TOPOLOGICAL RECONSTRUCTION OF GRAYSCALE IMAGES

By

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I dedicate this to my parents and incredible fiancée who are gracious enough to allow me to achieve this spectacular milestone and are the only ones who can truly appreciate how important this work is to me.

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Topological data analysis (TDA) is a powerful tool to solve classification problems using the underlying shape of data. Furthermore, TDA is useful in shape reconstruction of compact manifolds embedded in two and three dimensional Euclidean space. Unfortunately, these reconstructions can be computationally taxing. Many in the field are searching for solutions which are more computationally efficient and theoretical guarantees for the minimal amount of information for reconstructions within a given tolerance of error.

Our approach is to study TDA on voxel data to offer both a computationally and memory efficient solution in practice. Assuming a lattice structure for our cubical geometric realization we are able to prove a sharp upper bound for all dimensions of 2^d for the number of persistence diagrams generated from sublevel set filtrations required to reconstruct grayscale digital images. By improving topological reconstructions, the authors believe that TDA may become a pragmatic tool to apply to computerized tomography images for both storage and potentially classification for automated diagnosis of particular diseases such as melanoma.

CHAPTER 1
INTRODUCTION AND OVERVIEW OF MATHEMATICAL CONCEPTS

1.1 Introduction

The goal of this thesis is to establish an application of Topological Data Analysis towards image processing by utilizing cubical complexes rather than simplicial complexes. In particular using connections between Persistent Homology and Discrete Morse Theory, we demonstrate that topological information is sufficient to store digital images. Further, there exists a metric used to compare said topological summaries which has demonstrated possible machine learning applications. To this end we provide a different method for storing digital images which takes advantage of the image's underlying topology. We conclude by providing code as a proof of concept which transforms and restores digital images using this new storage method.

1.2 Overview of Mathematical Concepts

This section presents a high level description of the mathematical concepts used to prove the main results of the thesis. A more in depth description of all topics can be found in Chapter 3.

1.2.1 Cubical Complexes and Cubical Homology

A cubical complex is a set consisting of points, line segments, squares, cubes, and their d-dimensional counterparts. Since voxels naturally have a cubical geometric realization we compute cubical homology rather than the standard simplicial homology. We justify this by noting simplicial and cubical homology are naturally isomorphic Eilenberg and MacLane (1953).

1.2.2 Möbius Inversion

The theory of Möbius functions from combinatorics can be thought of as a generalization of the inclusion-exclusion principle. Möbius functions and their inversions when evaluated on partially ordered sets (posets) have a particularly nice Algebraic Topology interpretation. For example, the Möbius inversion may be viewed as a reduced Euler characteristic and has ties to Homology groups. When certain hypothesis are strengthened Möbius inversion can yield interpretations of geometric realizations of posets as simplicial complexes and an analog to Alexander Duality for Homology and Cohomology groups.

For our particular application, the proposed storage array can be represented as a collection of pairs of integers ($\mathbb{Z} \times \mathbb{Z}$) where the first entry represents a spatial location and the second the Möbius inversion function value of the associated vertex. This second value is constructed using the standard partial ordering of $\mathbb{R}^d$ (using $\leq$ relation component-wise) and inverting an associated value of critical points.

1.2.3 Euler Calculus

Euler Calculus is an integral calculus built with the Euler characteristic as a measure and o-minimal sets as measurable sets. A constructible function on a manifold is an integer valued function which is constant on a stratification. Under special circumstances one can recover a constructible function from its Radon transform. The Radon transform is a function which transforms a constructible function on an o-minimal set to a constructible function on a second o-minimal set using pullback and pushforward maps from the product space.

1.2.4 Persistent Homology

Persistent Homology quantifies topological features of a function. It defines the birth and death of homology classes at critical points, identifies pairs of these (persistence pairs), and provides a quantitative notion of their stability (persistence).

1.2.5 Discrete Morse Theory

Discrete Morse Theory provides combinatorial equivalents of several core concepts of classical Morse theory, such as discrete Morse functions, discrete gradient vector field, critical points, and a cancellation theorem for the elimination of critical points of a vector field. Further discrete Morse theory provides explicit and canonical constructions that would become quite complicated in the smooth setting while maintaining the intuition behind the tie established between Geometry and Topology established in the smooth setting. A key difference between the two topics is Morse theory makes statements about the homotopy type of the sublevel sets of a function, whereas persistence is concerned with their homology. While homology is an invariant of homotopy equivalences, the converse is not true: not every map inducing an isomorphism in homology is a homotopy equivalence.

CHAPTER 2
LITERATURE REVIEW OF TOPOLOGICAL SHAPE RECONSTRUCTION

Topological Data Analysis (TDA) is a recent field of mathematics which emerged from applied algebraic topology and computational geometry. TDA is motivated by the idea that using topology and geometry one can study "the shape of data" to infer robust qualitative information about the structure of said data. TDA provides mathematical, statistical, and algorithmic methods to infer, analyze, and utilize the complex topological and geometric structures underlying data. TDA has contributed to a range of statistical and computer science questions including: boundary reconstruction Du et al. (2018), classification Bubenik (2015), Crawford et al. (2016), Bendich et al. (2016), compression Gonzalez-Diaz et al. (2017), dimensionality reduction and high dimensional data visualization Singh et al. (2007), shape reconstruction Turner et al. (2014), Schapira (1995), image segmentation Assaf et al. (2018), Qaiser et al. (2018), reconstruction of graphical networks Dey et al. (2017), and video analysis Jimenez et al. (2016). This thesis is primarily concerned with contributions to the shape reconstruction realm but believes that the results and the approach offered may have implications in compression and classification as well.

Topological shape reconstruction asks the question: "Is it possible to reconstruct a shape using a mixture of global information such as connectivity and local information such as whether a point is an 'interior' or a 'boundary' point?" This question arose in early attempts to utilize computed tomography, magnetic resonance images, and ultrasound images. Images of these types generate cross sectional views of an object which can be combined to form what Topologists call a "height filtration" of the shape in question. The reader can view a sublevel set filtration as a sequence (possibly infinite) of spaces where each previous space includes into the next. Since inclusion is a continuous map, these types of objects are often studied in the field of topology.

2.1 Simplicial Complex Reconstructions

Triangle meshes are common data structures in computer imaging. Early discoveries of representing surfaces as simplices (sets of points with a particular geometric realization such as vertices, edges, trianges, tetrahedra, etc.) and stepwise constructions Braid et al. (1978) led to the early dominance of simplicial representations for shapes and surfaces. Continuous three dimensional surfaces can be approximated using simplicial complexes by taking the limit as the area of the simplices goes to 0 and the number of triangles needed to approximate the shape goes to infinity. By choosing a maximum and minimum area for our simplices we can obtain a variety of different resolutions. Due to the prevalence of simplicial representations computer scientists have refined algorithms for post processing simplicial meshes which improves the reconstruction. Recent advancements in computer science improve piecewise simplicial approximations utilizing a variety of post processing techniques such as '$\sqrt{3}$-division'. $\sqrt{3}$-division is a technique which intelligently subdivides piecewise linear simplicial complexes so that in the limit not only does the error of our simplicial representation go to zero with respect to the Hausdorff distance but further almost everywhere the simplicial representation is $\mathcal{C}^2$ continuous (with the exception of a special set of vertices at which the representation is $\mathcal{C}^1$ continuous) Kobbelt (2000). TDA offers alternative constructions to create a piecewise linear simplicial complex which has proven comparable in practice for reconstruction and classification tasks Reininghaus et al. (2015) and serves as a potential input for such post-processing algorithms.

2.1.1 Shape Reconstruction via Two Dimensional Cross Sections

One attempt at reconstruction was proposed in Bajaj et al. (1996). The authors considered reconstructing three dimensional images using two dimensional cross sections. The idea was to identify critical points along the boundary of the two dimensional cross sections and then identify amongst consecutive two dimensional slices which vertices were connected in order to obtain a simplicial approximation of the shape. This technique allows for the reconstruction of not just the exterior surface but also cavities via a "nested hull" approach. However, emulating this approach using TDA was later shown to be computationally taxing. In particular, Attali et al. (2012) proved that sublevel set reconstruction is NP-complete by showing through a series of related questions that for simplicial complexes embedded in $\mathbb{R}^3$, given a complex K and a subcomplex L it is not always possible to find a subcomplex of K that contains L with precisely the same homological features common to both L and K Attali et al. (2012). In other words, even with accurate reconstructions of the two dimensional cross sections (which Attali et al. (2012) also showed is always possible in polynomial time) it is extremely difficult to connect consecutive cross sections even for simplicial approximations of shapes.

2.1.2 Various Transforms

Other attempts at topological reconstructions include the persistent homology transform which considers a collection of sublevel set filtrations of special simplicial approximations (ones where no vertex v was coplanar with a set of three vertices which contain v in their convex hull) Turner et al. (2014). The advantage of this approach is that it would use these different sublevel filtrations to identify critical vertices and use the collection of multiple persistent homology diagrams to find a diagram in which they could quickly determine whether or not a simplex was built on a set of vertices. Initial problems with this approach is that persistent homology is also computationally taxing ($\mathcal{O}(n^3)$ where n is the number of critical vertices) and that they also required in theory infinitely many directions (one sublevel set filtration corresponding to a vector pointing every direction of $\mathbb{R}^3$. In practice, Turner et. al achieved practical results in three dimensions using roughly 108 filtrations. The theoretical hurdle was what Bajaj et al. (1996) would call the "shadow cast", or recently Belton et al. (2018) would call "indegree" of a critical vertex. For a critical vertex to be observable by a height filtration, there must exist a direction where locally the vertex has the smallest height (or possibly largest height if it completes a cycle). If we think of a regular *n*-polygon with its interior, as we take the limit as n goes to infinity, the measure of the set of angles for which a height filtration vector must point to detect a given vertex goes to 0. Therefore, to guarantee that the persistence information was sufficient to reconstruct the simplicial approximation Turner et al. (2014) required infinitely many directions.

One observation which led to a drastic speedup was the introduction of the Euler characteristic transform by Crawford et al. (2016). Similar to the persistent homology transform, the Euler characteristic Transform computes a weaker topological invariant from many different angles. This particular computation is an $\mathcal{O}(n)$ computation and the collection of Euler characteristic curves is still a sufficient statistic for the set of simplicial complexes whose vertices are in generic position. However, the question remained on how to reduce the theoretical limit of infinitely many filtrations required. A second direction would be to reduce the overall number of critical vertices by considering a more restrictive class of shapes.

If our goal is to make topological shape reconstruction more efficient using a transform method similar to those mentioned above, one could imagine attempting to improve multiple parts of the pipeline. One approach would be to sparsify or consider an approximation of our shape to reduce the number of critical points. This approach was taken by Gonzalez-Diaz et al. (2017) while attempting to answer a different question regarding sparsification. Gonzalez-Diaz et. al focused on efficient storage of well-composed polyhedral complexes. *Well-composed* refers to shapes which behave like manifolds on their boundaries (boundary points have open neighborhoods which are isomorphic to Euclidean half space for a given dimension). This approach works well for storage and reconstruction from an already stored image, but requires lengthy preprocessing and therefore, is not a direct answer to our question. Another approach would be to use a different type of approximation of a given shape altogether. So far all previous mentions have focused on simplicial approximations of images. One goal of this thesis is to convince the reader that a cubical approximation is a possible solution which merits further research.

2.2 Cubical Complex Resconstructions

One reason why cubical homology and cubical representations of images is appealing is that digital images already utilize a pixel or voxel format. Rather than considering a mesh or simplicial complex representation of our image, it is useful to develop a tool set which can utilize the already existing geometry of digital images. Further any surface can be rasterized and given a cubical representation by constructing a lattice of unit volume cubes and considering the collection of cubes which the surface intersects. This approximation is near lossless in the sense that the Hausdorff distance between the representation and the shape goes to zero as we consider smaller and smaller grid sizes for our lattice. The reason the persistent homology transform and Euler characteristic transform required infinitely many vectors is that for a generic embedding of both a simplicial and cubical complex, one does not have control of the where the set of vertices are embedded. By utilizing the additional structure of voxels, we can assume both that vertices have integer valued coordinates and further that every lower dimensional face of the cubical complex is a proper face of a top dimensional cube (cube of unit volume) contained in the set. This additional structure allows us to use discrete Morse theory, a theory which computes the homotopy type of a Morse complex (and can be used to compute homology via pairing birth death cells together), in order to compute persistence in $\mathcal{O}(n^2)$ rather than $\mathcal{O}(n^3)$ where n is the number of cubes via methodes found in Günther et al. (2012).

This approach does have a disadvantage in that the number of critical points strongly depends on the orientation. For example if a rectangular prism is embedded in three dimensions, the minimum number of critical points corresponds with an embedding where the faces of the prism are parallel with the hyperplanes formed by axes. If the rectangular prism were to be rotated so that the projection of one of its faces onto a hyperplane forms a 45 degree angle, every cube which is intersected corresponding to that face for the rasterization would contribute a critical vertex. However, it is the author's opinion that the advantages gained by such a representation (for example fixed indegree of vertices) and the existence of post processing smoothing algorithms such as 'marching cubes' make this a tractable approach.

2.3 Euler Calculus

Euler Calculus is an integral calculus built with the Euler characteristic as a measure and o-minimal sets as measurable sets. A constructible function on a manifold is an integer valued function which is constant on a stratification. Schapira demonstrated that under special circumstances one could recover a constructible function from its Radon transform Schapira (1995). The Radon transform is a function which transforms a constructible function on an o-minimal set to a constructible function on a second o-minimal set using pull backs and pushforwards from the product space. Since the Radon transform requires integration over a fiber, which is dependent on the o-minimal sets being considered, we return to the question of "how many directions are necessary to recovering either simplicial or cubical complexes if we consider the underlying object to be a piecewise linear manifold and the constructible function to be the indicator function for the set embedded in Euclidean space?" Recent developements by both R. Ghrist and Mai (2018) and Curry et al. (2018) have recently demonstrated (independently) that both the persistent homology transform and Euler characteristic transform are special cases of this Euler Calculus if we consider our constructible function to be the characteristic function by inverting the Radon transform against the Betti numbers, or Euler characteristic respectively. Euler Calculus appears to be the correct theoretical framework to address topological reconstructions arising from sublevel set filtrations which leaves us with the following problem, "how many directions are required to reconstruct a shape?"

2.4 Contributions to Shape Reconstruction

Curry et al. (2018) have proven an upper bound for the number of filtration directions of a general simplicial complex embedded Euclidean space which involves an analog of the in degree of the set of vertices as well as the number of vertices. Curry et. al. conjectured that different geometric realizations may offer different upper bounds (which are strictly better than those proposed in the original paper). This work is timely as one group of mathematicians Belton et al. (2018) already solved this problem for two dimensional representations using planar graphs (mathematical graphs which have an embedding in two dimensions such that no two edges intersect at a non-vertex point) have within months of question being posed. The main result of the thesis tackles this question with regards to full elementary cubical complexes. The author of the thesis also has proven that one can reconstruct full elementary cubical complexes embedded in a Euclidean space of finite dimension using analogs of the Euler Calculus and has written code as a proof of concept for dimensions two and three. Associated with this approach, the author shows in the discrete setting that the Euler characteristic transform when reduced to the minimum number of filtration vectors required is equivalent to another well-known result from mathematics, Möbius Inversion, an extremely powerful tool which has applications in enumerative combinatorics as well as number theory.

CHAPTER 3
MATHEMATICAL BACKGROUND

In this chapter we provide the mathematical framework required to establish the main results.

3.1 Elementary Cubical Complexes

We begin with some foundational definitions used in the study of voxels as mathematical objects.

Definition 3.1. *An **elementary interval** is a closed interval $I = [a, b] \subset \mathbb{R}$ such that $a \in \mathbb{Z}$ and $b \in \{a, a+1\}$. Elementary intervals that have zero length are **degenerate intervals** and those of unit length are **non-degenerate intervals**.*

Definition 3.2. *Let $d \geq 1$. An **elementary cube** C **in** $\mathbb{R}^d$ is a product of d elementary intervals,*

$$C := I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d.$$

*The **dimension of an elementary cube** is the number of non-degenerate intervals in the product. We denote the set of all elementary cubes in $\mathbb{R}^d$ of dimension i as $\mathcal{N}^i$ and refer to elements of $\mathcal{N}^i$ as **i-cubes**. A cube C is **full** if $C \in \mathcal{N}^d$. We define **the set of all elementary cubes in** $\mathbb{R}^d$ by $\mathcal{N}$, which is equal to the union $\bigcup_{i=0}^{d} \mathcal{N}^i$.*

Definition 3.3. *Two elementary cubes are **connected** if their intersection is non-empty.*

Definition 3.4. *Given two elementary cubes $\sigma = \prod_{j=1}^{d} I_j$ and $\tau = \prod_{j=1}^{d} I'_j$ we say σ **is a face of** τ, denoted $\sigma \subseteq \tau$, if $I_j \subseteq I'_j$ for all j.*

Definition 3.5. *An **elementary cubical complex** $\mathcal{K}$ is a set of elementary cubes in $\mathbb{R}^d$ such that for every $\tau \in \mathcal{K}$, if $\sigma \subseteq \tau$, then $\sigma \in \mathcal{K}$. A **full elementary cubical complex** is an elementary cubical complex such that for every $\sigma \in \mathcal{K}$, there exists a full cube $C \in \mathcal{K}$ which contains σ as a face.*

Definition 3.6. *Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$. Given $\sigma \in \mathcal{K}$, let $\mathcal{K}_\sigma$ be the set of elementary cubes in $\mathcal{K}$ which contain σ as a face. The **star of** σ is $st(\sigma) := \mathcal{K}_\sigma$. Given a set of elementary cubes $\mathcal{S} \subset \mathcal{K}$ the **closure of** $\mathcal{S}$ **in** $\mathcal{K}$, $\overline{\mathcal{S}}$, is the downwards closure of $\mathcal{S}$ in* *the face poset of $\mathcal{K}$. That is, $\overline{\mathcal{S}}$ is the set of all faces of cubes in $\mathcal{S}$. For a single elementary cube we will denote $\overline{\{\sigma\}} = \overline{\sigma}$.*

3.2 Euler Characteristic

The **Euler Characteristic** of a finite elementary cubical complex $\mathcal{K}$ in $\mathbb{R}^d$ is the alternating sum

$$\chi(\mathcal{K}) = \sum_{i=0}^{d} (-1)^i n_i$$

where n_i denotes the number of **i**-cubes in the complex. Further, the Euler characteristic can be computed using the ranks of the cubical homology groups (Section 3.4). The **Betti number** β_i is the rank of the *i*-th cubical homology group of a cubical complex. Since $\mathcal{K}$ is finite, the Euler characteristic can be equivalently defined as follows,

$$\chi(\mathcal{K}) = \sum_{i=0}^{d-1} (-1)^i \beta_i(\mathcal{K}).$$

Let $\mathcal{K}$ be a finite elementary cubical complex in $\mathbb{R}^d$. A primary focus of the thesis examines the largest elementary cubical complex which is a subset of $\mathcal{K}$ contained in a closed halfspace of $\mathbb{R}^d$. Let $f \in \mathbb{R}^d$, $t \in \mathbb{R}$ and $W_{f,t} = \{x \in \mathbb{R}^d \mid f \cdot x = t\}$ be a hyperplane in $\mathbb{R}^d$. Since a fixed elementary cube is a compact subset of $\mathbb{R}^d$, if σ is an elementary cube in $\mathbb{R}^d$ then all points of $x \in \sigma \subset \mathbb{R}^d$ will lie below the hyperplane $W_{f,t}$ if $f \cdot x \leq t$ for all $x \in \sigma$. Therefore, the largest elementary cubical complex in $\mathbb{R}^d$ which is a subset of $\mathcal{K}$ such that all elementary cubes lie beneath a fixed hyperplane $W_{f,t}$ is the set $\{\sigma \in \mathcal{K} \mid \max\{f \cdot x \mid x \in \sigma \subset \mathbb{R}^d\} \leq t\}$.

