MATHCOUNTS

2012 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Team Round problem with less that 10 sheets of scratch paper?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2012 MATHCOUNTS® State Competition. Though these solutions provide creative and concise ways of solving the problems from the competition, there are certainly numerous other solutions that also lead to the correct answer, and may even be more creative or more concise! We encourage you to find numerous solutions and representations for these MATHCOUNTS problems.

Special thanks to volunteer author Mady Bauer for sharing these solutions with us and the rest of the MATHCOUNTS community!

2012 State Competition

Sprint Round

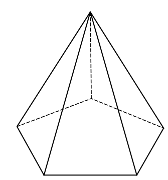
1. A bucket with a hole is filled with 10 gallons of water, but loses 2 gallons every 10 minutes. How many minutes does it take for the bucket to be empty?
   After 10 minutes the bucket will contain 8 gallons and after 20 minutes, 6 gallons, etc. It will lose 2 gallons 10/2 = 5 times, and 5 × 10 = 50 **Ans.**

2. The sum of the digits of a two-digit positive integer is 9. The difference between this integer and a two-digit integer containing the digits reversed is 27. We must find the product of the two digits.
   Let x and y be the two digits. Then the first number is $10x + y$ and the second number is $10y + x$. We also have
   $x + y = 9$. So, $x = 9 – y$ and
   $10x + y – (10y + x) = 27$.
   Substituting for x, we get
   $10(9 – y) + y – (10y + 9 – y) = 27$
   $90 – 10y + y – 10y – 9 + y = 27$
   $81 – 18y = 27$
   $18y = 54$
   $y = 3$. That means $x = 9 – y = 9 − 3 = 6$.
   The product of the two digits is 6 × 3 = 18 **Ans.**

3. How many diagonals does a convex octagon have?
   A convex octagon is an 8-sided figure. The number of diagonals in a polygon is $(1/2)n(n – 3)$, where n is the number of sides. In this case, $n = 8$. We have
   (1/2) × 8 × (8− 3) = 4 × 5 = 20 **Ans.**

4. Jean is twice as likely to make a free throw as she is to miss it. What is the probability that she will miss 3 times in a row?
   Being twice as likely to make a free throw as miss it means that Jean will make it 2 times out of every 3. So the probability that she misses is 1/3. Therefore, the probability that she will miss 3 times in a row is
   (1/3) × (1/3) × (1/3) = 1/27 **Ans.**

5. A pyramid has 6 vertices and 6 faces. How many edges does it have?
   That means the pyramid is a pentagonal pyramid and looks like this.

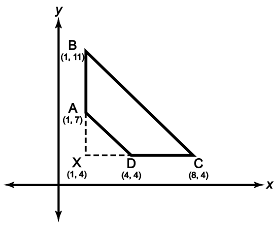
   

   There are 5 edges that extend from the apex to the vertices of the base. At the base there are another 5 edges. In other words, the number of edges in a pyramid is twice the number of edges of the base polygon. 5 + 5 = 10 **Ans.**

6. The product of the integers from 1 through 7 is equal to $2^j \cdot 2^k \cdot 5 \cdot 7$
   What is the value of $j – k$?
   $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 =$
   $2 \cdot 3 \cdot 2^2 \cdot 5 \cdot 2 \cdot 3 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$
   Therefore, $j = 4$ and $k = 2$ and
   $j – k = 4 – 2 = 2$ **Ans.**

7. A six-sided die was rolled repeatedly and 1 was rolled 4 times, 2 - 4 times, 3 - 3 times, 4 - 4 times, 5 - 2 times and 6 - 3 times. We need to find the mean of the 20 numbers that were rolled. The sum of the numbers rolled is (1 × 4) + (2 × 4) + (3 × 3) + (4 × 4) + (5 × 2) + (6 × 3) = 4 + 8 + 9 + 16 + 10 + 18 = 65. The mean number rolled is 65 ÷ 20 = 3.25 **Ans.**

8. Let's name the coordinates of the vertices of a trapezoid are A(1, 7), B(1, 11), C(8, 4) and D(4, 4). What is the area of the trapezoid?
Let's now extend one side of the trapezoid to create a segment with endpoints A(1, 11) and X(1, 4), and extend the other side of the trapezoid to create a segment with endpoints C(8, 4) and X(1, 4), as shown.

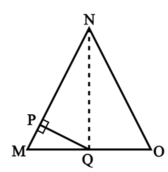


We create 2 right triangles, ΔAXD and ΔBXC. The area of the trapezoid is the area of ΔBXC less the area of ΔAXD. The area of ΔBXC is (1/2) × 7 × 7 = 49/2. The area of ΔAXD is (1/2) × 3 × 3 = 9/2. Therefore, the area of trapezoid ABCD is (49/2) – (9/2) = 40/2 = 20 **Ans.**

9. Malika ran 3 miles. She ran the first mile in 6 minutes and 45 seconds. Each of the remaining two miles after that took 1/9 longer than the previous mile. We must find, in seconds, how long it took Malika to run all 3 miles.
Converting 6 minutes and 45 seconds to seconds, we get (6 × 60) + 45 = 360 + 45 = 405. To find a time that is 1/9 longer we must multiply by 10/9 to get (10/9) × 405 = 10 × 45 = 450. Malika took 450 seconds to run the 2nd mile. For the final mile we have (10/9) × 450 = 10 × 50 = 500. It took Malika 500 seconds to run the 3rd mile. The total is 405 + 450 + 500 = 1355 **Ans.**

10. Four consecutive integers are substituted in every possible way for distinct values *a*, *b*, *c* and *d*. What is the positive difference between the smallest and largest possible values of (*ab* + *cd*)?
We could do this using *x*, *x*+1, *x*+2 and *x*+3 to represent the integers, but let's just choose 1, 2, 3 and 4 instead.
For *ab* we could have 1 × 2 = 2, 1 × 3 = 3, 1 × 4 = 4, 2 × 3 = 6, 2 × 4 = 8, or 3 × 4 = 12.
For *cd* we could have 3 × 4 = 12, 2 × 4 = 8, 2 × 3 = 6, 1 × 4 = 4, 1 × 3 = 3, or 1 × 2 = 2.
All possible sums *ab* + *cd* are 14, 11, 10, 10, 11, 14.
The smallest value of *ab* + *cd* is 10 and the largest value is 14. The difference is 14 − 10 = 4 **Ans.**

11. Triangle MNO is an isosceles triangle with MN = NO = 25. A line segment drawn from the midpoint of MO perpendicular to MN, intersects MN at point P with NP:PM = 4:1. We must find the length of the altitude drawn from point N to side MO.



