1^{st} ORDER O.D.E. EXAM QUESTIONS

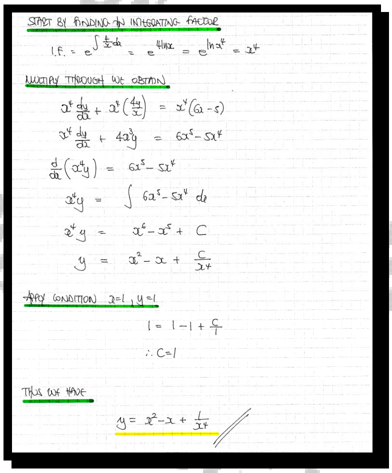
Question 1 **(**)**

$$\frac{dy}{dx}+\frac{4y}{x}=6x-5, \quad x>0.$$

Determine the solution of the above differential equation subject to the boundary condition is $y=1$ at $x=1$.

Give the answer in the form $y=f(x)$.

$\boxed{}$, $\boxed{y=x^2-x+\dfrac{1}{x^4}}$



START BY FINDING AN INTEGRATING FACTOR

$\text{I.F.} = e^{\int \frac{4}{x}dx} = e^{4\ln x} = e^{\ln x^4} = x^4$

MULTIPLY THROUGH WE OBTAIN

$x^4\frac{dy}{dx} + x^4\left(\frac{4y}{x}\right) = x^4(6x-5)$

$x^4\frac{dy}{dx} + 4x^3y = 6x^5-5x^4$

$\frac{d}{dx}\left(x^4y\right) = 6x^5-5x^4$

$x^4y = \int 6x^5-5x^4 \, dx$

$x^4y = x^6-x^5+C$

$y = x^2-x+\frac{C}{x^4}$

APPLY CONDITION $x=1, y=1$

$1 = 1-1+\frac{C}{1}$

$\therefore C=1$

THUS WE HAVE

$y = x^2-x+\frac{1}{x^4}$

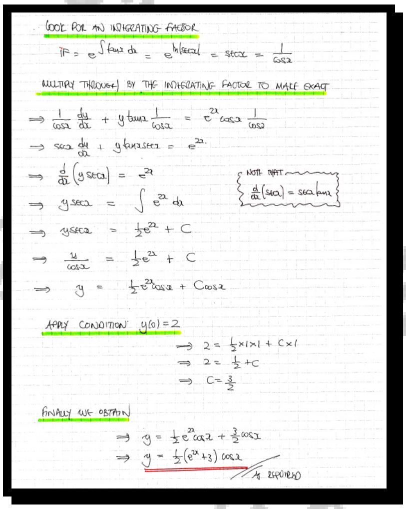
Question 2 (**+)

$$\frac{dy}{dx} + y \tan x = e^{2x} \cos x, \quad y(0) = 2.$$

Show that the solution of the above differential equation is

$$y = \frac{1}{2}\left(e^{2x} + 3\right)\cos x.$$

[ ] , [proof]


LOOK FOR AN INTEGRATING FACTOR

$\text{IF} = e^{\int \tan x \, dx} = e^{\ln|\sec x|} = \sec x = \frac{1}{\cos x}$

MULTIPLY THROUGH BY THE INTEGRATING FACTOR TO MAKE EXACT

$\Rightarrow \frac{1}{\cos x}\frac{dy}{dx} + y \tan x \frac{1}{\cos x} = e^{2x} \cos x \frac{1}{\cos x}$

$\Rightarrow \sec x \frac{dy}{dx} + y \tan x \sec x = e^{2x}$

$\Rightarrow \frac{d}{dx}(y \sec x) = e^{2x}$

NOTE THAT $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\Rightarrow y \sec x = \int e^{2x} \, dx$

$\Rightarrow y \sec x = \frac{1}{2}e^{2x} + C$

$\Rightarrow \frac{y}{\cos x} = \frac{1}{2}e^{2x} + C$

$\Rightarrow y = \frac{1}{2}e^{2x}\cos x + C\cos x$

APPLY CONDITION $y(0) = 2$

$\Rightarrow 2 = \frac{1}{2} \times 1 \times 1 + C \times 1$

$\Rightarrow 2 = \frac{1}{2} + C$

$\Rightarrow C = \frac{3}{2}$

FINALLY WE OBTAIN

$\Rightarrow y = \frac{1}{2}e^{2x}\cos x + \frac{3}{2}\cos x$

$\Rightarrow y = \frac{1}{2}(e^{2x} + 3)\cos x$ // AS REQUIRED

Question 3 (+)**

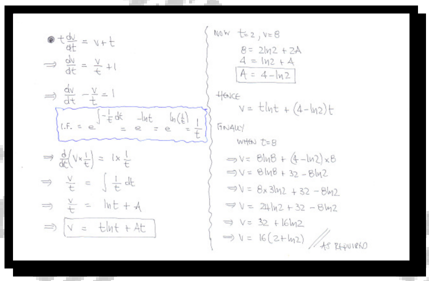
The velocity of a particle v ms^{-1} at time t s satisfies the differential equation

$$t\frac{dv}{dt} = v + t, \; t > 0.$$

Given that when $t = 2$, $v = 8$, show that when $t = 8$

$$v = 16(2 + \ln 2).$$

proof



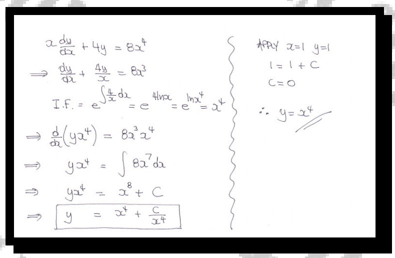
Question 4 (+)**

$$x\frac{dy}{dx} + 4y = 8x^4, \text{ subject to } y = 1 \text{ at } x = 1.$$

Show that the solution of the above differential equation is

$$y = x^4.$$

proof

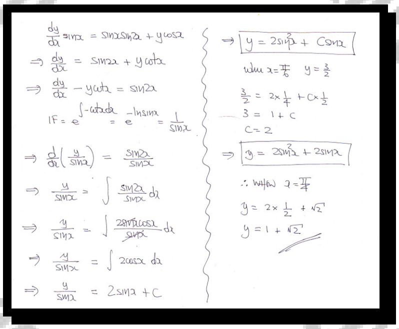


Question 5 (***)

$$\frac{dy}{dx}\sin x = \sin x \sin 2x + y\cos x .$$

Given that $y = \frac{3}{2}$ at $x = \frac{\pi}{6}$, find the exact value of y at $x = \frac{\pi}{4}$.

$\boxed{1+\sqrt{2}}$



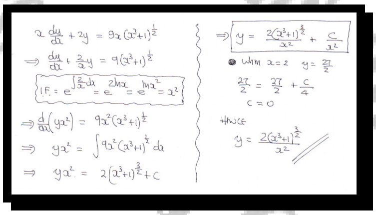
Question 6 (***)

$$x\frac{dy}{dx} + 2y = 9x\left(x^3+1\right)^{\frac{1}{2}}, \text{ with } y = \frac{27}{2} \text{ at } x = 2 .$$

Show that the solution of the above differential equation is

$$y = \frac{2}{x^2}\left(x^3+1\right)^{\frac{3}{2}} .$$

proof



Question 7 (***)

A trigonometric curve C satisfies the differential equation

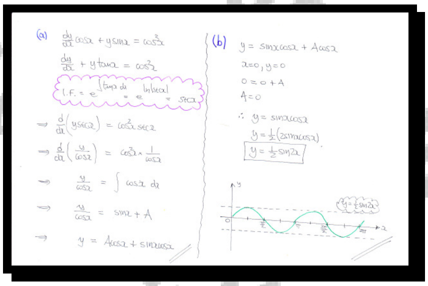
$$\frac{dy}{dx}\cos x + y\sin x = \cos^3 x .$$

a) Find a general solution of the above differential equation.

b) Given further that the curve passes through the Cartesian origin O, sketch the graph of C for $0 \le x \le 2\pi$.

The sketch must show clearly the coordinates of the points where the graph of C meets the x axis.

$$\boxed{y = \sin x\cos x + A\cos x}$$



Question 8 (***)

20 grams of salt are dissolved into a beaker containing 1 litre of a certain chemical.