Definition 3.7. *Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$ and $f \in \mathbb{R}^d$. We define the* ***Euler Characteristic Curve of $\mathcal{K}$ with respect to*** $f \in \mathbb{R}^d$, ECC$(\mathcal{K}, f)$ *to be the function*

$$\begin{aligned} ECC(\mathcal{K}, f) &: \mathbb{R} \to \mathbb{Z} \\ & t \mapsto \chi(\{\sigma \in \mathcal{K} \mid \max\{f \cdot x \mid x \in \sigma \subset \mathbb{R}^d\} \leq t\}). \end{aligned}$$

For our reconstruction it will be convenient to extend the definition of Euler characteristic to finite sets of elementary cubes.

Definition 3.8. *Let $\mathcal{S}$ be a finite set of elementary cubes in $\mathbb{R}^d$. We define $\chi(\mathcal{S}) := \sum_{i=0}^{d}(-1)^i s_i$ where s_i is the cardinality of the subset of i-cubes contained in $\mathcal{S}$.*

3.3 Möbius Inversion on Posets

Here we provide definitions for the functions δ, ζ, μ, and the convolution of two functions defined on closed intervals of locally finite partially order sets. The Euler characteristic can be viewed through the lens of enumerative combinatorics and has significant ties to the Möbius function μ. For additional information on these functions including an algebraic structure (the incidence algebra) of a locally finite partially ordered set we point the reader to Stanley (2011).

Definition 3.9. *Let $\mathcal{P}$ be a partially ordered set. An **induced subposet** of $\mathcal{P}$ is a subset $\mathcal{Q}$ of $\mathcal{P}$ and a partial order of $\mathcal{Q}$ such that for $s, t \in \mathcal{Q}$ we have $s \leq t$ in $\mathcal{Q}$ if and only if $s \leq t$ in $\mathcal{P}$.*

Definition 3.10. *Two partially ordered sets P and Q are **isomorphic**, if there exists an order-preserving bijection $\psi : P \to Q$ whose inverse is order-preserving; that is,*

$$s \leq t \text{ in } P \Leftrightarrow \psi(s) \leq \psi(t) \text{ in } Q.$$

Definition 3.11. *Let $\mathcal{P}$ be a partially ordered set. A **closed interval** in $\mathcal{P}$ is the set,*

$$[s, t] = \{u \in \mathcal{P} \mid s \leq u \leq t\}$$

*defined whenever $s \leq t$ (thus the empty set is not regarded as a closed interval.) We denote **the set of closed intervals of** $\mathcal{P}$ as Int($\mathcal{P}$).*

Definition 3.12. *Let $\mathcal{P}$ be a partially ordered set. We say $\mathcal{P}$ is **locally finite** if every closed interval of $\mathcal{P}$ is finite.*

Definition 3.13. *Let K be a field and f, g : Int($\mathcal{P}$) $\to K$ be functions defined on closed intervals of a locally finite partially order set $\mathcal{P}$. We define the **convolution of f and g** as:*

$$f * g([s, t]) = \sum_{s \leq p \leq t} f([s, p])g([p, t]).$$

Remark 3.14. *The above sum ($f * g$) is well-defined since $\mathcal{P}$ is locally finite.*

Since the thesis will exclusively consider convolutions of functions defined on closed intervals of a partially ordered set $\mathcal{P}$ we will denote $f([s,t]) = f(s,t)$ for convolution computations.

Lemma 3.15. *The convolution of functions defined on closed intervals of a locally finite partially order set $\mathcal{P}$ is associative.*

Proof. Let $\mathcal{P}$ be a locally finite partially ordered set and K be a field. Suppose f, g, h : $\text{Int}(\mathcal{P}) \to K$. We compute $(f * (g * h))(x, y)$ as follows

$$\begin{aligned}
(f * (g * h))(x, y) &= \sum_{x \le a \le y} f(x, a)(g * h)(a, y) \\
&= \sum_{x \le a \le y} f(x, a) \sum_{a \le b \le y} g(a, b)h(b, y) \\
&= \sum_{x \le a \le b \le y} f(x, a)g(a, b)h(b, y) \\
&= \sum_{x \le b \le y} \left[\sum_{x \le a \le b} f(x, a)g(a, b) \right] h(b, y) \\
&= \sum_{x \le b \le y} (f * g)(x, b)h(b, y) \\
&= ((f * g) * h)(x, y)
\end{aligned}$$

□

Definition 3.16. *Given a partially ordered set $\mathcal{P}$ and field K, we define the function $\delta(s,t)$: Int($\mathcal{P}$) $\to K$ on closed intervals of $\mathcal{P}$ by*

$$\delta(s,t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$$

where 1 is the multiplicative identity in the field K.

Definition 3.17. *Given a partially ordered set $\mathcal{P}$ and field K, we define the* ***zeta function*** *ζ : Int($\mathcal{P}$) $\to K$ on closed intervals of $\mathcal{P}$ by $\zeta(s,t) = 1$ where 1 is the multiplicative identity in the field K.*

Remark 3.18. *Given a function f : Int($\mathcal{P}$) $\to K$ defined on closed intervals of a locally finite partially order set $\mathcal{P}$, we have*

$$f * \zeta(x,y) = \sum_{x \leq a \leq y} f(x,a)\zeta(a,y) = \sum_{x \leq a \leq y} f(x,a).$$

Definition 3.19. *Given a locally finite partially ordered set $\mathcal{P}$ and field K, we define the* ***Möbius function*** μ : *Int*($\mathcal{P}$) $\to K$ *on closed intervals of $\mathcal{P}$ using the following recursive definition:*

$$\mu(s,t) = \begin{cases} 1 & \text{if } s = t \\ -\sum_{s \leq a < t} \mu(s,a) & \text{if } s < t \end{cases}$$

Example: Let $\mathcal{P}$ be the chain $(\mathbb{N}, \leq)$. It follows from the definition of the Möbius function that

$$\mu(a,b) = \begin{cases} 1 & \text{if } a = a \\ -1 & \text{if } b = a+1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.20. *Let $\mathcal{P}$ be a locally finite partially ordered set and K be a field. Then the functions μ, ζ, δ : Int($\mathcal{P}$) $\to K$ satisfy*

$$\mu * \zeta = \delta.$$

Proof. Let $s, t \in \mathcal{P}$ such that $s \leq t$. We compute the left side of the equation as follows:

$$\begin{aligned} \mu * \zeta(s,t) &= \sum_{s \leq a \leq t} \mu(s,a)\zeta(a,t) \\ &= \sum_{s \leq a \leq t} \mu(s,a) \end{aligned}$$

Now if $s = t$ then the sum expression simplifies to $\sum_{s \le a \le t} \mu(s, a) = \mu(s, s) = 1 = \delta(s, s)$. If $s < t$ then we can expand the sum using the definition of μ (Definition 3.19) as

$$\begin{aligned}\sum_{s \le a \le t} \mu(s, a) &= \mu(s, t) + \sum_{s \le a < t} \mu(s, a) \\ &= -\sum_{s \le a < t} \mu(s, a) + \sum_{s \le a < t} \mu(s, a) \\ &= 0 = \delta(s, t)\end{aligned}$$

□

Observe that the integer lattice $\mathbb{Z}^d$ is equal to the set of 0-elementary cubes in $\mathbb{R}^d$. For a finite full elementary cubical complex $\mathcal{K}$ in $\mathbb{R}^d$ it is often useful to assume without loss of generality that the vertex set $\mathcal{K}^0$ are vertices of a subposet of the product poset $(\mathbb{N}, \le)^d$. To this end we provide the following proposition proving that μ is multiplicative for product posets.

Proposition 3.21 (Stanley (2011) Proposition 3.8.2)**.** *Let P and Q be locally finite posets, and let $P \times Q$ be the product poset. If $(s, t) \le (s', t')$ in $P \times Q$ then*

$$\mu_{P \times Q}((s, t), (s', t')) = \mu_P(s, s')\mu_Q(t, t').$$

Proof. Let $(s, t) \le (s', t')$. Then

$$\begin{aligned}[\mu_P \mu_Q] * \zeta_{P \times Q}((s, t), (s', t')) &= \sum_{(s,t) \le (x,y) \le (s',t')} \mu_P(s, x)\mu_Q(t, y) \\ &= \sum_{s \le x \le s'} \mu_P(s, x) \sum_{t \le y \le t'} \mu_Q(t, y) \\ &= \delta_P(s, s')\delta_Q(t, t') \\ &= \delta_{P \times Q}((s, t), (s', t')) \\ &= \mu_{P \times Q} * \zeta_{P \times Q}((s, t), (s', t'))\end{aligned}$$

which by the uniqueness of inverses of nonvanishing functions ($f(s,s) \neq 0$ for all $s \in \mathcal{P}$) in the incidence algebra of functions defined on locally finite posets $\mathcal{P}$ under convolution (Proposition 3.6.2. Stanley (2011)), implies $\mu_{P\times Q} = \mu_P \mu_Q$. □

Definition 3.22. *If $s, t \in \mathcal{P}$ are elements in the partially ordered set $\mathcal{P}$, we say that t **covers** s **or** s **is covered by** t, if $s < t$ and no element $u \in \mathcal{P}$ satisfies $s < u < t$.*

Definition 3.23. *The **Hasse diagram of a finite poset** $\mathcal{P}$ is the directed graph whose vertices are the elements of $\mathcal{P}$. If $s, t \in \mathcal{P}$ such that s is covered by t then there exists a directed edge from s to t.*

3.4 Cubical Homology

In this section we define chain complexes, the boundary operator ∂, homology groups, and demonstrate functorality in the elementary cubical complex setting.

Definition 3.24. *Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$. Let $C_i(\mathcal{K})$ be the free abelian group on $\mathcal{K}^i$, the set of i-cubes of $\mathcal{K}$. The **chain complex** of $\mathcal{K}$ is the graded free abelian group $\{C_j(\mathcal{K})\}_{j\in\mathbb{Z}}$ with the homomorphisms $\partial_j : C_j(\mathcal{K}) \to C_{j-1}(\mathcal{K})$ called the **boundary operators** of the chain complex defined as follows.*

Definition 3.25. *Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$. The **cubical boundary operator** $\partial_j : C_j(\mathcal{K}) \to C_{j-1}(\mathcal{K})$ is defined on the generators of $C_j(\mathcal{K})$. Let $0 \leq j \leq d \in \mathbb{N}$, $\sigma = \prod_{k=1}^{d} I_k$ be an elementary j-cube where $I_k = [a_k, b_k]$, with $b_k = a_k$ or $b_k = a_k + 1$. Let I_a^b denote $\prod_{k=a}^{b} I_k$. We define*

$$\partial_j(\sigma) = \sum_{k=1}^{d} (-1)^{\dim(I_1^{k-1})} [I_1^{k-1} \times b_k \times I_{k+1}^d - I_1^{k-1} \times a_k \times I_{k+1}^d].$$

Lemma 3.26. *Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$. If $\sigma \in \mathcal{K}^j$ for $j \geq 1$ then*

$$\partial_{j-1}\partial_j(\sigma) = 0.$$

Proof. Let I_a^b denote $\prod_{k=a}^{b} I_k$ and $\sigma = I_1^d$ be an elementary j-cube. Then

$$\begin{aligned}
\partial_{j-1}\partial_j(\sigma) = \partial_{j-1}(\sum_{k=1}^{d}(-1)^{\dim(I_1^{k-1})}(I_1^{k-1}\times b_k \times I_{k+1}^d - I_1^{k-1}\times a_k\times I_{k+1}^d))& \\
= \sum_{1\le j<k\le d}(-1)^{\dim(I_1^{j-1})}(-1)^{\dim(I_1^{k-1})}(I_1^{j-1}\times b_j\times I_{j+1}^{k-1}\times b_k\times I_{k+1}^d& \\
- I_1^{j-1}\times a_j\times I_{j+1}^{k-1}\times a_k\times I_{k+1}^d)& \\
+ \sum_{1\le k<j\le d}(-1)^{\dim(I_1^{j-1})-1}(-1)^{\dim(I_1^{k-1})}(I_1^{k-1}\times b_k\times I_{k+1}^{j-1}\times b_j\times I_{j+1}^d& \\
- I_1^{k-1}\times a_k\times I_{k+1}^{j-1}\times a_j\times I_{j+1}^d)& \\
= 0&
\end{aligned}$$

since we can pull out a negative one from the second sum and observe that the two sums are the same sums up to reindexing. □

Definition 3.27. *We define the group of* ***boundaries*** *as $B_j(\mathcal{K}) :=$ im ∂_{j+1} and the group of* ***cycles*** *as $Z_j(\mathcal{K}) :=$ ker ∂_j. Since the boundary operators of a chain complex have the property that $\partial_{j-1}\partial_j = 0$ for all $1 \le j \le d \in \mathbb{N}$ (Lemma 3.26), the group of boundaries is a subgroup of the group of cycles. This allows us to define the* ***homology group*** *as $H_j(\mathcal{K}) := Z_j(\mathcal{K})/B_j(\mathcal{K})$.*

Let $\mathcal{K}_1, \mathcal{K}_2$ be elementary cubical complexes in $\mathbb{R}^d$. Suppose $f : \mathcal{K}_1 \to \mathcal{K}_2$ is a **cubical map** (a piecewise linear map which maps vertices to vertices) between elementary cubical complexes. There exists an induced map on chain complexes $f_* : C_j(\mathcal{K}_1) \to C_j(\mathcal{K}_2)$ defined by

$$f_*(\sum_n a_n\sigma_n) = \sum_{\substack{n \\ f(\sigma_n)\in\mathcal{K}_2^j}} a_n f(\sigma_n)$$

where $\sigma_n \in \mathcal{K}^j$ is a j-cube (f_* is also denoted $H_j(f)$).

Lemma 3.28. *Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a cubical map between elementary cubical complexes in $\mathbb{R}^d$. Then $\partial_j f_* = f_*\partial_j$.*

Proof. We prove the result by showing that the f_* and ∂ commute on the generators of j-th chain group. Let $\sigma \in \mathcal{K}_1^j$ be an elementary j-cube in $\mathbb{R}^d$. We compute the left hand side by considering two cases dependent on the dimension of the image of $f(\sigma)$.

If $f(\sigma) \notin \mathcal{K}_2^j$ i.e. $\dim(f(\sigma)) < j$ then $f_*(\sigma) = 0$ by definition so $\partial_* f_*(\sigma) = 0$. Further since $\dim(f(\sigma)) < j$, $\dim(f(\tau)) < j$ for all faces τ of σ, so $f_*(\partial_j \sigma) = 0$.

If $f(\sigma) \in \mathcal{K}_2^j$ then by the definition of a cubical map $\sigma = [a_1, b_1] \times \cdots \times [a_d, b_d]$ and $f(\sigma) = [f(a_1), f(b_1)] \times \cdots \times [f(a_d), f(b_d)]$. Let $I_k = [a_k, b_k]$, $J_k = [f(a_k), f(b_k)]$, $I_a^b = I_a \times \cdots \times I_b$, and $J_a^b = J_a \times \cdots \times J_b$. Then by the linearity of f

$$\begin{aligned}
\partial\sigma &= \sum_{k=1}^{d} (-1)^{\dim(I_1^{k-1})}(I_1^{k-1} \times b_k \times I_{k+1}^d - I_1^{k-1} \times a_k \times I_{k+1}^d) \\
f(\partial\sigma) &= \sum_{k=1}^{d} (-1)^{\dim(J_1^{k-1})}(J_1^{k-1} \times f(b_k) \times J_{k+1}^d - J_1^{k-1} \times f(a_k) \times J_{k+1}^d) \\
&= \partial(f\sigma).
\end{aligned}$$

□

By definition of the induced map and linearity of the boundary operator, $f_*\partial_j = \partial_j f_*$. Let $\mathbf{Cub}_d$ be the category of elementary cubical complexes in $\mathbb{R}^d$ whose morphisms are cubical maps. Let $\mathbf{1}$ be the identity morphism in $\mathbf{Cub}_d$. If $\sum_n a_n\sigma_n \in C_j(\mathcal{K}_1)$ is j-chain for $j \geq 0$, then

$$\mathbf{1}_*(\sum_n a_n\sigma_n) = \sum_n a_n\mathbf{1}(\sigma_n) = \sum_n a_n\sigma_n$$

so the induced map is the identity morphism in the category of chain complexes ($\mathbf{1}_* = \mathbf{1}$).

Similarly, given cubical maps $f : \mathcal{K}_1 \to \mathcal{K}_2$ and $g : \mathcal{K}_2 \to \mathcal{K}_3$, the composition of the induced maps gives

$$\begin{aligned}(g \circ f)_*(\sum_n a_n \sigma_n) &= \sum_{g(f(\sigma_n)) \in \mathcal{K}_3^j} a_n (g \circ f)(\sigma_n) \\ &= g_* \sum_{f(\sigma_n) \in \mathcal{K}_2^j} a_n f(\sigma_n) \\ &= g_* \circ f_* \sum_n a_n \sigma_n\end{aligned}$$

since the composition of cubical maps implies if $g(f(\sigma_n)) \in \mathcal{K}_3^j$ then $f(\sigma_n) \in \mathcal{K}_2^j$. Therefore, the assignments $\mathcal{K} \mapsto H_j(\mathcal{K})$ and $f \mapsto f_*$ define functor $H_j : \mathbf{Cub}_d \to \mathbf{Ab}$.

3.5 Filtrations and Persistent Homology

In order to reconstruct a finite elementary cubical complex $\mathcal{K}$ in $\mathbb{R}^d$ we consider the change of the cubical homology groups of collections of nested subcomplexes of $\mathcal{K}$. This is formally known as Persistent Homology. Using the Homology functor, each inclusion map between nested subcomplexes of $\mathcal{K}$ induces a linear map between the associated homology groups. In this manner we can study how voids are formed, merge, and are filled in order to better understand distinctions between the geometry and topology of $\mathcal{K}$.

Definition 3.29. *Given an elementary cubical complex $\mathcal{K}$ in $\mathbb{R}^d$, a (finite)* ***filtration of*** $\mathcal{K}$ *is a sequence of elementary cubical complexes $\emptyset = \mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \cdots \subseteq \mathcal{K}_n = \mathcal{K}$.*

Definition 3.30. *A* ***persistence module*** M *is a set of vector spaces $\{V_i\}_{i=1}^n$ over a field* ***k*** *and* ***k****-linear maps $\{i_a^b : V_a \to V_b\}_{a \leq b}$ such that:*

1. *for all a, $i_a^a : V_a \to V_a$ is the identity map,*
2. *for $a \leq b \leq c$, $i_a^c = i_b^c \circ i_a^b$.*

We focus on persistence modules which arise from sublevel filtrations of full elementary cubical complexes by applying the cubical homology functor from Section 3.4 (which is naturally isomorphic to singular homology Eilenberg and MacLane (1953)) in degree *j*, with coefficients in the field $\mathbb{Z}/2$. Since we are exclusively working over the field $\mathbb{Z}/2$ we will drop the field in the notation $H_j(\mathcal{K}_a, \mathbb{Z}/2)$ and instead write $H_j(\mathcal{K}_a)$ for the remainder of the paper. The vector space $C_i(\mathcal{K}_a)$ is generated by the set of j-cubes $\mathcal{K}_a^j$ and the induced linear maps are given by applying the homology functor to the inclusion maps $H_j(i_a^b) : H_j(\mathcal{K}_a) \to H_j(\mathcal{K}_b)$.

