2.3 The Existence and Uniqueness Theorem.

Suppose that $f(x,y)$ is continuous on the domain $\mathscr{D}$ and satisfies y-Lipschitz condition

$$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2| \qquad \forall (x,y_1),(x,y_2) \in \mathscr{D}.$$

We already know in this case that a solution passing through any given $(x_0,y_0) \in \mathscr{D}$ exists by Peano's Theorem, and is unique by Osgood's Theorem.

Theorem 4 (Existence and Uniqueness Theorem)**.** *Consider the initial value problem*

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}.$$

Let $\mathscr{D}$ be an open set in $\mathbb{R}^2$ that contains (x_0,y_0) and assume that $f : \mathscr{D} \to \mathbb{R}$ is continuous in t and Lipschtiz in y with Lipschitz constant K. Then there exists $a > 0$ so that the initial value problem has a solution on $(x_0 - a, x_0 + a)$ and this solution is unique.

We'll prove existence in two different ways and will prove uniqueness in two different ways. The first existence proof is constructive: we'll use a method of successive approximations — the Picard iterates — and we'll prove they converge to a solution. The second existence proof uses a fixed-point argument. Then we'll finish up by presenting two different proofs of uniqueness.

We start by relating the initial value problem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

to an *integral equation* instead of a differential equation. Namely, if we can find a continuous function y on $[\alpha,\beta]$ where $x_0 \in (\alpha,\beta)$ such that y satisfies

$$y(x) = y_0 + \int_{x_0}^{x} f(t,y(t))\,dt \tag{2.3.1}$$

then y on (α,β) is a solution of the initial value problem. Why? We know $f(x,y)$ is continuous hence $f(t,y(t))$ is continuous on $[\alpha,\beta]$. The fundamental theorem of Calculus then implies $y'(x) = f(x,y(x))$ for all $x \in (\alpha,\beta)$. Evaluating (2.3.1) at $x = x_0$ yields the desired $y(x_0) = y_0$.

Proof of Existence via Picard Iterates. *This is the same proof as found on pages 734-739 of "Ordinary Differential Equations" by M. Tenenbaum and H. Pollard.*

First, choose a rectangle $\mathscr{R}'$ that is centered at (x_0,y_0) such that $\mathscr{R}' \subset \mathscr{D}$:

$$\mathscr{R}' = [x_0 - A, x_0 + A] \times [y_0 - L, y_0 + L].$$

Since f is continuous,

$$|f(x,y)| \leq M \text{ for all } (x,y) \in \mathscr{R}'$$

for some $M > 0$. Using M we define the radius of the interval of existence:

$$a = \min\left\{\frac{L}{M}, A\right\}.$$

Let $\mathscr{R}$ be the (possibly narrower) rectangle

$$\mathscr{R} = [x_0 - a, x_0 + a] \times [y_0 - L, y_0 + L].$$

We will now show that there is a continuous function y on $[x_0 - a, x_0 + a]$ that satisfies the integral formulation (2.3.1). We construct a sequence of functions on $[x_0 - a, x_0 + a]$. The first function will be the constant function

$$y_1(x) \equiv y_0.$$

Starting with $y_0(x)$, we recursively define a sequence of continuous functions[1] on $[x_0 - a, x_0 + a]$:

$$\begin{aligned} y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt, \\ y_3(x) &= y_0 + \int_{x_0}^{x} f(t, y_2(t))\, dt, \\ &\vdots \\ y_{n+1}(x) &= y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt. \end{aligned}$$

Our goal will be to show that this sequence converges uniformly to some continuous function $y(x)$ and that the function y satisfies the integral equation (2.3.1).

First of all, we show that for each n, the graph of y_n is inside the rectangle $\mathscr{R}$. Obviously, this is true for $y_1(x) \equiv y_0$. Assume the graph of y_n is inside the rectangle $\mathscr{R}$. Using the definition of y_{n+1}, we have

$$|y_{n+1}(x) - y_0| \le \left|\int_{x_0}^{x} f(t, y_n(t))\, dt\right| \le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} \left|f(t, y_n(t))\right| dt \le M|x - x_0| \le Ma \le L$$

since $|f_n(x,y)|$ is bounded by M inside $\mathscr{R}$. This shows that the graph of y_{n+1} is inside $\mathscr{R}$. By induction, the graph of y_n is inside the rectangle $\mathscr{R}$ for all n.