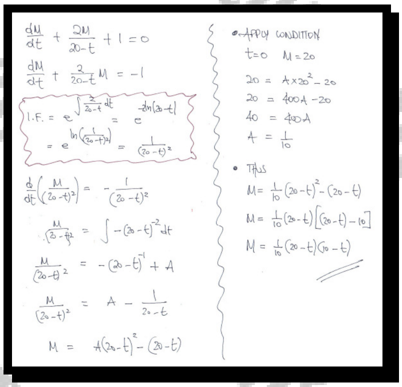
The mass of salt, M grams, which remains undissolved t seconds later, is modelled by the differential equation

$$\frac{dM}{dt}+\frac{2M}{20-t}+1=0, \ t \geq 0.$$

Show clearly that

$$M=\frac{1}{10}(10-t)(20-t).$$

proof



$$\frac{dM}{dt}+\frac{2M}{20-t}+1=0$$

$$\frac{dM}{dt}+\frac{2}{20-t}M=-1$$

$$\text{I.F.}=e^{\int \frac{2}{20-t}\,dt}=e^{-2\ln|20-t|}=e^{\ln\left(\frac{1}{(20-t)^2}\right)}=\frac{1}{(20-t)^2}$$

$$\frac{d}{dt}\left(\frac{M}{(20-t)^2}\right)=-\frac{1}{(20-t)^2}$$

$$\frac{M}{(20-t)^2}=\int -(20-t)^{-2}\,dt$$

$$\frac{M}{(20-t)^2}=-(20-t)^{-1}+A$$

$$\frac{M}{(20-t)^2}=A-\frac{1}{20-t}$$

$$M=A(20-t)^2-(20-t)$$

- APPLY CONDITION

$t=0 \quad M=20$

$$20=A\times 20^2-20$$

$$20=400A-20$$

$$40=400A$$

$$A=\frac{1}{10}$$

- THUS

$$M=\frac{1}{10}(20-t)^2-(20-t)$$

$$M=\frac{1}{10}(20-t)\left[(20-t)-10\right]$$

$$M=\frac{1}{10}(20-t)(10-t)$$

Question 9 (***+)

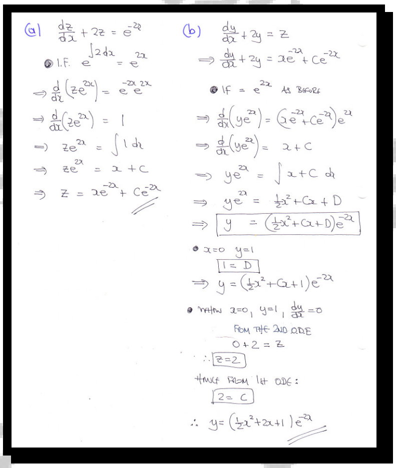
Given that $z = f(x)$ and $y = g(x)$ satisfy the following differential equations

$$\frac{dz}{dx} + 2z = e^{-2x} \quad \text{and} \quad \frac{dy}{dx} + 2y = z,$$

a) Find z in the form $z = f(x)$

b) Express y in the form $y = g(x)$, given further that at $x = 0$, $y = 1$, $\frac{dy}{dx} = 0$

$\boxed{z = (x + C)e^{-2x}}$, $\boxed{y = \left(\frac{1}{2}x^2 + 2x + 1\right)e^{-2x}}$


(a) $\frac{dz}{dx} + 2z = e^{-2x}$

- I.F. $e^{\int 2\,dx} = e^{2x}$

$\Rightarrow \frac{d}{dx}\left(ze^{2x}\right) = e^{-2x}e^{2x}$

$\Rightarrow \frac{d}{dx}\left(ze^{2x}\right) = 1$

$\Rightarrow ze^{2x} = \int 1\,dx$

$\Rightarrow ze^{2x} = x + C$

$\Rightarrow z = xe^{-2x} + Ce^{-2x}$

(b) $\frac{dy}{dx} + 2y = z$

$\Rightarrow \frac{dy}{dx} + 2y = xe^{-2x} + Ce^{-2x}$

- IF $= e^{2x}$ AS BEFORE

$\Rightarrow \frac{d}{dx}\left(ye^{2x}\right) = \left(xe^{-2x} + Ce^{-2x}\right)e^{2x}$

$\Rightarrow \frac{d}{dx}\left(ye^{2x}\right) = x + C$

$\Rightarrow ye^{2x} = \int x + C\,dx$

$\Rightarrow ye^{2x} = \frac{1}{2}x^2 + Cx + D$

$\Rightarrow \boxed{y = \left(\frac{1}{2}x^2 + Cx + D\right)e^{-2x}}$

- $x = 0$ $y = 1$

$\boxed{1 = D}$

$\Rightarrow y = \left(\frac{1}{2}x^2 + Cx + 1\right)e^{-2x}$

- WHEN $x = 0$, $y = 1$, $\frac{dy}{dx} = 0$

FROM THE 2ND ODE

$0 + 2 = z$

$\therefore \boxed{z = 2}$

HENCE FROM 1ST ODE:

$\boxed{2 = C}$

$\therefore y = \left(\frac{1}{2}x^2 + 2x + 1\right)e^{-2x}$

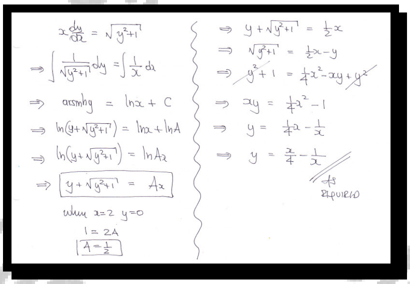
Question 10 (*+)**

$$x\frac{dy}{dx}=\sqrt{y^2+1}\ ,\ x>0\text{, with }\ y=0\ \text{ at }\ x=2\,.$$

Show that the solution of the above differential equation is

$$y=\frac{x}{4}-\frac{1}{x}.$$

proof



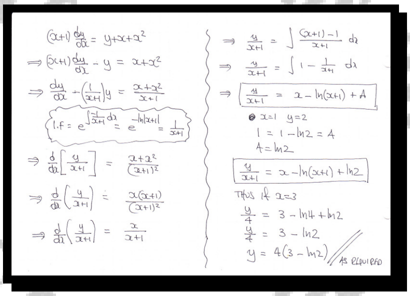
Question 11 (*+)**

$$(x+1)\frac{dy}{dx}=y+x+x^2,\ x>-1\,.$$

Given that $y=2$ at $x=1$, solve the above differential equation to show that

$$y=4(3-\ln 2)\ \text{ at }\ x=3\,.$$

proof

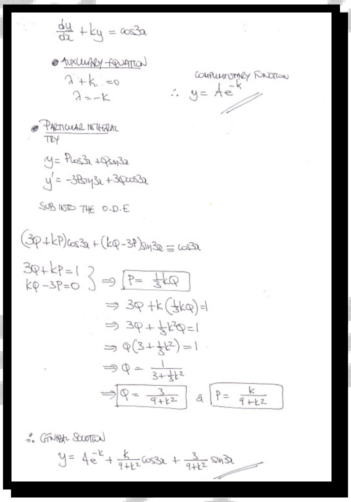


Question 12 (***+)

$$\frac{dy}{dx} + ky = \cos 3x \,,\ k \text{ is a non zero constant.}$$

By finding a complimentary function and a particular integral, or otherwise, find the general of the above differential equation.

$$\boxed{y = A\mathrm{e}^{-x} + \frac{k}{9+k^2}\cos 3x + \frac{3}{9+k^2}\sin 3x}$$


$\frac{dy}{dx} + ky = \cos 3x$

• AUXILIARY EQUATION

$\lambda + k = 0$

$\lambda = -k$

COMPLIMENTARY FUNCTION

$\therefore y = Ae^{-k}$

• PARTICULAR INTEGRAL

TRY

$y = P\cos 3x + Q\sin 3x$

$y' = -3P\sin 3x + 3Q\cos 3x$

SUB INTO THE O.D.E

$(3Q + kP)\cos 3x + (kQ - 3P)\sin 3x \equiv \cos 3x$

$\left.\begin{array}{l} 3Q + kP = 1 \\ kQ - 3P = 0 \end{array}\right\} \Rightarrow \boxed{P = \frac{1}{3}kQ}$

$\Rightarrow 3Q + k\left(\frac{1}{3}kQ\right) = 1$

$\Rightarrow 3Q + \frac{1}{3}k^2Q = 1$

$\Rightarrow Q\left(3 + \frac{1}{3}k^2\right) = 1$

$\Rightarrow Q = \frac{1}{3 + \frac{1}{3}k^2}$

$\Rightarrow \boxed{Q = \frac{3}{9+k^2}}$ & $\boxed{P = \frac{k}{9+k^2}}$

$\therefore$ GENERAL SOLUTION

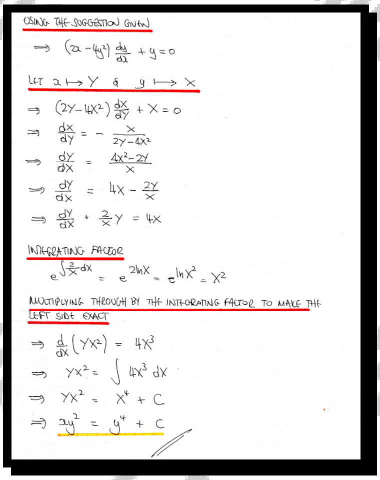
$y = Ae^{-k} + \frac{k}{9+k^2}\cos 3x + \frac{3}{9+k^2}\sin 3x$

Question 13 (***+)

$$\left(2x-4y^2\right)\frac{dy}{dx}+y=0.$$

By reversing the role of x and y in the above differential equation, or otherwise, find its general solution.