The *persistent homology groups* $H_j(a,b) := \text{im}(H_j(i_a^b))$ are the images of the induced maps under the homology functor. We say that a homology class $\alpha \in H_j(\mathcal{K}_b)$ is *born* at time b if $\alpha \notin H_j(a,b)$ for all $a < b$. For a homology class α born at time $b < d$, we say that α *dies* at time d if $H_j(i_b^d)(\alpha) = 0$ but for all $b \leq d' < d$, $H_j(i_b^{d'})(\alpha) \neq 0$.

Definition 3.31. *Let $\{\mathcal{K}_i\}_{i=0}^n$ be a filtration of a finite full elementary cubical complex $\mathcal{K}$. The **rank function** $\beta_j(\mathcal{K}) : \{(a,b) \in \mathbb{R}^2 | a \leq b\} \to \mathbb{R}$ defined by $\beta_j(\mathcal{K})(a,b) = \text{rank}(H_j(i_a^b))$, the rank of the induced map on homology from the inclusion map on spaces. This function is also refered to as the persistent Betti number.*

Definition 3.32 (Patel (2018) Theorem 4.1)**.** *The **i-th persistence diagram corresponding to a filtration of** $\mathcal{K}$, $\text{Dgm}_i(\mathcal{K})$ is the Möbius inversion of its i-th rank function. We denote* $\text{Dgm}(\mathcal{K}) := \coprod_{j=0}^{dim(\mathcal{K})} \text{Dgm}_j(\mathcal{K})$.

Definition 3.32 implies $\text{Dgm}_i(\mathcal{K})$ is a multi-set of points in $\mathbb{R}^2_{a<b} := \{(a,b) \in (-\infty \cup \mathbb{R}) \times (\mathbb{R} \cup \infty) : a < b\}$ such that the number of points (counting multiplicity) in $[-\infty, a] \times [b, \infty]$ is equal to the dimension of $H_i(a,b)$.

3.6 Discrete Morse Theory

Smooth Morse theory relates critical points of a generic smooth real-valued function (points where the gradient vanishes) on a manifold to the global topology of that manifold Milnor (1963). Forman extended this theory to cell complexes, which are discrete structures Forman (2002). In particular Theorem 3.40 establishes a connection between a discrete gradient vector field (Definition 3.36) on an elementary cubical complex $\mathcal{K}$ in $\mathbb{R}^d$ and the homotopy type of $\mathcal{K}$.

Definition 3.33. *Given two elementary cubes $\sigma, \tau \in \mathcal{N}$ in $\mathbb{R}^d$ we say σ **is a facet of** τ, denoted $\sigma \prec \tau$, if σ is a face of τ of codimension 1.*

Definition 3.34. *Let $\mathcal{S}$ be a set of elementary cubes. A* ***Morse pairing*** *in $\mathcal{S}$ is a set of pairs of cubes $M(\mathcal{S}) = \{(\sigma, \tau)\}$, with σ a facet of τ, such that each cube of $\mathcal{S}$ is contained in at most one pair. A cube $\sigma \in \mathcal{S}$* ***is critical with respect to*** *$M(\mathcal{S})$ if σ is not contained in any pair.*

Definition 3.35. *Given a Morse pairing $M(\mathcal{S})$, a* ***V-path*** *induced by $M(\mathcal{S})$ is a sequence in $\mathcal{S}$ $\tau_0, \sigma_1, \tau_1, ..., \sigma_l, \tau_l, \sigma_{l+1}$, where $(\sigma_i, \tau_i) \in M(\mathcal{S})$ for every $i = 1, ..., l$, $\sigma_i \neq \sigma_{i+1}$ for each $i = 1, ..., l$, and each σ_{i+1} is a facet of τ_i for each $i = 0, ..., l$. If $l = 0$, the V-path is* ***trivial****. A V-path is* ***cyclic*** *if $l > 0$ and $(\sigma_{l+1}, \tau_0) \in M(\mathcal{S})$. A V-path is* ***acyclic*** *if it has no cyclic subpaths. We say the Morse pairing $M(\mathcal{S})$ is* ***acyclic*** *if there is no cyclic V-path induced by $M(\mathcal{S})$.*

Definition 3.36. *Let $\mathcal{K}$ be an elementary cubical complex. A Morse pairing $M(\mathcal{K})$ is called a* ***discrete gradient vector field*** *if $M(\mathcal{K})$ is acyclic (Definition 3.35).*

Definition 3.37. *A function $f : \mathcal{K} \to \mathbb{R}$ on an elementary cubical complex $\mathcal{K}$ is a* ***discrete pseudo-Morse function*** *if there is a discrete gradient vector field (Definition 3.36) GVF($\mathcal{K}$) such that for all pairs (σ, τ) in $\mathcal{K}$ with σ a facet of τ,*

$$\text{if } (\sigma, \tau) \in \text{GVF}(\mathcal{K}) \text{ then } f(\sigma) \geq f(\tau)$$

$$\text{if } (\sigma, \tau) \notin \text{GVF}(\mathcal{K}) \text{ then } f(\sigma) \leq f(\tau)$$

In this case we say that GVF($\mathcal{K}$) and f are ***consistent****. Further, we call a discrete psuedo-Morse function a* ***flat pseudo-Morse function*** *if $f(\sigma) = f(\tau)$ if $(\sigma, \tau) \in$ GVF($\mathcal{K}$).*

Remark 3.38. *A discrete gradient vector field GVF($\mathcal{K}$) consistent with a discrete pseudo-Morse function f is not unique in general.*

Definition 3.39. *Given an elementary cubical complex $\mathcal{K}$ and a discrete gradient vector field on $\mathcal{K}$ GVF($\mathcal{K}$), the* ***modified Hasse diagram*** *is the directed graph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ whose vertex set $\mathcal{V}$ is equal to the set of faces of $\mathcal{K}$ which has one edge for each pair (σ, τ) where σ is a* *facet of τ and whose direction is given by*

$$\begin{cases} e(\tau,\sigma) \in \mathcal{E} & \textit{if } (\sigma,\tau) \notin \textit{GVF}(\mathcal{K}), \\ e(\sigma,\tau) \in \mathcal{E} & \textit{if } (\sigma,\tau) \in \textit{GVF}(\mathcal{K}). \end{cases}$$

*If $(\sigma,\tau) \in$ GVF($\mathcal{K}$) we say σ is a **tail of a modified arrow** and τ is a **head of a modified arrow** and σ **is paired with** τ.*

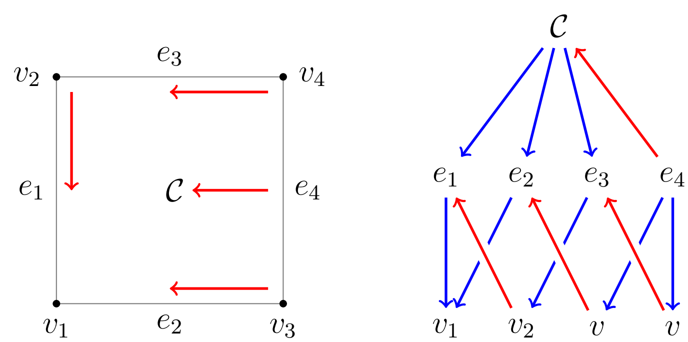

Figure 3-1. On the left is an elementary 2-cube $\mathcal{C}$ in $\mathbb{R}^2$ and a discrete gradient vector field GVF($\mathcal{C}$) $= \{(v_2, e_1), (v_3, e_2), (v_4, e_3), (e_4, \mathcal{C})\}$ indicated by arrows pointing from the facet to the paired face. On the right is the corresponding modified Hasse diagram of the face poset of $\mathcal{C}$ determined by GVF($\mathcal{C}$). The downwards arrows (blue) are unmatched faces pointing from an elementary cube to a facet. The upwards arrows (red) correspond to the matching from the discrete gradient vector field on the left. Observe that v_1 is critical as there is no available face to pair with on the left under the proposed gradient vector field which corresponds with v_1 being neither a head nor tail of a red arrow on the right.

Since every face of $\mathcal{K}$ is contained in at most one pair in a given discrete gradient vector field, we can partition the vertices of a modified Hasse diagram into three disjoint sets

$$\mathcal{V} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{Z}$$

where $\mathcal{A}$ is the set of faces which are heads of modified arrows, $\mathcal{B}$ is the set of faces which are tails of modified arrows, and $\mathcal{Z}$ is the set of critical faces (neither the head nor tail of a modified arrow) (Figure 3-1). This allows for a convenient interpretation of critical cubes of an elementary cubical complex. The following theorem establishes a correspondance between critical cubes of a GVF($\mathcal{K}$) and the homotopy type of $\mathcal{K}$.

Theorem 3.40 (Forman (2002) Theorem 2.5)**.** *Let GVF($\mathcal{K}$) be a discrete gradient vector field on a finite elementary cubical complex $\mathcal{K}$. Then $\mathcal{K}$ is homotopy equivalent to a CW complex with exactly one cell of dimension i for each critical i-cube.*

Corollary 3.41. *Let $\mathcal{K}$ be a finite elementary cubical complex and $\mathcal{S}$ be a finite set of elementary cubes such that $\mathcal{K} \coprod \mathcal{S}$ is an elementary cubical complex. Let $f : \mathcal{K} \coprod \mathcal{S} \to \mathbb{R}$, and $v \in \mathcal{K}^0$. If there exists an acyclic Morse pairing in $\mathcal{S}$ consistent with f with no critical cubes, then $\mathcal{K} \coprod \mathcal{S}$ is homotopy equivalent to $\mathcal{K}$.*

3.7 Euler Calculus

It is important to acknowledge that the collection of Euler charactistic curves, defined as the Euler characteristic transform can be derived from an older theory known as Euler Calculus. Euler Calculus is a calculus which uses the definable Euler characteristic as a measure. We provide the definitions necessary to include a theorem (first proven concurrently by Curry et al. (2018) and R. Ghrist and Mai (2018)) which establishes the injectivity of the Euler characteristic transform using Euler calculus.

Definition 3.42. *An* ***o-minimial structure*** $\mathcal{O} = \{\mathcal{O}_d\}$ *specifies for each $d \geq 0$, a collection of subsets $\mathcal{O}_d$ of $\mathbb{R}^d$ closed under intersection and complement. These collections are related to each other by the following rules:*

- *If $A \in \mathcal{O}_d$. then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ are both in $\mathcal{O}_{d+1}$; and*
- *If $A \in \mathcal{O}_{d+1}$, then $\pi(A) \in \mathcal{O}_d$ where $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is axis-alligned projection.*

We further require that $\mathcal{O}$ contains all algebraic sets and that $\mathcal{O}_1$ contains precisely all finite unions of elementary intervals (both degenerate and non-degenerate). Elements of $\mathcal{O}$ are called ***tame*** *or* ***definable*** *sets. A definable map $f : \mathbb{R}^d \to \mathbb{R}^n$ is one whose graph is definable.*

Full elementary cubical complexes are o-minimal structures since full elementary cubical complexes are instances of algebraic sets which are a subset of o-minimal structures. Definable sets play the role of measurable sets for an integration theory known as **Euler Calculus** R. Ghrist and Mai (2018).

Definition 3.43. *If $X \in \mathcal{O}$ is tame and $h : X \to \bigcup \sigma_i$ is a definable bijection with a collection of open simplices, then the* ***definable Euler characteristic*** *of X is*

$$\chi(X) := \sum_i (-1)^{\textit{dim}(\sigma_i)}$$

where dim(σ_i) denotes the dimension of the open simplex σ_i. We understand that $\chi(\emptyset) = 0$ corresponding to the empty sum.

Proposition 3.44. *For tame subsets $A, B \in \mathcal{O}$ we have*

$$\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B).$$

Definition 3.45. *A* ***constructible function*** *$\psi : X \to \mathbb{Z}$ is an integer-valued function on a tame set X with the property that every level set is tame. The set of constructible functions with domain X, denoted $CF(X)$, is closed under pointwise addition and multiplication.*

Definition 3.46. *The* ***Euler integral*** *of a constructible function $\psi : X \to \mathbb{Z}$ is the sum of the Euler characteristics of each of its level-sets,*

$$\int \psi \, \boldsymbol{d}\chi := \sum_{n=-\infty}^{\infty} \chi(\psi^{-1}(n)).$$

Definition 3.47. *Let $f : X \to Y$ be a tame mapping between definable sets. Let $\psi_Y : Y \to \mathbb{Z}$ be a constructible function on Y. The* ***pullback*** *of ψ_Y along f, $f^* : CF(Y) \to CF(X)$ is defined pointwise by*

$$f^*\psi_Y(x) = \psi_Y(f(x)).$$

Definition 3.48. *The* ***pushforward*** *of a constructible function $\psi_X : X \to \mathbb{Z}$ along a tame map $f : X \to Y$ is given by*

$$f_*\psi_X(y) = \int_{f^{-1}(y)} \psi_X \, \boldsymbol{d}\chi.$$

This defines a group homomorphism $f_* : CF(X) \to CF(Y)$.

Definition 3.49. *Suppose $S \subset X \times Y$ is a locally closed definable subset of the product of two definable sets. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the projections* *from the product onto the indicated factors. Let $\psi \in X \to \mathbb{Z}$ be a constructible function. The* ***Radon transform with respect to S*** *is the group homomorphism $\mathcal{R}_S : CF(X) \to CF(Y)$,*

$$\mathcal{R}_S(\psi) := \pi_{Y*}[(\pi_X^*\psi)1_S].$$

Proposition 3.50. *Let $\mathcal{S} = \{(x, W) | x \in W \subset \mathbb{R}^d\} \subset \mathbb{R}^d \times (S^{d-1} \times \mathbb{R})$. Then $\mathcal{R}_{\mathcal{S}}(\boldsymbol{1}_{\mathcal{N}})(W) = \chi(\mathcal{N} \cap W)$*

Proof.

$$\begin{aligned}
\mathcal{R}_S(\mathbf{1}_{\mathcal{N}})(W) &= (\pi_{S^{d-1}\times\mathbb{R}})_*[(\pi_{\mathbb{R}^d}^*\mathbf{1}_{\mathcal{N}})\mathbf{1}_{\mathcal{S}}](W) \\
&= \int_{(x,W)\in\mathcal{S}} (\pi_{\mathbb{R}^d}^*\mathbf{1}_{\mathcal{N}}) d\chi \\
&= \int_{x\in\mathcal{N}\cap W} \mathbf{1}_{\mathcal{N}}(x) d\chi \\
&= \chi(\mathcal{N} \cap W)
\end{aligned}$$

□

The collection of Euler characteristic curves for each sublevel filtration arising from a vector in S^{d-1}, $\{ECC(\mathcal{N}, f)\}_{f\in S^{d-1}}$ is refered to as **the Euler characteristic transform of** $\mathcal{N}$. Given such a collection of Euler characteristic curves, one can localize the Euler characteristic value to the contribution of the intersection of any hyperplane with $\mathcal{N}$.

Proposition 3.51 (Localization)**.** *Let $\mathcal{N} \subset \mathbb{R}^d$ be a geometric simplicial complex and $W = (f, t) \in AffGr^d = S^{d-1} \times \mathbb{R}$ be a $d-1$ dimensional hyperplane. The Euler characteristic transform of $\mathcal{N}$, $ECT(\mathcal{N})$, determines the value $\chi(\mathcal{N} \cap W)$.*

Proof. Using the inclusion-exclusion property of the Euler characteristic ($\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B)$) we compute the quantity $\chi(\mathcal{N} \cap W)$ as follows:

$$\begin{aligned}
\chi(\mathcal{N} \cap W) &= \chi(\{x \in \mathcal{N} | x \cdot f = t\}) \\
&= \chi(\{x \in \mathcal{N} | x \cdot f \leq t\} \cap \{x \in \mathcal{N} | x \cdot -f \geq -t\}) \\
&= \chi(\{x \in \mathcal{N} : x \cdot f \leq t\}) + \chi(\{x \in \mathcal{N} : x \cdot -f \geq -t\}) - \chi(\mathcal{N}) \\
&= ECC(\mathcal{N}, f)(t) + ECC(\mathcal{N}, -f)(-t) - ECC(\mathcal{N}, f)(\infty)
\end{aligned}$$

□

Since the Euler characteristic transform of $\mathcal{N}$ is equal to integrating $\int_{r \leq t} \chi(M \cap W = (v, r)) d\chi$, the Euler characteristic transform is implied by the Radon transform with the set $\mathcal{S} = \{(x, W) | x \in W \subset \mathbb{R}^d\} \subset \mathbb{R}^d \times (S^{d-1} \times \mathbb{R})$.

Theorem 3.52 (Theorem 3.1 in Schapira (1995))**.** *If $\mathcal{S} \subset X \times Y$ and $\mathcal{S}' \subset Y \times X$ have fibers $\mathcal{S}_x$ and $\mathcal{S}'_x$ in Y satisfying:*

- $\chi(\mathcal{S}_x \cap \mathcal{S}'_x) = \chi_1$ *for all $x \in X$, and*
- $\chi(\mathcal{S}_x \cap \mathcal{S}'_{x'}) = \chi_2$ *for all $x' \neq x \in X$,*

then for all $\psi \in CF(X)$,

$$(\mathcal{R}_{\mathcal{S}'} \circ \mathcal{R}_{\mathcal{S}})\psi = (\chi_1 - \chi_2)\psi + \chi_2 (\int_X \psi d\chi) \boldsymbol{1}_X.$$

Theorem 3.53 (Curry et al. (2018),R. Ghrist and Mai (2018))**.** *The Euler characteristic transform is injective on the set of o-minimal sets.*

Proof. Let $\mathcal{S} = \{(x, W) | x \in W \subset \mathbb{R}^d\} \subset \mathbb{R}^d \times (S^{d-1} \times \mathbb{R})$ and $\mathcal{S}' = \{(W, x) | x \in W \subset \mathbb{R}^d\} \subset (S^{d-1} \times \mathbb{R}) \times \mathbb{R}^d$. For $x = x' \in \mathbb{R}^d$ the fibers $\mathcal{S}_x \cap \mathcal{S}'_x$ is the set of hyperplanes that go through x and therefore, $\mathcal{S}_x \cap \mathcal{S}'_x = \mathbb{R}P^{d-1}$. Similarly, for $x \neq x' \in \mathbb{R}^d$ the fibers $\mathcal{S}_x \cap \mathcal{S}'_x$ are the set of hyperplanes that go through x and x' implying $\mathcal{S}_x \cap \mathcal{S}'_x = \mathbb{R}P^{d-2}$. The result follows from applying Theorem 3.52 and observing that the composition of Radon transforms for the sets applied to the characteristic function $\mathbf{1}_{\mathcal{N}}$ (where $\chi_1 = \frac{1}{2}(1 + (-1)^{d-1})$ and $\chi_2 = \frac{1}{2}(1 + (-1)^{d-2})$) is injective for an o-minimal set $\mathcal{N}$. $\square$

CHAPTER 4
MATHEMATICAL RESULTS

Let $\mathcal{K}$ be a finite full elementary cubical complex in $\mathbb{R}^d$ (Definition 3.5). In this chapter we restrict our view to discrete analogs of height filtrations of cubical complexes. In Section 4.1 we introduce the *lower star filtration* which is the sole filtration we consider for the remainder of the thesis. Section 4.2 presents a means to compute the Möbius inversion of the characteristic function on the set of full cubes from a collection of Euler characteristic curves. Further, we establish a sharp upper bound for the size of such a collection. Section 4.3 explores a geometric condition to classify critical vertices in a full elementary cubical complex which can be computed using the set of full elementary cubes.

