2. Kinematics, Decays and Reactions

Particle and Nuclear Physics

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In this section...

- Natural units
- Symmetries and conservation laws
- Relativistic kinematics
- Particle properties
- Decays
- Cross-sections
- Scattering
- Resonances

Units

The usual practice in particle and nuclear physics is to use Natural Units.

- Energies are measured in units of eV:

  Nuclear $\mathrm{keV}(10^3\ \mathrm{eV})$, $\mathrm{MeV}(10^6\ \mathrm{eV})$

  Particle $\mathrm{GeV}(10^9\ \mathrm{eV})$, $\mathrm{TeV}(10^{12}\ \mathrm{eV})$

- Masses are quoted in units of MeV/c^2 or GeV/c^2 (using $E = mc^2$)

  e.g. electron mass $m_e = 9.11 \times 10^{-31}\,\mathrm{kg} = (9.11 \times 10^{-31})(3 \times 10^8)^2\,\mathrm{J}/c^2$

  $= 8.20 \times 10^{-14}/1.602 \times 10^{-19}\ \mathrm{eV}/c^2 = 5.11 \times 10^5\ \mathrm{eV}/c^2 = 0.511\ \mathrm{MeV}/c^2$

- Atomic/nuclear masses are often quoted in unified (or atomic) mass units

  1 unified mass unit (u) = (mass of a $^{12}_{6}$C atom) / 12

  $1\,\mathrm{u} = 1\,\mathrm{g}/\mathrm{N}_A = 1.66 \times 10^{-27}\mathrm{kg} = 931.5\ \mathrm{MeV}/c^2$

- Cross-sections are usually quoted in barns: $1\mathrm{b} = 10^{-28}\,\mathrm{m}^2$.

Choose energy as the basic unit of measurement...		...and simplify by choosing $\hbar = c = 1$
Energy	GeV	GeV
Momentum	GeV/c	GeV
Mass	GeV/c^2	GeV
Time	$(\mathrm{GeV}/\hbar)^{-1}$	GeV^{-1}
Length	$(\mathrm{GeV}/\hbar c)^{-1}$	GeV^{-1}
Cross-section	$(\mathrm{GeV}/\hbar c)^{-2}$	GeV^{-2}

Reintroduce "missing" factors of $\hbar$ and c to convert back to SI units.

$\hbar c = 0.197\,\mathrm{GeV\,fm} = 1$ Energy $\longleftrightarrow$ Length

$\hbar = 6.6 \times 10^{-25}\,\mathrm{GeV\,s} = 1$ Energy $\longleftrightarrow$ Time

$c = 3.0 \times 10^{8}\,\mathrm{ms}^{-1} = 1$ Length $\longleftrightarrow$ Time

Units *Examples*

1. **cross-section $\boldsymbol{\sigma} = 2 \times 10^{-6}$ GeV^{-2} change into standard units**

   Need to change units of energy to length. Use $\hbar c = 0.197\,\mathrm{GeV fm} = 1$.

   $\mathrm{GeV}^{-1} = 0.197\,\mathrm{fm}$
   $\mathrm{GeV}^{-1} = 0.197 \times 10^{-15}\,\mathrm{m}$
   $\mathrm{GeV}^{-2} = 3.89 \times 10^{-32}\,\mathrm{m}^2$

   $$\sigma = 2 \times 10^{-6} \times (3.89 \times 10^{-32}\,\mathrm{m}^2)$$
   $$= 7.76 \times 10^{-38}\,\mathrm{m}^2$$

   And using $1\,\mathrm{b} = 10^{-28}\,\mathrm{m}^2$, $\sigma = 0.776\,\mathrm{nb}$

2. **lifetime $\boldsymbol{\tau} = 1/\Gamma = 0.5$ GeV^{-1} change into standard units**

   Need to change units of energy^{-1} to time. Use $\hbar = 6.6 \times 10^{-25}\,\mathrm{GeV\,s} = 1$.

   $\mathrm{GeV}^{-1} = 6.6 \times 10^{-25}\,\mathrm{s}$

   $$\tau = 0.5 \times (6.6 \times 10^{-25}\,\mathrm{s}) = 3.3 \times 10^{-25}\,\mathrm{s}$$

Also, can have Natural Units involving electric charge: $\epsilon_0 = \mu_0 = \hbar = c = 1$

3. **Fine structure constant** (dimensionless)

   $\alpha = \frac{\mathrm{e}^2}{4\pi\epsilon_0\hbar c} \sim \frac{1}{137}$ becomes $\alpha = \frac{\mathrm{e}^2}{4\pi} \sim \frac{1}{137}$ i.e. $e \sim 0.30(n.u.)$

Symmetries and conservation laws

The most elegant and powerful idea in physics

Noether's theorem:

every differentiable symmetry of the action of a physical system has a corresponding conservation law.

Symmetry	**Conserved current**
Time, t	Energy, E
Translational, x	Linear momentum, p
Rotational, θ	Angular momentum, L
Probability	Total probability always 1
Lorentz invariance	Charge Parity Time (CPT)
Gauge	charge (e.g. electric, colour, weak)

Lorentz invariance: laws of physics stay the same for all frames moving with a uniform velocity.
Gauge invariance: observable quantities unchanged (charge, E, v) when a field is transformed.

Nuclear reactions

Low energy, typically K.E. $\mathcal{O}(10\ \mathrm{MeV}) \ll$ nucleon rest energies.

$\Rightarrow$ non-relativistic formulae ok

Exception: always treat β-decay relativistically

$(m_e \sim 0.5\ \mathrm{MeV} < 1.3\ \mathrm{MeV} \sim m_n - m_p)$

Particle physics

High energy, typically K.E. $\mathcal{O}(100\ \mathrm{GeV}) \gg$ rest mass energies.

$\Rightarrow$ relativistic formulae usually essential.

Relativistic Kinematics *Special Relativity*

Recall the energy E and momentum p of a particle with mass m

$$E = \gamma m, \qquad |\vec{p}| = \gamma\beta m \qquad\qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c} = v$$

or $\quad \gamma = \dfrac{E}{m}, \qquad \beta = \dfrac{|\vec{p}|}{E}$ **and these are related by** $E^2 = \vec{p}^2 + m^2$

Interesting cases

- when a particle is at rest, $\vec{p} = 0, \;\; E = m$,
- when a particle is massless, $m = 0, \;\; E = |\vec{p}|$,
- when a particle is ultra-relativistic $E \gg m$, $E \sim |\vec{p}|$.

