AUSTRALIAN MATHS TRUST

2020 Australian Mathematical Olympiad

DAY 1
Tuesday, 4 February 2020
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

2. Amy and Bec play the following game. Initially, there are three piles, each containing 2020 stones. The players take turns to make a move, with Amy going first. Each move consists of choosing one of the piles available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to make a move.

   Prove that Bec can always win the game, no matter how Amy plays.

3. Let ABC be a triangle with $\angle ACB = 90°$. Suppose that the tangent line at C to the circle passing through A, B, C intersects the line AB at D. Let E be the midpoint of CD and let F be the point on the line EB such that AF is parallel to CD.

   Prove that the lines AB and CF are perpendicular.

4. Define the sequence $A_1, A_2, A_3, \ldots$ by $A_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$A_{n+1} = \frac{A_n + 2}{A_n + 1}.$$

   Define the sequence $B_1, B_2, B_3, \ldots$ by $B_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$B_{n+1} = \frac{B_n^2 + 2}{2B_n}.$$

   Prove that $B_{n+1} = A_{2^n}$ for all non-negative integers n.

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AUSTRALIAN MATHS TRUST

2020 Australian Mathematical Olympiad

DAY 2
Wednesday, 5 February 2020
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Each term of an infinite sequence $a_1, a_2, a_3, \ldots$ is equal to 0 or 1. For each positive integer n,

   (i) $a_n + a_{n+1} \neq a_{n+2} + a_{n+3}$, and

   (ii) $a_n + a_{n+1} + a_{n+2} \neq a_{n+3} + a_{n+4} + a_{n+5}$.

   Prove that if $a_1 = 0$, then $a_{2020} = 1$.

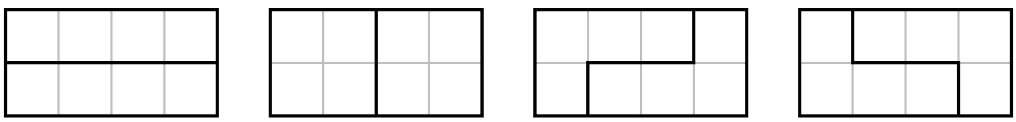
6. Let $ABCD$ be a square. For a point P inside $ABCD$, a *windmill* centred at P consists of two perpendicular lines ℓ_1 and ℓ_2 passing through P, such that

   - ℓ_1 intersects the sides AB and CD at W and Y, respectively, and
   - ℓ_2 intersects the sides BC and DA at X and Z, respectively.

   A windmill is called *round* if the quadrilateral $WXYZ$ is cyclic.

   Determine all points P inside $ABCD$ such that every windmill centred at P is round.

7. A *tetromino tile* is a tile that can be formed by gluing together four unit square tiles, edge to edge. For each positive integer n, consider a bathroom whose floor is in the shape of a $2 \times 2n$ rectangle. Let T_n be the number of ways to tile this bathroom floor with tetromino tiles. For example, $T_2 = 4$ since there are four ways to tile a 2×4 rectangular bathroom floor with tetromino tiles, as shown below.



   Prove that each of the numbers $T_1, T_2, T_3, \ldots$ is a perfect square.

8. Prove that for each integer k satisfying $2 \leq k \leq 100$, there are positive integers $b_2, b_3, \ldots, b_{101}$ such that

$$b_2^2 + b_3^3 + \cdots + b_k^k = b_{k+1}^{k+1} + b_{k+2}^{k+2} + \cdots + b_{101}^{101}.$$

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2020 Australian Mathematical Olympiad Solutions

1. Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

Solution 1 (Chris Wetherell)

Without loss of generality, we assume that $a \geq b$. Via the AM-GM inequality, AM $\geq$ GM, we must have

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

Doubling both sides and factorising gives

$$\begin{aligned} a - 2\sqrt{ab} + b &= 2 \\ (\sqrt{a} - \sqrt{b})^2 &= 2 \\ \sqrt{a} - \sqrt{b} &= \sqrt{2}. \end{aligned}$$

We ignore the negative root since $a \geq b$.

Solving for a and rearranging gives

$$\begin{aligned} \sqrt{a} &= \sqrt{b} + \sqrt{2} && (\dagger) \\ a &= b + 2\sqrt{2b} + 2 \\ 2\sqrt{2b} &= a - b - 2, \end{aligned}$$

hence $2\sqrt{2b}$ must be a non-negative integer, equal to $k \geq 0$, say.

Squaring gives $8b = k^2$ and it follows that k must be a multiple of 4, equal to $4n$, say. Hence $8b = (4n)^2 = 16n^2$, so $b = 2n^2$, where $n \geq 0$.

Substituting into $(\dagger)$ gives

$$\sqrt{a} = \sqrt{2n^2} + \sqrt{2} = n\sqrt{2} + \sqrt{2} = (n+1)\sqrt{2},$$

hence $a = 2(n+1)^2$.

Therefore the pair $\{a, b\}$ must be of the form $\{2(n+1)^2, 2n^2\}$ for some integer $n \geq 0$.

Finally, we verify that all such pairs do indeed have AM and GM which differ by 1:

$$\text{AM} = \frac{2(n+1)^2 + 2n^2}{2} = (n+1)^2 + n^2 = 2n^2 + 2n + 1,$$

and

$$\text{GM} = \sqrt{2(n+1)^2 \times 2n^2} = 2n(n+1) = 2n^2 + 2n = \text{AM} - 1.$$

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Solution 2 (Mike Clapper)

Let the two integers be $n+m$ and $n-m$, which have $AM = n$ and $GM = \sqrt{n^2 - m^2}$. We require that $n-1 = \sqrt{n^2 - m^2}$.

Squaring both sides and rearranging, we get $m^2 = 2n - 1$. This tells us that m is odd, so we can choose any odd number m and we get the two numbers $a = \frac{m^2+1}{2} + m = \frac{(m+1)^2}{2}$ and $b = \frac{m^2+1}{2} - m = \frac{(m-1)^2}{2}$ which satisfy the required condition.

Solution 3 (Alice Devillers)

Reorganising the equation we get $2\sqrt{ab} = a + b - 2$ thus after squaring: $4ab = a^2 + b^2 + 4 + 2ab - 4a - 4b$. Hence $0 = a^2 + b^2 + 4 - 2ab - 4a - 4b = (a-b)^2 + 4(1-a-b)$, so 4 divides $(a-b)^2$. It follows that $a-b$ is even and hence $1-a-b$ is odd. Thus $a - b = 4i + 2$ for some i. Then $(2i+1)^2 = a + b - 1 = 2b + 4i + 1$, and so $b = 2i^2$ and $a = 2i^2 + 4i + 2 = 2(i+1)^2$.

Solution 4 (Angelo Di Pasquale)

WLOG $a \leq b$. We have $\frac{a+b}{2} = \sqrt{ab} + 1$. Hence $\sqrt{ab}$ is rational. Since it is rational and a, b are integers, it follows that $\sqrt{ab}$ is an integer. Let $a = m^2k$ where k is the square-free part of a. Then $b = n^2k$ for some positive integer $n \geq m$. Putting this in yields

$$\frac{m^2k + n^2k}{2} = mnk + 1 \quad \Leftrightarrow \quad (n-m)^2k = 2.$$

It follows that $n = m + 1$ and $k = 2$. So $(a, b) = (2m^2, 2(m+1)^2)$ which works.

