2.1 Common Signals

Continuous-Time Unit Step Signals

Three slightly different definitions of the *unit step* signal can be found in engineering literature. Two commonly used definitions are

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$u_1(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

Note that in the second case the value of $u_1(0)$ is not defined.

The third definition, also known as Heaviside's unit step signal, is given by

$$u_h(t) = \begin{cases} 1, & t > 0 \\ 0.5, & t = 0 \\ 0, & t < 0 \end{cases}$$

The first two definitions are simpler and easier to work with than the third definition. It will be shown in Chapter 3 on Fourier analysis that the Heaviside definition of the unit step signal will be needed.

Having in mind that the primary concerns of this book are linear dynamic systems and for the reason of simplicity, we will use the definition of the unit step signal as given by $u(t)$, except where explicitly indicated that the presentation holds for the Heaviside unit step signal $u_h(t)$. In the entire Chapter 3 on Fourier analysis, the signal $u_h(t)$ will be used exclusively.

The graphical presentation of the unit step signal $u(t)$ is given in Figure 2.1.

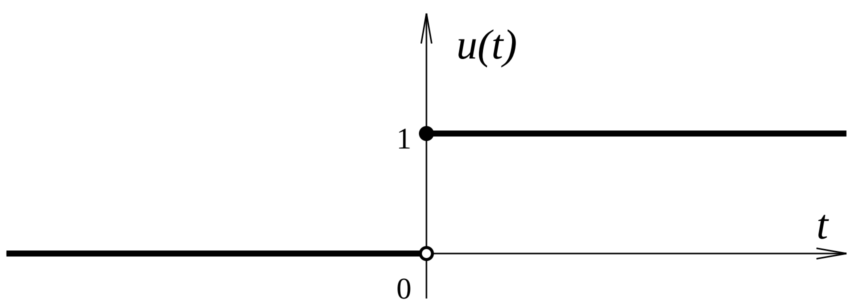

Figure 2.1: Continuous-time unit step signal $u(t)$

Continuous-Time Signum Signal—Sign Signal

The Heaviside unit step signal can be expressed in terms of sign function. The *sign function*, well known in mathematics, is defined by

$$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

The sign function is also known as the signum function. The formula of interest that will be used in the next chapter on the Fourier transform, relates the sign signal and the Heaviside unit step signal

$$u_h(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$

Discrete-Time Unit Step Signal

The discrete-time unit step signal can be obtained by sampling $u(t)$ with the sampling period T. The *discrete unit step signal* is defined by (see Figure 2.2)

$$u(kT) \triangleq u[k] = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

where kT stands for discrete time. Since T is a fixed positive constant, we will use throughout this book only k to indicate discrete-time instant kT, where k is any integer. That is, in our notation, unless explicitly indicated otherwise, for any discrete signal the following holds: $f(kT) \triangleq f[k]$.

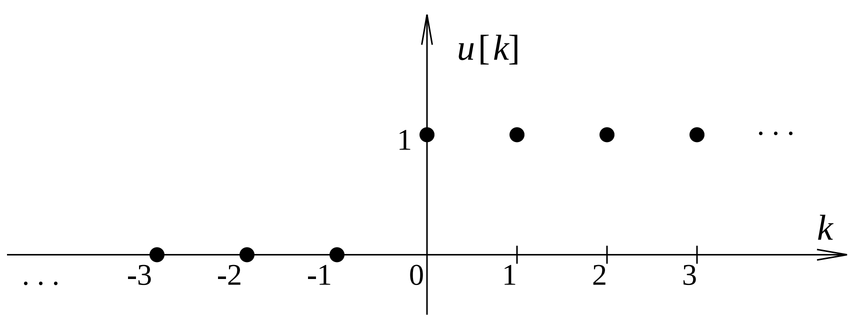

Figure 2.2: Discrete-time unit step signal $u[k]$

Continuous-Time Ramp Signals

The *unit ramp signal* is defined in the continuous-time domain by

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

The unit ramp signal has the slope equal to one for $t > 0$. We can also introduce the *ramp signal* that has an arbitrary slope α for $t > 0$ as $r_\alpha(t) = \alpha r(t)$.

It is easy to observe that in continuous time we have the following relationships between the unit step and unit ramp signals

$$u(t) = \frac{dr(t)}{dt}, \quad t \neq 0$$

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Note that the ramp signal is not differentiable at $t = 0$.

Discrete-Time Ramp Signals

Sampling the continuous-time unit ramp $r(t)$ signal we get its discrete counterpart as

$$r(kT) \triangleq r[k] = \begin{cases} k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

The corresponding graphical representations are given in Figure 2.3.

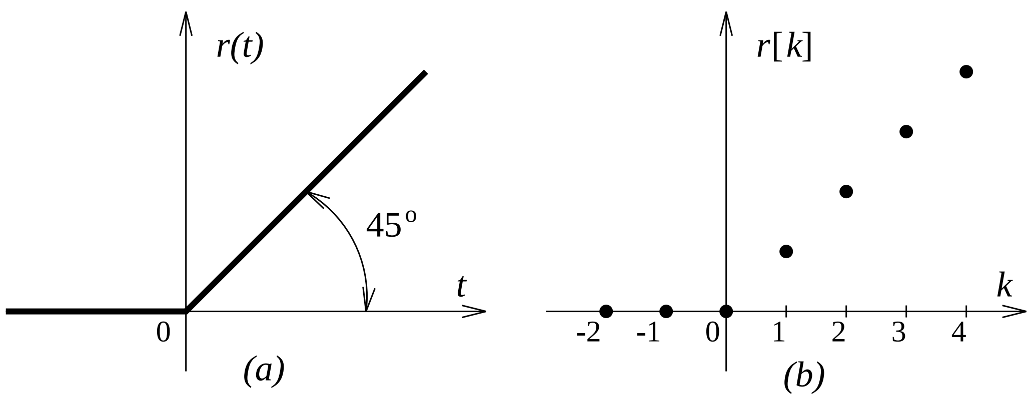

Figure 2.3: Continuous-time (a) and discrete-time (b) unit ramp signals

Continuous-Time Parabolic Signal

Similarly, we can introduce the *parabolic signal* as

$$f_p(t) = \begin{cases} t^2, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and in general, a family of signals of the form

$$f_n(t) = \begin{cases} t^n, & t \geq 0 \\ 0, & t < 0 \end{cases}, \quad n = 3, 4, 5, \ldots$$

These signals appear in some signal processing and control system applications.

