2019
State Competition Countdown Round Problems 1-80

This booklet contains problems to be used in the Countdown Round.

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NextThought

1. $\qquad$ If $n$ is a two-digit positive integer, what is the sum of the digits of $99 n$ ?
2. $\qquad$
3. $\qquad$ Let $B$ equal the sum of the positive integer multiples of 2 less than 40 , and let $Q$ equal the sum of the positive integer multiples of 4 less than 40 . What is the value of $B-Q$ ?
4. $\qquad$ How many of the first 100 positive integers are divisible by at least three distinct primes?
5. $\qquad$ An average orange weighs $\frac{1}{4}$ pound, and releases 1.6 ounces of juice when squeezed. If oranges cost $\$ 1.60$ per pound, how much does it cost to buy enough oranges to squeeze 8 ounces of orange juice?
6. $\qquad$ The value of $x \& y$ is defined as $\frac{1}{2} x y$. If $(a * a) \&(a ; s)=a^{3}$ and $a>0$, what is the value of $a$ ?
7. $\qquad$ The first six terms of an arithmetic sequence are $12, w, x, 33, y, z$. What is the value of $w+z$ ?
8. $\qquad$ The cash register at Puck's Grocery Store has pennies, nickels, dimes and quarters. If Magnus purchases $\$ 19.37$ worth of items and pays with a $\$ 20$ bill, what is the least number of coins he can receive as exact change?
9. $\qquad$ Consider all of the positive four-digit integers that can be formed using each of the digits $1,2,3$ and 4 exactly once. How many of these integers have a hundreds digit of 2 ?
10. $\qquad$ In how many ways can 48 identical lemon drops be divided into two or more groups containing at least two lemon drops each so that each group has the same number of lemon drops?
11. $\qquad$ A regular octagon with side length 1 unit has area $2+2 \sqrt{2}$ units ${ }^{2}$. The area of a regular octagon with side length 3 units can be expressed in simplest radical form as $a(b+\sqrt{c})$ units $^{2}$. What is the value of $a+b+c$ ?
12. $\qquad$ For a particular sequence, each term is the sum of the three preceding terms. If $a, b, c, d, e, 0,1,2,3$ are consecutive terms of this sequence, what is the value of $a+b+c+d+e$ ?
13. $\qquad$ Three positive integers have a sum of 6 . What is the least possible value of the sum of their reciprocals? Express your answer as a mixed number.
14. $\qquad$
Kei beat Roger in a $5-\mathrm{km}$ race by 13 seconds. If they run a $10-\mathrm{km}$ race, each at the same pace they ran the $5-\mathrm{km}$ race, by how many seconds will Kei beat Roger?
$\square$
,
15. $\qquad$ (dollars)
16. $\qquad$
17. $\qquad$ (circles)

Coplanar circles O and P both have radius 1 cm and are tangent to each other. How many circles of radius 2 cm are in this plane and are tangent to both circles O and P ?
18. $\qquad$ A box contains red, white and pink balls. The probability of randomly selecting a red ball from this box is $\frac{5}{16}$. The probability of randomly selecting a white ball is $\frac{3}{8}$. The probability of randomly selecting a pink ball is $\frac{5}{16}$. What is the fewest number of white balls that could be in the box?
19. $\qquad$ All of the items in Sully's store have two-digit positive integer prices. When the cashier accidentally switches the digits of one item's price, the total cost of the purchase increases by $\$ 18$. For how many different prices is this possible?
20. $\qquad$ $\left(\mathrm{cm}^{3}\right)$