Solution 5 (Angelo Di Pasquale)

(Shows that a rather mechanical solution can be found)

WLOG $a \leq b$ so that $a = b + d$ for some integer $d \geq 0$. Substitution yields

$$\frac{2a + d}{2} = \sqrt{a(a+d)} + 1 \quad \Leftrightarrow \quad 8a = (d-2)^2.$$

Thus d is even. Put $d = 2e$ to find $2a = (e-1)^2$. Thus e is odd. Put $e = 2f + 1$ to find $a = 2f^2$ and then $b = a + d = 2(f+1)^2$, which is easy to verify.

Solution 6 (Andrew Elvey Price)

Without loss of generality, $a \geq b$. Squaring both sides of $(a+b)/2 - 1 = \sqrt{ab}$ then rearranging yields

$$(a-b)^2 = 4a + 4b - 4.$$

It follows that $a-b$ is even, so we write $a - b = 2n$. Then $a + b = n^2 + 1$. Since a and b have the same parity is follows that $n = 2m + 1$ for some integer m. Solving these equations for a, b in terms of m yields $a = 2(m+1)^2$ and $b = 2m^2$. Finally we can check that this satisfies the conditions of the question for all non-negative integers m.

Solution 7 (Ivan Guo)

Since $\sqrt{ab}$ is rational, it must be an integer and ab is a perfect square. Write $g = gcd(a, b)$. Since g divides $(a+b) - 2\sqrt{ab} = 2$, we must have $g = 1$ or 2. If it is 1, then both a and b are perfect squares. But this contradicts $(\sqrt{a} - \sqrt{b})^2 = 2$ as 2 is not a perfect square. So $g = 2$. Then we can write $a = 2c, b = 2d$ (where c and d are coprime) to obtain $1 = c + d - 2\sqrt{cd} = (\sqrt{c} - \sqrt{d})^2$. Hence c and d must be perfect squares with a difference of 1, which lead to the required solution.

Solution 8 (Dan Mathews)

First note that $\sqrt{ab} = (a+b)/2 - 1$ must be an integer or half integer; but the square root of an integer is never a half integer, hence ab is a perfect square. If a, b have a prime factor $p > 2$ in common, then it must divide $\sqrt{ab}$ and $(a+b)/2$, but not 1, a contradiction. So the only possible common prime factor of a, b is 2. As ab is a perfect square then either $a = m^2, b = n^2$, or or $a = 2m^2, b = 2n^2$, for some non-negative m, n.

If $a = m^2$ and $b = n^2$, then $(a+b)/2 = \sqrt{ab} + 1$ simplifies to $(m-n)^2 = 2$, which has no solution in integers.

If $a = 2m^2$ and $b = 2n^2$, then $(a+b)/2 = \sqrt{ab} + 1$ simplifies to $(m-n)^2 = 1$, so $m = n \pm 1$. We verify that $a = 2m^2, b = 2(m \pm 1)$ is indeed a solution.

Solution 9 (Kevin McAvaney)

From $(a+b)/2 = 1 + 2\sqrt{ab}$, we have

$$a^2 - (4+2b)a + (b-2)^2 = 0.$$

Solving the quadratic for a gives

$$a = (2+b) \pm \sqrt{8b}.$$

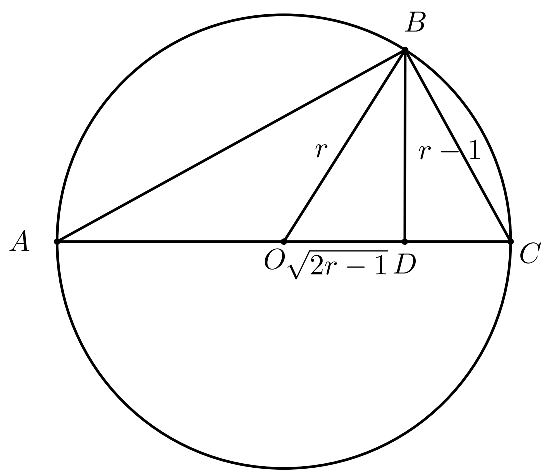
Since a is an integer, $b = 2c^2$ for any non-negative integer c, and

$$a = 2(c^2 \pm 2c + 1) = 2(c+1)^2 or 2(c-1)^2.$$

Solution 10 (Thanom Shaw)

Consider the line segment AC of length $a+b$ and a point D on the line segment such that $AD = a$ and $DC = b$. Let the perpendicular to AC through D meet the circle with diameter AC at B. We have that the radius of the circle is the arithmetic mean of a and b, and the length of BD is the geometric mean of a and b. (This can be shown using the similarity of triangles ABD and BCD).

Now, let the arithmetic mean be r, the radius of the circle, and the geometric mean one less, $r-1$. Geometrically, this is shown in the diagram below with $OB = r$ and $BD = r-1$. Applying Pythagoras' Theorem, we get that $OD = \sqrt{2r-1}$.



We then have that $AD = a = r + \sqrt{2r-1}$ and $DC = b = r - \sqrt{2r-1}$ (if $a > b$, say). But a and b are integers which means (the odd number) $2r-1$ must be a perfect square. Hence $2r - 1 = (2k+1)^2$ for some integer k and so $r = 2k^2 + 2k + 1$.

Hence, the pair of integers $(2k^2 + 2k + 1,\, 2k^2 + 2k)$ for all integers k are the only pairs of integers which have their arithmetic mean 1 more than their geometric mean.

Solution 11 (Ian Wanless)

(which may amount to the same thing, but feels different):

$$(a+b)/2 - 1 = \sqrt{ab}.$$

Double both sides, and square to get

$$a^2 + b^2 + 4 + 2ab - 4a - 4b = 4ab.$$

Hence

$$(a-b)^2 = 4(a+b-1)$$

Considering this mod 4 we see that $a - b$ is even. Hence $a + b$ is even so

$$(a-b)^2 \equiv 4 \pmod 8.$$

Thus $a - b \equiv 2 \pmod 4$. Let $t = a - b$. Then $a + b = t^2/4 + 1$. So $a = (t^2/4 + t + 1)/2$ and $b = (t^2/4 - t + 1)/2$ (both of which are integers, given that $t \equiv 2 \pmod 4$). Moreover, it is easy to check that a, b are solutions. So we have a parametrised family of solutions.

2. Amy and Bec play the following game. Initially, there are three piles, each containing 2020 stones. The players take turns to make a move, with Amy going first. Each move consists of choosing one of the piles available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to make a move.

   Prove that Bec can always win the game, no matter how Amy plays.

   Solution (Angelo Di Pasquale)

   Call a pile *perilous* if the number of stones in it is one more than a multiple of three, and *safe* otherwise. Ben has a winning strategy by ensuring that he only leaves Amy perilous piles. Ben wins because the number of stones is strictly decreasing, and eventually Amy will be left with two or three piles each with just one stone.

   To see that this is a winning strategy, we prove that Ben can always leave Amy with only perilous piles, and that under such circumstances, Amy must always leave Ben with at least one safe pile.

   On Amy's turn, whenever all piles are perilous it is impossible to choose one such perilous pile and divide it into two or three perilous piles by virtue of the fact that $1+1 \not\equiv 1 \pmod 3$ and $1+1+1 \not\equiv 1 \pmod 3$. Thus Amy must leave Ben with at least one safe pile.