Discrete-Time Parabolic Signal

The corresponding discrete-time equivalents can also be defined as follows

$$f_p[k] = \begin{cases} k^2, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

and

$$f_n[k] = \begin{cases} k^n, & k \geq 0 \\ 0, & k < 0 \end{cases}, \quad n = 3, 4, 5, \ldots$$

Rectangular Pulse

The *rectangular pulse* is mathematically defined by (see Figure 2.4a)

$$p_\tau(t) = \begin{cases} 1, & -\tau/2 \le t \le \tau/2 \\ 0, & \text{elsewhere} \end{cases}$$

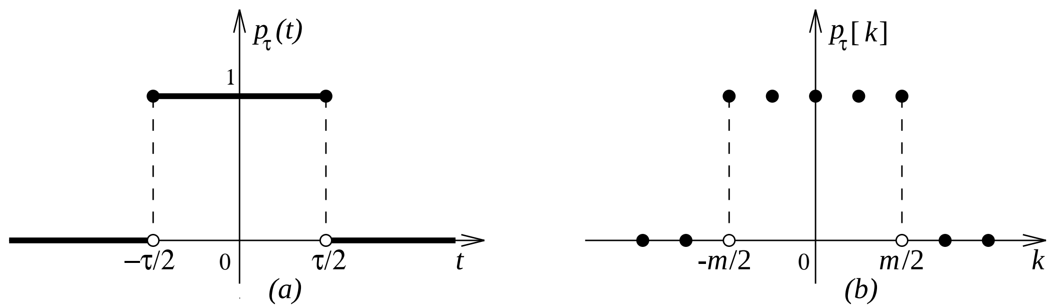

Figure 2.4: Rectangular pulses: continuous-time (a) and discrete-time (b)

Another definition of the (Heaviside) rectangular pulse

$$p_\tau^h(t) = \begin{cases} 1, & -\tau/2 < t < \tau/2 \\ 0.5, & t = \pm\tau/2 \\ 0, & \text{elsewhere} \end{cases}$$

Note that

$$p_\tau^h(t) = u_h\left(t + \frac{\tau}{2}\right) - u_h\left(t - \frac{\tau}{2}\right)$$

Train of Rectangular Pulses

It is defined by

$$\sum_{k=-\infty}^{\infty} p_\tau(t - kT_0)$$

In Problem 2.43 this train of recangular pulses is plotted for positive values of k using MATLAB (Figure 2.20 of the Solutions Manual)

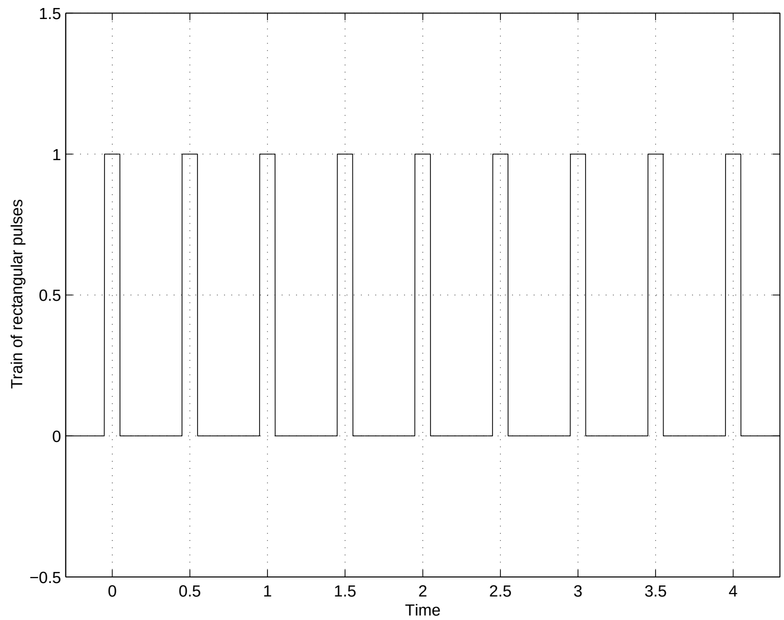

FIGURE 2.20 (from Solutions Manual)

Triangular Pulse

The *triangular pulse* is defined by

$$\Delta_\tau(t) = \begin{cases} 0, & t \le -\tau/2 \\ 1 + 2t/\tau, & -\tau/2 \le t \le 0 \\ 1 - 2t/\tau, & 0 \le t \le \tau/2 \\ 0, & \tau/2 \le t \end{cases}$$

The graphical presentation of this signal is given in Figure 2.5a.

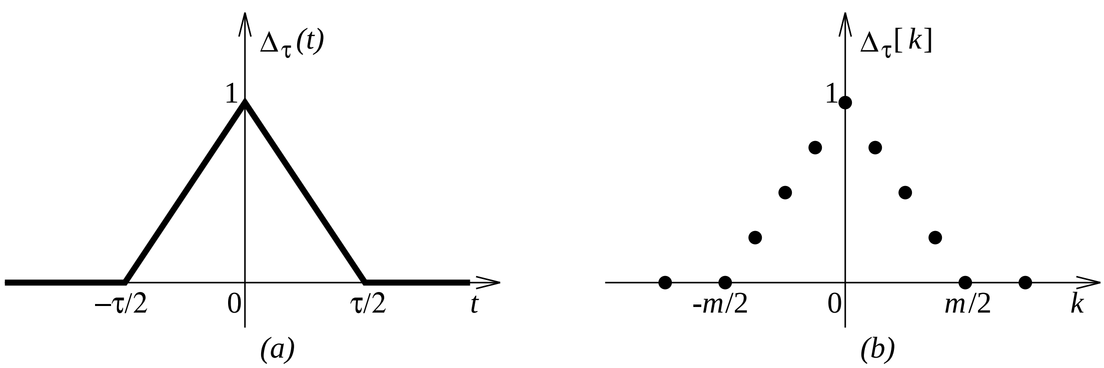

Figure 2.5: Triangular pulses: continuous-time (a) and discrete-time (b)