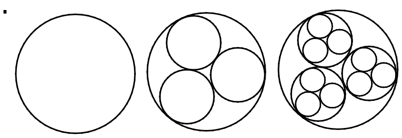
Let *h* = NQ, *x* = MQ and *y* = PQ. Given that NP:PM = 4:1 and MN = 25, it must be that that NP = 20 and PM = 5.
To reduce the number of unknowns, let's try creating equations for the hypotenuse of triangles MPQ, PNQ and MNQ using the Pythagorean Theorem.
For ΔMPQ, $5^2 + y^2 = x^2 \rightarrow 25 + y^2 = x^2$
For ΔPNQ, $y^2 + 20^2 = h^2 \rightarrow y^2 + 400 = h^2$
For ΔMNQ, $x^2 + h^2 = 25^2 \rightarrow x^2 + h^2 = 625$. Adding the first two equations we get $25 + y^2 + y^2 + 400 = x^2 + h^2 \rightarrow 2y^2 + 425 = x^2 + h^2$
Substituting 625 for $x^2 + h^2$, we get $2y^2 + 425 = 625 \rightarrow 2y^2 = 200 \rightarrow y^2 = 100$

and $y = 10$. Now let's find h,
$y^2 + 400 = h^2 \rightarrow 10^2 + 400 = h^2 \rightarrow$
$h^2 = 500 \rightarrow h = 10\sqrt{5}$ **Ans.**

12. $\{a,b,c,d\}$ is a set of numbers chosen from the first nine positive integers. If you add every possible pair of these four numbers you get 7, 9, 10, 12, 13 and 15. We must find the smallest possible product of these four numbers. There are four numbers from which to choose. This results in ${}_4C_2 = 6$ different pairings. Since we are given 6 different sums, we know that the four numbers are all different. The sum of 15 must come from the two largest numbers and can only come from {7, 8} or {6, 9}, resulting in products of 56 and 54, respectively. Therefore, we'd rather use {6, 9}, if possible. The sum of 7 must come from the two smallest numbers and can only come from {1, 6}, {2, 5} or {3, 4}, resulting in products of 6, 10 and 12, respectively. We don't need to test {1, 6, 6, 9} since the four chosen numbers must be distinct. Will {1, 6, 7, 8} work? No, there would need to be a sum of 8 for the 1, 7 pairing. Will {2, 5, 6, 9} work? No, again there would need to be a sum of 8 for the 2, 6 pairing. Will {2, 5, 7, 8} work? Yes, and the product of the four numbers is 560. The only other option we may want to test is {3, 4, 6, 9}. We see this works, too, but its product is 648. We know {3, 4, 7, 8} would result in the largest possible product even if it worked, so we don't need to test it. Our answer is 560 **Ans.**

13. In a sequence of positive integers, every term after the first two terms is the sum of the previous two terms of the sequence. The fifth term is 2012 so what is the maximum possible value of the first term?
Let x be the value of the first term and y be the value of the second term. We can then think of our sequence as: x, y, $x + y$, $x + 2y$ and $2x + 3y$. From this we have $2x + 3y = 2012$, and we want the maximum possible integer value for x, the first term. Therefore, we would like y to be as little as possible, while still remaining a positive integer. If $y = 1$, then $2x = 2009$, and x is not an integer. However, if $y = 2$, then $2x = 2006$, and $x = 1003$. **Ans.**

14. The figure shows the first three stages of a fractal, respectively. We must find how many circles in Stage 5 of the fractal.

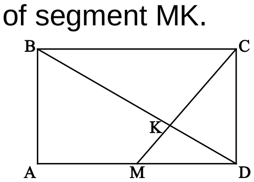


As we can see, in Stage 1, there is only 1 circle. In Stage 2, we have $1 + 3 = 4$.
In Stage 3, we have $1 + 3 + 9 = 13$.
We can write these 3 values as: 1, $1 + 3^1$, $1 + 3^1 + 3^2$
For Stage 4, each of the smaller circles would have 3 circles inscribed in each of the 9 medium sized circles of Stage 3. That's a total of 27 more circles, so Stage 4 has $1 + 3^1 + 3^2 + 3^3 = 13 + 27 = 40$ circles. We can see the pattern now and Stage 5 must have $1 + 3^1 + 3^2 + 3^3 + 3^4 = 40 + 81 = 121$ **Ans.**

15. We have a set of numbers {1, 2, 3, 4, 5} and we take products of three different numbers. We must find now many pairs of relatively prime numbers there are.
Let's list the products.
$1 \times 2 \times 3 = 6$
$1 \times 2 \times 4 = 8$
$1 \times 2 \times 5 = 10$
$1 \times 3 \times 4 = 12$

$1 \times 3 \times 5 = 15$
$1 \times 4 \times 5 = 20$
$2 \times 3 \times 4 = 24$
$2 \times 3 \times 5 = 30$
$2 \times 4 \times 5 = 40$
$3 \times 4 \times 5 = 60$
Two integers are said to be relatively prime if they have a GCF of 1.
Therefore, no 2 even numbers can be relatively prime since they will always have two as a common factor.
We have only one odd number and that's 15.
6 and 15 both have 3 as a divisor.
8 and 15 are relatively prime. That's 1 pair.
10 and 15 both have 3 as a divisor.
12 and 15 both have 3 as a divisor.
20 and 15 both have 4 as a divisor.
24 and 15 both have 3 as a divisor.
30 and 15 both have 3 as a divisor.
40 and 15 both have 5 as a divisor.
60 and 15 both have 4 as a divisor.
Look's like there is 1 pair. 1 **Ans.**

16. In rectangle ABCD, AB = 6 units. $m\angle DBC = 30°$, M is the midpoint of segment AD, and segments CM and BD intersect at point K. We must find the length of segment MK.