4.1 Lower Star Filtrations of Full Elementary Cubical Complexes

In this section we introduce the *lower star filtration* of a full elementary cubical complex in $\mathbb{R}^d$. The lower star filtration is a discrete analog to a height level filtration which maintains the cubical complex structure of each sublevel set. We also discuss geometric characterizations of level sets and its implications for computing the Euler characteristic curve of a full elementary cubical complex.

Given a filtration vector $f \in \mathbb{R}^d$, assign to each vertex $v = [r_1] \times [r_2] \times \cdots \times [r_d] \in \mathcal{K}^0$, the value $f(v) = f \cdot (r_1, \ldots, r_d)$. We call a vector $f = (f_1, \ldots, f_d) \in \mathbb{R}^d$ a **generic vector** if for all $c = (c_1, ..., c_d) \in \mathbb{Z}^d$, $\sum_{i=1}^{d} c_i f_i = 0$ implies $c_i = 0$ for all $i \in \{1, ..., d\}$. This definition implies that for all $v, w \in \mathbb{Z}^d$, $f(v) = f(w)$ if and only if $v = w$ by linearity of the dot product. Since all filtration values on the vertices are elements of $\mathbb{R}$, for a generic vector, the corresponding sublevel set filtration induces a total ordering of $\mathcal{K}^0 \subset \mathbb{Z}^d$ from $(\mathbb{R}, \leq)$.

Proposition 4.1. *The set of generic vectors of $\mathbb{R}^d$ has full d-dimensional Lebesgue measure.*

Proof. Suppose $r \in \mathbb{R}^d$ is a non-generic vector. This implies there exists a $c \neq 0 \in \mathbb{Z}^d$ such that $r \cdot c = 0$. Without loss of generality suppose the i-th coordinate of c be nonzero.

$$\sum_{j=1}^{d} c_j r_j = 0$$

$$\sum_{j \neq i} c_j r_j = -c_i r_i$$

$$-\sum_{j \neq i} \frac{c_j}{c_i} r_j = r_i.$$

This implies that for a non-generic vector there is at least one coordinate which is a $\mathbb{Z}^d$ linear sum of the other $d-1$ coordinates. Thus the set of non-generic vectors of $\mathbb{R}^d$ is a subset of $\mathbb{R}^{d-1} \times \mathbb{Z}^d$ which has d-dimensional Lebesgue measure 0. Therefore, the set of generic vectors (which is the complement of the set of non-generic vectors) has full measure. $\square$

We construct a filtration of a full elementary cubical complex $\mathcal{K}$ in $\mathbb{R}^d$ by extending our filtration values from the vertices of $\mathcal{K}$ to all higher dimensional elementary cubes. Each elementary cube $C \in \mathcal{K}$ is assigned the maximum value of the filtration values on its set of vertices, $f(C) = \max\{f(v_i) \mid v_i \in C \cap \mathcal{K}^0\}$. We call this the **lower star filtration of $\mathcal{K}$ under f**. For a filtration vector $f \in \mathbb{R}^d$ and real number $r \in \mathbb{R}$ we denote the full elementary cubical complexes $\mathcal{K}_{\leq r} := f^{-1}(-\infty, r]$ and $\mathcal{K}_{<r} := f^{-1}(-\infty, r)$ to be the respective preimages of the lower star filtration. We denote the persistence diagram (Definition 3.32) of $\mathcal{K}$ generated by the lower star filtration under f as Dgm$(\mathcal{K}, f)$.

In the next section we will use the following definition.

Definition 4.2. *Let $v \in \mathcal{N}^0$ in $\mathbb{R}^d$ and $f \in \mathbb{R}^d$ be a generic vector. The* ***full cube anchored at v in the direction of f*** *is the unique full cube in $st(v) \subset \mathcal{N}$ with the largest filtration value. We denote the full cube as $\mathcal{C}_v^f$. If f lies in the positive orthant, we just write $\mathcal{C}_v$.*

Proposition 4.3. *Let $\mathcal{K}$ be a full elementary cubical complex in $\mathbb{R}^d$. For all $f \in \mathbb{R}^d$, the lower star filtration of $\mathcal{K}$ under f is a flat pseudo-Morse function (Definition 3.37).*

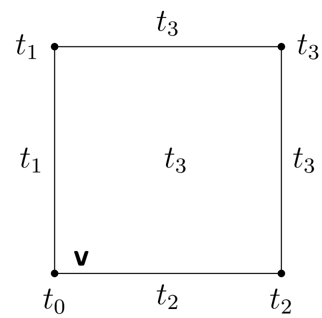

Figure 4-1. Filtration values of an elementary 2-cube in $\mathbb{R}^d$ anchored at **v** in the direction of $f = (f_x, f_y)$ where $f_x > f_y > 0$. The value $t_i < t_j$ when $i < j$.

Proof. Fix $f \in \mathbb{R}^d$. Observe that for all elementary cubes $\sigma, \tau \subset \mathcal{N}$ if σ is a face of τ then $\overline{\sigma} \cap \mathcal{N}^0 \subset \overline{\tau} \cap \mathcal{N}^0$ (Definition 3.6). Therefore, by definition of the lower star filtration, $f(\sigma) \leq f(\tau)$ which implies that the empty discrete gradient vector field (which pairs no elements of $\mathcal{N}$) is consistent with f. Therefore, f is a flat pseudo-Morse function. □

Definition 4.4. *Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$. We say that a vertex $v \in \mathcal{K}^0$ is a **critical vertex of** $\mathcal{K}$ **with respect to f** if a homology class of any degree α is born or dies at time $f(v)$. We say a vertex v is a **critical vertex of** $\mathcal{K}$ if there exists a generic vector $f \in \mathbb{R}^d$ such that v is a critical vertex of $\mathcal{K}$ with respect to f.*

Definition 4.5. *Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$. Given a vertex $v \in \mathcal{K}^0$ and a vector $f \in \mathbb{R}^d$, the **lower star of v with respect to f** denoted $st(v)^f_{\leq}$ is the set of elementary cubes in $st(v) \cap \mathcal{K}_{\leq f(v)}$ (recall the star of v, $st(v)$, Definition 3.6) .*

Remark 4.6. *Given a vertex of a full elementary cubical complex $v \in \mathcal{K}^0$ in $\mathbb{R}^d$, the lower star $st(v)^f_{\leq}$ is an elementary cubical complex if and only if $st(v)^f_{\leq} = \{v\}$ (otherwise the set of cubes will fail the downwards closure condition).*

Remark 4.7. *Let $\mathbf{1}_{\mathcal{K}}$ be the indicator function on an elementary cubical complex $\mathcal{K}$. Then $\mathbf{1}_{\mathcal{K}}(\mathcal{C}_v^{-f}) = 1 \Leftrightarrow st^f_{\leq}(v) \cap \mathcal{K}^d \neq \emptyset$.*

Proposition 4.8. *Let $\mathcal{K}$ be a full elementary cubical complex in $\mathbb{R}^d$. For a generic vector $f \in \mathbb{R}^d$ and $v \in \mathcal{K}^0$, the sets $\mathcal{K}_{\leq f(v)} - \mathcal{K}_{< f(v)} = st(v)^f_{\leq}$ are equal.*

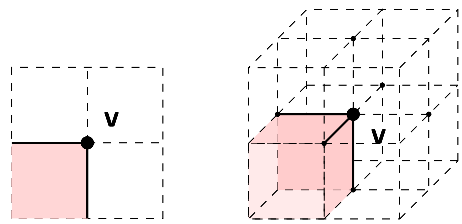

Figure 4-2. The lower star of v generated by some filtration vector f in the direction of the quadrant (+,+) and (-,+,+) respectively. The lower star $st(\mathbf{v})_{\leq f(\mathbf{v})}$ is depicted in red (shaded).

Proof. We proceed by showing containment in both directions. By definition $\mathcal{K}_{\leq f(v)} - \mathcal{K}_{<f(v)} = \{\tau \in \mathcal{K} | f(\tau) = f(v)\}$. Suppose $\tau \in \mathcal{K}_{\leq f(v)} - \mathcal{K}_{<f(v)}$. Since $f(\tau) = f(v)$, by the definition of the lower star filtration of $\mathcal{K}$ under f and the injectivity of f on $\mathcal{K}^0$, v is a vertex of τ. Therefore, $\tau \in st(v)$ and by assumption $\tau \in \mathcal{K}_{\leq f(v)}$ thus $\tau \in st(v) \cap \mathcal{K}_{\leq f(v)} = st(v)^f_{\leq}$. For the reverse containment suppose $\sigma \in st(v) \cap \mathcal{K}_{\leq f(v)}$. Since $\sigma \in \mathcal{K}_{\leq f(v)}$, $f(\sigma) \leq f(v)$. Since $\sigma \in st(v)$ by assumption, $f(\sigma) \geq f(v)$. Therefore, $f(\sigma) = f(v)$ and thus $\sigma \in \mathcal{K}_{\leq f(v)} - \mathcal{K}_{<f(v)}$. $\square$

Since f is a flat pseudo-Morse function (Proposition 4.3), by definition pairings of a discrete gradient vector field consistent with f may only occur between cubes within the same level set. Further if f is generic, Proposition 4.8 shows that a level set $f^{-1}(v)$ is equals the lower star of v for all $v \in \mathcal{K}^0$. Therefore, by the injectivity of our filtration on the vertices of $\mathcal{K}$ the birth and death of persistent homology correspond to filtration values of particular vertices. Therefore, to identify if $v \in \mathcal{K}^0$ is a critical vertex it is sufficient to observe whether or not there exists an unpaired elementary cube $\sigma \in st(v)^f_{\leq}$ for a generic $f \in \mathbb{R}^d$.

Remark 4.9. *A lower star filtration of $\mathcal{N}$ in $\mathbb{R}^d$ with respect to a generic $f \in \mathbb{R}^d$ is not obtainable as a filtration defined by values on the full cubes which are extended to all lower dimensional faces by assigning to an i-cube σ the minimum value of the set of filtration values of the full cubes which contain σ.*

Lemma 4.10. *Let $f \in \mathbb{R}^d$ be a generic vector and $\mathcal{K}$ be a finite full elementary cubical complex. The Euler characteristic curve (Definition 5.3) satisfies:*

$$ECC(\mathcal{K}, f)(t) = \sum_{\substack{v \in \mathcal{K}^0 \\ v \cdot f \leq t}} \chi(st^f_{\leq}(v)).$$

Proof. Proposition 4.8 demonstrates that the lower stars partition a full elementary cubical complex. Since the lower stars of every vertex are disjoint and $\mathcal{K}$ is finite, we can compute the Euler characteristic curve for any sublevel set $t \in \mathbb{R}$ as follows:

$$ECC(\mathcal{K}, f)(t) = \chi(\mathcal{K}_{\leq t}) = \chi(\bigcup_{\substack{v \in \mathcal{K}^0 \\ v \cdot f \leq t}} st^f_{\leq}(v)) = \sum_{\substack{v \in \mathcal{K}^0 \\ v \cdot f \leq t}} \chi(st^f_{\leq}(v)).$$

□

Here we provide a reference for the set of elementary cubes $\mathcal{N}$ in $\mathbb{R}^d$ (Definition 3.2) which is used in Lemma 4.11 found below. For a generic vector $f \in \mathbb{R}^d$, if $v \in \mathcal{N}^0$, $st(v)^f_{\leq} \subset \mathcal{N}$ is obtained by extending the function f defined on $\mathcal{N}^0$ to the entire set of elementary cubes $\mathcal{N}$.

Lemma 4.11. *If $f, r \in \mathbb{R}^d$ are two generic vectors such f and r point in the direction of the same orthant then $st(v)^f_{\leq} = st(v)^r_{\leq} \subset \mathcal{N}$ in $\mathbb{R}^d$ for all $v \in \mathcal{N}^0$.*

Proof. Let $f = (f_1, ..., f_d)$ and $r = (r_1, ..., r_d)$ be generic vectors such that $sgn(f_i) = sgn(r_i)$ for all $i \in \{1, ..., d\}$. Without loss of generality, assume that all coordinates $f_i, r_i > 0$ are positive. Since f and r are generic vectors their respective filtrations induce unique filtration values on the set of vertices $\mathcal{N}^0$. Choose $v \in \mathcal{N}^0$. Let $\mathcal{V}$ be the vertices of the unique full cube $\mathcal{C}$ such that $f(\mathcal{C}) = f(v)$ (which implies $r(\mathcal{C}) = r(v)$ since f and r point towards the same orthant).

Since the relative ordering on elementary cubes under the lower star filtration is translation invariant we assume without loss of generality that the element of $\mathcal{V}$ which has the smallest f-filtration and r-filtration value is the origin $0 \in \mathbb{R}^d$ and v is $(1, ..., 1) \in \mathbb{R}^d$. Each vertex in $z \in \mathcal{V}$ is expressible as $z = \sum_{i=0}^{d} c_i e_i$ where $c_i \in \{0, 1\}$ and e_i is the i^{th} standard basis vector of $\mathbb{R}^d$. Note that since all coordinates of f and r are positive, $f(v) \geq f(z)$ and $r(v) \geq r(z)$ for all $z \in \mathcal{V}$. Therefore, if $\sigma \in \mathcal{N}$ is an elementary cube which contains v as a face and whose vertices are a subset of $\mathcal{V}$, then $\sigma \in st(v)^f_{\leq}$ and $\sigma \in st(v)^r_{\leq}$.

Now suppose $\sigma' \in \mathcal{N}$ is an elementary cube which contains v as a face and contains a vertex $v' \notin \mathcal{V}$. Since $v' = (v'_1, ..., v'_d) \notin \mathcal{V}$ there exists a $v'_i > 1$ for some $i \in \mathcal{I}$. Since σ' contains both v and v' it must contain the vertex $m = (m_1, ..., m_d)$ where $m_i = \max\{v'_i, 1\}$. Therefore $f(v) < f(m)$ and $r(v) < r(m)$ which implies $f(\sigma') > f(v)$ and $r(\sigma') > r(v)$. Thus $\sigma' \notin st(v)^f_{\leq}$ and $\sigma' \notin st(v)^r_{\leq}$ which combined with the previous containment implies $st(v)^f_{\leq} = st(v)^r_{\leq}$. □

Lemma 4.11 demonstrates that the set of generic vectors of $\mathbb{R}^d$ are partitioned by the orthants in which they point. Any two generic vectors in the same orthant have the same lower star for a given elementary 0-cube. This immediately yields an upper bound, 2^d on the number of filtrations required to identify critical vertices. For this reason we will only filter our full elementary cubical complex $\mathcal{K}$ in $\mathbb{R}^d$ with a single generic vector from each orthant. We will call such a collection of vectors $\mathcal{F}$.

Corollary 4.12. *Given a full elementary cubical complex $\mathcal{K}$ in $\mathbb{R}^d$ and two generic vectors $f, g \in \mathbb{R}^d$ such that $sgn(f_i) = sgn(g_i)$ for all $i \in \{1, ..., d\}$, then*

$$ECC(\mathcal{K}, f) = ECC(\mathcal{K}, g).$$

We will show in the next section (Lemma 4.19) that 2^d is a sharp upper bound on the number of filtration vectors required to reconstruct a finite full elementary cubical complex in $\mathbb{R}^d$ from the collection of Euler characteristic curves arising from a lower star filtrations $\{ECC(\mathcal{K}, f)\}_{f \in \mathcal{F}}$.

4.2 Reconstructing Full Elementary Cubical Complexes

Schapira's inversion formula (Theorem 3.52) is used to demonstrate injectivity of the Euler characteristic transform (a collection Euler characteristic curves $\{ECC(\mathcal{N}, f)\}_{f \in S^{d-1}}$), however; it fails to give a computationally efficient inverse. More precisely to identify whether or not a full elementary cube was contained in $\mathcal{K}$, Schapira's inversion formula instructs us to integrate the constructible function $(\mathbf{1}_{\mathcal{M}})$ over the space of elementary cubes. Although theoretically possible, this would be equivalent to simply looking up the binary value of each entry of a matrix representation of a digital image. We instead choose to use a different approach which allows us to focus on the critical vertices from the Morse theoretic perspective as a means for reconstruction. Our approach is to use Möbius inversion on the characteristic function $\mathbf{1}_{\mathcal{K}}$ restricted to the set of full cubes and prove that the convolution value is obtainable from the set of Euler characteristic curves to reconstruct $\mathcal{K}$ (Theorem 4.17).

4.2.1 Recovering Möbius Inversion from Euler Characteristic Curves

Using the framework of Möbius inversion (found in Section 3.2), we define an indicator function for the set of full cubes $\mathcal{K}^d$ which is compatible with convolution with the Möbius function. Since $\mathcal{K}$ is a finite full elementary cubical complex we will without loss of generality assume that $\mathcal{K}^d \subset \mathbb{N}^d$. We will denote be the minimum element in the product poset $(\mathbb{N}, \leq)^d$ as $\underline{\mathbf{0}}$ and define the map $B_{\mathcal{K}} : \mathrm{Int}(\mathbb{N}^d, \leq) \to \mathbb{R}$ as

$$B_{\mathcal{K}}(s,t) := \begin{cases} \mathbf{1}_{\mathcal{K}}(\mathcal{C}_t) & \text{if } s \leq t \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbf{1}_{\mathcal{K}}$ is the indicator function for the set $\mathcal{K}$, and $\mathcal{C}_t$ is the full cube (Definition 4.2) anchored at t.

Using this new machinery we assign to each element in a poset $\mathcal{P} \subset (\mathbb{N}, \leq)^d$ the value $B_{\mathcal{K}} * \mu(\underline{\mathbf{0}}, p)$. Recall that the $\mu * \zeta(s,t) = \delta(s,t)$ (Lemma 3.20). The subsequent proposition establishes a recursive method for finding the binary value of a cube by evaluating $B_{\mathcal{K}} * \mu * \zeta(\underline{\mathbf{0}}, p)$.

