2-D SPATIAL FUNCTIONS

General case

$$f(x, y)$$

- The function *f* describes a *surface* in space
- The *height* of the surface at (x,y) is f(x,y)

Special cases

- Separable functions
- Radial functions

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2-D SEPARABLE FUNCTIONS

- 2-D function is separable, *if* it is product of two 1-D functions, one in x and the other in y

$$f(x, y) = f_1(x)f_2(y)$$

- Allows some important operations to be done in 1-D

- Example
  - 2-D rectangle function
  - Multiply two 1-D rectangle functions

$$rect\left(\frac{x}{a}, \frac{y}{b}\right) = rect\left(\frac{x}{a}\right)rect\left(\frac{y}{b}\right)$$

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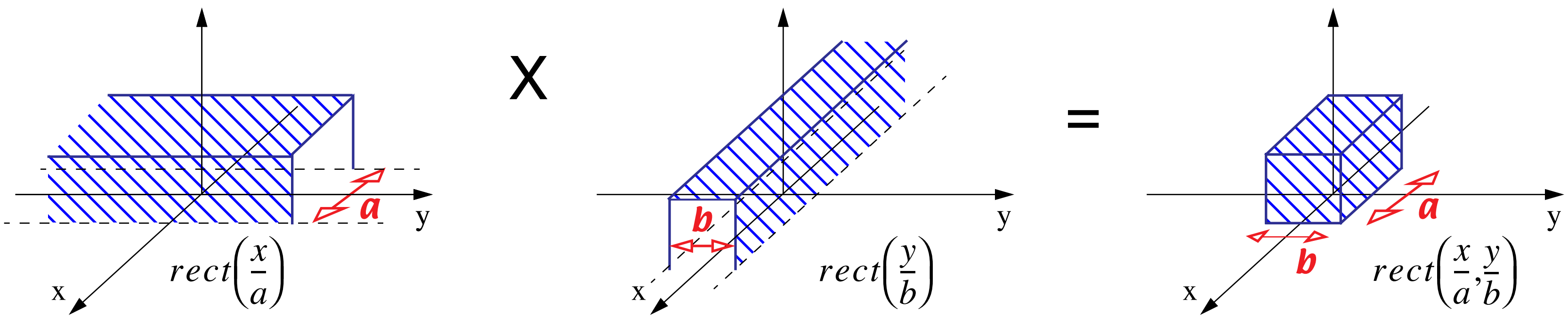
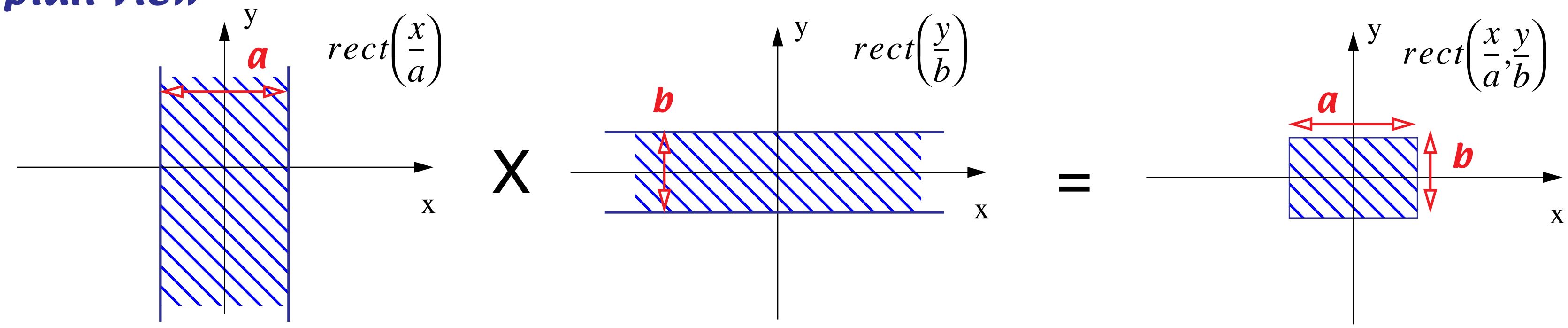
perspective view



