

18.03 LECTURE NOTES, SPRING 2014

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7. COMPLEX NUMBERS

Complex numbers are expressions of the form $x + yi$, where x and y are real numbers, and i is a new symbol. Multiplication of complex numbers will eventually be defined so that $i^2 = -1$. (Electrical engineers sometimes write j instead of i , because they want to reserve i for current, but everybody else thinks that's weird.) Just as the set of all real numbers is denoted \mathbb{R} , the set of all complex numbers is denoted \mathbb{C} .

Flashcard question: Is 9 a real number or a complex number?

Possible answers:

1. real number
2. complex number
3. both
4. neither

Answer: Both, because 9 can be identified with $9 + 0i$.

7.1. Operations on complex numbers.

real part $\operatorname{Re}(x + yi) := x$

imaginary part $\operatorname{Im}(x + yi) := y$ (**Note:** It is y , not yi , so $\operatorname{Im}(x + yi)$ is real)

complex conjugate $\overline{x + yi} := x - yi$ (negate the imaginary component)

One can **add**, **subtract**, **multiply**, and **divide** complex numbers (except for division by 0). Addition, subtraction, and multiplication are as for polynomials, except that after multiplication one should simplify by using $i^2 = -1$; for example,

$$\begin{aligned}(2 + 3i)(1 - 5i) &= 2 - 7i - 15i^2 \\ &= 17 - 7i.\end{aligned}$$

To divide z by w , multiply z/w by \bar{w}/\bar{w} so that the denominator becomes real; for example,

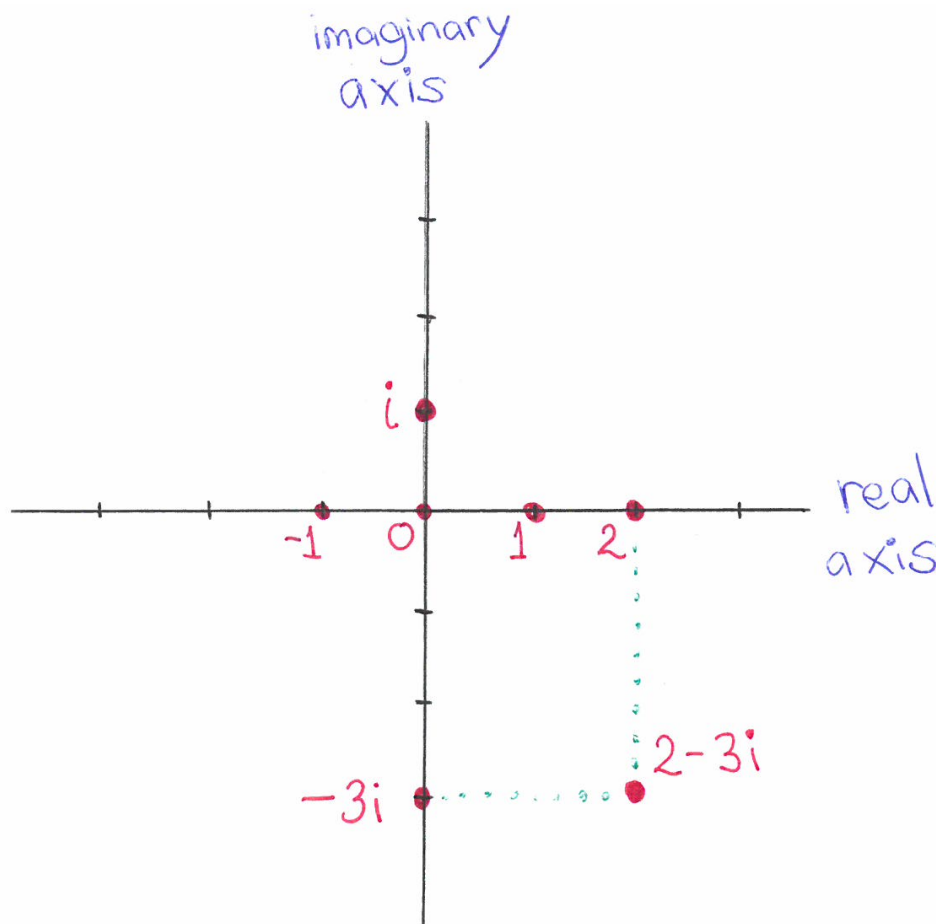
$$\frac{2 + 3i}{1 - 5i} = \frac{2 + 3i}{1 - 5i} \cdot \frac{1 + 5i}{1 + 5i} = \frac{2 + 13i + 15i^2}{1 - 25i^2} = \frac{-13 + 13i}{26} = -\frac{1}{2} + \frac{1}{2}i.$$