How many cubic centimeters are in the volume of a rectangular prism whose faces have areas $8 \mathrm{~cm}^{2}, 10 \mathrm{~cm}^{2}$ and $12 \mathrm{~cm}^{2}$ ? Express your answer in simplest radical form.
21. $\qquad$ If $x \neq 0$ and $x^{2}=y^{2}$, what is the sum of all possible values of $\frac{x+y}{x}$ ?
22. $\qquad$ If the integer $n$ is a perfect square from 1 to 2019, inclusive, what is the percent probability that $n$ is within 1 of a multiple of 5? Express your answer to the nearest whole number.
23. $\qquad$ Five babies were born in Strong Memorial Hospital. Betty weighed a pound more than Alec. Carly weighed five pounds less than twice Betty's weight. Dorothy and Ellen each weighed the average of Betty's and Carly's weights. If Alec weighed 6 pounds, what was the average weight, in pounds, of the five babies? Express your answer as a decimal to the nearest tenth.
24. $\qquad$ A recycle bin has an opening that is 8 inches high and 6 inches wide. What is the greatest possible diameter, in inches, of a rigid circular piece of cardboard that can be pushed through the opening?
25. $\qquad$ A three-digit number is formed using each of the digits 1,3 and 4 exactly once. What is the least such number that is divisible by 7 ?
26. $\qquad$ Four circles, each of radius 6 units, are tangent to each other, as shown.
 Perpendicular segments can be drawn from the center of each circle to the centers of its two tangent circles. If the area of the gray region can be expressed as $a+b \pi$ units $^{2}$, what is the value of $a-b$ ?
27. $\qquad$ (hours)

Tickets to the Glamorous Gaussian band performance cost $\$ 37.50$ each, or $\$ 110$ for a group of four. How many dollars could Johann save by taking advantage of the group deal to purchase a total of eight tickets?

How many positive integers less than or equal to 33 are divisible by 2,3 or 11 ?
28. $\qquad$
29. $\qquad$ (percent)
30. $\qquad$
31. $\qquad$
32. $\qquad$
33. $\qquad$ If two fair six-sided dice are rolled, what is the probability that the sum of the numbers rolled is divisible by 5 ? Express your answer as a common fraction.
34. $\qquad$ (inches)
35. $\qquad$
(liters)
36. $\qquad$ Numbers will be placed in the squares shown so that the number in each circle is the sum of the numbers in the two adjacent squares. When correctly filled in, what is the greatest of the numbers placed in the squares?

If $m=2^{5} \times 3^{7} \times 5^{9}$, what is the least positive integer $n$ for which the product $m n$ is a perfect cube?
38. $\qquad$ (cents)

Jill has two cylindrical cans of soup. The can of height 12 cm and diameter 8 cm costs $\$ 5.76$. The can of height 10 cm and diameter 6 cm costs $\$ 4.50$. What is the absolute difference in the number of cents charged per cubic centimeter between these two cans of soup? Express your answer as a common fraction in terms of $\pi$.
39. $\qquad$ If $x=\frac{1^{3}+2^{3}+3^{3}+4^{3}}{1+2+3+4}$, what is the value of $x$ ?
40. $\qquad$ What is the largest prime divisor of $3^{8}-1$ ?
41. $\qquad$ If $\frac{4 x^{2}-2}{3}=\sqrt{2}$, then the value of $\frac{600}{2 x^{2}-1}$ can be expressed in simplest radical form as $a \sqrt{b}$. What is the value of $a b$ ?
42. $\qquad$ Let $p$ equal half the circumference of a circle inscribed in equilateral triangle ABC as shown. Let $q$ equal the distance from vertex A to midpoint M of the opposite side. What is the value of the ratio $p: q$ ? Express your answer as a common fraction in terms of $\pi$.

Scott is thinking of a positive integer that is one more than, or one less than a multiple of 4. Quincy is thinking of an integer that is one more than, or one less than a multiple of 10 . Stella is thinking of an integer with an even number of positive integer divisors. If all three are thinking of the same integer $n$, what is the least possible value of $n$ ?
44. $\qquad$ If $x=\frac{1}{\sqrt{5}} \times \frac{3}{\sqrt{15}}+\frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{15}}$, what is the value of $\frac{1}{x^{2}}$ ?
45. $\qquad$
(units ${ }^{2}$ )
46. $\qquad$ If $a$ and $b$ are integers such that $a^{2}+b^{2}=61, a+b>0, a b<0$ and $|a|<|b|$, what is the value of $a$ ?
47. $\qquad$ Starr rolls two fair six-sided dice. When she adds the two numbers rolled, the sum is an odd number. What is the probability that the sum is 7? Express your answer as a common fraction.
48. $\qquad$
(dollars)
49. $\qquad$ The 3 rd and 9 th terms of a geometric sequence are 2 and 72 , respectively. What is the absolute difference between the 6th and 12th terms of this sequence?
50. $\qquad$ Andrea picks one vertex of a unit cube and measures its distance from each of the other vertices of the cube. What is the median of the distances that Andrea measured, in units? Express your answer as a decimal to the nearest tenth.
51. $\qquad$ Elsie is one of ten girls who took off two shoes and threw each of them, one at a time, into a pile. If Elsie is blindfolded and randomly selects two shoes from this pile, without replacement, what is the probability that Elsie will select the two shoes that she threw on the pile? Express your answer as a common fraction.
52. $\qquad$ What is the value of $(\sqrt{24}-\sqrt{6}) \times(\sqrt{48}+\sqrt{12})$ ? Express your answer in simplest radical form.
53. (integers)