Discrete-Time Rectangular and Triangular Pulses

Discrete versions of the rectangular and triangular pulses can be easily obtained by sampling. Two examples of these signals that are self explanatory are drawn in Figures 2.4b and 2.5b. In Figures 2.4b and 2.5b, m is defined by $m = 2[\tau/2T]$, where $[\,\cdot\,]$ stands for the integer part operation and T represents the sampling period. For example, the discrete-time rectangular pulse is analytically defined as follows

$$p_m[k] = \begin{cases} 1, & -m/2 \leq k \leq m/2 \\ 0, & \text{elsewhere} \end{cases}$$

Sinusoidal Signals

Sine and cosine signals are commonly used in engineering. These signals are very well known to all college students from basic high school courses. Note that

$$\cos(\theta t) = \sin\left(\theta t + \frac{\pi}{2}\right), \qquad \sin(\theta t) = \cos\left(\theta t - \frac{\pi}{2}\right)$$

Continuous-Time Sinc Signal

The *sinc signal* plays a very important role in Fourier analysis, communication systems, and signal processing. It is defined by (see Figure 2.6)

$$\mathbf{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

By the well known trigonometric limit $\sin(0)/0 = 1$, it follows that $\mathrm{sinc}(0) = 1$. For $t \neq 0$, the zeros of the sinc signal are at $t = \pm n, \;\; n = 1, 2, 3, \ldots$. When $t \to \pm\infty$ the sinc signal tends to zero. Also, $\mathrm{sinc}(-t) = \mathrm{sinc}(t)$.

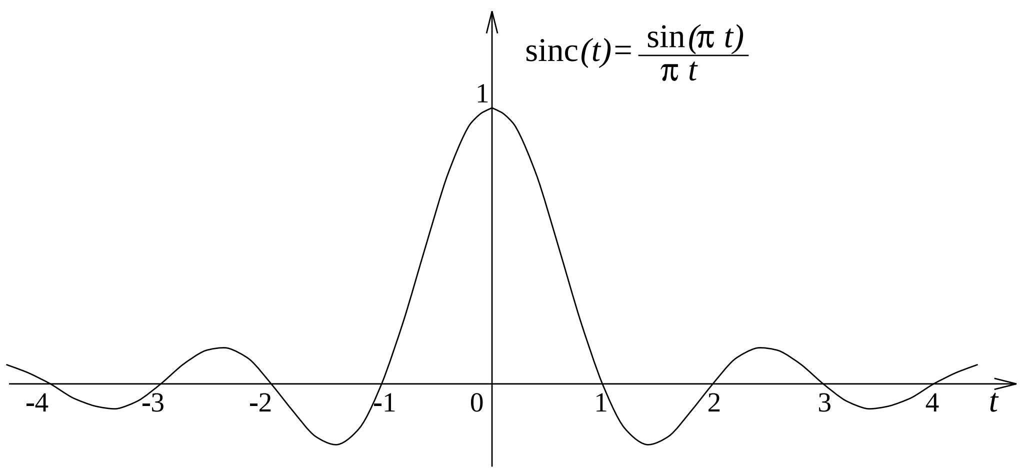

Figure 2.6: Continuous-time sinc signal

Discrete-Time Sinc Signal

The discrete-time sinc signal can be obtained by discretizing the corresponding continuous-time sinc signal. The discrete-time sinc signal plot, obtained by using MATLAB, is presented in Figure 2.7.

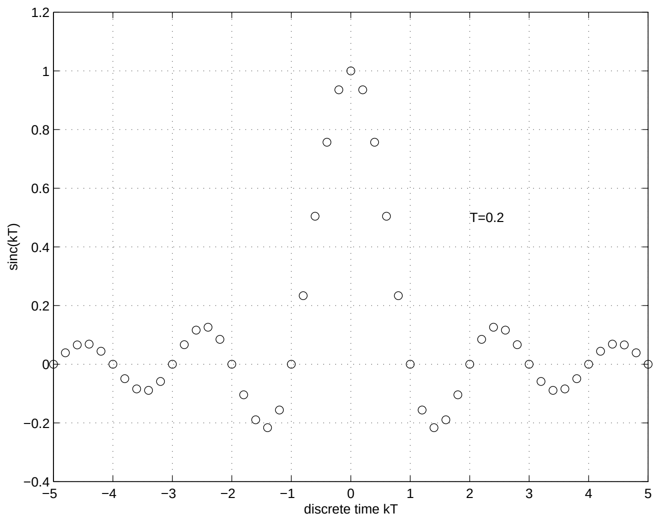

Figure 2.7 Discrete-time sinc signal obtained by MATLAB

The formal definition of the discrete-time sinc signal is given by

$$\mathrm{sinc}(kT) = \frac{\sin(\pi kT)}{\pi kT} \triangleq \frac{\sin(\pi[k])}{\pi[k]} \triangleq \mathrm{sinc}[k]$$

The Use of Basic Signals in Linear Systems

In the subsequent chapters, we will show that it is easy to find the response of a linear system due to basic input signals such as step and ramp signals. More complex signals can often be represented as linear combinations of step and ramp signals. This signal representation in terms of basic signals together with the linearity (superposition) and time invariance properties (introduced in Chapter 1) will help to easily obtain the linear system response due to the input signals that have complex waveform (shape).

The next example demonstrates how to represent a signal in terms of unit step and ramp signals.

Example 2.2: Consider the signal given in Figure 2.8.