$\boxed{}$, $\boxed{xy^2 = y^4 + C}$


USING THE SUGGESTION GIVEN

$\Rightarrow (2x-4y^2)\frac{dy}{dx}+y=0$

LET $x \mapsto Y$ & $y \mapsto X$

$\Rightarrow (2Y-4X^2)\frac{dX}{dY}+X=0$

$\Rightarrow \frac{dX}{dY} = -\frac{X}{2Y-4X^2}$

$\Rightarrow \frac{dY}{dX} = \frac{4X^2-2Y}{X}$

$\Rightarrow \frac{dY}{dX} = 4X - \frac{2Y}{X}$

$\Rightarrow \frac{dY}{dX} + \frac{2}{X}Y = 4X$

INTEGRATING FACTOR

$e^{\int \frac{2}{X}dX} = e^{2\ln X} = e^{\ln X^2} = X^2$

MULTIPLYING THROUGH BY THE INTEGRATING FACTOR TO MAKE THE LEFT SIDE EXACT

$\Rightarrow \frac{d}{dX}(YX^2) = 4X^3$

$\Rightarrow YX^2 = \int 4X^3\,dX$

$\Rightarrow YX^2 = X^4 + C$

$\Rightarrow xy^2 = y^4 + C$

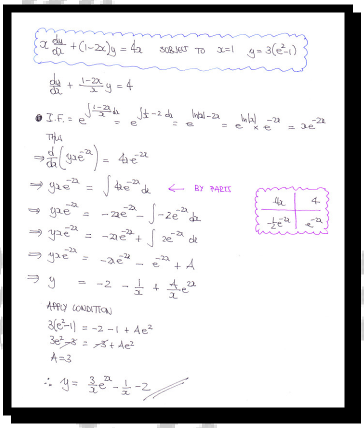
Question 14 **(****)**

The curve with equation $y = f(x)$ satisfies

$$x\frac{dy}{dx} + (1-2x)y = 4x, \quad x > 0, \quad f(1) = 3\left(e^2 - 1\right).$$

Determine an equation for $y = f(x)$.

$$\boxed{y = \frac{3}{x}e^x - \frac{1}{x} - 2}$$



$x\frac{dy}{dx} + (1-2x)y = 4x$ SUBJECT TO $x=1$ $y = 3(e^2-1)$

$\frac{dy}{dx} + \frac{1-2x}{x}y = 4$

- I.F. $= e^{\int \frac{1-2x}{x}dx} = e^{\int \frac{1}{x} - 2\,dx} = e^{\ln|x| - 2x} = e^{\ln|x|} \times e^{-2x} = xe^{-2x}$

THUS

$\Rightarrow \frac{d}{dx}\left(yxe^{-2x}\right) = 4xe^{-2x}$

$\Rightarrow yxe^{-2x} = \int 4xe^{-2x}\,dx$ ← BY PARTS

$4x$	4
$-\frac{1}{2}e^{-2x}$	e^{-2x}

$\Rightarrow yxe^{-2x} = -2xe^{-2x} - \int -2e^{-2x}\,dx$

$\Rightarrow yxe^{-2x} = -2xe^{-2x} + \int 2e^{-2x}\,dx$

$\Rightarrow yxe^{-2x} = -2xe^{-2x} - e^{-2x} + A$

$\Rightarrow y = -2 - \frac{1}{x} + \frac{A}{x}e^{2x}$

APPLY CONDITION

$3(e^2-1) = -2 - 1 + Ae^2$

$3e^2 - 3 = -3 + Ae^2$

$A = 3$

$\therefore y = \frac{3}{x}e^{2x} - \frac{1}{x} - 2$

Question 15 (****)

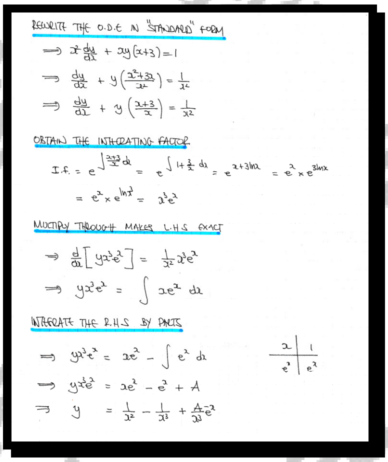
A curve C, with equation $y = f(x)$, passes through the points with coordinates $(1,1)$ and $(2,k)$, where k is a constant.

Given further that the equation of C satisfies the differential equation

$$x^2 \frac{dy}{dx} + xy(x+3) = 1,$$

determine the exact value of k.

$\boxed{}$, $\boxed{k = \dfrac{e+1}{8e}}$


REWRITE THE O.D.E IN "STANDARD" FORM

$\Rightarrow x^2 \frac{dy}{dx} + xy(x+3) = 1$

$\Rightarrow \frac{dy}{dx} + y\left(\frac{x^2+3x}{x^2}\right) = \frac{1}{x^2}$

$\Rightarrow \frac{dy}{dx} + y\left(\frac{x+3}{x}\right) = \frac{1}{x^2}$

OBTAIN THE INTEGRATING FACTOR

$\text{I.F.} = e^{\int \frac{x+3}{x} dx} = e^{\int 1 + \frac{3}{x} dx} = e^{x + 3\ln x} = e^x \times e^{3\ln x}$

$= e^x \times e^{\ln x^3} = x^3 e^x$

MULTIPLY THROUGH MAKES L.H.S EXACT

$\Rightarrow \frac{d}{dx}\left[y x^3 e^x\right] = \frac{1}{x^2} x^3 e^x$

$\Rightarrow y x^3 e^x = \int x e^x \, dx$

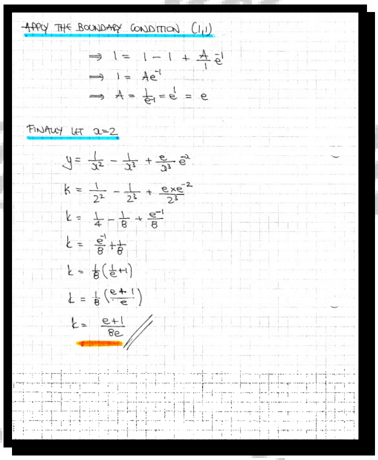
INTEGRATE THE R.H.S BY PARTS

$\Rightarrow y x^3 e^x = x e^x - \int e^x \, dx$

$\Rightarrow y x^3 e^x = x e^x - e^x + A$

$\Rightarrow y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{A}{x^3} e^{-x}$

