

## CHAPTER

# 15

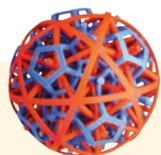
# Developing Fraction Concepts

## LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- 15.1** Describe and give examples for fractions constructs.
- 15.2** Name the types of fractions models and describe activities for each.
- 15.3** Explain foundational concepts of fractional parts, including iteration and partitioning, and connect these ideas to CCSS-M expectations.
- 15.4** Illustrate examples across fraction models for developing the concept of equivalence.
- 15.5** Compare fractions in a variety of ways and describe ways to teach this topic conceptually.
- 15.6** Synthesize how to effectively teach fraction concepts.

**F**ractions are one of the most important topics students need to understand in order to be successful in algebra and beyond, yet it is an area in which U.S. students struggle. NAEP test results have consistently shown that students have a weak understanding of fraction concepts (Sowder & Wearne, 2006; Wearne & Kouba, 2000). This lack of understanding is then translated into difficulties with fraction computation, decimal and percent concepts, and the use of fractions in other content areas, particularly algebra (Bailey, Hoard, Nugent, & Geary, 2012; Brown & Quinn, 2007; National Mathematics Advisory Panel, 2008). Therefore, it is absolutely critical that you teach fractions well, present fractions as interesting and important, and commit to helping students understand the big ideas.



## BIG IDEAS

- ◆ For students to really understand fractions, they must experience fractions across many constructs, including part of a whole, ratios, and division.
- ◆ Three categories of models exist for working with fractions—area (e.g.,  $\frac{1}{3}$  of a garden), length (e.g.,  $\frac{3}{4}$  of an inch), and set or quantity (e.g.,  $\frac{1}{2}$  of the class).
- ◆ Partitioning and iterating are ways for students to understand the meaning of fractions, especially numerators and denominators.
- ◆ Equal sharing is a way to build on whole-number knowledge to introduce fractional amounts.
- ◆ Equivalent fractions are ways of describing the same amount by using different-sized fractional parts.
- ◆ Fractions can be compared by reasoning about the relative size of the fractions. Estimation and reasoning are important in teaching understanding of fractions.



## Meanings of Fractions

Fraction understanding is developmental in nature. Fraction experiences should begin as early as first grade. In the Common Core State Standards students partition shapes and refer to the fractional amounts in grades 1 and 2 as “equal shares.” In grade 3, fractions are a major emphasis, with attention to using fraction symbols, exploring unit fractions (fractions with numerator 1), and comparing fractions. Grade 4 focuses on fraction equivalence and begins work on fraction operations (Chapter 16). This emphasis over years of time is an indication of both the complexity and the importance of fraction concepts. Students need significant time and experiences to develop a deep conceptual understanding of this important topic.

Understanding a fraction is much more than recognizing that  $\frac{3}{5}$  is three shaded parts of a shape partitioned into five sections. Fractions have numerous constructs and can be represented as areas, quantities, or on a number line. This section describes these big ideas. The next sections describe how to teach the concepts of fractions.

### Fraction Constructs

Understanding fractions means understanding all the possible concepts that fractions can represent. One of the commonly used meanings of fraction is part-whole. But many who research fraction understanding believe students would understand fractions better with more emphasis across other meanings of fractions (Clarke, Roche, & Mitchell, 2008; Lamon, 2012; Siebert & Gaskin, 2006).



### Pause & Reflect

Beyond shading a region of a shape, how else are fractions represented? Try to name three ideas. ●

**Part-Whole.** Using the part-whole construct is an effective starting point for building meaning of fractions (Cramer & Whitney, 2010). Part-whole can be shading a region, part of a group of people ( $\frac{3}{5}$  of the class went on the field trip), or part of a length (we walked  $3\frac{1}{2}$  miles).

**Measurement.** Measurement involves identifying a length and then using that length as a measurement piece to determine the length of an object. For example, in the fraction  $\frac{5}{8}$ , you can use the unit fraction  $\frac{1}{8}$  as the selected length and then count or measure to show that it takes five of those to reach  $\frac{5}{8}$ . This concept focuses on how much rather than how many parts, which is the case in part-whole situations (Behr, Lesh, Post, & Silver, 1983; Martinie, 2007).

**Division.** Consider the idea of sharing \$10 with 4 people. This is not a part-whole scenario, but it still means that each person will receive one-fourth ( $\frac{1}{4}$ ) of the money, or  $2\frac{1}{2}$  dollars. Division is often not connected to fractions, which is unfortunate. Students should understand and feel comfortable with the example here written as  $\frac{10}{4}$ ,  $4\overline{)10}$ ,  $10 \div 4$ ,  $2\frac{2}{4}$ , and  $2\frac{1}{2}$  (Flores, Samson, & Yanik, 2006).

**Operator.** Fractions can be used to indicate an operation, as in  $\frac{4}{5}$  of 20 square feet or  $\frac{2}{3}$  of the audience was holding banners. These situations indicate a fraction of a whole number, and students may be able to use mental math to determine the answer. This construct is not emphasized enough in school curricula (Usiskin, 2007). Just knowing how to represent fractions doesn’t mean students will know how to operate with fractions, which occurs in various other areas in mathematics (Johanning, 2008).

**Ratio.** Discussed at length in Chapter 18, the concept of ratio is yet another context in which fractions are used. For example, the fraction  $\frac{1}{4}$  can mean that the probability of an event is one in four. Ratios can be part-part or part-whole. For example, the ratio  $\frac{3}{4}$  could be the ratio of those wearing jackets (part) to those not wearing jackets (part), or it could be part-whole, meaning those wearing jackets (part) to those in the class (whole).

## Why Fractions Are Difficult

Students build on their prior knowledge, meaning that when they encounter situations with fractions, they naturally use what they know about whole numbers to solve the problems. Based on the research, there are a number of reasons students struggle with fractions. They include:

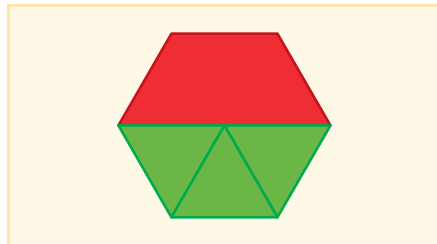
- There are many meanings of fractions (see later section “Fraction Constructs”).
- Fractions are written in a unique way.
- Students overgeneralize their whole-number knowledge (McNamara & Shaughnessy, 2010).

It is important for a teacher to help students see how fractions are alike and different from whole numbers. An explanation of common misapplications of whole-number knowledge to fractions follows, along with ways you can help.

**Misconception 1.** Students think that the numerator and denominator are separate values and have difficulty seeing them as a single value (Cramer & Whitney, 2010). It is hard for them to see that  $\frac{3}{4}$  is one number.

**How to Help:** Find fraction values on a number line. This can be a fun warm-up activity each day, with students placing particular values on a classroom number line or in their math journals. Measure with various levels of precision (e.g., to the nearest eighth-inch). Avoid the phrase “three *out of* four” (unless talking about ratios or probability) or “three over four.” Instead, say “three *fourths*” (Siebert & Gaskin, 2006).

**Misconception 2.** Students do not understand that  $\frac{2}{3}$  means two equal-sized parts (although not necessarily equal-shaped objects). For example, students may think that the following shape shows  $\frac{3}{4}$  green, rather than  $\frac{1}{2}$  green:



**How to Help:** Ask students to create their own representations of fractions across various manipulatives and on paper. Provide problems like the one illustrated here, in which all the partitions are not already drawn.

**Misconception 3.** Students think that a fraction such as  $\frac{1}{5}$  is smaller than a fraction such as  $\frac{1}{10}$  because 5 is less than 10. Conversely, students may be told the reverse—the bigger the denominator, the smaller the fraction. Teaching such rules without providing the reason may lead students to overgeneralize that  $\frac{1}{5}$  is more than  $\frac{7}{10}$ .

**How to Help:** Use many visuals *and* contexts that show parts of the whole. For example, ask students if they would rather go outside for  $\frac{1}{2}$  of an hour,  $\frac{1}{4}$  of an hour, or  $\frac{1}{10}$  of an hour. Use the idea of fair shares: Is it fair if Mary gets one-fourth of the pizza and Laura gets one-eighth? Ask students to explain why this is not fair and who gets the larger share.

**Misconception 4.** Students mistakenly use the operation “rules” for whole numbers to compute with fractions—for example,  $\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$ .

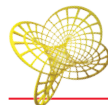
**How to Help:** Use many visuals *and* contexts. Emphasize estimation (see later section in this chapter) and focus on whether answers are reasonable or not.

Students who make these errors do not understand fractions. Until they understand fractions meaningfully, they will continue to make errors by overapplying whole-number concepts (Cramer & Whitney, 2010; Siegler et al., 2010). The most effective way to help students

reach higher levels of understanding is to use multiple representations, multiple approaches, and explanation and justification (Harvey, 2012; Pantziara & Philippou, 2012). This chapter is designed to help you help students deeply understand fractions.



### Complete Self-Check 15.1: Meanings of Fractions



## Models for Fractions

There is substantial evidence to suggest that the effective use of visuals in fraction tasks is important (Cramer & Henry, 2002; Siebert & Gaskin, 2006). Unfortunately, textbooks rarely incorporate manipulatives, and when they do, they tend to be only area models (Hodges, Cady, & Collins, 2008). This means that students often do not get to explore fractions with a variety of models and/or do not have sufficient time to connect the visuals to the related concepts. In fact, what appears to be critical in learning is that the use of physical tools leads to the use of mental models, which builds students' understanding of fractions (Cramer & Whitney, 2010; Petit, Laird, & Marsden, 2010).

Properly used, tools can help students clarify ideas that are often confused in a purely symbolic model. Sometimes it is useful to do the same activity with two different representations and ask students to make connections between them. Different representations offer different opportunities to learn. For example, an area model helps students visualize parts of the whole. A linear model shows that there is always another fraction to be found between any two numbers—an important concept that is underemphasized in the teaching of fractions. Some students are able to make sense of one representation, but not another. Importantly, students need to experience fractions in real-world contexts that are meaningful to them (Cramer & Whitney, 2010). These contexts may align well with one representation and not as well with another. For example, if students are being asked who walked the farthest, a linear model is more likely to support their thinking than an area model.

Table 15.1 provides an at-a-glance explanation of three types of models—area, length, and set—defining the wholes and their related parts for each model. Using appropriate representations and different categories of models broaden and deepen students' (and teachers') understanding of fractions.

An increasing number of Web resources are available to help represent fractions. One excellent source, though subscription based, is **Conceptua Fractions** (<https://www.youtube.com/watch?v=7OJTjYxWCIU>), developed by Conceptua Math. This site offers free tools that help students explore various fraction concepts using area, set, and length models (including the number line). The activities can be prescribed by the teacher and contain formative assessment resources.

**TABLE 15.1 MODELS FOR FRACTION CONCEPTS AND HOW THEY COMPARE**

Model	What Defines the Whole	What Defines the Parts	What the Fraction Means
Area	The area of the defined region	Equal area	The part of the area covered as it relates to the whole unit
Length or number line	The unit of distance or length	Equal distance/length	The location of a point in relation to 0 and other values on the number line
Set	Whatever value is determined as one set	Equal number of objects	The count of objects in the subset as it relates to the defined whole

Source: Based on Petit, Laird, & Marsden (2010).

**CCSS** Standards for Mathematical Practice

**MP5.** Use appropriate tools strategically.

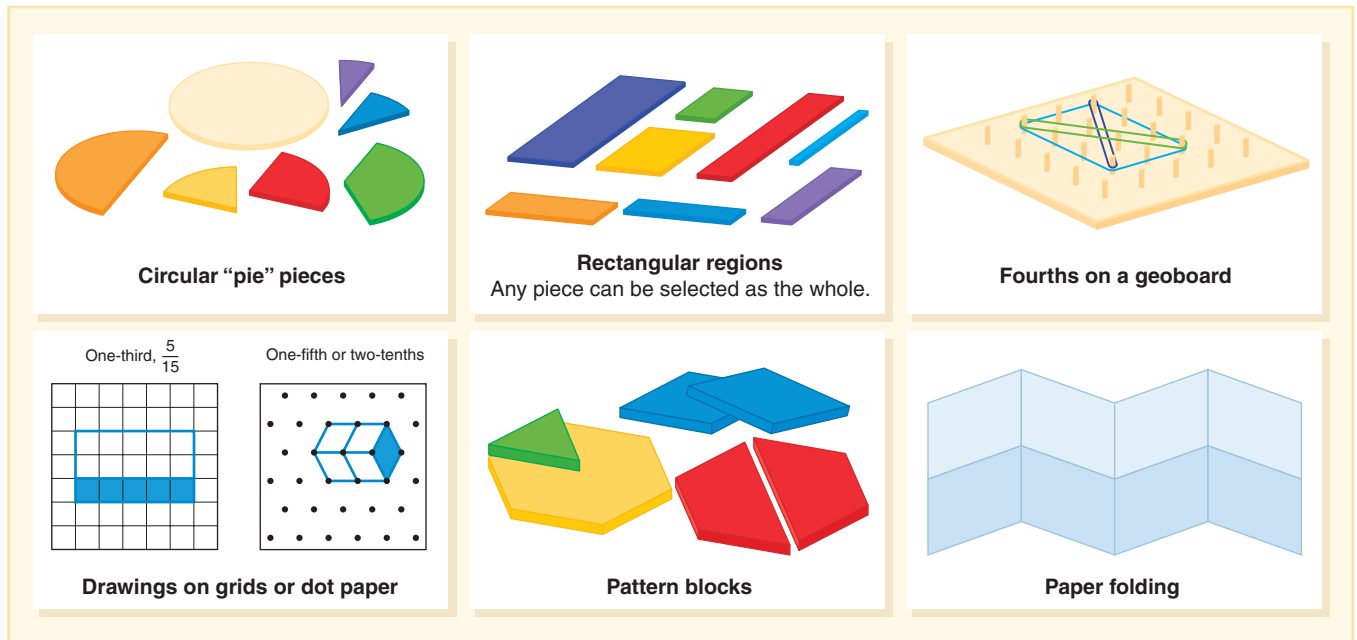


FIGURE 15.1 Area models for fractions.

### Area Models

With these visuals, fractions are based on parts of an area. See Figure 15.1 for examples. Area is a good place to begin fraction explorations because it lends itself to equal sharing and partitioning.

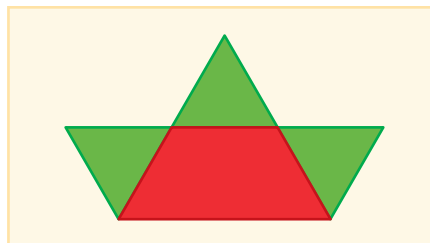
**Circular Fraction Pieces** are the most commonly used area model. One advantage of the circular model is that it emphasizes the part-whole concept of fractions and the meaning of the relative size of a part to the whole (Cramer, Wyberg, & Leavitt, 2008). Other area models in Figure 15.1 demonstrate how different shapes can be the whole. **Grid or Dot Paper** provides flexibility in selecting the size of the whole and the size of the parts (see Blackline Masters 5–11 for a selection). Many commercial versions of area manipulatives are available, including circular and rectangular pieces, pattern blocks, geoboards, and tangrams. Activity 15.1 (adapted from Roddick & Silvas-Centeno, 2007) uses pattern blocks to help students develop concepts of partitioning and iterating.