plan view



amplitude $= 1,\ |x|<a/2,\ |y|<b/2$

$= 1/2,\ |x|=a/2,\ |y|=b/2$

$= 0,$ *otherwise*

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Some 2-D separable functions

function	separable form
$\delta(x, y)$	$\delta(x)\delta(y)$
$\delta(x - x_0, y - y_0)$	$\delta(x - x_0)\delta(y - y_0)$
$rect\left(\frac{x}{a}, \frac{y}{b}\right)$	$rect\left(\frac{x}{a}\right)rect\left(\frac{y}{b}\right)$
$\mathrm{sinc}\left(\frac{x}{a}, \frac{y}{b}\right)$	$\mathrm{sinc}\left(\frac{x}{a}\right)\mathrm{sinc}\left(\frac{y}{b}\right)$
$tri\left(\frac{x}{a}, \frac{y}{b}\right)$	$tri\left(\frac{x}{a}\right)tri\left(\frac{y}{b}\right)$
$gaus\left(\frac{x}{a}, \frac{y}{b}\right)$	$gaus\left(\frac{x}{a}\right)gaus\left(\frac{y}{b}\right)$
$comb\left(\frac{x}{a}, \frac{y}{b}\right)$	$comb\left(\frac{x}{a}\right)comb\left(\frac{y}{b}\right)$

? Draw the profiles of tri(x/a,y/a) along the two axes, x and y, and along the 45° diagonal

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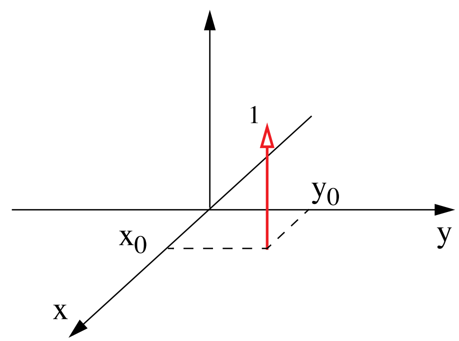
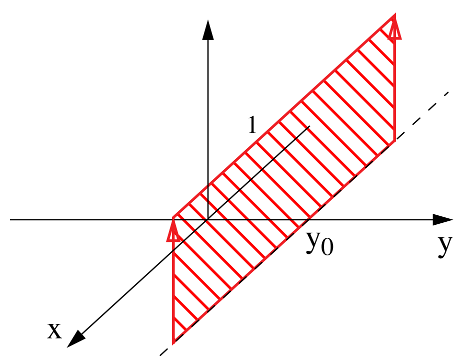
2-D IMPULSE AND RELATED FUNCTIONS

