MATHCOUNTS®

2019 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2019 MATHCOUNTS® State Competition. These solutions provide creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions
with the MATHCOUNTS community.

2019 State Sprint Round Solutions

1. Dividing, we get $\frac{380\ \cancel{\text{ft}}}{19\ \cancel{\text{ft}}} = 20$. So, Hyperion is **20** times the height of Paul's tree.

2. Substituting, we get $y = 2x + 7 = 2(2) + 7 = 4 + 7 = \mathbf{11}$.

3. There is a total of $1 + 2 + 3 + 4 = 10$ marbles, of which $1 + 2 = 3$ are black or white, making the probability of black or white $\mathbf{\frac{3}{10}}$.

4. According to the number line, Jessica earns \$36 for working 3 hours. So, for working 8 hours, she will earn $\frac{\$36}{3\ \cancel{\text{h}}} \times 8\ \cancel{\text{h}} = \$12 \times 8 = \$\mathbf{96}$ or **96.00**.

5. So far, 4 rounds have been played; playing 2 more yields a total of 6 rounds. To have an average of 20 $\frac{\text{pt}}{\text{rd}}$ requires ending up with 20 $\frac{\text{pt}}{\cancel{\text{rd}}} \times 6\ \cancel{\text{rd}} = 120$ points, of which $13 + 17 + 19 + 21 = 70$ have already been scored. To achieve a total of 120 points, Ricardo needs to score another $120 - 70 = \mathbf{50}$ points.

6. Let l and w represent the rectangles length and width, respectively. We know that $l = 12$ cm and $P = 2l + 2w = 2(12) + 2w = 24 + 2w$. Based on the first sentence we can we have $l = \frac{2}{5}P$. Now, substituting for l and P, we solve to get $12 = \frac{2}{5}(24 + 2w) \rightarrow 30 = 24 + 2w \rightarrow 2w = 6 \rightarrow w = \mathbf{3}$ cm.

7. Only one of them can be right about the number, so two of them must be wrong about the number, but right about the suit. So, the card must be a spade; otherwise, both Dee and Nia would be wrong about the suit, and both would be right about the number, which is impossible since their numbers are not the same. This contradicts the statement that nobody is wrong about both the suit and number. Furthermore, the number can not be 6 or 8 because that would make Dee or Nia correct about both the suit and number. This contradicts the statement that nobody is correct about both the suit and number. Therefore, the only option left for number, **7**, must be correct.

8. Let $\overline{\text{AB}}$ be the base of the triangle. Because the y-coordinates are equal, the base is horizontal. The area of the triangle is half the product of the base times the height, where the height is the perpendicular distance of the third vertex C from the base. The perpendicular to the horizontal is the vertical, so the height is the up-or-down distance of C from the base—a distance given to be 3 (the "one unit left" being irrelevant, as is the direction down versus up). The length of a horizontal line segment is the absolute difference between the x-coordinates, $|1 - 5| = |-4| = 4$. Therefore, the area of the triangle is $\frac{1}{2}(4)(3) = \mathbf{6}$ units^2.

9. The only score occurring more than once is 87, so that is the mode. There are 10 scores (an even number), so the median score is the average of the scores of ranks $\frac{10}{2} = 5$ and rank $\frac{10}{2} + 1 = 6$. They are 83 and 87, with an average of $83 + \frac{1}{2}(87 - 83) = 83 + 2 = 85$. The absolute difference of the median and the mean is $|85 - 87| = |-2| = \mathbf{2}$.

10. For $n \geq 2$, $n! < (n+1)! - n! < (n+1)!$ We have that $6! = 720 < 5040 = 7!$ (knowing this through either memorization of the lower factorials or a quick hand calculation). Therefore, n must be 6. We confirm this by checking: $7! - 6! = 5040 - 720 = 4320$. Thus, $n = \mathbf{6}$.

11. Ordinary calendar years have 365 days, which is 52 weeks and 1 day; however, nominally every fourth year (called leap years, those year numbers divisible by 4) has an extra day (called leap day—a 29th day added to February), giving leap years 52 weeks and 2 days. [NOTE: The rules for a year being a leap year are actually more complicated than this, but the simple rule of divisibility by 4 will give you the correct result until 2100.] Therefore, to get to the first day of 2019, one starts off 2018 (which is an ordinary year) on a Monday. When a whole number of weeks pass (such as 52 weeks), the next day will be the same day of the week we started on, in this case Monday. However, there is 1 more day in the year, resulting in the next year starting on a Tuesday, so 2019 starts on a Tuesday. Since 2019 is an ordinary year, 2020 starts 1 day of the week later, Wednesday. Now 2020 is divisible by 4 and is thus a leap year, so we must advance 2 days of the week for 2021, which starts on a Friday. Since 2021 is an ordinary year, 2022 starts 1 day of the week later, Saturday. Since 2022 is an ordinary year, 2023 starts 1 day of the week later, Sunday. Since 2023 is an ordinary year, 2024 starts 1 day of the week later, Monday, which is our goal. Therefore, the answer is **2024**.

12. Sally rides 5 weekdays at a rate of m miles/day, and she rides 2 weekend days at a rate of $(m + 5)$ miles/day. Therefore, the total for one week is $5m + 2(m + 5) = 5m + 2m + 10 = (7m + 10)$ miles. Since Sally rides 94 miles each week, that means $7m + 10 = 94 \rightarrow 7m = 84 \rightarrow m = \mathbf{12}$ miles.

