2 Sequences

2.7 Limits of Sequences & 2.8 A Discussion about Proofs

Sequences of numbers are the fundamental tool of our approach to analysis.

Definition 2.1. A *sequence* is a function s with domain $\{n \in \mathbb{Z} : n \geq m\}$ for some integer m. Alternatively, a sequence is an ordered set:

$$(s_n)_{n=m}^{\infty} = (s_m, s_{m+1}, s_{m+2}, \ldots)$$

- This is strictly the definition of an *infinite* sequence. We won't consider finite sequences.
- Most commonly $m = 0$ or 1 so that the initial term of the sequence is s_0 or s_1.
- If the domain is understood or not relevant, we might simply refer to the sequence (s_n).
- The codomain of a sequence can be any set. In elementary analysis, typically every s_n is a real number: in such a case we will say that "(s_n) is a sequence of real numbers." Towards the end of the course, se shall consider sequences of functions (e.g. examples 2 & 3 below).

Examples

1. For each $n \in \mathbb{N}$, let $s_n = \left(1 + \frac{1}{n}\right)^n$. Then

$$s_1 = 2, \quad s_2 = \frac{9}{4}, \quad s_3 = \frac{64}{27}, \quad \ldots$$

2. For each $n \in \mathbb{N}_0$, let s_n be the function $s_n : [0,1] \to \mathbb{R}$ defined by

$$s_n(x) = nx^n(1-x)$$

3. For each $n \in \mathbb{N}$, define the function $s_n : \mathbb{R} \to \mathbb{R}$ by

$$s_0 \equiv 1, \qquad s_{n+1} = 1 + \int_0^x s_n(t)\,\mathrm{d}t$$

   so that

$$s_1(x) = 1 + x, \quad s_2(x) = 1 + x + \frac{1}{2}x^2, \quad s_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3, \quad \ldots$$

Limits

We want to describe what it means for the terms of a sequence to approach arbitrarily close to some value. In a calculus class you should have become used to writing expressions such as

$$\lim_{n\to\infty} \frac{2n^2 + 3n - 1}{3n^2 - 2} = \frac{2}{3} \qquad \text{and} \qquad \lim_{n\to\infty} \sqrt{n^2+4} - n = \lim_{n\to\infty} \frac{4}{\sqrt{n^2+4}+n} = 0$$

Our first order of business is to make this logically watertight.

Definition 2.2. Let (s_n) be a sequence of real numbers and let $s \in \mathbb{R}$.

We say that (s_n) *converges to* s, if

$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N \implies |s_n - s| < \epsilon$$

We call s the *limit* of (s_n) and write $\lim s_n = s$ or simply $s_n \to s$ (read s_n *approaches* s).

We say that (s_n) *converges* if it has a limit, and that it *diverges* otherwise.

- A limit must be *finite*! We shall discuss sequences which *diverge to infinity* later.
- It is *your choice* whether to insist that N be an integer or to allow it to be a (general) real number; the definitions are equivalent.[1] Unless stated otherwise, we'll assume $N \in \mathbb{R}$. You should certainly state $N \in \mathbb{N}$ if something in your answer requires it!
- It is common but unnecessary to see $n \to \infty$ written: e.g. $\lim\limits_{n\to\infty} s_n = s$ or $s_n \xrightarrow[n\to\infty]{} s$. Feel free to do so if you feel it useful.

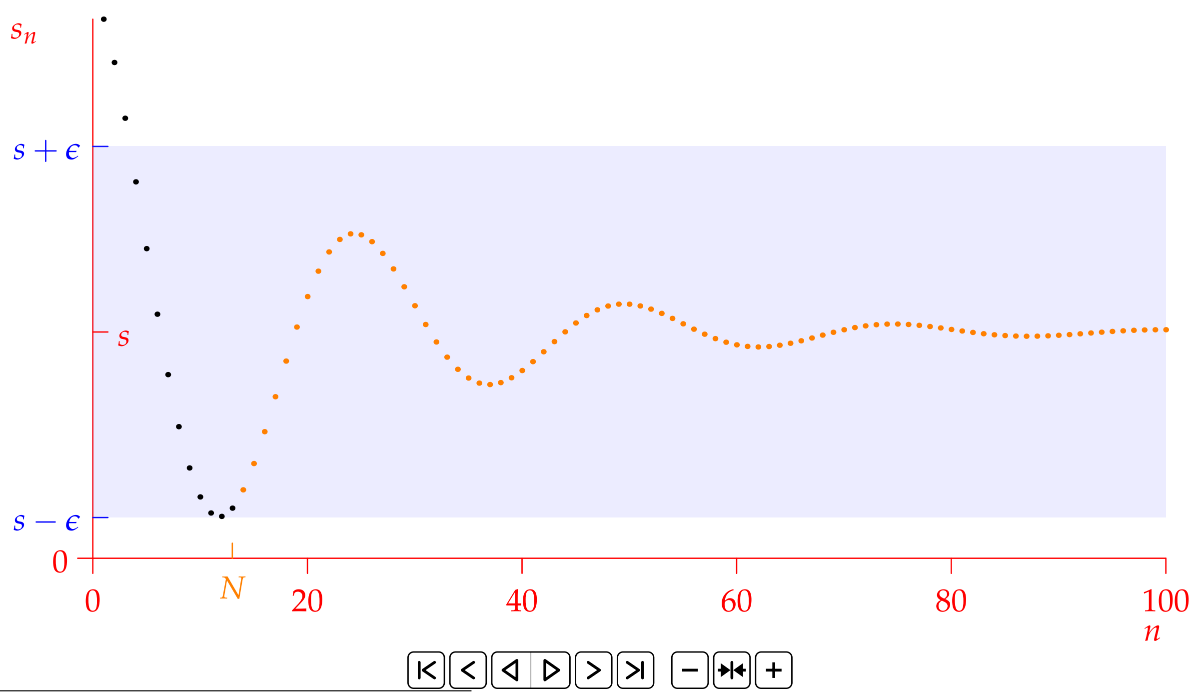
Below is a clickable version[2] of the limit definition for the sequence with n^{th} term

$$s_n = 1 + \frac{3}{2}e^{-n/20}\cos\frac{n}{4}$$

You should believe without proof that $s = \lim s_n = 1$. Try viewing the definition as a game:

Given $\epsilon > 0$, we *choose* N so that all terms s_n coming *after* N are closer to s than ϵ.

A proof amounts to a strategy that shows you will always win the game! We'll not give an explicit proof here (try it later once you've seen more examples...). Instead use the animation to help you understand that, as ϵ gets smaller, we're forced to choose N larger in order to satisfy the definition.



[1]This follows from the Archimidean principle: if $\exists N \in \mathbb{R}$ satisfying the definition, then $\exists \tilde{N} \in \mathbb{N}$ such that $\tilde{N} \geq N$. Certainly $n > \tilde{N} \implies n > N\ldots$

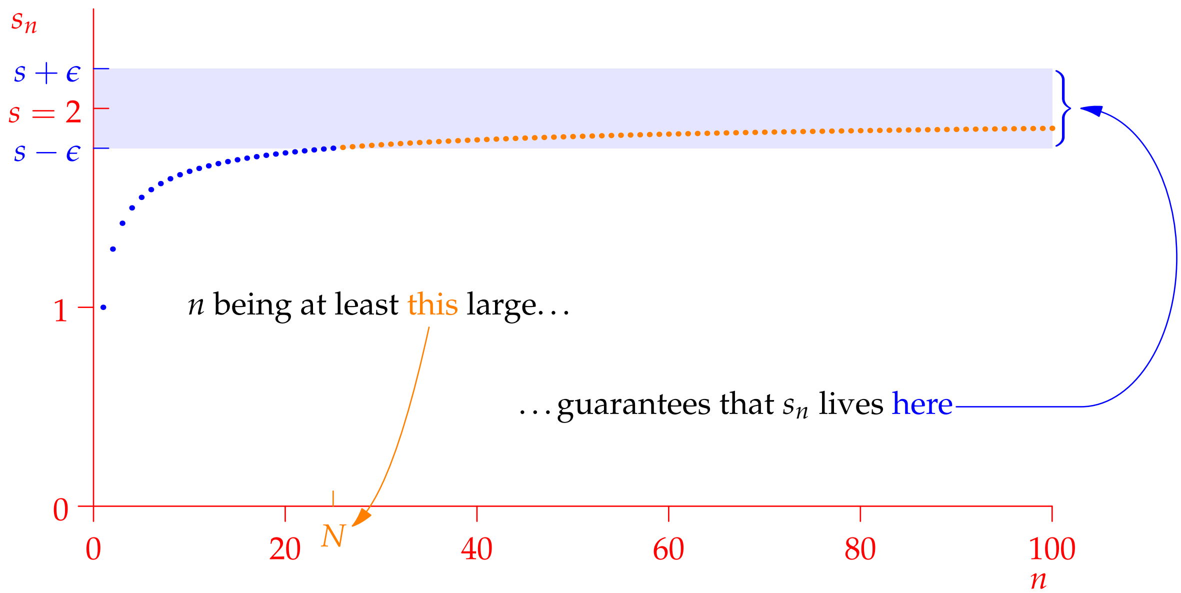
[2]If you want to the picture to move, you'll need to open these notes in a full-function pdf reader such as Acrobat. A lightweight pdf viewer or a web-broswer will likely only show a single still frame.

A Fully Worked Example

We *prove* that the sequence defined by $s_n = 2 - \frac{1}{\sqrt{n}}$ converges to $s = 2$.

The definition requires us to show that a 'for all' statement is true. Our proof should therefore have the following structure:

- Start by supposing that $\epsilon > 0$ has been given to us.
- Describe how to *choose* a number N (dependent on ϵ).
- Check (usually a direct proof with simple algebra) that if $n > N$ then $|s_n - s| < \epsilon$.



Scratch work. It is usually difficult to choose a suitable N, so it is a good idea to start with what you want to be true and let it inspire you.

- We want $n > N \implies \left|\left(2 - \frac{1}{\sqrt{n}}\right) - 2\right| < \epsilon$.
- This requires $\left|\frac{1}{\sqrt{n}}\right| < \epsilon$, which is equivalent to $n > \frac{1}{\epsilon^2}$.
- Choosing $N = \frac{1}{\epsilon^2}$ should be enough to complete the proof!

Warning! We do not yet have a proof! If your argument finishes "$\ldots \implies N = \frac{1}{\epsilon^2}$" then your conclusion is incorrect. Rearrange your scratch work to make it clear that you've satisfied the definition!

Proof. Let $\epsilon > 0$ be given. Let $N = \frac{1}{\epsilon^2}$. Then

$$n > N \implies n > \frac{1}{\epsilon^2} \implies \frac{1}{\sqrt{n}} < \epsilon$$
$$\implies |s_n - s| = \left|2 - \frac{1}{\sqrt{n}} - 2\right| < \epsilon$$

Thus $s_n \to 2$ as required. ∎

With practice, you might be able to produce a correct argument immediately for such a simple example. However, in most cases even experts expect to first need some scratch work.