We now compare successive Picard iterates. For $n \ge 2$, we have

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &= \left|\int_{x_0}^{x} f(t, y_n(t))\, dt - \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt\right| \\ &\le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} |f(t, y_n(t)) - f(t, y_{n-1}(t))|\, dt \\ &\le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} K|y_n(t) - y_{n-1}(t)|\, dt, \end{aligned}$$

[1] See pages 144-146 of the third edition of *Differential Equations, Dynamical Systems, and an Introduction to Chaos* by Hirsch, Smale, and Devaney for concrete examples of the Picard iterates for $y' = y$ and for $\vec{X}' = [0, 1; -1, 0]\vec{X}$.

where in the last inequality we used the y-Lipschitz assumption on f. In short, we showed that

$$|y_{n+1}(x)-y_n(x)| \le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} K|y_n(t)-y_{n-1}(t)|\,dt \tag{2.3.2}$$

for $n \ge 2$. We will now use this inequality iteratively, and our starting points will be the inequality

$$|y_2(x)-y_1(x)| = \left|\int_{x_0}^{x} f(t,y_0)\,dt\right| \le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} |f(t,y_0)|\,dt \le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} M\,dt = M|x-x_0|.$$

Using this in (2.3.2) for $n=2$ gives

$$|y_3(x)-y_2(x)| \le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} KM|t-x_0|\,dt = KM\frac{|x-x_0|^2}{2}.$$

Now, using this in (2.3.2) for $n=3$ gives

$$|y_4(x)-y_3(x)| \le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} K^2M\frac{|t-x_0|^2}{2}\,dt = K^2M\frac{|x-x_0|^3}{3!}.$$

The pattern is becoming clear, so we now use induction on n. Assume that

$$|y_{n+1}(x)-y_n(x)| \le K^{n-1}M\frac{|x-x_0|^n}{n!}. \tag{2.3.3}$$

Plugging this into (2.3.2), we get

$$|y_{n+2}(x)-y_{n+1}(x)| \le K^nM\int_{\min\{x,x_0\}}^{\max\{x,x_0\}} \frac{|t-x_0|^n}{n!}\,dt = K^nM\frac{|x-x_0|^{n+1}}{(n+1)!}.$$

This proves that (2.3.3) holds for all $n \ge 1$. It follows that

$$|y_{n+1}(x)-y_n(x)| \le K^{n-1}M\frac{a^n}{n!} = \frac{M}{K}\frac{(Ka)^n}{n!} \qquad \forall n \ge 1, \quad \forall x \in [x_0-a, x_0+a].$$

We use this to show that we have a Cauchy sequence in the uniform norm.

$$\begin{aligned}|y_m(x)-y_n(x)| &\le |y_m(x)-y_{m-1}(x)|+|y_{m-1}(x)-y_{m-2}(x)|+\ldots|y_{n+1}(x)-y_n(x)| \\ &\le \frac{M}{K}\sum_{k=n}^{m-1}\frac{(Ka)^k}{k!} \quad \forall x \in [x_0-a,x_0+a].\end{aligned}$$

Because

$$e^{Ka} = \sum_{k=0}^{\infty}\frac{(Ka)^k}{k!} < \infty$$

we know that we can make the tail as small as we want by choosing N sufficiently large. Given $\varepsilon > 0$ there exists N_0 so that

$$N \ge N_0 \quad \Longrightarrow \quad \frac{M}{K}\sum_{k=N}^{\infty}\frac{(Ka)^k}{k!} < \varepsilon.$$

This means that for any $m,n \ge N_0$ we have $\|y_m - y_n\| < \varepsilon$ — that is $\{y_n\}$ is a Cauchy sequence in the uniform norm. It follows that the sequence converges uniformly to some continuous function y.