x	1
e^x	e^x


APPLY THE BOUNDARY CONDITION $(1,1)$

$\Rightarrow 1 = 1 - 1 + \frac{A}{1} e^{-1}$

$\Rightarrow 1 = A e^{-1}$

$\Rightarrow A = \frac{1}{e^{-1}} = e^{1} = e$

FINALLY LET $x = 2$

$y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{e}{x^3} e^{-x}$

$k = \frac{1}{2^2} - \frac{1}{2^3} + \frac{e \times e^{-2}}{2^3}$

$k = \frac{1}{4} - \frac{1}{8} + \frac{e^{-1}}{8}$

$k = \frac{e^{-1}}{8} + \frac{1}{8}$

$k = \frac{1}{8}\left(\frac{1}{e} + 1\right)$

$k = \frac{1}{8}\left(\frac{e+1}{e}\right)$

$k = \frac{e+1}{8e}$

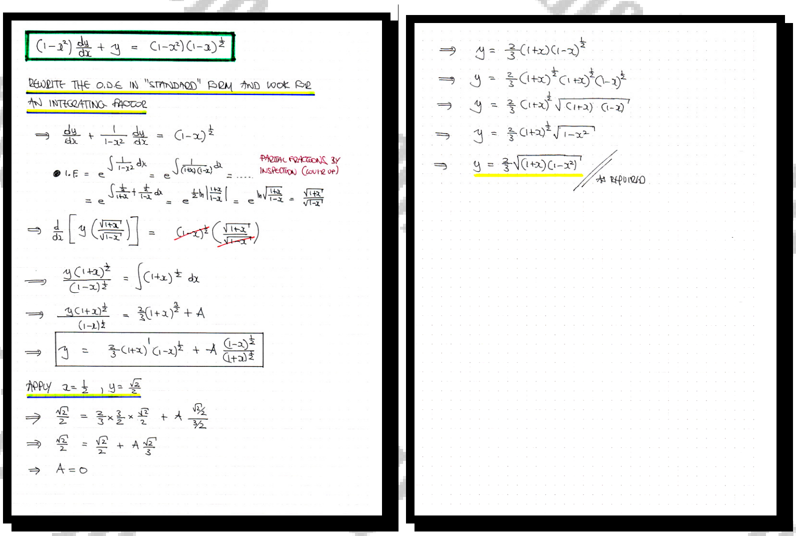
Question 16 (****)

$$\left(1-x^2\right)\frac{dy}{dx}+y=\left(1-x^2\right)\left(1-x\right)^{\frac{1}{2}}\ ,\ -1<x<1.$$

Given that $y=\frac{\sqrt{2}}{2}$ at $x=\frac{1}{2}$, show that the solution of the above differential equation can be written as

$$y=\frac{2}{3}\sqrt{\left(1-x^2\right)\left(1+x\right)}.$$

[ ], [proof]



$$\left(1-x^2\right)\frac{dy}{dx}+y=\left(1-x^2\right)\left(1-x\right)^{\frac{1}{2}}$$

REWRITE THE O.D.E IN "STANDARD" FORM AND LOOK FOR AN INTEGRATING FACTOR

$$\Rightarrow \frac{dy}{dx}+\frac{1}{1-x^2}\frac{dy}{dx}=(1-x)^{\frac{1}{2}}$$

- I.F $= e^{\int \frac{1}{1-x^2}dx} = e^{\int \frac{1}{(1+x)(1-x)}dx} = \dots$ PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$= e^{\int \frac{\frac{1}{2}}{1+x}+\frac{\frac{1}{2}}{1-x}dx} = e^{\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|} = e^{\ln\sqrt{\frac{1+x}{1-x}}} = \frac{\sqrt{1+x}}{\sqrt{1-x}}$$

$$\Rightarrow \frac{d}{dx}\left[y\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right)\right] = (1-x)^{\frac{1}{2}}\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right)$$

$$\Rightarrow \frac{y(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = \int (1+x)^{\frac{1}{2}}\,dx$$

$$\Rightarrow \frac{y(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = \frac{2}{3}(1+x)^{\frac{3}{2}}+A$$

$$\Rightarrow y = \frac{2}{3}(1+x)^{1}(1-x)^{\frac{1}{2}} + A\frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}$$

APPLY $x=\frac{1}{2}$, $y=\frac{\sqrt{2}}{2}$

$$\Rightarrow \frac{\sqrt{2}}{2} = \frac{2}{3}\times\frac{3}{2}\times\frac{\sqrt{2}}{2} + A\frac{\sqrt{2}/2}{3/2}$$

$$\Rightarrow \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + A\frac{\sqrt{2}}{3}$$

$$\Rightarrow A=0$$

$$\Rightarrow y=\frac{2}{3}(1+x)(1-x)^{\frac{1}{2}}$$

$$\Rightarrow y=\frac{2}{3}(1+x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$$

$$\Rightarrow y=\frac{2}{3}(1+x)^{\frac{1}{2}}\sqrt{(1+x)(1-x)}$$

$$\Rightarrow y=\frac{2}{3}(1+x)^{\frac{1}{2}}\sqrt{1-x^2}$$

$$\Rightarrow y=\frac{2}{3}\sqrt{(1+x)(1-x^2)}$$

// AS REQUIRED

Question 17 (****)

A curve C, with equation $y = f(x)$, meets the y axis the point with coordinates $(0,1)$.

It is further given that the equation of C satisfies the differential equation

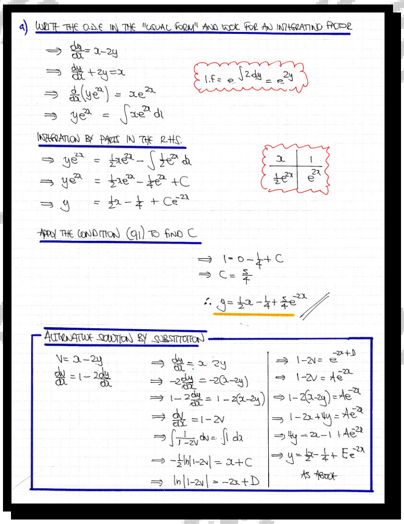
$$\frac{dy}{dx} = x - 2y.$$

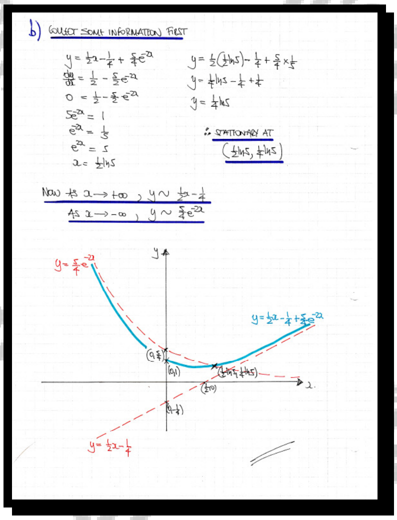
a) Determine an equation of C.

b) Sketch the graph of C.

The graph must include in exact simplified form the coordinates of the stationary point of the curve and the equation of its asymptote.

$\boxed{}$, $\boxed{y = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x}}$





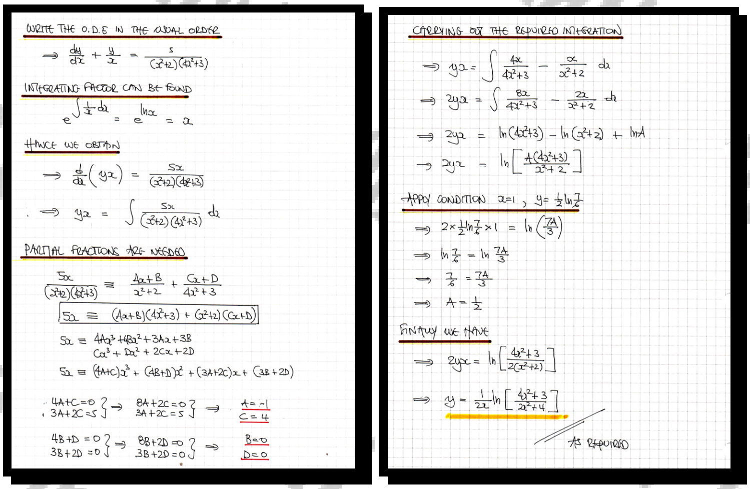
Question 18 (****)

$$\frac{dy}{dx}+\frac{y}{x}=\frac{5}{\left(x^2+2\right)\left(4x^2+3\right)},\quad x>0.$$

Given that $y=\frac{1}{2}\ln\frac{7}{6}$ at $x=1$, show that the solution of the above differential equation can be written as

$$y=\frac{1}{2x}\ln\left(\frac{4x^2+3}{2x^2+4}\right).$$

[   ], [proof]



WRITE THE O.D.E IN THE USUAL ORDER

$\Rightarrow \frac{dy}{dx}+\frac{y}{x}=\frac{5}{(x^2+2)(4x^2+3)}$

INTEGRATING FACTOR CAN BE FOUND

$e^{\int \frac{1}{x}\,dx}=e^{\ln x}=x$

HENCE WE OBTAIN

$\Rightarrow \frac{d}{dx}(yx)=\frac{5x}{(x^2+2)(4x^2+3)}$

$\Rightarrow yx=\int \frac{5x}{(x^2+2)(4x^2+3)}\,dx$

PARTIAL FRACTIONS ARE NEEDED

$\frac{5x}{(x^2+2)(4x^2+3)}\equiv\frac{Ax+B}{x^2+2}+\frac{Cx+D}{4x^2+3}$

$5x\equiv(Ax+B)(4x^2+3)+(x^2+2)(Cx+D)$

$5x\equiv 4Ax^3+4Bx^2+3Ax+3B$
$\quad\; Cx^3+Dx^2+2Cx+2D$

$5x\equiv(4A+C)x^3+(4B+D)x^2+(3A+2C)x+(3B+2D)$

$\left.\begin{array}{l}4A+C=0\\3A+2C=5\end{array}\right\}\Rightarrow\left.\begin{array}{l}8A+2C=0\\3A+2C=5\end{array}\right\}\Rightarrow \begin{array}{l}A=-1\\C=4\end{array}$