Uniqueness of Limit As suggested by the definite article (…call s *the* limit…) in Definition 2.2…

Theorem 2.3. *If* (s_n) *converges, then its limit is unique.*

Proof. Suppose s and t are two limits. Take $\epsilon = \frac{|s-t|}{2}$ in the definition of limit. Then $\exists N_1, N_2$ such that

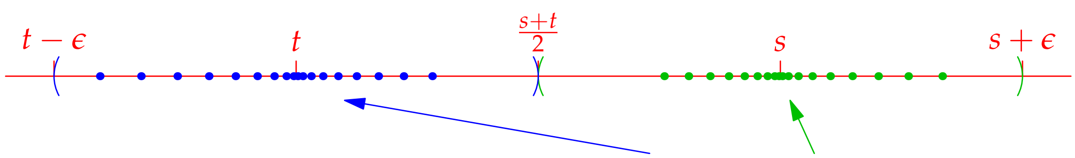
$$n > N_1 \implies |s_n - s| < \frac{|s-t|}{2} \quad \text{and} \quad n > N_2 \implies |s_n - t| < \frac{|s-t|}{2}$$

Let $n > \max\{N_1, N_2\}$. Then,

$$\begin{aligned} |s-t| = |s - s_n + s_n - t| &\le |s_n - s| + |s_n - t| && (\triangle\text{-inequality}) \\ &< \frac{|s-t|}{2} + \frac{|s-t|}{2} \\ &= |s-t| \end{aligned}$$

Contradiction. ∎

The idea of the proof is very simple: there exists a *tail* of the sequence (all terms s_n coming *after* some N) all of whose terms are close to *both limits*: this is complete nonsense!



For all $n > N$, s_n must lie both here and here!

Further Examples We give several more examples of using the limit definition. In all cases, *only* the formal argument needs to be presented. The challenge is figuring out what to write, so we present varying amounts of scratch work first.

1. Generalizing our previous example, we show that, for any $k \in \mathbb{R}^+$ the sequence defined by

   $$s_n = \frac{1}{n^k} \quad \text{has} \quad s_n \to 0$$

   Again a little scratch work, but faster this time. Given $\epsilon > 0$, we want to choose N such that

   $$n > N \implies \frac{1}{n^k} < \epsilon$$

   This amounts to having $n > \frac{1}{\epsilon^{1/k}}$. We can now write a formal argument:

   Proof. Let $\epsilon > 0$ be given. Let $N = \frac{1}{\epsilon^{1/k}}$. Then

   $$\begin{aligned} n > N \implies n > \frac{1}{\epsilon^{1/k}} &\implies \frac{1}{n} < \epsilon^{1/k} \\ &\implies \left| \frac{1}{n^k} - 0 \right| < \epsilon \end{aligned}$$

   We conclude that $\frac{1}{n^k} \to 0$. ∎

2. We prove that $s_n = \frac{2n+1}{3n-7}$ converges to $\frac{2}{3}$.

First some scratch work: we must conclude $\left|\frac{2n+1}{3n-7} - \frac{2}{3}\right| < \epsilon$, or equivalently $\left|\frac{17}{3(3n-7)}\right| < \epsilon$. For large n everything is positive, so it is sufficient for us to have

$$3n - 7 > \frac{17}{\epsilon} \iff n > \frac{7}{3} + \frac{17}{9\epsilon}$$

We now have enough for a proof:

Proof 1. Let $\epsilon > 0$ be given and let $N = \frac{7}{3} + \frac{17}{9\epsilon}$. Then

$$\begin{aligned} n > N &\implies n > \frac{7}{3} + \frac{17}{9\epsilon} \implies 0 < \frac{17}{3(3n-7)} < \epsilon \\ &\implies \left|\frac{2n+1}{3n-7} - \frac{2}{3}\right| = \left|\frac{3(2n+1) - 2(3n-7)}{3(3n-7)}\right| = \frac{17}{3(3n-7)} < \epsilon \end{aligned}$$

where the absolute values are dropped since $n > \frac{7}{3}$. We conclude that $\frac{2n+1}{3n-7} \to \frac{2}{3}$. ∎

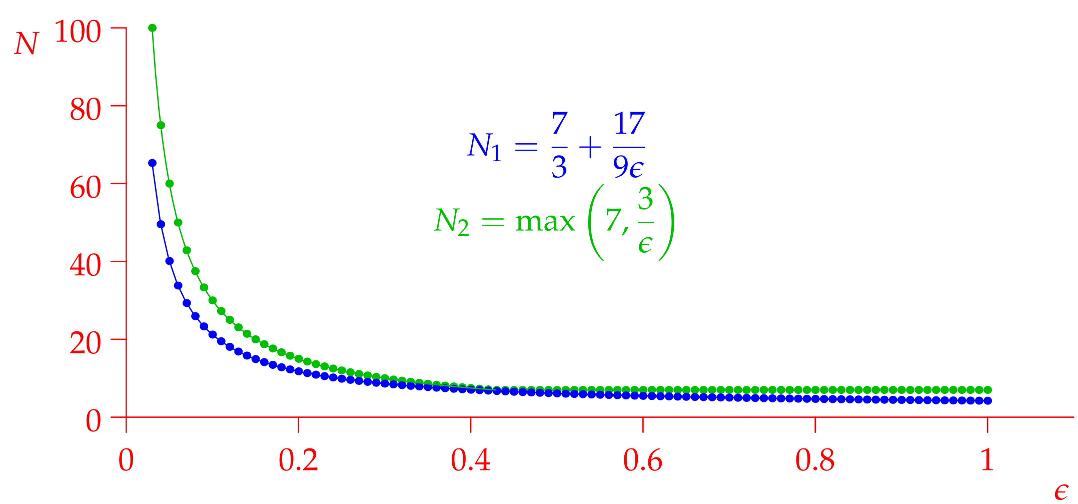
Here is an alternative proof where we use a simpler expression for N.

Proof 2. Let $\epsilon > 0$ be given and let $N = \max\left\{7, \frac{3}{\epsilon}\right\}$. Then

$$\begin{aligned} n > N \implies \left|\frac{2n+1}{3n-7} - \frac{2}{3}\right| &= \left|\frac{17}{3(3n-7)}\right| < \left|\frac{17}{6n}\right| && \text{(since } n > 7 \implies 3n - 7 > 2n\text{)} \\ &< \frac{3}{n} < \frac{3}{N} \le \epsilon && \text{(since } N \ge \tfrac{3}{\epsilon}\text{)} \end{aligned}$$

Hence result. ∎

The plot illustrates the two choices of N as functions of ϵ. Note that the second is always larger than the first! This is fine: if a particular choice of $N = N_1(\epsilon)$ works in a proof, so will any other $N_2(\epsilon)$ which is larger than N_1! Use this to your advantage to produce simpler arguments.



3. We prove that $s_n = \dfrac{4n^4 - 5n + 1}{3n^4 + 2n^2 + 3}$ converges to $\dfrac{4}{3}$.

   We want to conclude that

$$\left| \frac{4n^4 - 5n + 1}{3n^4 + 2n^2 + 3} - \frac{4}{3} \right| = \left| \frac{-8n^2 - 15n - 9}{3(3n^4 + 2n^2 + 3)} \right| < \epsilon$$

   Attempting to solve for n (as in the first method in Example 2) is crazy! Instead, as in the second approach, we simplify by observing that if n is sufficiently large, then

$$\left| \frac{-8n^2 - 15n - 9}{3(3n^4 + 2n^2 + 3)} \right| < \left| \frac{9n^2}{9n^4} \right| = \frac{1}{n^2}$$

   Indeed it is enough to have

$$9n^2 \geq 8n^2 + 15n + 9$$

   which, by solving the quadratic, holds when $n \geq \frac{15+\sqrt{261}}{2}$. Round this up to 16 and we have enough for the proof.

   Proof 1. Let $\epsilon > 0$ be given and let $N = \max\{16, \frac{1}{\sqrt{\epsilon}}\}$. Then

$$n > N \implies \left| s_n - \frac{4}{3} \right| = \left| \frac{-8n^2 - 15n - 9}{3(3n^4 + 2n^2 + 3)} \right| < \left| \frac{9n^2}{9n^4} \right| \qquad \text{(since } n > 16\text{)}$$
$$= \frac{1}{n^2} < \frac{1}{N^2} \leq \epsilon$$

   ∎

   If this were a formal answer, it would be wise to give a little scratch work to justify why $n > 16$ is sufficient for the inequality.

   Here is an alternative: this time we include a little of the scratch work in the answer, as you might do for a homework submission.

   Proof 2. Suppose that $n \geq 24$, then

$$n \geq 15 + \frac{9}{n} \implies n^2 \geq 15n + 9 \implies 9n^2 \geq 8n^2 + 15n + 9$$

   Let $\epsilon > 0$ be given, and let $N = \max\{24, \frac{1}{\sqrt{\epsilon}}\}$. Then

$$n > N \implies \left| s_n - \frac{4}{3} \right| = \left| \frac{4n^4 - 5n + 1}{3n^4 + 2n^2 + 3} - \frac{4}{3} \right| = \left| \frac{-8n^2 - 15n - 9}{3(3n^4 + 2n^2 + 3)} \right| < \left| \frac{9n^2}{9n^4} \right|$$
$$= \frac{1}{n^2} < \frac{1}{N^2} \leq \epsilon \qquad \text{(since } N \geq \tfrac{1}{\sqrt{\epsilon}}\text{)}$$

   ∎

   Therefore $s_n \to \dfrac{4}{3}$, as required.

Divergent sequences

Definition 2.4. Negating Definition 2.2 says that a sequence (s_n) *does not converge to s* if,

$$\exists \epsilon > 0 \text{ such that } \forall N,\ \exists n > N \text{ with } |s_n - s| \geq \epsilon$$

Furthermore, we say that (s_n) is *divergent* if it does not converge to any limit $s \in \mathbb{R}$. Otherwise said,

$$\forall s \in \mathbb{R},\ \exists \epsilon > 0 \text{ such that } \forall N,\ \exists n > N \text{ with } |s_n - s| \geq \epsilon$$

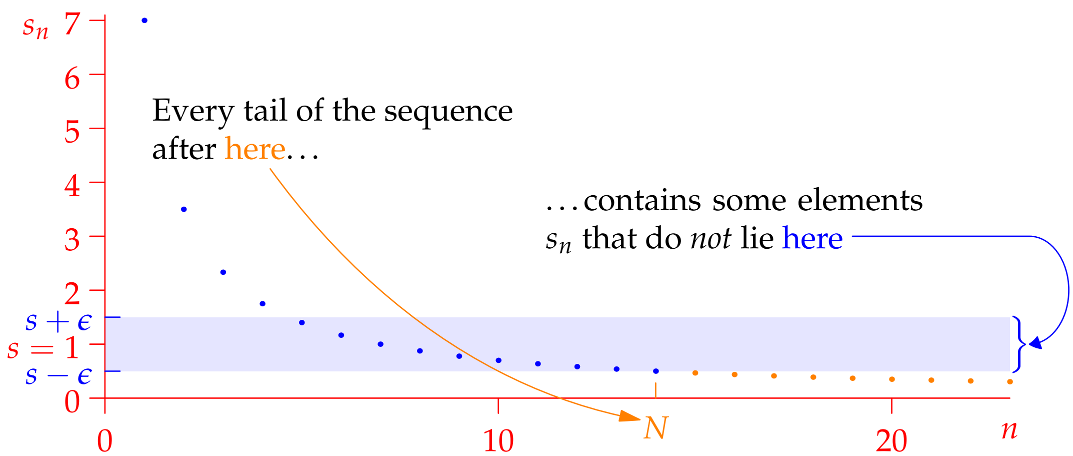
It can be helpful when proving divergence to assume $N \in \mathbb{N}$ so that you can quickly define n in terms of N. At the bottom of the page we'll explain what happens if you don't...