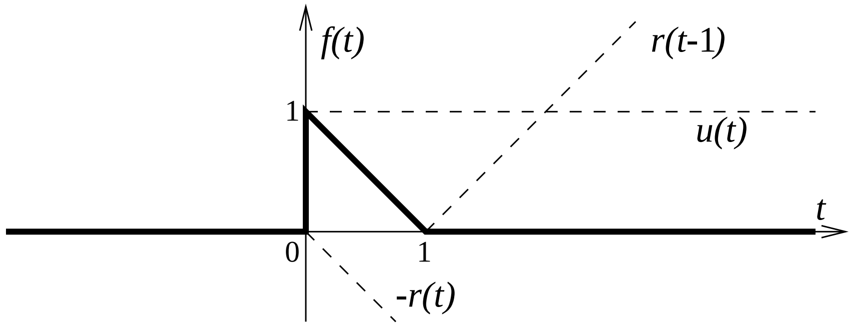

Figure 2.8: A simple signal

It is easy to observe that the signal presented by the solid lines in Figure 2.8 can be represented in terms of unit step and unit ramp signals as follows

$$f(t) = u(t) - r(t) + r(t-1)$$

The elementary signals $u(t), -r(t),$ and $r(t-1)$ are represented in the same figure using dashed lines. Note that this signal representation is not unique, hence other representations are possible, for example

$$f(t) = r(-t+1) - r(-t) - u(-t)$$

2.1.1 Impulse Delta Signal

The need for a new class of signals was for the first time observed by the end of the nineteenth century by O. Heaviside while analyzing electrical circuits (see Problems 2.18 and 2.19 and formulas (11.20) and (11.30) in Chapter 11 on electrical circuits). In the 1920s P. Dirac came to the same conclusion studying some problems in relativistic mechanics. The new class of functions—the so-called *distributions or singular functions* that, together with ordinary functions, forms the set of *generalized functions*. The generalized functions play a very important role in analysis of linear dynamic systems, and they are used in almost all engineering and scientific disciplines.

The impulse delta function is extremely important for linear system theory. Loosely speaking, this “strange” function has no time structure. It is equal to zero everywhere else except at zero, where it is equal to ∞.

However, its integral is well behaved, and it is defined by

$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$

The impulse delta signal can be visualized as a mathematical artifice of the rectangular pulse, represented in Figure 2.11a, in the limit when the width of the pulse tends to zero. In Figure 2.11b, we give also the symbolic notation for $\delta(t)$.

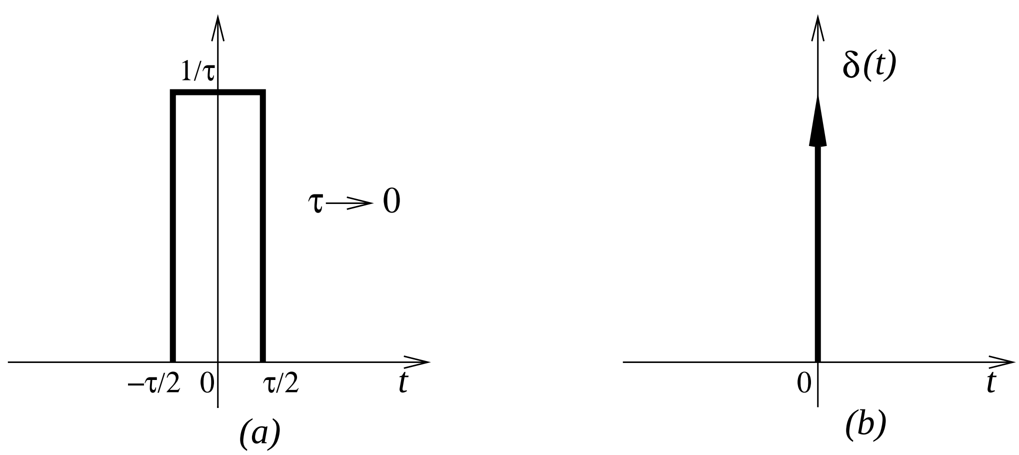

Figure 2.11: Approximation of the impulse delta signal (a) and symbolic representation for $\delta(t)$ (b)

Notice that the area of the rectangular pulse is always equal to one. The impulse delta signal is obtained in the limit when $\tau \rightarrow 0$, that is

$$\delta(t) = \lim_{\tau \to 0} \left\{ \frac{1}{\tau} p_\tau(t) \right\}$$

In the literature, the impulse delta signal is also called the *Dirac impulse function*, in honor of the great physicist and mathematician P. Dirac.

Example 2.4: Note that from the definition of the impulse delta signal it follows that

$$\int_{-5}^{3} \delta(t-4)dt = 0, \quad \int_{-5}^{4^+} \delta(t-4)dt = 1$$

In the first case the impulse delta signal is located outside of the integration limits, whereas in the second case it is within the integration limits.

The shifted impulse delta signal is defined by

$$\delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(t-t_0)dt=\int_{t_0^-}^{t_0^+}\delta(t-t_0)dt=1$$

This signal is represented in Figure 2.12.

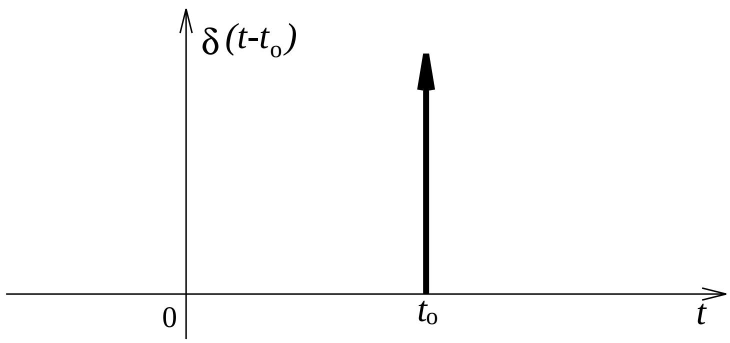

Figure 2.12: Shifted continuous-time impulse delta signal

Mathematical Definition of the Impulse Delta Signal (*Sifting Property*)

The impulse delta signal (function) in *mathematics* is defined by the integral

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

where $f(t)$ is an ordinary function continuous at $t = 0$. In engineering, we prefer to call this mathematical definition ***the sifting property*** of the impulse delta function since the effect of the impulse delta function in this integral is to take out (sift) a particular value of the function $f(t)$ at $t = t_0$.