Kinetic energy (K.E., or T) is the extra energy due to motion

$$T = E - m = (\gamma - 1)m$$

in the non-relativistic limit $\beta \ll 1, \quad T = \frac{1}{2}mv^2$

Relativistic Kinematics *Four-Vectors*

The kinematics of a particle can be expressed as a four-vector, e.g.

$$p_\mu = (E, -\vec{p}), \quad p^\mu = (E, \vec{p}) \qquad \text{and} \qquad x_\mu = (t, -\vec{x}), \quad x^\mu = (t, \vec{x})$$

$\mu : 0 \rightarrow 3$

multiply by a metric tensor to raise/lower indices

$$p_\mu = g_{\mu\nu} p^\nu, \qquad p^\mu = g^{\mu\nu} p_\nu \qquad\qquad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Scalar product of two four-vectors $A^\mu = (A^0, \vec{A})$, $B^\mu = (B^0, \vec{B})$ is invariant:

$$A^\mu B_\mu = A.B = A^0 B^0 - \vec{A}.\vec{B}$$

or $$p^\mu p_\mu = p^\mu g_{\mu\nu} p^\nu = \sum_{\mu=0,3} \sum_{\nu=0,3} p^\mu g_{\mu\nu} p^\nu = g_{00} p_0^2 + g_{11} p_1^2 + g_{22} p_2^2 + g_{33} p_3^2$$

$$= E^2 - |\vec{p}|^2 = m^2 \quad \text{invariant mass}$$

$(t, \vec{x})$ and $(E, \vec{p})$ transform between frames of reference, but

$d^2 = t^2 - \vec{x}^2$ Invariant interval is constant

$m^2 = E^2 - \vec{p}^2$ Invariant mass is constant

A common technique to identify particles is to form the invariant mass from their decay products.

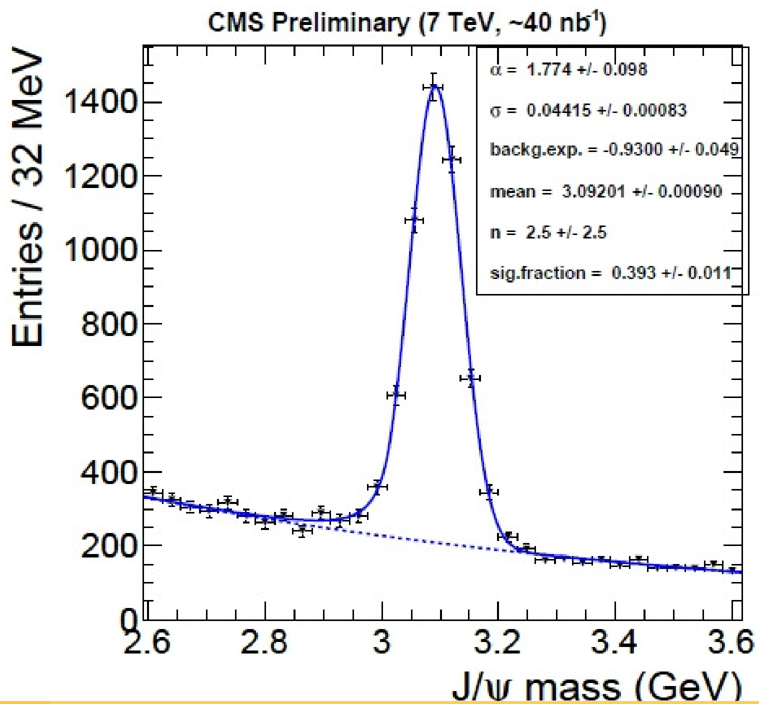
Remember, for a single particle $m^2 = E^2 - \vec{p}^2$.

For a system of particles, where $X \to 1 + 2 + 3 \ldots n$:

$$M_X^2 = ((E_1, \vec{p}_1) + (E_2, \vec{p}_2) + \ldots)^2 = \left(\sum_{i=1}^{n} E_i\right)^2 - \left(\sum_{i=1}^{n} \vec{p}_i\right)^2$$

In the specific (and common) case of a two-body decay, $X \to 1 + 2$, this reduces to

$$M_X^2 = m_1^2 + m_2^2 + 2\left(E_1 E_2 - |\vec{p}_1||\vec{p}_2| \cos\theta\right)$$



n.b. sometimes invariant mass M is called "centre-of-mass energy" E_{CM}, or $\sqrt{s}$

Relativistic Kinematics *Decay Example*

Consider a charged pion decaying at rest in the lab frame $\pi^- \to \mu^- \bar{\nu}_\mu$

Find the momenta of the decay products

How do we study particles and forces?

- **Static Properties**
  What particles/states exist?
  Mass, spin and parity (J^P), magnetic moments, bound states
- **Particle Decays**
  Most particles and nuclei are unstable.
  Allowed/forbidden decays $\rightarrow$ Conservation Laws.
- **Particle Scattering**
  Direct production of new massive particles in matter-antimatter annihilation.
  Study of particle interaction cross-sections.
  Use high-energies to study forces at short distances.

Force	Typical Lifetime [s]	Typical cross-section [mb]
Strong	10^{-23}	10
Electromagnetic	10^{-20}	10^{-2}
Weak	10^{-8}	10^{-13}

Particle Decays *Reminder*

Most particles are transient states – only a few live forever (e^-, p, ν, γ...).

- **Number** of particles remaining at time t

$$N(t) = N(0)p(t) = N(0)\mathrm{e}^{-\lambda t}$$

where $N(0)$ is the number at time $t = 0$.

- **Rate of decays** $$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N(0)\mathrm{e}^{-\lambda t} = -\lambda N(t)$$

Assuming the nuclei only decay. More complicated if they are also being created.

- **Activity** $$A(t) = \left|\frac{\mathrm{d}N}{\mathrm{d}t}\right| = \lambda N(t)$$

- It's rather common in nuclear physics to use the **half-life** (i.e. the time over which 50% of the particles decay). In particle physics, we usually quote the mean life. They are simply related:

$$N(\tau_{1/2}) = \frac{N(0)}{2} = N(0)\mathrm{e}^{-\lambda\tau_{1/2}} \quad \Rightarrow \quad \tau_{1/2} = \frac{\ln 2}{\lambda} = 0.693\tau$$

Decay Chains frequently occur in nuclear physics

$$N_1 \xrightarrow{\lambda_1} N_2 \xrightarrow{\lambda_2} N_3 \rightarrow \cdots$$

Parent *Daughter* *Granddaughter*

e.g. $^{235}\text{U} \rightarrow {}^{231}\text{Th} \rightarrow {}^{231}\text{Pa}$

$$\tau_{1/2}(^{235}\text{U}) = 7.1 \times 10^8 \text{ years}$$
$$\tau_{1/2}(^{231}\text{Th}) = 26 \text{ hours}$$

Activity (i.e. rate of decay) of the daughter is $\lambda_2 N_2(t)$.
Rate of change of population of the daughter

$$\frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = \lambda_1 N_1(t) - \lambda_2 N_2(t)$$

Units of Radioactivity are defined as the number of decays per unit time.

Becquerel (Bq) $=$ 1 decay per second

Curie (Ci) $= 3.7 \times 10^{10}$ decays per second.

Particle Decays

A decay is the transition from one quantum state (initial state) to another (final or daughter).

The transition rate is given by Fermi's Golden Rule:

$$\Gamma(i \to f) = \lambda = 2\pi \left|M_{fi}\right|^2 \rho(E_f) \qquad\qquad \hbar = 1$$

where λ is the number of transitions per unit time
M_{fi} is the transition matrix element
$\rho(E_f)$ is the density of final states.

$\Rightarrow \lambda\,\mathrm{d}t$ is the (constant) probability a particle will decay in time $\mathrm{d}t$.