Examples

1. We prove that the sequence with $s_n = \frac{7}{n}$ does not converge to $s = 1$. We need to show that

$$\exists \epsilon > 0 \text{ such that } \forall N,\ \exists n > N \text{ with } \left|\frac{7}{n} - 1\right| \geq \epsilon \qquad (*)$$

This is easy to *visualize*: we know that $s_n \to 0$, so the sequence must eventually be nearly a distance 1 from $s = 1$. Any value of ϵ smaller that 1 should satisfy $(*)$. We prove twice: once using the Definition directly, and once by contradiction.



Direct Proof. Let $\epsilon = \frac{1}{2}$. We need to force $\left|\frac{7}{n} - 1\right| \geq \frac{1}{2}$. Since we are only concerned with *large* values of n, the term in the absolute value is *negative*,

$$\left|\frac{7}{n} - 1\right| \geq \frac{1}{2} \iff 1 - \frac{7}{n} \geq \frac{1}{2} \iff \frac{7}{n} \leq \frac{1}{2} \iff n \geq 14$$

Given $N \in \mathbb{N}$, let $n = \max\{14, N+1\}$ to see that $\left|\frac{7}{n} - 1\right| \geq \epsilon$. We conclude that $s_n \not\to 1$. ■

If we had only assumed that N were *real*, then the definition of n fails to be an *integer.* This can be fixed in a couple of ways:

- Define $n = \max\{14, \lceil N \rceil + 1\}$ using the ceiling function.
- Appeal to the Archimidean property to show that $\exists n \in \mathbb{N}$ such that $n > \max\{13, N\}$.

Restricting to $N \in \mathbb{N}$ makes the argument easier to follow: just remember to state it to help the reader!

Contradiction Proof. Suppose $s_n \to 1$. Then

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n > N \implies \left|\frac{7}{n} - 1\right| < \epsilon$$

In particular, this should hold for $\epsilon = \frac{1}{2}$. But then, for all large n, we would require

$$\left|\frac{7}{n} - 1\right| < \frac{1}{2} \iff \frac{1}{2} < \frac{7}{n} < \frac{3}{2} \iff n < 14 \text{ and } n > \frac{14}{3}$$

Simply let $n = \max\{14, N+1\}$ for a contradiction. ■

2. The sequence defined by $s_n = (-1)^n$ is divergent. We prove by contradiction and, for variety, this time we invoke Archimedes.

   Proof. Suppose that $s_n \to s$ and let $\epsilon = 1$ in the definition of limit. Then $\exists N \in \mathbb{R}$ such that

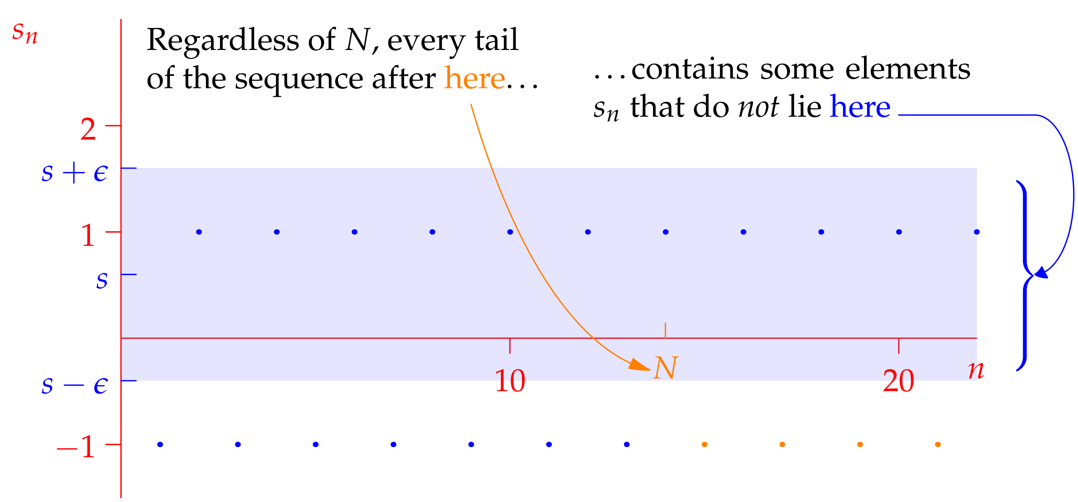
   $$n > N \implies |(-1)^n - s| < 1$$

   However, there exist (Archimedes[3]) both even $n_e > N$ and odd $n_o > N$. There are two cases:

   - If $s \geq 0$ then $|(-1)^{n_o} - s| = |-1 - s| = s + 1 \geq \epsilon$.
   - If $s < 0$ then $|(-1)^{n_e} - s| = |1 - s| = 1 - s \geq \epsilon$.

   Either way we have a contradiction and we conclude that (s_n) is divergent. ■

   The picture shows what happens when $s \geq 0$.



   We chose $\epsilon = 1$ because it is easy to work with, and because at least some of the elements of any tail of the sequence are $s_n = -1$, which are at least a distance of 1 from s.

[3] $\exists n \in \mathbb{N}$ such that $n > N$. Clearly n is even or odd; $n + 1$ provides the other.

3. The sequence define by $s_n = \ln n$ diverges.

   Before embarking on a proof, first visualize the sequence: you should recall from calculus that logarithms increase unboundedly. It therefore seems reasonable that, for any purported limit s, letting $\epsilon = 1$ will cause trouble, for eventually $\ln n \geq s + 1$. Again we present two proofs, the first verifies Definition 2.4 directly, the second is by contradiction.

   Proof 1. Suppose $s \in \mathbb{R}$ is given. Let $\epsilon = 1$ and suppose that $N \in \mathbb{N}$ is given. We define $n = \max\{N+1, e^{s+1}\}$. Then $n > N$ and

   $$\ln n \geq \ln(e^{s+1}) = s + 1$$

   In particular,

   $$|s_n - s| = |\ln n - s| \geq \epsilon$$

   whence (s_n) is divergent. ∎

   Proof 2. Suppose that (s_n) converges to $s \in \mathbb{R}$. Let $\epsilon = 1$: we may therefore assume $N \in \mathbb{N}$ exists satisfying the limit definition (Definition 2.2). Now define $n = \max\{N+1, e^{s+1}\}$. But then

   $$n > N \quad \text{and} \quad \ln n > s + 1 \implies |\ln n - s| > 1 = \epsilon$$

   Contradiction. We conclude that (s_n) diverges. ∎

   From now on we'll typically prefer contradiction arguments: these have the advantage of only having to remember one definition!

A Little Abstraction

Working explicitly with the limit definition is tedious. In the next section we'll develop the *limit laws* so we can combine limits of sequences without providing new ϵ-proofs. Of course, all the limit laws must first be *proved* based on the definition! To build up to this, here are three general results.

Lemma 2.5. *Suppose that $s_n \to s$. Then $s_n^2 \to s^2$.*

The challenge here is that we want to bound $|s_n^2 - s^2| = |s_n - s|\,|s_n + s|$, which means we need some control over $|s_n + s|$. There are several ways to do this: for instance by the triangle-inequality,

$$|s_n + s| = |s_n - s + 2s| \leq |s_n - s| + 2\,|s|$$

We can now begin a proof.

Proof. Let $\epsilon > 0$ be given and let $\delta = \min\{1, \frac{\epsilon}{1+2|s|}\}$. Since $s_n \to s$, $\exists N$ such that

$$n > N \implies |s_n - s| < \delta$$

But then

$$\begin{aligned} n > N \implies |s_n^2 - s^2| &= |s_n - s|\,|s_n + s| \leq |s_n - s|\,(|s_n - s| + 2\,|s|) && (\triangle\text{-inequality}) \\ &< \delta(1 + 2\,|s|) && (\text{since } |s_n - s| < \delta \leq 1) \\ &\leq \epsilon \end{aligned}$$

∎

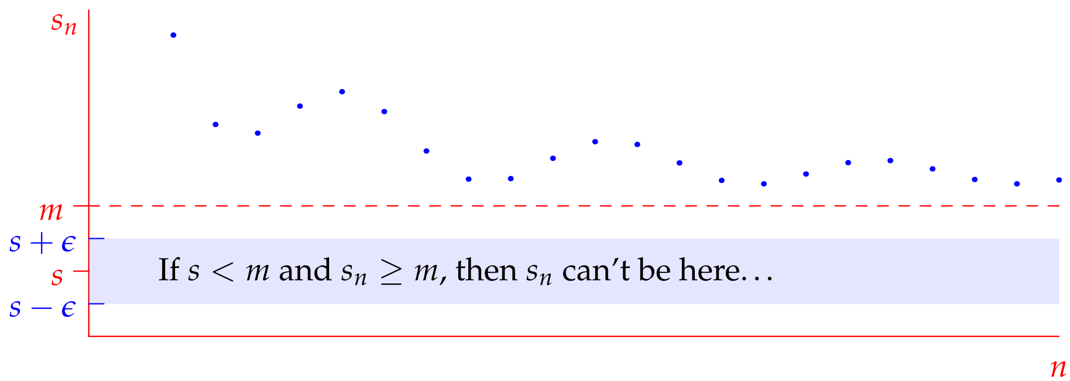
Theorem 2.6. *Suppose that $s_n \to s$ where (s_n) is bounded below by m. Then $s \geq m$.*

Proof. Suppose that $s < m$ and let $\epsilon = \frac{m-s}{2} > 0$. Then $\exists N$ such that

$$n > N \implies |s_n - s| < \frac{m-s}{2} \implies s_n - s < \frac{m-s}{2} \qquad (|x| < y \iff -y < x < y \ldots)$$

$$\implies s_n - m < \frac{s-m}{2} < 0 \qquad \text{(add } s-m \text{ to both sides)}$$

Contradiction. ∎



The picture should make clear the contradicton in the proof. There are several simple variations on the Theorem.

Strict Lower Bounds The same proof (*and conclusion*!) is valid when (s_n) has a strict lower bound.

For example the sequence with $s_n = \frac{1}{n}$ satisfies

$$\forall n \in \mathbb{N},\ s_n > 0, \quad \text{and } \lim s_n = 0$$

precisely in accordance with the Theorem. In particular, we *cannot* conclude that $\lim s_n > 0$.

Upper Bounds The corresponding result for sequences bounded above should be clear:

If $s_n \to s$ and $\forall n,\ s_n \leq M$ then $s \leq M$

Sequence Tails We need only assume that $s_n \geq m$ *for all but finitely many* s_n. In such a situation there must exist a final $s_k < m$, and the proof can easily be modified:

$$n > \max\{N, k\} \implies |s_n - s| < \frac{m-s}{2} \implies \cdots \text{ etc.}$$

For example, the sequence (s_n) with n^{th} term

$$s_n = \frac{10}{n^2} - \frac{1}{n} = \frac{10-n}{n^2}$$

is bounded above by $M = 0$ whenever $n \geq 10$. Theorem 2.6 confirms our belief that $\lim s_n \leq 0$ (clearly $\lim s_n = 0$ in this case!).

The caveats *for all large n* and *for some tail of the sequence* are equivalent, and often used. Many theorems can be modified this way; in the interests of brevity, it is common to avoid explicitly stating such, and even more common to ignore the caveat in the proof. Here is another famous example...