It remains to show that y satisfies the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

To show this, we use that

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt$$

and let $n \to \infty$. The left hand side converges to $y(x)$, so we only need to check that the integral on the right hand side converges to $\int_{x_0}^{x} f(t, y(t))\, dt$. This again follows from the Lipschitz condition (the graph of y is in $\mathscr{R} \subset \mathscr{D}$ because the graph of y_n is in $\mathscr{R}$ for all n):

$$\begin{aligned}\left| \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt - \int_{x_0}^{x} f(t, y(t))\, dt \right| &\le \int_{\min\{x, x_0\}}^{\max\{x, x_0\}} K |y_{n-1}(t) - y(t)|\, dt \\ &\le Ka \sup_{x_0 - a \le x \le x_0 + a} |y_{n-1}(x) - y(x)| = Ka \|y_{n-1} - y\| \to 0\end{aligned}$$

as $n \to \infty$, because we proved that $y_{n-1}(x)$ converges to $y(x)$ uniformly on $[x_0 - a, x_0 + a]$. This proves that y is a solution of (2.3.1), as desired. □

The above proof is similar to, but different from, the proof on pages 390-394 of of the third edition of *Differential Equations, Dynamical Systems, and an Introduction to Chaos* by Hirsch, Smale, and Devaney. They use a less-delicate method for bounding $|y_{n+1}(x) - y_n(x)|$ and, as a result, they end up studying a geometric series $\sum (aK)^n$. This leads to the additional requirement that $a < 1/K$ for the series to converge. The above proof is more careful in the bounding, leading to the series for the exponential function — it has an infinite radius of convergence and so no additional assumption is needed on a.

Also, in the above there was a second way of knowing that I could take the limit inside the integral. You just need to prove that if $y_n \to y$ uniformly and $f(x, y)$ is continuous then $f(x, y_n) \to f(x, y)$ uniformly.

Proof of Existence using Banach Fixed Point Theorem. First, choose a rectangle $\mathscr{R}'$ that is centered at (x_0, y_0) such that $\mathscr{R}' \subset \mathscr{D}$:

$$\mathscr{R}' = [x_0 - A, x_0 + A] \times [y_0 - L, y_0 + L].$$

Since f is continuous,

$$|f(x, y)| \le M \text{ for all } (x, y) \in \mathscr{R}'$$

for some $M > 0$. Using M we define the radius of the interval of existence:

$$a < \min \left\{ \frac{L}{M}, A, \frac{1}{K} \right\}.$$

Let $\mathscr{R}$ be (possibly narrower) rectangle

$$\mathscr{R} = [x_0 - a, x_0 + a] \times [y_0 - L, y_0 + L].$$

(*Note that a may be smaller in this proof than in the Picard Iterates proof because we also require that* $aK < 1$.)

We define the following subset of $C([x_0-a,x_0+a])$:

$$\mathscr{X} := \{y \in C([x_0-a,x_0+a]) \mid ||y-y_0|| \le L\}.$$

By construction, $y \in \mathscr{X}$ implies that the graph of y is contained in $\mathscr{R}$. This, in turn, implies that $|f(x,y(x))| \le M$ for all $x \in [x_0-a,x_0+a]$.

We introduce the Picard mapping $\Gamma : C([x_0-a,x_0+a]) \to C([x_0-a,x_0+a])$

$$\Gamma(y)(x) = y_0 + \int_{x_0}^{x} f(s,y(s))\, ds.$$

If we can find $y \in C([x_0-a,x_0+a])$ such that $\Gamma(y) \equiv y$ then we have found a function that satisfies (2.3.1) and have found a solution. Our strategy is:

Step 1: Show that Γ maps $\mathscr{X}$ into $\mathscr{X}$. Let $y \in \mathscr{X}$. Then

$$|\Gamma(y)(x)-y_0| = |\int_{x_0}^{x} f(s,y(s))\, ds| \le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} |f(s,y(s))|\, ds \le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} M\, ds = M|x-x_0| \le Ma \le L.$$

This is true for all $x \in [x_0-a,x_0+a]$, hence $\|y-y_0\| \le L$, as desired.

Step 2: Show that for all y and z in $\mathscr{X}$ we have

$$\|\Gamma(y)-\Gamma(z)\| \le aK\,\|y-z\|.$$

Because we chose a so that $aK < 1$, the Banach Fixed Point Theorem then implies that there exists a unique $y \in \mathscr{X}$ so that $\Gamma(y) = y$.