## Activity 15.1

CCSSM: 1.G.A.3; 2.G.A.3; 3.NF.A.1

### Playground Fractions

Create this “playground” with your pattern blocks (see **Pattern Block Playground** Activity Page). It is the whole. For each fraction below, find the pieces of the playground and draw it on your paper. For grades 1 and 2 use words, not fraction symbols (e.g., half of, one-half, or four-thirds).



- $\frac{1}{2}$  playground
- $1\frac{1}{2}$  playgrounds
- 2 playgrounds
- $\frac{1}{3}$  playground
- $\frac{2}{3}$  playground
- $\frac{4}{3}$  playground

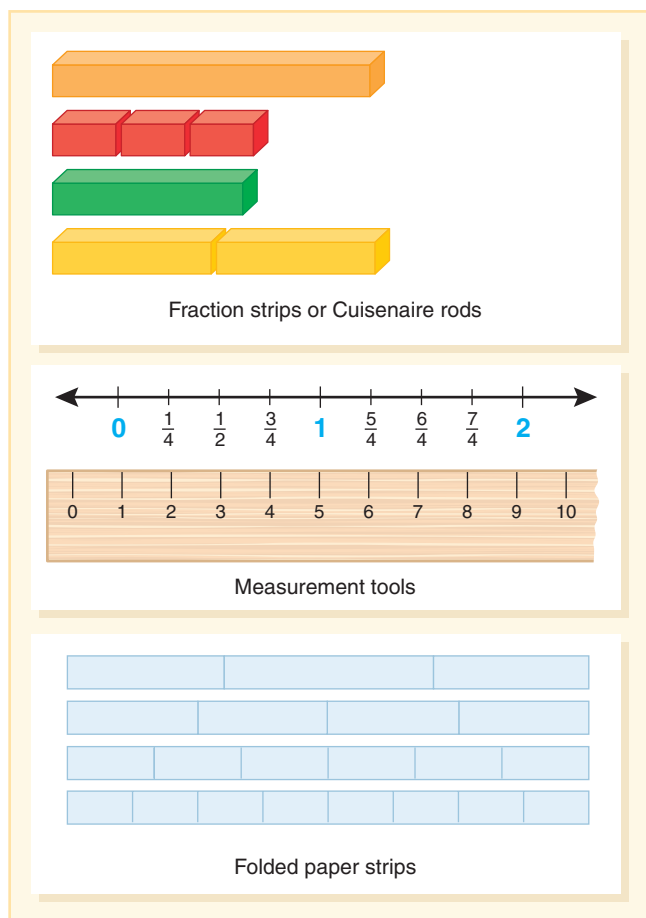
## Length Models

With length models, lengths or measurements are compared instead of areas. Either physical materials are compared on the basis of length or number lines are subdivided, as shown in Figure 15.2. Length models are very important in developing student understanding of fractions, yet they are not widely used in classrooms. Recent reviews of research on fractions (Petit et al., 2010; Siegler et al., 2010) report that the number line helps students understand a fraction as a number (rather than one number over another number) and helps develop other fraction concepts. In a report completed by the Institute of Educational Sciences (IES), the researchers prepared recommendations for supporting the learning of fractions (Siegler et al., 2010), advising teachers to:


Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades on. (p. 1)

Linear models are closely connected to the real-world contexts in which fractions are commonly used, such as measuring. Music, for example, is an excellent opportunity to explore  $\frac{1}{2}$ ,  $\frac{1}{4}$ ,  $\frac{1}{8}$ , and  $\frac{1}{16}$  in the context of notes (Goral & Wiest, 2007). In fact, Courey, Balogh, Siker, and Paik (2012) found that connecting fractions to measures in music significantly improved student understanding of fractions.

One length manipulative, Cuisenaire rods, has pieces in lengths of 1 to 10 measured in terms of the smallest strip or rod. Each length is a different color for ease of identification. Strips of paper or adding-machine tape, also a length model, can be folded to produce student-made fraction strips.

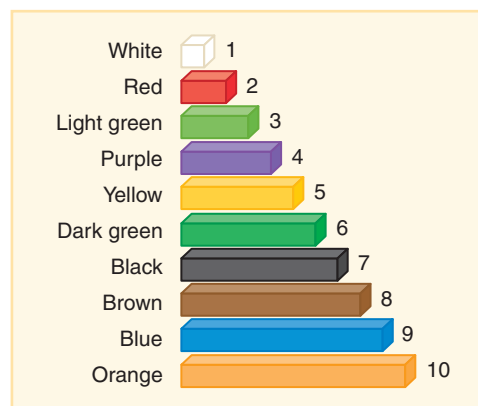


**FIGURE 15.2** Length or measurement models for fractions.

 **TECHNOLOGY Note.** Virtual Cuisenaire rods and accompanying activities can be found at online at various websites such as the University of Cambridge's NRich Project. ■

Rods or strips provide flexibility because any length can represent the whole. For example, if you wanted students to work with  $\frac{1}{4}$  and  $\frac{1}{8}$ , select the brown Cuisenaire rod, which is 8 units long. Therefore, the four rod (purple) becomes  $\frac{1}{2}$ , the two rod (red) becomes  $\frac{1}{4}$ , and the one rod (white) becomes  $\frac{1}{8}$ . For exploring twelfths, put the orange rod and red rod together to make a whole that is 12 units long.

Cuisenaire rods consist of the following colors and lengths:



The number line is a significantly more sophisticated length model than the physical tools described previously (Bright, Behr, Post, & Wachsmuth, 1988), but it is an essential model that needs to be emphasized more in the teaching of fractions (Clarke et al., 2008; Flores et al., 2006; Siegler et al., 2010; Usiskin, 2007; Watanabe, 2006).

Like with whole numbers, the number line is used to compare the relative size of numbers. Importantly, the number line reinforces that fact that there is always one more fraction to be found between two fractions. The following activity (based on Bay-Williams & Martinie, 2003) is a fun way to use a real-world context to engage students in thinking about fractions through a linear model.

## Activity 15.2

CCSSM: 3.NF.A.2a, b; 3.NF.A.3a, b, d

### Who Is Winning?

Use **Who Is Winning?** Activity Page and give students paper strips or ask them to draw a number line. This activity can be done two ways (depending on your lesson goals). First, ask students to use reasoning to answer the question “Who is winning?” Students can use reasoning strategies to compare and decide. Second, students can locate each person’s position on a number line. Explain that the friends below are playing “Red Light, Green Light.” The fractions tell how much of the distance they have already moved. Can you place these friends on a line to show where they are between the start and finish? Second, rather than place them, ask students to use reasoning to answer the question “Who is winning?”

Mary:  $\frac{3}{4}$     Harry:  $\frac{1}{2}$     Larry:  $\frac{5}{6}$   
Han:  $\frac{5}{8}$     Miguel:  $\frac{5}{9}$     Angela:  $\frac{2}{3}$

This game can be differentiated by changing the value of the fractions or the number of friends (fractions). The game of “Red Light, Green Light” may not be familiar to ELLs. Modeling the game with people in the class and using estimation are good ways to build background and support students with disabilities.



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## Set Models

In set models, the whole is understood to be a set of objects, and subsets of the whole make up fractional parts. For example, 3 objects are one-fourth of a set of 12 objects. The set of 12 in this example represents the unit, the whole or 1. The idea of referring to a collection of counters as a single entity makes set models difficult for some students. Putting a piece of yarn in a loop around the objects in the set to help students “see” the whole. Figure 15.3 illustrates several set models for fractions.

A common misconception with set models is to focus on the size of a subset rather than the number of equal sets in the whole. For example, if 12 counters make a whole, then a subset of 4 counters is one-third, not one-fourth, because 3 equal sets make the whole. However, the set model helps establish important connections with many real-world uses of fractions and with ratio concepts.

Two color counters are an effective set manipulative. Counters can be flipped to change their color to model various fractional parts of a whole set. Any countable objects (e.g., a box of crayons) can be a set model (with one box being the unit or whole). The following activity uses your students as the whole set. It can be done as an energizer, warm-up, or full lesson.

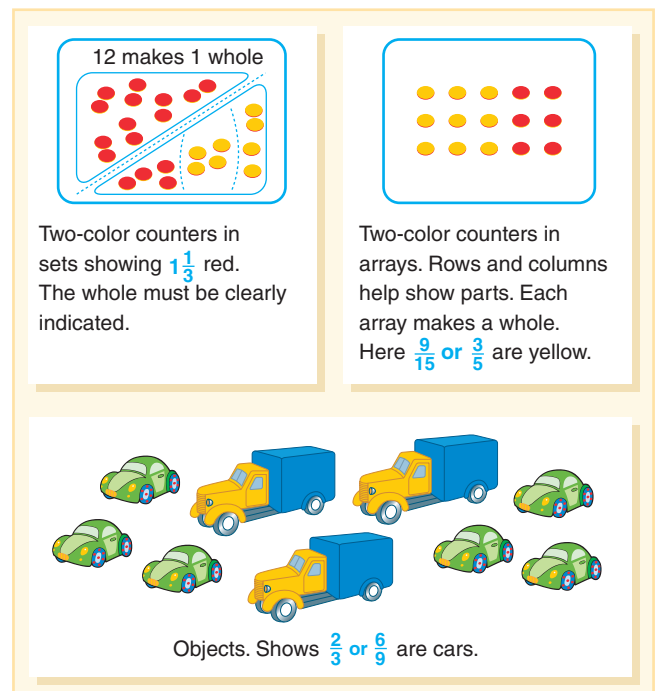


FIGURE 15.3 Set models for fractions.

## Activity 15.3

CCSSM: 3.NF.A.1; 3.NF.A.3b

## Class Fractions

Use a group of students as the whole—for example, six students if you want to work on halves, thirds, and sixths. Invite them to come to the front of the room. Say to the group, “If you [are wearing tennis shoes, have brown hair, etc.], move to the right. If not, move to the left.” Ask everyone, “What fraction of our friends [are wearing tennis shoes]?” Invite them to whisper to a partner, then share with the class.

Change the number of selected students in the whole. You can also ask, “How many friends in one-half of this group? One-fourth of this group?” Connect to the symbols for fractions (e.g.,  $\frac{1}{2}$ ,  $\frac{1}{3}$ , and  $\frac{1}{6}$ ), asking students to write the fraction that tells how many students [are wearing tennis shoes]. Also, if you have six students in your group, and you have three in your subgroup, then students are likely to say “three-sixths” and “one-half.” Discuss that these values are equivalent.

Students must be able to explore fractions across the three area, length, and set models. As a teacher, you will not know whether they really understand the meaning of a fraction such as  $\frac{1}{4}$  unless you have seen a student represent one-fourth using all three models.



**FORMATIVE ASSESSMENT Notes.** A straightforward way to assess students’ knowledge of a fractional amount is to give them a piece of paper folded into thirds. Write *area*, *length*, and *set* at the top of each section and give them a fractional value (e.g.,  $\frac{3}{4}$ ). Observe as they (1) draw a picture and (2) write a sentence describing a context or example for the selected fraction. This can be done exactly for commonly used fractions or can be an estimation activity with fractions like  $\frac{31}{58}$ . ■



**TECHNOLOGY Note.** Virtual manipulatives are available for all three models. Virtual manipulatives have been found to positively affect student achievement, especially when they are paired with using the actual manipulatives (Moyer-Packenham, Ulmer, & Anderson, 2012). Recommended sites include:

**Cyberchase (PBS):** *Cyberchase* is a popular television series. Their website offers videos that model fractions with real-world connections and activities such as “Thirteen Ways of Looking at a Half” (fractions of geometric shapes) and “Make a Match” (concept of equivalent fractions).

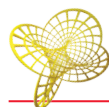
**Illuminations (NCTM) Fractions Model:** Explore length, area, region, and set models of fractions, including fractions greater than one, mixed numbers, decimals, and percentages.

**Math Playground Fraction Bars:** On this site you can explore fractional parts, the concepts of numerator and denominator, and equivalence.

**National Library of Virtual Manipulatives:** This site offers numerous models for exploring fractions, including fraction bars and fraction pieces. There is also an applet for comparing and visualizing fractions. ■



**Complete Self-Check 15.2: Models for Fractions**



## Fractional Parts

The first goal in the development of fractions should be to help students construct the idea of *fractional parts of the whole*—the parts that result when the whole or unit has been partitioned into *equal-sized portions or fair shares*. (Recall that Table 15.1 describes the meanings of parts and wholes across each type of model.)

Students understand the idea of separating a quantity into two or more parts to be shared fairly among friends. This is the beginning of understanding fractions and in the CCSS-M

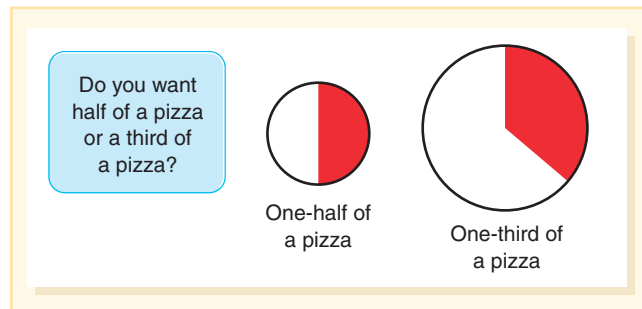


occurs in grades 1 and 2. In grade 3 and beyond, students make connections between the idea of fair (equal) shares and fractional parts. The next three sections describe ideas foundational to finding equal shares.

### Fraction Size Is Relative

A fraction by itself does not describe the size of the whole or the size of the parts. A fraction tells us only about the *relationship between* the part and the whole. Consider the following situation:

**Pizza Fallacy:** Mark is offered the choice of a third of a pizza or a half of a pizza. Because he is hungry and likes pizza, he chooses the half. His friend Jane gets a third of a pizza but ends up with more than Mark. How can that be?



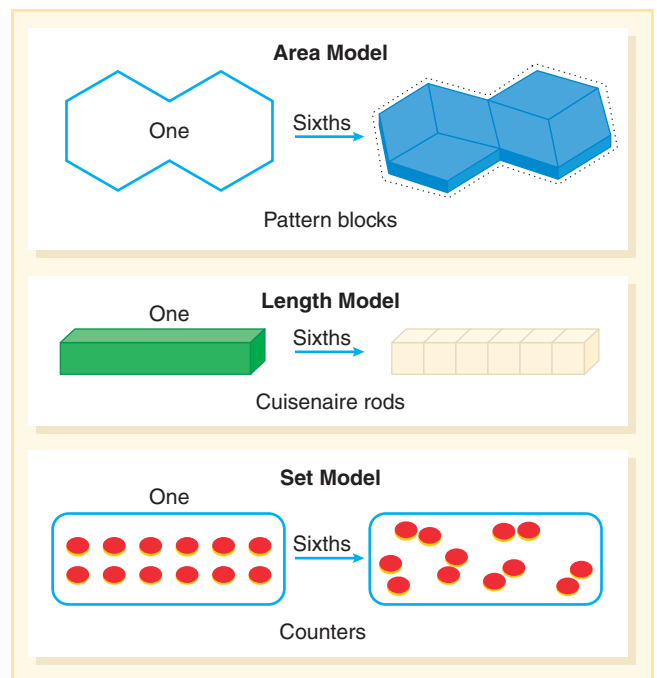
The visual illustrates how Mark got misdirected in his choice. The point of the “pizza fallacy” is that whenever two or more fractions are discussed in the same context, one cannot assume (as Mark did) that the fractions are all parts of the same size whole. Teachers can help students understand fractional parts if they regularly ask, “What is the whole?” or “What is the unit?”