Theorem 2.7 (Squeeze Theorem)**.** *Suppose that three sequences satisfy* $a_n \le s_n \le b_n$ *(for all large* n*) and that* (a_n) *and* (b_n) *both converge to* s*. Then* $s_n \to s$*.*

Proof. Since $a_n \le s_n \le b_n$, it is immediate that

$$a_n - s \le s_n - s \le b_n - s \implies |s_n - s| \le \max\{|a_n - s|, |b_n - s|\}$$

We now bound the RHS by ϵ: let $\epsilon > 0$ be given, then there exists N_a, N_b such that

$$n > N_a \implies |a_n - s| < \epsilon \quad \text{and} \quad n > N_b \implies |b_n - s| < \epsilon$$

Let $N = \max\{N_a, N_b\}$. Then

$$n > N \implies |s_n - s| \le \max\{|a_n - s|, |b_n - s|\} < \epsilon$$

■

2.9 Limit Theorems for Sequences

Our immediate goal is to be able to calculate limits naturally, without using ϵ-N proofs: these results are often known as the *limit laws.* We start with a result that allows us to compute the limit of any rational sequence.

Theorem 2.8. *Suppose* (s_n) *and* (t_n) *converge, to* s *and* t *respectively, and that* $k \in \mathbb{R}$ *is constant. Then*

(a) $\lim ks_n = ks$

(b) $\lim(s_n + t_n) = s + t$

(c) $\lim(s_n t_n) = st$ *(this extends Lemma 2.5: by induction we now have* $s_n^k \to s^k$ *for any* $k \in \mathbb{N}$*)*

(d) If $t \neq 0$ *then* $\lim \frac{s_n}{t_n} = \frac{s}{t}$

Before proving this, here is an example of its power.

$$\begin{aligned}
\lim \frac{3n^2 + 2n - 1}{5n^2 - 2} &= \lim \frac{3 + \frac{2}{n} - \frac{1}{n^2}}{5 - \frac{2}{n^2}} = \frac{\lim\left(3 + \frac{2}{n} - \frac{1}{n^2}\right)}{\lim\left(5 - \frac{2}{n^2}\right)} && \text{(part (d))} \\
&= \frac{\lim 3 + \lim \frac{2}{n} - \lim \frac{1}{n^2}}{\lim 5 - \lim \frac{2}{n^2}} && \text{(part (b))} \\
&= \frac{3 + 0 - 0}{5 - 0} = \frac{3}{5} && \text{(part (a) and example 1, page 4)}
\end{aligned}$$

This involves some (generally accepted) sleight of hand; one shouldn't really write $\lim s_n$ until one knows it exists!

Proving Theorem 2.8 requires a little work. We start by recalling the notion of *boundedness.*

Lemma 2.9. (s_n) *convergent* $\implies$ (s_n) *bounded (*$\exists M$ *such that* $\forall n, |s_n| \le M$*).*

Proof. Suppose $s_n \to s$ and let $\epsilon = 1$ in the definition of limit. Then $\exists N$ such that

$$n > N \implies |s_n - s| < 1 \implies s - 1 < s_n < s + 1 \implies |s_n| < \max\{|s - 1|, |s + 1|\}$$

The RHS bounds the tail of the sequence where $n > N$. We may therefore define the bound

$$M = \max\{|s - 1|, |s + 1|, |s_n| : n \le N\}$$

■

Note that the converse to this is *false*! For instance, $s_n = (-1)^n$ is bounded but *not* convergent!

Proof of Theorem 2.8. These arguments will likely be difficult to follow at first read. A crucial observation, used in all four parts, is that we can replace ϵ in the limit definition with *any positive number*: for instance $\frac{\epsilon}{|k|}$ in part (a). Compare with how we introduced δ in the proof of Lemma 2.5: at the cost of more symbols, all these arguments could be rephrased similarly.

(a) If $k = 0$, the result is trivial. Otherwise,[4] let $\epsilon > 0$ be given. Since $\frac{\epsilon}{|k|} > 0$, $\exists N$ such that

$$n > N \implies |s_n - s| < \frac{\epsilon}{|k|} \implies |ks_n - ks| = |k|\,|s_n - s| < \epsilon$$

(b) Let $\epsilon > 0$ be given. Then $\exists N$ such that

$$n > N \implies |s_n - s|\,, |t_n - t| < \frac{\epsilon}{2}$$

Apply the $\triangle$-inequality to see that

$$n > N \implies |s_n + t_n - (s+t)| \le |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(c) Let $\epsilon > 0$ be given. Since $s_n \to s$ and $t_n \to t$, there exists N such that

$$n > N \implies |s_n - s| < \frac{\epsilon}{2\,|t|} \quad \text{and} \quad |t_n - t| < \frac{\epsilon}{2M}$$

where M is a (positive) bound for (s_n) (Lemma 2.9). But now

$$\begin{aligned} |s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| &\overset{\triangle}{\le} |s_n|\,|t_n - t| + |t|\,|s_n - s| \\ &\le M\,|t_n - t| + |t|\,|s_n - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

In the exceptional case of $t = 0$, instead choose N such that $n > N \implies |t_n| < \frac{\epsilon}{M}$.

(d) Since $t_n \to t \neq 0$, we see[5] that $\exists N_1$ such that

$$n > N_1 \implies |t_n - t| < \frac{|t|}{2} \implies |t_n| > \frac{|t|}{2}$$

Now let $\epsilon > 0$ be given, whence $\exists N_2$ such that

$$n > N_2 \implies |t_n - t| < \frac{|t|^2\,\epsilon}{2}$$

Let $N = \max\{N_1, N_2\}$ to see that

$$n > N \implies \left|\frac{1}{t_n} - \frac{1}{t}\right| = \frac{|t - t_n|}{|t|\,|t_n|} < \frac{2\,|t - t_n|}{|t|^2} < \epsilon$$

whence $\frac{1}{t_n} \to \frac{1}{t}$. An appeal to part (c) completes the proof. ∎

[4] It is tempting to apply the squeeze theorem rather than working with ϵ: consider

$$0 \le |ks_n - ks| = |k|\,|s_n - s|$$

Unfortunately, showing that $|k|\,|s_n - s| \to 0$ requires the very statement we're trying to prove! You really need an ϵ-proof.

[5] Take $\epsilon = \frac{|t|}{2}$ in the definition of limit.

The next result tells us how to take limits of powers.

Theorem 2.10. *1. If $k > 0$ then $\frac{1}{n^k} \to 0$*

2. *If $|a| < 1$ then $a^n \to 0$*
3. *If $a > 0$ then $a^{1/n} \to 1$*
4. *$n^{1/n} \to 1$*

We give only a sketch proof: you should try to formalize these arguments as much as you can.

Sketch Proof. 1. This is covered as an example on page 4: given $\epsilon > 0$, let $N = \epsilon^{-1/k}\ldots$

2. The $a = 0$ case is trivial. Otherwise: given $\epsilon > 0$, let $N = \log_{|a|} \epsilon\ldots$

3. WLOG[6] suppose $a > 1$. We want to show that $s_n := a^{1/n} - 1 \to 0$. Since $s_n > 0$, the Binomial Theorem shows that

$$a = (1 + s_n)^n \geq 1 + ns_n \implies s_n \leq \frac{a-1}{n}$$

The squeeze theorem (or explicitly choosing $N = \frac{a-1}{\epsilon}$) completes the argument.

4. We must show that $s_n = n^{1/n} - 1 \to 0$. Again apply the Binomial Theorem: since $s_n > 0$,

$$n = (s_n + 1)^n = \sum_{k=0}^{n} \binom{n}{k} s_n^k \geq 1 + ns_n + \frac{1}{2}n(n-1)s_n^2 > \frac{1}{2}n(n-1)s_n^2 \implies s_n < \sqrt{\frac{2}{n-1}}$$

The squeeze theorem finishes things off (or choose $N = 2\epsilon^{-2} + 1$ if you prefer). ∎

We need one last result in order to compute all limits of sequences involving algebraic functions:

Corollary 2.11 (Limits of Roots)**.** *Suppose $s_n \to s$. If $k \in \mathbb{N}$ then $\sqrt[k]{s_n} \to \sqrt[k]{s}$ (k even only if $s_n \geq 0$).*

We omit the proof: see if you can complete it yourself, using the following factorization/inequality (valid when $s_n, s > 0$)

$$\left| s_n^{1/k} - s^{1/k} \right| = \frac{|s_n - s|}{\left| s_n^{\frac{k-1}{k}} + s_n^{\frac{k-2}{k}} s^{\frac{1}{k}} + \cdots + s^{\frac{k-1}{k}} \right|} < \frac{|s_n - s|}{s^{\frac{k-1}{k}}}$$

Examples

1. $\lim (3n)^{2/n} = (\lim 3^{1/n})^2 (\lim n^{1/n})^2 = 1$.
2. $\lim \dfrac{n^{2/n} + \left(3 - n^{-1}\sin n\right)^{1/5}}{4n^{-3/2} + 7} = \dfrac{1 + \sqrt[5]{3}}{7}$, where $\dfrac{\sin n}{n} \to 0$ follows from the squeeze theorem.

[6]The $a = 1$ case is trivial. If $a < 1$, then $b = \frac{1}{a} > 1$ has $a^{1/n} = \frac{1}{b^{1/n}} \to 1$ courtesy of Theorem 2.8 (d).

Divergence laws

We now consider unbounded sequences.

Definition 2.12. We say that (s_n) *diverges to* ∞ if,

$$\forall M > 0,\ \exists N \text{ such that } n > N \implies s_n > M$$

We write $s_n \to \infty$ or $\lim s_n = \infty$. The definition for $s_n \to -\infty$ is similar.

If (s_n) neither converges nor diverges to $\pm\infty$, we say that it *diverges by oscillation*. In such cases $\lim s_n$ is meaningless, though it is common to write $\lim s_n = \text{DNE}$ for 'does not exist.'

Examples

1. Prove that $n^2 + 4n \to \infty$.

   Let $M > 0$ be given,[7] and let $N = \sqrt{M}$. Then

   $$n > N \implies n^2 + 4n > n^2 > N^2 = M$$

2. Prove that $s_n = n^5 - n^4 - 2n + 1 \to \infty$.

   This is trickier, and not just because of the fifth power. We cannot simply ignore the lower order terms and concentrate on the highest power, since the extra terms are not all summed. Instead, note that[8]

   $$s_n > \frac{1}{2}n^5 \iff n^5 > 2(n^4 + 2n - 1) \iff n > 2 + \frac{4}{n^3} - \frac{1}{n^4}$$

   Certainly this holds if $n > 6$: we can now provide a proof.

   Let $M > 0$ be given, and let $N = \max\{6, \sqrt[5]{2M}\}$. Then

   $$n > N \implies s_n > \frac{1}{2}n^5 > \frac{1}{2}(2M) = M$$

3. Prove that the sequence defined by $s_n = n^2 - n^3$ diverges to $-\infty$.

   For some scratch work here, consider

   $$s_n = n^2(1-n) < -\frac{1}{2}n^3 \iff 1 - n < -\frac{1}{2}n \iff n \geq 2$$

   Now let $M > 0$ be given[9] and define $N = \max\{2, \sqrt[3]{2M}\}$. Then

   $$n > N \implies n > 2 \implies s_n < -\frac{1}{2}n^3 < -\frac{1}{2}N^3 \leq -M$$

   We conclude that $s_n \to -\infty$

---

[7] Try some scratch work first! We want $n^2 + 4n > M$ for large n which is certainly true if $n > \sqrt{M}\ldots$

[8] Compare this trick with the second proof of example 3 on page 6.