Let $p(t)$ be the probability that a particle still exists at time t, given that it was known to exist at $t = 0$.

Probability for particle decay in the next time interval $\mathrm{d}t$ is $= p(t)\lambda\,\mathrm{d}t$
Probability that particle survives the next is $= p(t + \mathrm{d}t) = p(t)(1 - \lambda\,\mathrm{d}t)$

$$p(t)(1 - \lambda\,\mathrm{d}t) = p(t + \mathrm{d}t) = p(t) + \frac{\mathrm{d}p}{\mathrm{d}t}\mathrm{d}t$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p(t)\lambda$$

$$\int_1^p \frac{\mathrm{d}p}{p} = -\int_0^t \lambda\,\mathrm{d}t$$

$$\Rightarrow p(t) = \mathrm{e}^{-\lambda t} \qquad \text{Exponential Decay Law}$$

Probability that a particle lives until time t and then decays in time $\mathrm{d}t$ is

$$p(t)\lambda\,\mathrm{d}t = \lambda\mathrm{e}^{-\lambda t}\,\mathrm{d}t$$

- The average lifetime of the particle

$$\tau = \langle t \rangle = \int_0^\infty t\lambda e^{-\lambda t}\,dt = \left[-te^{-\lambda t}\right]_0^\infty + \int_0^\infty e^{-\lambda t}\,dt = \left[-\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty = \frac{1}{\lambda}$$

$$\tau = \frac{1}{\lambda} \qquad p(t) = e^{-t/\tau}$$

- Finite lifetime $\Rightarrow$ uncertain energy ΔE, (c.f. Resonances, Breit-Wigner)
- Decaying states do not correspond to a single energy – they have a width ΔE

$$\Delta E.\tau \sim \hbar \quad \Rightarrow \quad \Delta E \sim \frac{\hbar}{\tau} = \hbar\lambda \qquad \hbar = 1 \text{ (n.u.)}$$

- The width, ΔE, of a particle state is therefore
  - Inversely proportional to the lifetime τ
  - Proportional to the decay rate λ (or equal in natural units)

Decay of Resonances

QM description of decaying states

Consider a state formed at $t = 0$ with energy E_0 and mean lifetime τ

$$\psi(t) = \psi(0)\mathrm{e}^{-iE_0 t}\mathrm{e}^{-t/2\tau} \qquad\qquad |\psi(t)|^2 = |\psi(0)|^2\,\mathrm{e}^{-t/\tau}$$

i.e. the probability density decays exponentially (as required).

The frequencies (i.e. energies, since $E = \omega$ if $\hbar = 1$) present in the wavefunction are given by the Fourier transform of $\psi(t)$, i.e.

$$f(\omega) = f(E) = \int_0^\infty \psi(t)\mathrm{e}^{iEt}\,\mathrm{d}t = \int_0^\infty \psi(0)\mathrm{e}^{-t(iE_0+\frac{1}{2\tau})}\mathrm{e}^{iEt}\,\mathrm{d}t$$

$$= \int_0^\infty \psi(0)\mathrm{e}^{-t(i(E_0-E)+\frac{1}{2\tau})}\,\mathrm{d}t = \frac{i\psi(0)}{(E_0 - E) - \frac{i}{2\tau}}$$

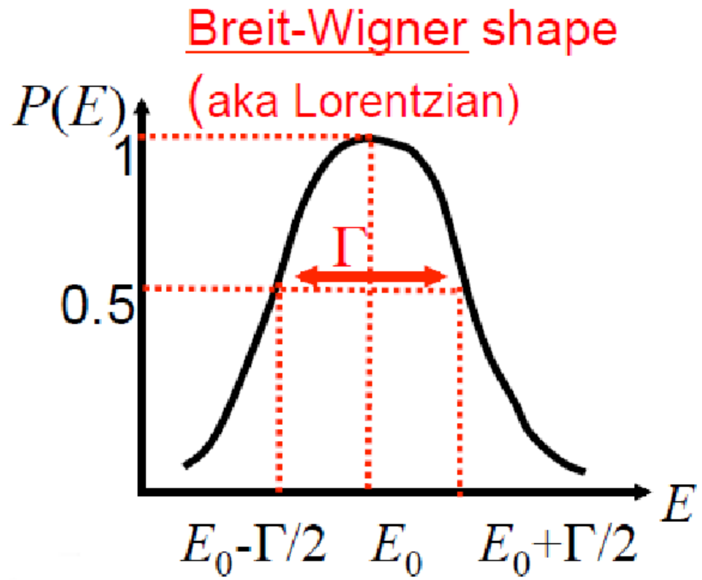
Probability of finding state with energy $E = f(E) * f(E)$ is

$$P(E) = |f(E)|^2 = \frac{|\psi(0)|^2}{(E_0 - E)^2 + \frac{1}{4\tau^2}}$$

Decay of Resonances *Breit-Wigner*

Probability for producing the decaying state has this energy dependence, i.e. resonant when $E = E_0$

$$P(E) \propto \frac{1}{(E_0 - E)^2 + 1/4\tau^2}$$



Consider full-width at half-maximum Γ

$$P(E = E_0) \propto 4\tau^2$$

$$P(E = E_0 \pm \frac{1}{2}\Gamma) \propto \frac{1}{(E_0 - E_0 \mp \frac{1}{2}\Gamma)^2 + 1/4\tau^2} = \frac{1}{\frac{\Gamma^2}{4} + \frac{1}{4\tau^2}}$$

$$P(E = E_0 \pm \frac{1}{2}\Gamma) = \frac{1}{2}P(E = E_0), \quad \Rightarrow \frac{1}{\frac{\Gamma^2}{4} + \frac{1}{4\tau^2}} = 2\tau^2$$

Total width (using natural units) $\Gamma = \frac{1}{\tau} = \lambda$

Partial Decay Widths

Particles can often decay with more than one decay mode
e.g. $Z \to e^+e^-$, or $\mu^+\mu^-$, or $q\bar{q}$ etc, each with its own transition rate,
i.e. from initial state i to final state f: $\lambda_f = 2\pi \left|M_{fi}\right|^2 \rho(E_f)$

The total decay rate is given by $\lambda = \sum_f \lambda_f$

This determines the average lifetime $\tau = \frac{1}{\lambda}$

The total width of a particle state $\Gamma = \lambda = \sum_f \lambda_f$

is defined by the partial widths $\Gamma_f = \lambda_f$

The proportion of decays to a particular decay mode is called the branching fraction or branching ratio $B_f = \frac{\Gamma_f}{\Gamma}, \quad \sum_f B_f = 1$

Reactions and Cross-sections

The strength of a particular reaction between two particles is specified by the interaction cross-section.

Cross-section σ – the effective target area presented to the incoming particle for it to cause the reaction.