Proposition 4.13. *Let p be a vertex of a finite full elementary cubical complex $\mathcal{K}$ in $\mathbb{R}^d$. The binary value of the cubical complex anchored at p, $\mathcal{C}_p$ is*

$$\boldsymbol{1}_{\mathcal{K}}(\mathcal{C}_p) = B_{\mathcal{K}} * \mu * \zeta(\underline{\boldsymbol{0}}, p) = \sum_{r \leq p \in \mathcal{S}} B_{\mathcal{K}} * \mu(\underline{\boldsymbol{0}}, r).$$

Proof. Since $\mathcal{K}$ is a finite elementary cubical complex, the vertices of $\mathcal{K}$ are elements of an induced poset of $\mathbb{Z}^d$ (Definition 3.9) which is order isomorphic to $\mathbb{N}^d$ (Definition 3.10). The rightmost equality holds by the associativity of convolution (Lemma 3.15) and Remark 3.18. For all $\underline{\mathbf{0}} \leq p$ using the definition of convolution, μ, and δ (namely $\mu * \zeta = \delta$), we obtain the following,

$$\begin{aligned} B_{\mathcal{K}} * \mu * \zeta(\underline{\mathbf{0}}, p) &= B_{\mathcal{K}} * (\mu * \zeta))(\underline{\mathbf{0}}, p) \\ &= B_{\mathcal{K}} * \delta(\underline{\mathbf{0}}, p) = \sum_{\underline{\mathbf{0}} \leq x \leq p} B_{\mathcal{K}}(\underline{\mathbf{0}}, x)\delta(x, p) \\ &= B_{\mathcal{K}}(\underline{\mathbf{0}}, p) = \mathbf{1}_{\mathcal{K}}(\mathcal{C}_p) \end{aligned}$$

□

Proposition 4.13 allows us to reconstruct a full elementary cubical complex by summing the nonzero convolutional values $B_{\mathcal{K}} * \mu$ for closed intervals of the form $[\underline{\mathbf{0}}, p]$. also known as the support of $B_{\mathcal{K}} * \mu$. Further, the calculation of the convolutional value $B_{\mathcal{K}} * \mu$ of a full elementary cubical complex is computed using exclusively the set of full cubes $\mathcal{K}^d$. Further we proceed by demonstrating an explicit tie between the Möbius inversion of the characteristic function on the set of full cubes and topological information encoded in the Euler characteristic curve.

Definition 4.14. *Let $f = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$. We define* ***the sign of*** f *to be* $\text{sgn}(f) := \prod_{i=1}^d \text{sgn}(x_i)$.

Proposition 4.15. *Let $\mathcal{K}$ be a finite full elementary cubical complex in $\mathbb{R}^d$. Let $\mathcal{F}$ be a collection of 2^d generic vectors with one vector pointing in the direction of each orthant of $\mathbb{R}^d$. Given a vertex $v \in \mathcal{K}^0$ and μ the Möbius function on the poset $(\mathbb{N}, \leq)^d$, we have*

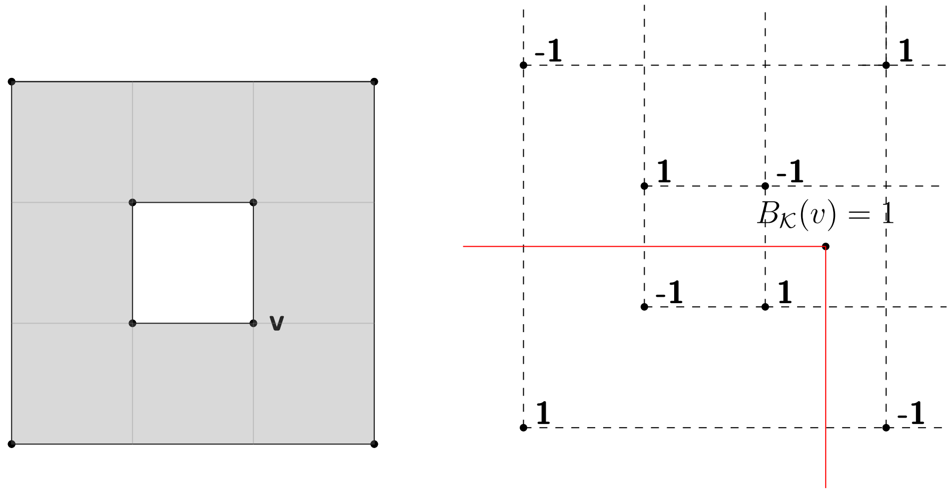

Figure 4-3. Reconstructing the cubical complex on the left using critical vertices and local information on the right in the direction $(+,+)$. On the left we have a full elementary cubical complex $\mathcal{K}$ in $\mathbb{R}^2$ (shaded) and its critical vertices, one of which is labeled v. On the right we have the support of the convolution values of $B_{\mathcal{K}} * \mu$ assigned to each vertex on the grid $\mathcal{N}$. The dashed lines emerging from a fixed vertex x indicates the region $R \subset \mathbb{R}^d$ such that if $y \in R$ then $x <_{\mathbb{N}^d} y$. We recover the value of $\mathbf{1}_{\mathcal{K}}(\mathcal{C}_v) = B_{\mathcal{K}}(\underline{\mathbf{0}}, v)$ by convolving with ζ which is equivalent to summing the value of $B_{\mathcal{K}} * \mu(\mathbf{0}, x)$ for all vertices which are in the downwards region enclosed by the lines emerging from the barycenter of the full cube $\mathcal{C}_v$ (Remark 3.18).

$B_{\mathcal{K}} * \mu(\underline{\boldsymbol{0}}, v) = \sum_{f \in \mathcal{F}} \mathrm{sgn}(f) \boldsymbol{1}_{\mathcal{K}}(\mathcal{C}_v^{-f})$, *the alternating sum of the binary values of top dimensional cubes contained in* $st(v)$.

Proof. Since $\mathcal{K}$ is a finite full elementary cubical complex we assume without loss of generality that $\mathcal{K}^0 \subset \mathbb{N}^d$. Suppose $s = (a_1, a_2, ..., a_d), t = (b_1, b_2, ..., b_d) \in \mathbb{N}^d$. Since μ is multiplicative (Proposition 3.21), $\mu(s,t) = \prod_{i=1}^d \mu_{\mathbb{N}}(a_i, b_i)$ where $\mu_{\mathbb{N}}(a_i, b_i)$ is the Möbius function evaluation inside the poset $(\mathbb{N}, \leq)$. Using Example 3.3 and the fact $\mathbb{N}^d$ is an induced subposet of $\mathbb{Z}^d$, $\mu(a_i, b_i) = 0$ for all $b_i \notin \{a_i, a_i + 1\}$. Therefore, if e_i denotes the *i*-th standard basis vector and $c_i \in \{0, 1\}$,

$$\mu(s,t) = \begin{cases} -1^{\Sigma(c)} & \text{if } s = t - \sum_{i=0}^d c_i e_i \\ 0 & \text{otherwise} \end{cases}$$

Observe that for a fixed $t \in \mathbb{N}^d$ and $f \in \mathbb{R}^d$ pointing in the direction of the first orthant, the intervals $s \le t$ with $\mu(s,t) \ne 0$ correspond precisely (via the map $(s,t) \mapsto s$) to the vertices of the unique full elementary cube $\mathcal{C} \in \mathcal{N}^d$ such that $f(\mathcal{C}) = f(t)$. This is the cube anchored at t in the direction of $-f$, $\mathcal{C}_t^{-f}$ (Definition 4.2). Call this set of vertices $\mathcal{V}(\mathcal{C}_t^{-f})$. For a fixed $s = t - \sum_{i=0}^d c_i e_i \in \mathcal{V}(\mathcal{C}_t^{-f})$ let $w(s)$ be the vector $w(s)_i = 1 - 2c_i$ (See Figure 4-4). Then the full cube anchored at $\mathcal{C}_s = \mathcal{C}_v^{w(s)} \in \mathcal{N}$. Further by construction of $w(s)$, $\text{sgn}(w(s)) = \mu(s,t)$.

Using the above computation for $\mu(s,t)$ and observing that the collection of $\{w(s)\}_{s \in \mathcal{V}(\mathcal{C}_t^{-f})}$ is a collection of vectors with one vector pointing in the direction of each orthant of $\mathbb{R}^d$, we compute the convolution value as follows:

$$
\begin{aligned}
B_{\mathcal{K}} * \mu(\underline{\mathbf{0}}, v) &= \sum_{\underline{\mathbf{0}} \le x \le v} B_{\mathcal{K}}(\underline{\mathbf{0}}, x)\mu(x, v) \\
&= \sum_{x \in \mathcal{V}(\mathcal{C}_v^{-f})} B_{\mathcal{K}}(\underline{\mathbf{0}}, x)\mu(x, v) \\
&= \sum_{x \in \mathcal{V}(\mathcal{C}_v^{-f})} \text{sgn}(w(x))\mathbf{1}_{\mathcal{K}}(\mathcal{C}_v^{w(x)}) \\
&= \sum_{f \in \mathcal{F}} \text{sgn}(f)\mathbf{1}_{\mathcal{K}}(\mathcal{C}_v^{-f}).
\end{aligned}
$$

□

The subsequent theorem will establish a tie between the Euler characteristics generated from sublevel set filtrations corresponding with the filtration vectors in $\mathcal{F}$ and the convolutional value $B_{\mathcal{K}} * \mu$.

Theorem 4.16. *Let $\mathcal{K}$ be a finite full elementary cubical complex in $\mathbb{R}^d$. Let $\mathcal{F}$ be a collection of 2^d generic vectors with one vector pointing in the direction of each orthant of $\mathbb{R}^d$. Given a vertex $v \in \mathcal{K}^0$ and μ the Möbius function on the poset $(\mathbb{N}, \le)^d$, then $B_{\mathcal{K}} * \mu(\underline{\mathbf{0}}, v) = \sum_{f \in \mathcal{F}} \text{sgn}(f)\chi(st_{\le}^f(v))$.*

Proof. Since $\mathcal{K}$ is a finite full elementary cubical complex we assume without loss of generality that $\mathcal{K}^0 \subset \mathbb{N}^d$. First we will show that each i cube σ is contained in 2^{d-i} lower stars. We complete the proof by showing if $i \ne d$, then the subset $\mathcal{S} \subset \mathcal{F}$ such that σ is in the lower star

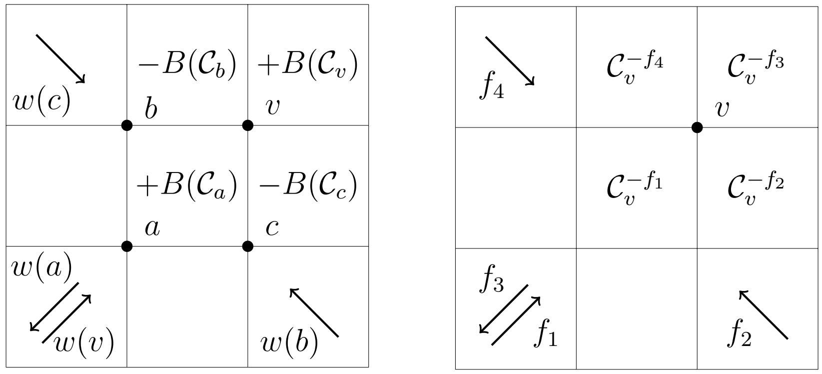

Figure 4-4. On the left hand side we label the vertices from the vertex set $\mathcal{V} = \{a, b, c, v\}$ described in the proof of Proposition 4.15 and their corresponding $B * \mu$ values on their full cubes respectively. For a fixed $x = v - \sum_{i=1}^{d} c_i e_i \in \mathcal{V}$ where e_i is a standard basis vector and $c_i \in \{0, 1\}$, the vector $w(x)_i = (1 - 2c_i)$. On the right hand side we depict each filtration vector f_i with an arrow and show the correspondence of each full cube $\mathcal{C} \in st(v)$ and the filtration which witnesses it anchored at v. Observe that the plus or minus on the left hand side corresponds with the sign of the f_i which witnesses $\mathcal{C}$ anchored at v.

of a vector in $\mathcal{S}$ consists of 2^{d-i-1} vectors with positive sign and 2^{d-i-1} vectors with negative sign.

Let $v = (v_1, ..., v_d) \in \mathcal{K}^0$ be a vertex in $\mathbb{R}^d$ and U be a non-full elementary *i*-cube such that $U \in st(v)$. We proceed by enumerating the number of full cubes containing U within $st(v)$ and demonstrate that the contribution of U to the right hand side is 0. Since $U = \prod_{k=1}^{d} [a_k, b_k]$ is an elementary i-cube there exists i non-degenerate intervals in the product. Let $\mathcal{J} \subset \{1, ..., d\}$ be the indices of the non-degenerate elementary intervals in the product of U. Further, since $U \in st(v)$, v must be a face of U so $v_k \in \{a_k, b_k\}$ for all k. Observe that any full cube $\mathcal{C}$ in the star of v is of the form $\mathcal{C} = \prod_{k=1}^{d} [c_k, d_k]$ where $[c_k, d_k] = [v_k, v_k + 1]$ or $[v_k - 1, v_k]$. Therefore, for U to be a face of $\mathcal{C}$, $[a_k, b_k] = [c_k, d_k]$ if $k \in \mathcal{J}$ is an index of a non-degenerate elementary interval in the product of U. This leaves precisely $2^{d-|\mathcal{J}|} = 2^{d-i}$ full cubes which are in the star of v and contain U as a face. We proceed to show that precisely half of these full cubes are in lower stars of v with respect to a filtrations vector $f \in \mathbb{R}^d$ with $\text{sgn}(f) = 1$.

Observe that the full cube $\prod_{k=1}^{d}[v_k, v_k \pm 1]$ is contained in the lower star of the v_k with respect to the vector $(\mp_1 1, ..., \mp_d 1) \in \mathbb{R}^d$ where $\mp_k$ is the opposite sign of the choice of the k-th elementary interval $[v_k, v_k \pm 1]$. Since we are only considering the set of full cubes in the star of v which contain U as a face and sgn is invariant under permutation of the coordinates of a vector, we only need to choose a matching between vectors of the form $(\pm 1, ... \pm 1, r_{d-i+1}, ..., r_d) \in \mathbb{R}^d$ where $r_k \neq 0$ for $k \in \{d-i+1, ..., d\}$ which differ in sign. We proceed by induction on the number of non-fixed $\pm$.

For the base case, there are two vectors of the form $(\pm 1, r_2, ..., r_d) \in \mathbb{R}^d$ where $r_k \neq 0$ for all $k \in \{1, ..., d\}$. We match the two vectors and note sgn$((1, r_2, ..., r_d)) =$ $-$sgn$((-1, r_2, ..., r_d))$. Now assume there exists a matching of the vectors of the form $(\pm 1, ..., \pm 1, r_n, ..., r_d) \in \mathbb{R}^d$ where $r_k \neq 0$ for all $k \in \{n, ..., d\}$ where precisely half the vectors have sgn$(r) = 1$. Fix $\pm_j 1$ for $j \in \{1, ..., n-1\}$ and match $f_1 = (\pm_1 1, ..., \pm_{n-1} 1, 1, r_{n+1}, ..., r_d)$ with $f_2 = (\pm_1 1, ..., \pm_{n-1} 1, -1, r_{n+1}, ..., r_d)$. Then sgn$(f_1) = -sgn(f_2)$. By the inductive hypothesis there exists matchings of vectors of the forms $(\pm 1, ..., \pm 1, 1, r_{n+1}, ..., r_d)$ and $(\pm 1, ..., \pm 1, -1, r_{n+1}, ..., r_d)$ respectively where the matching changes the sgn of the vectors. Therefore, let g_1 and g_2 be the unique vectors which matched with f_1 and f_2 under the assumed matchings respectively. By assumption sgn$(f_1) = -$sgn$(g_1) = -$sgn$(f_2) =$ $-(-$sgn$(g_2))$. Therefore, we can match g_1 and g_2 (since they differ in sgn) to extend union of the two matchings (changing only the assignments on the f_i and g_i) to n choices of $\pm$ which completes the proof by induction. Since sgn is invariant under permutation of the coordinates of a vector, we can permute the matching to respect vectors of the form $r = (\pm_1 1, ..., \pm_d 1)$ where $\pm_k 1$ is fixed so that U is in the lower star of v with respect to r. Therefore, for all i-cubes U where $0 \leq i < d$, the contribution of U to the signed sum of Euler characteristics is 0.

To complete the proof, observe that each full cube in $st(v)$ is contained in exactly one lower star. The sign of the contribution of each of the 2^d full cubes is determined by taking the sum of the coefficients of the element representing the reflection of the smallest valued cube under the filtration f_1 to the full cube in question. This implies that every full cube differs in sign from every other full cube which shares a face of codimension 1. Therefore using Remark 4.7 and Proposition 4.15, we can rewrite the sum on the right as

$$\sum_{f \in \mathcal{F}} \mathrm{sgn}(f)\chi(st^f_{\leq}(v)) = \sum_{f \in \mathcal{F}} \mathrm{sgn}(f)\mathbf{1}_{\mathcal{K}}(\mathcal{C}_v^{-f}) = B_{\mathcal{K}} * \mu(\underline{\mathbf{0}}, v)$$

. □

Theorem 4.17. *Let $\mathcal{K}$ be a finite full elementary cubical complex in $\mathbb{R}^d$ and $\mathcal{F}$ be a collection of 2^d generic vectors with one vector pointing in the direction of each orthant of $\mathbb{R}^d$. We may reconstruct $\mathcal{K}$ from the set of Euler characteristic curves of $\mathcal{K}$ generated by the lower star filtrations of the filtration vectors in $\mathcal{F}$.*

Proof. Since $\mathcal{K}$ is a finite full elementary cubical complex we assume without loss of generality that $\mathcal{K}^0 \subset \mathbb{N}^d$. By definition of the Euler characteristic curve, the Euler characteristic curve of full elementary cubical complexes is piecewise constant. Suppose $f \in \mathbb{R}^d$ is a generic vector. Since generic vectors induce a total ordering on the set of vertices of $\mathcal{N}^0$, let $V = \{..., v_{-i}, ..., v_{-1}, v_0, v_1, ...v_i, ...\}_{i \in \mathbb{N}}$ be an ordered set where for all $j \in \mathbb{Z}$, $f(v_j) \leq f(v_{j+1})$. Lemma 4.10 implies that for all $v_j \in V$ and fixed $f \in \mathcal{F}$,

$$ECC(\mathcal{K}, f)(f(v_j)) - ECC(\mathcal{K}, f)(f(v_{j-1})) = \chi(st^f_{\leq}(v_j)).$$

By computing the set $\{\chi(st^f_{\leq}(v)) \mid f \in \mathcal{F}, v \in \mathcal{K}^0\}$ the theorem follows immediately from the results of Theorem 4.16 and Proposition 4.13. □

Theorem 4.18. *Let $\mathcal{K}$ be a finite full elementary cubical complex in $\mathbb{R}^d$ and $\mathcal{F}$ be a collection of 2^d generic vectors with one vector pointing in the direction of each orthant of $\mathbb{R}^d$. We may reconstruct $\mathcal{K}$ from the set of Persistence Diagrams of $\mathcal{K}$ generated by lower star filtrations of filtration vectors in $\mathcal{F}$.*

Proof. Since $\mathcal{K}$ is a finite full elementary cubical complex we assume without loss of generality that $\mathcal{K}^0 \subset \mathbb{N}^d$. Recall that we can express the Euler characteristic of a sublevel set as the alternating sum of the ranks of the associated homology groups $\chi(\mathcal{K}) = \sum_{i \in \mathbb{K}} (-1)^i \beta_i(\mathcal{M})$. Therefore, the persistence diagram for a full elementary cubical complex filtered using sublevel set filtrations of a generic vector f, $\text{Dgm}(\mathcal{K}, f)$, determines the associated Euler characteristic curve, $ECC(\mathcal{K}, f)$. The result then follows from Theorem 4.17. □

Lemma 4.19. *Let $\mathcal{G}$ be any collection of generic vectors of $\mathbb{R}^d$. If $|\mathcal{G}| < 2^d$ then the collection of persistence diagrams generated from sublevel set filtrations $\{\text{Dgm}(\mathcal{K}, f)\}_{f \in \mathcal{G}}$, is not injective on the set of full elementary cubical complexes in $\mathbb{R}^d$.*