[9] The notion that $s_n \to -\infty$ can be phrased in multiple ways: some people prefer

$$\forall m < 0,\ \exists N \text{ such that } n > N \implies s_n < m$$

It should be clear that our M is simply $-m$.

Several of the limit laws can be adapted to sequences which diverge to $\pm\infty$.

Theorem 2.13. *Suppose $s_n \to \infty$. (Corresponding statements when $s_n \to -\infty$ should be clear.)*

(a) If $t_n \geq s_n$ for all n, then $t_n \to \infty$

(b) If $t_n \to t$ (finite), then $s_n + t_n \to \infty$.

(c) If $t_n \to t > 0$ then $s_n t_n \to \infty$.

(d) $\dfrac{1}{s_n} \to 0$

(e) If $t_n > 0$ satisfies $t_n \to 0$, then $\dfrac{1}{t_n} \to \infty$

Proof. We prove two of the results: try the rest yourself.

(b) Since (t_n) converges, it is bounded, whence $\exists A$ such that $\forall n,\ t_n \geq A$. Let M be given: since $s_n \to \infty$, $\exists N$ such that

$$n > N \implies s_n > M - A \implies s_n + t_n < M - A + A = M$$

(d) Let $\epsilon > 0$ be given, and let $M = \frac{1}{\epsilon}$. Then $\exists N$ such that

$$n > N \implies s_n > M = \frac{1}{\epsilon} \implies \frac{1}{s_n} < \epsilon$$

■

Rational Functions We can now find the limit of any rational sequence: $\frac{p_n}{q_n}$ where (p_n), (q_n) are polynomials in n. For example

$$\frac{3n^3+4}{2n^2-1} = \frac{3n+4n^{-2}}{2-n^{-2}} = (3n+4n^{-2}) \cdot \frac{1}{2-n^{-2}} \to \infty$$

by applying Theorem 2.13 (c) to

$$s_n := 3n + 4n^{-2} \to \infty \quad \text{and} \quad t_n = \frac{1}{2-n^{-2}} \to \frac{1}{2}$$

Indeed, you should be able to confirm the familiar result from elementary calculus:

Corollary 2.14. *If p_n, q_n are polynomials in n with leading coefficients p, q respectively then*

$$\lim \frac{p_n}{q_n} = \begin{cases} 0 & \text{if } \deg(p_n) < \deg(q_n) \\ \dfrac{p}{q} & \text{if } \deg(p_n) = \deg(q_n) \\ \operatorname{sgn}(\frac{p}{q})\infty & \text{if } \deg(p_n) > \deg(q_n) \end{cases}$$

2.10 Monotone and Cauchy Sequences

The first goal of this section is to address a difficulty with the definition of convergence: How do we show that a sequence is convergent without first knowing its limit? Monotone and Cauchy sequences are two classes of sequences where one has convergence without having to know the limit. The existence of limits for both types of sequences depends crucially on the completeness axiom. As a byproduct, we obtain an alternative construction of the real numbers.

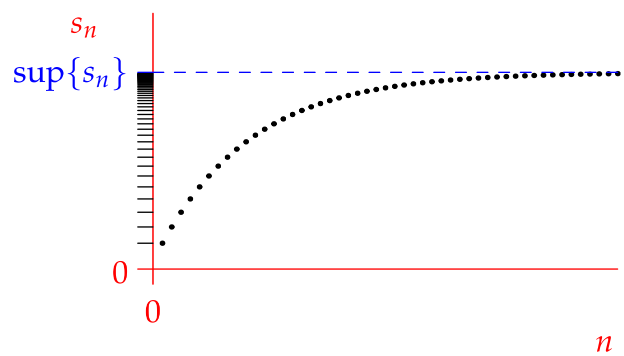
Definition 2.15.
- (s_n) is *non-decreasing* or *monotone-up* if $s_{n+1} \geq s_n$ for all n.
- (s_n) is *non-increasing* or *monotone-down* if $s_{n+1} \leq s_n$ for all n.
- (s_n) is *monotone* or *monotonic* if either of the above is true.

For example $s_n = \frac{7}{n} + 4$ is monotone-down/non-increasing.

Theorem 2.16 (Monotone Convergence)**.** *Bounded monotone sequences are convergent. Specifically:*

(a) If (s_n) is bounded above and non-decreasing, then $\lim s_n = \sup\{s_n\}$.

(b) If (s_n) is bounded below and non-increasing, then $\lim s_n = \inf\{s_n\}$.



Proof. Suppose (s_n) is non-decreasing and bounded above. Let $s = \sup\{s_n\}$; this exists by the completeness axiom and is finite since (s_n) is bounded.

Let $\epsilon > 0$ be given. Since s is the supremum, there exists some element $s_N > s - \epsilon$. The non-decreasing property means that

$$n > N \implies s_n \geq s_N > s - \epsilon \implies |s - s_n| < \epsilon$$

The non-increasing case is similar. ∎

Examples

1. Suppose (s_n) is defined by $s_n = 1$ and $s_{n+1} = \frac{1}{5}(s_n + 8)$. Then:
   - (Bounded above) $s_n < 2 \implies s_{n+1} < \frac{1}{5}[2 + 8] = 2$. By induction, (s_n) is bounded above by 2.
   - (Monotone-up) $s_{n+1} - s_n = \frac{4}{5}[2 - s_n] > 0$ since $s_n < 2$.

   We conclude that (s_n) converges. Indeed, if $s = \lim s_n$, then the limit laws show that s satisfies

$$s = \lim s_{n+1} = \frac{1}{5}(\lim s_n + 8) = \frac{1}{5}(s + 8) \implies s = 2$$

2. Define a sequence (s_n) by $s_0 = 2$ and

$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right) \tag{$*$}$$

The AM-GM inequality[10] says that $s_{n+1} \geq \sqrt{2}$ for all n, whence the sequence is bounded below. Moreover,

$$s_n - s_{n+1} = \frac{1}{2}\left(s_n - \frac{2}{s_n}\right) = \frac{s_n^2 - 2}{2s_n} \geq 0$$

since $s_n \geq \sqrt{2}$. We have a monotone-down sequence which is bounded below; it thus converges to some limit s. Indeed taking limits of $(*)$ yields

$$s = \frac{1}{2}\left(s + \frac{2}{s}\right) \implies s^2 = 2 \implies s = \sqrt{2}$$

This example shows why we need completeness in the proof: (s_n) is a monotone, bounded sequence of *rational* numbers, but it doesn't converge in $\mathbb{Q}$.

3. It can be shown that $s_n = \left(1 + \frac{1}{n}\right)^n$ defines a monotone-up sequence which is bounded above (see the worksheet on the class website). This provides one of the many *definitions* of e:

$$e := \lim \left(1 + \frac{1}{n}\right)^n$$

Theorem 2.17. *If (s_n) is unbounded and non-decreasing then $s_n \to \infty$. Similarly, if (s_n) is unbounded and non-decreasing then $s_n \to -\infty$.*

Proof. Since (s_n) is unbounded, given M, $\exists s_N > M$. Since (s_n) is non-decreasing we see that

$$n > N \implies s_n \geq s_N > M$$

∎

It now makes sense to write $\lim s_n = \sup\{s_n\}$ for any non-decreasing sequence even if this is ∞.

Limits Superior and Inferior

When analyzing a sequence, one is primarily interested in its *long-term* behavior: what can we say about the values s_n when n is very large? We currently have two tools at our disposal:

Limits Unfortulately, most sequences diverge by oscillation, so $\lim s_n$ is usually meaningless.

Suprema/Infima These are also unhelpful for discussing the long-term behaviour of most sequences. For example, consider the sequences defined by

$$s_n = \frac{1}{n} \quad \text{and} \quad t_n = \begin{cases} 1000 & \text{if } n \leq 1,000,000 \\ \frac{1}{n} & \text{if } n > 1,000,000 \end{cases}$$

when $n \geq 1$. These sequences clearly have the same long-term behavior ($\lim s_n = \lim t_n = 0$), but due to the fact that the first million terms are different, they have different suprema: $\sup\{t_n\} = 1000 > 1 = \sup\{s_n\}$.

Combining these concepts, however, turns out to pack a bigger punch...

[10] $\sqrt{xy} \leq \frac{x+y}{2}$ for all real x, y, with equality if and only if $x = y$. To prove this, start by expanding $(x-y)^2 \geq 0$...

Definition 2.18. Let (s_n) be a sequence. We define its *limit superior* $\limsup s_n$ and *limit inferior* $\liminf s_n$ as follows:

1. (s_n) is bounded above, define $v_N = \sup\{s_n : n > N\}$ and

$$\limsup s_n = \lim_{N\to\infty} v_N$$

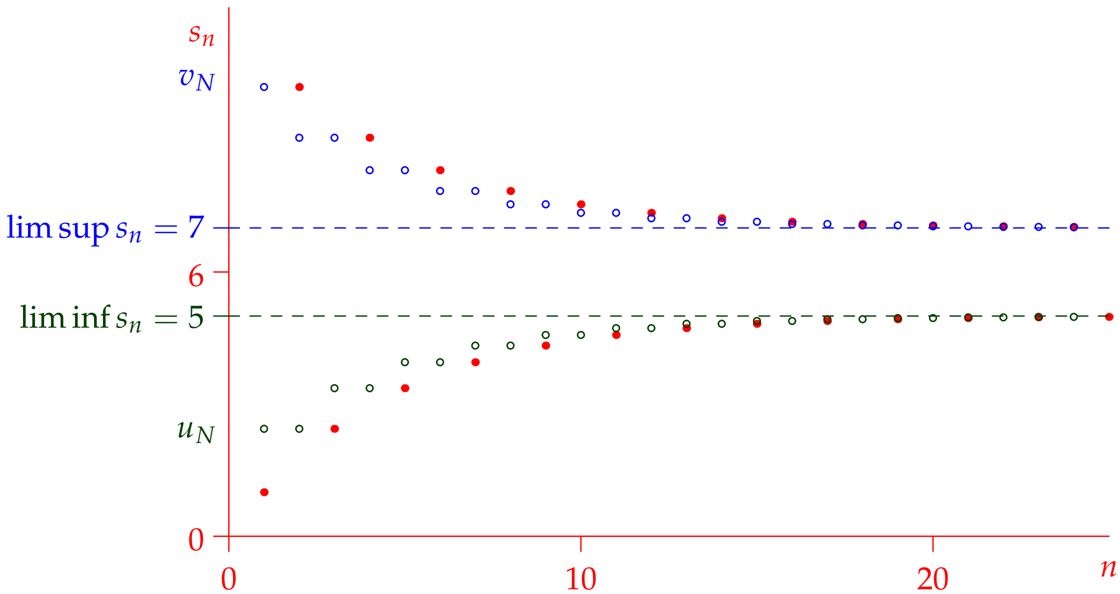
2. (s_n) is unbounded above, define $\limsup s_n = \infty$.
3. (s_n) is bounded below, define $u_N = \inf\{s_n : n > N\}$ and

$$\liminf s_n = \lim_{N\to\infty} u_N$$

4. (s_n) is unbounded below, define $\liminf s_n = -\infty$.