Since $m\angle DBC = 30°$ and $m\angle BCD = 90°$, it follows that $m\angle CDB = 180 - (30 + 90) = 180 - 120 = 60°$.
Therefore, triangle ΔBCD is a 30-60-90 right triangle. Because ABCD is a rectangle, CD = AB = 6, and the angle opposite side CD has measure 30°. Therefore, BD = 2 × 6 = 12. Let x = BC. Using the Pythagorean Theorem, we get $x^2 + 6^2 = 12^2 \rightarrow x^2 + 36 = 144 \rightarrow x^2 = 108 \rightarrow x = \sqrt{108} \rightarrow x = 6\sqrt{3}$.
We know that MD = (1/2)AD and AD = BC. So MD = $3\sqrt{3}$. We can now solve for the length of segment MC.
Let y represent the length of segment MC. Using the Pythagorean Theorem we have $6^2 + (3\sqrt{3})^2 = y^2 \rightarrow 36 + 27 = y^2 \rightarrow y = \sqrt{63} \rightarrow y = 3\sqrt{7}$.
Now we have to determine where segments BD and CM intersect.
Since $m\angle KDM = m\angle KBC$ and $m\angle MKD = m\angle CKB$, $\Delta MDK \sim \Delta CBK$. That means

$$\frac{MD}{BC} = \frac{MK}{CK} \rightarrow \frac{3\sqrt{3}}{6\sqrt{3}} = \frac{1}{2} = \frac{MK}{CK}$$ and

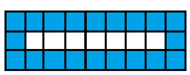
CK = 2MK.
Let z represent MK. Then $MK + CK = CM \rightarrow z + 2z = 3\sqrt{7} \rightarrow z = \sqrt{7}$ **Ans.**

17. Jack and Jill drove the same distance. Jill drove 20% faster than Jack and she arrived half an hour earlier. We must find how many hours Jack drove.
Let d be the distance they each drove, let s be the speed (mph) that Jack drove and let t be the time it took Jack to get there. Then the speed for Jill is $(6/5)s$ and her time to get there is $t - (1/2)$. For Jack, $d = st$, and for Jill, $d = (6/5)s \times (t - (1/2)) \rightarrow st = (6/5)st - (3/5)s \rightarrow t = (6/5)t - (3/5) \rightarrow 5t = 6t - 3 \rightarrow t = 3$ **Ans.**

18. What is the largest five-digit integer such that the product of the digits is 2520?
If we factor 2520, we see $2520 = 2^3 \times 3^2 \times 5 \times 7$. If we can start our five-digit integer with a 9, that would be great. Using the 3^2, we get the 9. We still have $2^3 \times 5 \times 7$ to work with. We can't make another 9 out of these, but the 2^3 can give us an 8, so we now have 98,_ _ _

with 5 × 7 left. One may think there are only two digits left and three blanks to fill, but remember any of those blanks can be filled with a 1 without changing the product of the digits. The answer is then 98,751 **Ans.**

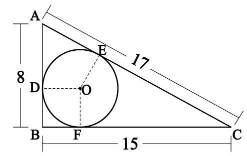
19. A rectangular prism is composed of unit cubes. The outside faces of the prism are painted blue and the seven unit cubes in the interior are unpainted. We must find how many unit cubes have exactly one painted face.
First we have to figure out what the dimensions of this prism are. Since 7 is prime, the only way to have 7 unpainted faces is for the prism to be 9 × 3 × 3. Then the middle layer of the prism looks like this:



Each of the shaded cubes has at least one side painted blue. The 4 corner cubes will have 2 sides painted blue leaving 27 – 7 – 4 = 16 cubes with one side painted blue. Now, consider the top or bottom layers. Since all cubes are exposed to the outside they all have at least one side painted blue. But all edge cubes have more than one side painted blue (side plus top or bottom – at least). Only the 7 inner cubes have exactly one painted face. So the top and bottom layers each have 7 cubes with only one side painted blue. That's a total of 16 + 7 + 7 = 30 **Ans.**

20. $f(x) = x^2 + 5$
$g(x) = 2(f(x))$
What is the greatest possible value of $f(x + 1)$ when $g(x) = 108$?
When $g(x) = 108$, we can write 108 = $2(f(x))$. Substituting we have 108 = $2(x^2 + 5)$. We can simplify to get $54 = x^2 + 5 \rightarrow 49 = x^2 \rightarrow x = 7$, and $x + 1 = 8$. Substituting $x = 8$ in the function $f(x) = x^2 + 5$, we get $f(8) = 8^2 + 5 = 69$ **Ans.**

21. A right triangle has sides with lengths 8, 15 and 17. A circle is inscribed in the triangle, as shown, and we must find the radius of the circle.

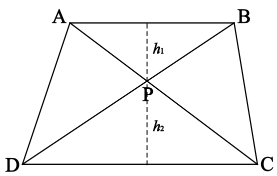


Draw a line from the center of the circle, point O, to side AB where the segment is tangent to the circle (point D). Do the same for side AC (point E) and for side BC (point F). Notice that $\overline{OD}$, $\overline{OE}$ and $\overline{OF}$ are radii of the circle. Since OE = OD, ADOE is a kite, and AD = AE. Similarly, since DE = OF, CEOF is also a kite, meaning EC = FC. Let r be the radius of the circle. Then AD = AE = $8 - r$, and CF = CE = $15 - r$. We know that AC = AE + CE , so substituting we get $8 - r + 15 - r = 17 \rightarrow 23 - 2r = 17 \rightarrow 2r = 6 \rightarrow r = 3$ **Ans.**

22. Given that $x \neq 0$, what quantity can be added to $\frac{x+1}{x}$ or multiplied by $\frac{x+1}{x}$ to give the same result?
Let y represent the value that satisfies the requirement, and let $z = \frac{x+1}{x}$. So we must find y such that $z + y = zy$. Solving for y, we get $zy - y = z \rightarrow$
$y(z - 1) = z \rightarrow y = \frac{z}{z-1}$.
Now let's substitute $\frac{x+1}{x}$ for z and

simplify. We get $y = \dfrac{z}{z-1} = \dfrac{\frac{x+1}{x}}{\frac{x+1}{x}-1} = \dfrac{x+1}{x+1-x} = \dfrac{x+1}{1}$ = x + 1 **Ans.**

23. In trapezoid ABCD segments AB and CD are parallel. Point P is the intersection of diagonals AC and BD. The area of ΔPAB is 16 and ΔPCD is 25. We must find the area of the trapezoid.
Start by drawing the height through the intersection of the diagonals, as shown.