Comparing two fractions with any representation can be made only if both fractions are parts of the same size whole. For example, when using Cuisenaire rods,  $\frac{2}{3}$  of a light green strip cannot be compared to  $\frac{2}{5}$  of an orange strip.

### Partitioning

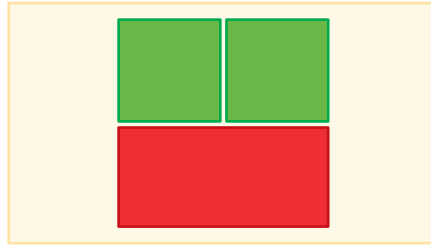
Sectioning a shape into equal-sized parts is called *partitioning*. When a brownie (or other area) has been partitioned into four equal shares, the parts are called *fourths*. Explain to students, “We call these *fourths*. The whole is cut into four parts. All of the parts are the same size—fourths.” The words for fractional parts (e.g., *halves*, *thirds*, *fourths*, *eighths*, and so on) are introduced before the symbols. In the CCSS-M the words for fractional parts (as they relate to equal shares) are introduced in grades 1 and 2; the symbols for fractions are introduced in grade 3. Figure 15.4 illustrates sixths across area, length, and set models.

**Partitioning with Area Models.** When partitioning an area into fractional parts, students need to be aware that (1) the fractional parts must be the same size, though not necessarily the same shape; and (2) the number of equal-sized parts that can be partitioned within the unit determines the fractional amount (e.g., partitioning into 4 parts means each



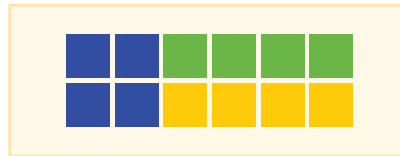
**FIGURE 15.4** Which of these shapes are partitioned into sixths? Explain why or why not for each.

part is *one-fourth* of the unit). It is important for students to understand, however, that sometimes visuals do not *show* all the partitions. For example, consider the following picture:



Referring back to the two criteria, a student might think, “If I partitioned this so that all pieces were the same size, then there will be four parts; therefore, the smaller partitioned region represents one-fourth”—not one-third, as many students without a conceptual understanding of fractional parts might suggest.

Some manipulatives, like fraction bars or fraction circles, can mislead students to believe that fractional parts must be the same *shape* as well as the same size. Color tiles can be used to create rectangles that address this misconception. Ask students to describe the fractional parts in a rectangle, such as the one illustrated here:



Students who recognize that each color represents thirds understand that fractional parts must be the same size, and that the shape of the thirds may be different.

Area models are the first types of models to use in teaching fractional parts. Young students, in particular, tend to focus on shape, when the focus should be on equal-sized parts. Activity 15.1 is an example of how you can use pattern blocks to focus on partitioning into equal-sized parts. You can build on this activity by building other shapes that use different pattern block pieces and then have students figure out how much each piece is of the whole. [View](#) how a classroom teacher helps his students understand parts of the whole using pattern blocks.

Activity 15.4 uses partitioned drawings to develop the concept of fractional parts.

## Activity 15.4

CCSSM: 1.G.A.3; 2.G.A.3; 3.NF.A.1

### Partitioning: Fourths or Not Fourths?

Use [Fourths or Not Fourths](#) Activity Page showing examples and nonexamples (which are very important to use with students with disabilities) of fourths (see Figure 15.5).


Ask students to identify the wholes that are correctly divided into fourths (equal shares) and those that are not. For each response, have students explain their reasoning. Repeat with other fractional parts, such as thirds or eighths. To challenge students, ask them to draw shapes that fit each of the four categories listed on the next page for other fractional parts, such as sixths. (See [Sixths or Not Sixths](#) Activity Page.)

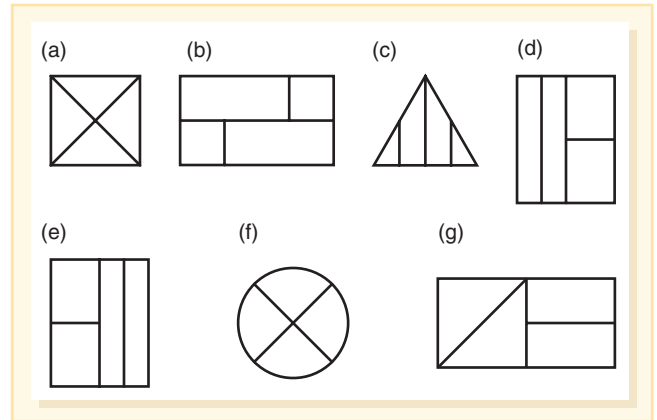


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In the preceding activity, the shapes fall in each of the following categories:

1. Same shape, same size: (a) and (f) [equivalent]
2. Different shape, same size: (e) and (g) [equivalent]
3. Different shape, different size: (b) and (c) [not equivalent]
4. Same shape, different size: (d) [not equivalent]

 **FORMATIVE ASSESSMENT Notes.** Activity 15.4 is a good diagnostic interview to assess whether students understand that it is the *size* that matters, not the shape. If students get all correct except (e) and (g), they hold the misconception that parts should be the same shape. Future tasks are needed that focus on equivalence. For example, you can ask students to take a square and subdivide a picture themselves, as in Activity 15.5. ■



**FIGURE 15.5** Given a whole, find fractional parts.

## Activity 15.5

CCSSM: 1.G.A.3; 2.G.A.3; 3.NF.A.1

### Finding (All the) Fair Shares

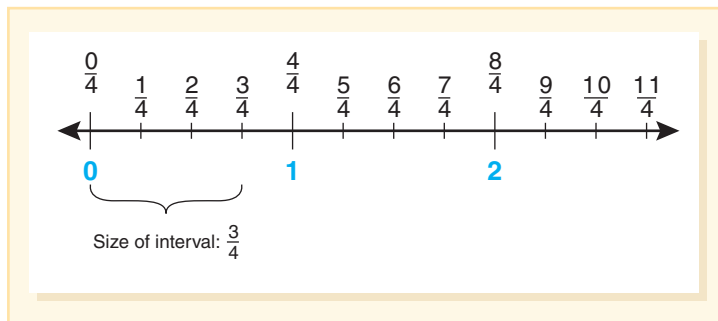
Give students dot paper and ask them to enclose a region that lends itself to partitioning with a particular fractional part. For example, they might enclose a 3-by-6 rectangle if they are going to partition into thirds. Ask students to find a way to partition the rectangle into thirds. Then redraw another rectangle that is the same size whole and partition it a different way to show thirds. Ask students to find a way to show thirds where the thirds are different shapes. See how many ways they can find. For ELLs, fraction parts sound like whole numbers (e.g., *fourths* and *fours*). Be sure to emphasize the *th* on the end and explicitly discuss the difference between four areas and a *fourth* of an area.



**Partitioning with Length Models.** The explanation of partitioning in the CCSS-M may be difficult to interpret. For example:

3.NF.A.2b: Represent a fraction  $a/b$  on a number line diagram by marking off  $a$  lengths  $1/b$  from 0. Recognize that the resulting interval has size  $a/b$  and that its endpoint locates the number  $a/b$  on the number line. (CCSSO, 2010, p. 24)

Put more simply, students need to be able to partition a number line into fourths and realize that each section is one-fourth:



Number lines are difficult for students. Students may ignore the size of the interval (McNamara & Shaughnessy, 2010; Petit et al., 2010). Students can develop an understanding of the number line by folding paper strips. Provide examples where the shaded sections are in

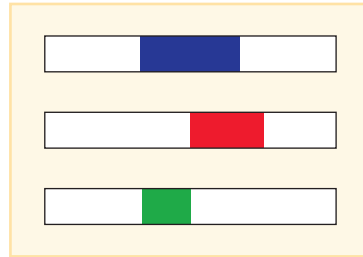
different positions and where partitioning isn't already shown to strengthen students' understanding of equal parts. Activity 15.6 and Activity 15.7 provide such opportunities, one with paper strips and one with number lines.

## Activity 15.6

CCSSM: 3.NF.A.1; 3.NF.A.2a, b

### What Fraction Is Colored?

Prepare a set of paper strips prior to doing this activity (you can cut 1-inch wide pieces of 8.5" by 11" paper and shade, or cut pieces of adding machine tape). Color the strips so that they have a fractional amount shaded in various positions (not just left justified!) (Sarazen, 2012). Here are a few examples:



Explain that the strip represents one whole. Give each student a paper strip. Ask students to explain what fraction is colored and explain how they know. A common misconception is for students to count parts and call each of these one-third. If a student makes this error, ask if the parts are the same-sized and if not to partition to make same-sized parts. Use toothpicks or uncooked spaghetti to illustrate the partitions:



Students can also justify their reasoning by measuring the length of each partition.

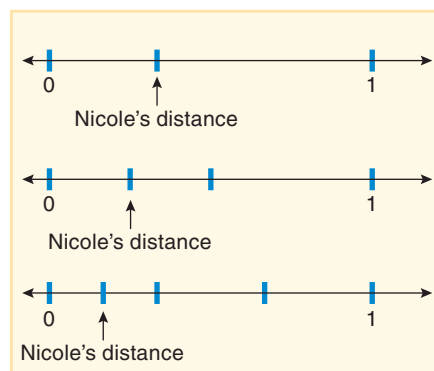
Using paper strips can help students better understand the number line, the focus of the next activity.

## Activity 15.7

CCSSM: 3.NF.A.1; 3.NF.A.2a, b

### How Far Did Nicole Go?

Give students number lines partitioned such that only some of the partitions are showing. Use a context such as walking to school. For each number line, ask, "How far has Nicole gone? How do you know?"



Students can justify their reasoning by measuring the length of each partition.

Locating a fractional value on a number line is particularly challenging but very important for students to be able to do. Shaughnessy (2011) found four common errors students make in working with the number line: They use incorrect notation, change the unit (whole), count the tick marks rather than the space between the marks, and count the ticks marks that appear without noticing any missing ones. This is evidence that we must use number lines more extensively in exploring fractions (most real-life contexts for fractions are measurement related).

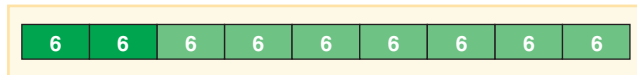
Partitioning is a strategy commonly used in Singapore (a high-performing country on international mathematics assessments) as a way to solve story problems. Consider the following story problem (England, 2010):

---

A nurse has 54 bandages. Of those,  $\frac{2}{9}$  are white and the rest are brown. How many of them are brown?

---

A bar diagram can be used as a tool for solving the problem. A student first partitions a strip into nine parts and then figures out the equal shares of bandages for each partition:



Did you notice that this is an example of fraction as operator? These types of partitioning tasks are good building blocks for multiplying with fractions.

**Partitioning with Set Models.** Students can partition sets of objects such as coins, counters, or baseball cards. When partitioning sets, students may confuse the number of counters in a share with the name of the share. In the example in Figure 15.4, the 12 counters are partitioned into 6 sets—*sixths*. Each share or part has two counters, but it is the number of shares that makes the partition show *sixths*. As with the other models, when the equal parts are not already figured out, then students may not see how to partition. Students seeing a picture of two cats and four dogs might think  $\frac{2}{4}$  are cats (Bamberger, Oberdorf, & Schultz-Ferrell, 2010). Consider the following problem:

---

Eloise has 6 trading cards, Andre has 4 trading cards, and Lu has 2 trading cards. What fraction of the trading cards does Lu have?

---

A student who answers “one-third” is not thinking about equal shares but about the number of people with trading cards.

Understanding that parts of a whole must be partitioned into equal-sized parts across different models is an important step in conceptualizing fractions and provides a foundation for exploring sharing and equivalence tasks, all of which are prerequisites to performing fraction operations (Cramer & Whitney, 2010).

## Sharing Tasks

An important recommendation by the IES research team on ways to help students learn fractions states, “Build on students’ informal understanding of sharing and proportionality to develop initial fraction concepts” (Siegler et al., 2010, p. 1). In particular, they suggest using equal-sharing activities to develop the concepts of fraction, equivalence, and ordering of fractions. See [Equal Sharing Stories Expanded Lesson](#) for a lesson designed for grades 1 or 2.

Students in the early grades partition by thinking about fair shares (division). Sharing tasks are generally posed in the form of a simple story problem. *Four friends are sharing two cookies. How many cookies will each friend get?* Then problems become slightly more difficult: *Suppose there are four cookies to be shared fairly among three children. How much will each child get?* See how [Eduardo](#) reasons about this sharing situation. Students initially perform sharing tasks by

distributing items one at a time. When this process leaves leftover pieces, students must figure out how to subdivide so that every group (or person) gets a fair share. Contexts that lend to subdividing an area include cookies, brownies, sandwiches, pizzas, and so on.

Pattern blocks are a good tool to focus on equal shares because each piece is *not* an equal share, so creating shapes with pattern blocks and asking about equal shares helps students focus on the important idea of fair (equal) shares. Ask students to create a “cookie” using the six different pattern block shapes and ask, “Can this cookie be shared fairly with 6 people?” (Ellington & Whitenack, 2010). The answer is “no.” Then, ask students to build a cookie that can be shared fairly.

Sharing brownies is a classic activity that focuses on partitioning to make equal shares (see, for example, Empson, 2002). Using concrete tools such as dough can make sharing accessible even to kindergartners (Cwikla, 2014).

## Activity 15.8

CCSSM: 1.G.A.3; 2.G.A.3

### Cookie Dough: Cut Me a Fair Share!

Give students a ball of dough and a plastic knife. Explain that they are going to be finding a way to share each group of cookies fairly with a group of students. Start with an example that is not too difficult. For example:

Four friends want to share ten brownies so that each friend gets the same amount of brownies. How much will each friend get?

Invite students to shape their dough into squares for brownies and then show how to share them fairly with four friends, using a paper knife if necessary. Encourage students to share their ways of thinking about this problem. A strategy many students will use for this problem is to deal out two brownies to each child and then halve each of the remaining brownies (see Figure 15.6).

Then, offer a selection of other sharing tasks with different numbers of brownies and different number of sharers (see additional examples below).

“Kids and Cookies” is an excellent online tool for sharing cookies (both round and rectangular). Display the situations on an interactive whiteboard and ask for different ways to share fairly (you can begin with whole numbers and increase in difficulty) (Center for Technology and Teacher Education, n.d.).