Example 2.5: By using the sifting property of the impulse delta signal the following integral can be calculated

$$\int_{-\infty}^{\infty} \left\{\left[e^{-5t}\cos(2t) + t^2\right]\delta(t) + (2t+1)\delta(t-2)\right\}dt$$

$$= [1 + 0] + [4 + 1] = 6$$

What can be said about the integral

$$\int_{-\infty}^{t_0} f(t)\delta(t-t_0)dt = ?$$

with an ordinary function $f(t)$ being continuous at $t = t_0$? Note that the impulse delta function is located exactly at the integral upper limit. This integral sometimes appears in actual derivations (such as in the well-known paper written by Athans and Tse in 1967). The following result was used by Athans and Tse to derive one of the classic results of linear control theory:

$$\int_{-\infty}^{t_0} f(t)\delta(t-t_0)dt = \frac{1}{2}f(t_0)$$

Hence

$$\int_{0^-}^{0} \delta(t)dt = \frac{1}{2}, \qquad \int_{-1}^{1} \delta(t-1)f(t)dt = \frac{1}{2}f(1)$$

Derivatives of the Impulse Delta Signal

Another visualization of the impulse delta signal can be obtained by considering the triangular pulse in the limit when $\tau \to 0$, Figure 2.13a.

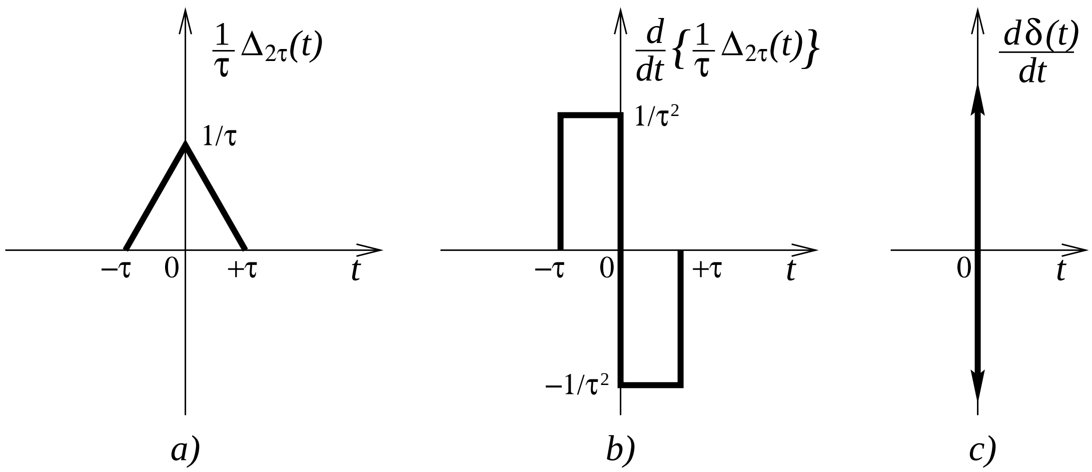

Figure 2.13: Approximations of the impulse delta signal (a), its derivative (b), and the symbolic representation of the derivative of the impulse delta signal (c)

It is obvious from this figure that

$$\delta(t) = \lim_{\tau \to 0} \left\{ \frac{1}{\tau} \Delta_{2\tau}(t) \right\}$$

This representation of the delta signal helps to visualize the *derivative of the impulse delta signal* as the limit of the signal represented in Figure 2.13b when $\tau \to 0$, that is

$$\frac{d\delta(t)}{dt} = \lim_{\tau \to 0} \left\{ \frac{1}{\tau^2} p_\tau \left(t + \frac{\tau}{2} \right) - \frac{1}{\tau^2} p_\tau \left(t - \frac{\tau}{2} \right) \right\}$$

The corresponding derivative is symbolically represented in Figure 2.13c with two impulses of width zero that tend to plus and minus infinity. The derivative of the unit impulse delta signal is also known in the literature as the ***unit-doublet***. Note that the unit-doublet $d\delta(t)/dt$ evaluated at $t = 0^+$ and $t = 0^-$ produces zero values like the impulse delta signal.

Mathematically, we can define the derivative of the delta impulse signal using its integral representation as

$$\int_{-\infty}^{\infty} f(t) \frac{d\delta(t - t_0)}{dt} dt$$

Similarly, we can define the second and higher order derivatives of the impulse delta signal by the following integrals

$$\int_{-\infty}^{\infty} f(t)\frac{d^i\delta(t-t_0)}{dt^i}dt, \qquad i=1,2,...$$

The second derivative of $\delta(t)$ is called the ***unit-triplet***. The $(n-1)$th derivative of $\delta(t)$ is called the ***unit-n-tuplet***. The unit triplet and unit-n-tuplets are also graphically represented using exactly the same plot as the one given in Figure 2.13c for the derivative of the unit impulse delta signal. Note that all $d^i\delta(t)/dt^i,\ i=1,2,...,$ evaluated at $t=0^+$ and $t=0^-$ produce zero values.

Other Properties of the Delta Impulse Signal

Time scaling property:

$$\int_{-\infty}^{\infty} f(t)\delta(at-t_0)dt = \frac{1}{|a|}f\left(\frac{t_0}{a}\right) \Rightarrow \delta(at-t_0) = \frac{1}{|a|}\delta\left(t-\frac{t_0}{a}\right)$$

Proof: Consider first the case when $a > 0$ and introduce the change of variables

$$at - t_0 = \sigma \Rightarrow t = \frac{\sigma + t_0}{a}, \quad dt = \frac{d\sigma}{a}$$

which implies

$$\int_{t=-\infty}^{t=\infty} f(t)\delta(at - t_0)dt = \frac{1}{a}\int_{\sigma=-\infty}^{\sigma=\infty} f\left(\frac{\sigma + t_0}{a}\right)\delta(\sigma)d\sigma = \frac{1}{a}f\left(\frac{t_0}{a}\right)$$

For $a < 0$ the same change of variables implies that the integral defined on the left side has the form

$$\frac{1}{a}\int_{\sigma=\infty}^{\sigma=-\infty} f\left(\frac{\sigma + t_0}{a}\right)\delta(\sigma)d\sigma = -\frac{1}{a}\int_{\sigma=-\infty}^{\sigma=\infty} f\left(\frac{\sigma + t_0}{a}\right)\delta(\sigma)d\sigma = -\frac{1}{a}f\left(\frac{t_0}{a}\right)$$

Putting together both cases $a > 0$ and $a < 0$, we obtain the stated time scaling result.