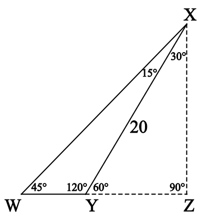
Let h_1 be the height of ΔPAB, h_2 be the height of ΔPCD and h be the height of the trapezoid. Notice that $h = h_1 + h_2$. Let b_1 = AB and b_2 = DC. The area of ΔPAB is 16 so $(½)b_1h_1 = 16 \rightarrow b_1h_1 = 32$. The area of ΔPCD is 25 so $(½)b_2h_2 = 25 \rightarrow b_2h_2 = 50$. The area of the trapezoid is $½(b_1 + b_2)h = ½(b_1 + b_2)(h_1 + h_2) = ½(b_1h_1 + b_1h_2 + b_2h_1 + b_2h_2)$. Substituting, we get $½(32 + b_1h_2 + b_2h_1 + 50) = ½(82 + b_1h_2 + b_2h_1)$. Since ΔPAB ~ ΔPCD, $\dfrac{b_1}{b_2} = \dfrac{h_1}{h_2}$. Cross-multiplying, we see $b_1h_2 = b_2h_1$. Now we can rewrite the expression for the area of the trapezoid as $½(82 + 2b_1h_2)$. It looks like we need to find b_1h_2, and we know that $b_1h_1 = 32$ and $b_2h_2 = 50$. Let's try multiplying these expressions. We have $(b_1h_1)(b_2h_2) = (32)(50) \rightarrow b_1h_1b_2h_2 = 1600$, which can be rewritten $(b_1h_2)(b_2h_1) = 1600$. But we know that $b_1h_2 = b_2h_1$, so we have $(b_1h_2)^2 = 1600 \rightarrow b_1h_2 = 40$. Now substituting this value into the equation for the area of the trapezoid, we have $½(82 + 2b_1h_2) = ½(82 + 80) = ½(162) = 81$ **Ans.**

Note that this is the derivation of the following formula for the area of the trapezoid: $A + B + 2\sqrt{AB}$, where A and B are the areas of the two triangles whose bases are parallel to each other, which makes this problem quite a bit easier!

24. The sum of the squares of two positive numbers is 20 and the sum of their reciprocals is 2. We must find their product. Let x and y be the two numbers. We are told $x^2 + y^2 = 20$ and $\dfrac{1}{x} + \dfrac{1}{y} = 2$. Simplifying the second equation, we get $\dfrac{1}{x} + \dfrac{1}{y} = 2 \rightarrow$ $\dfrac{y}{xy} + \dfrac{x}{xy} = 2 \rightarrow \dfrac{x+y}{xy} = 2 \rightarrow x + y = 2xy$.
We need a way to relate these equations to determine $x + y$. Squaring the quantity $x + y$, we get $(x + y)^2 = x^2 + 2xy + y^2$. Substituting for $x^2 + y^2$ and $2xy$, this can be rewritten as $(x + y)^2 = 20 + x + y \rightarrow (x + y)^2 - (x + y) - 20 = 0$. We can make this quadratic equation easier to solve if we let A = $x + y$. We have $A^2 - 2A - 20 = 0$. Factoring, we see that $(A - 5)(A + 4) = 0$, and A = 5 or A = −4. Since we know that x and y are positive, it must be that A = 5. That means $x + y = 5$. Substitute 5 for $x + y$ in the equation $x + y = 2xy$ to get $5 = 2xy$ $\rightarrow xy = \dfrac{5}{2}$ **Ans.**

25. A triangle has angles measuring 15°, 45° and 120°. The side opposite the 45° angle is 20 units. The area of the

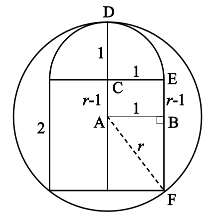
triangle can be expressed as $m - n\sqrt{q}$ and we must find the sum $m + n + q$. Suppose we draw the external altitude of the triangle from the 15° angle perpendicular to the line that contains the side opposite that the 15° angle, as shown.



Now we need to determine the height and length of the base of ΔWXY to determine its area. Since ∠XYZ is the supplement to ∠XYW, we know that $m\angle XYZ = 180 - 120 = 60°$. That means ΔXYZ is a 30-60-90 right triangle. For 30-60-90 right triangles then length of the shorter leg is ½ the length of the hypotenuse. That means YZ = ½(20) = 10. The length of the longer leg is $\sqrt{3}$ times the length of the shorter leg. Therefore, $XZ = 10\sqrt{3}$.
Notice that ΔWXZ is a 45-45-90 right triangle. That means $WZ = XZ = 10\sqrt{3}$ and $WY = 10\sqrt{3} - 10$. Therefore, the area of ΔWXY is $\frac{1}{2}(10\sqrt{3} - 10)(10\sqrt{3}) = (5\sqrt{3} - 5)(10\sqrt{3}) = 150 - 50\sqrt{3}$. So, $m = 150$, $n = 50$ and $q = 3$, and the sum $m + n + q = 150 + 50 + 3 = 203$ **Ans.**

26. A semicircle is positioned above a square. The diameter of the semicircle is 2 units. We must find the radius, r, of the smallest circle that contains this figure.

    If the diameter of the semicircle is 2, then its radius is 1 and each side of the square is 2. Segment AD is a radius of the circle, so AD = r. Thus, AC = $r - 1$ Drawing a perpendicular segment from point A to create rectangle ABEC, as shown, we see that BE = $r - 1$, too.