The arithmetic operations on complex numbers satisfy the same properties as for real numbers ($zw = wz$ and so on). The mathematical jargon for this is that \mathbb{C} , like \mathbb{R} , is a **field**. In particular,

for any complex number z and integer n , the n^{th} power z^n can be defined in the usual way (need $z \neq 0$ if $n < 0$); e.g., $z^3 := zzz$, $z^0 := 1$, $z^{-3} := 1/z^3$. (**Warning:** Although there is a way to define z^n also for a *complex* number n , when $z \neq 0$, it turns out that z^n has more than one possible value for non-integral n , so it is ambiguous notation. Anyway, the most important cases are e^z , and z^n for integers n ; the other cases won't even come up in this class.)

If you change every i in the universe to $-i$ (that is, take the complex conjugate everywhere), then all true statements remain true. For example, $i^2 = -1$ becomes $(-i)^2 = -1$. Another example: If $z = vw$, then $\bar{z} = \bar{v} \bar{w}$.

7.2. The complex plane. Just as real numbers can be visualized as points on a line, complex numbers can be visualized as points in a plane: plot $x + yi$ at the point (x, y) .

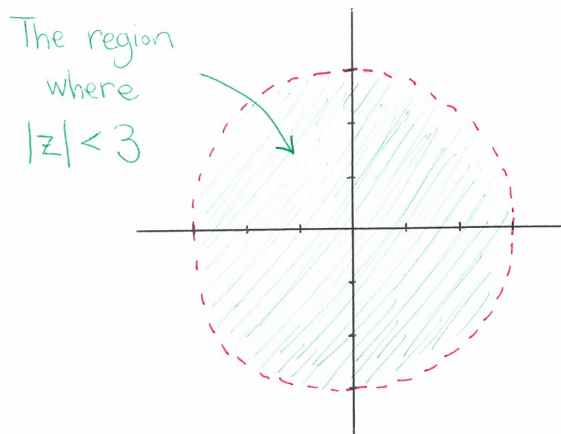


Addition and subtraction of complex numbers has the same geometric interpretation as for vectors. The same holds for scalar multiplication of a complex number by a *real* number. (The geometric interpretation of multiplication by a *complex* number is different; we'll explain it soon.) Complex conjugation reflects a complex number in the real axis.

The **absolute value** (or **magnitude** or **modulus**) $|z|$ of a complex number $z = x + iy$ is its distance to the origin:

$$|x + yi| := \sqrt{x^2 + y^2} \quad (\text{this is a real number}).$$

For a complex number z , inequalities like $z < 3$ do not make sense, but inequalities like $|z| < 3$ do, because $|z|$ is a real number. The complex numbers satisfying $|z| < 3$ are those in the open disk of radius 3 centered at 0 in the complex plane. (**Open** disk means the disk without its boundary.)



7.3. Some useful identities. The following are true for all complex numbers z :

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}, \quad \overline{\bar{z}} = z, \quad z\bar{z} = |z|^2.$$

Also, for any *real* number a and complex number z ,

$$\operatorname{Re}(az) = a \operatorname{Re} z, \quad \operatorname{Im}(az) = a \operatorname{Im} z.$$

(These can fail if a is not real.)

Proof of the first identity: Write z as $x + yi$. Then $\operatorname{Re} z = x$ and $\frac{z + \bar{z}}{2} = \frac{(x+yi) + (x-yi)}{2} = x$ too.

The proofs of the others are similar.

7.4. Complex roots of polynomials.

real polynomial: polynomial with real coefficients

complex polynomial: polynomial with complex coefficients

Example 7.1. How many roots does the polynomial $z^3 - 3z^2 + 4$ have? It factors as $(z - 2)(z - 2)(z + 1)$, so it has only two distinct roots (2 and -1). But if we count 2 twice, then the number of roots *counted with multiplicity* is 3, equal to the degree of the polynomial.

Some real polynomials, like $z^2 + 9$, cannot be factored completely into degree 1 real polynomials, but do factor into degree 1 complex polynomials: $(z + 3i)(z - 3i)$. In fact, *every* complex polynomial factors completely into degree 1 complex polynomials — this is proved in advanced courses in complex analysis. This implies the following:

Fundamental theorem of algebra. *Every degree n complex polynomial $f(z)$ has exactly n complex roots, if counted with multiplicity.*

Since real polynomials are special cases of complex polynomials, the fundamental theorem of algebra applies to them too. For real polynomials, the non-real roots can be paired off with their complex conjugates.