Units: σ 1 barn (b) $= 10^{-28} m^2$ Area

σ is defined as the reaction rate per target particle Γ, per unit incident flux Φ

$$\Gamma = \Phi\sigma$$

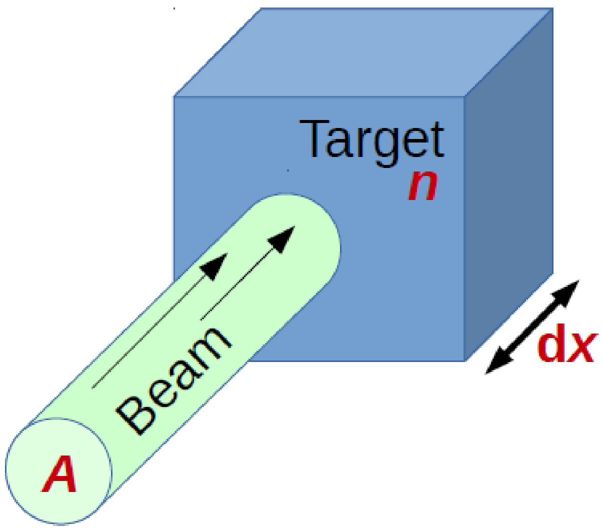
where the flux Φ is the number of beam particles passing through unit area per second.

Γ is given by Fermi's Golden Rule (previously used λ).

Scattering with a beam

Consider a beam of particles incident upon a target:

Beam of N particles per unit time in an area A



Target of n nuclei per unit volume

Target thickness $\mathrm{d}x$ is small

Number of target particles in area A, $N_T = nA\,\mathrm{d}x$
Effective area for absorption $= \sigma N_T = \sigma nA\,\mathrm{d}x$
Incident flux $\Phi = N/A$
Number of particles scattered per unit time

$$= -dN = \Phi\sigma N_T = \tfrac{N}{A}\sigma nA\,\mathrm{d}x$$

$$\sigma = \frac{-dN}{nN\,\mathrm{d}x}$$

Attenuation of a beam

Beam attenuation in a target of thickness L:

- Thick target $\sigma n L \gg 1$:

$$\int_{N_0}^{N} -\frac{dN}{N} = \int_0^L \sigma n \, \mathrm{d}x$$

$$N = N_0 \mathrm{e}^{-\sigma n L}$$

This is exact.

i.e. the beam attenuates *exponentially*.

- Thin target $\sigma n L << 1$, $\quad \mathrm{e}^{-\sigma n L} \sim 1 - \sigma n L$

$$N = N_0(1 - \sigma n L)$$

Useful approximation for thin targets.

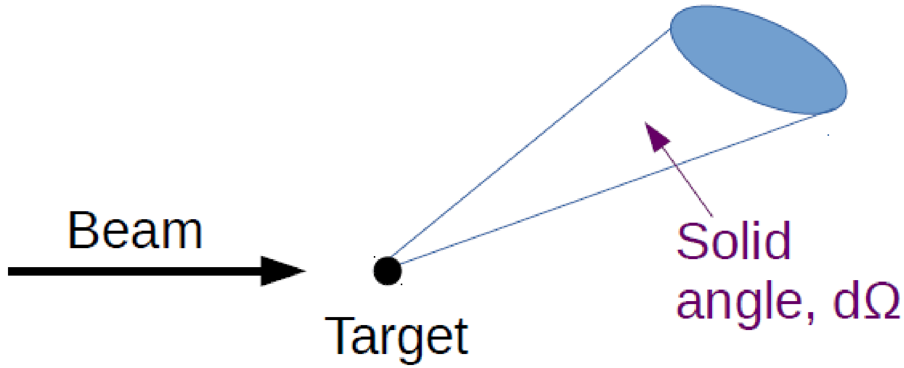
Or, the number scattered $= N_0 - N = N_0 \sigma n L$

Mean free path between interactions $= 1/n\sigma$
often referred to as "interaction length".

Differential Cross-section

The angular distribution of the scattered particles is not necessarily uniform

** n.b. $\mathrm{d}\Omega$ can be considered in position space, or momentum space **



Number of particles scattered per unit time into $\mathrm{d}\Omega$ is $\mathrm{d}N_{\mathrm{d}\Omega} = \mathrm{d}\sigma\,\Phi N_T$

Differential cross-section

units: area/steradian

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}N_{\mathrm{d}\Omega}}{(\Phi \times N_T \times \mathrm{d}\Omega)}$$

The differential cross-section is the number of particles scattered per unit time and solid angle, divided by the incident flux and by the number of target nuclei, N_T, defined by the beam area.

Most experiments do not cover 4π solid angle, and in general we measure $\mathrm{d}\sigma/\mathrm{d}\Omega$.

Angular distributions provide more information than the total cross-section about the mechanism of the interaction, e.g. angular momentum.

Partial Cross-section

Different types of interaction can occur between particles

e.g. $e^+e^- \to \gamma$, or $e^+e^- \to Z\ldots$

$$\sigma_{\text{tot}} = \sum_i \sigma_i$$

where the σ_i are called partial cross-sections for different final states.

Types of interaction

- Elastic scattering: $a + b \to a + b$

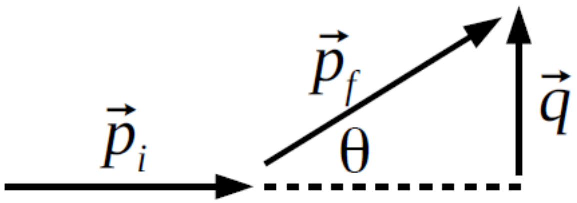
  only the momenta of a and b change

- Inelastic scattering: $a + b \to c + d$

  final state is not the same as initial state

Scattering in QM

Consider a beam of particles scattering from a fixed potential $V(r)$:



$\vec{q} = \vec{p}_f - \vec{p}_i$
"momentum transfer"

NOTE: using natural units $\vec{p} = \hbar\vec{k} \quad \rightarrow \quad \vec{p} = \vec{k}$ etc

The scattering rate is characterised by the interaction cross-section

$$\sigma = \frac{\Gamma}{\Phi} = \frac{\text{Number of particles scattered per unit time}}{\text{Incident flux}}$$

How can we calculate the cross-section?

Use Fermi's Golden Rule to get the transition rate

$$\Gamma = 2\pi |M_{fi}|^2 \rho(E_f)$$

where M_{fi} is the matrix element and $\rho(E_f)$ is the density of final states.

Scattering in QM

1^{st} order Perturbation Theory using plane wave solutions of form

$$\psi = N\mathrm{e}^{-i(Et-\vec{p}.\vec{r})}$$

Require:

1. Wave-function normalisation
2. Matrix element in perturbation theory M_{fi}
3. Expression for incident flux Φ
4. Expression for density of states $\rho(E_f)$

1. **Normalisation**

   Normalise wave-functions to one particle in a box of side L:

$$|\psi|^2 = N^2 = 1/L^3$$

$$N = (1/L)^{3/2}$$

Scattering in QM

2 **Matrix Element**

This contains the interesting physics of the interaction:

$$M_{fi} = \langle \psi_f | \hat{H} | \psi_i \rangle = \int \psi_f^* \hat{H} \psi_i \, \mathrm{d}^3\vec{r} \quad = \int N e^{-i\vec{p}_f.\vec{r}} V(\vec{r}) N e^{i\vec{p}_i.\vec{r}} \, \mathrm{d}^3\vec{r}$$

$$M_{fi} = \frac{1}{L^3} \int e^{-i\vec{q}.\vec{r}} V(\vec{r}) \, \mathrm{d}^3\vec{r} \qquad \text{where} \quad \vec{q} = \vec{p}_f - \vec{p}_i$$

3 **Incident Flux**

Consider a “target” of area A and a beam of particles travelling at velocity v_i towards the target. Any incident particle within a volume $v_i A$ will cross the target area every second.

$$\Phi = \frac{v_i A}{A} n = v_i n$$

where n is the number density of incident particles $= 1$ per L^3

Flux $=$ number of incident particles crossing unit area per second

$$\Phi = v_i / L^3$$

Scattering in QM

4. **Density of States** *also known as "phase space"*

For a box of side L, states are given by the periodic boundary conditions:

$$\vec{p} = (p_x, p_y, p_z) = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Each state occupies a volume $(2\pi/L)^3$ in p space (neglecting spin).