Proof. Let $v \in \mathcal{N}^0$ be a vertex. Let $\mathcal{K}_1 = \overline{st(v)} \subset \mathcal{N}$ in $\mathbb{R}^d$ be the downward closure of the star of v in $\mathcal{N}$ (Definition 3.6). Since $\mathcal{G}$ has fewer than 2^d filtration vectors there exists an orthant $\mathcal{O}$ in which no vector $f \in \mathcal{G}$ points. Suppose the vector $r \in \mathbb{R}^d$ points in the direction of $\mathcal{O}$. Let $\mathcal{K}_2 = \overline{st(v) - \mathcal{C}_v^{-r}}$ in $\mathbb{R}^d$. For all filtration vectors $f \in \mathcal{G}$, the sublevel sets of $\mathcal{K}_1$ and $\mathcal{K}_2$ are either empty or homotopic to a point. Therefore, the persistence diagrams $\text{Dgm}(\mathcal{K}_1, f)$ and $\text{Dgm}(\mathcal{K}_2, f)$ consist of precisely one essential birth. The essential birth is born at the filtration value of the anchor of the full cubes with smallest filtration values in the direction of f contained in $st(v)$. Let these full cubes be $\mathcal{E}_{\mathcal{K}_1}$ and $\mathcal{E}_{\mathcal{K}_2}$ respectively. By construction of $\mathcal{K}_i$, $\mathcal{E}_{\mathcal{K}_1} = \mathcal{E}_{\mathcal{K}_2}$ for all f and therefore have the same anchor $\mathcal{E}_v^f \in \mathcal{K}_1^0, \mathcal{K}_2^0$. Therefore, for all $f \in \mathcal{G}$, $\text{Dgm}(\mathcal{K}_1, f) = \text{Dgm}(\mathcal{K}_2, f)$. □

Theorem 4.20. *If $\mathcal{K}$ is a finite full elementary complex in $\mathbb{R}^d$ then 2^d is a sharp upper bound on the number of generic filtration vectors f required to reconstruct $\mathcal{K}$ from a collection of Euler Characteristic Curves $\{ECC(\mathcal{K}, f)\}_f$.*

Proof. The theorem follows immediately from the lower bound established in Lemma 4.19 and upper bound established in Theorem 4.17. □

The sharp bound of Theorem 4.20 is lower than the bound obtained by subdividing the cubical complex to construct a simplicial complex and using the computational geometry approach utilized by Curry et al. (2018). The reason for this improvement was the choice to allow for error in the reconstruction by choosing a cubical geometric realization via

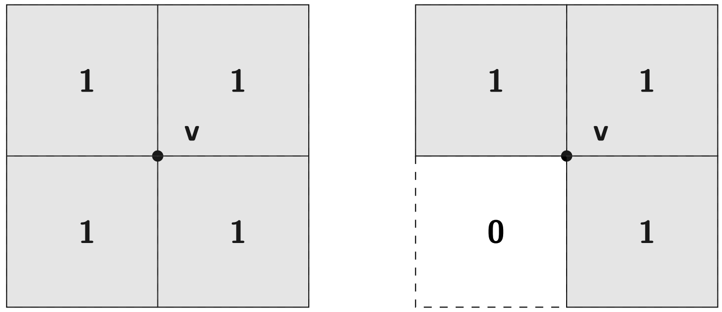

Figure 4-5. This figure demonstrates that for an insufficient number of filtration directions ($|\mathcal{F}| < 2^d$), the collection of persistence diagrams is not injective on the set of full elementary cubical complexes embedded in $\mathbb{R}^d$. The left and right full elementary cubical complexes depict $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively referenced in the proof of Lemma 4.19 for dimension $d = 2$ when $\mathcal{F}$ does not contain a vector which points in the direction of the first quadrant (+,+).

rasterization (which can be made ϵ close with respect to the Hausdorff metric). The set of vertices of a full elementary cubical complex $\mathcal{K}$ has fixed indegree (discussed in subsection 2.1.2) for a given dimension. This in turn implies that the orbit for almost any vector $f \in \mathbb{R}^d$ under the group action of pointwise multiplication by $(\mathbb{Z}/2)^d$ forms a collection $\mathcal{F}$ which is sufficient to reconstruct $\mathcal{K}$ for reconstruction (Corollary 4.12 and Theorem 4.17).

It is important to note that during this reconstruction we only required the knowledge of whether a vertex would be assigned a nonzero value under the convolution $B_{\mathcal{K}} * \mu$. For dimensions $d \geq 3$ the support of $B_{\mathcal{K}} * \mu$ is a proper subset of the set of critical vertices of a full elementary cubical complex (see figure 4-6). Therefore, even though the convolution value yields a powerful tie between topological information and geometric information (namely the embedding) of full elementary cubical complexes, it does not provide a method to speed up persistence calculations nor critical vertex identification. However, one positive of the inclusion of the support of $(B * \mu)$ into the set of critical vertices of full elementary cubical complexes is that it may offer a means to sparsify digital images without fully computing homology groups or the medial axis.

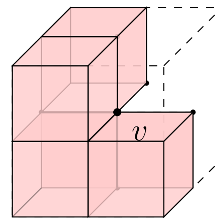

Figure 4-6. An example of a star $st(v)$ in $\mathbb{R}^3$ where the vertex v evaluates to 0 under $B_{\mathcal{K}} * \mu$. Observe that v is a critical vertex with respects to the two filtrations coming from the two back right octants (+,+,-) and (-,+,+), however they are both deaths of a connected component and the corresponding filtration vectors differ in sign.

4.2.2 Reconstruction via Euler Calculus

The right hand side of the Lemma 4.16 can be rewritten using Euler Calculus as a composition of Radon transforms utilizing the ring structure of constructible functions. The associated computations can be found in Lemma 4.21.

Lemma 4.21. $B_{\mathcal{K}} * \mu(\underline{\boldsymbol{0}}, v) = \sum_{f \in \mathcal{F}} \mathsf{sgn}(f)\chi(st^{f}_{\leq}(v)) = \mathcal{R}_{S'}([(-1)^{\mathsf{sgn}(*)}\mathcal{R}_S(\boldsymbol{1}_{\mathcal{K}})])$

Proof. The leftmost equality is established in the proof of Theorem 4.16. Therefore we focus on the rightmost equality $\sum_{f \in \mathcal{F}} \mathsf{sgn}(f)\chi(st^{f}_{\leq}(v)) = \mathcal{R}_{S'}([(-1)^{\mathsf{sgn}(*)}\mathcal{R}_S(\mathbf{1}_{\mathcal{K}})])$. Let $f \subset \mathbb{R}^d$ be a generic vector and $(\mathbb{Z}/2)^d f$ be the orbit of f acted on by the multiplicative group $(\mathbb{Z}/2)^d$ via pointwise multiplication. Let $S = \{(\sigma, v, t) \mid \max\{x \cdot v | x \in \sigma \subset \mathbb{R}^d\} = t\} \subset \mathcal{N} \times ((\mathbb{Z}/2^d)f \times \mathbb{R})$ and $S' = \{(v, t, \sigma) \mid \sigma = z \in \mathbb{Z}^d, z \cdot v = t\} \subset ((\mathbb{Z}/2^d)f \times \mathbb{R}) \times \mathcal{N}$.

We first compute the radon transform of the characteristic function $\mathcal{R}_S(\mathbf{1}_{\mathcal{K}})$:

$$\begin{aligned}
\mathcal{R}_S(\mathbf{1}_{\mathcal{K}})(v,t) &= \int_{(\sigma,v,t)\in\pi^{-1}_{(\mathbb{Z}/2)^d f \times \mathbb{R}}(v,t)} \mathbf{1}_{\mathcal{K}}(\pi_{\mathcal{N}}(\sigma,v,t))\mathbf{1}_S(\pi_{\mathcal{N}}(\sigma,v,t))d\chi \\
&= \int_{(\sigma,v,t)\in\pi^{-1}_{(\mathbb{Z}/2)^d f(\sigma) \times \mathbb{R}}(v,t)\cap S} \mathbf{1}_{\mathcal{K}}(\pi_{\mathcal{N}}(\sigma,v,t))d\chi \\
&= \int_{\tau \in st_{\leq}(z) | z \in (v,t)\cap\mathbb{Z}^d} \mathbf{1}_{\mathcal{K}}(\tau)d\chi \\
&= \chi(st_{\leq}(z) \cap \mathcal{K})
\end{aligned}$$

From here we compute the second radon transform as follows with the understanding that we define $\mathsf{sgn}(v,t) := \mathsf{sgn}(v)$:

$$
\begin{aligned}
\mathcal{R}_{S'}([(-1)^{\mathsf{sgn}(*)}\chi(st_{\leq}(z)\cap\mathcal{K})]) &= \int_{(\sigma,v,t)\in\pi_{\mathcal{N}}^{-1}(z)} \pi^{*}_{(\mathbb{Z}/2)^d f\times\mathbb{R}}[(-1)^{\mathsf{sgn}(\cdot)}\chi(st_{\leq}(\cdot\cap\mathcal{K}))]\mathbf{1}_{S}(\sigma,v,t)d\chi \\
&= \int_{(v,t)\in(\mathbb{Z}/2)^d f\times\mathbb{R}|z\cdot v=t} (-1)^{\mathsf{sgn}(v)}\chi(st_{\leq}(z\cap\mathcal{K}))d\chi \\
&= \sum_{f\in(\mathbb{Z}/2)^d f} \mathsf{sgn}(f)\chi(st^{f}_{\leq}(z))
\end{aligned}
$$

□

4.3 Geometric Condition for Recognizing critical vertices

This section presents a necessary condition (Proposition 4.31) and a sufficient condition (Proposition 4.32) for criticality of a vertex in an elementary cubical complex filtered by sublevel sets of generic vectors. We begin by presenting a motivating example proving that the Euler characteristic is sufficient to identify all critical vertices for dimensions $d \leq 3$ (Proposition 4.24). Using discrete Morse theory we establish a geometric condition for identifying critical cells of full elementary cubical complexes filtered by sublevel sets of height functions for all dimensions. In Proposition 4.28 we establish a total ordering on the set of elementary cubes which we use to construct perfect acyclic Morse pairings in the lower star of each vertex (Definitions 3.35 and 4.22). We conclude by presenting a conjecture which relates the necessary and sufficient conditions.

Definition 4.22. *Let $\mathcal{S}$ be a finite set of elementary cubes. We say a Morse pairing in $\mathcal{S}$ (Definition 3.34) is **perfect** if the number of critical cubes with respect to $M(\mathcal{S})$ (Definition 3.34) is the minimum number of critical cubes amongst the set of all Morse pairings in $\mathcal{S}$.*

Definition 4.23. *Let $\mathcal{K}$ be a finite elementary cubical complex and f be a flat pseudo-Morse function on $\mathcal{K}$ (Definition 3.37). We say a discrete gradient vector field $M(\mathcal{K})$ (Definition 3.36) consistent with f is **locally perfect** or **a locally perfect matching** if the restriction of the Morse pairing $M(\mathcal{K})$ is perfect (Definition 4.22) in each level set of f.*

Proposition 4.24. *Let $\mathcal{K}$ be a full elementary cubical complex in $\mathbb{R}^d$, $f \in \mathbb{R}^d$ be a generic vector, and $v \in \mathcal{K}^0$. Assume $d \leq 3$. Then $\chi(st^f_{\leq}(v)) = 0$ iff there exists a perfect acyclic Morse pairing (Definitions 3.35 and 4.22) in $st^f_{\leq}(v)$ with no critical cubes.*

Proof. Assume $d = 3$ and let $\mathcal{K} \subset \mathcal{N}^3$, $f \in \mathbb{R}^3$ be a generic vector, and $v \in \mathcal{K}^0$. Let k_i be the number of i-cubes in $st^f_{\leq}(v)$. Then $\chi(st^f_{\leq}(v)) = 1 - k_1 + k_2 - k_3$ where $k_1 \in \{0, 1, 2, 3\}$, $k_2 \in \{0, 1, 2, 3\}$ and $k_3 \in \{0, 1\}$. Suppose $\chi(st^f_{\leq}(v)) = 0$. The possible solutions for (k_1, k_2, k_3) which satisfy the equation and the downwards closure constraint of a cubical complex are $(1, 0, 0), (2, 1, 0), (3, 2, 0)$, and $(3, 3, 1)$. Each of these solutions correspond with matchings in Figure 4-7 containing 1,2,3, and 4 modified arrows in the modified Hasse diagram (Definition 3.39) respectively. If $\chi(st^f_{\leq}(v)) \neq 0$ then the number of cubes of even dimension does not equal the number of cubes of odd dimension implies every Morse pairing in $st^f_{\leq}(v)$ must contain at least one critical cube. The perfect Morse pairings for cases $d = 0, 1, 2$ are restrictions of the Morse pairing from Figure 4-7 corresponding to the face posets $\overline{v}$, $\overline{e_1}$, and $\overline{s_1}$ respectively. $\square$

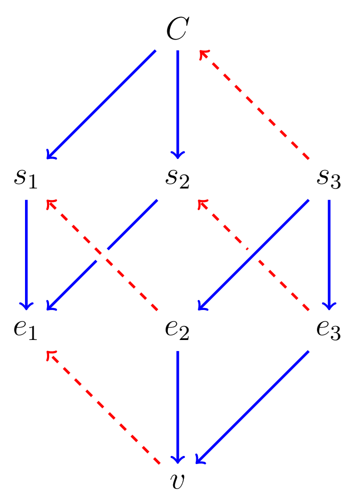

Figure 4-7. The product matching (as described in the proofs of Lemma 4.24 and Lemma 4.29) for the faces of a full cube of dimension $d = 3$ in $st_{\leq}(v)$ represented by a modified Hasse diagram. Red edges (dashed) are modified edges indicating a matching.

It is important to note that this result strongly depends on the dimensionality of the ambient space. For example the Euler characteristic of a critical point need not be nonzero for dimensions $d \geq 4$ since for all such $\mathbb{R}^d$ there exist full elementary cubical complexes where multiple cubes are adjoined in the star of a critical vertex where the parity of said cubes differ. The Euler-Poincare formula offers an alternative way to calculate the Euler characteristic namely

$$\chi(\mathcal{K}) = \sum_{i=0}^{d} (-1)^i \beta_i(\mathcal{K})$$

where β_i is the i^{th} Betti number; it is clear the homology groups can change between sublevel sets while Euler characteristic remains unchanged.

Definition 4.25. *Let $\sigma \in \mathcal{N}$ be an elementary cube in $\mathbb{R}^d$. Let $f \in \mathbb{R}^d$ be a generic vector and $v = (x_1, ..., x_d)$ be the unique vertex such that $\sigma \in st^f_{\leq}(v)$. Then if $\sigma = \prod_{j=1}^{d} I_j$ is an expression of σ as a product of elementary intervals (Definition 3.1) of the form $[x_j]$ and $[x_j - 1, x_j]$, then* ***we define $I_\sigma \subseteq \{1, ..., d\}$ to be the set of indices of non-degenerate elementary intervals in the product*** *$\sigma = \prod_{j=1}^{d} I_j$.*

Definition 4.26. *Let $v \in \mathcal{N}^0$ in $\mathbb{R}^d$. We define an order $\leq_{st}$ on the set of elementary cubes in the lower star of v such that for all $\sigma \in \mathcal{N}$, $\sigma \leq_{st} \sigma$ and for all $\sigma \neq \tau$:*

$$\sigma \leq_{st} \tau \Leftrightarrow \max(I_\sigma \setminus I_\tau) < \max(I_\tau \setminus I_\sigma)$$

where we say that max$(\emptyset) := -\infty$.

Remark 4.27. *For a fixed vertex $v \in \mathcal{N}^0$ and generic $f \in \mathbb{R}^d$, $\leq_{st}$ is a total order on $st^f_{\leq}(v)$.*

Proof. Observe that we could first map each subset $S \subset \{1, ..., d\}$ to the ordered set (S) where elements are written in decreasing order (see Figure 4-8). The relation $\leq_{st}$ is equivalent to ordering the ordered subsets $(S), (T)$ using lexicographic ordering and therefore is a total order. ∎

Proposition 4.28. *Let $f \in \mathbb{R}^d$ be a generic vector. There exists a total ordering on the set of elementary cubes $\mathcal{N}$ in $\mathbb{R}^d$ which is order preserving with respect to the partial ordering on $\mathcal{N}$ generated by the lower star filtration of $\mathcal{N}$ under f.*

Proof. Recall that the level sets of the lower star filtration are precisely the lower stars of vertices $v \in \mathcal{N}^0$ (Proposition 4.8). Since f is a generic vector, f is injective on the set $\mathcal{N}^0$. Further, for a fixed vertex and generic vector f, $\leq_{st}$ is a total order on the set of elementary cubes in the lower star (Remark 4.27). Therefore, we can extend $\leq_{st}$ to a total order on $\mathcal{N}$ using lexicographic order on the ordered pair $(f(\sigma), o(\sigma))$, where $o(\sigma) : st_{\leq}(v) \to \mathbb{N}$ is a $\leq_{st}$ order preserving map. $\square$

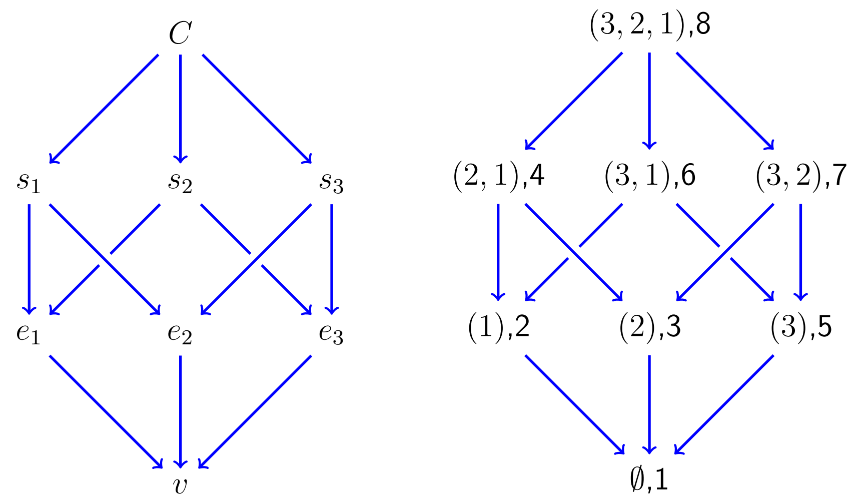

Figure 4-8. The figure on the left depicts the Hasse diagram of elementary cubes of the lower star. The figure on the right depicts the ordered set (decreasing) of the indices of the non-degenerate cubes $\sigma \mapsto (I_\sigma)) \subset \{1, ..., d\}$ referenced in Proposition 4.28 (right). The second number in labeling of the vertices on the right hand side corresponds to the ordering of the corresponding face of the unique full cube in the lower star of v under the order $\leq_{st}$. $\leq_{st}$ is equivalent to writing the elements of I_σ in decreasing order and then ordering the ordered sets using lexicographic order.