Example 7.2. The degree 3 polynomial $z^3 + z^2 - z + 15$ factors as $(z + 3)(z - 1 - 2i)(z - 1 + 2i)$, so it has three distinct roots: -3 , $1 + 2i$, and $1 - 2i$. Of these roots, -3 is real, and $1 + 2i$ and $1 - 2i$ form a complex conjugate pair.

Example 7.3. Want a fourth root of i ? The fundamental theorem of algebra guarantees that $z^4 - i = 0$ has a complex solution (in fact, four of them). We'll soon learn how to find them.

The fundamental theorem of algebra will be useful for constructing solutions to higher order linear ODEs with constant coefficients, and for discussing eigenvalues.

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7.5. Real and imaginary parts of complex-valued functions. Suppose that $y(t)$ is a complex-valued function of a real variable t . Then

$$y(t) = f(t) + i g(t)$$

for some *real-valued* functions of t . Here $f(t) := \operatorname{Re} y(t)$ and $g(t) := \operatorname{Im} y(t)$. Differentiation and integration can be done component-wise:

$$\begin{aligned} y'(t) &= f'(t) + i g'(t) \\ \int y(t) dt &= \int f(t) dt + i \int g(t) dt. \end{aligned}$$

Example 7.4. Suppose that $y(t) = \frac{2 + 3i}{1 + it}$. Then

$$y(t) = \frac{2 + 3i}{1 + it} = \frac{2 + 3i}{1 + it} \cdot \frac{1 - it}{1 - it} = \frac{(2 + 3t) + i(3 - 2t)}{1 + t^2} = \underbrace{\left(\frac{2 + 3t}{1 + t^2} \right)}_{f(t)} + i \underbrace{\left(\frac{3 - 2t}{1 + t^2} \right)}_{g(t)}.$$

The functions in parentheses labelled $f(t)$ and $g(t)$ are real-valued, so these are the real and imaginary parts of the function $y(t)$. \square

7.6. The complex exponential function. Derivatives and DEs make sense for complex-valued functions of a complex variable z , and work in a similar way. In particular, the existence and uniqueness theorem shows that there is a unique such function $f(z)$ satisfying

$$f'(z) = f(z), \quad f(0) = 1.$$

This function is called the **complex exponential function** e^z .

The number e is *defined* as the value of e^z at $z = 1$. But it is the function e^z , not the number e , that is truly important. Defining e without defining e^z first is a little unnatural. And even if e were defined first, one could not use it to define e^z , because “ e raised to a complex number” has no *a priori* meaning.

Theorem 7.5. *The complex exponential function e^z has the following properties:*

- (a) *The derivative of e^z is e^z .*
- (b) $e^0 = 1$.
- (c) $e^{z+w} = e^z e^w$ for all complex numbers z and w .
- (d) $(e^z)^n = e^{nz}$ for every complex number z and integer n . The $n = -1$ case says $\frac{1}{e^z} = (e^z)^{-1} = e^{-z}$.
- (e) **Euler's identity:**

$$\boxed{e^{it} = \cos t + i \sin t} \quad \text{for every real number } t.$$

- (f) *More generally,*

$$e^{x+yi} = e^x(\cos y + i \sin y) \quad \text{for all real numbers } x \text{ and } y. \quad (1)$$

- (g) $e^{-it} = \overline{e^{it}} = \cos t - i \sin t$ for every real number t .

- (h) $|e^{it}| = 1$ for every real number t .

Of lesser importance is the power series representation

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (2)$$

This formula can be deduced by using Taylor's theorem with remainder, or by showing that the right hand side satisfies the DE and initial condition. Some books use (1) or (2) as the *definition* of the complex exponential function, but the DE definition we gave is less contrived and focuses on what makes the function useful.

Proof of Theorem 7.5.

- (a) True by definition.
- (b) True by definition.
- (c) As a warm-up, consider the special case in which $w = 3$. By the chain rule, e^{z+3} is the solution to the DE with initial condition

$$f'(z) = f(z), \quad f(0) = e^3.$$

The function $e^z e^3$ satisfies the *same* DE with initial condition. By uniqueness, the two functions are the same: $e^{z+3} = e^z e^3$. The same argument works for any other complex constant w in place of 3, so $e^{z+w} = e^z e^w$.