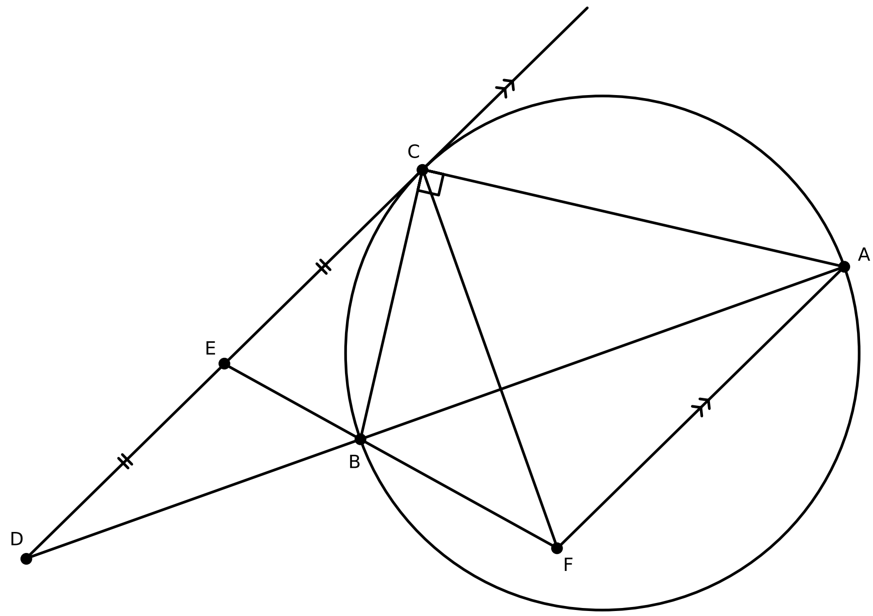
   On Ben's turn, whenever one of the piles is safe, he can divide it into two or three piles, each of which are safe, by virtue of the fact that $2 \equiv 1+1 \pmod 3$ and $0 \equiv 1+1+1 \pmod 3$.

3. Let ABC be a triangle with $\angle ACB = 90°$. Suppose that the tangent line at C to the circle passing through A, B, C intersects the line AB at D. Let E be the midpoint of CD and let F be the point on the line EB such that AF is parallel to CD.

   Prove that the lines AB and CF are perpendicular.

Solution 1 (Alan Offer)

Let BC and AF meet at H. Since CD and AH are parallel, triangles BCE and BHF are similar, and so too are triangles BDE and BAF. Hence, $AF/DE = BF/BE = FH/EC = FH/DE$, where the last equation holds since E is the midpoint of CD. It follows that $AF = FH$.



Since $\angle ACH$ is a right angle, the circumcircle of triangle ACH has AH as diameter and so F as centre. Hence, $FC = FH$ as these are both radii. It follows that triangle CFH is isosceles, so $\angle BCF = \angle BHF = \angle BCD = \angle BAC$, where the second equation follows from the alternate angle theorem and the last equation from the the alternate segment theorem.

Letting G be the point of intersection between AB and CF, it follows that triangles ABC and CBG are similar, as two pairs of corresponding angles are equal. Hence, $\angle BGC = \angle BCA$, which is a right angle, as required.

Solution 2 (Alice Devillers)

The diameter of the circumcircle is AB, and WLOG we can assume the radius is 1. So we can set $A = (-1, 0)$, $B = (1, 0)$ and $C = (\cos\alpha, \sin\alpha)$ where $\alpha \in (0, \pi) \setminus \{\pi/2\}$. We then successively get (some computations omitted, use parametric form of lines) $D = (\sec\alpha, 0)$, $E = (\frac{1}{2}\sec\alpha + \frac{1}{2}\cos\alpha, \frac{1}{2}\sin\alpha)$, $F = (\cos\alpha, \frac{\cos\alpha\sin\alpha}{\cos\alpha - 1})$. Hence C and F have the same first component, and so CF is orthogonal to AB.

Solution 3 (Angelo Di Pasquale)

(Trying to prove that CB bisects $\angle ECF$ via angle bisector theorem.)

Let G be the intersection of lines AB and CF. Since E is the midpoint of CD and $AF \parallel CD$, it follows that lines $(FD, FC; FE, FA)$ form a harmonic pencil. Hence $(D, G; B, A)$ are harmonic points. Thus the circle with diameter AB is a circle of Apollonius for points G and D. Hence AC is the external bisector of $\angle DCG$. Let Z be any point on the extension of ray DC beyond C. Using $CD \parallel AF$, we find

$$\angle FCA = \angle ACZ = \angle CAF.$$

Hence $FC = FA$. Since $\triangle BED \sim \triangle BAF$ we compute

$$\frac{EB}{BF} = \frac{DE}{FA} = \frac{EC}{FC}.$$

Hence by the angle bisector theorem, BC bisects $\angle ECF$. Let $\angle BCF = \angle ECB = x$. By the alternate segment theorem we have $\angle CAB = x$. The angle sum in $\triangle ABC$ yields $\angle ABC = 90° - x$. Finally, using the angle sum in $\triangle BCG$ yields $\angle CGB = 90°$.

Solution 4 (Angelo Di Pasquale)

Menelaus' theorem applied to $\triangle CDG$ with transversal line EBF yields

$$\frac{GB}{BD} \cdot \frac{DE}{EC} \cdot \frac{CF}{GF} = 1. \tag{1}$$

Note that
$$\frac{CF}{GF} = \frac{CG + GF}{GF} = \frac{CG}{GF} + 1 = \frac{DG}{GA} + = \frac{DG + GA}{GA} = \frac{DA}{GA}.$$

Putting this into (1), using $DE = EC$, and rearranging yields

$$GB \cdot DA = GA \cdot BD. \tag{2}$$

Let O be the centre of circle ABC, and r its radius. Then (2) is equivalent to

$$(r - OG)(r + OD) = (r + OG)(OD - r) \quad \Leftrightarrow \quad OG \cdot OD = r^2 \quad \Leftrightarrow \quad \frac{OG}{OC} = \frac{OC}{OD}.$$

The last equality above implies $\triangle OGC \sim \triangle OCD$ (PAP). Hence $\angle OGC = \angle OCD = 90°$.

Solution 5 (Angelo Di Pasquale)

As in my first alternative solution, $(D, G; B, A)$ are harmonic points. Hence lines $(CD, CG; CB, CA)$ form a harmonic pencil. Projecting from C onto the circle shows that $ACBK$ is a harmonic quadrilateral, where K is the second intersection point of line CG with the circle. Hence $CB/CA = KB/KA$. Since $\angle AKB = 90°$ (AB is a diameter of the circle), we have $\triangle BCA \sim \triangle BKA$ (PAP). But since AB is common, these triangles are congruent and thus related by a reflection in AB. Since C and K are symmetric across AB, it follows that $CK \perp AB$, as desired.

Solution 6 (Angelo Di Pasquale)

(A faster version of previous alternative solution)

As in my first alternative solution, $(D, G; B, A)$ are harmonic points. So the polar of D with respect to circle ABC passes through G. But it also passes through C as DC is tangent to circle ABC at C. Hence the polar of D is the line CG. Since AB is a diameter of circle ABC, we have $AB \perp CG$.

Solution 7 (Angelo Di Pasquale)

(Reverse reconstruction)

Let F' be the intersection of the line through C that is perpendicular to AB with the line through A that is parallel to CD. Let G' be the intersection of CF' and AB. By the alternate segment theorem we have

$$\angle DCB = \angle CAB = 90^\circ - \angle G'CA = \angle BCG'.$$

Hence BC bisects $\angle DCG'$.