The picture below shows the sequences (s_n), (u_N) and (v_N) when

$$s_n = 6 + 5\left(\frac{4}{5}\right)^n + (-1)^n$$



Computing $\limsup s_n = 7$ and $\liminf s_n = 5$ directly is a little messy, so we omit the calculation. What should be plausible from the picture is the the sequence (s_n) consists of two subsequences, one decreasing towards 7 and the other increasing towards 5: from this observation, the construction of the sequences (u_N) and (v_N) should be clear.

It should be clear from the definitions that, whenever they exist,

(u_N) is *monotone-up*, (v_N) *monotone-down*, and $u_N \le v_N$.

These facts and the Monotone Convergence Theorem combine for a little housekeeping:

Lemma 2.19. *1.* $\limsup s_n$ *and* $\liminf s_n$ *exist for any sequence (they might be infinite).*

2. $\liminf s_n \le \liminf s_n$.

Examples

1. Let $s_n = (-1)^n$. Then

$$\forall N \in \mathbb{N}, \quad u_N = \inf\{s_n : n > N\} = -1 \quad \text{and} \quad v_N = \sup\{s_n : n > N\} = 1$$

   Therefore $\limsup s_n = 1$ and $\liminf s_n = -1$.

2. Let $s_n = \dfrac{(-1)^n}{n}$. Then

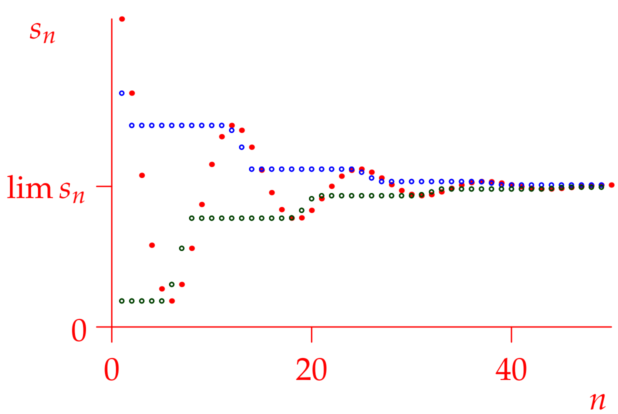
$$u_N = \inf\{s_n : n > N\} = \begin{cases} -\frac{1}{N+2} & \text{if } N \text{ odd} \\ -\frac{1}{N+1} & \text{if } N \text{ even} \end{cases} \qquad \text{and} \quad v_N = \begin{cases} \frac{1}{N+1} & \text{if } N \text{ odd} \\ \frac{1}{N+2} & \text{if } N \text{ even} \end{cases}$$

   Clearly $\liminf s_n = 0 = \limsup s_n$.

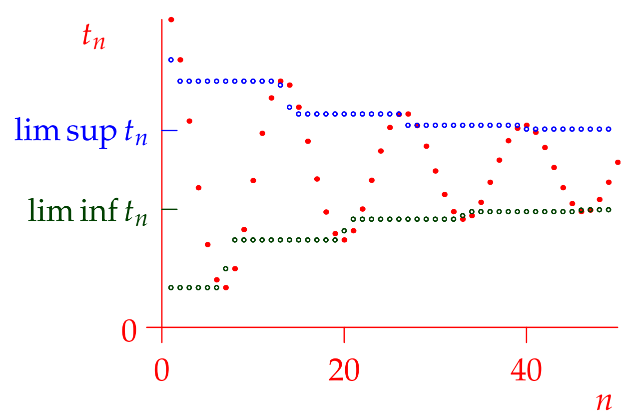
These examples should suggest a result:

Theorem 2.20. *Let (s_n) be a sequence. Then $\liminf s_n = \limsup s_n$ if and only if $\lim s_n = s$ for some $s \in [-\infty, \infty]$, in which case all three expressions equal s.*

Before proving this, here are two pictures to help visualize the concepts.[11]



Convergence: $\liminf s_n = \lim s_n = \limsup s_n$



Divergence: $\liminf t_n < \limsup t_n$

In both pictures, the sequence (u_N) is in green and (v_N) in blue. It should be clear from the definition that for all N we have

$$u_N = \inf\{s_n : n > N\} \le s_{N+1} \le v_N = \sup\{s_n : n > N\}$$

so that the original sequence is *almost* trapped between (u_N) and (v_N). A minor redefinition could remove the word '*almost*,' though the cost of fixing several inequalities in later proofs makes this counter-productive.

[11] In case you're interested, the explicit sequences are $s_n = 2 + 3e^{-\frac{n}{10}} \cos \frac{n}{2}$ with limit 2, and $t_n = 4 + \sin \frac{n}{2} + 4e^{-\frac{n}{20}} \cos \frac{n}{2}$ which diverges by oscillation.

Proof of Theorem. We first prove the $\Rightarrow$ direction: there are three cases.

(a) Suppose $\limsup s_n = \liminf s_n = s$ is finite. Since $u_{n-1} \le s_n \le v_{n-1}$ for all n, the squeeze theorem tells us that $s_n \to s$.

(b) Suppose $\limsup s_n = \liminf s_n = \infty$. Then $u_{n-1} \le s_n$ for all n with $u_{n-1} \to \infty$. Theorem 2.13(a) shows that $s_n \to \infty$.

(c) $\limsup s_n = \liminf s_n = -\infty$ is similar.

Now for the $\Leftarrow$ direction: again there are three cases.

(a) Suppose $\lim s_n = s$ is finite. Then $\lim s_n = s$ says that, for all $\epsilon > 0$, $\exists M$ such that

$$\begin{aligned} N > M &\implies |s_N - s| < \epsilon \implies s_N < s + \epsilon \\ &\implies v_N = \sup\{s_n : n > N\} \le s + \epsilon && \text{(definition of supremum)} \\ &\implies \limsup s_n = \lim_{N\to\infty} v_N \le s + \epsilon && \text{(Theorem 2.6)} \end{aligned}$$

Since this holds for every $\epsilon > 0$ we conclude[12] that $\limsup s_n \le s$.

Similarly $\liminf s_n \ge s$. Combining with Lemma 2.19 we obtain

$$s \le \liminf s_n \le \limsup s_n \le s$$

whence all terms are equal.

(b) Suppose $\lim s_n = \infty$. Then $\forall M > 0$, $\exists N$ such that $n > N \implies s_n > M$. But then

$$u_N = \inf\{s_n : n > N\} \ge M$$

whence $u_N \to \infty$ and so $\liminf s_n = \infty$. Clearly $\limsup s_n = \infty$ also.

(c) Again, $\lim s_n = -\infty$ is similar.

■

Cauchy Sequences

A sequence is Cauchy[13] when terms in the tails of the sequence are constrained to stay close to one another. This will shortly provide an alternative way of describing *convergence.*

Definition 2.21. (s_n) is a *Cauchy sequence* if

$$\forall \epsilon > 0,\ \exists N \text{ such that } m, n > N \implies |s_n - s_m| < \epsilon$$

Examples

1. Let $s_n = \frac{1}{n}$. Let $\epsilon > 0$ be given and let $N = \frac{1}{\epsilon}$. Then[14]

$$m \ge n > N \implies |s_m - s_n| = \frac{1}{n} - \frac{1}{m} \le \frac{1}{n} < \frac{1}{N} = \epsilon$$

Thus (s_n) is Cauchy. A similar argument works for any $s_n = \frac{1}{n^k}$ for positive k.

---

[12] In case this makes you nervous. . . Suppose $a \le b + \epsilon$ for all $\epsilon > 0$. If $a > b$, let $\epsilon = \frac{1}{2}(a - b)$ to get a contradiction. . .

[13] Augustin-Louis Cauchy (1789–1857) was a French mathematician, responsible (in part) for the ϵ-definition of limit.

[14] WLOG we may assume $m \ge n$. This assumption is very common!

2. Let $(s_n)_{n=0}^{\infty}$ be the sequence defined inductively as follows:

$$s_0 = 1, \quad s_{n+1} = \begin{cases} s_n + 3^{-n} & \text{if } n \text{ even} \\ s_n - 4^{-n} & \text{if } n \text{ odd} \end{cases} \quad \text{that is } (s_n) = \left(1, 2, \frac{5}{3}, \frac{67}{36}, \ldots\right)$$

Then $|s_{n+1} - s_n| \leq 3^{-n}$, whence

$$m > n \implies |s_m - s_n| \leq \sum_{k=n}^{m-1} 3^{-k} = \frac{3^{-n} - 3^{-m}}{1 - \frac{1}{3}} < \frac{3}{2} \cdot 3^{-n}$$

where we used the familiar formula for geometric series from calculus. Now let $\epsilon > 0$ be given and let $N = -\log_3 \frac{2}{3}\epsilon$, whence

$$m > n > N \implies |s_m - s_n| \leq \frac{3}{2} \cdot 3^{-n} < \frac{3}{2} \cdot 3^{-N} = \epsilon$$

We conclude that (s_n) is Cauchy.

Theorem 2.22 (Cauchy Completeness)**.** *A sequence of real numbers is convergent if and only if it is Cauchy.*

Proof. ($\Rightarrow$) Suppose $s_n \to s$. Given $\epsilon > 0$ we may choose N such that

$$\begin{aligned} m, n > N &\implies |s_n - s| < \frac{\epsilon}{2} \quad \text{and} \quad |s_m - s| < \frac{\epsilon}{2} \\ &\implies |s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m| < \epsilon \end{aligned}$$

whence (s_n) is Cauchy.

($\Leftarrow$) To discuss the convergence of (s_n) we first need a potential limit! In view of Theorem 2.20, the obvious candidates are $\limsup s_n$ and $\liminf s_n$. We have two goals: show that (s_n) is bounded, whence the limits superior and inferior are *finite,* and then show that they are *equal.*

(Boundedness of (s_n)) Take $\epsilon = 1$ in the definition of Cauchy to see that $\exists N \in \mathbb{N}$ such that

$$m, n > N \implies |s_n - s_m| < 1$$

It follows that

$$n > N \implies |s_n - s_{N+1}| < 1 \implies s_n < s_{N+1} + 1$$

Thus (s_n) is bounded above. Similarly (s_n) is bounded below.

($\limsup s_n = \liminf s_n$) Given $\epsilon > 0$, $\exists N$ such that

$$m, n > N \implies |s_n - s_m| < \epsilon \implies s_n < s_m + \epsilon$$

But then

$$\begin{aligned} m > N &\implies v_N \leq s_m + \epsilon && (\text{since } v_N = \sup\{s_n : n > N\}) \\ &\implies v_N \leq u_N + \epsilon && (\text{since } u_N = \inf\{s_n : n > N\}) \end{aligned}$$

Since (v_N) is monotone-down and (u_N) monotone-up, we see that

$$\forall \epsilon > 0, \ \limsup s_n \leq v_N \leq u_N + \epsilon \leq \liminf s_n + \epsilon$$

whence $\limsup s_n \leq \liminf s_n$. By Lemma 2.19 we have equality.

By Theorem 2.20 we conclude that (s_n) converges to $\limsup s_n = \liminf s_n$. ∎

Now that we have the Theorem, the above examples are seen to converge. Clearly example 1 converges to zero! Example 2 can be shown to converge to $\frac{223}{120} = 1.8583333\ldots$ using geometric series.

The main point of the Cauchy Completeness Theorem is easy to miss. To show that (s_n) is convergent using the original Definition (2.2) *one must already know the limit*! We are now in the position of (hopefully) being able to show that a sequence is Cauchy (and thus convergent) *without* first knowing its limit. There are many applications of this idea, here is a simple example.