The relationship between the number of things to be shared and the number of sharers determines problem difficulty. Students’ initial strategies for sharing involve halving, so a good place to begin is with two, four, or even eight sharers. Here are some examples:

- 5 brownies shared with 2 children
- 5 brownies shared with 4 children
- 7 brownies shared with 4 children
- 2 brownies shared with 4 children
- 4 brownies shared with 8 children
- 3 brownies shared with 4 children

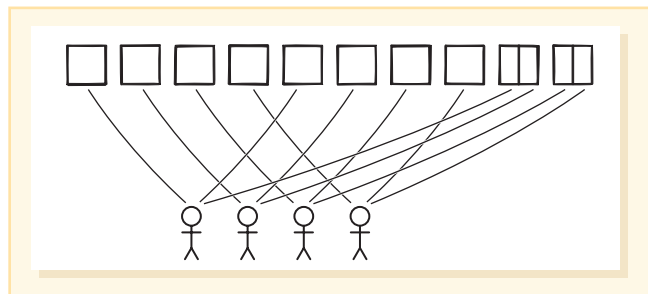


FIGURE 15.6 Ten brownies shared with four children.

These can be prepared as [Brownie Sharing Cards](#). The last example, three brownies shared with four children, is more challenging because there are more sharers than items, and it involves more than just finding halves. One strategy is to partition each brownie into four parts and give each child one-fourth from each brownie—a total of three-fourths. Students (even adults) are surprised at the relationship between the problem and the answer. [Felisha](#) explains the fractional amount each of 5 children get when sharing 2 cookies, but loses track of what the whole is in determining each person’s share.

Tier 1 task: For students who still need experience with halving	Tier 2 task: For students comfortable with halving and ready to try other strategies	Tier 3 task: For students ready to solve tasks in which students combine halving with new strategies
How can 2 people share 3 brownies?	How can 4 people share 3 brownies?	How can 3 people share 5 brownies?
How can 2 people share 5 brownies?	How can 3 people share 4 brownies?	How can 3 people share 2 brownies?
How can 4 people share 3 brownies?	How can 3 people share 5 brownies?	How can 6 people share 4 brownies?
How can 3 people share 4 brownies?	How can 6 people share 4 brownies?	How can 5 people share 4 brownies?

**FIGURE 15.7** Example of a tiered lesson for the brownie-sharing problem.

## Pause & Reflect

Is this pattern (that three brownies shared among four children means each gets three-fourths) true for any sharing tasks? Why is this true? ●

Because the level of difficulty of these sharing tasks varies, it is useful for creating a tiered lesson. In a tiered lesson, the goal (sharing) is the same, but the specific tasks vary in their challenge. Figure 15.7 shows how one teacher offers these three tiers for her lesson on sharing brownies (Williams, 2008).

Sharing into thirds or sixths is more challenging because students cannot rely on halving to get to the answer. Here are some examples:

- 4 pizzas shared with 3 children
- 7 pizzas shared with 6 children
- 5 pizzas shared with 3 children
- 4 pizzas shared with 6 children

Figure 15.8 shows how a student partitioned to solve “5 pizzas shared with 3 children.” This took much guess and check, at which point the teacher asked, “Can you see a pattern in how you have divided the pizza and how many people are sharing?” At this point, the student noticed a pattern: If there are three people, the remaining pizzas need to be partitioned into thirds.

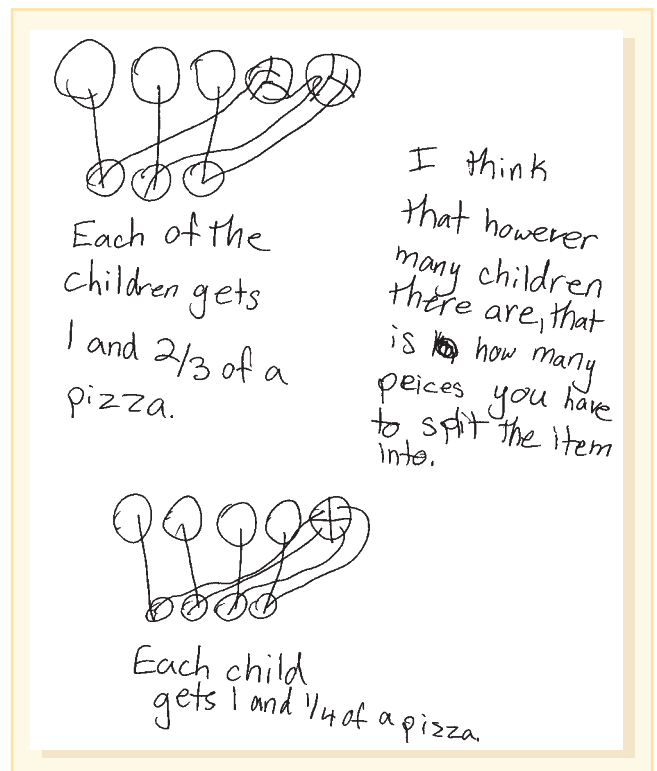
As students report their answers, it is important to emphasize the equivalence of different representations (Flores & Klein, 2005). For example, in the case of three people sharing four pizzas, the answer might be noted on the board this way:

$$\frac{4}{3} = 1\frac{1}{3} = 1 + \frac{1}{3}$$

## Iterating

In whole-number learning, counting precedes and helps students to add and later subtract. This is also true with fractions. Counting fractional parts, or *iterating*, helps students understand the relationship between the parts (the numerator) and the whole (the denominator). The iterative concept is most clear when focusing on these two ideas about fraction symbols:

- The top number (numerator) *counts*.
- The bottom number (denominator) tells *what is being counted*.



**FIGURE 15.8** Elizabeth explains a pattern for finding equal shares of a pizza.

Students need to understand that  $\frac{3}{4}$ , for example, can be thought of as a count of three parts called *fourths* (Post, Wachsmuth, Lesh, & Behr, 1985; Siebert & Gaskin, 2006; Tzur, 1999).

If you know the kind of part you are counting, you can tell when you get to one, when you get to two, and so on. Students should be able to answer the question, “How many fifths are in one whole?” just as they know how many ones are in ten. However, the 2008 National Assessment of Education Progress (NAEP) results indicated that only 44 percent of students answered this question correctly (Rampey, Dion, & Donahue, 2009). This is the focus of Activities 15.9 through 15.11.

## Activity 15.9

CCSSM: 3.NF.A.1; 3.NF.A.2a, b

### More, Less, or Equal to One Whole

Give students a collection of fractional parts (all the same-size pieces) and indicate the kind of fractional part they have. For example, if done with Cuisenaire rods, the collection might have seven light green rods/strips with a caption or note indicating “each piece is  $\frac{1}{8}$ .” The task is to decide if the collection is less than one whole, equal to one whole, or more than one whole. Ask students to draw pictures or use symbols to explain their answer. As students count each collection of parts, discuss the relationship to one whole. Ask questions that help students focus on the meaning of the numerator and denominator, such as “Why did we get almost two wholes with seven-fourths, and yet we don’t even have one whole with ten-twelfths?”

After exploring Activity 15.9 with same-sized pieces, try Activity 15.10, which returns to using the pattern blocks to help students focus on the size of the parts, not the number of pieces or partitions (Champion & Wheeler, 2014; Ellington & Whitenack, 2010).

## Activity 15.10

CCSSM: 3.NF.A.1; 3.NF.A.2a, b

### Pattern Block Creatures

Ask students to build a Pattern Block Creature that fits with a set of rules (a creature represents one-whole). These rules can begin with just stating a fractional quantity for a color, such as “The red trapezoid is one-fourth of the creature.” But, more constraints can be added to the rules. For example:

The blue parallelogram is one-sixth of the creature. Use at least two colors to build your creature.

The yellow hexagon is one-half of the creature. Use three colors to build your creature.

Green triangles are one-third of your creature. Use four different colors to build your creature.

After a student creates their creature, they can sketch the creature on paper and write the rule below it. Other students can critique the creature to see if it follows the rules it was given. Alternatively, the student can write their rule as “The red trapezoid is \_\_\_\_\_ of my creature” and trade it with another student to see if they can figure out the fractional amount.

## Activity 15.11

CCSSM: 3.NF.A.1; 3.NF.A.3a, c

### Calculator Fraction Counting

Many calculators, like the TI-15, display fractions in correct fraction format and offer a choice of showing results as mixed numbers or simple fractions. Ask students to type in a fraction (e.g.,  $\frac{1}{4}$ ) and then  $+$  and the fraction again. To count, press 0 **Op1**, **Op1**, **Op1**, repeating to get the number of fourths wanted. The display will show the counts by fourths and also the number of times that the **Op1** key has been pressed. Ask students questions such as the following: “How many fourths to get to 3?” “How many fifths to get to 2?” These can get increasingly more challenging: “How many fourths to get to  $4\frac{1}{2}$ ?” “How many two-thirds to get to 6? Estimate and then count by two-thirds on the calculator.” Students, particularly students with disabilities, should coordinate their counts with fraction models, adding a new fourths piece to the pile with each count.

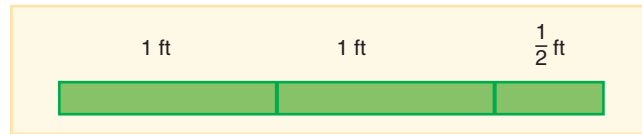


STUDENTS  
with  
SPECIAL  
NEEDS

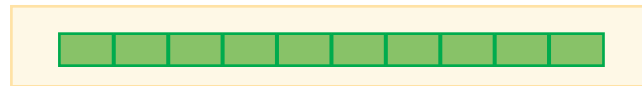


The TI-15 display can be switched back and forth from mixed number to fractions, reinforcing the equivalence of values such as  $1\frac{2}{3}$  and  $\frac{5}{3}$ . A variation on Activity 15.11 is to show students a mixed number such as  $3\frac{1}{8}$  and ask how many counts of  $\frac{1}{8}$  on the calculator it will take to count that high. The students should try to stop at the correct number,  $\frac{25}{8}$ , before pressing the mixed-number key.

Iterating applies to all models but is particularly connected with length models because iteration is much like measuring. Consider that you have  $2\frac{1}{2}$  yards of ribbon and are trying to figure out how many fourths you can cut. You can draw a strip and start counting (iterating) the fourths:



Using a ribbon that is  $\frac{1}{4}$  of a yard long as a measuring tool, a student marks off ten fourths:



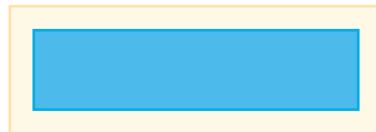
Students can participate in many tasks that involve iterating lengths, including ones where they are asked to find what the whole or unit is.

## Activity 15.12

CCSSM: 3.NF.A.1; 3.NF.A.2a, b

### A Whole Lot of Fun

Use [A Whole Lot of Fun Activity Page](#) and a strip of paper like the one here:



Tell students that this strip is three-fourths of one whole (unit). Ask students to sketch strips of the other lengths on their paper (e.g.,  $\frac{5}{2}$ ). You can repeat this activity by selecting other values for the starting amount and selecting different fractional values to sketch. A context, such as walking, is effective in helping students make sense of the situation. Be sure to use fractions less than and greater than 1 and mixed numbers.

Notice that to solve the task in Activity 15.12, students first partition the piece into three sections to find  $\frac{1}{4}$  and then iterate the  $\frac{1}{4}$  to find the other lengths.

Iterating can be done with area models. Display some circular fractional pieces in groups as shown in Figure 15.9. For each collection, tell students what type of piece is being shown and simply count them together: “One-fourth, two-fourths, three-fourths, four-fourths, five-fourths.” Ask, “If we have five-fourths, is that more than one whole, less than one whole, or the same as one whole?” To reinforce the piece size even more, you can slightly alter your language to say, “One one-fourth, two one-fourths, three one-fourths,” and so on.

Iteration can also be done with set models. For example, show a collection of two-color counters and ask questions such as, “If 5 counters is one-fourth of the whole, how much of the whole is 15 counters?” These problems can be framed as engaging puzzles for students.

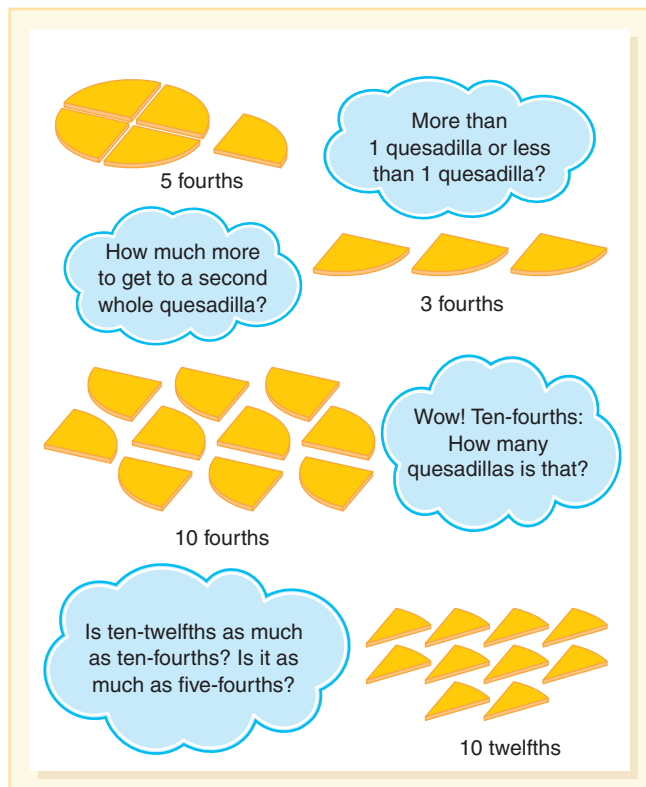


FIGURE 15.9 Iterating fractional parts in an area model.

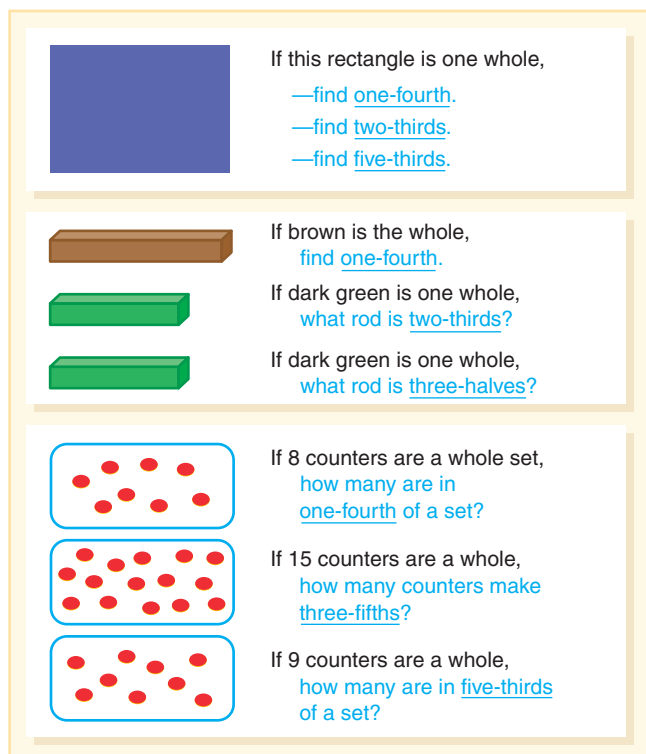


FIGURE 15.10 Given the whole and the fraction, find the part.