From the angle sum in $\triangle CDG'$, we have $\angle G'DC = 90^\circ - 2x$. And since $AF' \parallel CD$, we have $\angle DAF' = 90^\circ - 2x$. Hence $\angle CAF' = 90^\circ - x = \angle F'CA$. Thus $CF' = AF'$.

Aiming to apply Menelaus' theorem to $\triangle CDG'$ we compute

$$\frac{DE}{EC} \cdot \frac{CF'}{G'F'} \cdot \frac{G'B}{BD} = \frac{AF'}{G'F'} \cdot \frac{G'B}{BD} = \frac{CD}{CG'} \cdot \frac{AF'}{G'F'} = 1.$$

The first equality is due to $DE = EC$ and $CF' = AF'$, the second is due to $\triangle DCG \sim \triangle AF'G'$, and the last is from the angle bisector theorem in $\triangle CDG'$. Hence by Menelaus' theorem, points E, B, and F' are collinear. Hence $F' = F$, and $G' = G$, which implies the result.

Solution 8 (Ivan Guo)

Let $AF \cap CD$ be the point of infinity I. Let $H = AB \cap CF$ and let CF meet the circle again at G.

Since E is a midpoint, $(C, D; E, I) = -1$ (cross ratio, harmonic conjugates). Projecting them through F onto the line AB, we have $(H, D; B, A) = -1$. Projecting those through C onto the circle, we have $(G, C; B, A) = -1$. Since AB is a diameter, C and G must be symmetric about AB. Thus CG is perpendicular to AB as required.

Solution 9 (Angelo Di Pasquale)

Let P be the point where CF and AB meet, and let Q be the point 'at infinity' on the line CD. Since E is the midpoint of CD, the range (C, D, E, Q) is harmonic. Projecting from F, it follows that (P, D, B, A) is harmonic. Hence CF is the polar of D and so is perpendicular to the diameter AB.

Solution 10 (Chaitanya Rao)

Let O be the midpoint of AB. We firstly show that $\triangle AOF \sim \triangle DBC$. Note that since AF and CD are parallel, $\angle OAF = \angle BDC$ (alternate angles) and $\triangle AFB \sim \triangle DEB$ (two pairs of alternate angles). Then

$$\begin{aligned}\frac{AF}{AO} &= \frac{2AF}{AB} \quad \text{(as } O \text{ is the midpoint of } AB\text{)}\\ &= \frac{2DE}{DB} \quad \text{(as } \triangle AFB \sim \triangle DEB\text{)}\\ &= \frac{DC}{DB} \quad \text{(as } E \text{ is the midpoint of } CD\text{).}\end{aligned}$$

As two sides of $\triangle AOF$ and $\triangle DBC$ have lengths in the same ratio ($\frac{AF}{AO} = \frac{DC}{DB}$) and the angles between them are equal ($\angle OAF = \angle BDC$), $\triangle AOF$ and $\triangle DBC$ are similar.

Let FO extended meet AC at P. Then

$$\begin{aligned}\angle CAB &= \angle DCB \quad \text{(alternate segment theorem).}\\ &= \angle AFO \quad \text{(as } \triangle AOF \sim \triangle DBC\text{)}\\ &= \angle AFP \quad \text{(by construction of } P\text{)}\end{aligned}$$

Next, through an angle chase we find

$$\begin{aligned}\angle FAP &= \angle FAB + \angle BAC\\ &= \angle BDC + \angle BCE \quad \text{(alternate angles and alternate segment theorem)}\\ &= \angle ABC \quad \text{(exterior angle equals the sum of interior opposite angles of } \triangle BCD\text{).}\end{aligned}$$

It follows that $\triangle ABC \sim \triangle FAP$ (two pairs of corresponding angles are equal). Therefore, $\angle OPA = \angle FPA = \angle ACB = 90^\circ$ and we conclude that OP is parallel to BC. Then $\triangle AOP$ and $\triangle ABC$ are also similar (equal corresponding pairs of angles) and as O is the midpoint of AB, P is the midpoint of AC.

Since altitude FP bisects side AC, $\triangle ACF$ is isosceles with $\triangle FAP$ congruent to $\triangle FCP$. Hence $\triangle ABC$ is additionally similar to $\triangle FCP$ so corresponding sides have the same angle between them. As $AC \perp FP$ it follows that $AB \perp FC$ as required.

4. Define the sequence $A_1, A_2, A_3, \ldots$ by $A_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$A_{n+1} = \frac{A_n + 2}{A_n + 1}.$$

Define the sequence $B_1, B_2, B_3, \ldots$ by $B_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$B_{n+1} = \frac{B_n^2 + 2}{2B_n}.$$

Prove that $B_{n+1} = A_{2^n}$ for all non-negative integers n.

Solution 1 (Chris Wetherell)

The proof is by induction on n. Checking $n = 0$, we have

$$B_{0+1} = B_1 = 1 = A_1 = A_{2^0},$$

so the base case is true.

Define functions

$$f(x) = \frac{x+k}{x+1} \qquad \text{and} \qquad g(x) = \frac{x^2+k}{2x},$$

so that $A_{n+1} = f(A_n)$ and $B_{n+1} = g(B_n)$. We are required to prove that

$$g^{(n)}(1) = f^{(2^n-1)}(1) \quad \text{for all } n \geq 0.$$

(The base case $n = 0$ can be established in this setting by interpreting both $f^{(0)}(x)$ and $g^{(0)}(x)$ as the identity function, so that both sides of the equation above equal 1.)

To apply the inductive argument which follows, we also need to establish the case $n = 1$:

$$g^{(1)}(1) = g(1) = \frac{1^2 + k}{2 \times 1} = \frac{1+k}{1+1} = f(1) = f^{(2^1-1)}(1).$$

Next we prove that $g(f(x)) = f(f(g(x)))$:

$$\begin{aligned}
g(f(x)) &= \frac{f(x)^2 + k}{2f(x)} \\
&= \frac{\left(\dfrac{x+k}{x+1}\right)^2 + k}{2\dfrac{x+k}{x+1}} \\
&= \frac{(x+k)^2 + k(x+1)^2}{2(x+k)(x+1)} \\
&= \frac{(k+1)x^2 + 4kx + k(k+1)}{2x^2 + 2(k+1)x + 2k},
\end{aligned}$$

$$
\begin{aligned}
f(f(g(x)) &= \frac{f(g(x)) + k}{f(g(x)) + 1} \\
&= \frac{\dfrac{g(x)+k}{g(x)+1} + k}{\dfrac{g(x)+k}{g(x)+1} + 1} \\
&= \frac{(k+1)g(x) + 2k}{2g(x) + k + 1} \\
&= \frac{(k+1)\dfrac{x^2+k}{2x} + 2k}{2\dfrac{x^2+k}{2x} + k + 1} \\
&= \frac{(k+1)x^2 + 4kx + k(k+1)}{2x^2 + 2(k+1)x + 2k} \\
&= g(f(x)).
\end{aligned}
$$

Applying this identity m times, for some $m \geq 1$, we have

$$
\begin{aligned}
g(f^{(m)}(x)) &= g(\underbrace{f(f(f(\cdots f}_{m}(x)\cdots)))) \\
&= f^{(2)}(g(\underbrace{f(f(\cdots f}_{m-1}(x)\cdots)))) \\
&= f^{(4)}(g(\underbrace{f(\cdots f}_{m-2}(x)\cdots))) \\
&= \cdots \\
&= f^{2(m-1)}(g(f(x)) \\
&= f^{(2m)}(g(x)).
\end{aligned}
$$

Finally, assume by induction that $g^{(n)}(1) = f^{(2^n-1)}(1)$. Then, via the above generalised identity with $m = 2^n - 1$ and the fact that $f(1) = g(1)$, we have

$$
\begin{aligned}
g^{(n+1)}(1) &= g(g^{(n)}(1)) \\
&= g(f^{(2^n-1)}(1)) \\
&= f^{(2(2^n-1))}(g(1)) \\
&= f^{(2^{n+1}-2)}(f(1)) \\
&= f^{(2^{n+1}-1)}(1).
\end{aligned}
$$

This completes the proof.