Let $y,z \in \mathscr{X}$. Then

$$\begin{aligned}|\Gamma(y)(x)-\Gamma(z)(x)| &= |\int_{x_0}^{x} f(s,y(s))-f(s,z(s))\, ds| \le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} |f(s,y(s))-f(s,z(s))|\, ds \\ &\le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} K|y(s)-z(s)|\, ds \le \|y-z\| \int_{\min\{x,x_0\}}^{\max\{x,x_0\}} K\, ds = M|x-x_0|\,\|y-z\| \le Ma\|y-z\|\end{aligned}$$

□

The proof via fixed point argument is quick and clean and gives us uniqueness for free. On the other hand, it may give a smaller interval of existence and it uses higher-powered machinery such as the Banach Fixed Point theorem and understanding $C([x_0-a,x_0+a])$ as a complete metric space. One of the results of the Fixed Point theorem is that one can choose any function in $\mathscr{X}$ as the first function $y_1(x)$ and then construct a sequence via $y_{n+1} = \Gamma(y_n)$ and that this sequence will converge to a fixed point of Γ (and hence a solution of the ODE). That construction is precisely the Picard Iterates that we used in the first proof except that we took $y_1(x) \equiv y_0$.

Theorem 5 (Banach Fixed Point Theorem)**.** *Let (X,d) be a non-empty complete metric space with a mapping $\Gamma : X \to X$ that satisfies*

$$d(\Gamma(x),\Gamma(y)) \le q\, d(x,y)$$

for all $x,y \in X$ for some $q \in [0,1)$. Then Γ has a unique fixed point x^* in X. This fixed point can be found as follows: choose an arbitrary $x_0 \in X$ and define a sequence via $x_{n+1} = \Gamma(x_n)$. This sequence converges to x^*.

First Uniqueness Proof. *This is the same proof as found on pages 739-740 of "Ordinary Differential Equations" by M. Tenenbaum and H. Pollard.*

Suppose we have another solution z on $[x_0-a,x_0+a]$ such that $z(x_0)=y_0$. Then the graph of z is inside $\mathscr{R}$. (*Make sure you understand why this must be true!*) Then

$$\begin{aligned} |y(x)-z(x)| &= \Big|\int_{x_0}^{x}\big(f(t,y(t))-f(t,z(t))\big)\,dt\Big| \\ &\le \int_{\min\{x,x_0\}}^{\max\{x,x_0\}}\big|f(t,y(t))-f(t,z(t))\big|\,dt \le K\int_{\min\{x,x_0\}}^{\max\{x,x_0\}}|y(t)-z(t)|\,dt \\ &\le K\int_{\min\{x,x_0\}}^{\max\{x,x_0\}}2L\,dt = 2LK|x-x_0|. \end{aligned} \tag{2.3.4}$$

We proceed by the same induction argument we did earlier for existence, by plugging the bound recursively into the integral in (2.3.4). Namely,

$$\begin{gathered} |y(x)-z(x)| \le K\int_{\min\{x,x_0\}}^{\max\{x,x_0\}}2L\,dt \Longrightarrow |y(x)-z(x)| \le 2LK|x-x_0|, \\ |y(x)-z(x)| \le K\int_{\min\{x,x_0\}}^{\max\{x,x_0\}}2LK|t-x_0|\,dt \Longrightarrow |y(x)-z(x)| \le \frac{2L(K|x-x_0|)^2}{2}, \\ \vdots \\ |y(x)-z(x)| \le K\int_{\min\{x,x_0\}}^{\max\{x,x_0\}}\frac{2LK^{n-1}|t-x_0|^{n-1}}{(n-1)!}\,dt \Longrightarrow |y(x)-z(x)| \le \frac{2L(K|x-x_0|)^n}{n!}. \end{gathered}$$

We know that $(K|x-x_0|)^n/n! \to 0$ as $n\to\infty$. If there were a point $\tilde{x}\in[x_0-a,x_0+a]$ where $|y(\tilde{x})-z(\tilde{x})|>0$ then this would cause a contradiction because for n sufficiently large the inequality would be violated. For this reason there can be no such $\tilde{x}$ and $y\equiv z$ on $[x_0-a,x_0+a]$, as desired. $\square$

Second Uniqueness Proof. Suppose we have another solution z on $[x_0-a,x_0+a]$ such that $z(x_0)=y_0$. By the same arguments as above, the inequality (2.3.4) holds.