delta $\delta(x - x_0, y - y_0)$



blade $\delta(y - y_0)$



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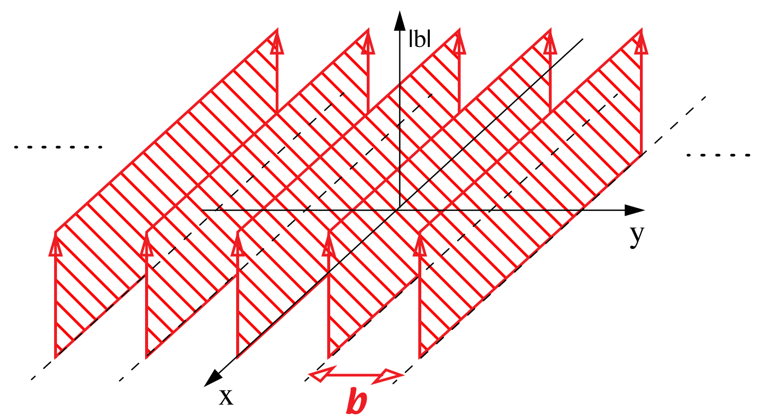
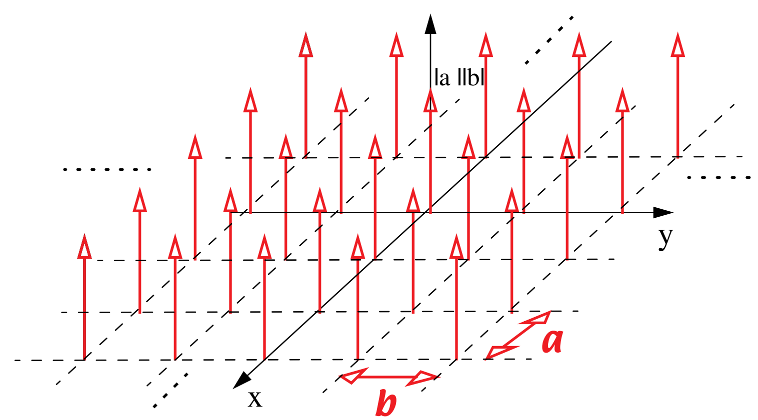
grill $comb\left(\frac{y}{b}\right)$



grid $comb\left(\frac{x}{a},\frac{y}{b}\right) = |a||b| \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x-na, y-mb)$



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2-D RADIAL FUNCTIONS

Special case

$$f(x, y) = f\left(\sqrt{x^2 + y^2}\right) = f(r)$$

- Equivalent to a 1-D function of radius, r
- Generate 2-D surface by rotating profile f(r) around origin
  - f(r) called the "generating function"

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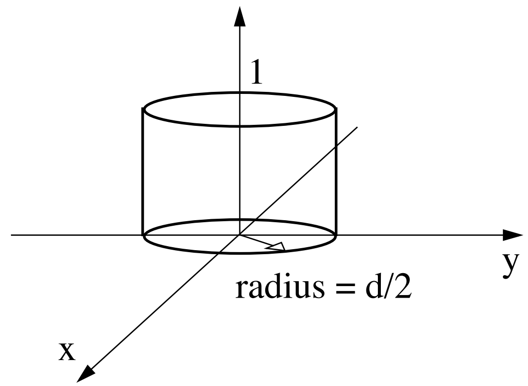
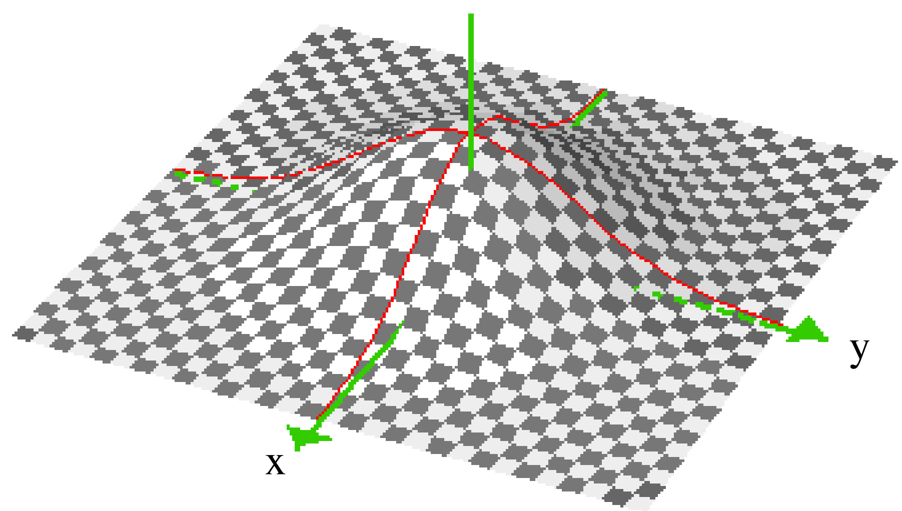
Examples of radial functions

- ***cylinder: cyl(r/d)***



- ***Gaussian:*** $gaus\left(\frac{r}{d}\right) = e^{-\pi(r/d)^2}$



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- **Sombrero** $$somb\left(\frac{r}{d}\right) = \frac{2J_1\left(\frac{\pi r}{d}\right)}{\frac{\pi r}{d}}$$
  - J_1 is a first-order Bessel function of the first kind.
  - zeros of $somb(r/d)$ are at r/d = 1.22, 2.23, 3.24, . . .