Number of states between p and $p + \mathrm{d}p$ in solid angle $\mathrm{d}\Omega$

$$\mathrm{d}N = \left(\frac{L}{2\pi}\right)^3 \mathrm{d}^3\vec{p} = \left(\frac{L}{2\pi}\right)^3 p^2 \,\mathrm{d}p\,\mathrm{d}\Omega \qquad (\mathrm{d}^3\vec{p} = p^2\,\mathrm{d}p\,\mathrm{d}\Omega)$$

$$\therefore \rho(p) = \frac{\mathrm{d}N}{\mathrm{d}p} = \left(\frac{L}{2\pi}\right)^3 p^2\,\mathrm{d}\Omega$$

Density of states in energy $E^2 = p^2 + m^2 \;\Rightarrow 2E\,\mathrm{d}E = 2p\,\mathrm{d}p \;\Rightarrow \frac{\mathrm{d}E}{\mathrm{d}p} = \frac{p}{E}$

$$\rho(E) = \frac{\mathrm{d}N}{\mathrm{d}E} = \frac{\mathrm{d}N}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}E} = \left(\frac{L}{2\pi}\right)^3 p^2 \frac{E}{p}\,\mathrm{d}\Omega$$

For relativistic scattering ($E \sim p$) $\qquad \rho(E) = \left(\frac{L}{2\pi}\right)^3 E^2\,\mathrm{d}\Omega$

Scattering in QM

Putting all the parts together:

$$\mathrm{d}\sigma = \frac{1}{\Phi} 2\pi \left| M_{fi} \right|^2 \rho(E_f) \quad = \frac{L^3}{v_i} 2\pi \left| \frac{1}{L^3} \int \mathrm{e}^{-i\vec{q}.\vec{r}} V(\vec{r})\, \mathrm{d}^3\vec{r} \right|^2 \left(\frac{L}{2\pi} \right)^3 p_f E_f \, \mathrm{d}\Omega$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(2\pi)^2 v_i} \left| \int \mathrm{e}^{-i\vec{q}.\vec{r}} V(\vec{r})\, \mathrm{d}^3\vec{r} \right|^2 p_f E_f$$

For relativistic scattering, $v_i = c = 1$ and $p \sim E$

Born approximation for the differential cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{E^2}{(2\pi)^2} \left| \int \mathrm{e}^{-i\vec{q}.\vec{r}} V(\vec{r})\, \mathrm{d}^3\vec{r} \right|^2$$

n.b. may have seen the *non-relativistic* version, using m^2 instead of E^2

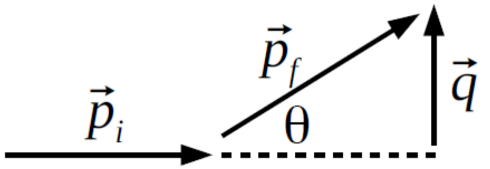
Rutherford Scattering

Consider relativistic elastic scattering in a Coulomb potential

$$V(\vec{r}) = -\frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{Z\alpha}{r}$$

Special case of Yukawa potential $V = g e^{-mr}/r$
with $g = Z\alpha$ and $m = 0$ (see Appendix C)