13. For 33!, we have $1/3$ of the integers from 1 to 33, inclusive, divisible by 3, or $33/3 = 11$ integers. So, 11 of the factors are contributing a factor of 3. Of those 11, every third one is contributing a second factor of 3, or $11/3 = 3$ integers (must round down because we have not reached the fourth, which would be 36). Of those 3, every third one contributes a third factor of 3, or $3/3 = 1$ integer. If that resulted in a count of at least 3, we would keep repeating the process until we got to less than 3. Totaling all of these gives us $11 + 3 + 1 = 15$ factors of 3 in 33!, so $p = \mathbf{15}$.

14. The expression $\frac{5!+6!}{4!+3!}$ can be rewritten as $\frac{5!+6\times5!}{4\times3!+3!}$. Simplifying, we get $\frac{7\times5!}{5\times3!} = \frac{7\times4!}{3!} = 7 \times 4 = \mathbf{28}$.

15. If the original rectangle has dimensions $l \times w$, then the new rectangle has dimensions $(l + 3) \times (w + 3)$. The area of the original rectangle is lw, and the area of the new rectangle is $lw + 3(l + w) + 9$. We are told that the new rectangle's area is three times that of the original rectangle. So, $3lw = lw + 3(l + w) + 9 \rightarrow 2lw - 3(l + w) - 9 = 0$. Since we know that $l = 2w$, we can substitute to get $2(2w^2) - 3(3w) - 9 = 0 \rightarrow 4w^2 - 9w - 9 = 0$. We can solve this equation by factoring the quadratic expression as follows: $(4w + 3)(w - 3) = 0$. So, $4w + 3 = 0 \rightarrow 4w = -3 \rightarrow w = -3/4$, or $w - 3 = 0 \rightarrow w = 3$. Since width must be a positive value, it follows that the original rectangle has width 3 feet and length $2(3) = 6$ feet. Recall that the new rectangle's length is 3 feet more than the original rectangle's length, or $6 + 3 = \mathbf{9}$ feet.

16. The original six consecutive integers are 30, 31, 32, 33, 34 and 35, with sum is $\frac{6}{2}(30 + 35) = 195$. The five consecutive integers we're looking for must have $\frac{195}{5} = 39$ as the mean and median. Therefore, the greatest integer is 2 more than 39, or **41**.

17. Simplifying, the left side of the equations yields $1 - \frac{1}{324} = \frac{323}{324} = \frac{(18+1)(18-1)}{18^2} = \frac{17}{18} \times \frac{19}{18}$. When we want to minimize a sum for a given product, in general the strategy is to make the values as close together as possible, and, indeed, $17 + 19 = \mathbf{36}$ works.

18. Let's work with just the rightmost two digits. For the units digit, $8 \times 1 = 8$ does not impact the tens digit. To get the tens digit of the product, we need to cross-multiply the units and tens digits of the two factors: $C \times 1 + 8 \times C = 9C$ must end in 5. The only digit C for which that works is $C = \mathbf{5}$.

19. The graph of $\frac{(x-3)(y-7)}{(x+1)(2y-5)} = \frac{1}{2}$ has two "missing" points at $x = -1$ and $y = 5/2$ because division by 0 is undefined. If we cross-multiply, we get $2(x-3)(y-7) = (x+1)(2y-5)$. For either of the two "problematic" values, the right side of the equation is 0, so the left side must be likewise 0. When $x = -1$, that means $y = 7$; the only other option would be $x = 3$, but that cannot be the case when x is already −1. Similarly, when $y = \frac{5}{2}$, that means $x = 3$. Therefore, the two problematic points are $(-1, 7)$ and $\left(3, \frac{5}{2}\right)$. These two points are problematic because of division by 0, but would otherwise have been on the line; therefore, the slope of the line in question is the same as the slope of the line containing these two points: $\frac{\frac{5}{2}-7}{3-(-1)} = \frac{5-7\times 2}{4\times 2} = -\mathbf{\frac{9}{8}}$.

20. For a parabola given by $Ax^2 + Bx + C = 0$, the axis of symmetry is the line $x = -\frac{A}{2B}$. Consider the parabola whose equation is $y = x^2 + 4x$, where $A = 1$, $B = 4$ and $C = 0$. The vertical axis of symmetry for this parabola is at $x = -\frac{4}{2\times 1} \rightarrow x = -2$. We quickly see that $x = 0$ satisfies the inequality since $0^2 + 4 \times 0 \leq 1$, but $x = 1$ does not since $1^2 + 4 \times 1 > 1$. Therefore, the greatest integer solution occurs when $x = 0$, which is two more than $x = -2$; by symmetry the least integer solution occurs when $x = -4$, which is 2 less than $x = -2$. Integer solutions occur for all x in the set {−4, −3, −2, −1, 0}. Their sum is −4 + (−3) + (−2) + (−1) + 0 = **−10**.