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2-D CONVOLUTION

Extension of 1-D convolution

$$g(x, y) = f(x, y) \ast\ast h(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha, \beta) h(x-\alpha, y-\beta) d\alpha d\beta$$

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Properties

- generally same as 1-D convolution (commutative, distributive, associative - see earlier notes)

2-D convolution simplifies for separable functions

- General case $g(x, y) = f(x, y) \circledast \circledast h(x, y)$

- Case I: *h* (or *f*) separable

$$
\begin{aligned}
g(x, y) &= f(x, y) \circledast \circledast [h_1(x) h_2(y)] \\
&= f(x, y) \circledast h_1(x) \circledast h_2(y)
\end{aligned}
$$

two 1-D convolutions

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- **Case II: *f* and *h* separable**

$$g(x, y) = [f_1(x) \ast h_1(x)][f_2(y) \ast h_2(y)]$$
$$= g_1(x)g_2(y)$$

g(x,y) is also separable

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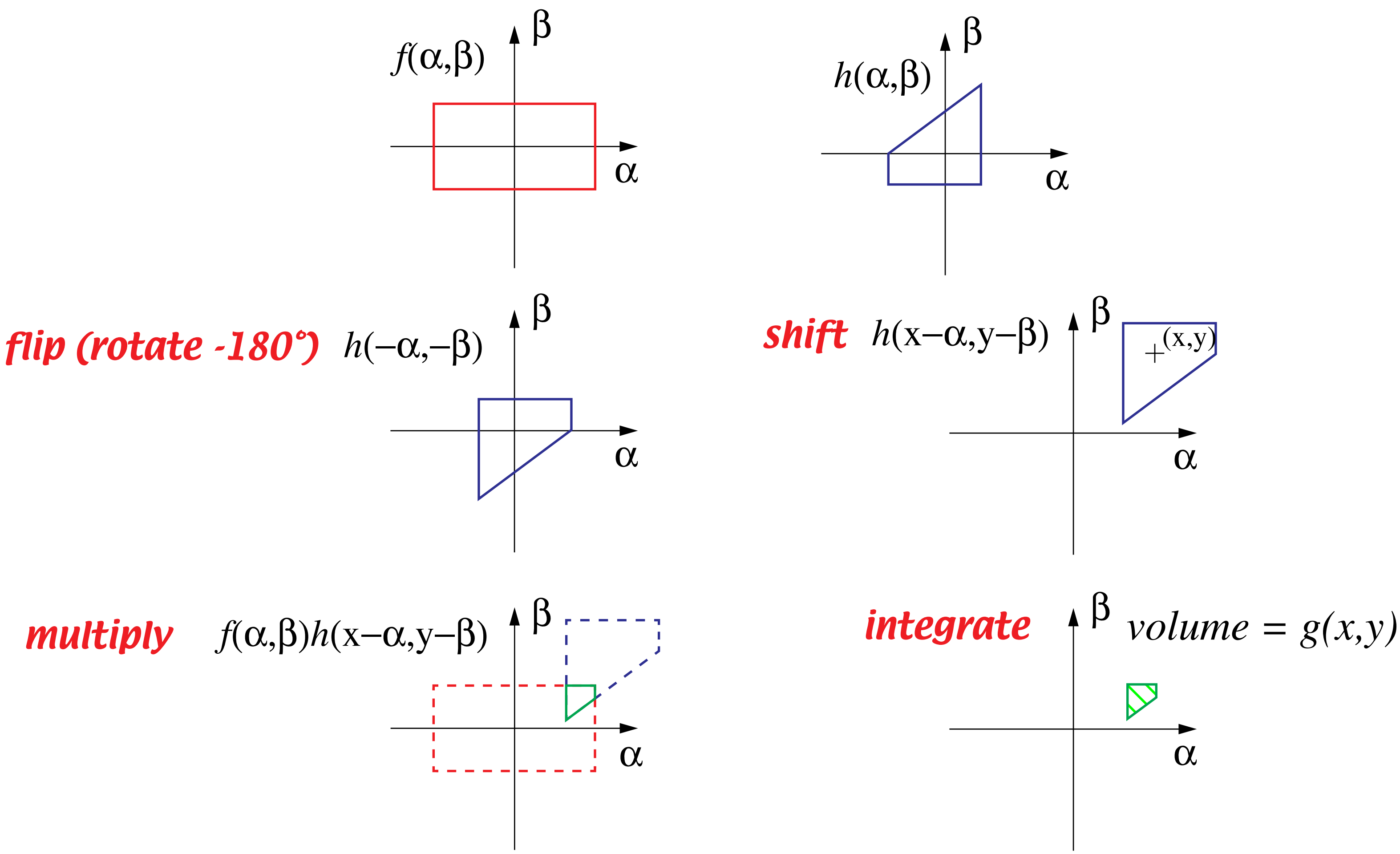
Example 2-D convolution

