhoff plate theory is often called the theory of thin plates. We note that if the plate is relatively thick then the transverse shear deformation can be considered too. The relevant plate solution is provided by the Mindlin plate theory [1].

14.2.1 Displacement field

Based on Fig.14.1 we investigate the displacement of a point of the midplane of an elastic flat plate [2,3]. The displacement field can be captured by three components: the transverse displacement along z and the rotations about x and y, i.e.:

$$\underline{u} = \begin{bmatrix} \beta z \\ -\alpha z \\ w \end{bmatrix}, \qquad (14.1)$$

where $\alpha = \alpha(x,y)$ is the rotation about axis x, $\beta = \beta(x,y)$ is the rotation about axis y and $w = w(x,y)$ is the transverse displacement.

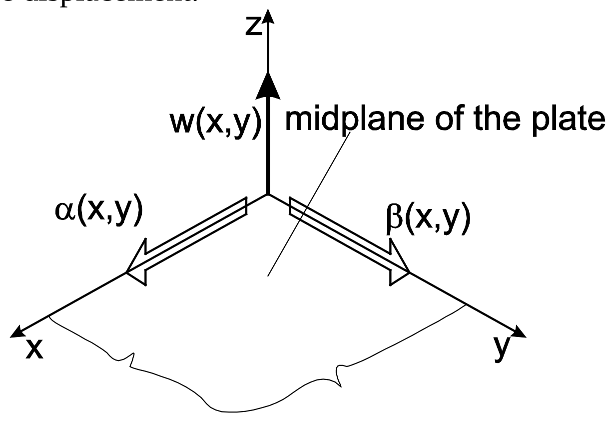

Fig.14.1.Displacement of a point in the midplane of a flat plate.

14.2.2 Strain components

Assuming small strains we can calculate the strain components by using the strain-displacement equation defined in section 11 by Eq.(11.14) [1,4]:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \beta_{,x} z\,,\ \varepsilon_y = \frac{\partial v}{\partial y} = -\alpha_{,y} z\,,\ \varepsilon_z = 0\,, \qquad (14.2)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = (\beta_{,y} - \alpha_{,x})z\,,\ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \beta + w_{,x}\,,\ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\alpha + w_{,y}\,,$$

where – for the sake of simplicity - the derivatives with respect to x and y are indicated in the subscript. In the sequel we assume that the cross section planes remain flat and the outward normal of each cross section is perpendicular to the cross section plane after the deformation. This assumption is called Kirchhoff-Love hypothesis [1]. From the latter it follows that in the planes perpendicular to the midplane of the plate the shear strains are equal to zero:

$$\gamma_{xz} = \gamma_{yz} = 0 \Rightarrow \beta = -w_{,x} \text{ and } \alpha = w_{,y}\,. \qquad (14.3)$$

Utilizing the former we obtain from Eq.(14.1) that:

$$\underline{u} = \begin{bmatrix} -w_{,x} \cdot z \\ -w_{,y} \cdot z \\ w \end{bmatrix}. \qquad (14.4)$$

The strain components become:

$$\varepsilon_x = -w_{,xx} \cdot z\,,\ \varepsilon_y = -w_{,yy} \cdot z\,,\ \gamma_{xy} = -2w_{,xy} \cdot z\,. \qquad (14.5)$$

Consequently in the midplane points $\varepsilon_z = 0$. According to the Kirchhoff plate theory under the assumption of small strains the components of the displacement and strain field can be defined by $w(x,y)$.

14.2.3 Stress field, forces and moments in the midplane

Assuming plane stress state we express the stress components by Eqs.(11.18) and (14.5):

$$\sigma_x = \frac{E}{1-\nu^2}\left[\varepsilon_x + \nu\varepsilon_y\right] = -E_1(w_{,xx} + \nu \cdot w_{,yy}) \cdot z = A \cdot z\,, \qquad (14.6)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left[\varepsilon_y + \nu\varepsilon_x\right] = -E_1(w_{,yy} + \nu \cdot w_{,xx}) \cdot z = B \cdot z\,,$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = -E_1(1-\nu)w_{,xy} \cdot z = C \cdot z\,,$$

where $E_1 = E/(1-\nu^2)$, A, B and C are constants. Similarly to the theory of beams subjected to bending the stress distributions are given by linear functions along the thickness direction, as it is shown by Fig.14.2.

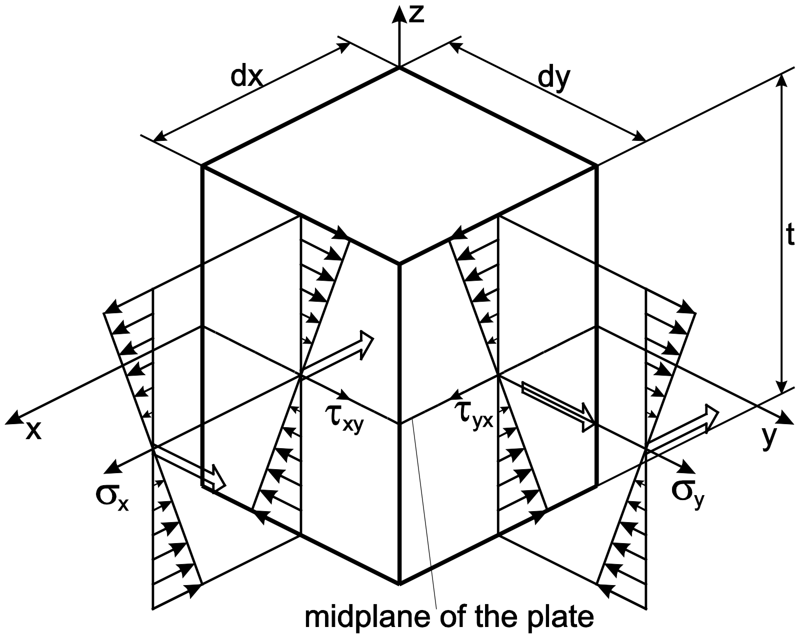

Fig.14.2. Distribution of the stresses along the thickness direction of a differential plate element.