Derivative property: Let $\delta^{(1)}(t)$ denote $d\delta(t)/dt$, then

$$\int_{-\infty}^{\infty} f(t)\delta^{(1)}(t-t_0)dt = -f^{(1)}(t_0)$$

where $f^{(1)}(t_0)$ stands for $df(t)/dt$ evaluated at $t = t_0$. In general, it can also be shown that for the nth derivative we have

$$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t-t_0)dt = (-1)^n f^{(n)}(t_0)$$

Proof: Integrating by parts, we obtain

$$\int_{-\infty}^{\infty} f(t)\delta^{(1)}(t-t_0)dt = f(\infty)\delta(\infty) - f(-\infty)\delta(-\infty)$$

$$-\int_{-\infty}^{\infty} f^{(1)}(t)\delta(t-t_0)dt = 0 - 0 - f^{(1)}(t_0)$$

Similarly, integrating n–times by parts, we get the general formula for the nth derivative.

From the sifting property the impulse delta signal we can get *additional properties of the impulse delta signal* as

$$f(t)\delta(t) = f(0)\delta(t); \qquad f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$

This property can be proved as follows. Let $\varphi(t)$ be another ordinary function continuous at $t=0$. Then, by using the sifting property we have

$$\int_{-\infty}^{\infty} f(t)\varphi(t)\delta(t)dt = f(0)\varphi(0) = f(0)\int_{-\infty}^{\infty} \varphi(t)\delta(t)dt$$

$$= \int_{-\infty}^{\infty} f(0)\varphi(t)\delta(t)dt$$

Comparing the first and the last integral in the above expression, the property follows. Using the above property, a pretty interesting result follows: $t^n\delta(t)=0$, n is any positive real number.

From the sifting and derivative properties the following property can be established

$$\delta^{(2n)}(-t) = \delta^{(2n)}(t), \quad \delta^{(2n+1)}(-t) = -\delta^{(2n+1)}(t), \quad n = 0, 1, 2, ...$$

It follows from this property that for $n = 0$ the following is satisfied $\delta(-t) = \delta(t)$, hence $\delta(t)$ is an even function.

Example 2.6: Using the properties of the impulse delta signal we can evaluate the following integrals as

$$\int_{-\infty}^{\infty} \delta(2t-1)e^{-3t}\sin(\pi t)dt = \frac{1}{2}e^{-\frac{3}{2}}\sin\left(\frac{\pi}{2}\right) = \frac{1}{2}e^{-\frac{3}{2}}$$

$$\int_{-\infty}^{3} \left(t^3 + 2\sin(\pi t) - 2\right)\delta^{(1)}(t-1)dt = (-1)\frac{d}{dt}\left(t^3 + 2\sin(\pi t) - 2\right)_{|t=1}$$

$$= -(3 + 2\pi\cos(\pi)) = -3$$

Applications of the Delta Impulse Function to Solving Differential Equations

Consider the linear system defined in Problem 1.13

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{df(t)}{dt}+3f(t), \qquad f(t)=e^{-5t}, \quad t\geq 0$$

The forcing function (system input) can be represented by $f(t)=e^{-5t}u(t)$ so that the right-hand side of this differential equation is

$$\frac{df(t)}{dt}+3f(t)=-5e^{-5t}u(t)+e^{-5t}\delta(t)+3e^{-5t}u(t)=-2e^{-5t}u(t)+\delta(t)$$

By the linearity principle the system response will have two components coming from the forcing function: due to $-2e^{-5t}u(t)$ and $\delta(t)$. The problem of finding the system response to the delta impulse function input is one of the central problems of linear systems theory. It will be considered in Chapters 3, 4, 6 and 8.

Discrete-Time Impulse Delta Signal

The sampling technique makes no sense in an attempt to obtain the discrete-time impulse delta signal from the continuous-time impulse delta signal since the continuous-time impulse delta signal has no time structure. The discrete-time impulse delta signal is defined as a very nice signal which is equal to 1 at $k = 0$ and 0 everywhere else, that is

$$\delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

The discrete-time impulse delta signal is also called the Kronecker delta function. This form for the impulse delta signal in the discrete-time domain can be justified by using a discrete version of the sifting property

$$\sum_{k=-\infty}^{k=\infty} f[k]\delta[k-k_0] = f[k_0]$$

Since this infinite sum has to produce only $f[k_0]$, the discrete-time impulse delta signal must be zero everywhere except at k_0 where it must be equal to one.

The shifted version of the *discrete-time impulse delta signal* is defined by

$$\delta[k-k_0] = \begin{cases} 1, & k = k_0 \\ 0, & k \neq k_0 \end{cases}$$

Note that it follows from the definition of the discrete-time impulse delta signal that the following property holds

$$f[k]\delta[k] = f[0]\delta[k] \quad \text{or} \quad f[k]\delta[k-k_0] = f[k_0]\delta[k-k_0]$$

The discrete-time impulse delta signal and its shifted version are presented in Figure 2.14.