- ***For simple graphics, let f and h be binary "mask" functions, equal to one within the boundary and zero elsewhere:***



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2-D CORRELATION

Identical to 2-D convolution, except neither function is "flipped"

$$g(x, y) = f(x, y) \star\star h(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha, \beta)h(\alpha - x, \beta - y)d\alpha d\beta$$

Important difference

- *not* commutative

$$f \star\star h \neq h \star\star f$$

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2-D FOURIER TRANSFORMS

Forward transform

$$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(xu + yv)}dxdy$$

Inverse transform

$$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v)e^{j2\pi(xu + yv)}dudv$$

Properties

- **1-D transform properties generally also apply to 2-D transform**
- **2-D transform is separable**

$$F(u, v) = \int_{-\infty}^{\infty} e^{-j2\pi yv}\left(\int_{-\infty}^{\infty} f(x, y)e^{-j2\pi xu}dx\right)dy$$

- **Therefore, can apply as two, sequential 1-D transforms in x and y (often done this way on a computer)**

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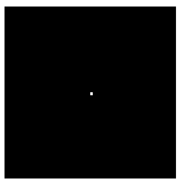
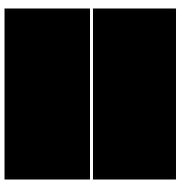
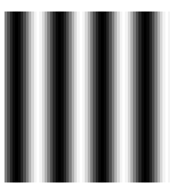
2-D FOURIER TRANSFORM PAIRS

descriptive name of f(x,y)	picture of f(x,y)	f(x,y)	F(u,v)	descriptive name of F(u,v)	picture of |F(u,v)|
delta		$\delta(x, y)$	1	constant	
blade		$\delta(x)$	$\delta(v)$	blade	
ripple		$\cos(2\pi ax)$	$\frac{1}{2\|a\|}\delta\delta\left(\frac{u}{a}\right)\delta(v)$	double delta	

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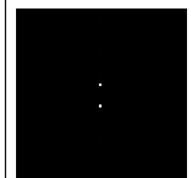
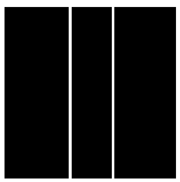
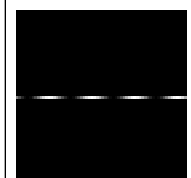
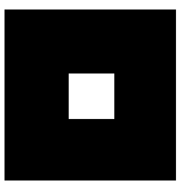
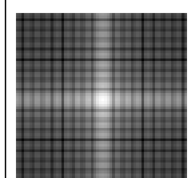
descriptive name of f(x,y)	picture of f(x,y)	f(x,y)	F(u,v)	descriptive name of F(u,v)	picture of |F(u,v)|
ripple		$\cos(2\pi ay)$	$\frac{1}{2\|a\|}\delta\delta\left(\frac{v}{a}\right)\delta(u)$	double delta	
double blade		$\frac{1}{2\|a\|}\delta\delta\left(\frac{x}{a}\right)$	$\cos(2\pi au)\delta(v)$		
box		$rect(x, y)$	$\mathrm{sinc}(u, v)$		