The stress couples in the midplane of the plate are calculated by integrating the stresses over the thickness [3]:

$$M_x = \int_{-t/2}^{t/2} \sigma_x z dz = \int_{-t/2}^{t/2} A z^2 dz = -I_1 E_1 (w_{,xx} + \nu \cdot w_{,yy}), \qquad (14.7)$$

$$M_y = \int_{-t/2}^{t/2} \sigma_y z dz = \int_{-t/2}^{t/2} B z^2 dz = -I_1 E_1 (w_{,yy} + \nu \cdot w_{,xx}),$$

$$M_{xy} = -\int_{-t/2}^{t/2} \tau_{xy} z dz = -\int_{-t/2}^{t/2} C z^2 dz = I_1 E_1 (1-\nu) w_{,xy},$$

$$M_{yx} = \int_{-t/2}^{t/2} \tau_{yx} z dz = \int_{-t/2}^{t/2} C z^2 dz = -I_1 E_1 (1-\nu) w_{,xy},$$

where M_x is the bending moment along axis x, M_y is the bending moment along axis y, M_{xy} and M_{yx} are the twisting moments. Moreover:

$$I_1 = \frac{t^3}{12}, \qquad (14.8)$$

which is – similarly to beams – the second order moment of inertia of the cross section. The stress couples in the midplane of the plate are demonstrated in Fig.14.3a. The relationships between stresses and stress couples (bending and twisting moments) based on Eqs.(14.6) and (14.7) are:

$$\sigma_x = \frac{M_x}{I_1} z, \sigma_y = \frac{M_y}{I_1} z, \tau_{xy} = \frac{M_{xy}}{I_1} z. \qquad (14.9)$$

For the equilibrium of a differential plate element transverse shear forces are required. Transverse shear forces are shown by Fig.14.3b and they can be calculated using the following formulae [1,3]:

$$Q_x = \int_{-t/2}^{t/2} \tau_{xz} dz \,,\; Q_y = \int_{-t/2}^{t/2} \tau_{yz} dz \,. \tag{14.10}$$

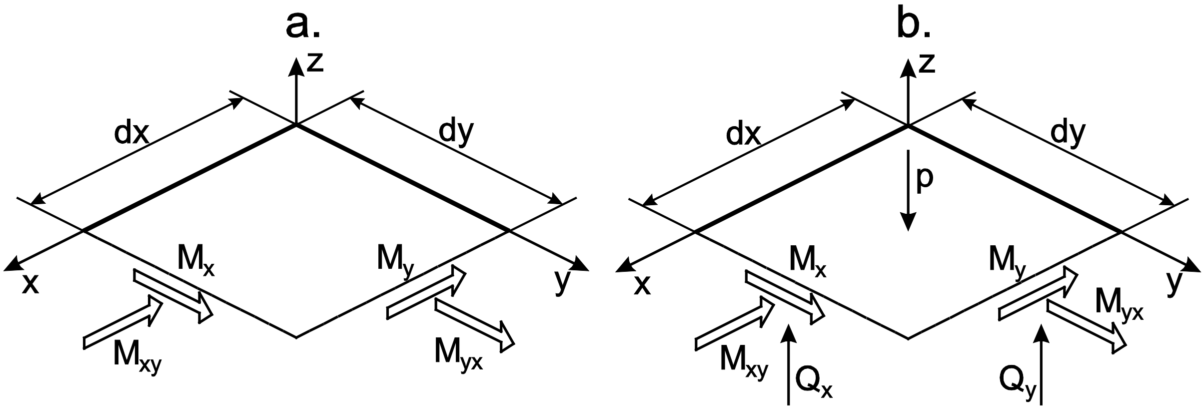

Fig.14.3. Stress couples in the midplane of a thin differential plate element (a) and its equilibrium in the case of transverse shear forces and distributed load (b).

14.2.4 The equilibrium and governing equation of thin plates

The homogeneous equilibrium equation with respect to the stress field has already been introduced in section 11. [4]:

$$\underline{\underline{\sigma}} \cdot \nabla = \underline{0} \,, \tag{14.11}$$

of which first component equation is:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \,. \tag{14.12}$$

Integrating the equation with respect to z yields:

$$\int \frac{\partial}{\partial x} Azdz + \int \frac{\partial}{\partial y} Czdz + \tau_{xz} - \tau_{xz}^0 = 0 \,, \tag{14.13}$$

and:

$$\frac{\partial}{\partial x}\left(\frac{1}{2} Az^2\right) + \frac{\partial}{\partial y}\left(\frac{1}{2} Cz^2\right) = \tau_{xz}^0 - \tau_{xz} \,, \tag{14.14}$$

finally:

$$\frac{\partial}{\partial x}\left(\frac{1}{2} \sigma_x z\right) + \frac{\partial}{\partial y}\left(\frac{1}{2} \tau_{xy} z\right) = \tau_{xz}^0 - \tau_{xz} \,. \tag{14.15}$$