We introduce $u(x)=|y(x)-z(x)|$ and assume $x>x_0$. In this case (2.3.4) can be written as

$$u(x) \le K\int_{x_0}^{x}u(t)\,dt.$$

If we denote the integral on the right hand side as $U(x)=\int_{x_0}^{x}u(t)\,dt$ then $U'(x)=u(x)$ and the inequality can be written as

$$U'(x) \le KU(x). \tag{2.3.5}$$

If this was equality instead of inequality, we could solve this separable differential equation to get $U(x)=U(x_0)e^{K(x-x_0)}$ for all $x\in[x_0,x_0+a]$. Recalling that $U\ge 0$ and $U(x_0)=0$, it would follow that $U\equiv 0$ on $[x_0,x_0+a]$. This would then imply $u\equiv 0$ and thus $y\equiv z$ on $[x_0,x_0+a]$, as desired.

In fact, we can use the differential inequality (2.3.5) to prove that $U(x)\le U(x_0)e^{K(x-x_0)}$ for all $x\in[x_0,x_0+a]$. This implies that $U\equiv 0$ on $[x_0,x_0+a]$ which then implies $y\equiv z$ on $[x_0,x_0+a]$, as desired.

To see that (2.3.5) implies $U(x)\le U(x_0)e^{K(x-x_0)}=0$ for all $x\in[x_0,x_0+a]$, divide $U(x)$ by $e^{K(x-x_0)}$ and compute the derivative of this ratio:

$$\big(U(x)e^{-K(x-x_0)}\big)' = U'(x)e^{-K(x-x_0)} - KU(x)e^{-K(x-x_0)} = \big(U'(x) - KU(x)\big)e^{-K(x-x_0)} \le 0.$$

Therefore, this ratio is nonincreasing and

$$U(x)e^{-K(x-x_0)} \le U(x_0)e^{-K(x_0-x_0)} = U(x_0).$$

This implies that $U(x) \le U(x_0)e^{K(x-x_0)}$, as desired.

It remains to show that $y \equiv z$ on $[x_0 - a, x_0]$. For $x < x_0$, (2.3.4) is

$$u(x) \le K\int_x^{x_0} u(t)\,dt.$$

Small modifications of the above argument lead to the desired result. □

In the above proof, we proved that $U'(x) \le KU(x)$ implies $U(x) \le U(x_0)e^{K(x-x_0)}$. This can be a useful thing to know, simply at the level of solutions of differential inequalities. In fact, it was more than we needed to finish off the uniqueness proof — the moment we knew that $0 \le U(x)e^{-K(x-x_0)} \le U(x_0) = 0$ we were done.

The second uniqueness proof is a classic method of proving uniqueness. The differential inequality is a Grönwall's Inequality. Here is a slightly more general form of Grönwall's inequality, when K is a function of x rather than a constant.

Theorem 6 (Grönwall's Inequality)**.** *Suppose that $U'(x) \le K(x)U(x)$, where $K(x)$ is continuous and $U(x)$ is differentiable for $x \ge x_0$. Then*

$$U(x) \le U(x_0)e^{\int_{x_0}^x K(t)\,dt}.$$

In other words, $U(x)$ is bounded by the solution of the differential equation $U'(x) = K(x)U(x)$.

Can you see how to generalize the previous proof to prove this theorem?

From the first proof of the existence theorem, we saw that the interval of existence $(x_0 - a, x_0 + a)$ of the solution was determined by $a = \min\{L/M, A\}$ where the rectangle $[x_0 - A, x_0 + A] \times [y_0 - L, y_0 + L]$ is contained in the region where $f(x,y)$ is continuous and Lipschitz. What does this mean in practice? What happens when the ODE is one which has a maximal interval of existence which isn't all of $\mathbb{R}$?