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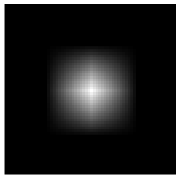
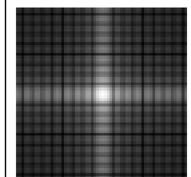
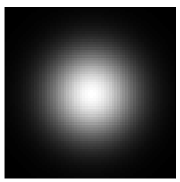
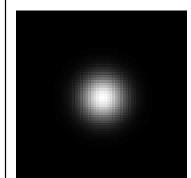
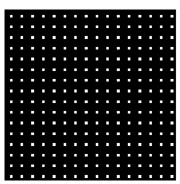
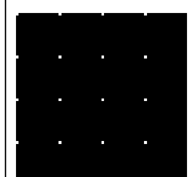
descriptive name of f(x,y)	**picture of f(x,y)**	**f(x,y)**	**F(u,v)**	**descriptive name of F(u,v)**	**picture of |F(u,v)|**
		$tri(x, y)$	$\mathrm{sinc}^2(u, v)$		
		$gaus(x, y)$	$gaus(u, v)$		
grid or bed-of-nails		$comb(x, y)$	$comb(u, v)$	grid or bed-of-nails	

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2-D FOURIER TRANSFORM PROPERTIES

name	f(x,y)	F(u,v)
separability	$f_1(x)f_2(y)$	$F_1(u)F_2(v)$
scaling	$f(x/a, y/b)$	$\|a\|\|b\|F(au, bv)$
shifting	$f(x \pm a, y \pm b)$	$e^{\pm j2\pi(au + bv)}F(u, v)$
linearity (superposition)	$af_1(x, y) + bf_2(x, y)$	$aF_1(u, v) + bF_2(u, v)$
convolution	$f_1(x, y) ** f_2(x, y)$	$F_1(u, v) \cdot F_2(u, v)$
	$f_1(x, y) \cdot f_2(x, y)$	$F_1(u, v) ** F_2(u, v)$
affine	$f(ax + by + c, dx + ey + f)$	$\frac{1}{\|\Delta\|}e^{-j\left(\frac{2\pi}{\Delta}\right)[(bf - ec)u + (dc - af)v]}F\left(\frac{eu - dv}{\Delta}, \frac{-bu + av}{\Delta}\right)$, where $\Delta = ae - db$

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? Show that the affine property reduces to the shifting property or the scaling property with appropriate parameters

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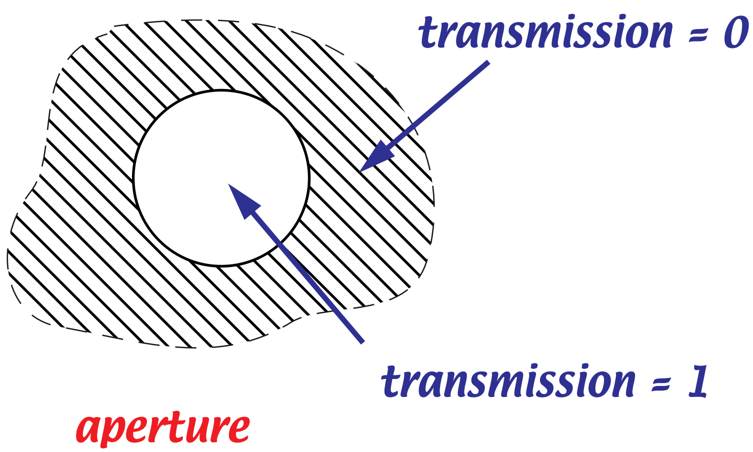
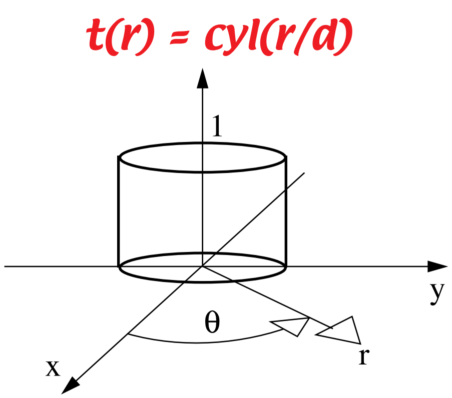
2-D FOURIER TRANSFORMS IN POLAR COORDINATES