Next, we integrate Eq.(14.15) within the ranges of $-t/2$ and $t/2$:

$$\frac{\partial}{\partial x}\left(\int_{-t/2}^{t/2} \sigma_x zdz\right) + \frac{\partial}{\partial y}\left(\int_{-t/2}^{t/2} \tau_{xy} zdz\right) = 2\int_{-t/2}^{t/2} (\tau_{xz}^0 - \tau_{xz}(z))dz \,, \tag{14.16}$$

where τ^0_{xz} is an integration constant. A possible solution for τ_{xz}, which satisfies even the dynamic boundary conditions is [3]:

$$\tau_{xz} = \tau_{xz}^0 \left(1 - \frac{4z^2}{t^2}\right). \tag{14.17}$$

In fact Eq.(14.17) gives the difference between the area under a rectangle and a parabola, which is 1/3 of the total area. Accordingly, if it is multiplied by two, then mathematically we obtain the area under the parabola, that is, from Eq.(14.16) we have:

$$2\int_{-t/2}^{t/2}(\tau_{xz}^{0}-\tau_{xz}(z))dz=\int_{-t/2}^{t/2}\tau_{xz}zdz=Q_x\,, \qquad (14.18)$$

which is not else than the shear force along axis x given by Eq.(14.10). Taking Eqs.(14.6), (14.9) and (14.18) back into the equilibrium equation we obtain:

$$\frac{\partial M_x}{\partial x}-\frac{\partial M_{xy}}{\partial y}-Q_x=0\,. \qquad (14.19)$$

The second component equation and the corresponding equilibrium equation in terms of the stress couples and transverse shear force are:

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}=0\,, \qquad (14.20)$$

$$\frac{\partial M_{yx}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=0\,,$$

and:

$$M_{xy}=-M_{yx}\,. \qquad (14.21)$$

From the third component equation of Eq.(14.11) we obtain the following:

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}=0\,. \qquad (14.22)$$

We integrate Eq.(14.22) within the ranges of –*t*/2 and *t*/2 with respect to *z*:

$$\int_{-t/2}^{t/2}\frac{\partial \tau_{xz}}{\partial x}dz+\int_{-t/2}^{t/2}\frac{\partial \tau_{yz}}{\partial y}dz+\int_{-t/2}^{t/2}\frac{\partial \sigma_z}{\partial z}dz=0\,, \qquad (14.23)$$

and:

$$\frac{\partial}{\partial x}\int_{-t/2}^{t/2}\tau_{xz}dz+\frac{\partial}{\partial y}\int_{-t/2}^{t/2}\tau_{yz}dz+\int_{-t/2}^{t/2}d\sigma_z=0\,. \qquad (14.24)$$

Based on Eq.(14.10) the first two terms are the shear forces Q_x and Q_y, the third one is – in accordance with the dynamic boundary condition – the intensity of the distributed load, *p*, perpendicularly to the midplane, i.e.:

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+p=0\,. \qquad (14.25)$$

Summarizing the equilibrium equations we have:

$$M_{x,x}-M_{xy,y}-Q_x=0\,, \qquad (14.26)$$

$$M_{y,y}+M_{yx,x}-Q_y=0\,,$$

$$Q_{x,x}+Q_{y,y}+p=0\,.$$

To derive the plate equation we rearrange the first two equations:

$$Q_{x,x}=M_{x,xx}-M_{xy,yx}\,, \qquad (14.27)$$

$$Q_{y,y}=M_{y,yy}+M_{yx,xy}\,.$$

Taking them back into the third of Eq.(14.26) we obtain the following:

$$M_{x,xx}-2M_{xy,xy}+M_{y,yy}+p=0\,. \qquad (14.28)$$

By the help of Eq.(14.7) we have:

$$-I_1E_1(w_{,xxxx}+\nu\cdot w_{,yyxx}+w_{,yyyy}+\nu\cdot w_{,xxyy}+2(1-\nu)w_{,xyxy})=-p\,, \qquad (14.29)$$

which, after a simple rearrangement have the form of [5]:

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{p}{I_1 E_1}, \qquad (14.30)$$

or:

$$\nabla^2\nabla^2 w(x,y)=\frac{p}{I_1 E_1}. \qquad (14.31)$$

Consequently the governing equation is a fourth order partial differential equation with the proper kinematic and dynamic boundary conditions. That means that the problem of plates subjected to bending is a boundary value problem.

14.3 Finite element equations of thin plates

For the finite element solution of the problem of thin plates subjected to bending we collect the strain and stress field components into vectors and we assume plane stress state [1,6]:

$$\underline{\varepsilon}^T=[\varepsilon_x,\varepsilon_y,\gamma_{xy}], \qquad (14.32)$$
$$\underline{\sigma}^T=[\sigma_x,\sigma_y,\tau_{xy}].$$

Based on Eq.(14.5) the strain components can be written as:

$$\underline{\varepsilon}^T=-z\underline{\kappa}, \qquad (14.33)$$

where $\underline{\kappa}$ is the vector of curvatures:

$$\underline{\kappa}=\begin{bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{bmatrix}=\begin{bmatrix}w_{,xx}\\ w_{,yy}\\ 2w_{,xy}\end{bmatrix}. \qquad (14.34)$$

Incorporating the material law we formulate the vector of stress components as:

$$\underline{\sigma}=\underline{\underline{C}}^{str}\underline{\varepsilon}. \qquad (14.35)$$