Decimals and the Real Numbers

What should a decimal expression mean? It is clear what a *terminating* decimal means, since every such can be written as a rational number; for instance

$$12.31452 = \frac{1231452}{10000}$$

What about a decimal that does not terminate? We can instead view the decimal as representing a *sequence of rational numbers*; for example

$$3.14159\cdots \text{ represents the sequence } (s_n) = \left(3, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \ldots\right)$$

Naturally, we'd like every such sequence to converge!

Generally, it is enough to consider decimals of the form

$$s = 0.d_1d_2d_3\cdots \tag{$*$}$$

Consider the sequence (s_n) where

$$s_n = 0.d_1d_2\cdots d_n = \sum_{k=1}^{n} 10^{-k}d_k$$

is the rational number comprising the first n decimal places of s. We prove that (s_n) is Cauchy:

Let $\epsilon > 0$ be given and choose $N = -\log_{10}\epsilon$. Then

$$\begin{aligned} m > n > N \implies |s_n - s_m| &= \sum_{k=n+1}^{m} 10^{-k}d_k && (= 0.0\cdots 0d_{n+1}\cdots d_m000\cdots) \\ &< 10^{-n} < 10^{-N} = \epsilon \end{aligned}$$

The sequence (s_n) is Cauchy and thus converges to some real number s. This limit is precisely what it meant by the expression $(*)$. The upshot is that every decimal represents a single real number.

Aside: An alternative definition of $\mathbb{R}$ We can moreover use this approach to give another *definition* of the real numbers which does not rely on Dedekind cuts.

Consider the set $\mathcal{C}$ of all Cauchy sequences of rational numbers and define an equivalence relation on $\mathcal{C}$:

$$(s_n) \sim (t_n) \iff |s_n - t_n| \to 0$$

We may then define $\mathbb{R} = {}^{\mathcal{C}}\!/\!_{\sim}$. Intuitively, (s_n) and (t_n) have the same limit, though this notion is not required in order to make the definition rigorous. Some work is still required to define $+, \cdot, \leq$, etc., and to check all the axioms of an ordered field.

2.11 Subsequences

Most often a sequence does not exhibit any general properties (convergence, etc.). However, if we delete some of the sequence we may obtain a *subsequence* with interesting behavior.

Definition 2.23. (s_{n_k}) is a *subsequence* of (s_n) if it is a subset $(s_{n_k}) \subseteq (s_n)$, and

$$n_1 < n_2 < n_3 < \cdots$$

A subsequence is simply an infinite subset, ordered the same as the original sequence.

Example Take $s_n = (-1)^n$ and $s_{n_k} = 1$ (where $n_k = 2k$). Note that (s_n) is a non-convergent sequence with a convergent subsequence. Indeed our main goal for this section is to prove the famous Bolzano–Weierstrass Theorem, that *all* bounded sequences possess a convergent subsequence.

Lemma 2.24. *Every subsequence of a convergent sequence converges to the same limit:* $s_n \to s \implies s_{n_k} \to s$.

Proof. Let $\epsilon > 0$ be given. Then $\exists N$ such that

$$n > N \implies |s_n - s| < \epsilon$$

Now $n_k \geq k$ for all k, whence

$$k > N \implies n_k > N \implies |s_{n_k} - s| < \epsilon$$

∎

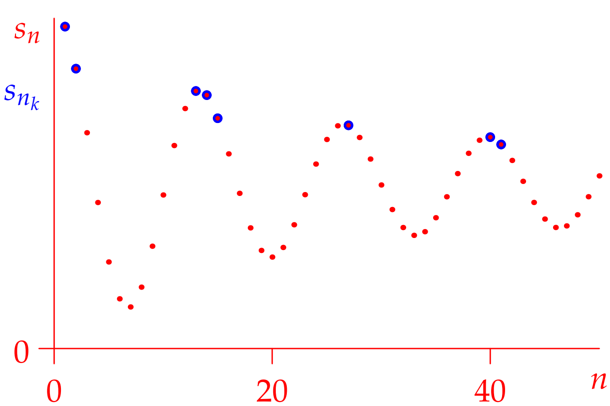
Theorem 2.25. *Every sequence has a monotonic subsequence.*

Proof. Let (s_n) be a sequence. We call s_n 'dominant' if $m > n \implies s_m < s_n$. There are two cases:

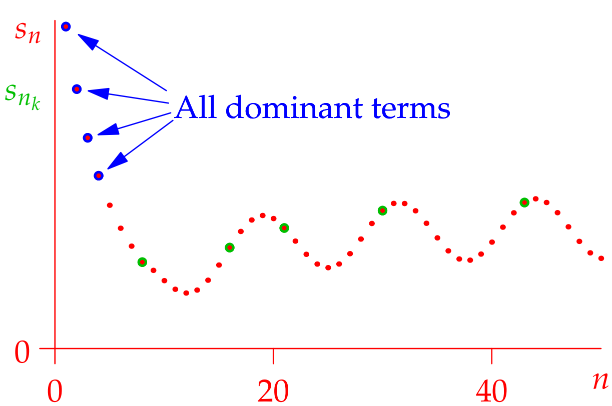
1. There are infinitely many dominant terms. The subsequence of dominant terms is decreasing. Moreover if s_n is dominant, then $v_{n-1} = \sup\{s_k : k > n-1\} = s_n$, whence the subsequence of dominant terms converges to $\limsup s_n$ (or diverges to $\limsup s_n = -\infty$ in the special case).

2. There are finitely many dominant terms. Choose s_{n_1} to come after all dominant terms. Similarly, since s_{n_1} is not dominant, $\exists n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$. Repeat this process to obtain a non-decreasing subsequence.

∎



Case 1: dominant subsequence
monotone-down



Case 2: Finitely many dominant terms
Monotone-up subsequence

Theorem 2.26. *Given a sequence (s_n), there exist subsequences (s_{n_k}) and (s_{n_l}) such that*

$$\lim s_{n_k} = \limsup s_n \quad \textit{and} \quad \lim s_{n_l} = \liminf s_n$$

Proof. We prove only the claim regarding $\limsup s_n$, since the other is similar. There are three cases to consider; visualizing the third is particularly difficult and may take several readings...

1. $\limsup s_n = \infty$: Define a subsequence (s_{n_k}) inductively via

   $$n_1 = \min\{n \in \mathbb{N} : s_{n_1} > 1\} \qquad n_k = \min\{n \in \mathbb{N} : n_k > n_{k-1},\ s_{n_k} > k\}$$

   Since $\limsup s_n = \infty$, the sequence (s_n) is unbounded above; for any $k > 0$, there exist *infinitely many terms* s_n greater than k. At each step in the creation of (s_{n_k}) we are taking the minimum of a *non-empty* set of natural numbers; (s_{n_k}) is therefore well-defined. Clearly

   $$s_{n_k} > k \quad \text{whence} \quad s_{n_k} \to \infty = \limsup s_n$$

2. $\limsup s_n = -\infty$: Since $\liminf s_n \le \limsup s_n = -\infty$, we conclude (Theorem 2.20) that $\lim s_n = -\infty$. It follows that (s_n) is itself a suitable subsequence.

3. $\limsup s_n = v$ is finite: Let $n_1 = 1$. For each $k \ge 2$, perform a dual construction:

   - Since (v_N) is monotone-down and converges to v,

     $$\exists N_k \ge n_{k-1} \text{ such that } v \le v_{N_k} < v + \frac{1}{2k}$$

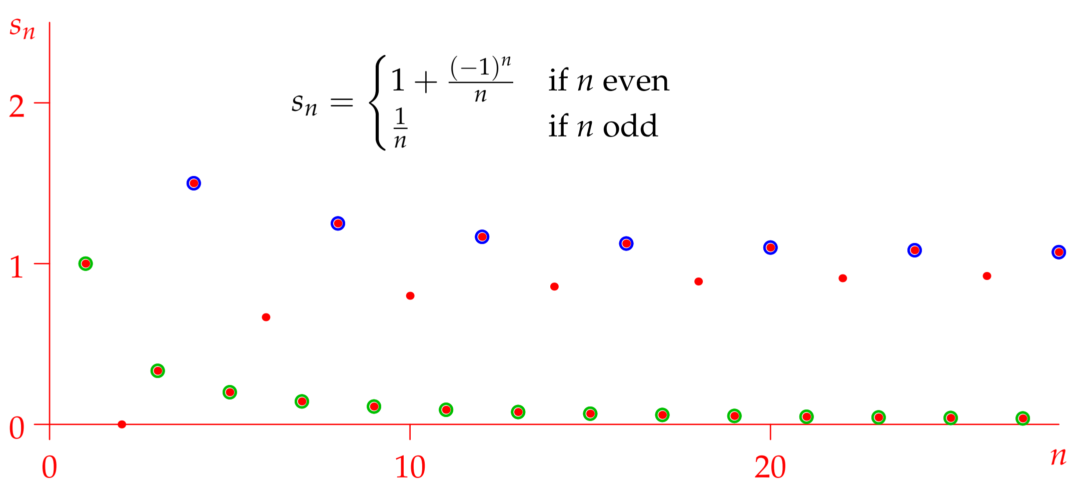
   - Since $v_{N_k} = \sup\{s_n : n > N_k\}$,

     $$\exists n_k > N_k \text{ such that } v_{N_k} - s_{n_k} < \frac{1}{2k}$$

   But then $|v - s_{n_k}| < \frac{1}{k}$, whence s_{n_k} is a subsequence convergent to v. ■

Corollary 2.27. *There exists a monotonic subsequence $s_{n_k} \to \limsup s_n$ (to $\liminf s_n$ similarly).*

Proof. By Theorem 2.26, $\exists (s_{n_l})$ such that $s_{n_l} \to \limsup s_n$. This subsequence has a monotonic subsequence by Theorem 2.25, which must converge to the same limit $\limsup s_n$ by Lemma 2.24. ■



Example: Monotonic sequences converging to $\limsup s_n = 1$ and $\liminf s_n = 0$

Theorem 2.28 (Bolzano–Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

We give three proofs! The first two are corollaries of the above discussion; the third is the classic proof and is independent of the discussion of limits superior/inferior.

Proof 1. Theorem 2.25 says there exists a monotone subsequence. This is bounded and thus converges by the monotone convergence theorem. ∎

Proof 2. By Theorem 2.26, there exists a subsequence converging to the *finite* value $\limsup s_n$. ∎

Proof 3. Suppose (s_n) is bounded by M. One of the intervals $[-M, 0]$ or $[0, M]$ must contain infinitely many terms of the sequence (perhaps both!). Call this interval E_0 and define

$$n_0 = \min\{n \in \mathbb{N} : s_{n_0} \in E_0\}$$

Now repeat. Split E_0 into left and right half-intervals. One of these intervals must contain infinitely many terms of the subsequence

$$(s_n \in E_0 : n > n_0)$$

Call this half-interval E_1 and choose $n_2 = \min\{n \in \mathbb{N} : n_1 > n_0, s_{n_1} \in E_1\}$. Repeat this process *ad infinitum*, we obtain a family of nested intervals

$$E_0 \supset E_1 \supset E_2 \supset \cdots \qquad \text{of width} \quad |E_k| = \frac{M}{2^k}$$

and a subsequence (s_{n_k}) where each $s_{n_k} \in E_k$.