$\left.\begin{array}{l}4B+D=0\\3B+2D=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}8B+2D=0\\3B+2D=0\end{array}\right\}\Rightarrow \begin{array}{l}B=0\\D=0\end{array}$

CARRYING OUT THE REQUIRED INTEGRATION

$\Rightarrow yx=\int \frac{4x}{4x^2+3}-\frac{x}{x^2+2}\,dx$

$\Rightarrow 2yx=\int \frac{8x}{4x^2+3}-\frac{2x}{x^2+2}\,dx$

$\Rightarrow 2yx=\ln(4x^2+3)-\ln(x^2+2)+\ln A$

$\Rightarrow 2yx=\ln\left[\frac{A(4x^2+3)}{x^2+2}\right]$

APPLY CONDITION $x=1$, $y=\frac{1}{2}\ln\frac{7}{6}$

$\Rightarrow 2\times\frac{1}{2}\ln\frac{7}{6}\times 1=\ln\left(\frac{7A}{3}\right)$

$\Rightarrow \ln\frac{7}{6}=\ln\frac{7A}{3}$

$\Rightarrow \frac{7}{6}=\frac{7A}{3}$

$\Rightarrow A=\frac{1}{2}$

FINALLY WE HAVE

$\Rightarrow 2yx=\ln\left[\frac{4x^2+3}{2(x^2+2)}\right]$

$\Rightarrow y=\frac{1}{2x}\ln\left[\frac{4x^2+3}{2x^2+4}\right]$

AS REQUIRED

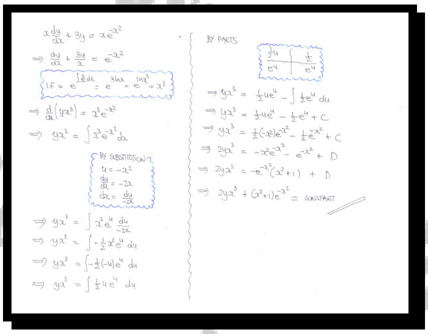
Question 19 (****)

$$x\frac{dy}{dx}+3y=x\mathrm{e}^{-x^2}, \quad x>0.$$

Show clearly that the general solution of the above differential equation can be written in the form

$$2yx^3+\left(x^2+1\right)\mathrm{e}^{-x^2}=\text{constant}.$$

proof



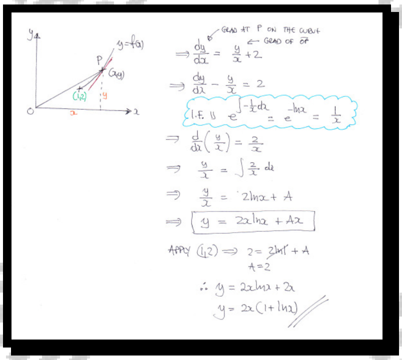
Question 20 (****)

The general point P lies on the curve with equation $y = f(x)$.

The gradient of the curve at P is 2 more than the gradient of the straight line segment OP.

Given further that the curve passes through $Q(1,2)$, express y in terms of x.

$$\boxed{y = 2x(1+\ln x)}$$



Question 21 (****+)

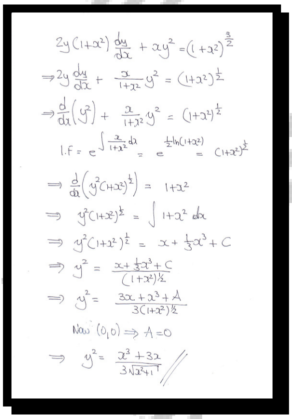
A curve with equation $y = f(x)$ passes through the origin and satisfies the differential equation

$$2y\left(1+x^2\right)\frac{dy}{dx} + xy^2 = \left(1+x^2\right)^{\frac{3}{2}}.$$

By finding a suitable integrating factor, or otherwise, show clearly that

$$y^2 = \frac{x^3+3x}{3\sqrt{x^2+1}}.$$

, proof


$2y(1+x^2)\frac{dy}{dx} + xy^2 = (1+x^2)^{\frac{3}{2}}$

$\Rightarrow 2y\frac{dy}{dx} + \frac{x}{1+x^2}y^2 = (1+x^2)^{\frac{1}{2}}$

$\Rightarrow \frac{d}{dx}(y^2) + \frac{x}{1+x^2}y^2 = (1+x^2)^{\frac{1}{2}}$

$\text{I.F} = e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ln(1+x^2)} = (1+x^2)^{\frac{1}{2}}$

$\Rightarrow \frac{d}{dx}\left(y^2(1+x^2)^{\frac{1}{2}}\right) = 1+x^2$

$\Rightarrow y^2(1+x^2)^{\frac{1}{2}} = \int 1+x^2\,dx$

$\Rightarrow y^2(1+x^2)^{\frac{1}{2}} = x + \frac{1}{3}x^3 + C$

$\Rightarrow y^2 = \frac{x+\frac{1}{3}x^3+C}{(1+x^2)^{\frac{1}{2}}}$

$\Rightarrow y^2 = \frac{3x+x^3+A}{3(1+x^2)^{\frac{1}{2}}}$

Now $(0,0) \Rightarrow A=0$

$\Rightarrow y^2 = \frac{x^3+3x}{3\sqrt{x^2+1}}$

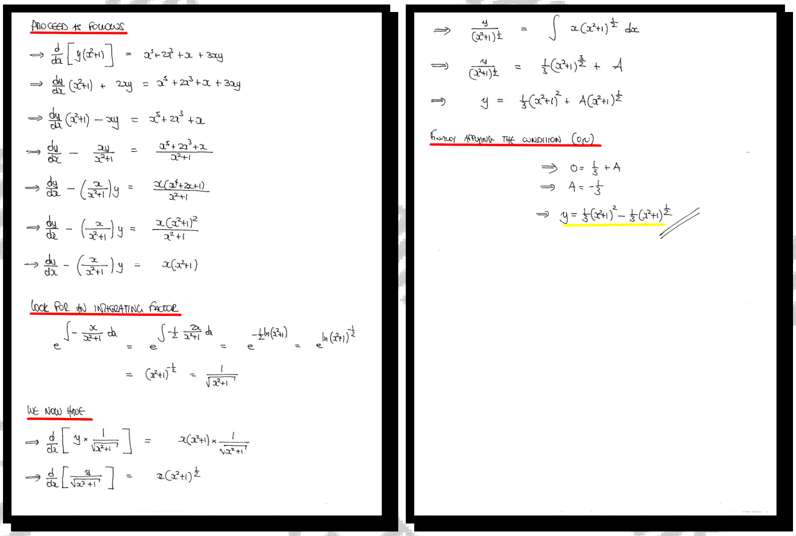
Question 22 (****+)

The curve with equation $y = f(x)$ passes through the origin, and satisfies the relationship

$$\frac{d}{dx}\left[y\left(x^2+1\right)\right] = x^5 + 2x^3 + x + 3xy .$$

Determine a simplified expression for the equation of the curve.

$$y = \tfrac{1}{3}\left(x^2+1\right)^2 - \tfrac{1}{3}\left(x^2+1\right)^{\frac{1}{2}}$$


PROCEED AS FOLLOWS

$\Rightarrow \frac{d}{dx}\left[y(x^2+1)\right] = x^5+2x^3+x+3xy$

$\Rightarrow \frac{dy}{dx}(x^2+1) + 2xy = x^5+2x^3+x+3xy$

$\Rightarrow \frac{dy}{dx}(x^2+1) - xy = x^5+2x^3+x$

$\Rightarrow \frac{dy}{dx} - \frac{xy}{x^2+1} = \frac{x^5+2x^3+x}{x^2+1}$

$\Rightarrow \frac{dy}{dx} - \left(\frac{x}{x^2+1}\right)y = \frac{x(x^4+2x+1)}{x^2+1}$

$\Rightarrow \frac{dy}{dx} - \left(\frac{x}{x^2+1}\right)y = \frac{x(x^2+1)^2}{x^2+1}$