Solution 2 (Alice Devillers)

It is obvious that all A_n and B_n are positive rational numbers. We claim that for all n if $A_n = \frac{c}{d}$ then $A_{2n} = \frac{c^2+2d^2}{2cd}$. This holds for $n = 1$ as $A_1 = 1$ and $A_2 = \frac{3}{2}$. Assume the claim holds for n we now show it holds for $n+1$. We easily compute $A_{n+1} = \frac{c/d+2}{c/d+1} = \frac{c+2d}{c+d}$, $A_{2n+1} = \frac{c^2+2d^2+4cd}{c^2+2d^2+2cd}$,

$$A_{2n+2} = A_{2(n+1)} = \frac{3c^2+6d^2+8cd}{2c^2+4d^2+6cd} = \frac{(c+2d)^2+2(c+d)^2}{2(c+2d)(c+d)},$$

which proves the claim. It is now very easy to prove what is asked in the question, also by induction. It is true for $n=0$ as $B_{0+1} = 1 = A_{2^0}$. Assume $B_{n+1} = A_{2^n} = \frac{c}{d}$, we show it is true for $n+1$:

$$B_{n+2} = \frac{B_{n+1}^2+2}{2B_{n+1}} = \frac{(c/d)^2+2}{2(c/d)} = \frac{c^2+2d^2}{2cd} = A_{2\times 2^n} = A_{2^{n+1}}$$

by the claim above.

Solution 3 (Angelo Di Pasquale)

We may introduce x_n and y_n which satisfy $A_n = \frac{x_n}{y_n}$ for all positive integers n. Using the recurrence $A_{n+1} = \frac{A_n+2}{A_n+1}$ we want x_n and y_n to satisfy

$$x_{n+1} = x_n + 2y_n \qquad \text{and} \qquad y_{n+1} = x_n + y_n.$$

Putting $x_n = y_{n+1} - y_n$ into the first recurrence and $y_n = \frac{1}{2}(x_{n+1} - x_n)$ into the second recurrence and tidying up yields

$$x_{n+2} - 2x_{n+1} - x_n \qquad \text{and} \qquad y_{n+2} - 2y_{n+1} - y_n$$

for each positive integer n. The initial values $x_1 = y_1 = 1$, $x_2 = 3$ and $y_2 = 2$ get us off the ground. Tracing the recurrence backwards to define $x_0 = 1$ and $y_0 = 0$ makes the computations a bit easier, but either way using standard recurrence methods we deduce

$$x_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2} \qquad \text{and} \qquad y_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}. \tag{1}$$

The conclusion of the problem statement gives us a clue as to what to do next. For example if $n=3$, we hope that

$$B_4 = \frac{B_3^2+2}{2B_n} \qquad \Leftrightarrow \qquad A_8 = \frac{A_4^2+2}{2A_4} \qquad \Leftrightarrow \qquad \frac{x_8}{y_8} = \frac{x_4^2+2y_4^2}{2x_4y_4}.$$

Checking a few small values for n seems to confirm that

$$x_{2n} = x_n^2 + 2y_n^2 \qquad \text{and} \qquad y_{2n} = 2x_ny_n. \tag{2}$$

It is straightforward to confirm that the equalities in (2) are true by using the equalities in (1). It follows from (2) that

$$A_{2n} = \frac{x_{2n}}{y_{2n}} = \frac{x_n^2+2y_n^2}{2x_ny_n} = \frac{A_n^2+2}{2A_n}. \tag{3}$$

The result now follows inductively from (3) because $B_1 = A_{2^0}$, and if $B_n = A_{2^{n-1}}$, then

$$B_{n+1} = \frac{B_n^2+2}{2B_n} = \frac{A_{2^{n-1}}^2+2}{2A_{2^{n-1}}} = A_{2^n},$$

as desired.

Solution 4 (Andrew Elvey Price)

Writing out the first few terms A_n, we recognise the numerators and denominators as the coefficients appearing in the expansion $(1+\sqrt{2})^n$, that is, $(1+\sqrt{2})^n = B_n + c_n\sqrt{2}$, where $A_n = B_n/c_n$. Based on this we conjecture that

$$A_n = \sqrt{2}\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{(1+\sqrt{2})^n - (1-\sqrt{2})^n}.$$

This can be proven by induction. The base case $A_1 = 1$ is true. for the inductive step, we assume the equation above and aim to prove the same equation with n replaced by $n+1$. $A_n + 1$ and $A_n + 2$ can be calculated as follows

$$A_n + 1 = \frac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^n - (1-\sqrt{2})^n},$$

and

$$A_n + 2 = \sqrt{2}\frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^n - (1-\sqrt{2})^n}.$$

It follows that

$$A_{n+1} = \frac{A_n + 2}{A_n + 1} = \sqrt{2}\frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}},$$

which completes the induction.

Now it suffices to prove that

$$B_n = \sqrt{2}\frac{(1+\sqrt{2})^{2^{n-1}} + (1-\sqrt{2})^{2^{n-1}}}{(1+\sqrt{2})^{2^{n-1}} - (1-\sqrt{2})^{2^{n-1}}}.$$

Again, we can prove this by induction. The base cases $B_1 = 1$ and $B_2 = 3/2$ can easily be checked. For the inductive step, we assume that the equation above holds for some $n \geq 2$, and we will show that it also holds for $n+1$. For convenience we will write $c = (1+\sqrt{2})^{2^{n-1}}$, then we have

$$B_n = \sqrt{2}\frac{c + c^{-1}}{c - c^{-1}},$$

and we want to prove that

$$B_{n+1} = \sqrt{2}\frac{c^2 + c^{-2}}{c^2 - c^{-2}}.$$

This follows immediately by substituting the formula for B_n into the equation

$$B_{n+1} = \frac{B_n^2 + 2}{2B_n}.$$

Solution 5 (Ivan Guo)

Here is a solution motivated by noticing that both sequences converge to $\sqrt{2}$ and the theory of Pell's equation. Define the integer sequences $\{p_n\}$ and $\{q_n\}$ by

$$p_k + q_k\sqrt{2} = (1+\sqrt{2})^k.$$

We claim that $A_n = p_n/q_n$ for all $n \geq 1$ and $B_{n+1} = p_{2^n}/q_{2^n}$ for all $n \geq 0$.