Now let $\epsilon > 0$ be given and let $N \in \mathbb{N}$ satisfy $N > \log_2 \frac{M}{\epsilon}$. Then

$$k, l > N \implies s_{n_k}, s_{n_l} \in E_N \implies |s_{n_k} - s_{n_l}| \leq \frac{M}{2^N} < \epsilon$$

The subsequence (s_{n_k}) is Cauchy, and thus converges. ∎

The advantage of the final proof is that it generalizes to higher dimensions: rather than intervals, a family of shrinking boxes is constructed...

Divergence by Oscillation Recall Definition 2.12, where we stated that a sequence (s_n) *diverges by oscillation* if it neither converges nor diverges to $\pm\infty$. We can now give a more positive statement which gives light to the notion of oscillation.

Corollary 2.29. *Let (s_n) be a sequence. The following are equivalent:*

- (s_n) diverges by oscillation
- $\liminf s_n \neq \limsup s_n$
- (s_n) *has at least two subsequences which converges to different limits.*

We omit a proof, though it requires nothing more than putting together some of the previous results. The word *oscillation* comes from the third interpretation: if $s_1 \neq s_2$ are the limits of the two subsequences, then in any tail of the sequence $\{s_n : n > N\}$ there are infinitely many terms arbitrarily close to s_1 and infinitely many (different!) terms arbitrarily close to s_2. In this sense the original sequence *oscillates* between the neighborhoods of s_1 and s_2. Of course the sequence could have many other *subsequential limits.*

Subsequential Limits & Closed Sets - non-examinable

Definition 2.30. We call $s \in \mathbb{R} \cup \{\pm\infty\}$ a *subsequential limit* of a sequence (s_n), if there exists a subsequence (s_{n_k}) such that $s_{n_k} \to s$.

Examples

1. The sequence defined by $s_n = \frac{1}{n}$ has only one subsequential limit, namely zero. Recall Lemma 2.24: $s_n \to 0$ implies that every subsequence also converges to 0.
2. If $s_n = (-1)^n$, then the subsequential limits of (s_n) are ± 1.
3. The sequence $s_n = n^2(1 + (-1)^n)$ has subsequential limits 0 and ∞.
4. $(s_n) = (2, 4, 2, 6, 4, 2, 8, 6, 4, 2, 10, \ldots)$ has all positive even numbers as subsequential limits.

Denseness and the countability of $\mathbb{Q}$ The set of subsequential limits of a sequence can be surprisingly large, as we now show. You have seen in a previous class that the rational numbers are a countable set: otherwise said, $\exists f : \mathbb{N} \to \mathbb{Q}$ bijective. This means that there exists a sequence (r_n) defined by $r_n = f(n)$ which lists every rational number: here is a concrete example

$$(r_n) = \Big(\frac{0}{1}, \underbrace{\frac{1}{1}, -\frac{1}{1}}_{|p|+q=2}, \underbrace{\frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}}_{|p|+q=3}, \underbrace{\frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1}}_{|p|+q=4}, \underbrace{\frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{2}, -\frac{3}{2}, \frac{4}{1}, -\frac{4}{1}}_{|p|+q=5}, \ldots\Big)$$

where $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}, \gcd(p,q) = 1\}$ and terms are grouped by increasing $|p| + q$. The following consequence of this should seem truly bizarre. . .

Theorem 2.31. *Let $a \in \mathbb{R}$. Then (r_n) has a subsequence which converges to a.*

Proof. Define a subsequence (r_{n_k}) of rational numbers inductively:

$$n_1 := \min\{n \in \mathbb{N} : |r_n - a| < 1\} \qquad n_k := \min\{n \in \mathbb{N} : n > n_{k-1} \text{ and } |r_n - a| < \tfrac{1}{k}\}$$

- The interval $(a - 1, a + 1)$ contains infinitely many rational numbers (Archimedes), thus r_{n_1} is well-defined.
- For some fixed $k \geq 2$, suppose $r_{n_1}, \ldots, r_{n_{k-1}}$ have been defined, where

  $$n_1 < n_2 < \ldots < n_{k-1} \text{ and } \forall j \leq k-1, \left|r_{n_j} - a\right| < \frac{1}{j}$$

  The interval $(a - \frac{1}{k}, a + \frac{1}{k})$ contains infinitely many rational numbers. Since $r_1, \ldots, r_{n_{k-1}}$ is a *finite* list, there is at least one (indeed infinitely many) rational $r_{n_k} \in (a - \frac{1}{k}, a + \frac{1}{k})$ such that $n_k > n_{k-1}$. Thus r_{n_k} is well-defined.
- By induction, the subsequence (r_{n_k}) is well-defined. Clearly $|r_{n_k} - a| < \frac{1}{k} \implies r_{n_k} \to a$.

■

We already know that every real number is the limit of some sequence of rational numbers. The Theorem goes further: every real number is the limit of some subsequence of a *particular* sequence!

Theorem 2.32. *Let (s_n) be a sequence in $\mathbb{R}$ and let S be its set of subsequential limits. Then*

1. *S is non-empty (as a subset of $\mathbb{R} \cup \{\pm\infty\}$).*

2. $\sup S = \limsup s_n$ *and* $\inf S = \liminf s_n$.

3. $\lim s_n$ *exists iff S has only one element: namely* $\lim s_n$.

Proof. 1. By Theorem 2.26, $\limsup s_n \in S$.

2. By 1, $\limsup s_n \leq \sup S$. For any subsequence (s_{n_k}), we have $n_k \geq k$, whence

$$\forall N, \ \{s_{n_k} : k > N\} \subseteq \{s_n : n > N\} \implies \lim s_{n_k} = \limsup s_{n_k} \leq \limsup s_n$$

This holds for *every* convergent subsequence, whence $\sup S \leq \limsup s_n$, and we have equality. The result for $\inf S$ is similar.

3. Applying Theorem 2.20, we see that $\lim s_n$ exists if and only if

$$\begin{aligned}\limsup s_n = \liminf s_n &\iff \sup S = \inf S \\ &\iff S \text{ has only one element}\end{aligned}$$

■

Closed Sets You've used the notion of a closed interval for years. Here is the sequential *definition* of a closed set.

Definition 2.33. A subset $A \subseteq \mathbb{R}$ is *closed* if every convergent sequence in A has its limit in A.

Examples

1. The interval $[0,1]$ is closed. If $(s_n) \subseteq [0,1]$ is a convergent sequence $s_n \to s$, then

$$0 \leq s_n \leq 1 \implies s \in [0,1]$$

More generally, every closed interval $[a,b]$ is closed, as are *finite* unions of closed intervals, e.g. $[1,5] \cup [7,11]$.

2. The interval $(0,1]$ is not closed. In particular, the sequence $s_n = \frac{1}{n}$ lies entirely in the interval but has limit lying outside.

Theorem 2.34. *If (s_n) is a sequence, then its set of (finite) subsequential limits is closed.*

We omit the proof: it is not difficult, but involves unpleasantly many subscripts (subsequences of subsequences...). The theorem essentially says that one can make a set closed by throwing in all its *sequential limits.*

2.12 Lim sup's and Lim inf's

In this section we collect a couple of useful results, mostly for later use. Firstly we observe that the limit laws do not work as tightly for limits superior and inferior.

Theorem 2.35. 1. *For any bounded sequences $(s_n), (t_n)$ we have*

$$\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$$

In general we do not expect equality.

2. *If, in addition, $s_n \to s$ is convergent, then we have equality*

$$\limsup(s_n + t_n) = s + \limsup t_n$$

Careful modifications can be made for unbounded sequences.

Proof. 1. For any N, observe that $\{s_n + t_n : n > N\}$ has upper bound

$$\sup\{s_n : n > N\} + \sup\{t_n : n > N\}$$

from which

$$\sup\{s_n + t_n : n > N\} \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$$

Now take limits as $N \to \infty$.

To see that equality is unlikely, consider the sequences $s_n = (-1)^n = -t_n$. Then

$$\limsup(s_n + t_n) = 0 < 2 = \limsup s_n + \limsup t_n$$

2. By Theorem 2.26 $\exists t_{n_k} \to \limsup t_n$. Therefore

$$s_{n_k} + t_{n_k} \to s + \limsup t_n$$

By Theorem 2.32, $\limsup(s_n + t_n)$ is the supremum of the set of subsequential limits of $(s_n + t_n)$, whence

$$s + \limsup t_n \le \limsup(s_n + t_n)$$

Combining with part 1 gives the result.

∎

A similar result is available for products:

Corollary 2.36. 1. *For any bounded non-negative sequences $(s_n), (t_n)$ we have*

$$\limsup(s_n t_n) \le (\limsup s_n)(\limsup t_n)$$

with no expectation of equality.

2. *If, in addition, $s_n \to s > 0$ is convergent, then*

$$\limsup(s_n t_n) = s \limsup t_n$$

The next result will be critical when we study infinite series.

Theorem 2.37. *Let (s_n) be a non-zero sequence. Then*

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{1/n} \le \limsup |s_n|^{1/n} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

Proof. We prove only the first inequality. Suppose $\liminf \left|\frac{s_{n+1}}{s_n}\right| > 0$, for otherwise the inequality is trivial, and assume that $0 < L < \liminf \left|\frac{s_{n+1}}{s_n}\right|$. Then

$$\lim_{N\to\infty} \inf\left\{\left|\frac{s_{n+1}}{s_n}\right| : n > N\right\} > L \implies \exists N \text{ such that } \inf\left\{\left|\frac{s_{n+1}}{s_n}\right| : n > N\right\} > L$$
$$\implies \left|\frac{s_{n+1}}{s_n}\right| > L, \quad \forall n > N$$

If $n > N$, we then have

$$|s_n| > L^{n-N}|s_N| \implies |s_n|^{1/n} > L\left(L^{-N}|s_N|\right)^{1/n} \to L \qquad \text{(Theorem 2.10, (c))}$$

Therefore $\liminf |s_n|^{1/n} \ge L$ for all $L < \liminf \left|\frac{s_{n+1}}{s_n}\right|$, which establishes the first inequality. ∎

Corollary 2.38. $\lim \left|\frac{s_{n+1}}{s_n}\right| = L \implies \lim |s_n|^{1/n} = L$

Examples

1. Here is a quick proof that $\lim n^{1/n} = 1$ (recall Theorem 2.10, (d)): let $s_n = n$, then

$$\lim \left|\frac{s_{n+1}}{s_n}\right| = \lim \frac{n+1}{n} = 1 \implies \lim n^{1/n} = \lim |s_n|^{1/n} = 1$$

2. $\lim(n!)^{1/n} = \infty$. Simply let $s_n = n!$ and apply the corollary:

$$\lim \left|\frac{s_{n+1}}{s_n}\right| = \lim(n+1) = \infty$$

3. We compute $\lim\left(\frac{(2n)!}{(n!)^2}\right)^{1/n} = 4$. Taking $s_n = \frac{(2n)!}{(n!)^2}$, we obtain

$$\lim |s_n|^{1/n} = \lim \left|\frac{s_{n+1}}{s_n}\right| = \lim \frac{(2n+2)!(n!)^2}{(2n)!(n+1)!^2} = \frac{(2n+2)(2n+1)}{(n+1)^2} = 4$$

4. Note that the converse to the corollary is false! For example, consider

$$(s_n) = \left(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots\right), \text{ where } s_{2n-1} = s_{2n} = 2^{1-n}$$

and check that this satisfies

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| = \frac{1}{2} < \lim |s_n|^{1/n} = \frac{1}{\sqrt{2}} < \limsup \left|\frac{s_{n+1}}{s_n}\right| = 1$$