For example: “Three counters represent  $\frac{1}{8}$  of my set; how big is my set?” If the fraction is not a unit fraction, then students first partition and then iterate. For example, “Twenty counters represents  $\frac{2}{3}$  of my set; how big is my set?” first requires finding  $\frac{1}{3}$  (10 counters), then iterating that three times to get 30 counters in three-thirds (one whole). **Counting Counters: Find the Part** Activity Page and **Counting Counters: Find the Whole** Activity Page provides such problems for students to solve.

## Pause & Reflect

Work through the exercises in Figures 15.10 and 15.11. If you do not have access to Cuisenaire rods or counters, just draw lines. What can you learn about student understanding of partitioning and iterating if they are able to solve problems in Figure 15.10 but not 15.11? If students are stuck, what contexts for each model can be used to support their thinking? ●

The partitioning and iterating questions are challenging yet very effective at helping students reflect on the meanings of the numerator and denominator and understand the meaning of fractions.

**FORMATIVE ASSESSMENT Notes.** The tasks in Figures 15.10 and 15.11 can be used as performance assessments. If students are able to solve these types of tasks, they can partition and iterate. That means they are ready to do equivalence and comparison tasks. If they are not able to solve problems such as these, provide a range of similar tasks, using real-life contexts and involving area, length, and set models. ■

## Fraction Notation

The way that we write fractions with a top and a bottom number and a bar between is a convention—an arbitrary agreement for how to represent fractions. However, understanding of the convention can be clarified by giving explicit attention to the meaning of the numerator and the denominator as part of iterating activities. Students can understand the idea of halves and fourths, yet not understand the meaning of the symbols  $\frac{1}{2}$  and  $\frac{1}{4}$ . In the CCSS-M, understanding the symbols for fractions is an emphasis in grade 3. Fractions should include examples that are less than 1, equal to one (e.g.,  $\frac{4}{4}$ ), and greater than 1 (e.g.,  $\frac{4}{8}$  and  $\frac{4}{3}$ ). Engage students in iterating tasks and using symbols, then pose questions to make sense of the symbols, such as:

- What does the numerator in a fraction tell us?
- What does the denominator in a fraction tell us?
- What might a fraction equal to 1 look like?
- How do you know if a fraction is greater than or less than 1? Greater than or less than 2?

Here are some likely explanations for the top and bottom numbers from third graders:

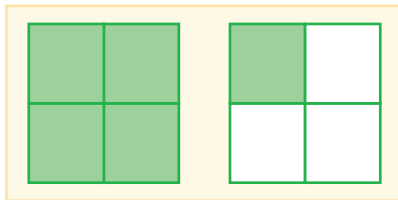
- The numerator is the counting number. It tells how many shares or parts we have. It tells how many have been counted.
- The denominator tells what size piece is being counted. For example, if there are four parts in a whole, then we are counting *fourths*.

Making sense of symbols requires connections to visuals. Illustrating what  $\frac{5}{4}$  looks like in terms of pizzas (area), on a number line (length), or connected to filling bags with objects (set) will help students make sense of this value.

One of the best things we can do for students is to emphasize equivalence and different ways to write fractional amounts.

## Pause & Reflect

What fraction notation might you use for the visual here (the large square represents one unit)? ●



There are (at least) three ways to notate this quantity:

$$\frac{5}{4} \quad 1\frac{1}{4} \quad \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

Do you think that students would be able to describe this quantity in all three ways? In the fourth National Assessment of Educational Progress (NAEP), fewer than half of the seventh graders assessed knew that  $5\frac{1}{4}$  was the same as  $5 + \frac{1}{4}$  (Kouba et al., 1988).

Throughout this chapter we have been including fractions less than 1 and fractions greater than 1. This helps students develop understanding of fractions as values that come between whole numbers (or can be equivalent to whole numbers). Too often, students aren't exposed to numbers equal to or greater than 1 (e.g.,  $\frac{6}{6}$ ,  $\frac{5}{2}$  or  $4\frac{1}{4}$ ), so when these values are added into the mix (no pun intended!), students find them confusing.

The term *improper fraction* is used to describe fractions that are greater than one, such as  $\frac{5}{4}$ . This term can be a source of confusion as the word *improper* implies that this representation is not acceptable, which is not the case at all—in fact, it is often the preferred representation in algebra. Instead, try not to use this phrase and instead use “fraction” or “fraction greater than 1.” Note that the word *improper* is not used in the CCSS-M content standards.

If you have counted fractional parts beyond a whole, as discussed in the previous section, your students already know how to write  $\frac{13}{6}$  or  $\frac{13}{5}$ . Ask students to use a model to illustrate these values and find equivalent representations using wholes and fractions (mixed numbers). Using connecting cubes was the most effective way to help students see both forms for recording fractions greater than 1 (Neumer, 2007) (see Figure 15.12). Students identify one cube as the unit fraction ( $\frac{1}{5}$ ) for the problem ( $\frac{13}{5}$ ). They count out 12 fifths and build wholes. Conversely, they could start with the mixed number, build it, and find out how many total cubes (or fifths) were used. Repeated experiences in building and solving these tasks will help students to notice a pattern that actually explains the algorithm for moving between mixed numbers and fractions greater than 1.

Help students move from physical models to mental images. Challenge students to figure out the two equivalent forms by just picturing the stacks in their heads. A good explanation for

**Box 1:** Three blue rectangles stacked vertically. The top rectangle is shaded. Question: If this rectangle is one-third, what could the whole look like?

**Box 2:** Three blue rectangles stacked vertically. The top two rectangles are shaded. Question: If this rectangle is three-fourths, draw a shape that could be the whole.

**Box 3:** Three blue rectangles stacked vertically. The top two rectangles are shaded. Question: If this rectangle is four-thirds, what rectangle could be the whole?

**Box 4:** Three rods stacked vertically. The top rod is purple, the middle is dark green, and the bottom is yellow. Question: If purple is one-third, what rods are the whole?

**Box 5:** Three rods stacked vertically. The top rod is dark green, the middle is purple, and the bottom is yellow. Question: If dark green is two-thirds, what rod is the whole?

**Box 6:** Three rods stacked vertically. The top rod is yellow, the middle is purple, and the bottom is dark green. Question: If yellow is five-fourths, what rod is one whole?

**Box 7:** A set of 4 red counters in a 2x2 grid. Question: If 4 counters are one-half of a set, how big is the set?

**Box 8:** A set of 12 red counters in a 3x4 grid. Question: If 12 counters are three-fourths of a set, how many counters are in the full set?

**Box 9:** A set of 10 red counters in a 2x5 grid. Question: If 10 counters are five-halves of a set, how many counters are in one set?

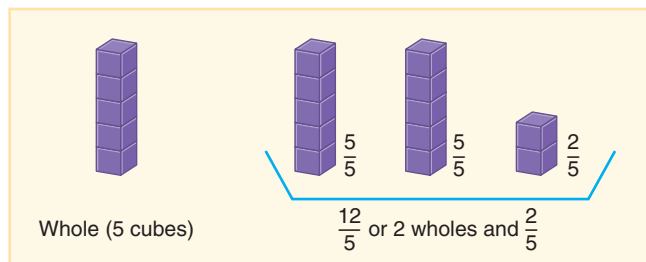
**FIGURE 15.11** Given the part and the fraction, find the whole.

### CCSS Standards for Mathematical Practice

**MP1.** Make sense of problems and persevere in solving them.

### CCSS Standards for Mathematical Practice

**MP8.** Look for and express regularity in repeated reasoning.



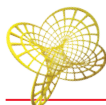
**FIGURE 15.12** Connecting cubes are used to represent the equivalence of  $\frac{12}{5}$  and  $2\frac{2}{5}$ .

$3\frac{1}{4}$  might be that there are 4 fourths in one whole, so there are 8 fourths in two wholes and 12 fourths in three wholes. The extra fourth makes 13 fourths in all, or  $\frac{13}{4}$ . (Note the iteration concept playing a role.)

Do not push the standard algorithm (multiply the bottom by the whole number and add the top), as it can interfere with students making sense of the relationship between the two and their equivalence.



### Complete Self-Check 15.3: Fractional Parts



## Equivalent Fractions

As discussed in Chapter 14, equivalence is a critical but often poorly understood concept. This is particularly true with fraction equivalence. In the CCSS-M, fraction equivalence and comparisons are emphasized in grade 3 and applied in grade 4 (and beyond) as students engage in computation with fractions. Students cannot be successful in fraction computation without a strong understanding of fraction equivalence.

### Conceptual Focus on Equivalence



### Pause & Reflect

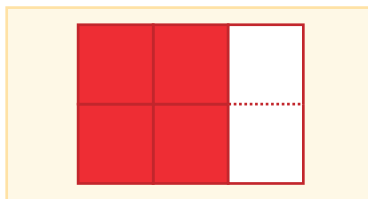
How do you know that  $\frac{4}{6} = \frac{2}{3}$ ? Before reading further, think of at least two different explanations. ●

Here are some possible answers to the preceding question:

1. They are the same because you can simplify  $\frac{4}{6}$  and get  $\frac{2}{3}$ .
2. If you have a set of 6 items and you take 4 of them, that would be  $\frac{4}{6}$ . But you can make the 6 into 3 groups, and the 4 would be 2 groups out of the 3 groups. That means it's  $\frac{2}{3}$ .



3. If you start with  $\frac{2}{3}$ , you can multiply the top and the bottom numbers by 2, and that will give you  $\frac{4}{6}$ , so they are equal.
4. If you had a square cut into 3 parts and you shaded 2, that would be  $\frac{2}{3}$  shaded. If you cut all 3 of these parts in half, that would be 4 parts shaded and 6 parts in all. That's  $\frac{4}{6}$ , and it would be the same amount.



All of these answers are correct. But let's think about what they tell us. Responses 2 and 4 are conceptual, although not as efficient. The procedural responses, 1 and 3, are efficient but do not indicate conceptual understanding. All students should eventually be able to write an equivalent fraction for a given fraction. At the same time, the procedures should never be taught or used until the students understand what the result means. Consider how different the procedure and the concept appear to be:

**Concept:** Two fractions are equivalent if they are representations for the same amount or quantity—if they are the same number.

**Procedure:** To get an equivalent fraction, multiply (or divide) the top and bottom numbers by the same nonzero number.

Rushing too quickly to the algorithm can impede students’ conceptual understanding of fractions and fraction equivalence. Be patient!

### Equivalent Fraction Models

The general approach to helping students create an understanding of equivalent fractions is to have them use contexts and models to find different names for a fraction (see Figure 15.13 for examples). This is the first time in students’ experience that they are seeing that a fixed quantity can have more than one name (actually an infinite number of names). Area models are a good place to begin understanding equivalence.

## Activity 15.13

CCSSM: 3.NF.A.1; 3.NF.3a, b, c

### Making Stacks

Select a manipulative that is designed for exploring fractions (e.g., pattern blocks, tangrams, fraction strips, or fraction circles). Prepare fraction cards with different fractional amounts, such as  $\frac{2}{3}$ ,  $\frac{1}{2}$ ,  $\frac{3}{4}$ , 1,  $\frac{3}{2}$ ,  $\frac{4}{3}$ , and 2. (Note: you may want to begin with fractions less than 1, then move to fractions equal to and greater than 1.) Working individually or with a partner, students first identify the whole. Then they see how many stacks they can make on top of the whole. A stack must use the same-sized piece. Ask students to record all the possibilities they find (they can color the shapes and write the fraction). After completing several examples, have students look at the fractions they wrote for a stack and describe or write about the patterns they notice.

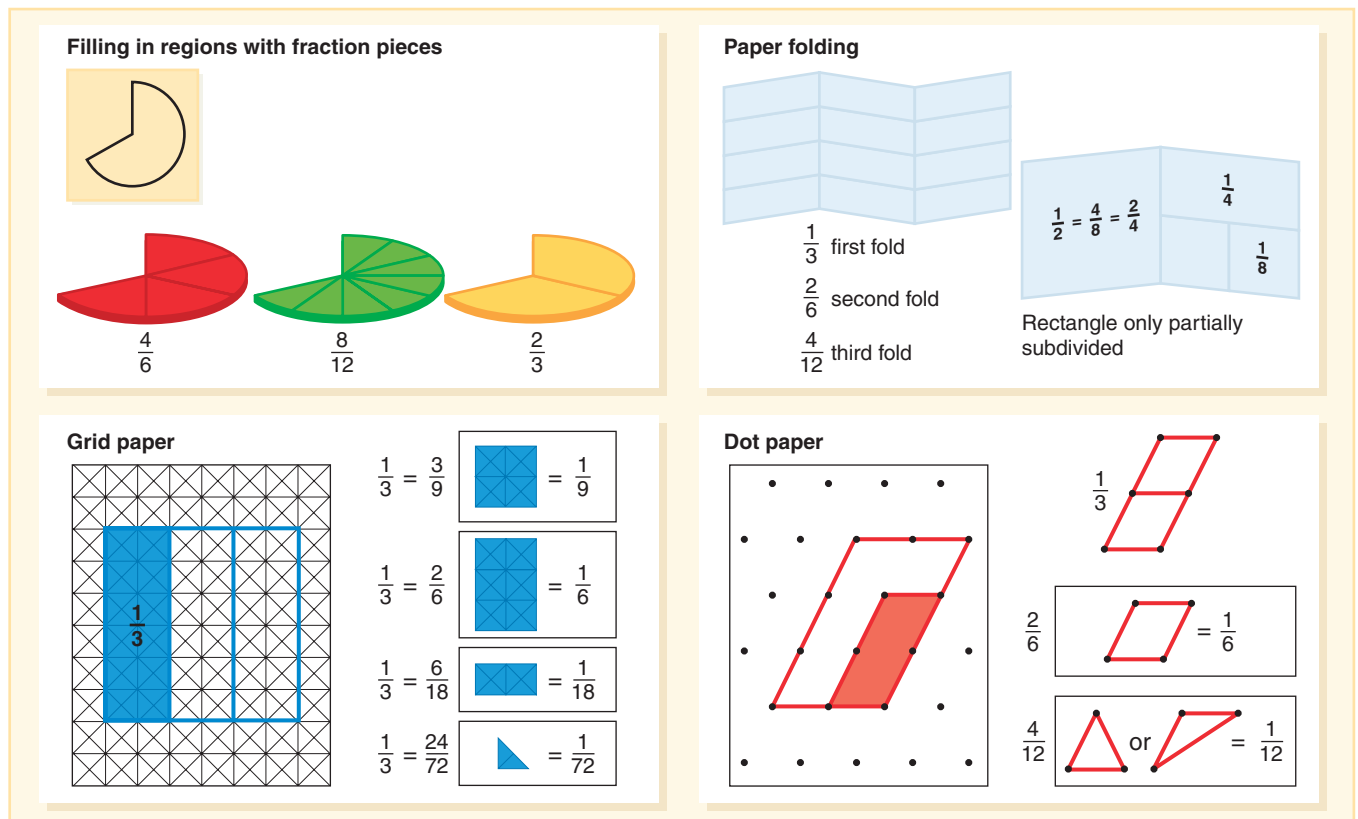


FIGURE 15.13 Area models for equivalent fractions.