$\Rightarrow \frac{dy}{dx} - \left(\frac{x}{x^2+1}\right)y = x(x^2+1)$

LOOK FOR AN INTEGRATING FACTOR

$e^{\int -\frac{x}{x^2+1}\,dx} = e^{\int -\frac{1}{2}\frac{2x}{x^2+1}\,dx} = e^{-\frac{1}{2}\ln(x^2+1)} = e^{\ln(x^2+1)^{-\frac{1}{2}}}$

$= (x^2+1)^{-\frac{1}{2}} = \frac{1}{\sqrt{x^2+1}}$

WE NOW HAVE

$\Rightarrow \frac{d}{dx}\left[y \times \frac{1}{\sqrt{x^2+1}}\right] = x(x^2+1) \times \frac{1}{\sqrt{x^2+1}}$

$\Rightarrow \frac{d}{dx}\left[\frac{y}{\sqrt{x^2+1}}\right] = x(x^2+1)^{\frac{1}{2}}$

$\Rightarrow \frac{y}{(x^2+1)^{\frac{1}{2}}} = \int x(x^2+1)^{\frac{1}{2}}\,dx$

$\Rightarrow \frac{y}{(x^2+1)^{\frac{1}{2}}} = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + A$

$\Rightarrow y = \frac{1}{3}(x^2+1)^2 + A(x^2+1)^{\frac{1}{2}}$

FINALLY APPLYING THE CONDITION (0,0)

$\Rightarrow 0 = \frac{1}{3} + A$

$\Rightarrow A = -\frac{1}{3}$

$\Rightarrow y = \frac{1}{3}(x^2+1)^2 - \frac{1}{3}(x^2+1)^{\frac{1}{2}}$

Question 23 (****+)

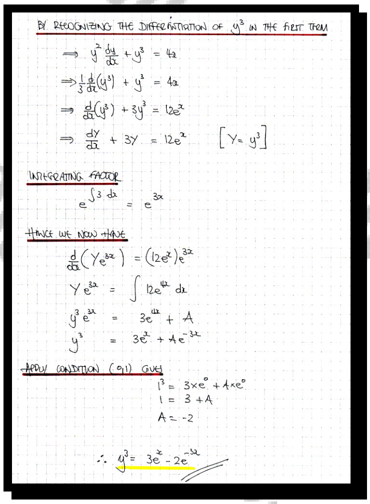
A curve with equation $y = f(x)$ passes through the point with coordinates $(0,1)$ and satisfies the differential equation

$$y^2 \frac{dy}{dx} + y^3 = 4\mathrm{e}^x .$$

By finding a suitable integrating factor, solve the differential equation to show that

$$y^3 = 3\mathrm{e}^x - 2\mathrm{e}^{-3x} .$$

[ ] , [proof]



BY RECOGNIZING THE DIFFERENTIATION OF y^3 IN THE FIRST TERM

$$\Rightarrow y^2 \frac{dy}{dx} + y^3 = 4e^x$$
$$\Rightarrow \frac{1}{3}\frac{d}{dx}(y^3) + y^3 = 4e^x$$
$$\Rightarrow \frac{d}{dx}(y^3) + 3y^3 = 12e^x$$
$$\Rightarrow \frac{dY}{dx} + 3Y = 12e^x \qquad [Y = y^3]$$

INTEGRATING FACTOR

$$e^{\int 3\,dx} = e^{3x}$$

HENCE WE NOW HAVE

$$\frac{d}{dx}\left(Ye^{3x}\right) = \left(12e^x\right)e^{3x}$$
$$Ye^{3x} = \int 12e^{4x}\,dx$$
$$y^3 e^{3x} = 3e^{4x} + A$$
$$y^3 = 3e^x + Ae^{-3x}$$

APPLY CONDITION (0,1) GIVES

$$1^3 = 3 \times e^0 + A \times e^0$$
$$1 = 3 + A$$
$$A = -2$$

$$\therefore y^3 = 3e^x - 2e^{-3x}$$

Question 24 **(****+)**

It is given that a curve with equation $y = f(x)$ passes through the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ and satisfies the differential equation

$$\left(\frac{dy}{dx} - \sqrt{\tan x}\right)\sin 2x = y .$$

Find an equation for the curve in the form $y = f(x)$.

SP, $\boxed{y = x\sqrt{\tan x}}$

MULTIPLY OUT FIRST & REARRANGE

$$\Rightarrow \left[\frac{dy}{dx} - \sqrt{\tan x}\right]\sin 2x = y$$

$$\Rightarrow \frac{dy}{dx}\sin 2x - \sin 2x\sqrt{\tan x} = y$$

$$\Rightarrow \frac{dy}{dx}\sin 2x - y = \sin 2x\sqrt{\tan x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{\sin 2x} = \sqrt{\tan x}$$

LOOK FOR AN INTEGRATING FACTOR

$$e^{\int -\frac{1}{\sin 2x}\,dx} = e^{\int -\operatorname{cosec} 2x\,dx} = e^{\frac{1}{2}\ln|\operatorname{cosec}2x + \cot 2x|}$$

$$= e^{\ln(\operatorname{cosec}2x + \cot 2x)^{\frac{1}{2}}} = (\operatorname{cosec}2x + \cot 2x)^{\frac{1}{2}}$$

$$= \left(\frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x}\right)^{\frac{1}{2}} = \left(\frac{1+\cos 2x}{\sin 2x}\right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{1 + (2\cos^2 x - 1)}{2\sin x\cos x}} = \sqrt{\frac{2\cos^2 x}{2\sin x\cos x}}$$

$$= \sqrt{\frac{\cos x}{\sin x}} = \sqrt{\cot x}$$

RETURNING TO THE O.D.E

$$\Rightarrow \frac{d}{dx}\left(y\sqrt{\cot x}\right) = \sqrt{\tan x}\sqrt{\cot x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{\sqrt{\tan x}}\right) = 1$$

$$\Rightarrow \frac{y}{\sqrt{\tan x}} = \int 1\,dx$$

$$\Rightarrow \frac{y}{\sqrt{\tan x}} = x + C$$

$$\Rightarrow y = x\sqrt{\tan x} + C\sqrt{\tan x}$$

APPLY BOUNDARY CONDITION $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

$$\Rightarrow \frac{\pi}{4} = \frac{\pi}{4}\sqrt{\tan\frac{\pi}{4}} + C\sqrt{\tan\frac{\pi}{4}}$$

$$\Rightarrow \frac{\pi}{4} = \frac{\pi}{4}\times 1 + C\times 1$$

$$\Rightarrow \frac{\pi}{4} = \frac{\pi}{4} + C$$

$$\Rightarrow C = 0$$

$$\therefore y = x\sqrt{\tan x}$$

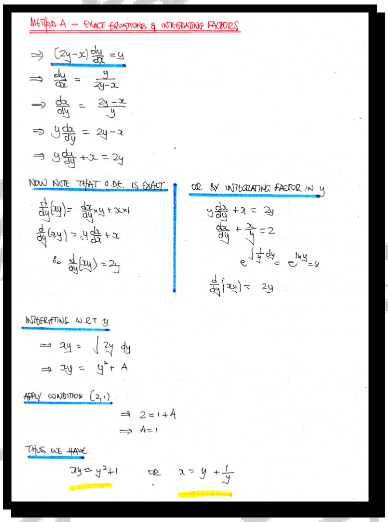
Question 25

The variables x and y satisfy

$$(2y - x)\frac{dy}{dx} = y\,, \quad y > 0\,,\ x > 0\,.$$

If $y = 1$ at $x = 2$, show that $x = y + \frac{1}{y}$.