Lemma 4.29. *Let $v \in \mathcal{N}^0$ and $f \in \mathbb{R}^d$ be a generic vector. If C is an elementary cube in $\mathbb{R}^d$ then there exists a perfect acyclic Morse pairing (Definitions 3.35 and 4.22) in $\overline{C} \cap st^f_{\leq}(v)$.*

Proof. Let $v = [v_1, ..., v_d]$ and $\mathcal{S} = \overline{C} \cap st^f_{\leq}(v)$. Let $\leq_{st}$ be the total order of $st^f_{\leq}(v)$ (Definition 4.27). Let $\mathcal{E} = \{E_i\}_{i=1}^d$ denote the ordered set of 1-cubes in $st^f_{\leq}(v)$, such that $E_i \leq_{st} E_{i+1}$. Observe that $\mathcal{S}$ is a lattice since every cube $s \in \mathcal{S}$ can be uniquely identified with a subset of $\mathcal{E} \cap \mathcal{S}$ and the poset of subsets of a set under containment forms a lattice. If $\mathcal{S} = \emptyset$ then the statement is trivially true. If $\mathcal{S} = v$, the emptyset is a perfect acyclic Morse pairing (Definition 3.34) in $\mathcal{S}$. Now assume $\dim(\mathcal{S}) \geq 1$. We begin by constructing a Morse pairing in $\mathcal{S}$, $M(\mathcal{S})$ using the lattice structure of the face poset of $\overline{S}$. We then proceed to prove that $M(\mathcal{S})$ is acyclic (Definition 3.35) with no critical cubes.

Let E_k denote the smallest indexed $E_i \in \mathcal{E} \cap \mathcal{S}$ and e_k be the unique non-degenerate elementary interval in the product decomposition of E_k. We partition the set $\mathcal{S}$ into the disjoint union $\mathcal{S} = \mathcal{S}_1 \coprod \mathcal{S}_2$ where $\mathcal{S}_1 := \{\sigma \mid E_k \nsubseteq \sigma\}$ and $\mathcal{S}_2 := \{\tau \mid E_k \subseteq \tau\}$. Denote the elementary interval decomposition of $\sigma \in \mathcal{S}$ as $\sigma = I_1 \times \cdots \times I_d$. Observe that if $\sigma \in \mathcal{S}_1$ then $\sigma = I_1 \times \cdots I_{k-1} \times [v_k] \times I_{k+1} \times \cdots \times I_d$. Let $\tau := I_1 \times \cdots \times I_{k-1} \times e_k \times I_{k+1} \times I_d$. Then $\tau \in \mathcal{S}_2$ and τ is the join of E_k and σ in the face poset of $\mathcal{S}$ which is a lattice.

We define the map $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ by $\sigma \mapsto I_1 \times \cdots \times I_{k-1} \times e_k \times I_{k+1} \times I_d$. Let $\varphi^{-1} : \mathcal{S}_2 \to \mathcal{S}_2$ be the map defined by $\tau \mapsto I_1 \times \cdots \times I_{k-1} \times [v_k] \times I_{k+1} \times I_d$. Then given a $\sigma \in \mathcal{S}_1$, $\varphi^{-1} \circ \varphi(\sigma) = \varphi^{-1}(I_1 \times \cdots \times I_{k-1} \times e_k \times I_{k+1} \times I_d) = I_1 \times \cdots \times I_{k-1} \times [v_k] \times I_{k+1} \times I_d = \sigma$. Similarly $\varphi \circ \varphi^{-1}(\tau) = \tau$. Therefore, φ is a bijection between the sets $\mathcal{S}_1$ and $\mathcal{S}_2$. Further, by definition of $\mathcal{S}_1$ and φ, $\dim(\varphi(\sigma)) = \dim(\sigma) + 1$ and $\sigma \subset \varphi(\sigma)$ so σ is a facet of $\varphi(\sigma)$. Observe that since $\mathcal{S} = \mathcal{S}_1 \coprod \mathcal{S}_2$, $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ is a bijection, and $\varphi^{-1}(\tau)$ is a facet of τ, $M(\mathcal{S})$ is a perfect Morse pairing.

To show that the pairing is acyclic assume to the contrary that $\tau_0, \sigma_1, ..., \tau_l, \sigma_{l+1}$ is a minimal cyclic V-path in $M(\mathcal{S})$. Then by definition of cyclic V-paths, $(\sigma_{l+1}, \tau_0) \in M(\mathcal{S})$ which implies $\tau_0 = \prod_{j=1}^{k-1} I_j(\sigma_{l+1}) \times e_k \times \prod_{j=k+1}^{d} I_j(\sigma_{l+1})$. Since σ_1 is a facet of τ_0, $\sigma_1 \subset \prod_{j=1}^{k-1} I_j(\sigma_{l+1}) \times e_k \times \prod_{j=k+1}^{d} I_j(\sigma_{l+1})$. Since $(\sigma_1, \tau_1) \in M(\mathcal{S})$, $E_k \nsubseteq \sigma_1$ which implies by dimensionality $\sigma_1 = \prod_{j=1}^{k-1} I_j(\sigma_{l+1}) \times [v_k] \times \prod_{j=k+1}^{d} I_j(\sigma_{l+1}) = \sigma_{l+1}$. If $l = 1$ then $\sigma_1 = \sigma_2$ contradicts the definition of V-path ($\sigma_i \neq \sigma_{i+1}$). If $l \geq 2$ then since $\sigma_{l+1} = \sigma_1$ and

$(\sigma_1, \tau_1) \in M(\mathcal{S})$, the path $\tau_1, \sigma_2, ..., \tau_l, \sigma_{l+1}$ is a shorter cyclic V-path which contradicts the minimality assumption of $\tau_0, \sigma_1, ..., \tau_l, \sigma_{l+1}$. Therefore, no cyclic V-paths in $M(\mathcal{S})$ exist and therefore $M(\mathcal{S})$ is a perfect acyclic Morse pairing in $\mathcal{S}$. $\square$

Definition 4.30. *Let $\mathcal{K}$ in $\mathbb{R}^d$ be an elementary cubical complex, $f \in \mathbb{R}^d$ be a generic vector, and $v \in \mathcal{K}^0$. We say an elementary cube $\sigma \in \mathcal{K}$ is a* ***maximal cube of*** $\mathcal{K}$ ***with respect to*** f ***and*** v *if σ is a maximal element of the face poset of $st^f_{\leq}(v)$.*

Proposition 4.31 (Extension)**.** *Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$, $f \in \mathbb{R}^d$ be a generic vector, and $v \in \mathcal{K}^0$. Let $\mathcal{M}$ be the set of maximal cubes of $\mathcal{K}$ with respect to f and v. If $\bigcap_{m \in \mathcal{M}} \overline{m} \cap st^f_{\leq}(v) \supsetneq \{v\}$ then there exists a perfect acyclic Morse pairing in $st^f_{\leq}(v)$ which contains no critical cubes. Thus v is not a critical point of $\mathcal{K}$ with respect to f.*

Proof. Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$, $f \in \mathbb{R}^d$ be a generic vector, and $v \in \mathcal{K}^0$. Let $\mathcal{M}$ be the set of maximal cubes of $\mathcal{K}$ with respect to f and v. Observe by Corollary 3.41, the the existence of a perfect acyclic Morse pairing in $st^f_{\leq}(v)$ which contains no critical cubes implies that $\mathcal{K}_{\leq f(v)}$ is homotopy equivalent to $\mathcal{K}_{<f(v)}$. Therefore, the existence of a perfect acyclic Morse pairing in $st^f_{\leq}(v)$ which contains no critical cubes implies v is not a critical cube of $\mathcal{K}$ with respect to f. We produce a perfect acyclic Morse pairing in $st^f_{\leq}(v)$ which contains no critical cubes by induction on the cardinality of $\mathcal{M}$.

For the base case, assume $\mathcal{M} = \{m_1, m_2\}$. Since m_1, m_2 are both maximal cubes of $\mathcal{K}$ with respect to f and v, $\dim(m_i) \geq 1$ for $i \in \{1, 2\}$. Suppose $\overline{m_1} \cap \overline{m_2} \cap st^f_{\leq}(v) \supsetneq \{v\}$. Since $\dim(m_1) \geq 1$, by Lemma 4.29 there exists a perfect acyclic Morse pairing in $\overline{m_1} \cap st^f_{\leq}(v)$, $M(\overline{m_1} \cap st^f_{\leq}(v))$, which contains no critical cubes. Since m_1, m_2 are elementary cubes there exists a $D \in \mathcal{K}$ such that $\dim(D) \geq 1$ and $\overline{m_1} \cap \overline{m_2} \cap st^f_{\leq}(v) = \overline{D} \cap st^f_{\leq}(v)$. We extend $M_1 = M(\overline{m_1} \cap st^f_{\leq}(v))$ to a perfect acyclic Morse pairing in the union $\overline{m_1} \cap \overline{m_2} \cap st^f_{\leq}(v)$ as follows.

Let $M_2 = M(\overline{m_2} \cap st^f_{\leq}(v))$ be the perfect acyclic Morse pairing induced by the join operation as in the proof of Lemma 4.29. Let $\mathcal{E} = \{E_i\}_{i=1}^d$ denote the ordered set of 1-cubes in $st^f_{\leq}(v)$, such that $E_i \leq_{st} E_{i+1}$. Let e_i be the unique non-degenerate elementary interval in the product decomposition of E_i. Let E_k denote the smallest indexed $E_i \in \mathcal{E} \cap \overline{D}$. For an elementary cube $\sigma \in \mathcal{N}$ let $I_j(\sigma)$ denote the j-th elementary interval in the product decomposition of σ. Since D is an elementary cube, the face poset of $st^f_{\leq}(v) \cap \overline{D}$ forms a lattice. Further since the dim$(D) \geq 1$, if $\sigma \in \overline{m_2} - \overline{D}$ does not contain E_k as a face then $\prod_{j=1}^{k-1} I_j(\sigma) \times e_k \times \prod_{j=k+1}^{d} I_j(\sigma) \in \overline{m_2} - \overline{D}$. Note the pairing $\{(\sigma, \prod_{j=1}^{k-1} I_j(\sigma) \times e_k \times \prod_{j=k+1}^{d} I_j(\sigma))\}_{\sigma \in (\overline{m_2} - \overline{D}) \cap st^f_{\leq}}$ is precisely the Morse pairing M_2 restricted to $(\overline{m_2} - \overline{D}) \cap st^f_{\leq}(v)$ up to relabeling the minimum 1-cube in $\mathcal{E} \cap \overline{m_2}$ with the minimum 1-cube in $\mathcal{E} \cap \overline{m_1}$ for all faces in $\overline{m_2} \cap st^f_{\leq}(v)$. Let M_2' be the relabeled Morse pairing of M_2 provided above. Then Morse pairing

$$M((\overline{m_1} \cup \overline{m_2}) \cap st^f_{\leq}(v)) = M_1 \cup M_2' \restriction_{(\overline{m_2} - \overline{D}) \cap st^f_{\leq}(v)}$$

is a perfect acyclic Morse pairing in $(\overline{m_1} \cup \overline{m_2}) \cap st^f_{\leq}(v)$ consistent with f which has no critical cubes.

For the inductive step assume for $|\mathcal{M}| = n \in \mathbb{N}$, if $\bigcap_{m \in \mathcal{M}} \overline{m} \cap st^f_{\leq}(v) \supsetneq \{v\}$ then there exists a perfect acyclic Morse pairing in $st^f_{\leq}(v)$ which contains no critical cubes. Let $|\mathcal{M}| = n$ and C be an elementary cube such that $C \notin \mathcal{M}$ and $C \in st^f_{\leq}(v)$. For notational purposes let $\mathcal{S} = (\bigcup_{m \in \mathcal{M}} \overline{m} \cup \overline{C}) \cap st^f_{\leq}(v)$. Suppose $\bigcap_{s \in \mathcal{S}} \overline{s} \cap st^f_{\leq}(v) \supsetneq \{v\}$. This implies that $\bigcap_{m \in \mathcal{M}} \overline{m} \cap st^f_{\leq}(v) \supsetneq \{v\}$ which by the inductive hypothesis implies there exists a perfect acyclic Morse pairing in $\bigcup_{m \in \mathcal{M}} \overline{m} \cap st^f_{\leq}(v)$, given by restrictions and relabelings of Lemma 4.29.

Observe for each $A \subset \mathcal{M}$, $\bigcap_{a \in A} \overline{a} \cap \overline{C} \cap st^f_{\leq}(v) = \overline{D_A} \cap st^f_{\leq}(v)$ for some $D_A \in \mathcal{K}^i$ with $i \geq 1$. Therefore, $(\overline{C} - \bigcup_{m \in \mathcal{M}} \overline{m}) \cap st^f_{\leq}(v) = (\overline{C} - \bigcup_{A \subset \mathcal{M}} \overline{D_A}) \cap st^f_{\leq}(v)$. Let $\mathcal{E} = \{E_i\}_{i=1}^d$ denote the ordered set of 1-cubes in $st^f_{\leq}(v)$, such that $E_i \leq_{st} E_{i+1}$. Let E_k denote the smallest indexed $E_i \in \mathcal{E} \cap \bigcap_{s \in \mathcal{S}} (\overline{s})$. We can partition $(\overline{C} - \bigcup_{A \subset \mathcal{M}} \overline{D_A}) \cap st^f_{\leq}(v) = S_1 \coprod S_2$ such that $S_1 := \{\sigma \mid E_k \nsubseteq \sigma\}$ and $S_2 := \{\tau \mid E_k \subseteq \tau\}$. Since each $\overline{D_A} \cap st^f_{\leq}(v)$ is a sublattice of $\overline{C} \cap st^f_{\leq}(v)$, $(\overline{C} - \bigcup_{A \subset \mathcal{M}} \overline{D_A}) \cap st^f_{\leq}(v)$ is closed under joins with E_k. Therefore for each element $\sigma \in S_1$, there exists a unique $\sigma \vee E_k \in S_2$ showing $|\mathcal{S}_1|=|\mathcal{S}_2|$. Since $C \notin \mathcal{M}$, $C \in \mathcal{S}_2$ and the unique face $\sigma \subset C$ with codimension 1 which does not contain E_k is in $\mathcal{S}_1$. Therefore we extend the existing perfect acyclic Morse pairing in $\bigcap_{m\in\mathcal{M}} m \cap st^f_{\leq}(v)$ to a Morse pairing in $\mathcal{S}$ with no critical cubes by using the join of elements of S_1 and E_k,

$$M(\mathcal{S}) = M(\bigcap_{m\in\mathcal{M}} \overline{m} \cap st^f_{\leq}(v)) \cup \{(\sigma, \sigma \vee E_k)\}_{\sigma\in S_1}.$$

Further, since $M(\bigcup_{m\in\mathcal{M}} \overline{m} \cap st^f_{\leq}(v))$ is given by compatible restrictions and relabelings of Lemma 4.29, and the extension uses the join of the minimal 1-cube E_k in the intersection $\bigcap_{m\in\mathcal{M}} \overline{m} \cap st^f_{\leq}(v)$, $M(\mathcal{S})$ is acyclic by an analogous argument to the one provided in Lemma 4.29. Therefore, there exists a perfect acyclic Morse pairing in $(\bigcup_{m\in\mathcal{M}} \overline{m} \cup \overline{C}) \cap st^f_{\leq}(v)$ which contains no critical cubes which implies v is not a critical point of $\mathcal{K}$ with respect to f. $\square$

Proposition 4.32 (Corner)**.** *Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$, $f \in \mathbb{R}^d$ be a generic vector, and $v \in \mathcal{N}^0$. Let $\mathcal{M}$ be the set of maximal cubes of $\mathcal{K}$ with respect to f and v. If $\bigcap_{m\in\mathcal{M}} \overline{m} \cap st^f_{\leq}(v) = \{v\}$ and for all $\mathcal{A} \subsetneq \mathcal{M}$, $\cap_{a\in\mathcal{A}} \overline{a} \cap st^f_{\leq}(v) \neq \{v\}$ then v is a critical point of $\mathcal{K}$ with respect to f (Definition 4.4).*

Proof. Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$, $f \in \mathbb{R}^d$ be a generic vector, and $v \in \mathcal{K}^0$. Let $\mathcal{M}$ be the set of maximal cubes of $\mathcal{K}$ with respect to f and v. We prove the result by induction on the cardinality of $\mathcal{M}$.

For the base case, assume $\mathcal{M} = \{m_1, m_2\}$. Since m_1, m_2 are both maximal cubes of $\mathcal{K}$ with respect to f and v, $\dim(m_i) \geq 1$ for $i \in \{1, 2\}$. Suppose $\overline{m_1} \cap \overline{m_2} \cap st^f_{\leq}(v) = \{v\}$. We can compute the parity of the number of cubes in the lower star as follows:

$$\begin{aligned}
|st^f_{\leq}(v)| &= |(\overline{m_1} \cap st^f_{\leq}(v)) \cup (\overline{m_1} \cap st^f_{\leq}(v))| = |\overline{m_1} \cup \overline{m_2} \cap st^f_{\leq}(v)| \\
&= |\overline{m_1} \cap st^f_{\leq}(v)| + |\overline{m_2} \cap st^f_{\leq}(v)| - |\overline{m_1} \cap \overline{m_2} \cap st^f_{\leq}(v)| \\
&= 2^{\dim(m_1)} + 2^{\dim(m_2)} - 1 \\
&\equiv_2 1.
\end{aligned}$$

Since the parity of the number of cubes in the lower star is odd, every perfect acyclic Morse pairing in $\overline{m_1} \cup \overline{m_2} \cap st^f_{\leq}(v)$ consistent with f contains a critical cube. Thus, $\overline{m_1} \cap \overline{m_2} \cap st^f_{\leq}(v) = \{v\}$ implies v is a critical point of $\mathcal{K}$ with respect to f.