    Since segment EF is a side of the square, EF = 2, and BF = 2 − (r − 1) = 3 − r. Using the Pythagorean Theorem, we have $r^2 = 1^2 + (3 - r)^2 \rightarrow r^2 = 1 + 9 - 6r + r^2 \rightarrow 6r = 10 \rightarrow r = 5/3$ **Ans.**

27. In how many ways can 6 different gifts be given to five different children with each child receiving at least one gift and each gift being given to exactly one child?
    In any case one child can get 2 gifts while the other children each get only one gift. Since there are 5 kids there are 5 different kids who can get the 2 gifts. For the kid who gets the 2 gifts there are (6 × 5)/2 = 15 combinations of 2 gifts. There are 4! = 24 possibilities to assign single gifts to the other 4 kids. Therefore, the total number of ways is 5 × 15 × 24 = 1800 **Ans.**

28. If the cost of a dozen eggs is reduced by x cents, a buyer will pay one cent less for $x + 1$ eggs than if the cost of a dozen eggs is increased by x cents. What is the value of x?
    Let c be the cost of a dozen eggs in cents. Then the cost of one egg is c/12 cents. If we reduce that price by x cents, the cost of one egg is $\frac{c-x}{12}$. If we add

x cents, then the cost of one egg is $\frac{c+x}{12}$. So we have the following:

$\frac{c-x}{12}\times(x+1)+1=\frac{c+x}{12}\times(x+1)$ →

$\frac{cx+c-x^2-x}{12}+1=\frac{cx+c+x^2+x}{12}$ →

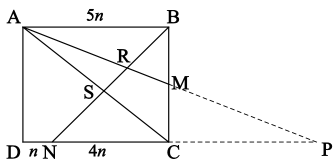
$\frac{cx+c-x^2-x+12}{12}=\frac{cx+c+x^2+x}{12}$ →

$cx+c-x^2-x+12 = cx+c+x^2+x$ → $-x^2 - x + 12 = x^2 + x$ → $2x^2 + 2x - 12 = 0$ → $(x + 3)(2x - 4) = 0$. So $x + 3 = 0$ and $x = -3$. But x can't be negative. So we have $2x - 4 = 0$ → $2x = 4$ → $x = 2$ **Ans.**

29. For how many two-element subsets $\{a,b\}$ of the set $\{1, 2, 3, \ldots, 36\}$ is the product of ab a perfect square?
Consider 1 as the value for a. Then any value less than or equal to 36 that is also a perfect square will work. So, there are **5** subsets available with these possible values of b: 4, 9, 16, 25 and 36.
Consider 2 as the value for a. Any odd power of 2 multiplied by a perfect square will work for b: 8, 18 and 32 for a total of **3** subsets.
Consider 3 for *a.* The values for b are 12, 27 for a total of **2** subsets.
Consider 4 for *a.* Only other perfect squares will work for b: 9, 16, 25 and 36 for a total of **4** subsets.
Consider 5 for *a.* Any odd power of 5 multiplied by a perfect square will work for b, and 20 is the only one in our original set. So that's **1** subset.
Consider 6 for a. The product of any odd power of 2 and any odd power of 3 will work for b, and 24 is the only one in our original set. So that's **1** subset.
Consider 7 for a. Any odd power of 7 multiplied by a perfect square works for b: 28 is the only one in our original set. So that's **1** subset.
Consider 8 for a. Any odd power of 2 multiplied by a perfect square will work for b: 2 (already counted with $a = 2$), 18 and 32, for a total of **2** subsets.
Consider 9 for a. Only other perfect squares will work for b: 4 (already counted for $a = 4$), 16, 25 and 36 for a total of **3** subsets.
Consider 10 for a. No subsets within our original set.
Consider 11 for a. No subsets.
Consider 12 for a. Any odd power of 3 multiplied by a perfect square works for b: 27 is the creates the only **1** subset
Consider 13 for a. No subsets.
Consider 14 for a. No subsets within our original set.
Consider 15 for a. No subsets within our original set.
Consider 16 for a. 25 and 36 work for b for a total of **2** subsets.
Consider 17 for a. No subsets.
Consider 18 for a. Any odd power of 2 multiplied by a perfect square works for b: 8 (already counted for $a = 8$) and 32, for a total of **1** subset.
Considering 19-24 for a. No subsets.
Consider 25 for *a.* As a perfect square, only another perfect square would work for *b.* The only one not yet counted is 36 for a total of **1** subset. The total is $5 + 3 + 2 + 4 + 1 + 1 + 1 + 2 + 3 + 1 + 2 + 1 + 1 = 10 + 6 + 6 + 5 = 27$ **Ans.**

30. Point M of rectangle ABCD is the midpoint of side BC and point N lies on CD such that DN:NC = 1:4. Segment BN intersects AM and AC at points R and S. If NS:SR:RB = x:y:z, what is the minimum possible value of $x + y + z$?
Let AB = $5n$ then NC = $4n$. We know that ΔABS ~ ΔCNS since $m\angle$BAS = $m\angle$NCS and $m\angle$ASB = $m\angle$CSN. That

means BS:NS = 5:4. Thus NS = (4/9)BN and BS = (5/9)BN. If we extend segment AM beyond point M, and extend segment DC beyond point C the segments will intersect at point P, as shown.



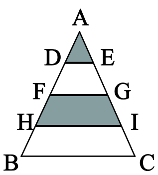
Now ΔABM ≅ ΔPCM, so CP = AB = 5*n* and NP = 4*n* + 5*n* = 9*n*. Also ΔABR ~ ΔPNR and AB:PN = 5:9 = BR:RN. Thus, BR = (5/14)BN and RN = (9/14)BN. It follows that SR = BS − BR = [(5/9) – (5/14)]BN = (25/126)BN. We now have all three segments expressed in terms of BN. Namely, NS:SR:RB = (4/9):(25/126):(5/14) = (56/126):(25/126):(45/126) = 56:25:45. The minimum sum is 56 + 25 + 45 = 126 **Ans.**

Target Round

1. Dr. Gru took 30 seconds to pump \$9.00 of gasoline. It took a total of 105 seconds to pump 15 gallons of gas. We must find the cost for one gallon of gas at the station.
   If 15 gallons are pumped in 105 seconds, then one gallon is pumped in 105/15 = 7 seconds. In 30 seconds you can pump 30/7 gallons and this costs \$9.00. Let *x* represent the cost of a gallon of gas. Then (30/7)*x* = 9 → 30*x* = 63 → *x* = 2.10 **Ans.**