V, [ ], proof



METHOD A – EXACT EQUATIONS & INTEGRATING FACTORS

$\Rightarrow (2y-x)\frac{dy}{dx} = y$

$\Rightarrow \frac{dy}{dx} = \frac{y}{2y-x}$

$\Rightarrow \frac{dx}{dy} = \frac{2y-x}{y}$

$\Rightarrow y\frac{dx}{dy} = 2y - x$

$\Rightarrow y\frac{dx}{dy} + x = 2y$

NOW NOTE THAT O.D.E. IS EXACT

$\frac{d}{dy}(xy) = \frac{dx}{dy}\times y + x \times 1$

$\frac{d}{dy}(xy) = y\frac{dx}{dy} + x$

$\therefore \frac{d}{dy}(xy) = 2y$

OR BY INTEGRATING FACTOR IN y

$y\frac{dx}{dy} + x = 2y$

$\frac{dx}{dy} + \frac{x}{y} = 2$

$e^{\int \frac{1}{y}dy} = e^{\ln y} = y$

$\frac{d}{dy}(xy) = 2y$

INTEGRATING W.R.T y

$\Rightarrow xy = \int 2y \, dy$

$\Rightarrow xy = y^2 + A$

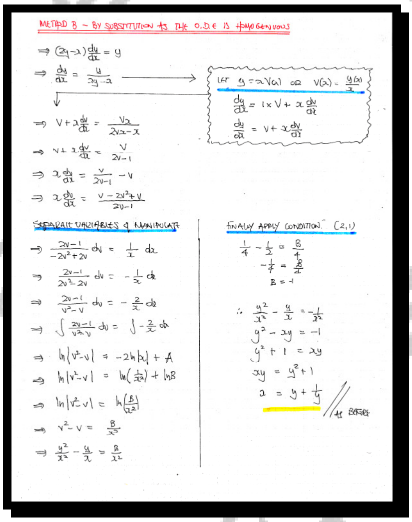
APPLY CONDITION $(2,1)$

$\Rightarrow 2 = 1 + A$

$\Rightarrow A = 1$

THUS WE HAVE

$xy = y^2 + 1$ OR $x = y + \frac{1}{y}$



METHOD B – BY SUBSTITUTION AS THE O.D.E IS HOMOGENEOUS

$\Rightarrow (2y-x)\frac{dy}{dx} = y$

$\Rightarrow \frac{dy}{dx} = \frac{y}{2y-x}$

LET $y = xV(x)$ OR $V(x) = \frac{y(x)}{x}$

$\frac{dy}{dx} = 1\times V + x\frac{dV}{dx}$

$\frac{dy}{dx} = V + x\frac{dV}{dx}$

$\Rightarrow V + x\frac{dV}{dx} = \frac{Vx}{2Vx - x}$

$\Rightarrow V + x\frac{dV}{dx} = \frac{V}{2V-1}$

$\Rightarrow x\frac{dV}{dx} = \frac{V}{2V-1} - V$

$\Rightarrow x\frac{dV}{dx} = \frac{V - 2V^2 + V}{2V-1}$

SEPARATE VARIABLES & MANIPULATE

$\Rightarrow \frac{2V-1}{-2V^2+2V}\,dV = \frac{1}{x}\,dx$

$\Rightarrow \frac{2V-1}{2V^2-2V}\,dV = -\frac{1}{x}\,dx$

$\Rightarrow \frac{2V-1}{V^2-V}\,dV = -\frac{2}{x}\,dx$

$\Rightarrow \int \frac{2V-1}{V^2-V}\,dV = \int -\frac{2}{x}\,dx$

$\Rightarrow \ln|V^2-V| = -2\ln|x| + A$

$\Rightarrow \ln|V^2-V| = \ln\left(\frac{1}{x^2}\right) + \ln B$

$\Rightarrow \ln|V^2-V| = \ln\left(\frac{B}{x^2}\right)$

$\Rightarrow V^2 - V = \frac{B}{x^2}$

$\Rightarrow \frac{y^2}{x^2} - \frac{y}{x} = \frac{B}{x^2}$

FINALLY APPLY CONDITION $(2,1)$

$\frac{1}{4} - \frac{1}{2} = \frac{B}{4}$

$-\frac{1}{4} = \frac{B}{4}$

$B = -1$

$\therefore \frac{y^2}{x^2} - \frac{y}{x} = -\frac{1}{x^2}$

$y^2 - xy = -1$

$y^2 + 1 = xy$

$xy = y^2 + 1$

$x = y + \frac{1}{y}$ // AS BEFORE

Question 26

The variables x and y satisfy

$$\frac{dy}{dx} = \frac{y(y+1)}{y-x-xy-1}, \quad y>0.$$

If $y=1$ at $x=1-\ln 4$, show that $y+\ln(y+1)=0$ at $x=3$.

V, , proof

START MANIPULATING AS FOLLOWS

$$\frac{dy}{dx} = \frac{y(y+1)}{y-1-xy-x} = \frac{y(y+1)}{(y-1)-x(y+1)}$$

"INVERTING" TO SWAP DEPENDENT & INDEPENDENT VARIABLES

$$\Rightarrow \frac{dx}{dy} = \frac{(y-1)-x(y+1)}{y(y+1)}$$

SPLITTING R.H.S

$$\Rightarrow \frac{dx}{dy} = \frac{y-1}{y(y+1)} - \frac{x}{y} \quad (y>0)$$

$$\Rightarrow y\frac{dx}{dy} = \frac{y-1}{y+1} - x$$

$$\Rightarrow y\frac{dx}{dy} + x = \frac{y-1}{y+1}$$

NOW THE L.H.S IS EXACT IN y (OR INTEGRATING FACTOR)

- $\frac{d}{dy}(xy) = \frac{dx}{dy}\cdot y + x\cdot 1 = y\frac{dx}{dy} + x$

  $\therefore \frac{d}{dy}(xy) = \frac{y-1}{y+1}$

- $y\frac{dx}{dy} + x = \frac{y-1}{y+1}$

  $\frac{dx}{dy} + \frac{x}{y} = \frac{y-1}{y(y+1)}$

  $e^{\int \frac{1}{y}\,dy} = e^{\ln y} = y$

  $\therefore \frac{d}{dy}(xy) = \frac{y(y-1)}{y(y+1)} \quad (y>0)$

  $\frac{d}{dy}(xy) = \frac{y-1}{y+1}$ AS OPPOSITE

INTEGRATING W.R.T y

$$\Rightarrow xy = \int \frac{y-1}{y+1}\,dy$$

$$\Rightarrow xy = \int \frac{(y+1)-2}{y+1}\,dy$$

$$\Rightarrow xy = \int 1 - \frac{2}{y+1}\,dy$$

$$\Rightarrow xy = y - 2\ln(y+1) + A \qquad y>0 \text{ NO MODULI NEEDED}$$

APPLY BOUNDARY CONDITION GIVEN

$x = 1-\ln 4$, $y=1$

$$\Rightarrow (1-\ln 4)\times 1 = 1 - 2\ln 2 + A$$

$$\Rightarrow 1-\ln 4 = 1-\ln 4 + A$$

$$\Rightarrow A = 0$$

$$\therefore xy = y - 2\ln(y+1)$$

WHEN $x=3$

$$\Rightarrow 3y = y - 2\ln(y+1)$$

$$\Rightarrow 2y = -2\ln(y+1)$$

$$\Rightarrow y = -\ln(y+1)$$

$$\Rightarrow y + \ln(y+1) = 0$$

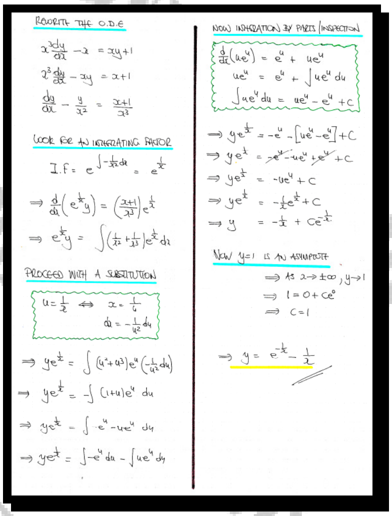
Question 27 (*****)

The curve with equation $y = f(x)$ has the line $y = 1$ as an asymptote and satisfies the differential equation

$$x^3 \frac{dy}{dx} - x = xy + 1, \quad x \neq 0.$$

Solve the above differential equation, giving the solution in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = e^{-\frac{1}{x}} - \frac{1}{x}}$



REWRITE THE O.D.E

$$x^3\frac{dy}{dx} - x = xy + 1$$
$$x^3\frac{dy}{dx} - xy = x + 1$$
$$\frac{dy}{dx} - \frac{y}{x^2} = \frac{x+1}{x^3}$$

LOOK FOR AN INTEGRATING FACTOR

$$\text{I.F.} = e^{\int -\frac{1}{x^2}dx} = e^{\frac{1}{x}}$$
$$\Rightarrow \frac{d}{dx}\left(e^{\frac{1}{x}}y\right) = \left(\frac{x+1}{x^3}\right)e^{\frac{1}{x}}$$
$$\Rightarrow e^{\frac{1}{x}}y = \int\left(\frac{1}{x^2}+\frac{1}{x^3}\right)e^{\frac{1}{x}}\,dx$$