The strain components can be obtained by a two-variable function $w(x,y)$, the finite element interpolation of the $w(x,y)$ function depends on the element type and the chosen degrees of freedom, but it can always be formulated in the form below:

$$w(x,y)=\underline{A}^T\underline{\lambda}, \qquad (14.36)$$

where $\underline{A}$ is the vector of unknown coefficients, $\underline{\lambda}$ is the vector of basis polynomials. The vector of nodal displacements is:

$$\underline{u}_e=\underline{\underline{M}}\,\underline{A}, \qquad (14.37)$$

which, for example in the case of a triangle element with three nodes becomes:

$$\underline{u}_e^T=[w_1 \quad \alpha_1 \quad \beta_1 \quad w_2 \quad \alpha_2 \quad \beta_2 \quad w_3 \quad \alpha_3 \quad \beta_3]. \qquad (14.38)$$

In Eq.(14.37) matrix $\underline{\underline{M}}$ can be calculated based on the approximate $w(x,y)$ function and Eq.(14.1). The α_i and β_i parameters are the rotations about the axes x and y in the corresponding nodes, where i = 1, 2, 3. From Eq.(14.37) we have:

$$\underline{A}=\underline{\underline{M}}^{-1}\underline{u}_e. \qquad (14.39)$$

Generally speaking, the vector of strain components can be determined using the strain-displacement matrix:

$$\underline{\varepsilon}=\underline{\underline{B}}\,\underline{u}_e, \qquad (14.40)$$

where Eq.(14.40) can be reformulated utilizing Eqs.(14.5), (14.37) and (14.39) as follows:

$$\underline{\varepsilon} = \underline{\underline{R}}\,\underline{A} = \underline{\underline{R}}\,\underline{\underline{M}}^{-1}\underline{u}_e, \tag{14.41}$$

where matrix $\underline{\underline{R}}$ establishes the relationship between the vector of strain components and the vector of unknown coefficients, its dimension is element dependent. Consequently we have:

$$\underline{\underline{B}} = \underline{\underline{R}}\,\underline{\underline{M}}^{-1}. \tag{14.42}$$

Following the definition by Eq.(12.9) we formulate the element stiffness matrix as:

$$\underline{\underline{K}}_e = \int_{V_e} \underline{\underline{B}}^T \underline{\underline{C}}^{str^T} \underline{\underline{B}} dV. \tag{14.43}$$

The dimension of the element stiffness matrix depends on the number of nodes and the number of nodal degrees of freedom. Similarly to the plane membrane element, the vector of forces is composed as the sum of several terms. The most common is the distributed (surface) load and concentrated force. By formulating the work of external forces we derive the force vector related to the distributed load:

$$W_e = \int_{A_{pe}} p \cdot w(x,y) dA = \underline{u}_e^T \underline{F}_{ep}, \tag{14.44}$$

where p is the intensity of the distributed load perpendicularly to the midplane of the plate, $w(x,y)$ is the approximate function of the deflection surface according to Eq.(14.36). The vector $\underline{F}_{ep}$ can be determined based on the vector of nodal displacements. In the case of concentrated loads, considering e.g. a triangular shape plate element with three nodes, at each node there can be a force perpendicularly to the plate surface and even concentrated moments acting about the x and y axes, respectively:

$$\underline{F}_{ec}^T = \begin{bmatrix} F_{z1} & M_{x1} & M_{y1} & F_{z2} & M_{x2} & M_{y2} & F_{z3} & M_{x3} & M_{y3} \end{bmatrix}. \tag{14.45}$$

Thus, the vector of forces becomes:

$$\underline{F}_e = \underline{F}_{ep} + \underline{F}_{ec}. \tag{14.46}$$

Eventually, the finite element equilibrium equation for a single element and for the whole structure is:

$$\underline{\underline{K}}_e \underline{u}_e = \underline{F}_e, \underline{\underline{K}}\,\underline{U} = \underline{F}. \tag{14.47}$$

Similarly to the plane membrane elements there is large number of plate bending elements. These elements will be reviewed in section 15.

14.4 Basic equations of the technical theory of thin shells

In that case when the midplane of a thin-walled structure is not flat but curved, then we talk about shells. The analytical investigation of shells requires considerably complicated mathematical computations. Therefore in the sequel only the most important equations will be presented.

14.4.1 Geometrical equations

Due to the fact that the midsurface of shells is curved, we need to introduce curvilinear coordinate systems, as it is shown by Fig.14.4.

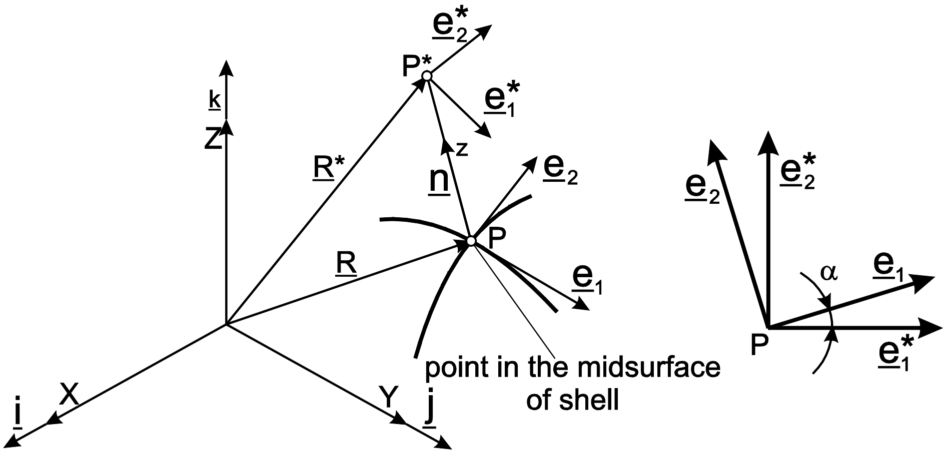

Fig.14.4. Coordinate lines and unit basis vectors of the midsurface of a shell.