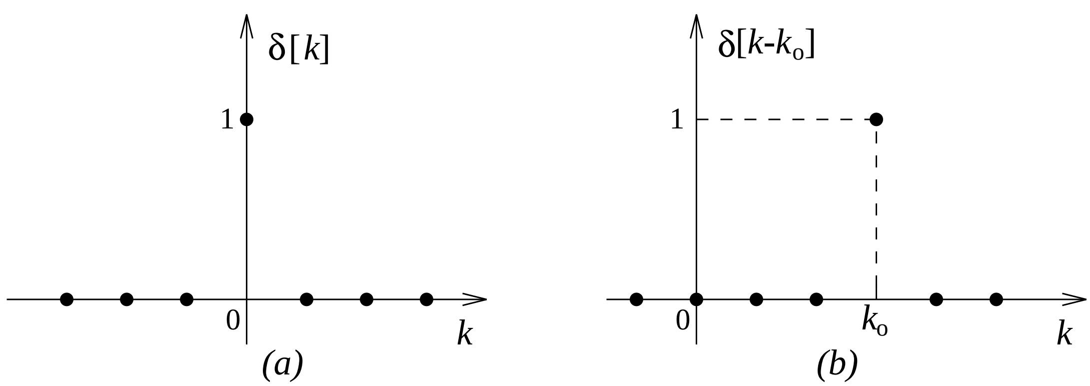

Figure 2.14: Discrete-time impulse delta signal (a) and its shifted version (b)

Generalized Derivatives

The impulse delta function can be used to define the *generalized derivative*. At the point of jump discontinuity ($f\left(t_1^+\right) \neq f\left(t_1^-\right)$) the function $f(t)$ has no derivative in the ordinary sense. From the geometric point of view, since the derivative stands for a slope of the tangent at the given point, t_1, we can say that the derivative at the point of jump discontinuity is equal to infinity. Since the shifted impulse delta signal $\delta(t - t_1)$ is equal to infinity at $t = t_1$, we can use the impulse delta function in order to define the generalized derivative.

Definition 2.1: Consider a function $f(t)$ that has jump discontinuities at the points $t_1, t_2, ...t_j$. The generalized derivative of $f(t)$ is defined by

$$\frac{Df(t)}{Dt} = \sum_{i=1}^{j} \left(f\left(t_i^+\right) - f\left(t_i^-\right)\right)\delta(t - t_i) + \frac{df(t)}{dt}\Big|_{t \neq t_1, t_2, \ldots, t_j}$$

where $Df(t)/Dt$ stands for the generalized derivative, and df/dt is the ordinary derivative at the points where it exists.

Example 2.7: Find the generalized derivative of the signal in Figure 2.15.

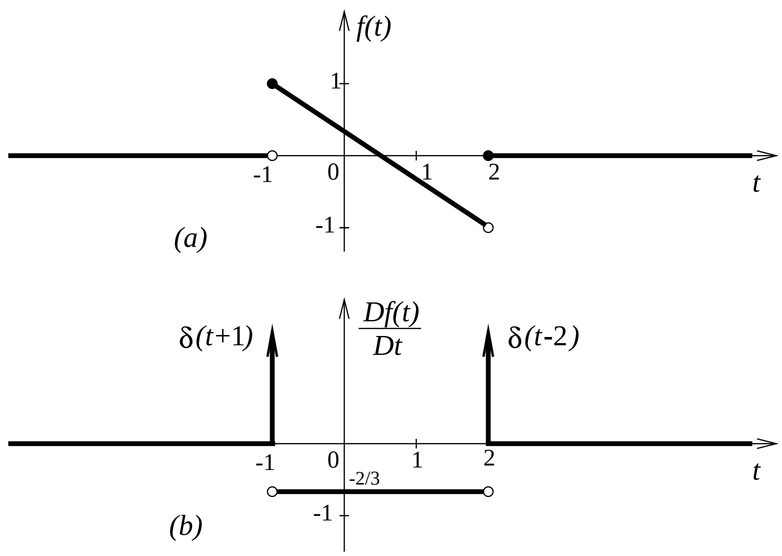

Figure 2.15: Continuous-time signal and its generalized derivative

In this example we have two points of jump discontinuities at -1 and 2 so that the generalized derivative according to Definition 2.1 is obtained as

$$\frac{Df(t)}{Dt} = \left[f\left(-1^{+}\right) - f\left(-1^{-}\right)\right]\delta(t+1) + \left[f\left(2^{+}\right) - f\left(2^{-}\right)\right]\delta(t-2)$$

$$+\frac{df(t)}{dt}[u(t+1) - u(t-2)] = \delta(t+1) + \delta(t-2) - \frac{2}{3}[u(t+1) - u(t-2)]$$

2.2 Signal Operations

Signals are mathematical functions. All known mathematical operations with functions are applicable to signals. A very important signal operation is convolution.

Definition 2.2: Continuous-Time Convolution

For continuous-time signals $g(t)$ and $v(t)$, the convolution is defined by

$$g(t) * v(t) = \int_{-\infty}^{\infty} v(\tau)g(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} v(t-\tau)g(\tau)d\tau = v(t) * g(t), \qquad -\infty \leq t \leq \infty$$

where the star denotes the convolution operator. The equality of two formulas in can be easily established by using a simple change of variables. Note that in the convolution integral t is a parameter and τ is a dummy variable of integration.

This formula also states the commutativity property of the convolution. Other properties of the continuous-time convolution follows from the properties of integrals. They will be studied in detail in Chapter 6. The use of convolution in the analysis of linear time invariant systems will be considered in Chapters 3–6, 8.

Definition 2.3: Discrete-Time Convolution

Given discrete-time signals $g[k]$ and $v[k]$, the discrete-time convolution is defined by

$$g[k] * v[k] = \sum_{m=-\infty}^{m=\infty} g[m]v[k-m]$$

$$= \sum_{m=-\infty}^{m=\infty} g[k-m]v[m] = v[k] * g[k], \qquad -\infty \leq k \leq \infty$$

where k is a parameter and m is a dummy variable of summation.