Assume the statement is true for $|\mathcal{M}| = n$. Now assume $|\mathcal{M}| = n+1$ and for all $\mathcal{A} \subsetneq \mathcal{M}$, $\bigcap_{a \in \mathcal{A}} \overline{a} \cap st^f_{\leq}(v) \neq \{v\}$. Choose $C \in \mathcal{M}$ and let $\mathcal{M}' = \mathcal{M} - \{C\}$. By assumption $\bigcap_{m \in M'} \overline{m} \cap st^f_{\leq}(v) \supsetneq \{v\}$ so by Proposition 4.31 there exists a perfect Morse pairing on $\bigcup_{m \in M'} \overline{m} \cap st^f_{\leq}(v)$ with no critical cubes. Thus $|\bigcup_{m \in M'} \overline{m} \cap st^f_{\leq}(v)| \equiv_2 0$. Suppose $\bigcap_{m \in \mathcal{M}} (\overline{m}) \cap st^f_{\leq}(v) = \{v\}$. We can compute the parity of the number of cubes in the lower star using inclusion-exclusion as follows:

$$\begin{aligned}
|st^f_{\leq}(v)| &= |(\bigcup_{m \in M'} \overline{m} \cap st^f_{\leq}(v)) \cup (\overline{C} \cap st^f_{\leq}(v))| \\
&= |\bigcup_{m \in M'} \overline{m} \cap st^f_{\leq}(v)| + |\overline{C} \cap st^f_{\leq}(v)| - |\bigcup_{m \in \mathcal{M}'} \overline{m} \cap \overline{C} \cap st^f_{\leq}(v)| \\
&\equiv_2 |\bigcup_{m \in \mathcal{M}'} (\overline{m} \cap \overline{C}) \cap st^f_{\leq}(v)| \\
&= \sum_{B \subseteq \mathcal{M}'} (-1)^{|B|-1} 2^{\dim(\cap_{b \in B} (\overline{b} \cap \overline{C} \cap st^f_{\leq}(v)))} \\
&\equiv_2 1.
\end{aligned}$$

Since the parity of the number of cubes in the lower star is odd, every perfect acyclic Morse pairing in $(\bigcup_{m \in \mathcal{M}} \overline{m} \cup \overline{C}) \cap st^f_{\leq}(v)$ consistent with f contains a critical cube. Thus, for all $\mathcal{M}$ with $|\mathcal{M}| \leq n+1$ and for all $\mathcal{A} \subsetneq \mathcal{M}$, $\cap_{a \in \mathcal{A}} \overline{a} \cap st^f_{\leq}(v) \neq \{v\}$, $\bigcap_{m \in \mathcal{M}} \overline{m} \cap st^f_{\leq}(v) = \{v\}$ implies v is a critical point of $\mathcal{K}$ with respect to f. $\square$

Conjecture 4.33. *Let $\mathcal{K}$ be an elementary cubical complex in $\mathbb{R}^d$, $f \in \mathbb{R}^d$ be a generic vector, and $v \in \mathcal{N}^0$. Let $\mathcal{M}$ be the set of maximal cubes of $\mathcal{K}$ with respect to f and v. Then $\bigcap_{m \in \mathcal{M}} \overline{m} \cap st^f_{\leq}(v) = \{v\}$ iff v is a critical point of $\mathcal{K}$ with respect to f (Definition 4.4).*

Let $\mathcal{K}$ be a full elementary cubical complex and $f \in \mathbb{R}^d$ be generic. For each $v \in \mathcal{K}^0$ choose a perfect acyclic Morse pairing $M(st^f_{\leq}(v))$ which exists by definition. Let $M(\mathcal{K}) = \{M(st^f_{\leq}(v))\}_{v \in \mathcal{K}^0}$ generated by the union of perfect acyclic Morse pairings (Definitions

3.35 and 4.22) on the lower stars, is a locally perfect discrete gradient vector field on $\mathcal{K}$. Furthermore, each $M(st^f_{\leq}(v))$ pairs elements within a lower star, which implies that $M(\mathcal{K})$ is consistent with f (thus f is a flat pseudo-Morse function on $\mathcal{K}$). Proposition 4.31 yields a combinatorial necessary condition to determine whether a vertex v is a critical vertex of $\mathcal{K}$ with respect to f (Definition 4.4) with the reverse direction in Conjecture 4.33. We map all elements of the face poset of $st^f_{\leq}(v)$ to their corresponding subsets of $\{1, ..., d\}$ (the explicit correspondence can be found in Proposition 4.28 and is depicted in Figure 4-8). Compute the intersection of the subsets corresponding to the maximal cubes of $\mathcal{K}$ with respect to f and v. If the intersection is not empty v is not critical. Proposition 4.32 handles a subcase of the forwards direction of Conjecture 4.33. We believe with further examination of the set differences of the face posets of elementary cubes intersect a lower star (which posess a Borel algebra structure) that one could show the general case for the forwards direction of the conjecture by examining the Euler characteristic of consecutive set differences of maximal cubes introduced using the total ordering from Proposition 4.28.

CHAPTER 5
IMPLEMENTATION

This chapter includes code to implement the entire pipeline of a topological storage and reconstruction based on the theory found in Chapter 4. We also include Corollaries 5.4 and 5.5 which extend the results of Chapter 4 to grayscale images.

5.1 Grayscale Images and Weighted Euler Characteristic

From an applied standpoint we would like to extend the results from the previous chapter to grayscale images in an efficient manner. Möbius Inversion works in the more general setting of integer valued functions with compact support. Therefore, in the actual implementation of the code rather than viewing a *d*-dimensional gray scale image which can be stored as a $d+1$ dimensional binary value by viewing the gray scale as a separate channel where the height corresponds with the original intensity, we use a weighted Euler characteristic in order to quickly compute integer values other than 0 or 1 without introducing an extra 2^d number of filtration directions. We will make this notion of "weighted Euler Characteristic" precise in the following subsection.

The weighted Euler Characteristic allows us to view a grayscale image as a disjoint union of finitely many *d*-dimensional images rather than a $(d+1)$-dimensional image. A computational issue with considering the height of a $(d+1)$-dimensional image to be the grayscale value is that there exists a deformation retract via projection onto the *d*-dimensional underlying shape. In other words, there is no interesting homological information in a $(d+1)$-dimensional representation of a grayscale image which cannot be witnessed in some number of axis-aligned *d*-dimensional slices. The idea behind the weighted Euler characteristic is to weight homological features by the number of *d*-dimensional slices in a $(d+1)$-dimensional representation which witness them.

Definition 5.1. *Given a finite full elementary cubical complex $\mathcal{K}$ and a positive integer valued function $\mathcal{G} : \mathcal{K} \to \mathbb{N}$ with the property that $\mathcal{G}(\sigma) = max\{\mathcal{G}(\tau) \in \mathcal{K} | \sigma, \tau \in \mathcal{K}, \sigma \subset \tau\}$, we define a **grayscale digital image** to be the pair $(\mathcal{K}, \mathcal{G})$.*

Definition 5.2. *Given a grayscale digital image* $(\mathcal{K}, \mathcal{G})$*, the* ***weighted Euler characteristic*** *is defined as the sum*

$$\chi_{\mathcal{G}}(\mathcal{K}) = \sum_{i=0}^{d} \sum_{\sigma \in \mathcal{K}^i} (-1)^i \mathcal{G}(\sigma).$$

Definition 5.3. *Let* $(\mathcal{K}, \mathcal{G})$ *be a grayscale digital image in* $\mathbb{R}^d$ *and* $f \in \mathbb{R}^d$*. We define the* ***Euler Characteristic Curve of*** $(\mathcal{K}, \mathcal{G})$ ***with respect to*** $f \in \mathbb{R}^d$, $\mathsf{ECC}(\mathcal{K}, f)$ *to be the function*

$$\begin{aligned} &ECC(\mathcal{K}, f) : \mathbb{R} \to \mathbb{Z} \\ &\qquad t \mapsto \chi_{\mathcal{G}}(\{\sigma \in \mathcal{K} \mid \max\{f \cdot x \mid x \in \sigma \subset \mathbb{R}^d\} \leq t\}). \end{aligned}$$

With these new definitions we state a corollary of the reconstruction result (Theorem 4.16) which extends the theoretical framework (Theorem 4.17) to grayscale images.

Corollary 5.4 (Theorem 4.16)**.** *Let* $(\mathcal{K}, \mathcal{G})$ *be a grayscale digital image and* $\mathcal{T}$ *be the poset whose elements are vertices of* $\mathcal{K}^0$ *under the product order* $(\mathbb{Z}, \leq)^d$ *and* $\mathcal{F}$ *be a collection of generic vectors with one vector pointing in the direction of each orthant of* $\mathbb{R}^d$*. Then*

$$\mathcal{G}(\mathcal{C}_v) B_{\mathcal{K}} * \mu(\underline{\boldsymbol{0}}, v) = \sum_{f \in \mathcal{F}} \mathsf{sgn}(f) \chi_{\mathcal{G}}(st_{\leq}^{f}(v)).$$

Proof. Since the function $\mathcal{G}$ is independent of the filtration direction, evaluating the weighted Euler characteristic is equivalent to replacing the evaluation of a full cube by the characteristic function $\mathbf{1}_{\mathcal{K}}(\mathcal{C}_v^f)$ with the weighted characteristic function $\mathcal{G}(\mathcal{C}_v^f)\mathbf{1}_{\mathcal{K}}(\mathcal{C}_v^f)$. □

Corollary 5.5 (Theorem 4.17)**.** *Let* $(\mathcal{K}, \mathcal{G})$ *be a grayscale digital image in* $\mathbb{R}^d$ *and* $\mathcal{F}$ *be a collection of* 2^d *generic vectors with one vector pointing in the direction of each orthant of* $\mathbb{R}^d$*. We may reconstruct* $(\mathcal{K}, \mathcal{G})$ *from the set of weighted Euler characteristic curves of* $(\mathcal{K}, \mathcal{G})$ *generated by the lower star filtrations of the filtration vectors in* $\mathcal{F}$*.*

Proof. There exists a bijection between the weighted Euler Characteristic Curves and the set of weighted Euler characteristics of the lower stars by analogous computations to those found in

Lemma 4.10. Using this bijection the Corollary follows immediately from the results of corollary 5.4 and multiplying all sides of Proposition 4.13 by $\mathcal{G}(\mathcal{C}_v)$. $\square$

5.2 Pipeline

The code associated with the reconstruction has been packaged and is available at the github repository found at https://github.com/lbetthauser/Topological-Reconstruction.git. The pipeline consists of two stages. The first stage takes a grayscale digital image as input which is then transformed into a variety of inputs. For dimension $d = 2$ the output is either a Möbius inverted matrix which stores stores the Möbius inverted value of each vertex, an Euler array which is a collection of 2^d Euler characteristic curves corresponding to $\mathbb{Z} \times \mathbb{Z}$ arrays where elements are of the form $[n(v), B_{\mathcal{K}} * \mu(\underline{\mathbf{0}}, v)]$ (where $n : \mathcal{K}^0 \to \mathbb{N}$ is an injective map), or a collection of barcodes corresponding to the persistence diagram of each filtration. For dimension $d = 3$ the output is either the Möbius inverted matrix or Euler array. The second stage of the pipeline takes in a Möbius inverted matrix ($d = 2, 3$), Euler array ($d = 2, 3$), or collection of barcodes ($d = 2$) and returns the original grayscale image.

In order to compute the persistent homology barcodes, we utilize a beta version of the code *Perseus* written by Mischaikow and Nanda (2013). This particular version of *Perseus* is one of the few persistence software packages which is capable of taking in an elementary cubical filtration which is not equivalent to specifying a filtration on the top dimensional cubes.

5.2.1 Storage

Compressing a grayscale image represents the first half of the pipeline. Given a *.jpg* representation of a grayscale image, we compute an Euler characteristic curve for a vector pointing towards each orthant (which is equivalent information to the Euler characteristic transform by Lemma 4.11). Each Euler characteristic curve is stored as a $\mathbb{Z} \times \mathbb{Z}$ array.

5.2.2 Reconstruction Algorithm

The reconstruction algorithm represents the second half of the pipeline. Given a collection of Euler characteristic curves corresponding $\mathbb{Z} \times \mathbb{Z}$ arrays where elements are of the form $[n(v), B * \mu(v)]$ (where $n : \mathcal{K}^0 \to \mathbb{N}$ is an injective map), we utilize the results of Corollary

```
>>> ecc_f0
array([[  0.00000000e+00,   2.55000000e+02],
       [  1.86102000e+05,   1.00000000e+00],
       [  1.86543000e+05,   1.00000000e+00],
       ...,
       [  2.62459000e+05,  -1.00000000e+00],
       [  2.62483000e+05,  -1.00000000e+00],
       [  2.62916000e+05,  -1.00000000e+00]])
```

Figure 5-1. The array *ecc_f0* is the Euler characteristic curve corresponding to a vector f_0 pointing the direction of the first orthant of the letter-E (depicted in Figure A-2.) The first column consists of integer values storing the location of a vertex v, such that the weighted Euler characteristic of the lower star $\chi_{\mathcal{G}}(st_{\leq}^{f_0}(v)) \neq 0$. The second column stores the value $\chi_{\mathcal{G}}(st_{\leq}^{f_0}(v))$ in order to recover Möbius inversion of the grayscale image via the method described in Corollary 5.4.

5.4 to reconstruct the Möbius inverted matrix. Once the Möbius inverted matrix has been recovered, the original grayscale image is returned via convolution with the ζ-function per Proposition 4.13. These results easily generalize to vector integer valued functions on $\mathcal{K}$ (for example color channels) where the reconstruction is simply applied to each individual component of the vector and then is concatenated with the knowledge that the underlying cubical complex is fixed. Images corresponding to the Möbius inverted matrix can be found in the Appendix as Figures A-1 and A-2.

Algorithm 5.6. *Compute Möbius inversion values from Euler Characteristic Curves*

```
procedure RECONSTRUCT (K, G)({ECC((K, G), f)}_{f∈F})
    Construct Möbius_Matrix                          ▷ represents relative position of vertices
    for filtration in F do
        for position in ECC((K, G), f)[:,0]: do       ▷ position is in bijection with vertex
            Möbius_Matrix[position] += sgn(filtration)ECC((K, G), f)[position, 1]   ▷ B_K * μ(vertex)
        end for
    end for
    for voxel in Voxel_Matrix do
        voxel = Convolve with Zeta(Möbius_Matrix[Anchor])    ▷ grayscale G(1_{K^d})
    end for
    return Voxel_Matrix                               ▷ grayscale matrix of image
end procedure
```

CHAPTER 6
SUMMARY AND CONCLUSIONS

In this thesis we provide an approach to reconstruct a grayscale digital image (full elementary cubical complex) using only vertices where persistent homology changes via a convolution of the Möbius Function and its connection to Euler characteristic curves. A future direction of this work is to explore storing binary voxel data as an array $\mathbb{Z} \times \mathbb{Z}$. One could store a generic filtration vector f in memory, and construct such an array using the injective birth time of critical vertex and convolution with the Möbius function. Recent advancements have been made in computing the Euler Characteristic of cubical complexes Heiss and Wagner (2017) which makes this approach efficient. Additionally, computing persistent homology of a full elementary cubical complex can be done in $\mathcal{O}(n^2)$ rather than $\mathcal{O}(n^3)$ which is the complexity for simplicial complexes Günther et al. (2012). Therefore, it is feasible to compute persistence quickly and use the sum of the bottleneck distances to establish a metric to compare digital images for classification tasks.

APPENDIX
PIPELINE FIGURES

The Appendix contains two figures which demonstrate the Möbius inversion of a grayscale image. The composition of the processes depicted by Figure A-1 and Figure A-2 is a lossless process. We believe for quantized images of the inputs where thresholding values are selected via persistence Chung and Day (2018), Möbius inversion combined with other smoothing techniques may offer a lossy compression for digital images.

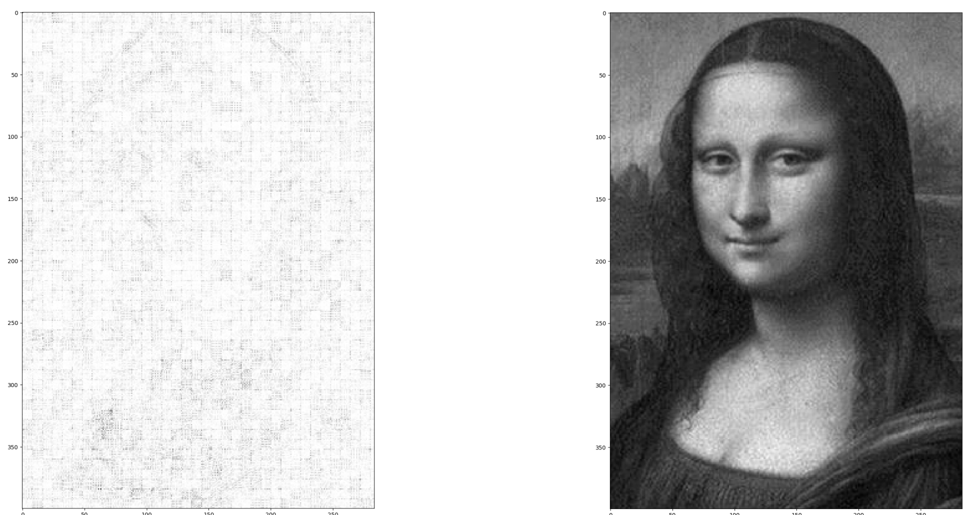

Figure A-1. Example of the input and output of the second half of the reconstruction pipeline described in Algorithm 5.6. A collection of 2^2 Euler characteristic curves which corresponds to the support of the Möbius inversion of a grayscale image (shown above) is inputted. After convolution with the ζ-function, the original gray scale image (displayed below) is returned. The composition of the processes depicted by Figure A-2 and Figure A-1 is lossless.

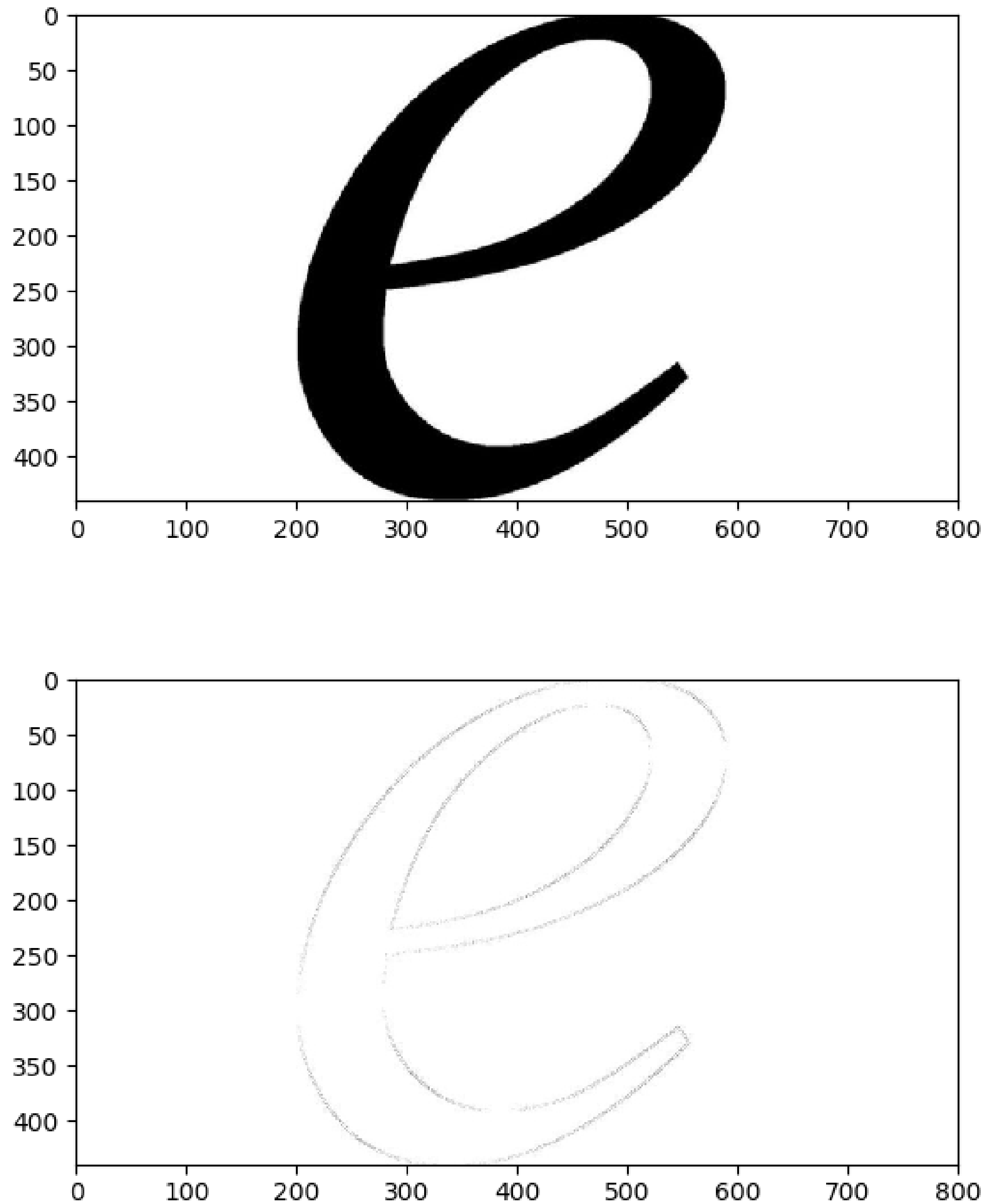

Figure A-2. Example of the input and output of the first half of the reconstruction pipeline. The letter-E (shown above) is compressed by storing the support of the Möbius inversion of the grayscale image (displayed on the bottom) which in turn is stored as a collection of 2^2 Euler characteristic curves (such as the one in Figure 5-1).

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