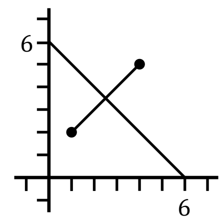
21. Because the interval is symmetric about 0, the probability of being on any one side of 0 is 1/2. The probability of being exactly at 0 is regarded as exactly 0, so we do not need to account for that one point. If an event cannot happen, the probability of that event happening is indeed 0; however, the converse is *not* necessarily true, as in this case—choosing 0 is indeed a possible outcome but it has probability 0, the same as choosing any other specific point in the interval. The probability of the first point being negative and the second point positive is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; the same occurs for the first point being positive and the second point negative. Therefore, the total probability is $\frac{1}{4} + \frac{1}{4} = \mathbf{\frac{1}{2}}$. [NOTE: Some might argue that the problem is more trivial than that—namely, that the second point is either on the same side or the opposite side, each with probability 1/2, and we want the case of the opposite side. However, that works only for the case of the breakpoint being at the center of the interval, as happens in this problem that 0 is at the midpoint between −1 and +1. This solution is more general and works for asymmetric cases as well, where the answer is not 1/2. The computation with this solution is slightly slower, which is offset by not spending time verifying that the conditions required for the faster solution are met. It is a matter of individual preference.]

22. Average speed equals total distance divided by total time. There are three parts to the trip. Going east and going north are perpendicular motions, so those two motions form the legs of a right triangle. We are given that the legs are 4000 m and 3000 m, thus forming a 3-4-5 right triangle with scale factor 1000 m. The third part of the trip is a hypotenuse of $5 \times 1000\text{ m} = 5000\text{ m}$. This makes the total distance for the trip $4000 + 3000 + 5000 = 12{,}000\text{ m}$. The time for the first part is $\frac{400\cancel{0}\ \cancel{\text{m}}}{2\cancel{0}\ \cancel{\text{m}}/\text{s}} = 200\text{ s}$ and for the second part is $\frac{300\cancel{0}\ \cancel{\text{m}}}{3\cancel{0}\ \cancel{\text{m}}/\text{s}} = 100\text{ s}$. We are told that the third part takes 100 s. So, the total time is $200 + 100 + 100 = 400\text{ s}$. The result is an average speed of $\frac{12\ \cancel{000}\text{ m}}{4\cancel{00}\text{ s}} = \mathbf{30}\text{ m/s}$.

23. Adding the two equations $a - bc = 19$ and $a + bc = 99$ yields $2a = 118$. So, $a = 59$, which means $bc = 40$. The sum of two numbers is minimized for a fixed product when the two numbers are as close in value as possible. Neither 6 nor 7 divides 40, so $5 \times 8 = 40$ is the best we can do in the domain of the integers. That gives us the minimum sum $59 + 5 + 8 = \mathbf{72}$.

24. We have for the six faces the sum of the integers 1 through 7, inclusive, except for the one missing value m, and that sum is given to be 24. Therefore, $24 = 1 + 2 + 3 + 4 + 5 + 6 + 7 - m = 28 - m$. So, $m = 4$, and the six faces are 1, 2, 3, 5, 6, 7, of which four values (2, 3, 5, 7) are prime, making the probability of rolling a prime to be $\mathbf{\frac{2}{3}}$.

25. The goal is to find the largest circle that contains the points (1, 2) and (4, 5), and is tangent to either the x-axis or the y-axis, whichever is closer. The center of the circle must be along the perpendicular bisector of the two given points. The slope of the given line segment is $\frac{5-2}{4-1} = 1$. So, the slope of the perpendicular bisector is the negative reciprocal of that, or −1. The perpendicular bisector must go through the midpoint of the given segment, $\left(\frac{1+4}{2},\ \frac{2+5}{2}\right) = \left(\frac{5}{2},\ \frac{7}{2}\right)$. The center of the circle, then, must be on the line $y = 6 - x$, and, for the circle to be in the first quadrant, $0 < x < 6$. The square of the distance between $(x,\ 6 - x)$ and each of (1, 2) and (4, 5) is $(x - 1)^2 + (x - 4)^2 = 2x^2 - 10x + 17$. For the circle to be tangent to the nearest axis, this value needs to match the square of the distance to the nearest axis. The nearest axis is the y-axis when $0 < x < 3$ and the square of that distance is x^2. The nearest axis is the x-axis when $3 < x < 6$ and the square of that distance is $(x - 6)^2 = x^2 - 12x + 36$. We can observe from the figure that the given line segment is above the line $y = x$ that bisects the first quadrant, so there is more room to place a larger circle between that segment and the nearest axis below the segment than above the segment. So, the second case will yield the larger circle. Thus, we need $2x^2 - 10x + 17 = x^2 - 12x + 36$, so $x^2 + 2x - 19 = 0$. The quadratic formula yields $x = \frac{-2\pm\sqrt{4+76}}{2} = -1 \pm 2\sqrt{5}$. Only the + option yields the requisite $3 < x < 6$. Therefore, $x = -1 + 2\sqrt{5}$ and $y = 6 - x = 7 - 2\sqrt{5}$. The y-component is the needed radius, so the result is $\mathbf{7 - 2\sqrt{5}}$ units.



26. A handy fact to know for MATHCOUNTS is that the volume of a regular tetrahedron with edge length s is given by $V = \frac{\sqrt{2}}{12}s^3$. In general, the volume of a tetrahedron with a triangular face of area A and perpendicular height with respect to that base h is given by $V = \frac{1}{3}hA$. Recall for a regular tetrahedron that each face is an equilateral triangle satisfying with area $A = \frac{\sqrt{3}}{4}s^2$. We can rewrite the formula $\frac{1}{3}hA$ as $\frac{1}{3} \times h \times \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{12}s^2h$. We can equate the two expressions for the volume of this tetrahedron to get $\frac{\sqrt{2}}{12}s^3 = \frac{\sqrt{3}}{12}s^2h$. Simplifying, we see that $h = \frac{\sqrt{6}}{3}s$. Since this tetrahedron has edge length s = 5 cm, we can substitute to get $h = \mathbf{\frac{5\sqrt{6}}{3}}$ cm.