Both can be proven easily by induction. The base cases are clear. For the first claim, it suffices to note that

$$p_{k+1} + q_{k+1}\sqrt{2} = (p_k + q_k\sqrt{2})(1+\sqrt{2}) = (p_k + 2q_k) + (p_k + q_k)\sqrt{2},$$

which implies

$$A_{k+1} = \frac{A_k + 2}{A_k + 1} = \frac{p_k/q_k + 2}{p_k/q_k + 1} = \frac{p_k + 2q_k}{p_k + q_k} = \frac{p_{k+1}}{q_{k+1}}.$$

For the second claim, it suffices to note that

$$p_{2^k} + q_{2^k}\sqrt{2} = (p_{2^{k-1}} + q_{2^{k-1}}\sqrt{2})^2 = (p_{2^{k-1}}^2 + 2q_{2^{k-1}}^2) + 2p_{2^{k-1}}q_{2^{k-1}}\sqrt{2},$$

which implies

$$B_{k+1} = \frac{B_k^2 + 2}{2B_k} = \frac{(p_{2^{k-1}}/q_{2^{k-1}})^2 + 2}{2p_{2^{k-1}}/q_{2^{k-1}}} = \frac{p_{2^{k-1}}^2 + 2q_{2^{k-1}}}{2p_{2^{k-1}}q_{2^{k-1}}} = \frac{p_{2^k}}{q_{2^k}}.$$

Therefore $B_{n+1} = p_{2^n}/q_{2^n} = A_{2^n}$, as required.

Solution 6 (Daniel Mathews)

We observe that the sequence $A_n - 1$ is 0/1, 1/2, 2/5, 5/12, 12/29, which leads us to define a sequence f_n by $f_0 = 0$, $f_1 = 1$ and $f_n = 2f_{n-1} + f_{n-2}$ for $n \geq 2$. Indeed one can then verify that $A_n = 1 + f_{n-1}/f_n$ for $n \geq 1$. For if we define $g_n = 1 + f_{n-1}/f_n$, then $g_1 = 0 = A_1$, and

$$\begin{aligned}
(g_n + 2)/(g_n + 1) &= (3 + f_{n-1}/f_n)/(2 + f_{n-1}/f_n) \\
&= (3f_n + f_{n-1})/(2f_n + f_{n-1}) \\
&= 1 + f_n/f_{n+1} \quad (\text{since } f_{n+1} = 2f_n + f_{n-1}) \\
&= g_{n+1},
\end{aligned}$$

so that g_n and A_n satisfy the same recursion, hence $g_n = A_n$ for all n.

We can solve the linear recurrence for f_n by standard methods and we find

$$f_n = 2^{-3/2}[(1+\sqrt{2})^n - (1-\sqrt{2})^n].$$

Thus we have A_n as

$$A_n = 1 + [(1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}]/[(1+\sqrt{2})^n - (1-\sqrt{2})^n]$$

which simplifies to

$$A_n = \sqrt{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]/[(1+\sqrt{2})^n - (1-\sqrt{2})^n].$$

We now show that $A_{2^n} = B_{n+1}$. For $n = 1$ we verify $A_{2^1} = A_2 = 3/2 = B_2$. It remains to show that A_{2^n} satisfies the same recursion as B_{n+1}, i.e. that $A_{2^{n+1}} = (A_{2^n}^2 + 2)/(2A_{2^n})$.

We verify this directly: upon clearing denominators in

$$(A_{2^n}^2 + 2)/(2A_{2^n})$$

and cancelling constant factors we obtain

$$\frac{[((1+\sqrt{2})^{2^n}+(1-\sqrt{2})^{2^n})^2+((1+\sqrt{2})^{2^n}-(1-\sqrt{2})^{2^n})^2]}{[\sqrt{2}[(1+\sqrt{2})^{2^n}+(1-\sqrt{2})^{2^n}][(1+\sqrt{2})^{2^n}-(1-\sqrt{2})^{2^n}]]}.$$

The numerator simplifies using the identity $(x+y)^2+(x-y)^2=2x^2+2y^2$, and the denominator simplifies using the identity $(x+y)(x-y)=x^2-y^2$. We end up with

$$\sqrt{2}[(1+\sqrt{2})^{2^{n+1}}+(1-\sqrt{2})^{2^{n+1}}]/[(1+\sqrt{2})^{2^{n+1}}-(1-\sqrt{2})^{2^{n+1}}]$$

which is $A_{2^{n+1}}$ as desired.

Solution 7 (Alan Offer)

Define the sequences $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ by $p_1 = q_1 = 1$, $p_{n+1} = p_n + 2q_n$ and $q_{n+1} = p_n + q_n$. Then $A_1 = p_1/q_1$. Furthermore, if $A_n = p_n/q_n$ then

$$A_{n+1} = \frac{\frac{p_n}{q_n}+2}{\frac{p_n}{q_n}+1} = \frac{p_n+2q_n}{p_n+q_n} = \frac{p_{n+1}}{q_{n+1}}.$$

It follows by mathematical induction that $A_n = p_n/q_n$ for all $n \geq 1$.

Similarly, define sequences $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$ by $u_1 = v_1 = 1$, $u_{n+1} = u_n^2 + 2v_n^2$ and $v_{n+1} = 2u_nv_n$. Then $B_1 = u_1/v_1$. Furthermore, if $B_n = u_n/v_n$ then

$$B_{n+1} = \frac{\left(\frac{u_n}{v_n}\right)^2+2}{\frac{2u_n}{v_n}} = \frac{u_n^2+2v_n^2}{2u_nv_n} = \frac{u_{n+1}}{v_{n+1}}.$$

It follows by mathematical induction that $B_n = u_n/v_n$ for all $n \geq 1$.

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. We claim that $A^n = \begin{pmatrix} p_n & 2q_n \\ q_n & p_n \end{pmatrix}$ for $n \geq 1$. This certainly holds for $n = 1$. Suppose the claim holds for some n. Then

$$A^{n+1} = A \cdot A^n = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p_n & 2q_n \\ q_n & p_n \end{pmatrix} = \begin{pmatrix} p_n+2q_n & 2q_n+2p_n \\ p_n+q_n & 2q_n+p_n \end{pmatrix} = \begin{pmatrix} p_{n+1} & 2q_{n+1} \\ q_{n+1} & p_{n+1} \end{pmatrix}.$$

Thus the claim is true by mathematical induction.

We also claim that $A^{2^n} = \begin{pmatrix} u_{n+1} & 2v_{n+1} \\ v_{n+1} & u_{n+1} \end{pmatrix}$ for all $n \geq 0$. This certainly holds for $n = 0$. Suppose the claim holds for some n. Then

$$\begin{aligned} A^{2^{n+1}} = \left(A^{2^n}\right)^2 &= \begin{pmatrix} u_{n+1} & 2v_{n+1} \\ v_{n+1} & u_{n+1} \end{pmatrix}\begin{pmatrix} u_{n+1} & 2v_{n+1} \\ v_{n+1} & u_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} u_{n+1}^2+2v_{n+1}^2 & 4u_{n+1}v_{n+1} \\ 2u_{n+1}v_{n+1} & 2v_{n+1}^2+u_{n+1}^2 \end{pmatrix} \\ &= \begin{pmatrix} u_{n+2} & 2v_{n+2} \\ v_{n+2} & u_{n+2} \end{pmatrix}. \end{aligned}$$

Thus the claim is true by mathematical induction.