(d) If $n = 0$, then this is $1 = 1$ by definition. If $n > 0$,

$$(e^z)^n = \overbrace{e^z e^z \cdots e^z}^{n \text{ copies}} \stackrel{\text{(c) repeatedly}}{=} \overbrace{e^{z+z+\cdots+z}}^{n \text{ copies}} = e^{nz}.$$

If $n = -m < 0$, then

$$(e^z)^{-m} = \frac{1}{(e^z)^m} \stackrel{\text{(just shown)}}{=} \frac{1}{e^{mz}} = e^{-mz}$$

since $e^{mz} e^{-mz} = e^{mz+(-mz)} = e^0 = 1$.

(e) The calculation

$$\begin{aligned} \frac{d}{dt} (\cos t + i \sin t) &= -\sin t + i \cos t \\ &= i(\cos t + i \sin t) \end{aligned}$$

shows that the function $\cos t + i \sin t$ is the solution to the DE with initial condition

$$f'(t) = if(t), \quad f(0) = 1.$$

But e^{it} is a solution too, by the chain rule. By uniqueness, the two functions are the same (the existence and uniqueness theorem applies also to complex-valued functions of a real variable t).

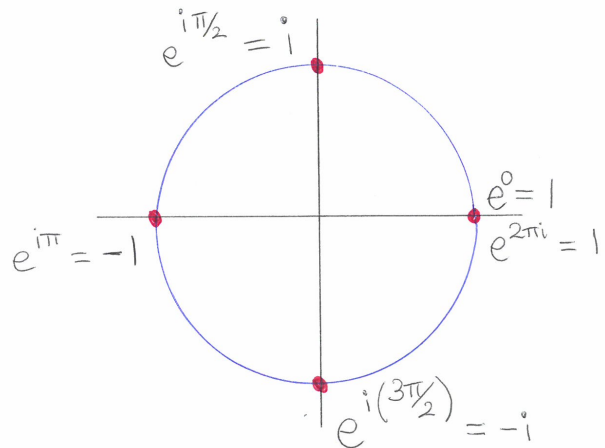
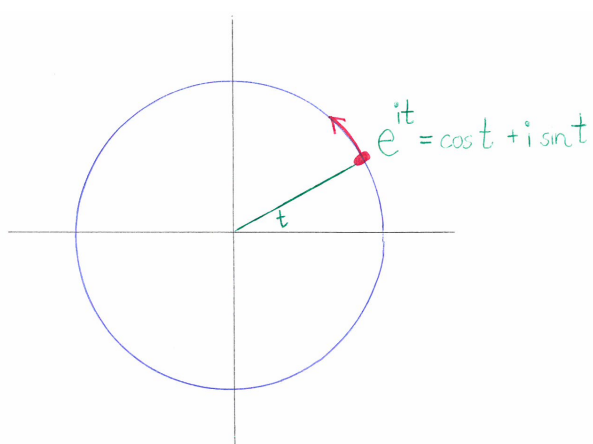
(f) By (c) and (e), $e^{x+yi} = e^x e^{iy} = e^x (\cos y + i \sin y)$.

(g) Changing every i in the universe to $-i$ transforms $e^{it} = \cos t + i \sin t$ into $e^{-it} = \cos t - i \sin t$. (Substituting $-t$ for t would do it too.) On the other hand, applying complex conjugation to both sides of $e^{it} = \cos t + i \sin t$ gives $\overline{e^{it}} = \cos t - i \sin t$.

(h) By (e), $|e^{it}| = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1$. □

Remark 7.6. Some older books use the awful abbreviation $\text{cist } t := \cos t + i \sin t$, but this belongs in a cispool [*sic*], since e^{it} is a more useful expression for the same thing.

As t increases, the complex number $e^{it} = \cos t + i \sin t$ travels counterclockwise around the unit circle.



7.7. Polar form of a complex number. Given a nonzero complex number $z = x + yi$, we can express the point (x, y) in polar coordinates r and θ :

$$x = r \cos \theta, \quad y = r \sin \theta.$$

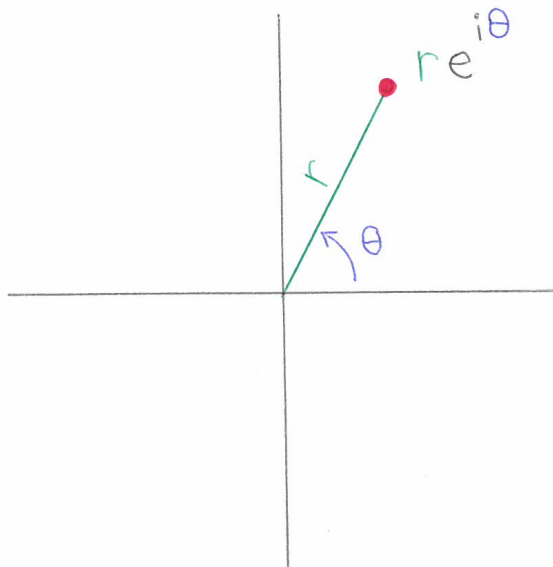
Then

$$x + yi = (r \cos \theta) + (r \sin \theta)i = r(\cos \theta + i \sin \theta).$$