27. The height of the original cone is $\left(\frac{1}{3/4}\right)15 = \left(\frac{4}{3}\right)15 = 20$ cm. The radius of the original cone is given to be 12 cm. Therefore, the volume of the original cone is $\frac{1}{3} \times \pi \times 12^2 \times 20 = 960\pi$ cm^3. The volume of two similar figures is proportional to the cube of the relative linear scaling. The smaller cone is $\frac{3}{4}$ as tall as the original cone, so the volume of the smaller cone is $\left(\frac{3}{4}\right)^3 = \frac{27}{64}$ the volume of the original cone. The volume of the frustum is the rest of the volume of the original cone, so $\left(1 - \frac{27}{64}\right) 960\pi = \frac{37}{64} \times 960\pi = \frac{37}{8}\ 120\pi = 37 \times 15\pi = \mathbf{555\pi}$ cm^3.

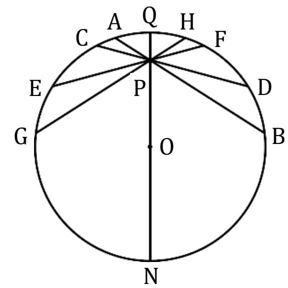
28. The equations of the two circles are $169 = x^2 + y^2$ and $225 = x^2 + (y - 14)^2 = x^2 + y^2 - 28y + 196$. When the second equation is subtracted from the first we get $-56 = 28y - 196$, so $28y = 140$ and $y = 5$. Thus, $y = 5$ is the equation of the line containing the common chord. Substituting into the first equation yields $169 = x^2 + 25$, so $x^2 = 144$ and $x = \pm 12$ for the endpoints of the common chord. Because both endpoints have the same y-coordinates, the length of the chord is the absolute difference between the two x-coordinates: $|12 - (-12)| = \mathbf{24}$ units.

29. Since 12 of the 16 coin flips, land heads up, it follows that 4 must land tails up. These 4 tails can occur in a mix of various patterns: as the first flips, as the last flips, or as separators between runs of heads up. With up to 4 such separators, there cannot be more than 5 runs of heads up. Because there are 12 flips that land heads up but no runs of more than 4 heads, there must be at least 3 runs of heads up. We need to account for the number of permutations of each pattern of runs of heads up and similarly of each pattern of runs of tails up. We will account for the runs of heads up by setting up patterns systematically starting with the longest possible runs and working down in sizes from right to left. For tails, we will look at how many flips land tails up before the first that lands heads up (may be 0, 1 or 2), how many flips land tails up between consecutive runs of heads (may be 1, 2 or 3), and how many flips land tails up after the last flip that lands heads up (may be 0, 1 or 2), with the sum of these values needing to be exactly 4. When there are 3 runs of heads up, we have the following possibilities: 2 1 1 0; 1 1 1 1; 0 1 1 2; 1 2 1 0; 1 1 2 0; 0 2 1 1; 0 1 2 1; 0 3 1 0; 0 2 2 0; 0 1 3 0—a total of <u>10</u> possibilities. When there are 4 runs of heads up, we have the following possibilities: 1 1 1 1 0; 0 1 1 1 1; 0 2 1 1 0; 0 1 2 1 0; 0 1 1 2 0—a total of <u>5</u> possibilities. When there are 5 runs of heads up, we have only the <u>1</u> possibility: 0 1 1 1 1 0. Now let's set up an organized table with the patterns of runs of heads up, the number of permutations of each pattern, the count of patterns of associated runs of tails up based on findings above, and the overall count of arrangements of those patterns:

heads up pattern	# permutations of pattern	# associated tails up patterns	total # arrangements
4 4 4	$\frac{3!}{3!} = 1$	10	10
4 4 3 1	$\frac{4!}{2!1!1!} = 12$	5	60
4 4 2 2	$\frac{4!}{2!2!} = 6$	5	30
4 4 2 1 1	$\frac{5!}{2!1!2!} = 30$	1	30
4 3 3 2	$\frac{4!}{1!2!1!} = 12$	5	60
4 3 3 1 1	$\frac{5!}{1!2!2!} = 30$	1	30
4 3 2 2 1	$\frac{5!}{1!1!2!1!} = 60$	1	60
4 2 2 2 2	$\frac{5!}{1!4!} = 5$	1	5
3 3 3 3	$\frac{4!}{4!} = 1$	5	5
3 3 3 2 1	$\frac{5!}{3!1!1!} = 20$	1	20
3 3 2 2 2	$\frac{5!}{2!3!} = 10$	1	10
Total	—	—	320

Thus, 320 possible cases meet our criteria. There are a total of $\frac{16!}{12!4!} = \frac{16\times 15\times 14\times 13}{4\times 3\times 2\times 1} = 2 \times 5 \times 14 \times 13 =$ 1820 total cases in all. So, the desired probability is $\frac{320}{1820} = \mathbf{\frac{16}{91}}$.