Useful for optical apertures

- Many apertures in optics have circular symmetry
- cyl(r/d) describes the transmission function of a circular aperture





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Derive a special Fourier transform for such functions

$$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(xu + yv)} dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\sqrt{x^2 + y^2}\right) e^{-j2\pi(xu + yv)} dxdy$$

- **Substitute:**

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \quad \text{space domain} \qquad \begin{aligned} u &= \rho\cos\phi \\ v &= \rho\sin\phi \end{aligned} \quad \text{spatial frequency domain}$$

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Hankel Transform

- **2-D Fourier transform in polar coordinates**

$$F(\rho\cos\phi, \rho\sin\phi) = \int_0^{\infty} r\,dr \int_0^{2\pi} d\theta f(r) e^{-j2\pi\rho r\cos(\theta - \phi)}$$

$$= \int_0^{\infty} f(r) r\,dr [2\pi J_0(2\pi\rho r)]$$

where $J_0(\,)$ is the zero-order Bessel function of the first kind.

- **Simplifying, we have the zero-order Hankel transform,**

$$F(\rho\cos\phi, \rho\sin\phi) = 2\pi\int_0^{\infty} f(r) J_0(2\pi\rho r) r\,dr$$

- **If $f(r,\theta) = f(r)$ (circularly symmetric), then $F(\rho,\phi) = F(\rho)$**

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Inverse Hankel transform

$$f(r) = 2\pi \int_0^{\infty} F(\rho) J_0(2\pi r \rho) d\rho$$

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HANKEL TRANSFORM PROPERTIES AND PAIRS

Properties

name	f(r)	F(ρ)
scaling	$f\left(\frac{r}{b}\right)$	$\lvert b\rvert^2 F(b\rho)$
convolution	$f(r) ** h(r)$	$F(\rho)H(\rho)$

Pairs

$f(r)$	$F(\rho)$
$\frac{\delta(r)}{\pi r}$	1
$1/r$	$1/\rho$

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$f(\mathbf{r})$	$F(\rho)$
cyl(r)	$\frac{\pi}{4} somb(\rho)$
e^{-r}	$\frac{2\pi}{(4\pi^2\rho^2 + 1)^{3/2}}$
gaus(r)	gaus(ρ)

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SUMMARY: 1-D AND 2-D FUNCTIONS

Two applications

- models for optical image formation
  - continuous functions, convolution and Fourier transforms
- digital image processing
  - discrete functions, convolution and Fourier transforms

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We need both for digital imaging systems

1-D functions

- see other courses in time signals and systems
- don't worry about causality in spatial coordinates

2-D functions

- Many are extension of 1-D functions
- Separable 2-D functions are easier to manipulate
- Radial functions are special case

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- ***Summary table***

case	convolution	Fourier transform
general	$f(x, y) \circledast\circledast h(x, y)$	$F(u, v)H(u, v)$
h separable	$f(x, y) \circledast h_1(x) \circledast h_2(y)$	$F(u, v)H_1(u)H_2(v)$
both separable	$[f_1(x) \circledast h_1(x)][f_2(y) \circledast h_2(y)]$	$F_1(u)H_1(u)F_2(v)H_2(v)$

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