2. If $kx + 12 = 3k$, for how many integer values of *k* is *x* a positive integer?
   Solving the given equation for *x*, we get $kx + 12 = 3k \rightarrow kx = 3k - 12 \rightarrow \frac{3k-12}{k} \rightarrow 3 - \frac{12}{k}$. Any positive value for $k \leq 4$ results in $\frac{12}{k} \geq 3$ and, thus, $x \leq 0$. Any negative value for $k < -12$ results in a fraction value for $\frac{12}{k}$, not an integer. Therefore, the values of *k* that result in positive integer values for *x* are −12, −6, −4, −3, −2, −1, 6 and 12 for a total of 8 values. 8 **Ans.**

3. In ΔABC, segments AB and AC have each been divided into four congruent segments. We must find the fraction of the triangle that is shaded.



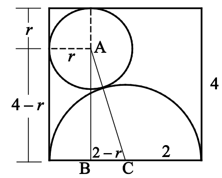
   Let *x* be the length of BC and let *h* be the altitude from vertex A to side BC. Then the area of the triangle is (1/2)*xh*. Let's compute the area of ΔADE. Because segments AB and AC have each been divided into four congruent segments it follows that DE = (1/4)*x*, the altitude from A to segment DE must be (1/4)*h* and the area of ΔADE can be written as (1/2) × (1/4)*x* × (1/4)*h* = (1/32)*xh.* Now we find the area of ΔAHI. We know HI = (3/4)*x* and the altitude is (3/4)*h*. Therefore, its area is (1/2) × (3/4)*x* × (3/4)*h* = (9/32)*xh.* Now to find the area of trapezoid FGIH we must subtract the area of ΔAFG from the area of ΔAHI. Since FG = (1/2)*x* and the altitude is (1/2)*h*, the area of ΔAFG is (1/2) × (1/2)*x* × (1/2)*h* = (1/8)*xh* Therefore, the area of the trapezoid is

$(9/32)xh - (1/8)xh = (5/32)xh.$
The area of the two shaded portions is $(1/32)xh + (5/32)xh = (6/32)xh = (3/16)xh.$
The fraction of the triangle that is shaded then is:
$$\frac{\frac{3}{16}xh}{\frac{1}{2}xh} = \frac{\frac{3}{16}xh}{\frac{8}{16}xh} = \frac{3}{8}$$ **<u>Ans.</u>**

4. The sum of the first n terms of a sequence, $a_1 + a_2 + \cdots + a_n$, is given by the formula $S_n = n^2 + 4n + 8$
What is the value of a_6?
No terms are given for this sequence, but we know that the difference between the sum of the first six terms and the sum of the first five terms, $S_6 - S_5$, equals a_6. Using the given formula, we have $S_6 = 6^2 + (4 \times 6) + 8 = 68$, and $S_5 = 5^2 + (4 \times 5) + 8 = 53$. Therefore, $a_6 = S_6 - S_5 = 68 - 53 = 15$ **<u>Ans.</u>**

5. The first and last initials of the 348 students form a unique ordered letter pair. We must find how many more students are required to guarantee that there are two students whose initials form the same ordered letter pair. Since there are 26 letters of the alphabet we can have 26 × 26 = 676 unique combinations. Therefore, we must have 677 students to ensure that two students have the same initials. The number of additional students needed is 677 − 348 = 329 **<u>Ans.</u>**

6. A semicircle and circle are placed inside a square with sides of length 4. The circle is tangent to two adjacent sides of the square and to the semicircle. The diameter of the semicircle is a side of the square (or 4). We must find the radius of the circle. Since the diameter of the semicircle is 4, its radius is 2. Let A be the center of the circle and let C be the center of the semicircle. Let's start by drawing a segment from point A perpendicular to the bottom of the square at point B, and drawing another segment from point A to point C, as shown.



Let r represent the radius of the circle. Then AB = $4 - r$, BC = $2 - r$, and AC = $r + 2$. Using the Pythagorean Theorem, we have $(4 - r)^2 + (2 - r)^2 = (r + 2)^2 \rightarrow (16 - 8r + r^2) + (4 - 4r + r^2) = r^2 + 4r + 4 \rightarrow 2r^2 - 12r + 20 = r^2 + 4r + 4 \rightarrow r^2 - 6r + 16 = 0$. Since this quadratic equation cannot be factored, we can use the quadratic formula,
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$, to solve for r.
Substituting $a = 1$, $b = -16$ $c = 16$ into the formula, we have
$$r = \frac{16 \pm \sqrt{16^2 - 4 \times 1 \times 16}}{2 \times 1}$$. Solving for r, we get $r = \frac{16 \pm \sqrt{256 - 64}}{2} = \frac{16 \pm \sqrt{192}}{2} = \frac{16 \pm 8\sqrt{3}}{2} = 8 \pm 4\sqrt{3}$. Since $8 + 4\sqrt{3} > 4$, this root will not work. Therefore, $r = 8 - 4\sqrt{3} \approx 8 - 6.928 \approx 1.07$ **<u>Ans.</u>**

7. The diameter of a spherical balloon is increased by 150%. We must find by what percent the volume increases. The volume of a sphere is $V = (4/3)\pi r^3$. If the diameter increases by 150% the new diameter is 2.5 times or 5/2 of the old diameter. It follows that the new

radius, $r' = (5/2)r$. The new volume is $(4/3)\pi(r')^3 = (4/3)\pi((5/2)r)^3 = (4/3)\pi(125/8)r^3 = (125/6)\pi r^3$. This is an increase of $(125/6) \div (4/3) = (125/6) \times (3/4) = 125/8 = 15.625 =$ 1562.5%. This is an increase of 1562.5% − 100% = 1462.5% **Ans.**