30. In order to obtain the least possible integer radius satisfying all of the specified criteria, we will need to have the figure be symmetric about the NQ axis so that chords $\overline{AB}$ and $\overline{HG}$ are congruent (and their corresponding subsegments), as are chords $\overline{CD}$ and $\overline{FE}$. The statement of the problem does not forbid such an outcome; otherwise, so many chords and segments all having to be different lengths would necessitate numbers with so many distinct factors as to fail being a least result. Therefore, we may ignore chords $\overline{EF}$ and $\overline{GH}$ as redundant. In order to have the radius (such as segment $\overline{OQ}$) be an integer, the diameter $\overline{NQ}$ must have an even integer measure, so that segments $\overline{NP}$ and $\overline{PQ}$ must both have even integer measures or both have odd integer measures. The measures of the various segments must be integers satisfying $PQ < PA = PH < PC = PF < PD = PE < PB = PG < PN$. Thus, we must find six distinct integer values satisfying all the specified conditions, as well as the intersecting chords theorem, which says that $PA \times PB = PC \times PD = PQ \times PN$. Thus, there needs to be an underlying integer that has at least six distinct factors. We must keep in mind our restriction that PQ and PN must both be odd or both be even, meaning we have two cases to contend with.



Case 1: When PQ and PN are both odd, the minimum such underlying integer must itself be odd. The least odd positive integer having six or more factors is $3^2 \times 5 = 45$, so that $PQ = 1$ and $PN = 45$, with the radius of the circle being $(1 + 45)/2 = 23$. Incidentally, $PA = 3$, $PB = 15$, $PC = 5$ and $PD = 9$.

Case 2: When PQ and PN are both even, the least factor we are regarding, PQ, is at least 2. However, 1 is always a factor as is its counterpart $PQ \times PN$. That means we are ignoring at least 2 factors besides the six that we are processing, so our underlying number must have at least eight factors. The least positive integer having eight or more factors is $2^3 \times 3 = 24$. The least even factor is 2, with its counterpart being $24/2 = 12$. That yields a radius of $(2 + 12)/2 = 7$. As a side note, $PA = 3$, $PB = 8$, $PC = 4$, and $PD = 6$.

Case 2 yields the least possible integer radius of circle O, which is **7** units.

2019 State Target Round Solutions

1. Because 1 ft $= 12$ in, the dimensions of the drawing are $\frac{3}{4}$ ft $\times$ 1 ft, for an area of $\frac{3}{4}$ ft^2. We are told that the actual area of 768 ft^2. The ratio of the actual area to the drawing area is $\frac{4}{3} \times 768 = 1024$, making the ratio for a linear dimension $\sqrt{1024} = 32$. Thus, the longer side of the drawing is 1 ft, which corresponds to an actual length of 32×1 ft $=$ **32** feet.

2. Here we are to find the area of a polygon knowing the x- and y-coordinates of the vertices in order around the polygon; in this case, we have a triangle, meaning three vertices. There is a great algorithm, over 200 years old, for handling this problem, namely, the Gauss area formula, also known as the surveyor's formula and, more recently and affectionately among the MATHCOUNTS world, the shoelace theorem. Arrange the coordinate pairs of the vertices in a vertical stack of (x, y) values. Start at any vertex and follow the edges of the polygon from vertex to vertex and record its coordinates and the coordinates of each vertex as encountered until you reach the starting vertex and record its coordinates once more. You may go either direction around the polygon, with the direction affecting only the sign of the result, so we express the formula with an absolute value to avoid caring about direction. It is important that you remember to have the last ordered pair in the stack be the same as the first ordered pair. In this case vertex O is at the origin; when such happens, it is best to start and end at that vertex. Doing so guarantees that the formula will contribute 0 at the first step and last step. For this particular triangle, let's take the vertices in the order O-B-A-O:
   (0, 0) [O]
   (12, 9) [B]
   (6, 12) [A]
   (0, 0) [O].
   The process is now to take every consecutive pair of vertex coordinates, referring to them generically as
   $(a, \ b)$
   $\times$
   $(c, \ d)$.
   Now cross-multiply ad and bc, and subtract to get $ad - bc$, calculate this for each consecutive pair, add those results, take absolute value and divide by 2 [the absolute value being only upon computing the sum of the values over the vertices, *not* for the value at each pair of vertices]:
   OB: $0 \times 9 - 0 \times 12 = 0$;
   BA: $12 \times 12 - 9 \times 6 = 144 - 54 = 90$;
   AO: $6 \times 0 - 12 \times 0 = 0$. [See why starting with a vertex at the origin simplifies calculations.]
   Sum: $0 + 90 + 0 = 90$, for which the absolute value is 90.
   Dividing by 2 yields a final area of $\frac{90}{2} =$ **45** units^2.

3. Because harvesting is allowed from September 1 through April 30, that means harvesting is not allowed May 1 through August 31 (must remember how many days are in each month). Since May, July and August each have 31 days, but June has 30 days, that means that an interval of $(3 \times 31 + 30) = 123$ days. There are 365 days in an ordinary calendar year. Therefore, the fraction of time in an ordinary year in which harvesting is not allowed is $\frac{123}{365} \approx 0.337 =$ **33.7** %. [NOTE: The varying number of days across the various months means that we cannot simply say it is 4 months out of 12 months for the level of accuracy needed.]

4. To have an odd number of positive factors requires a number to be a perfect square.
   Because $\sqrt{2018} = 44.9\ldots$, there are **44** perfect squares between 1 and 2018, inclusive.