In other words,

$$\boxed{z = re^{i\theta}}.$$

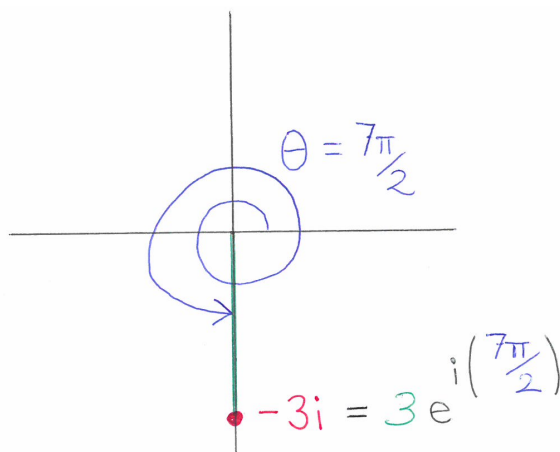
Here $re^{i\theta}$ is called a **polar form** of the complex number z . One has $r = |z|$; here r must be a *positive* real number (assuming $z \neq 0$).



Any possible θ for z (a possible value for the **angle** or **argument** of z) may be called **arg z** , but this is dangerously ambiguous notation since there are many values of θ for the same z : this means that **arg z** is not a function.

Example 7.7. Suppose that $z = -3i$. So z corresponds to the point $(0, -3)$. Then $r = |z| = 3$, but there are infinitely many possibilities for the angle θ . One possibility is $-\pi/2$; all the others are obtained by adding integer multiples of 2π :

$$\arg z = \dots, -5\pi/2, -\pi/2, 3\pi/2, 7\pi/2, \dots$$



So z has many polar forms:

$$\dots = 3e^{i(-5\pi/2)} = 3e^{-i\pi/2} = 3e^{i(3\pi/2)} = 3e^{i(7\pi/2)} = \dots \quad \square$$

To specify a *unique* polar form, we would have to restrict the range for θ to some interval of width 2π . The most common choice is to require $-\pi < \theta \leq \pi$. This special θ is called the **principal value of the argument**, and is denoted in various ways:

$$\theta = \text{Arg } z = \underset{\text{Mathematica}}{\text{Arg}[z]} = \underset{\text{Mathematica}}{\text{ArcTan}[x, y]} = \underset{\text{MATLAB}}{\text{atan2}(y, x)} .$$

Warning: The supplementary notes require $0 \leq \theta < 2\pi$ instead. **Warning:** In MATLAB, be careful to use (y, x) and not (x, y) . **Warning:** Although the principal value θ satisfies the “slope formula” $\tan \theta = y/x$ whenever $x \neq 0$, the formula $\theta = \tan^{-1}(y/x)$ is true only half the time. The problem is that there are *two* angles in the range $(-\pi, \pi]$ with the same value of \tan (there are two values of θ for each line through the origin, differing by π), and \tan^{-1} returns the one in the range $(-\pi/2, \pi/2)$. For example, $\tan^{-1}(y/x)$ evaluated at $(1, 1)$ and $(-1, -1)$ produces the same value $\pi/4$, but $(-1, -1)$ is actually at angle $-3\pi/4$. The “2-variable arctangent function” used above fixes this.

Test for equality of two nonzero complex numbers in polar form:

$$r_1 e^{i\theta_1} = r_2 e^{i\theta_2} \iff r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2\pi k \text{ for some integer } k.$$

(This assumes that r_1 and r_2 are positive real numbers, and that θ_1 and θ_2 are real numbers, as you would expect for polar coordinates.)