**CCSS** Standards for  
Mathematical  
Practice

**MP3.** Construct viable arguments and critique the reasoning of others.

In a classroom discussion following this activity, you can help students reason about equivalent fractions by asking them to consider what other equivalencies are possible (and justify their thinking). For example, ask, “What equivalent fractions could you find if we had six-tenths in our fraction kit? If you could have a piece of any size at all, what other fraction names are possible?”

The following activity moves from using manipulatives to sketches on paper.

## Activity 15.14

CCSSM: 3.NF.A.1; 3.NF.3a, b, c

### Dot Paper Equivalences

Use [Fraction Names](#) Activity Page, which includes three different grids with a fraction shaded (each enclosed area represents one whole). Ask students how many fraction names they think the first problem has. Then ask them to see how many they can find (working individually or in partners). Invite students to share and explain the fraction names they found for problem 1. Repeat for the next two problems. Alternatively, cut this page into three task cards, laminate the cards, and place each at a station along with an overhead pen. Have students rotate in partners to a station and see how many fraction names they can find for that shape (using the pen as needed to show their ways). Rotate to the next station. You may also want to refer to the [Dot Paper Equivalences Expanded Lesson](#).

To make additional pictures, create your own using your choice of [Grid or Dot Paper](#) (see Blackline Masters 5–11). (Figure 15.13 includes an example drawn on an isometric grid). The larger the size of the whole, the more names the activity will generate.

The Dot Paper Equivalences activity involves what Lamon (2012) calls “unitizing”—that is, given a quantity, finding different ways to chunk the quantity into parts in order to name it. She points out that this is a key ability related not only to equivalent fractions but also to proportional reasoning, especially in the comparison of ratios.

Length models should be used in activities similar to the Making Stacks task. Asking students to locate  $\frac{2}{5}$  and  $\frac{4}{10}$  on a number line, for example, can help them see that the two fractions are equivalent (Siegler et al., 2010). Rods or paper strips can be used to designate both a whole and a part, as illustrated in Figure 15.14. Students use smaller rods to find fraction names for the given part. To have larger wholes or values greater than one whole, use a train of two or three rods. Folding paper strips is another method of creating fraction names. In the example shown in Figure 15.14, one-half is subdivided by successive folding in half. Other folds would produce other names, and these possibilities should be discussed if no one tries to fold the strip in an odd number of parts.

## Activity 15.15

CCSSM: 3.NF.A.3a, b, d; 4.NF.A.2

### Stretching Number Lines

Using elastic strips has been effective in helping students understand equivalence of fractions and compare fractions (Harvey, 2012).

Cut strips of elastic (about 1 meter or yard in length). Hold the elastic taut and mark off ten partitions on each. Hand one out to each pair of students. Ask students to use their stretching number line to find the place on a table that represents the fraction of the distance across the table. For each pair, ask: “Which distance is greater, or are they equal?”

- $\frac{5}{10}$  of the distance across.
- $\frac{1}{2}$  of the distance across.
- $\frac{3}{10}$  of the distance across.
- $\frac{3}{8}$  of the distance across. (Note: they have to rethink the whole as 8 sections.)
- $\frac{3}{4}$  of the distance across. (Note: they have to rethink the whole as 8 sections.)
- $\frac{6}{8}$  of the distance across. (Note: they have to rethink the whole as 8 sections.)

For early finishers, invite them to find their own equivalencies using their elastic to test their ideas.

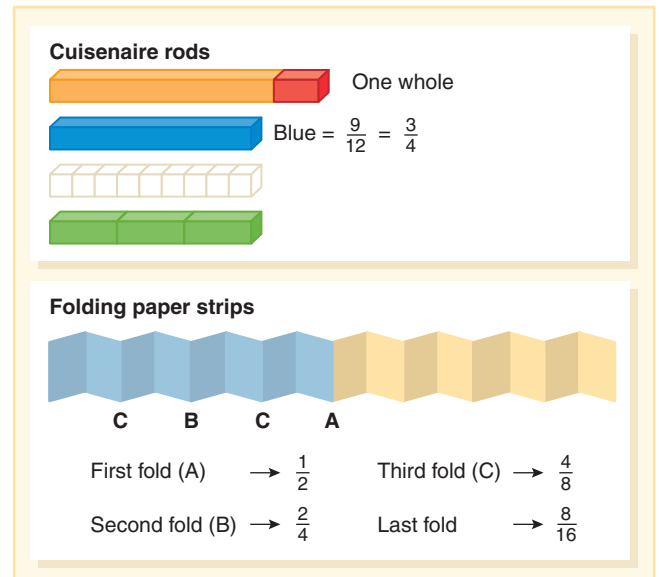
Set models can also be used to develop the concept of equivalence. Legos, a highly motivating manipulative, can help students learn to write fraction equivalencies (for an excellent elaboration on this idea, see Gould, 2011). Lego bricks and can be viewed as an area (array) or as a set (students can count the studs).

Two color counters are an effective tool for fraction equivalencies. Click here to listen to John Van de Walle discuss fractions equivalencies with two color counters. Activity 15.17 uses the context of apples, which can be modeled using two color counters.

In the activities so far, there has only been a hint of a rule for finding equivalent fractions. Activity 15.18 moves a bit closer, but should still be done before developing an algorithm.

Students who have learning disabilities and other students who struggle with mathematics may benefit from using clocks to do equivalence; for example, to find equivalent fractions for  $\frac{10}{12}$ ,  $\frac{3}{4}$ ,  $\frac{4}{6}$ , and so on (Chick, Tierney, & Storeygard, 2007).

NCTM's Illuminations website offers an excellent set of three units called "Fun with Fractions." Each unit uses one of the model types (area, length, or set) and focuses on comparing and ordering fractions and equivalences. The five to six lessons in each unit incorporate a range of manipulatives and engaging activities to support student learning.



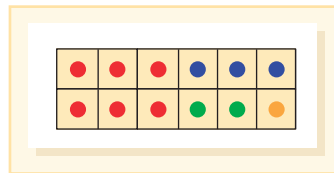
**FIGURE 15.14** Length models for equivalent fractions.

## Activity 15.16

CCSSM: 2.G.A.3; 3.NF.A.1; 3.NF.3a, b, c; 4.NF.B.3a, b

### Lego Land: Building Options

Hand out one 2-by-6 Lego to each student. Ask them to describe it (there are 12 studs, two rows of 6). For second grade or as a warm-up, ask students what same-colored pieces could cover their land (e.g., 6 of the 2-by-1 pieces). Ask students to imagine that 12 pieces of Legos represents one plot of land. It can be covered with various smaller pieces as shown here:



Ask students to build the plot of land using different Lego pieces (1-by-2, 1-by-3, 2-by-2, 1-by-1, 2-by-6, etc.). After they have completed their plot of land, ask students to tell the fraction of their land that is represented by a particular piece (e.g., the 2-by-6 is  $\frac{6}{12}$  as well as  $\frac{1}{2}$  and  $\frac{2}{4}$ ).

To focus on iteration and to build connections to addition (grade 4), students can write equations to describe their Lego Land. In the one pictured here, that would be

$$\frac{6}{12} + \frac{3}{12} + \frac{2}{12} + \frac{1}{12} \quad \text{OR} \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}$$

Note that students have misconceptions about how to name fractions parts, naming the blue part as  $\frac{1}{3}$  rather than  $\frac{1}{4}$  because they see three pieces (Wilkerson, Bryan, & Curry, 2012). This becomes a good topic for a classroom discussion: What is the fractional value of the blue pieces (simplified)?

## Activity 15.17

CCSSM: 3.NF.A.1; 3.NF.3a, b, c

ENGLISH  
LANGUAGE  
LEARNERS

## Apples and Bananas

Use [Apples and Bananas](#) Activity Page or just have students set out a specific number of counters in two colors—for example, 24 counters, with 16 of them red (apples) and 8 of them yellow (bananas). The 24 counters make up the whole. The task is to group the counters into different fractional parts of the whole and use the parts to create fraction names for the fractions that are apples and fractions that are bananas. Ask questions such as, “If we make groups of four, what part of the set is red?” to encourage students to think of different ways to form equal-sized groups. In Figure 15.15, 24 counters are arranged in different groups. You might also suggest arrays (see Figure 15.16). ELLs may not know what the term *group* means because when used in classrooms, the word usually refers to arranging students. Spend time before the activity modeling what it means to group objects.

## Activity 15.18

CCSSM: 3.NF.3a, b, c; 4.NF.A.1

## Missing-Number Equivalences

Use [Missing-Number Equivalences](#) Activity Page or give students an equation expressing an equivalence between two fractions, but with an unknown value. Ask students to use counters or rectangles to illustrate and find the equivalent fraction. Example equations:

$$\frac{5}{3} = \frac{\quad}{6} \quad \frac{2}{3} = \frac{6}{\quad} \quad \frac{8}{12} = \frac{\quad}{3} \quad \frac{9}{12} = \frac{3}{\quad}$$

The missing value can be in a numerator or a denominator; the missing number can be either larger or smaller than the corresponding part of the equivalent fraction. (All four possibilities are represented in the examples.) Figure 15.17 illustrates how Zachary represented the equivalences with equations and partitioning rectangles. The examples shown involve simple whole-number multiples between equivalent fractions. Next, consider pairs such as  $\frac{6}{8} = \frac{\quad}{12}$  or  $\frac{9}{12} = \frac{6}{\quad}$ . In these equivalences, one denominator or numerator is not a whole-number multiple of the other. In addition, include equivalencies for whole numbers and fractions greater than one:  $\frac{8}{x} = \frac{6}{6}$   $\frac{10}{3} = \frac{x}{9}$

**CCSS** Standards for  
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**MP1.** Make sense of problems and persevere in solving them.

## Developing an Equivalent-Fraction Algorithm

When students understand that fractions can have different (but equivalent) names, they are ready to develop a method for finding equivalent names for a particular value. An area model is a good visual for connecting the concept of equivalence to the standard algorithm for finding equivalent fractions (multiply both the top and bottom numbers by the same number to get an equivalent fraction). The approach suggested here is to look for a pattern in the way that the fractional parts in both the part and the whole are counted. Activity 15.21 is a beginning, but a good class discussion following the activity will also be required.

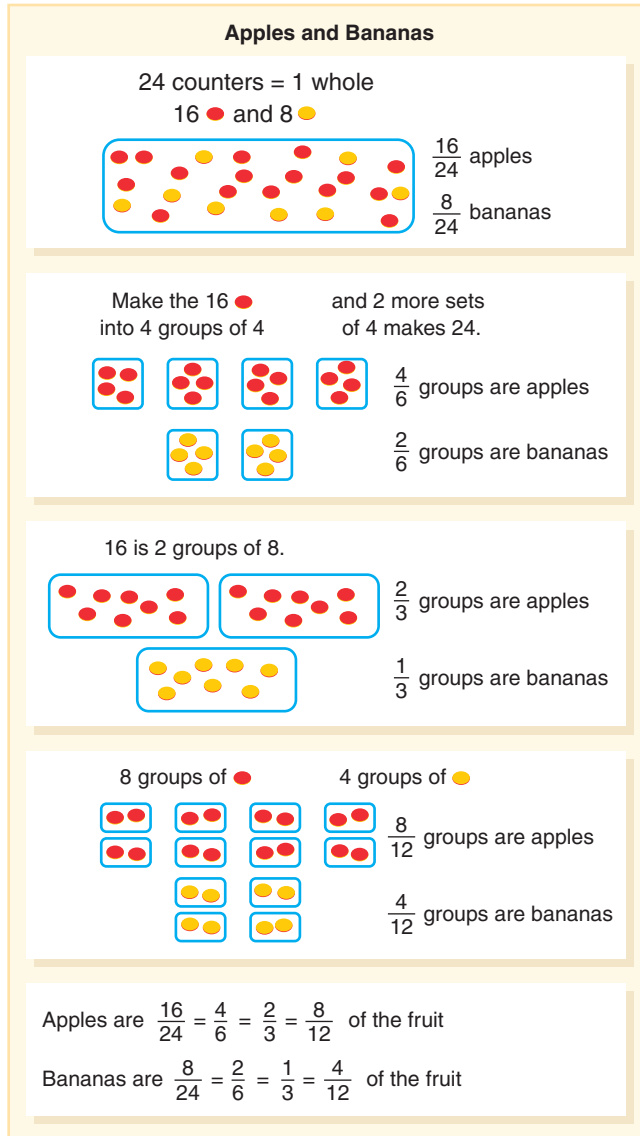
## Activity 15.19

CCSSM: 3.NF.3b; 4.NF.A.1

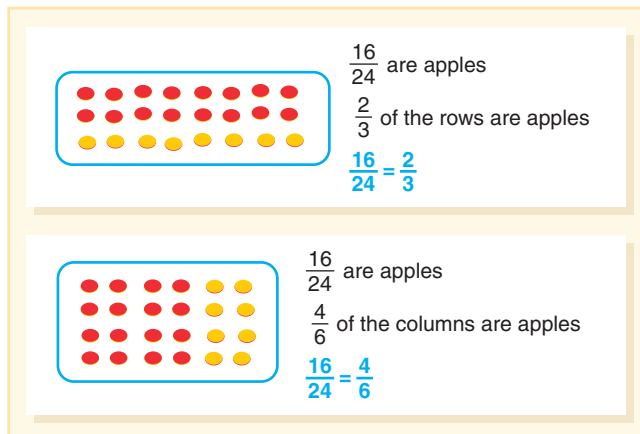
## Garden Plots

Have students draw a square “garden” on blank paper, or give each student a square of paper (like origami paper). Begin by explaining that the garden is divided into rows of various vegetables. In the first example, you might illustrate four rows (fourths) and designate  $\frac{3}{4}$  as corn. Ask students to partition their square into four rows and shade three-fourths as in Figure 15.18. Then explain that the garden is going to be shared with family and friends in a way that each person gets a harvest that is  $\frac{3}{4}$  corn. Show how the garden can be partitioned horizontally to represent two people sharing the corn (i.e.,  $\frac{6}{8}$ ). Ask what fraction of the newly divided garden is corn. Next, tell students to come up with other ways that friends can share the garden (they can choose how many friends, or you can). For each newly divided garden, ask students to record an equation showing the equivalent fractions.