Therefore, for all non-negative integers n, we have

$$B_{n+1} = \frac{u_{n+1}}{v_{n+1}} = \frac{p_{2^n}}{q_{2^n}} = A_{2^n}.$$

Solution 8 (Chaitanya Rao)

After writing initial terms we derive a recurrence that defines the numerator and denominator of A_n and that leads us to the following conjecture:

The solution to the recurrence for A_n is

$$c_n := \sqrt{2}\frac{\alpha^n + \beta^n}{\alpha^n - \beta^n}$$

where $\alpha = (1+\sqrt{2})$ and $\beta = (1-\sqrt{2})$. To prove this, for $n = 1$,

$$c_1 = \sqrt{2}\frac{\alpha^1 + \beta^1}{\alpha^1 - \beta^1} \quad = \sqrt{2}\frac{2}{2\sqrt{2}} = 1$$

and

$$\begin{aligned}
\frac{c_n+2}{c_n+1} &= 1 + \frac{1}{c_n+1} \\
&= 1 + \frac{1}{\sqrt{2}\frac{\alpha^n+\beta^n}{\alpha^n-\beta^n} + 1} \\
&= 1 + \frac{\alpha^n - \beta^n}{\sqrt{2}(\alpha^n+\beta^n) + (\alpha^n - \beta^n)} \\
&= \frac{\sqrt{2}(\alpha^n+\beta^n) + (\alpha^n-\beta^n) + \alpha^n - \beta^n}{\sqrt{2}(\alpha^n+\beta^n) + (\alpha^n-\beta^n)} \\
&= \frac{\sqrt{2}\left[(1+\sqrt{2})\alpha^n + (1-\sqrt{2})\beta^n\right]}{(1+\sqrt{2})\alpha^n - (1-\sqrt{2})\beta^n} \\
&= \sqrt{2}\frac{\alpha^{n+1}+\beta^{n+1}}{\alpha^{n+1}-\beta^{n+1}} \\
&= c_{n+1}.
\end{aligned}$$

Hence c_n satisfies the first order recurrence as A_n with the same initial value and we conclude $c_n = A_n$ for $n = 1, 2, 3 \ldots$. Next we note that

$$\begin{aligned}
A_{2n} &= \sqrt{2}\frac{\alpha^{2n}+\beta^{2n}}{\alpha^{2n}-\beta^{2n}} \\
&= \sqrt{2}\frac{(\alpha^n+\beta^n)^2 + (\alpha^n-\beta^n)^2}{2(\alpha^n+\beta^n)(\alpha^n-\beta^n)} \\
&= \sqrt{2}\frac{\alpha^n+\beta^n}{2(\alpha^n-\beta^n)} + \frac{\alpha^n-\beta^n}{\sqrt{2}(\alpha^n+\beta^n)} \\
&= \frac{A_n}{2} + \frac{1}{A_n} \\
&= \frac{A_n^2+2}{2A_n}.
\end{aligned}$$

It follows that $A_{2^n} = \frac{A_{2^{n-1}}^2+2}{2A_{2^{n-1}}}$. If we define $d_n = A_{2^{n-1}}$ for $n = 1, 2, \ldots$ we find that $d_1 = A_1 = 1$ and $d_{n+1} = \frac{d_n^2+2}{2d_n}$. This is the same recurrence that defines B_n and we conclude that $B_{n+1} = d_{n+1} = A_{2^n}$ for non-negative integers n.

Solution 10 (Ian Wanless)

Let $X(n) = (1+\sqrt{2})^n$ and $Y(n) = (1-\sqrt{2})^n$. We first prove by induction that $A_n = \sqrt{2}(X(n)+Y(n))/(X(n)-Y(n))$. In the base case of $n=1$, our formula holds since $\sqrt{2}(X(1)+Y(1))/(X(1)-Y(1)) = \sqrt{2}(2)/(2\sqrt{2}) = 1 = A_1$. Now suppose that our formula hold for $n=k$. Then

$$\begin{aligned}
A_{k+1} = \frac{A_k+2}{A_k+1} &= \frac{\sqrt{2}\frac{X(k)+Y(k)}{X(k)-Y(k)}+2}{\sqrt{2}\frac{X(k)+Y(k)}{X(k)-Y(k)}+1} \\
&= \frac{(\sqrt{2}+2)X(k)+(\sqrt{2}-2)Y(k)}{(\sqrt{2}+1)X(k)+(\sqrt{2}-1)Y(k)} \\
&= \sqrt{2}\frac{(1+\sqrt{2})X(k)+(1-\sqrt{2})Y(k)}{(1+\sqrt{2})X(k)-(1-\sqrt{2})Y(k)} \\
&= \sqrt{2}\frac{X(k+1)+Y(k+1)}{X(k+1)-Y(k+1)}
\end{aligned}$$

which means that our formula holds when $n=k+1$. By induction the formula holds for all positive integers n.

Next we use another induction to show that $B_{n+1} = A_{2^n}$. The base case, $n=0$, works because we are told that $B_1 = A_1$. Assuming that $B_{k+1} = A_{2^k}$, we find that

$$\begin{aligned}
B_{k+2} = \frac{(B_{k+1})^2+2}{2B_{k+1}} = \frac{(A_{2^k})^2+2}{2A_{2^k}} &= \frac{\left(\sqrt{2}\frac{X(2^k)+Y(2^k)}{X(2^k)-Y(2^k)}\right)^2+2}{2\sqrt{2}\frac{X(2^k)+Y(2^k)}{X(2^k)-Y(2^k)}} \\
&= \frac{(X(2^k)+Y(2^k))^2+(X(2^k)-Y(2^k))^2}{\sqrt{2}(X(2^k)+Y(2^k))(X(2^k)-Y(2^k))} \\
&= \frac{2X(2^k)^2+2Y(2^k)^2}{\sqrt{2}(X(2^k)^2-Y(2^k)^2)} \\
&= \frac{\sqrt{2}(X(2^{k+1})+Y(2^{k+1}))}{X(2^{k+1})-Y(2^{k+1})} = A_{2^{k+1}}.
\end{aligned}$$

The required result follows by induction.

5. Each term of an infinite sequence $a_1, a_2, a_3, \ldots$ is equal to 0 or 1. For each positive integer n,

   (i) $a_n + a_{n+1} \neq a_{n+2} + a_{n+3}$, and

   (ii) $a_n + a_{n+1} + a_{n+2} \neq a_{n+3} + a_{n+4} + a_{n+5}$.

   Prove that if $a_1 = 0$, then $a_{2020} = 1$.

Solution 1 (Jamie Simpson)

We show that $a_{2020} = 1$.

We will call $a_i + a_{i+1} \neq a_{i+2} + a_{i+3}$ and $a_i + a_{i+1} + a_{i+2} \neq a_{i+3} + a_{i+4} + a_{i+5}$ rules A and B, respectively. By Rule A we cannot have 4 consecutive terms of the sequence equal to each other, so the sequence must begin $0, 1$ or $0, 0, 1$ or $0, 0, 0, 1$.

Suppose first that $a_0 = 0$ and a_1=1. Since the sequence begins $0, 1$ Rule A means that a_3 and a_4 are equal. Suppose first that they both equal 0, so the sequence begins $0, 1, 0, 0$. Rule A implies the next members are 0 then 1 so the sequence begins $0, 1, 0, 0, 0, 1$; but this is impossible by Rule B.

We conclude that the sequence begins $0, 1, 1, 1$. Next must come 0 by rule A then 0 by rule B giving $0, 1, 1, 1, 0, 0$. Then 0 and then 1 by rule A giving $0, 1, 1, 1, 0, 0, 0, 1$. Then 1 by Rule B, then 1 by rule A giving $0, 1, 1, 1, 0, 0, 0, 1, 1, 1$. Now the process can be repeated so that

$$a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}, a_{i+5} = 1, 1, 1, 0, 0, 0$$

whenever $i \equiv 2 \mod 6$. Since $2018 \equiv 2 \mod 6$ we have $a_{2018}, a_{2019}, a_{2020} = 1, 1, 1$. So $a_{2020} = 1$.