Let's consider the initial value problem

$$\begin{cases} y' = y^2 \\ y(x_0) = y_0 > 0 \end{cases}$$

In this case, $f(x,y) = y^2$; this is continuous on $\mathbb{R}^2$ and is Lipschitz on infinite horizontal strips $\mathbb{R} \times [C,D]$ where $C, D \in \mathbb{R}$. For this reason, we can take the rectangle $[x_0 - A, x_0 + A] \times [y_0 - L, y_0 + L]$ in the existence proof to be whatever size we want. Once the rectangle is fixed, the upper bound M is determined: $M = (y_0 + L)^2$. Then $a = \min\{L/(y_0 + L)^2, A\}$. Let's assume A has been taken large and so $a = L/(y_0 + L)^2$. We see that $a \to 0$ as $L \to 0$. Also, $a \to 0$ as $L \to \infty$. The largest interval of existence guaranteed by the existence proof occurs when we take L to be "just right". Specifically, $L = y_0$; this results in $a_{\text{opt}} = 1/(4y_0)$.

We now consider
$$\begin{cases} y' = y^2 \\ y(x_0) = 1 \end{cases} \implies y = \frac{1}{1-x}, \quad x \in (-\infty, 1)$$
Applying the existence and uniqueness theorem once, we get a solution on $(x_0 - a_0, x_0 + a_0) = (-1/4, 1/4)$. We'd like to continue the solution to the right and so we consider
$$\begin{cases} y' = y^2 \\ y(1/4) = 1/(1-1/4) = 4/3 \end{cases}$$
That is, we're applying the existence theorem about the point $(x_1, y_1) = (1/4, 4/3)$. The interval of existence is $(x_1 - a_1, x + 1 + a_1)$ where the optimal choice of a is $a_1 = 3/16$. *Pause to think here! We started with a solution on* $(-1/4, 1/4)$. *We then found a solution on* $(1/4 - 3/16, 1/4 + 3/16)$. *Because these two intervals overlap and because we have uniqueness of solutions, we know that the first solution equals the second solution in the overlap region. This allows us to use the two solutions to create a solution on* $(-1/4, 1/4 + 3/16)$ *via "use the first solution on* $(-1/4, 1/4)$ *and use the second solution on* $[1/4, 1/4 + 3/16)$.

We'd like to continue the solution further to the right and so we consider
$$\begin{cases} y' = y^2 \\ y(7/16) = 1/(1-7/16) = 16/9 \end{cases}$$
That is, we're applying the existence theorem about the point $(x_2, y_2) = (7/16, 16/9)$. The interval of existence is $(x_2 - a_2, x + 2 + a_2)$ where the optimal choice of a is $a_2 = 9/64$.

We started with a solution on $(-1/4, 1/4)$. At this point, we've continued the solution to the right twice and have a solution on $(-1/4, 37/64)$. We can keep doing this. We'll find
$$x_k = 1 - (3/4)^k, \qquad y_k = (4/3)^k, \qquad \text{and} \qquad a_k = 1/4\,(3/4)^k.$$
As $k \to \infty$ we have the interval of existence shrinking ($a_k \to 0$) and $x_k \uparrow 1$ as $y_k \uparrow \infty$. However many times we use the existence theorem to continue the solution to the right, we can't get past $x = 1$.

<u>Caveat</u> The above is a circular argument. I knew the exact solution $y(x) = 1/(1-x)$ and I used this exact solution in evaluating the y_k that went into the initial value problems. These then went into the a_k which then led to the x_k which then failed to get past $x = 1$ as $k \to \infty$. In some sense it's a bit, "Put rabbit into hat, reach into hat, Look! A rabbit!"

But the real point is: if we find ourselves studying
$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$
and we're able to find δ so that $\delta < a$ for all (x_0, y_0) then this implies that the maximal interval of existence is $\mathbb{R}$. Similarly, if we're able to find an upper bound for a where the upper bound goes to zero as x_0 increases then, if the upper bound goes to zero sufficiently fast, this may imply that the solution cannot be continued past some $x = x^*$. The art is in finding such estimates when one doesn't have an explicit solution to work with.