Signal Correlation

In Chapters 9 and 10 we will introduce an operation on signals that is important for both digital signal processing and communication systems known as the *signal correlation*. The signal correlation has the similar form to the signal convolution in both continuous- and discrete-time domains, but it has completely different physical meaning. The signal correlation will be used to determine the energy distribution in the signal. Continuous- and discrete-time signal correlations are respectively defined by

$$R_{vg}(t) = \int_{-\infty}^{\infty} v(\tau)g(\tau + t)d\tau, \qquad -\infty \leq t \leq \infty$$

and

$$R_{vg}[k] = \sum_{m=-\infty}^{m=\infty} v[m]g[m+k], \qquad -\infty \leq k \leq \infty$$

Definition 2.4: Forward Difference (Discrete-Time "Derivative")

The discrete time counter part of the derivative is the forward difference. Consider a discrete-time signal $f[k]$ defined in some discrete-time interval $k \in [k_1 \quad k_2]$. Then, the forward difference (discrete-time "derivative") of $f[k]$ in the given interval is defined by

$$\Delta f[k] = f[k+1] - f[k]$$

This definition can be justified by using the following reasoning. The continuous-time derivative geometrically represents the slope of the tangent at the given point, hence the derivative can be approximated as

$$\frac{df(t)}{dt} \approx \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Taking $t = k\Delta t$, where k is an integer, we have

$$\Delta t \frac{df(k\Delta t)}{dt} \approx f((k+1)\Delta t) - f(k\Delta t)$$

It can be seen that the approximation of the continuous-time derivative is proportional to the forward difference. In this textbook, we will call the forward difference "the discrete-time derivative".

Example 2.9: Using definitions of the discrete-time unit step and ramp signals given in (2.5) and (2.7), it can be observed that

$$\Delta r[k] = r[k+1] - r[k] = u[k]$$

that is, the discrete-time unit step signal is the "discrete-time derivative" of the discrete-time unit ramp signal.

Example 2.10: It is easy to see that

$$\Delta u[k-1] = u[k] - u[k-1] = \left\{ \begin{matrix} 1, & k = 0 \\ \\ 0, & k \neq 0 \end{matrix} \right\} = \delta[k]$$

The discrete-time unit step signal can be expressed in terms of the discrete-time delta impulse signal as follows

$$u[k] = \sum_{m=-\infty}^{k} \delta[m]$$

where m is a dummy variable of summation. Since $\delta[m]$ is equal to one only for $m = 0$ and equal to zero for all other m's we see that if $k < 0$ the infinite summation does not include the delta impulse at zero, thus the sum is equal to zero. If $k \geq 0$ the summation will be always equal to one since the delta impulse at zero (the only signal equals to one) is included within the limits of summation.

Similarly, we can establish the following relationship

$$r[k] = \sum_{m=-\infty}^{k-1} u[m]$$

Note that the upper limit is $k - 1$, not k. Why? Iterate.

2.3 Signal Classification

Like mathematical functions signals can be classified as: periodic or aperiodic (nonperiodic), even or odd, real or complex, continuous-time or discrete-time, deterministic or stochastic (random), sinusoidal, exponential, and so on.

Let us review here that a *periodic signal* satisfies

$$f(t) = f(t + T_p), \qquad T_p < \infty$$

for all t and some T_p, where T_p is the period after which the signal repeats itself. For example, for sine and cosine functions $T_p = 2\pi$.

Even signals are symmetrical with respect to the vertical axis, that is

$$f(-t) = f(t)$$

For example, $\cos(t)$ and $\mathrm{sinc}(t)$ are even signals.

Odd signals are symmetrical with respect to the origin, hence they satisfy

$$f(-t) = -f(t)$$

For example, $\sin(t)$ is an odd signal.

Continuous-Time Signal Energy

Recall from elementary electrical engineering courses that the electrical energy developed on a resistor is proportional either to the square of the constant current through the resistor or to the square of the constant voltage on the resistor. Hence, the square of the signal serves as a measure of signal energy. In the case when the signal changes in time, we have to integrate (or sum in the case of discrete-time signals) over given time period of interest. For example, in the case of a time varying current $i(t)$, the energy developed (dissipated as heat) on the resistor during the time interval from t_1 to t_2 is given by

$$E_{[t_1,t_2]} = R\int_{t_1}^{t_2} i^2(t)dt$$

Definition 2.5: The *continuous-time signal energy* over the time interval $[t_1, t_2]$ of length $L = t_2 - t_1$ is defined by

$$E_L = \int_{t_1}^{t_2} |f(t)|^2 dt$$

The *total continuous-time signal energy* is given by

$$E_\infty = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

Note that these definitions are general and hold even for complex signals, in which case $f(t)f^*(t) = |f(t)|^2$, where $f^*(t)$ denotes the complex conjugate signal. In the case of real signals, the absolute values can be removed.

Continuous-Time Signal Power

Recall from elementary physics that the power is work (energy) over time (speed of work). In order to get the expression for the average signal power we must divide the corresponding expression for energy by the length of the time interval so that we have the following definition.

Definition 2.6: The *continuous-time signal average power* is defined by

$$P_\infty = \lim_{L\to\infty} \frac{1}{L} \int_{-L/2}^{L/2} |f(t)|^2 dt$$

Discrete-Time Signal Energy and Power

Definition 2.7: The *discrete-time signal energy* over the time interval $[k_1, k_2]$ of length $M = k_2 - k_1$ is defined by

$$E_M = \sum_{k=k_1}^{k=k_2} |f[k]|^2$$

The *total discrete-time signal energy* is given by

$$E_\infty = \sum_{k=-\infty}^{k=\infty} |f[k]|^2$$

Definition 2.8: The *discrete-time signal average power* is defined by

$$P = \lim_{M \to \infty} \frac{1}{2M+1} \sum_{k=-M}^{k=M} |f[k]|^2$$

Energy Signals and Power Signals

Based on their energy and power, signals are classified as follows:

(1) *Energy signals* have finite total energy, $E_\infty < \infty$, and zero average power, $P_\infty = 0$.

(2) *Power signals* have infinite total energy and finite average power, that is, $E_\infty = \infty$, $P_\infty < \infty$.

For example, the rectangular and triangular pulses are energy signals. Periodic signals have infinite energy and very often finite average power, thus, in most cases periodic signals are power signals.

Causal and Anticausal signals

Causal signals satisfy $f(t) = 0$ for all $t < 0$. If a signal is not causal, that is, if $f(t) \neq 0$ for some $t < 0$, the signal is called *anticausal*. Similarly, discrete-time signals $f[k] = 0$ for $k < 0$ are causal, otherwise signals are anticausal. Anticausal signals are common in signal processing. Signals encountered in real world dynamic systems are causal.