5. The average number of points per possession is $\frac{1019\times 3\text{ pt}+1797\times 2\text{ pt}+1421\times 1\text{ pt}}{6995\text{ poss}} = \frac{8072\text{ pt}}{6995\text{ poss}} \approx 1.153\,97\ \frac{\text{pt}}{\text{poss}}$. For 100 possession, the average number of points is 100 times this value, or **115.397** pt.

6. The area of square PQRS is less than the area of square ABCD by the sum of the areas of the 4 congruent right triangles APS, BQP, CRQ and DSR—a difference equal to $1156\text{ cm}^2 - 676\text{ cm}^2 = 480\text{ cm}^2$. The area of square EFGH is less than the area of square PQRS by the sum of the areas of the 4 congruent right triangles HSP, EPQ, FQR and GRS, which are congruent with the 4 right triangles APS, BQP, CRQ and DSR, making the difference in area between squares PQRS and EFGH also equal to 480 cm^2. Therefore, the area of square EFGH is $676\text{ cm}^2 - 480\text{ cm}^2 = \mathbf{196}\text{ cm}^2$.

7. When a cube is subdivided along each of the three face-centered axes into n congruent slabs, a block composed of n^3 congruent smaller cubes is formed. Each of those smaller cubes has 6 faces, resulting in a total of $6n^3$ faces. Only the outer surface—the 6 faces—of the original cube is painted. Each of the 6 faces of the original cube involves 1 face from each of the n^2 smaller cubes making up the larger face, yielding $6n^2$ smaller faces that are painted, with the remaining $6n^3 - 6n^2$ smaller faces unpainted. Removing the outer layer on each face yields an $(n-2)\times(n-2)\times(n-2)$ cube of totally unpainted smaller blocks, with $6(n-2)^3$ unpainted smaller faces. Thus, with each of the $6n^3 - 6n^2 = 6(n^3 - n^2)$ unpainted smaller faces equally likely, of which $6(n-2)^3$correspond to completely unpainted smaller cubes, the probability of landing on a completely unpainted small cube upon landing on an unpainted smaller face is $\frac{(n-2)^3}{n^3-n^2}$.
When $n = 8$, this probability is $\frac{6^3}{8^3-8^2} = \frac{216}{512-64} = \frac{216}{448} = \mathbf{\frac{27}{56}}$.

8. We are told that $a_1 = 20$ and $a_2 = 19$. We can derive the remainder of the terms of the sequence using the definition $a_n = |a_{n-1}| - |a_{n-2}|$. The table shows the first 12 terms of the sequence:

$a_1 = 20$	$a_2 = 19$	$a_3 = -1$
$a_4 = -18$	$a_5 = 17$	$a_6 = -1$
$a_7 = -16$	$a_8 = 15$	$a_9 = -1$
$a_{10} = -14$	$a_{11} = 13$	$a_{12} = -1$

We are taught to look for patterns, but we must be careful about assuming the "most obvious" pattern is correct, especially when dealing with non-linear behavior—absolute value might look to be close to linear, but it most certainly is not. The repeated increase by 2 in the left column (except for minor sign issue of the first term), decrease by 2 in the middle column and apparent constancy of the right column might not hold when the left or middle column crosses 0. Let's use this pattern to accelerate getting near this crossing:

$a_{28} = \lvert -1\rvert - \lvert 3\rvert = 1 - 3 = -2$	$a_{29} = \lvert -2\rvert - \lvert -1\rvert = 2 - 1 = 1$	$a_{30} = \lvert 1\rvert - \lvert -2\rvert = 1 - 2 = -1$
$a_{31} = \lvert -1\rvert - \lvert 1\rvert = 1 - 1 = 0$	$a_{32} = \lvert 0\rvert - \lvert -1\rvert = 0 - 1 = -1$	$a_{33} = \lvert -1\rvert - \lvert 0\rvert = 1 - 0 = 1$

Look at that sudden sign flip. Let's continue to see what happens.

$a_{34} = \lvert 1\rvert - \lvert -1\rvert = 1 - 1 = 0$	$a_{35} = \lvert 0\rvert - \lvert 1\rvert = 0 - 1 = -1$	$a_{36} = \lvert -1\rvert - \lvert 0\rvert = 1 - 0 = 1$

We have two consecutive rows with the exact same sequence of values. That will continue to hold. Because 2019 is divisible by 3, a_{2019} falls in the right column, with value **1**.

2019 State Team Round Solutions

1. We are told that $n^{12} = 216$, and we know that $n^4 = n^{12/3} = (n^{12})^{1/3}$. So, it follows that $(n^{12})^{1/3} = 216^{1/3} = \sqrt[3]{216} = \mathbf{6}$.

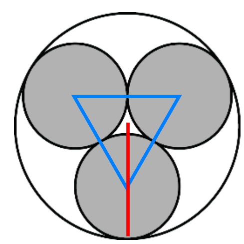
2. Let's try substituting prime numbers (in ascending order) for p in $2^p - 1$ and stop when the result is composite. For p = 2, we have $2^2 - 1 = 3$, which is prime. For p = 3, we have $2^3 - 1 = 7$, which is prime. For p = 5, we have $2^5 - 1 = 31$, which is prime. For p = 7, we have $2^7 - 1 = 127$, which is prime. For p = 11, we have $2^{11} - 1 = 2047$, which equals 23×89, so it must be composite. Thus, **11** is the answer.