$$|M_{if}|^2 = \frac{16\pi^2 Z^2\alpha^2}{q^4}$$



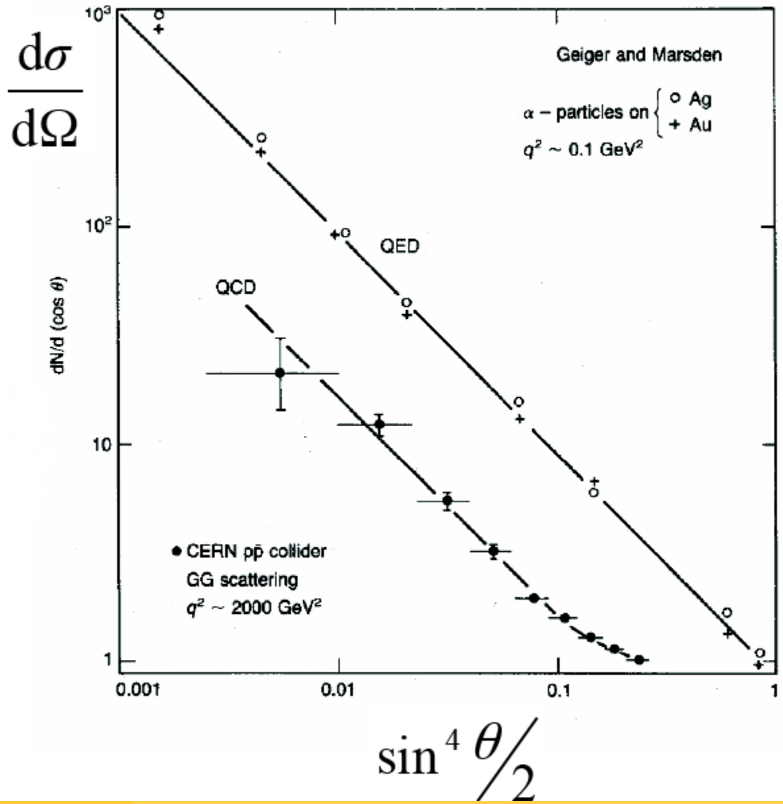
$$\vec{q} = \vec{p}_f - \vec{p}_i$$

$$|\vec{q}|^2 = |\vec{p}_i|^2 + |\vec{p}_f|^2 - 2\vec{p}_i.\vec{p}_f$$

elastic scattering, $|\vec{p}_i| = |\vec{p}_f| = |\vec{p}|$

$$= 2|\vec{p}|^2(1-\cos\theta) = 4E^2\sin^2\frac{\theta}{2}$$

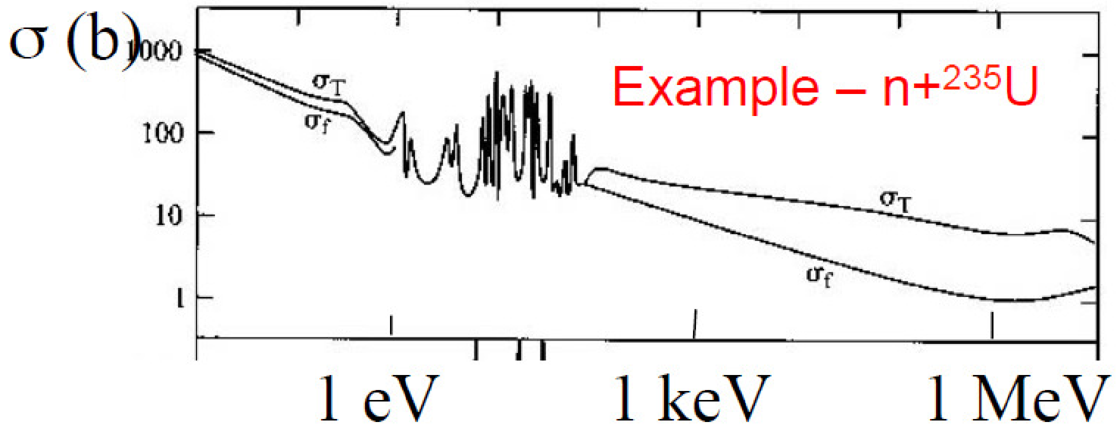
$$\frac{d\sigma}{d\Omega} = \frac{4E^2Z^2\alpha^2}{q^4} = \frac{4E^2Z^2\alpha^2}{16E^4\sin^4\frac{\theta}{2}} = \frac{Z^2\alpha^2}{4E^2\sin^4\frac{\theta}{2}}$$



Cross-section for Resonant Scattering

Some particle interactions take place via an intermediate resonant state which then decays

$$a + b \rightarrow Z^* \rightarrow c + d$$



Two-stage picture: (Bohr Model)

Formation $a + b \rightarrow Z^*$

Occurs when the collision energy $E_{CM} \sim$ the natural frequency (i.e. mass) of a resonant state.

Decay $Z^* \rightarrow c + d$

The decay of the resonance Z^* is independent of the mode of formation and depends only on the properties of the Z^*.
May be multiple decay modes.

Resonance Cross-Section

The resonance cross-section is given by

$$\sigma = \frac{\Gamma}{\Phi} \quad \text{with} \quad \Gamma = 2\pi\left|M_{fi}\right|^2 \rho(E_f)$$

$$\mathrm{d}\sigma = \frac{1}{\Phi} 2\pi \left|M_{fi}\right|^2 \rho(E_f) \quad **$$

$$= \frac{L^3}{v_i} 2\pi \left|M_{fi}\right|^2 \frac{p_f^2 L^3}{v_f (2\pi)^3}\,\mathrm{d}\Omega$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{p_f^2}{(2\pi)^2 v_i v_f}\left|M_{fi}\right|^2$$

factors of L cancel
as before, $M \propto 1/L^3$

** same as Born Approx.

$$\text{incident flux } \Phi = \frac{v_i}{L^3}$$

$$\text{density of states } \rho(p) = \frac{\mathrm{d}N}{\mathrm{d}p} = \left(\frac{L}{2\pi}\right)^3 p^2\,\mathrm{d}\Omega$$

Only need to account for $\rho(E)$ of one particle.
Energy conservation fixes the other.

$$\rightarrow \rho(E) = \frac{\mathrm{d}N}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}E} = \left(\frac{L}{2\pi}\right)^3 p^2 \frac{E}{p}\,\mathrm{d}\Omega$$

$$= \left(\frac{L}{2\pi}\right)^3 p^2 \frac{1}{v}\,\mathrm{d}\Omega$$

The matrix element M_{fi} is given by 2^{nd} order Perturbation Theory

$$M_{fi} = \sum_Z \frac{M_{iZ} M_{Zf}}{E - E_Z}$$

n.b. 2^{nd} order effects are large since $E - E_Z$ is small $\rightarrow$ large perturbation

where the sum runs over all intermediate states.

Near resonance, effectively only one state Z contributes.

Resonance Cross-Section

Consider one intermediate state described by

$$\psi(t) = \psi(0)\mathrm{e}^{-iE_0 t}\mathrm{e}^{-t/2\tau} = \psi(0)\mathrm{e}^{-i\left(E_0 - i\frac{\Gamma}{2}\right)t}$$

this describes a states with energy $= E_0 - i\Gamma/2$

$$|M_{fi}|^2 = \frac{|M_{iZ}|^2\,|M_{Zf}|^2}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$$

Rate of decay of Z:

$$\Gamma_{Z\to f} = 2\pi\,|M_{Zf}|^2\,\rho(E_f) = 2\pi\,|M_{Zf}|^2\,\frac{4\pi p_f^2}{(2\pi)^3\,v_f} = |M_{Zf}|^2\,\frac{p_f^2}{\pi v_f}$$

Rate of formation of Z:

$$\Gamma_{i\to Z} = 2\pi\,|M_{iZ}|^2\,\rho(E_i) = 2\pi\,|M_{iZ}|^2\,\frac{4\pi p_i^2}{(2\pi)^3\,v_i} = |M_{iZ}|^2\,\frac{p_i^2}{\pi v_i}$$

nb. $|M_{Zi}|^2 = |M_{iZ}|^2$.
Hence M_{iZ} and M_{Zf} can be expressed in terms of partial widths.

Resonance Cross-Section

Putting everything together: $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{p_f^2}{(2\pi)^2 v_i v_f} |M_{fi}|^2$$

$$\Rightarrow \sigma = \frac{4\pi p_f^2}{(2\pi)^2 v_i v_f} \frac{\pi v_f}{p_f^2} \frac{\pi v_i}{p_i^2} \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E-E_0)^2 + \frac{\Gamma^2}{4}} \quad = \frac{\pi}{p_i^2} \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$$

We need to include one more piece of information to account for spin...

Resonance Cross-Section

Breit-Wigner Cross-Section

$$\sigma = \frac{\pi g}{p_i^2} \cdot \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E - E_0)^2 + \frac{\Gamma^2}{4}}$$

The g factor takes into account the spin

$$\mathrm{a} + \mathrm{b} \to \mathrm{Z}^* \to \mathrm{c} + \mathrm{d}, \qquad g = \frac{2J_Z + 1}{(2J_\mathrm{a} + 1)(2J_\mathrm{b} + 1)}$$

and is the ratio of the number of spin states for the resonant state to the total number of spin states for the a+b system,
i.e. the probability that a+b collide in the correct spin state to form Z*.