7.8. **Operations in polar form.** Some arithmetic operations on complex numbers are easy in polar form:

multiplication: $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ (multiply absolute values, add angles)

reciprocal: $\frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta}$

division: $\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$ (divide absolute values, subtract angles)

n^{th} power: $(r e^{i\theta})^n = r^n e^{in\theta}$ for any integer n

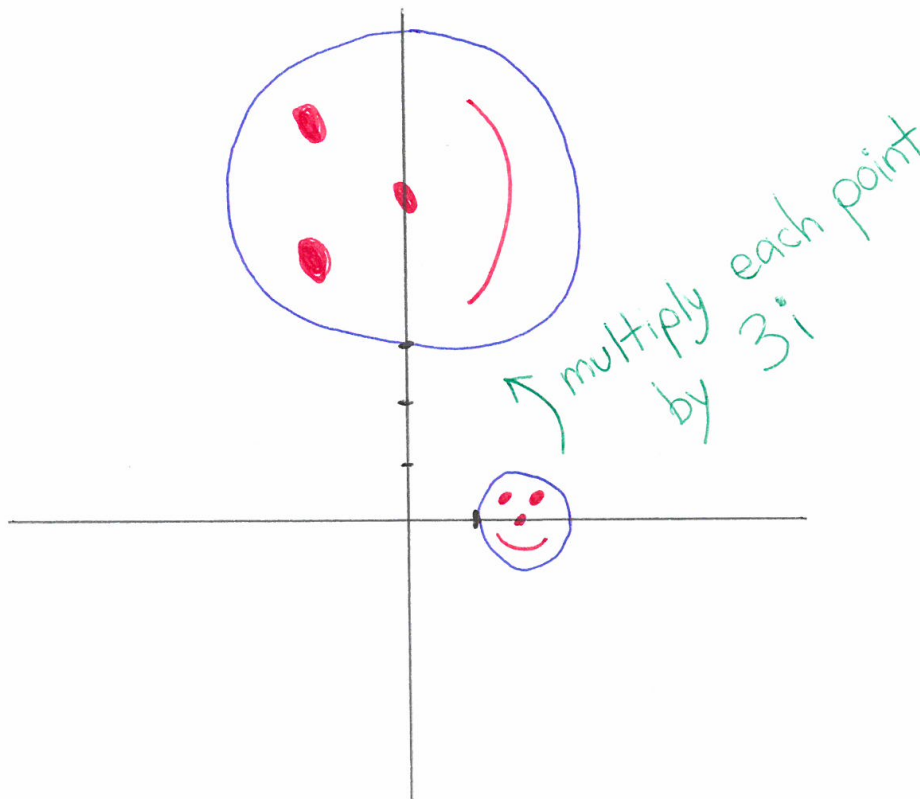
complex conjugation: $\overline{r e^{i\theta}} = r e^{-i\theta}$.

Taking absolute values gives identities:

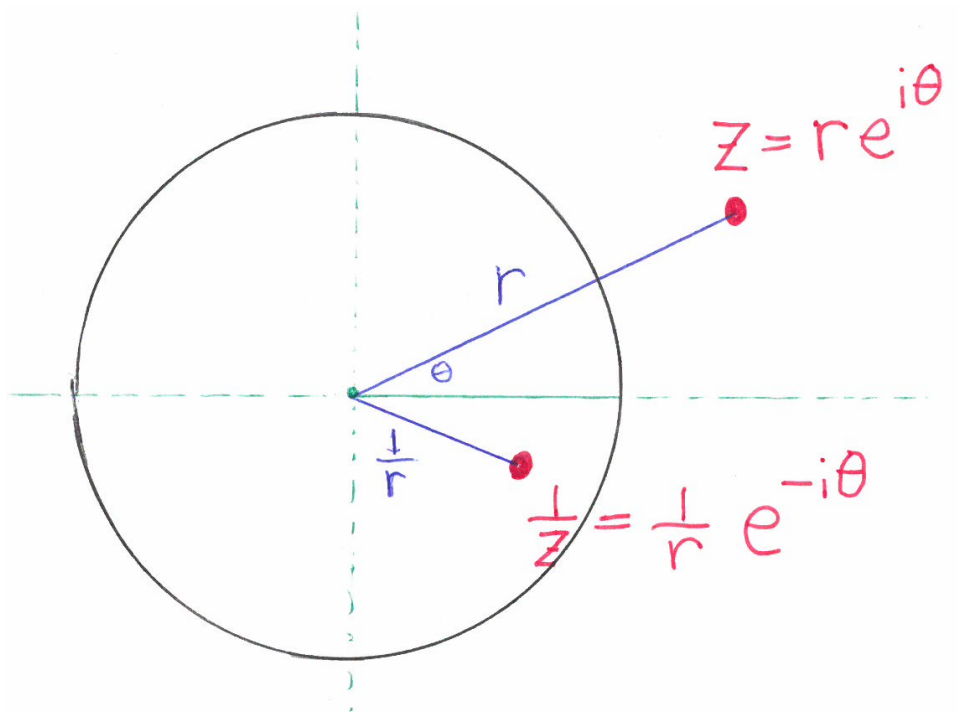
$$|z_1 z_2| = |z_1| |z_2|, \quad \left| \frac{1}{z} \right| = \frac{1}{|z|}, \quad \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad |z^n| = |z|^n, \quad |\bar{z}| = |z|.$$