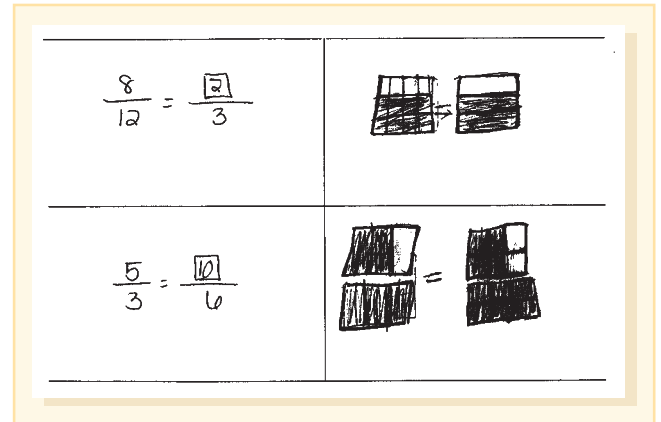




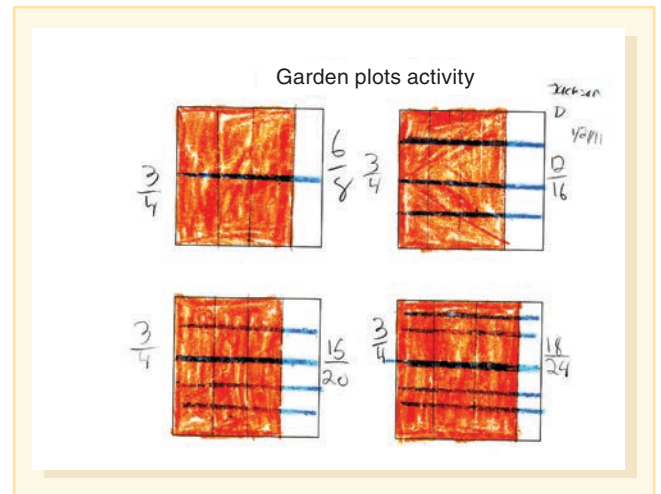
**FIGURE 15.15** Set models for illustrating equivalent fractions.



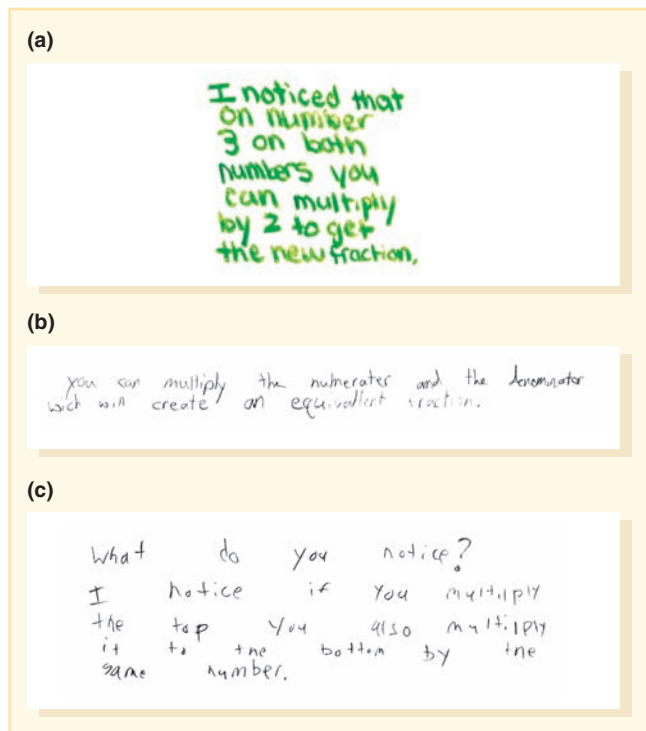
**FIGURE 15.16** Arrays for illustrating equivalent fractions.



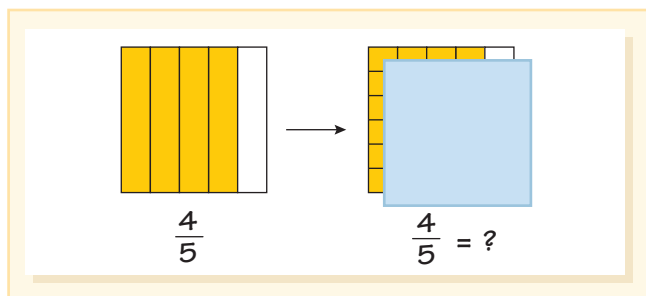
**FIGURE 15.17** A student illustrates equivalence fractions by partitioning rectangles.



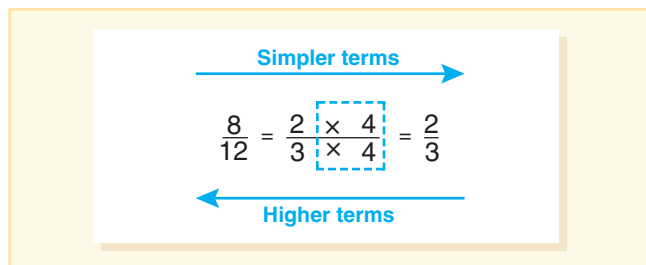
**FIGURE 15.18** A third grader partitions a garden to model fraction equivalences.



**FIGURE 15.19** Students explain what they notice about fraction equivalences based on partitioning “gardens” in different ways.



**FIGURE 15.20** How can you count the fractional parts if you cannot see them all?



**FIGURE 15.21** Using the equivalent-fraction algorithm to write fractions in simplest terms.

After students have prepared their own examples, provide time for them to look at their fractions and gardens and notice patterns about the fractions and the diagrams. Once they have time to do this individually, ask students to share. Figure 15.19 provides student explanations that illustrate the range of “noticing.”

As you can see, for some of these students more experiences are needed. You can also assist in helping students make the connection from the partitioned square to the procedure by displaying a square (for example, partitioned to show  $\frac{4}{5}$ ) (see Figure 15.20). Then, partition the square vertically into six parts, covering most of the square as shown in the figure. Ask, “What is the new name for my  $\frac{4}{5}$ ?”

The reason for this exercise is that it helps students see the connection to multiplication. With the covered square, students can see that there are four columns and six rows to the shaded part, so there must be  $4 \times 6$  parts shaded. Similarly, there must be  $5 \times 6$  parts in the whole. Therefore, the new name for  $\frac{4}{5}$  is  $\frac{4 \times 6}{5 \times 6}$ , or  $\frac{24}{30}$ .

Examine **examples** of equivalent fractions that have been generated with other visuals (e.g., a number line) and see if the rule of multiplying top and bottom numbers by the same number holds. Ask students to explain why. Ask students, “If the rule is correct, how can  $\frac{6}{8}$  and  $\frac{9}{12}$  be equivalent?”

**Writing Fractions in Simplest Terms.** The multiplication scheme for equivalent fractions produces fractions with larger denominators. But creating equivalent forms for  $\frac{6}{8}$  might involve multiplication to get  $\frac{12}{16}$  or division to get  $\frac{3}{4}$ . To write a fraction in *simplest terms* means to write it so that numerator and denominator have no common whole-number factors. One meaningful approach to this task of finding simplest terms is to reverse the earlier process, as illustrated in Figure 15.21. The search for a common factor or a simplified fraction should be connected to grouping. Texas Instruments offers a comparing fractions activity using the number line on their Classroom Activities Exchange.

Two additional things should be noted regarding fraction simplification:

1. Notice that the phrase *reducing fractions* was not used. Because this would imply that the fraction is being made smaller, this terminology should be avoided. Fractions are simplified, *not* reduced.
2. Teachers sometimes tell students that fraction answers are incorrect if not in simplest or lowest terms. This also misinforms students about the equivalence of fractions. When students add  $\frac{1}{6} + \frac{1}{2}$ , both  $\frac{2}{3}$  and  $\frac{4}{6}$  are correct. It is best to reinforce that they are both correct and are equivalent.

**Multiplying by One.** Mathematically, equivalence is based on the multiplicative identity (any number multiplied by 1 remains unchanged). Any fraction of the form  $\frac{a}{b}$  can be used as the identity element. Therefore,  $\frac{3}{4} = \frac{3}{4} \times 1 = \frac{3}{4} \times \frac{2}{2} = \frac{6}{8}$ . Furthermore, the numerator and denominator of the identity element can also be fractions. In this way,  $\frac{6}{12} = \frac{6}{12} \times \left(\frac{1/6}{1/6}\right) = \frac{1}{2}$ . Understanding this idea is an expectation in the CCSS-M in grade 4.



**TECHNOLOGY Note.** Developing the concept of equivalence can be supported with the use of technology. In the NCTM e-Examples, there is a fraction game (Fraction Track) for two players (Applet 5.1, *Communicating about Mathematics Using Games*). The game uses a number-line model, and knowledge of equivalent fractions plays a significant role. The Equivalent Fractions tool from NCTM's Illuminations website is designed to help students create equivalent fractions by dividing and shading square or circular regions and then matching each fraction to its location on a number line. Students can use the computer-generated fraction or build their own. Once the rectangular or circular shape is divided, the student fills in the parts or fractional region and then builds two models equivalent to the original fraction. The three equivalent fractions are displayed in a table and in the same location on a number line. ■



### Complete Self-Check 15.4: Equivalent Fractions



## Comparing Fractions

When students are looking to see whether two or more fractions are equivalent, they are comparing them. If they are not equivalent, then students can determine which ones are smaller and which ones are larger. As illustrated in [Ally's interview](#) about comparing fractions, students often have misconceptions about fractions and therefore are not able to compare—for example, thinking that bigger numbers in the denominator mean the fraction is bigger. The ideas described previously for equivalence across area, length, and set models are appropriate for comparing fractions. The use of contexts, models, and mental imagery can help students build a strong understanding of the relative size of fractions (Bray & Abreu-Sanchez, 2010; Petit et al., 2010). The next section offers ways to support students' understanding of the relative size of fractions.

### Comparing Fractions Using Number Sense

In the National Assessment of Educational Progress (NAEP) test, only 21 percent of fourth-grade students could explain why one unit fraction was larger or smaller than another—for example,  $\frac{1}{5}$  and  $\frac{1}{4}$  (Kloosterman et al., 2004). For eighth graders, only 41 percent were able to correctly put in order three fractions given in simplified form (Sowder, Wearne, Martin, & Strutchens, 2004).

**Comparing Unit Fractions.** As noted earlier, whole-number knowledge can interfere with comparing fractions. Students think, “Seven is more than four, so sevenths should be bigger than fourths” (Mack, 1995). The inverse relationship between number of parts and size of parts cannot be “told” but must be developed in each student through many experiences, including ones that were described earlier related to partitioning, iterating, estimation, and equivalence.

## Activity 15.20

CCSSM: 3.NF.A.3d; 4.NF.A.2

### Ordering Unit Fractions

List a set of unit fractions such as  $\frac{1}{3}$ ,  $\frac{1}{8}$ ,  $\frac{1}{5}$ , and  $\frac{1}{10}$  (assume same size whole for each fraction). Ask students to use reasoning to put the fractions in order from least to greatest. Challenge students to explain their reasoning with an area model (e.g., circles) **and** on a number line. Ask students to connect the two representations. (“What do you notice about  $\frac{1}{3}$  of the circle and  $\frac{1}{3}$  on the number line?”) Students with disabilities may need to use clothespins with the fractions written on them and place them on the line first.

Repeat with all numerators equal to some number other than 1.

Repeat with fractions that have different numerators and different denominators. You can vary how many fractions are being compared to differentiate the task.



STUDENTS  
with  
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Students may notice that larger bottom numbers mean smaller fractions (this is an important pattern to notice), but it only holds true when the numerators are the same. Still, it is conjecture that can be posed to the class for testing. Eventually they will find cases where it is not true (e.g.,  $\frac{1}{8} < \frac{1}{4}$  but  $\frac{3}{8} > \frac{1}{4}$ ).

**CCSS** Standards for  
Mathematical  
Practice

**MP2.** Reason abstractly and quantitatively.

**Comparing Any Fractions.** You have probably learned rules or algorithms for comparing two fractions. The usual approaches are finding common denominators and using cross-multiplication. These rules can be effective in getting correct answers but require no thought about the size of the fractions. If students are taught these rules before they have had the opportunity to think about the relative sizes of various fractions, they are less likely to develop number sense about fraction size. The goal is to select an efficient strategy for determining the larger fraction, and these two methods are not always the most efficient (CCSSO, 2010).

## Pause & Reflect

Assume for a moment that you do not know the common-denominator or cross-multiplication techniques. Now examine the pairs of fractions in Figure 15.22 and select the largest of each pair using a reasoning approach that a fourth grader might use. ●

The **Which Is Greater?** Activity Page can be used to do this activity with students. Figure 15.23 provides explanations from two students on the first column (A–F). Both students are able to reason to determine which is larger, though one student is better able to articulate those ideas.

The following list summarizes ways that the fractions in Figure 15.22 might have been compared:

1. *Same-size whole (same denominators).* To compare  $\frac{3}{8}$  and  $\frac{5}{8}$ , think about having 3 parts of something and also 5 parts of the same thing. (This method can be used for problems B and G.)
2. *Same number of parts (same numerators) but different-sized wholes.* Consider the case of  $\frac{3}{4}$  and  $\frac{3}{7}$ . If a whole is divided into 7 parts, the parts will certainly be smaller than if divided into only 4 parts. (This strategy can be used with problems A, D, and H.)
3. *More than/less than one-half or one.* The fraction pairs  $\frac{3}{7}$  versus  $\frac{5}{8}$  and  $\frac{5}{4}$  versus  $\frac{7}{8}$  do not lend themselves to either of the previous thought processes. In the first pair,  $\frac{3}{7}$  is less than half of the number of sevenths needed to make a whole, and so  $\frac{3}{7}$  is less than a half. Similarly,  $\frac{5}{8}$  is more than a half. Therefore,  $\frac{5}{8}$  is the larger fraction. The second pair is determined by noting that one fraction is greater than 1 and the other is less than 1. (This method could be used on problems A, D, F, G, and H.)

4. *Closeness to one-half or one.* Why is  $\frac{9}{10}$  greater than  $\frac{3}{4}$ ? Each is one fractional part away from one whole, and tenths are smaller than fourths. Similarly, notice that  $\frac{5}{8}$  is smaller than  $\frac{1}{6}$  because it is only one-eighth more than a half, whereas  $\frac{1}{6}$  is a sixth more than a half. Can you use this basic idea to compare  $\frac{3}{5}$  and  $\frac{5}{9}$ ? (*Hint:* Each is half of a fractional part more than  $\frac{1}{2}$ .) Also try  $\frac{5}{7}$  and  $\frac{7}{9}$ . (This is a good strategy for problems C, E, I, J, K, and L.)

How did your reasons for choosing fractions in Figure 15.22 compare to these ideas? It is important that you are comfortable with these informal comparison strategies as a major component of your own number sense as well as for helping students develop theirs. Notice that some of the comparisons, such as problems D and H, could have been solved using more than one of the strategies listed.

Tasks you design for your students should assist them in developing these methods of comparing two fractions. The ideas should emerge from your students' reasoning. To teach "the four ways to compare fractions" defeats the purpose of encouraging

Which fraction in each pair is greater?  
Give one or more reasons. Try not to use drawings or models.  
Do not use common denominators or cross-multiplication.