3. The fourth term, 200, is 2 times its predecessor, the third term, of 100. Therefore, the common ratio of this geometric sequence is 2, meaning that each term has a successor 2 times as big and a predecessor (except for the first term) 1/2 as big. The first term is the second predecessor of the third term, so its value is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)100 = \mathbf{25}$.

4. Just as 37 as a base-ten number means $3 \times 10^1 + 7 \times 10^0 = 3 \times 10 + 7 \times 1 = 37$, so 37 as a base-eight number means $3 \times 8^1 + 7 \times 8^0 = 3 \times 8 + 7 \times 1 = 24 + 7 = \mathbf{31}$ years in base ten.

5. When 1000 M&Ms are randomly drawn from the jar and 245 of them do not have the "m," that means our best estimator of the probability of a candy not having the "m" is 245/1000 = 0.245. Thus, if there are n M&Ms in total, then we expect that $0.245n$ M&Ms should not have the "m." But we know that the number of M&Ms without the "m" is 1000, so it follows that $0.245n \approx 1000$. Since there are an integer number of M&Ms, we see that $n \approx 1000/0.245 \approx \mathbf{4082}$ M&Ms are expected to be in the jar.

6. Let r and R represent the radius of the smaller circles and the large circle, respectively. The centers of the smaller circles form the vertices of an equilateral triangle whose centroid is the center of the large circle. The sides of the equilateral triangle are $2r$ so the altitude of the triangle (which is also a median of the triangle) has length $\sqrt{3}/2 \times 2r = \sqrt{3}r$. The radius of the large circle is one radius of a small circle plus the distance from a vertex to the centroid of the triangle. The distance from a triangle vertex to its centroid is 2/3 of the length of the median, in this case $(2/3)\sqrt{3}r$. Thus, $R = \left(1 + (2/3)\sqrt{3}\right)r$. The requested ratio is the area of three small circles divided by the area of the large circle, or $3\pi r^2/\pi R^2$. Substituting for R, we get



$$\frac{3\pi r^2}{\pi R^2} = \frac{3r^2}{\left(1 + \frac{2}{3}\sqrt{3}\right)^2 r^2} = \frac{3}{\left(1 + \frac{2}{3}\sqrt{3}\right)^2} = \frac{3\left(1 - \frac{2}{3}\sqrt{3}\right)^2}{\left(1 + \frac{2}{3}\sqrt{3}\right)^2\left(1 - \frac{2}{3}\sqrt{3}\right)^2} = \frac{3\left(1 - \frac{4}{3}\sqrt{3} + \frac{4}{3}\right)}{\left(1 - \frac{4}{3}\right)^2} = 9\left(3 - 4\sqrt{3} + 4\right)$$
$$= 63 - 36\sqrt{3}.$$

If $a + b\sqrt{c} = 63 - 36\sqrt{3}$, then $a = 63,\ b = -36,\ c = 3$. So, $a + b + c = 63 - 36 + 3 = \mathbf{30}$.

7. Since $a - b = 1$, we can let $a = c + \frac{1}{2}$ and let $b = c - \frac{1}{2}$. Then, using the binomial theorem, we have $a^5 - b^5 = \left(c + \frac{1}{2}\right)^5 - \left(c - \frac{1}{2}\right)^5 = c^5 + \frac{5}{2}c^4 + \frac{5}{2}c^3 + \frac{5}{4}c^2 + \frac{5}{16}c + \frac{1}{32} - \left(c^5 - \frac{5}{2}c^4 + \frac{5}{2}c^3 - \frac{5}{4}c^2 + \frac{5}{16}c - \frac{1}{32}\right) = 5c^4 + \frac{5}{2}c^2 + \frac{1}{16}$. The first two terms are always at least 0, with the least value 0 occurring if and only if $c = 0$. Thus, the least possible value of $a^5 - b^5$ is $\mathbf{\frac{1}{16}}$.

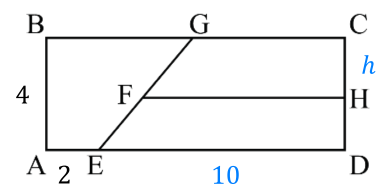
8. We need to find two positive factors n and d of $9! = 362{,}880 = 2^7 \times 3^4 \times 5 \times 7$ such that $nd = 9!$, with n and d as close to each other as possible. In order for $x = n/d$ to minimize $|x-1|$, we need $n \le d$, because $\left|\frac{\min(n,d)}{\max(n,d)} - 1\right| \le \left|\frac{\max(n,d)}{\min(n,d)} - 1\right|$, with equality holding if and only if $n = d$, and this holds for all positive real numbers n and d. Because $\sqrt{9!} = 602.395\ldots$, we thus need $n \le 602$ and $d \ge 603$. We need to be very careful in seeking the answer because 9! has $(7+1)\times(4+1)\times(1+1)\times(1+1) = 160$ factors, so that there are 80 pairs (n, d)—that is too many to determine all the pairs but we need to make sure we do not miss the key pair. There is only one factor of 5 and one of 7, so let's see whether they go in the numerator or denominator.