In base five, how many integers greater than 4 have the property that its digits are in strictly decreasing order?
54.
55.
56. $\qquad$ (pounds)
57. $\qquad$
58. $\qquad$
59. $\qquad$
60. $\qquad$ If $\frac{1+2+3+4+\cdots+n}{n}=8$, what is the value of $n$ ?
61. $\qquad$ The least common multiple of 63 and 30 equals $a$ and the least common multiple of 18 and 175 equals $b$. What is the greatest common factor of $a$ and $b$ ?
62. $\qquad$ Ten students are to be divided into two teams of five for a kickball game. If Jane, Jack and Jill all insist on playing on the same team while Bobby refuses to play on the same team as Billy, in how many ways can the two teams be formed?
63. $\qquad$ What is the value of $\frac{13!-12!}{2 \times 10!+11!-10!}$ ?
64. $\qquad$ If the 7 -digit number 3,A62,84B is divisible by both 4 and 11 , what is the greatest possible value of $\mathrm{A}+\mathrm{B}$ ?
65. $\qquad$ Sid the magician has a hat containing a rabbit, two identical apples, three identical scarves and four identical wands. Sid randomly pulls four items from the hat, without replacement. What is the probability that the four items are distinct? Express your answer as a common fraction.
66. $\qquad$ The sum of the positive square roots of two numbers is 16 . What is the sum of the arithmetic mean and the geometric mean of these two numbers?
67. $\qquad$ What is the greatest three-digit integer whose digits are all distinct and for which the geometric mean of its digits is an integer?
68. (integers)
69. $\qquad$
For how many three-digit odd integers are its digits in increasing order?
The graph of the equation $(x+3) y=x^{2}-9$ passes through distinct points with coordinates $(0, a),(b, 0)$ and $(c, 0)$. What is the value of $a+b+c$ ?
70. $\qquad$

71 $\qquad$ (km)
72. $\qquad$ If $a$ and $b$ are positive integers and $a^{2}+6 a b+9 b^{2}=100$, what is the sum of all possible values of $a$ ?
73. $\qquad$ Suppose $a, b$ and $c$ are distinct perfect squares with $a<b<c$ and $\sqrt{a b c}=12$. If $S=a+b+c$, what is the sum of all possible values of $S$ ?
74. $\qquad$ If $a, b, c$ and $d$ are values chosen randomly and without replacement from the set $\{1,2,3,4,5\}$, what is the probability that $a b-c d=18$ ? Express your answer as a common fraction.
75. $\qquad$ If $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}=\frac{7}{4}$, what is the value of $\frac{x}{y}$ ? Express your answer as a common fraction.

If $x+y=2$ and $\left(x^{2}+y^{2}\right)^{2}+2 x y\left(x^{2}+y^{2}\right)=40$, what is the value of $x y$ ?
77. $\qquad$ The Aurifeuillean factors of an integer that can be written as $4 n^{4}+1$ are often denoted as $L=2 n^{2}-2 n+1$ and $M=2 n^{2}+2 n+1$, where $L M=4 n^{4}+1$ and $n$ is a positive integer. What is the sum of the two Aurifeuillean factors of 40,001 ?
78. $\qquad$ If $(x, y)$ is a solution to the system of equations $3 x+10 y=29$ and $9 x+y=89$, what is the value of $5 x+7 y$ ?
79. $\qquad$ What is the least positive integer with at least ten positive integer divisors?
The line $y=k x$ is equidistant from the points with coordinates $(1,3)$ and $(5,4)$. What value of $k$ minimizes the distance to either point? Express your answer as a common fraction.