A.  $\frac{4}{5}$  or  $\frac{4}{9}$

B.  $\frac{4}{7}$  or  $\frac{5}{7}$

C.  $\frac{3}{8}$  or  $\frac{4}{10}$

D.  $\frac{5}{3}$  or  $\frac{5}{8}$

E.  $\frac{3}{4}$  or  $\frac{9}{10}$

F.  $\frac{3}{8}$  or  $\frac{4}{7}$

G.  $\frac{7}{12}$  or  $\frac{5}{12}$

H.  $\frac{3}{5}$  or  $\frac{3}{7}$

I.  $\frac{5}{8}$  or  $\frac{6}{10}$

J.  $\frac{9}{8}$  or  $\frac{4}{3}$

K.  $\frac{4}{6}$  or  $\frac{7}{12}$

L.  $\frac{8}{9}$  or  $\frac{7}{8}$

**FIGURE 15.22** Comparing fractions using reasoning strategies.

students to apply their number sense. Instead, select pairs of fractions that will likely elicit desired comparison strategies. On one day, for example, you might have pairs of fractions with the same numerators. Ask students to tell which is greater and why they think so. Ask them to give an example to convince you. On another day, you might pick fraction pairs in which each fraction is exactly one part away from a whole. Try to build strategies over several days by the strategic choice of fraction pairs.

The use of an area or number-line model may help students who are struggling to reason mentally. Place greater emphasis on students' reasoning and connect it to the visual models.

## Using Equivalent Fractions to Compare

Equivalent-fraction concepts can be used in making comparisons. Smith (2002) suggests that the comparison question to ask is, "Which of the following two (or more) fractions is greater, or are they equal?" (p. 9). He points out that this question leaves open the possibility that two fractions that may look different can, in fact, be equal.

In addition to this point, with equivalent-fraction concepts, students can adjust how a fraction looks so that they can use ideas that make sense to them. Burns (1999) describes how fifth graders compared  $\frac{6}{8}$  to  $\frac{4}{5}$ . (You might want to stop for a moment and think how you would compare these fractions.) One child changed the  $\frac{4}{5}$  to  $\frac{8}{10}$  so that both fractions would be two parts away from the whole and he reasoned from there. Another changed both fractions to a common *numerator* of 12.

Be absolutely certain to revisit the comparison activities and include pairs such as  $\frac{8}{12}$  and  $\frac{2}{3}$  in which the fractions are equal but do not appear to be.

## Estimating with Fractions

Number sense with fractions means that students have some intuitive feel about the relative size of fractions (knowing "about" how big a particular fraction is). As students are deciding about how big something is, they are comparing that fraction to benchmark numbers, such as  $0$ ,  $\frac{1}{2}$ , or  $1$ . As with whole numbers, students are less confident and less capable of estimating than they are at computing exact answers and a focus on estimation can strengthen their understanding of fractions (Clarke & Roche, 2009). Therefore, you need to provide many opportunities for students to estimate with fractions. In daily classroom discussions, ask questions like "About what fraction of your classmates are wearing sweaters?" Or after tallying survey data about a topic like favorite dinner, ask, "About what fraction of our class picked spaghetti?" Activity 15.13 offers examples of visual estimating activities with area and number lines.

### Activity 15.21

CCSSM: 3.NF.A.1; 3.NF.A.2a, b

#### About How Much?

Draw a picture like one of those in Figure 15.24 (or prepare some ahead of time for the overhead). Have each student write down a fraction that he or she thinks is a good estimate of the amount shown (or the indicated mark on the number line). Listen to the ideas of several students, and ask them whether a particular estimate is a good one. There is no single correct answer, but estimates should be in the "ballpark." If students have difficulty coming up with an estimate, ask whether they think the amount is closer to  $0$ ,  $\frac{1}{2}$ , or  $1$ . For students with disabilities, you may want to give them a set of cards showing possible options for estimates. Then they can match the card to one of the pictures.



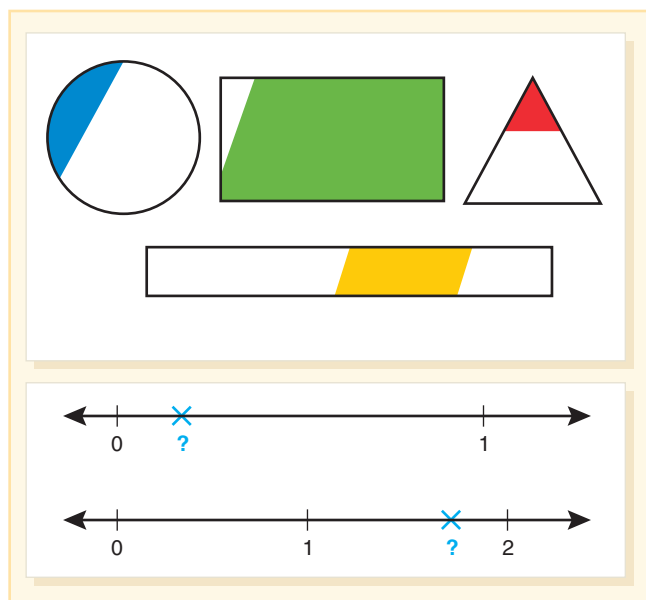
(a)

- A.  $\frac{4}{5}$  is larger because it is  $\frac{1}{5}$  from a whole +  $\frac{4}{4}$  is close to being  $\frac{1}{2}$ .
- B. The denominator is the same but the NUMERATOR of  $\frac{5}{7}$  is greater, making it a bigger fraction.
- C.  $\frac{4}{10}$  is greater because they are both 1 away from half, but tenths are smaller, therefore  $\frac{4}{10}$  is closer + bigger.
- D. Because  $\frac{5}{3}$  is greater than whole but  $\frac{5}{8}$  is only eighth bigger than only half.
- E.  $\frac{9}{10}$  is bigger because they are both 1 away from whole but tenths are smaller, making it closer to half.
- F.  $\frac{4}{7}$  is greater because it's denominator is smaller, + it's numerator is bigger.

(b)

- A.  $\frac{4}{5}$  is only one away from being a whole.  $\frac{4}{7}$  is closer to  $\frac{1}{2}$ .
- B.  $\frac{5}{7}$  is greater than  $\frac{4}{7}$  because  $\frac{5}{7}$  is closer to a whole.
- C.  $\frac{4}{10}$  is greater than  $\frac{3}{8}$  because  $\frac{4}{10}$  would have smaller slices.
- D.  $\frac{5}{3}$  is greater than  $\frac{5}{8}$  because  $\frac{5}{3}$  is greater than one whole.
- E.  $\frac{9}{10}$  is greater than  $\frac{3}{4}$  because  $\frac{9}{10}$  would have smaller slices.
- F.  $\frac{4}{7}$  is greater than  $\frac{3}{8}$  because  $\frac{4}{7}$  is closer to a whole.

FIGURE 15.23 Two students explain how they compared the fractions in problems A through F from Figure 15.22.



**FIGURE 15.24** About how much? Name a fraction for each drawing and explain why you chose that fraction.

The number line is a good model for helping students develop a better understanding for the relative size of a fraction (Petit, Laird, & Marsden, 2010). For example, if students think about where  $\frac{3}{20}$  might be by partitioning a number line between 0 and 1, they will see that  $\frac{3}{20}$  is close to 0, whereas  $\frac{9}{10}$  is quite close to 1. Number lines should also go beyond 1, asking students to tell a nearby benchmark fraction—for example, explaining that  $3\frac{3}{7}$  is almost  $3\frac{1}{2}$ .

After students have experience with visuals, they should continue to reason about the relative size of fractions using mental strategies or creating their own visuals to reason about the fractions.

Finally, comparing fractions can include finding fractions that fall between two given fractions. An important idea in fractions that there is always one more fraction between any two given numbers. **Fraction Find** is an activity that focuses on this idea. The fractions selected can be varied to meet individual students' needs.

## Activity 15.22

CCSSM: 3.NF.A.3d; 4.NF.A.2

### Zero, One-Half, or One

On a set of cards, write a collection of 10 to 15 fractions, one per card. A few should be greater than  $1\frac{9}{8}$  or  $\frac{11}{10}$ , with the others ranging from 0 to 1. Let students sort the fractions into three groups: those close to 0, close to  $\frac{1}{2}$ , and close to 1. For those close to  $\frac{1}{2}$ , have them decide whether the fraction is more or less than half. The difficulty of this task largely depends on the fractions you select. The first time you try this, use fractions that are very close to the three benchmarks, such as  $\frac{1}{20}$ ,  $\frac{53}{100}$ , or  $\frac{9}{10}$ . On subsequent days, mostly use fractions with denominators less than 20. You might include a few fractions that are exactly in between the benchmarks, such as  $\frac{2}{8}$  or  $\frac{3}{4}$ . Ask students to explain how they are using the numerator and denominator to decide. For ELLs, be sure the term **benchmark** is understood and encourage illustrations as well as explanations.

As an extension or alternative to differentiate this activity, ask students to create their own fractions close to each benchmark fraction.



Complete Self-Check 15.5: Comparing Fractions



## Teaching Considerations for Fraction Concepts

Because the teaching of fractions is so important, and because fractions are often not well understood even by adults, a recap of the big ideas is needed. Hopefully you have recognized that one reason fractions are not well understood is that there is a lot to know about them, from part-whole relationships to division constructs, and understanding includes representing across area, length, and set models and includes contexts that fit these models. Many of these strategies may not have been part of your own learning experience, but they must be part of your teaching experience so that your students can fully understand fractions and be successful in algebra and beyond.

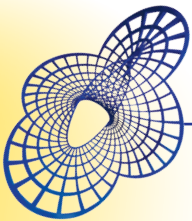
Iterating and partitioning must be a significant aspect of fraction instruction. Equivalence, including comparisons, is a central idea for which students must have sound understanding and skill. Connecting visuals with the procedure and not rushing the algorithm too soon are important aspects of the process.

Clarke and colleagues (2008) and Cramer and Whitney (2010), researchers of fraction teaching and learning, offer research-based recommendations that provide an effective summary of this chapter:

1. Give a greater emphasis to number sense and the meaning of fractions, rather than rote procedures for manipulating them.
2. Provide a variety of models and contexts to represent fractions.
3. Emphasize that fractions are numbers, making extensive use of number lines in representing fractions.
4. Spend whatever time is needed for students to understand equivalences (concretely and symbolically), including flexible naming of fractions.
5. Link fractions to key benchmarks and encourage estimation.



### Complete Self-Check 15.6: Teaching Considerations for Fraction Concepts



## REFLECTIONS ON CHAPTER 15

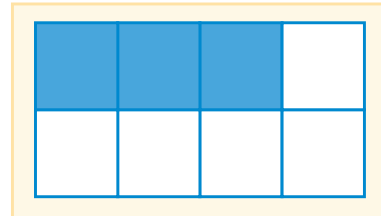
### WRITING TO LEARN

[Click here](#) to assess your understanding and application of chapter content.

1. What is the goal of activities involving the concept of sharing? When would you implement sharing activities?
2. Give examples of manipulatives and contexts that fall into each of the three categories of fraction models (area, length, and set).
3. What does partitioning mean? Explain and illustrate.
4. What does iteration mean? Explain and illustrate.
5. What are two ways to build the conceptual relationship between  $\frac{11}{4}$  and  $2\frac{3}{4}$ ?
6. Describe two ways to compare  $\frac{5}{12}$  and  $\frac{5}{8}$  (not using common denominator or cross-product methods).

### FOR DISCUSSION AND EXPLORATION

- ◆ A common error that students make is to write  $\frac{3}{5}$  for the fraction represented here:



- ◆ Why do you think they do this? What activity or strategy would you use to try to address this misconception?
- ◆ Fractions are often named by adults (and depicted in cartoons) as a dreaded math topic. Why do you think this is true? How might your fraction instruction alter this perception for your students?



## RESOURCES FOR CHAPTER 15

### LITERATURE CONNECTIONS

Context takes students away from rules and encourages them to explore ideas in a more open and meaningful manner. The way that students approach fraction concepts in these contexts may surprise you.

#### How Many Snails? A Counting Book *Giganti (1988)*

Each page of this book has a similar pattern of questions. For example, the narrator wonders how many clouds there are, how many of them are big and fluffy, and how many of them are big and fluffy and gray. Students can look at the pictures and find the fraction of the objects (e.g., clouds) that have the particular characteristic (big and fluffy). Whitin and Whitin (2006) describe how a class used this book to write their own stories in this pattern and record the fractions for each subset of the objects.

#### The Doorbell Rang *Hutchins (1986)*

Often used to investigate whole-number operations of multiplication and division, this book is also an excellent early introduction to fractions. The story is a simple tale of two children preparing to share a plate of 12 cookies. Just as they have figured out how to share the cookies, the doorbell rings and more children arrive. You can change the number of children to create a sharing situation that requires fractions (e.g., 5 children).

#### The Man Who Counted: A Collection of Mathematical Adventures *Tahan (1993)*

This book contains a story, “Beasts of Burden,” about a wise mathematician, Beremiz, and the narrator, who are traveling together on one camel. They are asked by three brothers to solve an argument. Their father has left them 35 camels to divide among them: half to one brother, one-third to another, and one-ninth to the third brother. The story provides an excellent context for discussing fractional parts of sets and how fractional parts change as the whole changes. However, if the whole is changed from 35 to, say, 36 or 34, the problem of the indicated shares remains unresolved. The sum of  $\frac{1}{2}$ ,  $\frac{1}{3}$ , and  $\frac{1}{9}$  will never be one whole, no matter how many camels are involved. Bresser (1995) describes three days of activities with his fifth graders.

#### Apple Fractions *Pallotta (2002)*

This book offers interesting facts about apples while introducing fractions as fair shares (of apples, a healthier option than books that focus on chocolate and cookies!). In addition, the

words for fractions are used and connected to fraction symbols, making it a good connection for fractions in grades 1–3.

### RECOMMENDED READINGS

#### Articles

Clarke, D. M., Roche, A., & Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School*, 13(7), 373–380.

*Ten excellent tips for teaching fractions are discussed and favorite activities are shared. An excellent overview of teaching fractions.*

Flores, A., & Klein, E. (2005). From students’ problem-solving strategies to connections in fractions. *Teaching Children Mathematics*, 11(9), 452–457.

*This article offers a very realistic view (complete with photos of student work) of how children develop initial fraction concepts and an understanding of notation as they engage in sharing tasks like those described in this chapter.*

#### Books

Burns, M. (2001). *Teaching arithmetic: Lessons for introducing fractions, grades 4–5*. Sausalito, CA: Math Solutions Publications.

*This book offers well-designed lessons with lots of details, sample student dialogue, and blackline masters. These are introductory ideas for fraction concepts. Five lessons cover one-half as a benchmark. Assessments are also included.*

Lamon, S. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies*. New York, NY: Taylor & Francis Group.

*As the title implies, this book has a wealth of information to help with better understanding fractions and teaching fractions well. Many rich tasks and student work are provided throughout.*

McNamara, J., & Shaughnessy, M. M. (2010). *Beyond pizzas and pies: 10 essential strategies for supporting fraction sense (grades 3–5)*. Sausalito, CA: Math Solutions Publications.

*This book has it all—classroom vignettes, discussion of research on teaching fractions, and many activities, including student work.*

#### Website

Rational Number Project (website housed at the University of Minnesota).

*This project offers numerous readings, activities for students, and other materials—a great collection of high quality resources for teaching fractions for understanding.*