PROCEED WITH A SUBSTITUTION

$$u = \frac{1}{x} \iff x = \frac{1}{u}, \quad dx = -\frac{1}{u^2}du$$

$$\Rightarrow ye^{\frac{1}{x}} = \int (u^2+u^3)e^u\left(-\frac{1}{u^2}du\right)$$
$$\Rightarrow ye^{\frac{1}{x}} = -\int (1+u)e^u\,du$$
$$\Rightarrow ye^{\frac{1}{x}} = \int -e^u - ue^u\,du$$
$$\Rightarrow ye^{\frac{1}{x}} = \int -e^u\,du - \int ue^u\,du$$

NOW INTEGRATION BY PARTS (INSPECTION)

$$\frac{d}{dx}(ue^u) = e^u + ue^u$$
$$ue^u = e^u + \int ue^u\,du$$
$$\int ue^u\,du = ue^u - e^u + C$$

$$\Rightarrow ye^{\frac{1}{x}} = -e^u - [ue^u - e^u] + C$$
$$\Rightarrow ye^{\frac{1}{x}} = -e^u - ue^u + e^u + C$$
$$\Rightarrow ye^{\frac{1}{x}} = -ue^u + C$$
$$\Rightarrow ye^{\frac{1}{x}} = -\frac{1}{x}e^{\frac{1}{x}} + C$$
$$\Rightarrow y = -\frac{1}{x} + Ce^{-\frac{1}{x}}$$

NOW $y=1$ IS AN ASYMPTOTE

$$\Rightarrow \text{As } x \to \pm\infty,\ y \to 1$$
$$\Rightarrow 1 = 0 + Ce^0$$
$$\Rightarrow C = 1$$

$$\Rightarrow y = e^{-\frac{1}{x}} - \frac{1}{x}$$

Question 28 (*****)

It is given that a curve with equation $x = f(y)$ passes through the point $\left(0, \frac{1}{2}\right)$ and satisfies the differential equation

$$(2y + 3x)\frac{dy}{dx} = y.$$

Find an equation for the curve in the form $x = f(y)$.

$\boxed{}$, $\boxed{x = 4y^3 - y}$

METHOD A

REARRANGE & TREAT y AS THE INDEPENDENT VARIABLE

$\Rightarrow (2y+3x)\frac{dy}{dx} = y$

$\Rightarrow 2y + 3x = y\frac{dx}{dy}$

$\Rightarrow y\frac{dx}{dy} - 3x = 2y$

$\Rightarrow \frac{dx}{dy} - \frac{3}{y}x = 2$

INTEGRATING FACTOR CAN NOW BE FOUND

$e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = e^{\ln\frac{1}{y^3}} = \frac{1}{y^3}$

HENCE WE NOW HAVE

$\Rightarrow \frac{d}{dy}\left(x \cdot \frac{1}{y^3}\right) = 2 \cdot \frac{1}{y^3}$

$\Rightarrow \frac{x}{y^3} = \int \frac{2}{y^3}\,dy$

$\Rightarrow \frac{x}{y^3} = -\frac{1}{y^2} + A$

$\Rightarrow x = Ay^3 - y$

APPLY CONDITION $(0, \frac{1}{2})$

$\Rightarrow 0 = A\left(\frac{1}{8}\right) - \frac{1}{2}$

$\Rightarrow 0 = A - 4$

$\Rightarrow A = 4$

$\therefore x = 4y^3 - y$

METHOD B

PROCEED BY A SUBSTITUTION

$y = xV(x)$

$\frac{dy}{dx} = V(x) + x\frac{dV}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{2y+3x}$

$\Rightarrow V + x\frac{dV}{dx} = \frac{xV}{2xV + 3x}$

$\Rightarrow x\frac{dV}{dx} = \frac{V}{2V+3} - V$

$\Rightarrow x\frac{dV}{dx} = \frac{V - 2V^2 - 3V}{2V+3} = \frac{-2V^2-2V}{2V+3} = \frac{-2V(V+1)}{2V+3}$

SEPARATING VARIABLES

$\Rightarrow \frac{2V+3}{V(V+1)}\,dV = -\frac{2}{x}\,dx$

$\Rightarrow \int \frac{3}{V} - \frac{1}{V+1}\,dV = \int -\frac{2}{x}\,dx$ (PARTIAL FRACTIONS BY INSPECTION)

$\Rightarrow 3\ln|V| - \ln|V+1| = -2\ln|x| + \ln A$

$\Rightarrow \ln\left|\frac{V^3}{V+1}\right| = \ln\left|\frac{A}{x^2}\right|$

$\Rightarrow \frac{V^3}{V+1} = \frac{A}{x^2}$

$\Rightarrow \frac{\frac{y^3}{x^3}}{\frac{y}{x}+1} = \frac{A}{x^2}$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION IN THE L.H.S. BY x^3

$\Rightarrow \frac{y^3}{yx^2 + x^3} = \frac{A}{x^2}$

MULTIPLY BOTH SIDES BY x^2

$\Rightarrow \frac{y^3}{y+x} = A$

APPLY THE CONDITION $(0, \frac{1}{2})$

$\Rightarrow \frac{\frac{1}{8}}{\frac{1}{2}+0} = A$

$\Rightarrow A = \frac{1}{4}$

$\therefore \frac{y^3}{y+x} = \frac{1}{4}$

$4y^3 = y + x$

$x = 4y^3 - y$ AS BEFORE

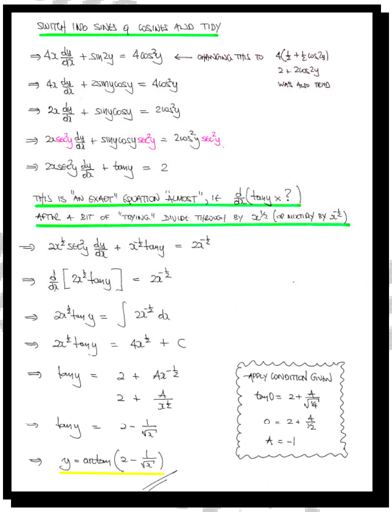
Question 29 (*****)

Use suitable manipulations to solve this **exact** differential equation.

$$4x\frac{dy}{dx}+\sin 2y = 4\cos^2 y\,, \quad y\left(\tfrac{1}{4}\right)=0\,.$$

Given the answer in the form $y = f(x)$.

$$y = \arctan\left[2 - \frac{1}{\sqrt{x}}\right]$$



SWITCH INTO SINES & COSINES AND TIDY

$\Rightarrow 4x\frac{dy}{dx} + \sin 2y = 4\cos^2 y$ ← CHANGING THIS TO $4\left(\frac{1}{2}+\frac{1}{2}\cos 2y\right)$ $2 + 2\cos 2y$ WAS ALSO TRIED

$\Rightarrow 4x\frac{dy}{dx} + 2\sin y\cos y = 4\cos^2 y$

$\Rightarrow 2x\frac{dy}{dx} + \sin y\cos y = 2\cos^2 y$

$\Rightarrow 2x\sec^2 y\frac{dy}{dx} + \sin y\cos y\sec^2 y = 2\cos^2 y\sec^2 y$

$\Rightarrow 2x\sec^2 y\frac{dy}{dx} + \tan y = 2$

THIS IS "AN EXACT" EQUATION "ALMOST", IF $\frac{d}{dx}(\tan y \times ?)$

AFTER A BIT OF "TRYING" DIVIDE THROUGH BY $x^{1/2}$ (OR MULTIPLY BY $x^{-\frac{1}{2}}$)

$\Rightarrow 2x^{\frac{1}{2}}\sec^2 y\frac{dy}{dx} + x^{-\frac{1}{2}}\tan y = 2x^{-\frac{1}{2}}$

$\Rightarrow \frac{d}{dx}\left[2x^{\frac{1}{2}}\tan y\right] = 2x^{-\frac{1}{2}}$

$\Rightarrow 2x^{\frac{1}{2}}\tan y = \int 2x^{-\frac{1}{2}}\,dx$

$\Rightarrow 2x^{\frac{1}{2}}\tan y = 4x^{\frac{1}{2}} + C$

$\Rightarrow \tan y = 2 + Ax^{-\frac{1}{2}}$

$2 + \frac{A}{x^{\frac{1}{2}}}$

APPLY CONDITION GIVEN

$\tan 0 = 2 + \frac{A}{\sqrt{1/4}}$

$0 = 2 + \frac{A}{1/2}$

$A = -1$

$\Rightarrow \tan y = 2 - \frac{1}{\sqrt{x}}$

$\Rightarrow y = \arctan\left(2 - \frac{1}{\sqrt{x}}\right)$