Useful points to remember:

- p_i is calculated in the centre-of-mass frame (σ is independent of frame of reference!)
- $p_i \sim$ lab momentum if the target is heavy (often true in nuclear physics, but not in particle physics).
- E is the total energy (if two particles colliding, $E = E_1 + E_2$)
- Γ is the total decay rate
- $\Gamma_{Z\to i}$ and $\Gamma_{Z\to f}$ are the partial decay rates

Resonance Cross-Section *Notes*

- Total cross-section $$\sigma_{\rm tot} = \sum_f \sigma(i \to f)$$ $$\sigma = \frac{\pi g}{p_i^2} \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$$

  Replace Γ_f by Γ in the Breit-Wigner formula

- Elastic cross-section $$\sigma_{\rm el} = \sigma(i \to i)$$

  so, $\Gamma_f = \Gamma_i$

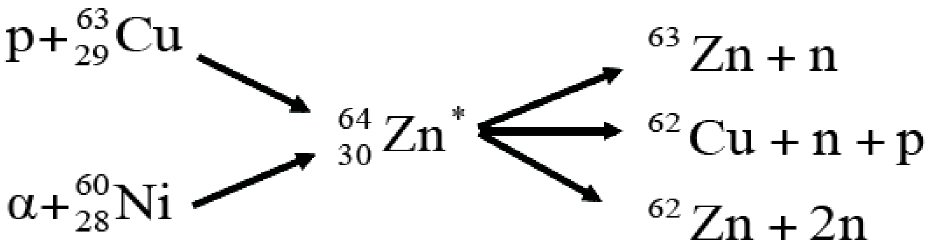
- On peak of resonance $(E = E_0)$ $$\sigma_{\rm peak} = \frac{4\pi g \Gamma_i \Gamma_f}{p_i^2 \Gamma^2}$$

  Thus $$\sigma_{\rm el} = \frac{4\pi g B_i^2}{p_i^2}, \quad \sigma_{\rm tot} = \frac{4\pi g B_i}{p_i^2}, \quad B_i = \frac{\Gamma_i}{\Gamma} = \frac{\sigma_{\rm el}}{\sigma_{\rm tot}}$$

By measuring $\sigma_{\rm tot}$ and $\sigma_{\rm el}$, can cancel B_i and infer g and hence the spin of the resonant state.

Resonances *Nuclear Physics Example*

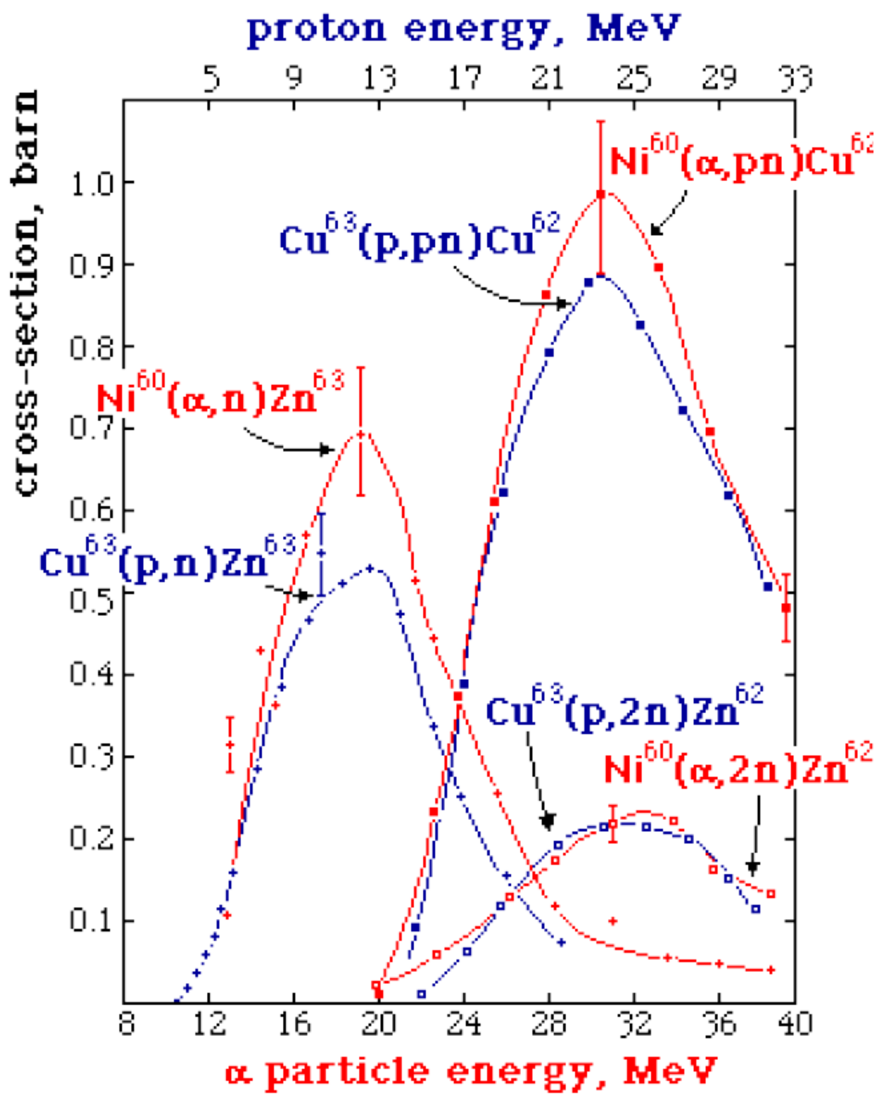
Can produce the same resonance from different initial states, decaying into various final states, e.g.



$$\sigma[^{60}\mathrm{Ni}(\alpha,n)^{63}Zn] \sim \sigma[^{63}\mathrm{Cu}(p,n)^{63}Zn]$$

n.b. common notation for nuclear reactions:

$a+A \rightarrow b+B \;\equiv\; A(a,b)B$



Energy of p selected to give same c.m. energy as for α interaction.