Question 7.8. What happens if you take a smiley in the complex plane and multiply each of its points by $3i$?

Solution: Since $i = e^{i\pi/2}$, multiplying by i adds $\pi/2$ to the angle of each point; that is, it rotates counterclockwise by 90° (around the origin). Next, multiplying by 3 does what you would expect: dilate by a factor of 3. Doing both leads to...



For example, the nose was originally on the real line, a little less than 2, so multiplying it by $3i$ produces a big nose close to $(3i)2 = 6i$. \square



Question 7.9. How do you trap a lion?

Answer: Build a cage in the shape of the unit circle $|z| = 1$. Get inside the cage. Make sure that the lion is outside the cage. Apply the function $1/z$ to the whole plane. Voilà! The lion is now inside the cage, and you are outside it. (Only problem: There's a lot of other stuff inside the cage too. Also, don't stand too close to $z = 0$ when you apply $1/z$.)

Question 7.10. Why not always write complex numbers in polar form?

Answer: Because addition and subtraction are difficult in polar form!

7.9. The function $e^{(a+bi)t}$. Fix a nonzero complex number $a + bi$. As the real number t increases, the complex number $(a + bi)t$ moves along a line through 0, and $e^{(a+bi)t}$ moves along part of a line, a circle, or a spiral, depending on the value of $a + bi$. Try the “Complex Exponential” mathlet

<http://mathlets.org/mathlets/complex-exponential/>

to see this.

Example 7.11. Consider $e^{(-5-2i)t} = e^{-5t}e^{i(-2t)}$ as $t \rightarrow \infty$. Its absolute value is e^{-5t} , which tends to 0, so the point is moving inward. Its angle is $-2t$, which is decreasing, so the point is moving clockwise. It's spiraling inwards clockwise.

7.10. Finding n^{th} roots.

7.10.1. *An example.*

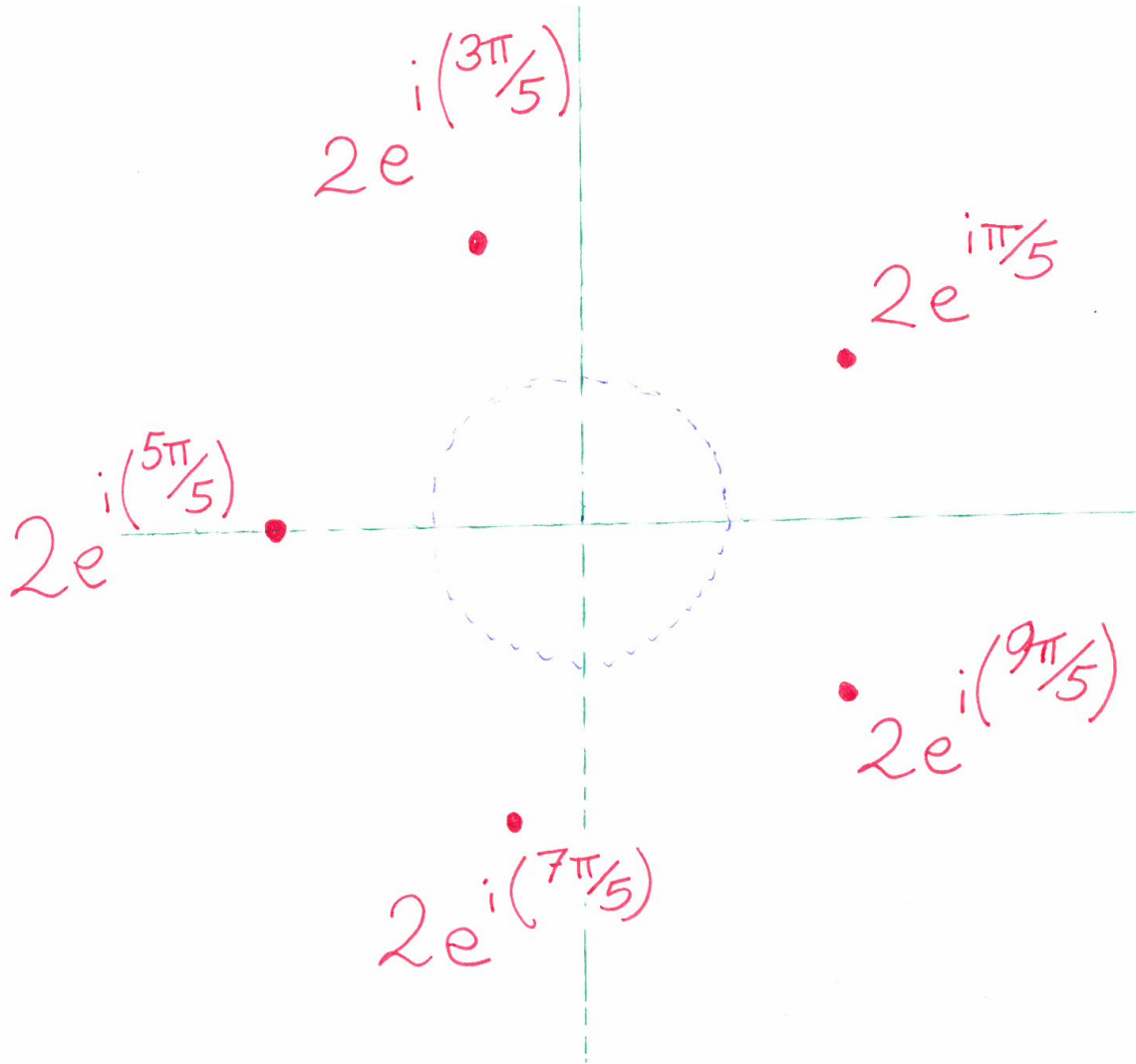
Problem 7.12. What are the complex solutions to $z^5 = -32$?