The two-parameter representation of the midsurface of shells can be formulated in the form of a vector equation [1,4]:

$$\underline{R} = \underline{R}(q_1, q_2)\,, \tag{14.48}$$

where:

$$X = X(q_1, q_2)\,,\ Y = Y(q_1, q_2)\,,\ Z = Z(q_1, q_2)\,, \tag{14.49}$$

are the global coordinates, $\underline{R}$ is the position vector of a point in the, q_1 and q_2 are the general or curvilinear coordinates of the surface (see Fig.14.4). If the parameters take on the values q_1 = constant and q_2 = constant, we obtain the q_1 and q_2 coordinate lines. The tangent unit vectors $\underline{e}_i$ and the arc lengths dS_i of the coordinate lines are:

$$\underline{e}_i = \frac{1}{H_i}\frac{\partial \underline{R}}{\partial q_i} = \frac{1}{H_i}\underline{R}_i\,,\ dS_i = H_i dq_i\,, \tag{14.50}$$

where:

$$H_i = \left|\underline{R}_{,i}\right|,\ i = 1,\,2, \tag{14.51}$$

are the so-called Lamé parameters [1] or metric coefficients [4]. In the followings we assume that the local coordinate axes are mutually perpendicular at each point, and the curvilinear system is orthogonal, i.e. $\underline{e}_1 \cdot \underline{e}_2 = 0$. The outward unit normal vector of the midsurface becomes:

$$\underline{n} = \underline{e}_1 \times \underline{e}_2\,. \tag{14.52}$$

The triad of unit orthogonal vectors [$\underline{e}_1$, $\underline{e}_2$, $\underline{n}$] determines an orthogonal curvilinear coordinate system at an actual point P. The curvature and the torsion of coordinate lines are given by the Frenet formulae [1,7]:

$$\frac{1}{R_i} = -\underline{n}\cdot\frac{\partial^2 \underline{R}}{\partial S_i^2} = -\underline{n}\cdot\frac{1}{H_i^2}\frac{\partial^2 \underline{R}}{\partial q_i^2} = -\frac{1}{H_i^2}\underline{n}\cdot\underline{R}_{,ii}\,,\ i = 1,2, \tag{14.53}$$

$$\frac{1}{R_{12}} = -\underline{n}\cdot\frac{\partial^2 \underline{R}}{\partial S_1 \partial S_2} = -\underline{n}\cdot\frac{1}{H_1 H_2}\underline{R}_{,12}\,,$$

where R_1 and R_2 are the radii of curvature. If $R_{12} = 0$, then the q_1 and q_2 lines are the lines of principal curvatures on the midsurface, moreover the directions of the unit basis vectors $\underline{e}_1$ and $\underline{e}_2$ are the principal directions. The curvature of the midsurface is a tensor quantity. If the directions of vectors $\underline{e}_1$' and $\underline{e}_2$' are not the principal directions, then the angle, which determines the principal directions can be obtained by:

$$tg2\alpha = \frac{2/R_{12}^{'}}{1/R_1^{'}+1/R_2^{'}}. \qquad (14.54)$$

The values of the principal curvatures are [1,7]:

$$\frac{1}{R_1} = \frac{\cos^2\alpha}{R_1^{'}} + \frac{\sin^2\alpha}{R_2^{'}} + \frac{\sin 2\alpha}{R_{12}^{'}}, \qquad (14.55)$$

$$\frac{1}{R_2} = \frac{\sin^2\alpha}{R_1^{'}} + \frac{\cos^2\alpha}{R_2^{'}} - \frac{\sin 2\alpha}{R_{12}^{'}},\ \frac{1}{R_{12}} = 0.$$

In the followings we investigate the special case, when the directions of unit basis vectors coincide with the principal directions. The derivatives of the unit basis vectors of the coordinate system on the midsurface are [1,4]:

$$\underline{e}_{i,i} = -\frac{H_{i,j}}{H_j}\underline{e}_j - \frac{H_i}{R_i}\underline{n},\ \underline{e}_{i,j} = -\frac{H_{j,i}}{H_i}\underline{e}_j,\ \underline{n}_{,j} = \frac{H_j}{R_j}\underline{e}_j,\ i \neq j,\ i,j = 1,2. \qquad (14.56)$$

Point P^* is located on a surface parallel to the midsurface and the distance of point P^* from point P is given by coordinate z measured along the normal vector $\underline{n}$. Based on Fig.14.4 the position vector of point P^* is:

$$\underline{R}^* = \underline{R} + z\underline{n}. \qquad (14.57)$$

The unit vectors are independent of coordinate z, viz.:

$$\underline{e}_i^* = \underline{e}_i. \qquad (14.58)$$

The derivative of the position vector of point P^* can be written as:

$$\underline{R}_{,i}^* = \underline{R}_{,i} + z\underline{n}_{,i} = H_i(1+\frac{z}{R_i})\underline{e}_i. \qquad (14.59)$$

Consider the followings:

$$H_i^* = H_i(1+\frac{z}{R_i}) \text{ and } dS_i^* = dS_i(1+\frac{z}{R_i}),\ i = 1,2. \qquad (14.60)$$

which are the Lamé parameters and arc lengths with respect to point P^*.

14.4.2 Stress resultants and couples, equilibrium equations

Fig.14.5 shows the stresses on the boundary planes of a differential shell element, while Fig.14.6 presents the stress resultants and couples (internal forces and moments) on the midsurface of the differential shell element with dimensions of $dS_1 x dS_2$.