The Z boson

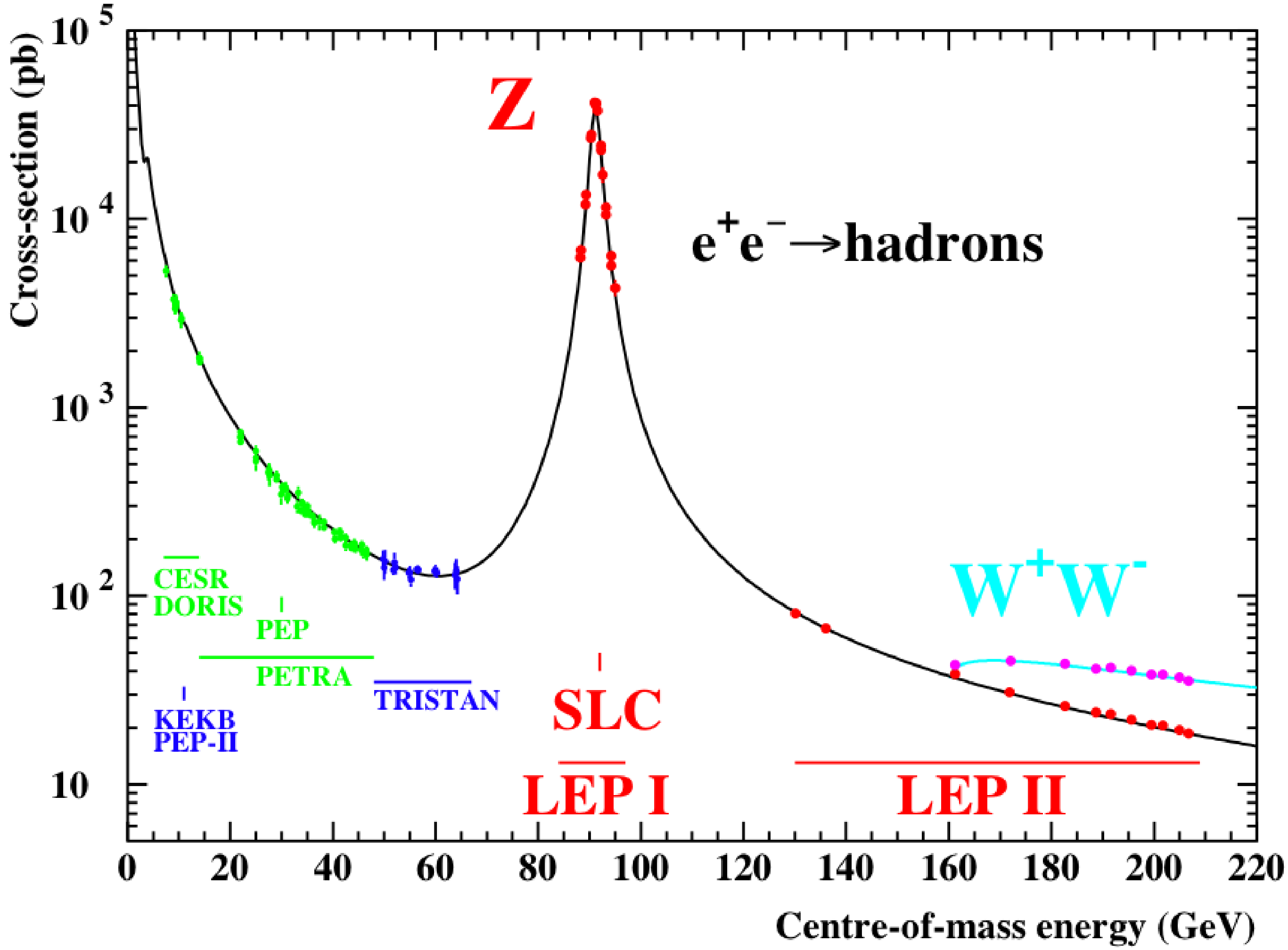
$$\Gamma_Z \sim 2.5 \text{ GeV}$$

$$\tau = \frac{1}{\Gamma_Z} = 0.4 \text{ GeV}^{-1}$$

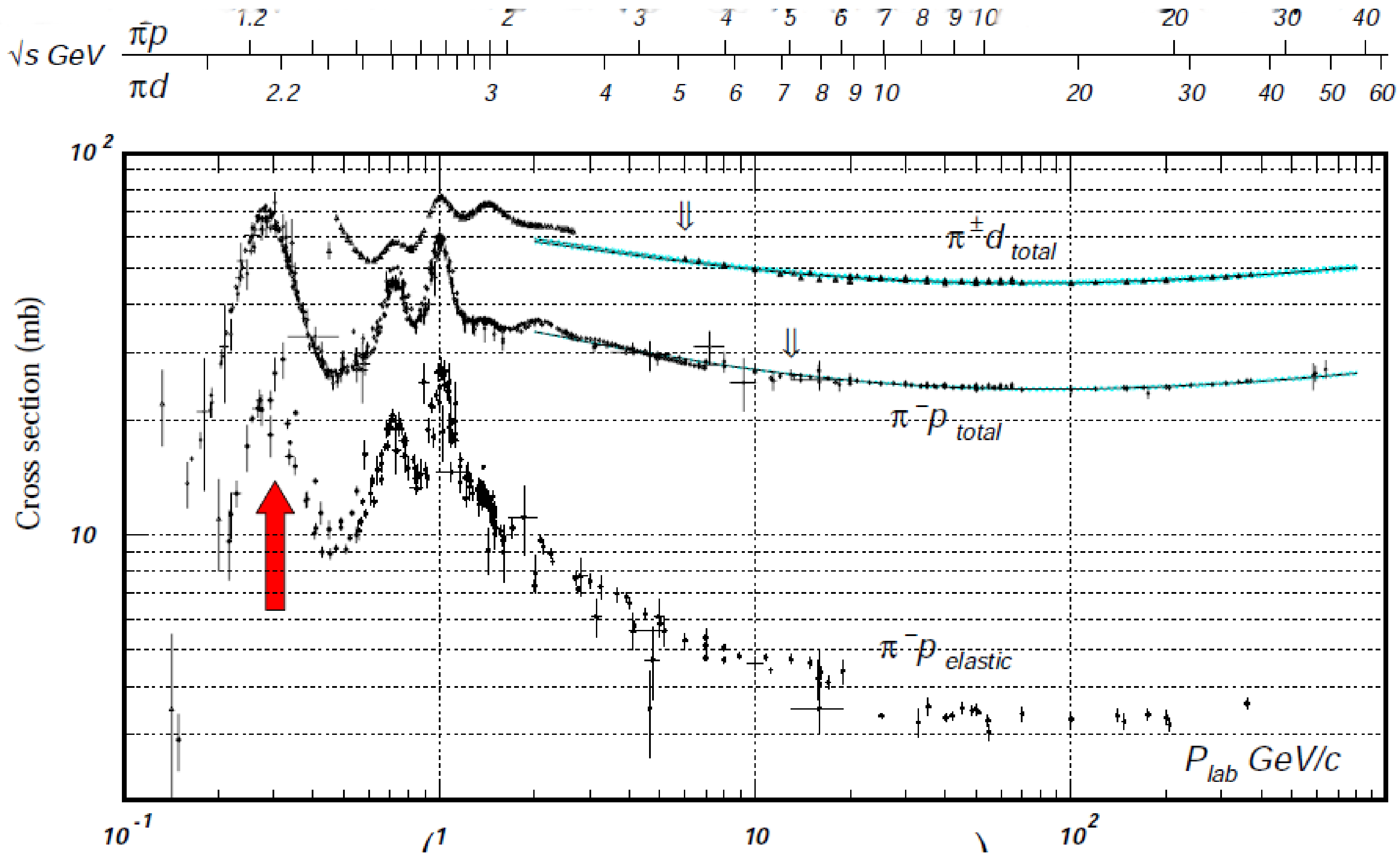
$$= 0.4 \times \hbar$$

$$= 2.5 \times 10^{-25} \text{ s}$$

$$(\hbar = 6.6 \times 10^{-25} \text{ GeV s})$$



Resonance observed at $p_\pi \sim 0.3$ GeV, $E_{CM} \sim 1.25$ GeV



$$\sigma_{\text{total}} = \sigma(\pi^- p \to R \to \text{anything}) \sim 72\,\text{mb}$$

$$\sigma_{\text{elastic}} = \sigma(\pi^- p \to R \to \pi^- p) \sim 28\,\text{mb}$$

Resonances *$\pi^- p$ scattering example*

Summary

- Units: MeV, GeV, barns
- Natural units: $\hbar = c = 1$
- Relativistic kinematics: most particle physics calculations require this!
- Revision of scattering theory: cross-section, Born Approximation.
- Resonant scattering
- Breit-Wigner formula (important in both nuclear and particle physics):

$$\sigma = \frac{\pi g}{p_i^2} \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E - E_0)^2 + \frac{\Gamma^2}{4}}$$

- Measure total and elastic σ to measure spin of resonance.

Up next...
Section 3: Colliders and Detectors