Solution: Rewrite the equation in polar form, using $z = re^{i\theta}$:

$$\begin{aligned}(re^{i\theta})^5 &= 32e^{i\pi} \\ r^5 e^{i(5\theta)} &= 32e^{i\pi} \\ r^5 &= 32 & 5\theta &= \pi + 2\pi k \text{ for some integer } k \\ \text{absolute values} & & \text{angles} & \\ r = 2 & \quad \text{and} \quad \theta = \frac{\pi}{5} + \frac{2\pi k}{5} \text{ for some integer } k \\ z &= 2e^{i\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right)} \text{ for some integer } k.\end{aligned}$$

These are numbers on a circle of radius 2; to get from one to the next (increasing k by 1), rotate by $2\pi/5$. Increasing k five times brings the number back to its original position. So it's enough to take $k = 0, 1, 2, 3, 4$. Answer:

$$2e^{i(\pi/5)}, 2e^{i(3\pi/5)}, 2e^{i(5\pi/5)}, 2e^{i(7\pi/5)}, 2e^{i(9\pi/5)}. \quad \square$$



Remark 7.13. The fundamental theorem of algebra predicts that the polynomial $z^5 + 32$ has 5 roots when counted with multiplicity. We found 5 roots, so each must have multiplicity 1.

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7.10.2. Roots of unity.

The same method shows that the n^{th} roots of unity (the solutions to $z^n = 1$) are the numbers $e^{i(\frac{2\pi k}{n})}$ for $k = 0, 1, 2, \dots, n - 1$. Taking $k = 1$ gives the number $\zeta := e^{2\pi i/n}$. In terms of ζ , the complete list of n^{th} roots of unity is

$$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$$

(after that they start to repeat: $\zeta^n = 1$).

7.10.3. *Finding n^{th} roots in general.*

Problem 7.14. Given $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, what are the solutions to $z^n = \alpha$?

Write α as $re^{i\theta}$. Then the solutions to $z^n = \alpha$ are $\sqrt[n]{r}e^{i(\frac{\theta}{n} + \frac{2\pi}{n}k)}$ for $k = 0, 1, 2, \dots, n-1$.

Another way to list the solutions: If z_0 is any particular solution, such as $\sqrt[n]{r}e^{i\theta/n}$, then the complete list of solutions is

$$z_0, \zeta z_0, \zeta^2 z_0, \dots, \zeta^{n-1} z_0. \quad \square$$

Try the “Complex Roots” mathlet

<http://mathlets.org/mathlets/complex-roots/>