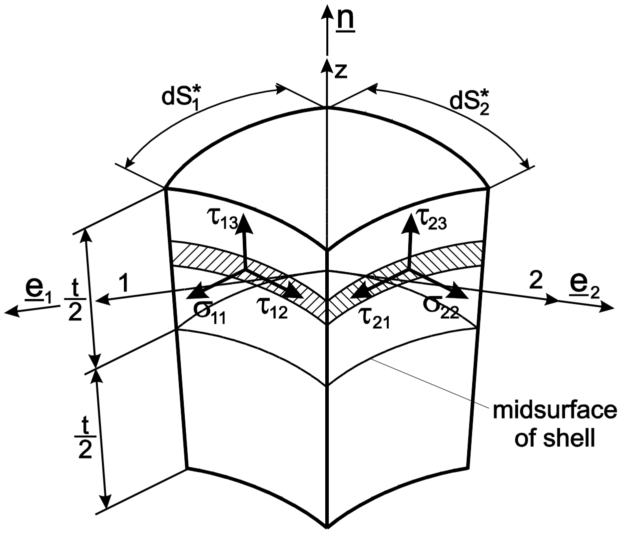

Fig.14.5. Stress components on the boundary planes of a differential shell element.

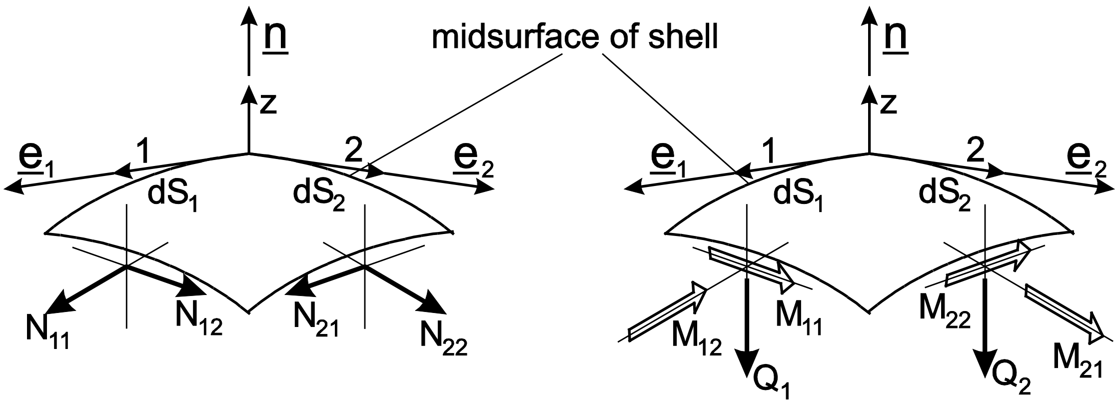

Fig.14.6. Internal forces and moments in the midsurface of a differential shell element.

We must consider the relationship between the arc lengths dS_i and dS_i^* given by Eq.(14.60) when we establish the relationship between the stresses acting on the differential shell element with thickness t and the internal forces, moments on the midsurface of the shell element. The stress resultants and stress couples acting on the curve with outward normal $\underline{e}_1$ are:

$$N_{11}=\int_{-t/2}^{t/2}\sigma_{11}(1+\frac{z}{R_2})dz\,,\ N_{12}=\int_{-t/2}^{t/2}\tau_{12}(1+\frac{z}{R_2})dz\neq N_{21}\,,\ Q_1=-\int_{-t/2}^{t/2}\tau_{13}(1+\frac{z}{R_2})dz\,,$$

$$M_{11}=\int_{-t/2}^{t/2}\sigma_{11}z(1+\frac{z}{R_2})dz\,,\ M_{12}=\int_{-t/2}^{t/2}\tau_{12}z(1+\frac{z}{R_2})dz\neq M_{21}\,,\ (14.61)$$

where N_{11} is the in-plane normal force, N_{12} and N_{21} are the in-plane shear forces, Q_1 is the transverse shear force, M_{11} is the bending moment, M_{12} and M_{21} are the twisting moments, respectively. It must be taken into consideration that although the reciprocity law of shear stresses implies τ_{12} = τ_{21}, in the equations above $N_{12} \neq N_{21}$ and $M_{12} \neq M_{21}$, which can be explained by the fact that the radii of curvatures are in general not equal to each other, i.e.: $R_1 \neq R_2$. The development of equilibrium equations establishing the equilibrium between external loads and internal forces and moments in the shell structure is also very complicated. Therefore we present only the resulting equations. The equilibrium equations in the case of stress resultants are [1,7]:

$$(H_2N_{11})_{,1}+(H_1N_{21})_{,2}+N_{12}H_{1,2}-N_{22}H_{2,1}-H_1H_2(\frac{Q_1}{R_1}+p_1)=0\,,$$

$$(H_2N_{12})_{,1}+(H_1N_{22})_{,2}+N_{21}H_{2,1}-N_{11}H_{1,2}-H_1H_2(\frac{Q_2}{R_2}+p_2)=0\,, \tag{14.62}$$

$$(H_2Q_1)_{,1}+(H_1Q_2)_{,2}+H_1H_2(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}-p_3)=0\,,$$

where p_1 and p_2 are the tangentially distributed loads along directions 1 and 2, p_3 is the distributed load perpendicularly to the shell midsurface. The equilibrium equations in the case of stress couples and moment of stress resultants are:

$$(H_2M_{12})_{,1}+(H_1M_{22})_{,2}+M_{21}H_{2,1}-M_{11}H_{1,2}-H_1H_2Q_2=0\,,$$

$$(H_2M_{11})_{,1}+(H_1M_{21})_{,2}+M_{12}H_{1,2}-M_{22}H_{2,1}+H_1H_2Q_1=0\,, \tag{14.63}$$

$$\frac{M_{12}}{R_1}-\frac{M_{21}}{R_2}+N_{12}-N_{21}=0\,, \tag{14.64}$$

where in the subscript the comma and the number refer to the differentiation with respect to the corresponding coordinate.