<u>Case 1:</u> Both 5 and 7 are in the numerator or both are in the denominator: Because $602/35$ is between 17 and 18, we know that $35 \times 17 = 595$ and $35 \times 18 = 630$ bound 602. Now, 630 is indeed a factor of 9! since $18 = 2^1 \times 3^2$ divides $2^7 \times 3^4$. However, the prime value 17 makes 595 not work because 17 is not a factor of 9!; the largest value that works is $35 \times 16 = 560$. This case generates two possible pairs: (576, 630) and (560, 648), with the two values in the first pair being the ones closer to equal.

<u>Case 2:</u> One of 5 and 7 is in the numerator and the other is in the denominator: The closest we can come to making the numerator and denominator equal in this case is to put all four factors of 3 with the 7 and all seven factors of 2 with the 5 to yield the pair (568, 640).

The components of (576, 630) in Case 1 are closer to equal than those of (568, 640) in Case 2. Therefore, $n = 576$, $d = 630$ and $x = \frac{576}{630} = \frac{32}{35}$, for which $|x-1| = \mathbf{\frac{3}{35}}$.

9. We are given that AB = 4, and that AE = 2, which combined with AD = BC = 12 means that ED = 10, and the area of the rectangle ABCD is $12 \times 4 = 48$. The area of trapezoid AEGB is given by the product of AB times the average of AE and BG; this area is given as being 1/3 the area of rectangle ABCD, or 16. Thus, $16 = 4 \times \frac{2+\text{BG}}{2}$, so $8 = 2 + \text{BG}$, meaning that



BG = 6 and GC = 6 since their sum is 12. The area of trapezoid DEGC is the area of rectangle ABCD minus the area of trapezoid AGEB, so 32. We are to find the vertical placement of $\overline{\text{FH}}$ so that trapezoids FHCG and EDHF each have area $32/2 = 16$. The vertical placement of $\overline{\text{FH}}$ is specified by the length h of $\overline{\text{CH}}$, where $0 < h < 4$. For $h = 0$, $\overline{\text{FH}}$ coincides with $\overline{\text{GC}}$ and has length 6; as h grows toward 4, $\overline{\text{FH}}$ lengthens linearly, becoming 10 when $\overline{\text{FH}}$ coincides with $\overline{\text{ED}}$ of length 10 at $h = 4$. This means $\text{FH} = 6 + h$. The area of each trapezoid is to be 16, but it is also the product of CH and the average of GC and FH. So, we have $16 = h\frac{6+(6+h)}{2} = h\left(6 + \frac{h}{2}\right) = \frac{1}{2}h^2 + 6h$, so $0 = \frac{1}{2}h^2 + 6h - 16$. Using the quadratic formula, we obtain: $h = \frac{-6 \pm \sqrt{6^2 - 4\left(\frac{1}{2}\right)(-16)}}{2(1/2)} = -6 \pm \sqrt{36+32} = -6 \pm \sqrt{68} = -6 \pm 2\sqrt{17}$. Because h is restricted to being positive, we must use the + sign, so $h = -6 + 2\sqrt{17}$. The length of $\overline{\text{FH}}$ is 6 greater, thus $\mathbf{2\sqrt{17}}$ units.

10. The probabilities for Amy and for Rob are not independent of one another: The probability that points are scored on any one roll by Amy or Rob is 1/2 or 2/5, respectively, but the probability that both score points on one roll is 1/10, not $\left(\frac{1}{2}\right)\left(\frac{2}{5}\right) = \frac{1}{5}$. This means that one cannot validly determine the probabilities for Amy and Rob separately and multiply them together. Let's focus on relative gain in points rather than absolute numbers of points (particularly on a roll of 2, where Amy wins 2 points while Rob wins 3 points, which is a net gain of 1 point for Rob). Let's set up a table of what can happen on any one roll:

Point impact	Relevant roll values	Probability	Event type symbol
Net gain of 2 for Amy	4, 6, 8, 10 [even but not prime]	0.4	W
None	1, 9 [neither even nor prime]	0.2	X
Net gain of 1 for Rob	2 [both even and prime]	0.1	Y
Net gain of 3 for Rob	3, 5, 7 [prime but not even]	0.3	Z

When the results of 4 rolls are combined, we want Amy to have a net positive number of points. We need to collect all of the mixes of 4 event types for which this occurs:

Event type mix	# net points for Amy	# permutations of mix	Probability of mix
W, W, W, W	8	$\frac{4!}{4!} = 1$	$1 \times 0.4^4 = 0.0256$
W, W, W, [XYZ]	6, 5, or 3	$\frac{4!}{3!\,1!} = 4$	$4 \times 0.4^3 \times 0.6 = 0.1536$
W, W, X, X	4	$\frac{4!}{2!\,2!} = 6$	$6 \times 0.4^2 \times 0.2^2 = 0.0384$
W, W, X, [YZ]	3 or 1	$\frac{4!}{2!\,1!\,1!} = 12$	$12 \times 0.4^2 \times 0.2 \times 0.4 = 0.1536$
W, W, Y, Y	2	$\frac{4!}{2!\,2!} = 6$	$6 \times 0.4^2 \times 0.1^2 = 0.0096$
W, X, X, X	2	$\frac{4!}{1!\,3!} = 4$	$4 \times 0.4 \times 0.2^3 = 0.0128$
W, X, X, Y	1	$\frac{4!}{1!\,2!\,1!} = 12$	$12 \times 0.4 \times 0.2^2 \times 0.1 = 0.0192$
Total	—	—	$0.4128 = \mathbf{41.28}\,\%$