8. In one roll of four standard six-sided dice, what is the probability of rolling exactly three different numbers?
   There are 6^4 different possible outcomes when rolling four dice. The ones that we are interested in are those where two of the dice have the same value and the other two have different unique values. We can represent the probability of this happening as $\frac{6}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{4}{6}=\frac{120}{6^4}$.
   Each possibility can occur 6 ways. For example, if S, D_1, and D_2 are the three unique values then the ways these four dice can be rolled are SSD_1D_2, SD_1SD_2, SD_1D_2S, D_1SD_2S, D_1D_2SS, D_1SSD_2. Note that we don't include forms like D_2SSD_1 here because that is the case where D_1 has the value of D_2 and D_2 has the value of D_1. We would end up double counting them.
   Therefore the probability is
   $\frac{120}{6^4}\times 6=\frac{120}{6^3}=\frac{20}{36}=\frac{5}{9}$ **Ans.**

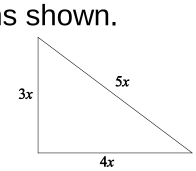
Team Round

1. Adult tickets are \$5 and student tickets are \$2. 5 times as many student tickets were sold as adult tickets for a total of \$1125. We must find the number of tickets sold.
   Let x be the number of adult tickets sold and let y be the number of student tickets sold. Then we have $5x + 2y = 1125$ and $y = 5x$. Substituting and solving for x, we get $5x + 2(5x) = 1125 \rightarrow 5x + 10x = 1125 \rightarrow 15x = 1125 \rightarrow x = 75$. So the total number of tickets sold is $75 + (5 \times 75) = 450$ **Ans.**

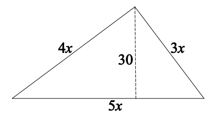
2. CDs sell for 3 different amounts. Three customers bought 3 CDs each but none bought three of the same price. The first customer spent \$4, the second spent \$9 and the third spent \$12. We must find the price of the most expensive CD. Prices of each CD are integers. Let's look at the first customer who spent \$4 for 3 CDs. This is only possible if one price for a CD is \$1 and the other price is \$2, since 1 + 1 + 2 = 4. The second customer bought 3 CDs and spent \$9. If the customer bought one of each type, the third price would be 9 − 1 – 2 = 6. But this won't work for the third customer who spent \$12 because there's no scenario using \$1, \$2 and \$6 that works. How about if the customer bought one expensive CD and two that cost \$2 each? Then the expensive CD would cost 9 – 2 – 2 = 5. This works if the third customer bought two \$5 CDs and one \$2 CD, since 5 + 5 + 2 = 12. But how about if the second customer bought one expensive CD and two \$1 CDs? Then the expensive CD would be 9 – 1 – 1 = 7. Does that work for the third customer? No, because there's no way to get 12 – 7 = 5 from a combination of 2 other CDs. Anything else? The second customer could have bought 2 expensive and 1 cheap CD. Then the expensive CD would be (9 – 1) ÷ 2 = 4. This only works if the third customer bought 3 CDs at the same price, which isn't allowed. So, the price of the most expensive CD is \$5 **Ans.**

3. When one integer is removed from a list of 5 integers, the mean of the remaining four integers is 3 less than the mean of the original 5 integers. So what is the positive difference between the mean of the original five integers and the integer that was removed?
Let x_1, x_2, x_3, x_4 and x_5 be the 5 integers. Then the mean of the integers, M, is
$M = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$ and
$5M = x_1 + x_2 + x_3 + x_4 + x_5$.
Now, suppose the integer we remove is x_5. We have
$M - 3 = \frac{x_1 + x_2 + x_3 + x_4}{4}$ and
$4(M - 3) = x_1 + x_2 + x_3 + x_4$.
Substituting $4(M - 3)$ for $x_1 + x_2 + x_3 + x_4$ in the first equation and simplifying, we get $4(M - 3) + x_5 = 5M \rightarrow 4M - 12 + x_5 = 5M \rightarrow x_5 - M = 12$. So, the positive difference between the mean of the original five integers and the integer that was removed is 12 **Ans.**

4. The legs of a right triangle are in the ratio 3:4. One of the altitudes is 30 ft. What is the greatest possible area of this triangle?
We have a right triangle with sides in the proportions shown.



There are 3 possible scenarios for this triangle. Since the height in each of these scenarios is 30, the triangle with the largest area will have the base of greatest length. In one scenario the altitude is the shorter leg, in which case $3x = 30$ and $x = 10$. This triangle has a base of $4x = 4(10) = 40$. In another case, the longer leg is the altitude, in which case $4x = 30$ and $x = 7.5$. This triangle has a base of length $3x = 3(7.5) = 22.5$. In the third scenario, the altitude is drawn perpendicular to the hypotenuse creating two similar triangles as shown.



That means $3x/30 = 5/4$. Cross-multiplying and solving for x, we get $3x(4) = 30(5) \rightarrow 12x = 150 \rightarrow x = 12.5$. This triangle has a base of length $5(12.5) = 62.5$. This triangle has the largest base. Therefore, the largest possible area is $(1/2)(62.5)(30) = (62.5)(15) = 937.5$ **Ans.**

5. The positive difference of the cubes of two consecutive positive integers is 111 less than 5 times the product of the two consecutive integers. We must find the sum of the two consecutive integers.
Let x and $x + 1$ be the two consecutive integers. Then we have $(x + 1)^3 - x^3 = 5x(x + 1) - 111$. Simplifying yields
$(x^2 + 2x + 1)(x + 1) - x^3 = 5x^2 + 5x - 111$
$x^3 + x^2 + 2x^2 + 2x + x + 1 - x^3 = 5x^2 + 5x - 111$
$3x^2 + 3x + 1 = 5x^2 + 5x - 111$
$2x^2 + 2x - 112 = 0$
$x^2 + x - 56 = 0$.
We can solve for x by factoring to get $(x + 8)(x - 7) = 0$. Since the integers must be positive we have $x = 7$, $x + 1 = 8$, and $7 + 8 = 15$ **Ans.**

6. In how many ways can 18 be written as the sum of four distinct positive integers?
For this problem, it's best to make an organized list. We must make sure that any sum we put down is not repeated in the list even by changing the order

because the problem does state that order counts differently. We know that if there are 4 numbers that satisfy the sum, there will be 4! = 24 different ways to write it. Next we start with the supposition that the first 3 numbers are the smallest possible.