14.4.3 Displacement field, strain components

Based on Fig.14.7 the vector of displacements and rotations in a point P on the shell midsurface can be written as:

$$\underline{u}=u\underline{e}_1+v\underline{e}_2+w\underline{n}\,,\ \underline{\beta}=\beta_1\underline{e}_1-\beta_2\underline{e}_2+\beta_3\underline{n}\,. \tag{14.65}$$

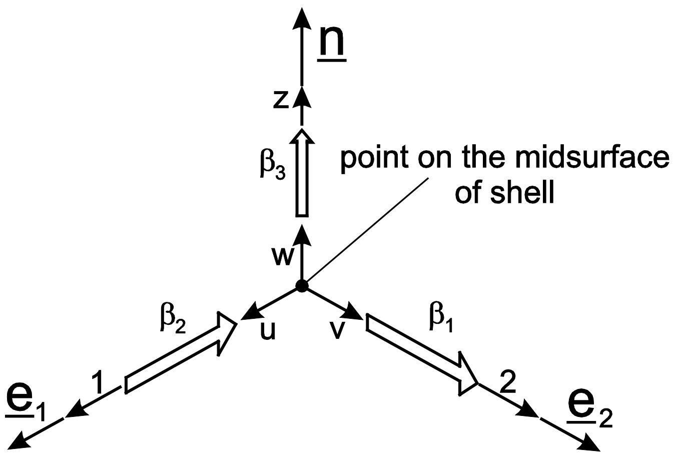

Fig.14.7. Displacement of a point on the midsurface of a thin shell.

In accordance with the kinematic hypothesis of the shell theory the components of vector $\underline{u}$ in a point P^* out of the midsurface are [1,7]:

$$u^*=u+\beta_1z\,,\ v^*=v+\beta_2z\,,\ w^*=w\,, \tag{14.66}$$

i.e. the line of material points, which is perpendicular to the shell midsurface remains perpendicular during the deformation. The equations describing the in-plane strains and changes in curvature are [1,7]:

$$\varepsilon_{11}=\frac{1}{H_1}u_{,1}+\frac{H_{1,2}}{H_1H_2}v+\frac{1}{R_1}w,\qquad(14.67)$$

$$\varepsilon_{22}=\frac{1}{H_2}v_{,2}+\frac{H_{2,1}}{H_1H_2}u+\frac{1}{R_2}w,$$

$$2\gamma_{12}=\frac{H_1}{H_2}\left(\frac{u}{H_1}\right)_{,2}+\frac{H_2}{H_1}\left(\frac{v}{H_2}\right)_{,1},$$

$$\kappa_{11}=\frac{1}{H_1}\beta_{1,1}+\frac{H_{1,2}}{H_1H_2}\beta_2,$$

$$\kappa_{22}=\frac{1}{H_2}\beta_{2,2}+\frac{H_{2,1}}{H_1H_2}\beta_1,$$

$$2\kappa_{12}=\frac{H_1}{H_2}\left(\frac{\beta_1}{H_2}\right)_{,2}+\frac{H_2}{H_1}\left(\frac{\beta_2}{H_1}\right)_{,1}+\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\beta_3,$$

where ε_1 and ε_2 are the in-plane strains in the directions of q_1 and q_2 coordinate lines, γ_{12} is the shear strain related to the change of angle between unit vectors $\underline{e}_1$ and $\underline{e}_2$ during the deformation, κ_{11} and κ_{22} are the changes in curvatures in the directions of q_1 and q_2 parameters, κ_{12} is the twisting curvature. The shear strains related to the unit normal and unit vectors $\underline{e}_1$, $\underline{e}_2$ are [1,7]:

$$\gamma_{13}=-\frac{u}{R_1}+\frac{1}{H_1}w_{,1}+\beta_1,\ \gamma_{23}=-\frac{v}{R_2}+\frac{1}{H_2}w_{,2}+\beta_2.\qquad(14.68)$$

We assume that during the deformation of shell an actual line of material points remain perpendicular to the curved shape of shell midsurface, accordingly the shear strains given by (14.68) are equal to zero. The kinematic hypothesis of shell theory together with the one mentioned before is called the Kirchhoff–Love hypothesis. Under theses assumptions we have:

$$\beta_1=\frac{u}{R_1}-\frac{1}{H_1}w_{,1},\ \beta_2=\frac{v}{R_1}-\frac{1}{H_2}w_{,2}.\qquad(14.69)$$

In other words the additional transverse shear deformation is neglected (similarly to Kirchhoff's theory of thin plates). The rotation about axis z can be formulated by the following expression [1,7]:

$$\beta_3=\frac{1}{2H_1H_2}[(H_2v)_{,1}-(H_1u)_{,2}].\qquad(14.70)$$

Nevertheless, in most of the cases the rotation about z is negligible; therefore it is not considered in the equations.

14.4.4 Approximations within the technical theory of thin shells

The shell is considered to be thin if the thickness is relatively small compared to the smaller radius of curvature, viz. [1]:

$$\frac{z}{R_2}<<1.\qquad(14.71)$$

Consequently, the Lamé parameters and the arc lengths on the midsurface and out of the midsurface are approximately equal, which leads to:

$$H_i^*\cong H_i\ \text{and:}\ dS_i^*\cong dS_i,\ i=1,2.\qquad(14.72)$$

Accordingly, Eq.(14.61) can be simplified significantly:

$$N_{11}=\int_{-t/2}^{t/2}\sigma_{11}dz\,,\ N_{12}=\int_{-t/2}^{t/2}\tau_{12}dz=N_{21}\,,\ Q_1=-\int_{-t/2}^{t/2}\tau_{13}dz\,, \qquad (14.73)$$

$$M_{11}=\int_{-t/2}^{t/2}\sigma_{11}zdz\,,\ M_{12}=\int_{-t/2}^{t/2}\tau_{12}zdz=M_{21}\,.$$

It is seen that in this case the transverse shear forces and torsional moments are equal to each other, which violates the equilibrium equations given by Eq. (14.64). This approximation is permitted within the technical theory of thin shells.

14.5 Major steps in the finite element modeling of shells

In the course of the finite element discretization of shells – similarly to the plane and plate problems – we proceed the interpolation of the geometry and the displacement field [1,7]. The vector of displacement and rotation components in a point located on the shell midsurface is:

$$\underline{u}^T=\begin{bmatrix}u & v & w\end{bmatrix}, \qquad (14.74)$$

$$\underline{\beta}^T=\begin{bmatrix}\beta_1 & \beta_2 & \beta_3\end{bmatrix}.$$

The components of these vectors are not independent of each other. From Eq.(14.67) we calculate the in-plane strains and the changes in curvature:

$$\underline{\varepsilon}^T=\begin{bmatrix}\varepsilon_{11} & \varepsilon_{22} & 2\gamma_{12}\end{bmatrix}, \qquad (14.75)$$

$$\underline{\kappa}^T=\begin{bmatrix}\kappa_{11} & \kappa_{22} & 2\kappa_{12}\end{bmatrix}.$$

We collect the in-plane forces and moments into a vector:

$$\underline{N}^T=\begin{bmatrix}N_{11} & N_{22} & N_{12}\end{bmatrix}, \qquad (14.76)$$

$$\underline{M}^T=\begin{bmatrix}M_{11} & M_{22} & M_{12}\end{bmatrix}.$$

Transverse shear forces Q_1, Q_2 are not considered in the calculation of the deformation. Finally the vectors of the surface loads and concentrated forces and moments are:

$$\underline{p}^T=\begin{bmatrix}p_1 & p_1 & p_3\end{bmatrix}, \qquad (14.77)$$

$$\underline{\rho}_N^T=\begin{bmatrix}N_1 & N_2 & -Q\end{bmatrix},$$

$$\underline{\rho}_M^T=\begin{bmatrix}M_1 & M_2 & 0\end{bmatrix},$$

where $\underline{p}$ contains the distributed loads in the directions of coordinate lines q_1 and q_2 and also the distributed load perpendicularly to the shell midsurface, $\underline{\rho}_N$ and $\underline{\rho}_M$ contain the concentrated forces and moments acting in the nodes. Using the vectors given by Eqs.(14.75)-(14.77) the total potential energy is formulated as:

$$\Pi_e=\frac{1}{2}\int_A(\underline{\varepsilon}^T\underline{N}+\underline{\kappa}^T\underline{M})H_1H_2dq_1dq_2-\int_A\underline{u}^T\underline{p}H_1H_2dq_1dq_2-\int_S(\underline{u}^T\underline{\rho}_N+\underline{\beta}^T\underline{\rho}_M)dS\,. \qquad (14.78)$$

We assume that the material of the thin shell is linear elastic, homogeneous and isotropic. Then, the vector of in-plane forces and vector of moments can be calculated as follows:

$$\underline{N}=t\underline{\underline{C}}^{str}\underline{\varepsilon}\,,\ \underline{M}=\frac{t^3}{12}\underline{\underline{C}}^{str}\underline{\kappa}\,, \qquad (14.79)$$

where the constitutive matrix assuming plane stress state is:

$$\underline{\underline{C}}^{str} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{14.80}$$

Accordingly, Eq.(14.78) becomes:

$$\begin{gathered}\Pi_e = \frac{1}{2}\int_A (t\underline{\varepsilon}^T \underline{\underline{C}}^{str} \underline{\varepsilon} + \frac{t^2}{12}\underline{\kappa}^T \underline{\underline{C}}^{str} \underline{\kappa}) H_1 H_2 dq_1 dq_2 + \\ -\int_A \underline{u}^T \underline{p} H_1 H_2 dq_1 dq_2 - \int_S (\underline{u}^T \underline{\rho}_N + \underline{\beta}^T \underline{\rho}_M) dS,\end{gathered} \tag{14.81}$$

Utilizing the definition of the element stiffness matrix and the vector of nodal forces we can derive the expression below:

$$\Pi_e = \frac{1}{2}\underline{u}_e^T \underline{\underline{K}}_e \underline{u}_e - \underline{u}_e^T \underline{F}_e, \tag{14.82}$$

from which the finite element equilibrium equation for a single element (the first of Eq.(14.47)) can be derived. As a next step we summarize the potential energy of each element:

$$\Pi = \sum \Pi_e = \frac{1}{2}\underline{U}^T \underline{\underline{K}}\underline{U} - \underline{U}^T \underline{F}, \tag{14.83}$$

and finally applying the minimum principle of the total potential energy we obtain the structural equilibrium equation:

$$\underline{\underline{K}}\underline{U} = \underline{F}. \tag{14.84}$$

For the finite element modeling of shells there is very large number of element types. Not only the flat shell elements, which give more accurate result under high mesh resolution, but also the curved (e.g. cylindrical shell element) and doubly-curved shell element types are available, which approximate better both the geometry and the displacement field using the same element number. The different plate and shell elements are discussed in sections 15-17.

14.6 Bibliography

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