1 + 2 + 3 + 12 = 18
1 + 2 + 4 + 11 = 18
1 + 2 + 5 + 10 = 18
1 + 2 + 6 + 9 = 18
1 + 2 + 7 + 8 = 18.

That's a total of 5 combinations using 1 and 2 as the first two numbers.

Now we go to 1 and 3 as the first two numbers.

1 + 3 + 4 + 10 = 18
1 + 3 + 5 + 9 = 18
1 + 3 + 6 + 8 = 18

and we're done with this one since each number is distinct. That's 3 combinations. We can also have

1 + 4 + 5 + 8 = 18
1 + 4 + 6 + 7 = 18.

That's 2 combinations.

That concludes the list of combinations having 1 as the smallest number in the sum. That's a total of 5 + 3 + 2 = **10** combinations.

Let's now start with 2 as the first number.

2 + 3 + 4 + 9 = 18
2 + 3 + 5 + 8 = 18
2 + 3 + 6 + 7 = 18

That's 3 combinations.

2 + 4 + 5 + 7 = 18

That's 1 combination.

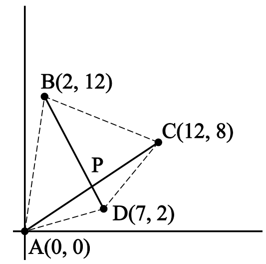
That's a total of 3 + 1 = **4** combinations with 2 as the smallest number.

Now, with 3 as the smallest number we only have 3 + 4 + 5 + 6 = 18.

That's **1** way and we've now listed all possible combinations of 4 distinct positive integers with a sum of 18.

That's a total of 10 + 4 + 1 = 15 combinations, each with 24 arrangements. 15 × 24 = 360 **Ans.**

7. Four towns are located at A(0,0), B(2,12), C(12,8), and D(7,2). A warehouse is built at point P so the sum of the distances PA + PB + PC + PD is minimized. We must find the coordinates of point P. Okay, let's graph the four points.



Notice that these points form a quadrilateral. The point P should be at the intersection of the diagonals, segments AC and BD, to minimize the sum of PA + PB + PC + PD. To find P, we must determine the equations for the two lines that contain segments AC and BD and find where they intersect.

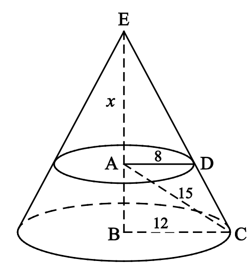
Since the slope of a line is $m = \frac{y_2 - y_1}{x_2 - x_1}$, for the first line containing segment AC, we have $m = \frac{8-0}{12-0} = \frac{8}{12} = \frac{2}{3}$. Recall the slope-intercept form $y = mx + b$. Using point A, we can substitute $x = 0$ and $y = 0$ to get $0 = 0(0) + b$ and $b = 0$. Therefore, the equation for the line that contains points A and C is $y = (2/3)x$. Now, let's determine the equation for the line that contains the points B and D. The slope of this line is $m = \frac{12-2}{2-7} = \frac{10}{-5} = -2$. Using point B, we can substitute $x = 2$ and $y = 12$ to get $12 = 2(-2) + b \rightarrow 12 = -4 + b \rightarrow b = 16$.

The equation that contains points B and D is $y = -2x + 16$. The lines intersect when $-2x + 16 = (2/3)x$. Solving for x yields $16 = (8/3)x \rightarrow x = 16(3/8) = 6$. When $x = 6$, $y = (2/3)(6) = 4$. The coordinates of point P are (6, 4) **Ans.**

8. A hot-air balloon descends at a constant rate of 15 ft per minute starting from 1200 ft above the ground. A helium-filled balloon is released at a height of 10 ft above the ground. It goes up at a rate of 5 ft per second. We must find how many minutes expire before the two balloons are at the same height.
First, note that the helium-filled balloon goes up at 5 ft per second. Let's convert that to minutes. 5 ft per second is the same as $5 \times 60 = 300$ ft per minute.
Let x be the number of minutes that it takes for the two balloons to reach the same height. We have
$1200 - 15x = 10 + 300x \rightarrow 1190 = 315x$.
So, $x = 1190/315 \approx 3.78$ **Ans.**

9. There is more than one four-digit positive integer in which the thousands digit is the number of 0s in the four-digit number, the hundreds digit is the number of 1s, the tens digit is the number of 2s and the units digit is the number of 3s. What is the sum of all such integers?
The thousands digit is the number of 0s. This means that there must be at least 1 digit that is 0, otherwise this wouldn't be a 4-digit number. Let's start with 1 zero. Then the first digit is 1. That makes the second digit a 1 but that can't be since the hundreds digit is the number of ones and we would be at 11_ _. So we would have 1210 and this makes sense because we have 1 zero, 2 ones and 1 two. That seems to be the only way we could have 1 zero. Now let's try 2 zeros. 2_1_ ≥ 2010 but that doesn't work. What about 2 twos? 2_2_ ≥ 2020 and that does work. It doesn't appear that there's another way we could have 2 zeroes. How about 3 zeroes? 3_ _1 won't work. So we have 1210 + 2020 = 3230 **Ans.**

10. Segments AD and BC are the radii of the top and bottom bases of the frustum. AD = 8, BC = 12 and AC = 15, what is the volume of the frustum?
First, notice that AB = 9 since triangle ABC is a right triangle.
Now let's draw the entire cone.



If we name the point at the top of the cone E, we have two similar triangles, ΔEAD and ΔEBC.

Let x = EA. Then $\frac{x}{x+9} = \frac{8}{12} = \frac{2}{3}$. Cross-multiplying yields $3x = 2(x + 9) \rightarrow 3x = 2x + 18 \rightarrow x = 18$. So, the height of the entire cone is $h = 18 + 9 = 27$.
The volume of the cone with height 18 (x = EA = 18) is $(1/3)\pi r^3 h = (1/3)\pi(8^2)(18) = 384\pi$. The volume of the cone with height 27 (h = EB = 27) is $(1/3)\pi(12^2)(27) = 1296\pi$. Thus, the volume of the frustum is $1296\pi - 384\pi = 912\pi$ **Ans.**