If the sequence begins $0, 0, 1$ then similar reasoning means that

$$a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}, a_{i+5} = 1, 1, 1, 0, 0, 0$$

whenever $i \equiv 3 \mod 6$. Since $2019 \equiv 3 \mod 6$ we have $a_{2019}, a_{2020}, a_{2021} = 1, 1, 1$. So again $a_{2020} = 1$. And if the sequence begins $0, 0, 0, 1$ then

$$a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}, a_{i+5} = 1, 1, 1, 0, 0, 0$$

whenever $i \equiv 4 \mod 6$. Since $2020 \equiv 4 \mod 6$ we have $a_{2020}, a_{2021}, a_{2022} = 1, 1, 1$ and once again $a_{2020} = 1$.

Solution 2 (Alice Devillers)

I first proved a few lemmas, all of which quite easy to prove using rules A and B. Lemma 1: no 4 consecutive of the same digit (immediate by A). Lemma 2: no 101 and no 010 (by contradiction, use A 3 times then B). Lemma 3: no 1001 and no 0110 (immediate by A). Then it follows that once you get to 1 in the sequence, it must go 111000 with a period 6. At the start we get either 1,2, or 3 zeros, then 111000 and repeat. Thus in all three cases, the sequence has a period of 6. Since $2020 \equiv 4 \pmod 6$, a_{2020} is the same as a_4 and in all three cases $a_4 = 1$.

Solution 3 (Angelo Di Pasquale)

Define *Rule 1*: $a_i + a_{i+1} \neq a_{i+2} + a_{i+3}$, and *Rule 2*: $a_i + a_{i+1} + a_{i+2} \neq a_{i+3} + a_{i+4} + a_{i+5}$.

The subsequence 010 is forbidden because tracing it forward (or backward) using rule 1 yields 010001 which violates rule 2. Since the transformation $x \to 1 - x$ preserves rules 1 and 2, it follows that the subsequence 101 is also forbidden.

Also the subsequences 0110, 1001, 1111 and 0000 are forbidden by rule 1.

Since there must be 1 somewhere near the middle of the sequence (to avoid 0000), it must be part of 111. To avoid the six forbidden subsequences, the 111 subsequence uniquely extends in each direction as a subsequence of the doubly infinite sequence $\ldots 000111000111000111 \ldots$ until we reach a_1 and a_{2020}. So the sequence has antiperiod 3. As $3 \mid 2019$, we have $a_1 \neq a_{2020}$, from which the result follows.

Solution 4 (Angelo Di Pasquale)

As in alternative solution 1, the six subsequences 010, 101, 0110, 1001, 1111 and 0000 are forbidden.

There is no way to replace the symbols u and v in $1uv1$ or in $0uv0$ without it containing one of the six forbidden subsequences. This shows that the sequence has antiperiod 3, and we conclude as in solution 1.

Solution 5 (Andrew Elvey Price)

We start by showing that there is no consecutive sequence 010. Suppose for the sake of contradiction that such a sequence occurs in the first 2010 terms. Then by successive applications of rule A, the following three terms of the sequence must be 001, but this contradicts rule B. Similarly, if the sequence 010 occurs in the last 20 terms, it must be preceeded by 100, which again contradicts rule B.

Note that by symmetry there is no sequence 101.

Now we will show that two terms of the sequence with exactly two terms separating them must be different i.e., $a_{i+3} = 1 - a_i$. Suppose for the sake of contradiction that they are the same. WLOG we assume that they are both 0's, so we have a sequence 0??0. By rule A the two numbers denoted by ? must be different, however this contradicts rule C.

We have proven that $a_{i+3} = 1 - a_i$ for all i. Hence,

$$a_{2020} = 1 - a_{2017} = a_{2014} = \ldots = 1 - a_1 = 1.$$

Solution 6 (Ivan Guo)

We first prove that $a_i \neq a_{i+3}$ by contradiction. WLOG, they are both 0 (otherwise we can take $b_i = 1 - a_i$ and argue the same way) and $i + 6 \leq 2020$ (otherwise we can reverse the sequence). Then there are four possibilities for the terms $a_i, \ldots, a_{i+3}$:

$$0000, 0110, 0100, 0010.$$

The first two are clearly bad by rule A. If we have 0100, the next two are forced by rule A to be 010001, but then this violates rule B. If we have 0010, the next three are forced by rule A to be 0010001 which again violates rule B. So we have the required contradiction. Hence the sequence alternates every 3 terms. Therefore $a_{2020} = 1 - a_{2017} = 1 - a_1 = 0$ as 2017 is 1 mod 6.

Solution 7 (Daniel Mathews)

First, there can never be three consecutive terms 0,1,0. For if there were, then the next term must be 0 (else 0+1=0+1), then 0 (else 1+0=0+1), then 1 (Else 0+0=0+0). But then we have six consecutive terms 0,1,0,0,0,1, a contradiction since 0+1+0=0+0+1.

The same argument with 0s and 1s reversed shows there can never be any three consecutive terms 1,0,1.

Consider the sequence as consisting of a block of 0s, then a block of 1s, then a block of 0s, and so on. If a block (other than the first) has size 1, then we have 3 consecutive terms 101 or 010, a contradiction. If a block (other than the first) has size 2, then we have 4 consecutive terms 0110 or 1001, a contradiction since 0+1=1+0 and 1+0=0+1. And if a block has size 4 or more, then we have 4 consecutive terms 0000 or 1111, a contradiction since 0+0=0+0 and 1+1=1+1.

Thus the first block has length at most 3, and every subsequent block has length exactly 3. Accordingly, for all i, a_{i+3} is different from a_i. Since 2020 - 1 = 2019 is an odd multiple of 3, a_1 is different from a_{2020}. So $a_1 = 0$ implies $a_{2020} = 1$.

Solution 8 (Kevin McAvaney)

Assume $a_{2020} = 0$. Then, using the given rules and working backwards from a_{2020}, we have the following sequence endings up to a_{2020}. The question marks indicate no possible term. In cases 2, 4, 6, the sequence 111000 recurs backwards.

```
1.        ?0001000
2.    111000111000
3.         ?000100
4.     11100011100
5.          ?00010
6.      1110001110
```

Since $2020 = 6*336 + 4$, sequence 2 starts with $1000111000\cdots$, sequence 4 starts with $11000111000\cdots$, sequence 6 starts with $111000111000\cdots$. In each case $a_1 = 1$, a contradiction.

Solution 9 (Alan Offer)

Refer to the two conditions as (1) and (2).

With the intention of arriving at a contradiction, suppose $a_{i+3} = a_i$. Then $a_{i+1} \neq a_{i+2}$ by (1), so $a_{i+1}+a_{i+2} = 1$. Hence $a_{i+3}+a_{i+4} \neq 1$, so by (1) again, we have $a_{i+4} = a_{i+3} = a_i$.

As exactly one of a_{i+1} and a_{i+2} is equal to a_i, while both of a_{i+3} and a_{i+4} are equal to a_i, it follows from (2) that $a_{i+5} = a_i$ as well.

Now exactly two of a_{i+1}, a_{i+2} and a_{i+3} are equal to a_i, while both of a_{i+4} and a_{i+5} are equal to a_i, so again by (2), it follows that $a_{i+6} = a_i$.

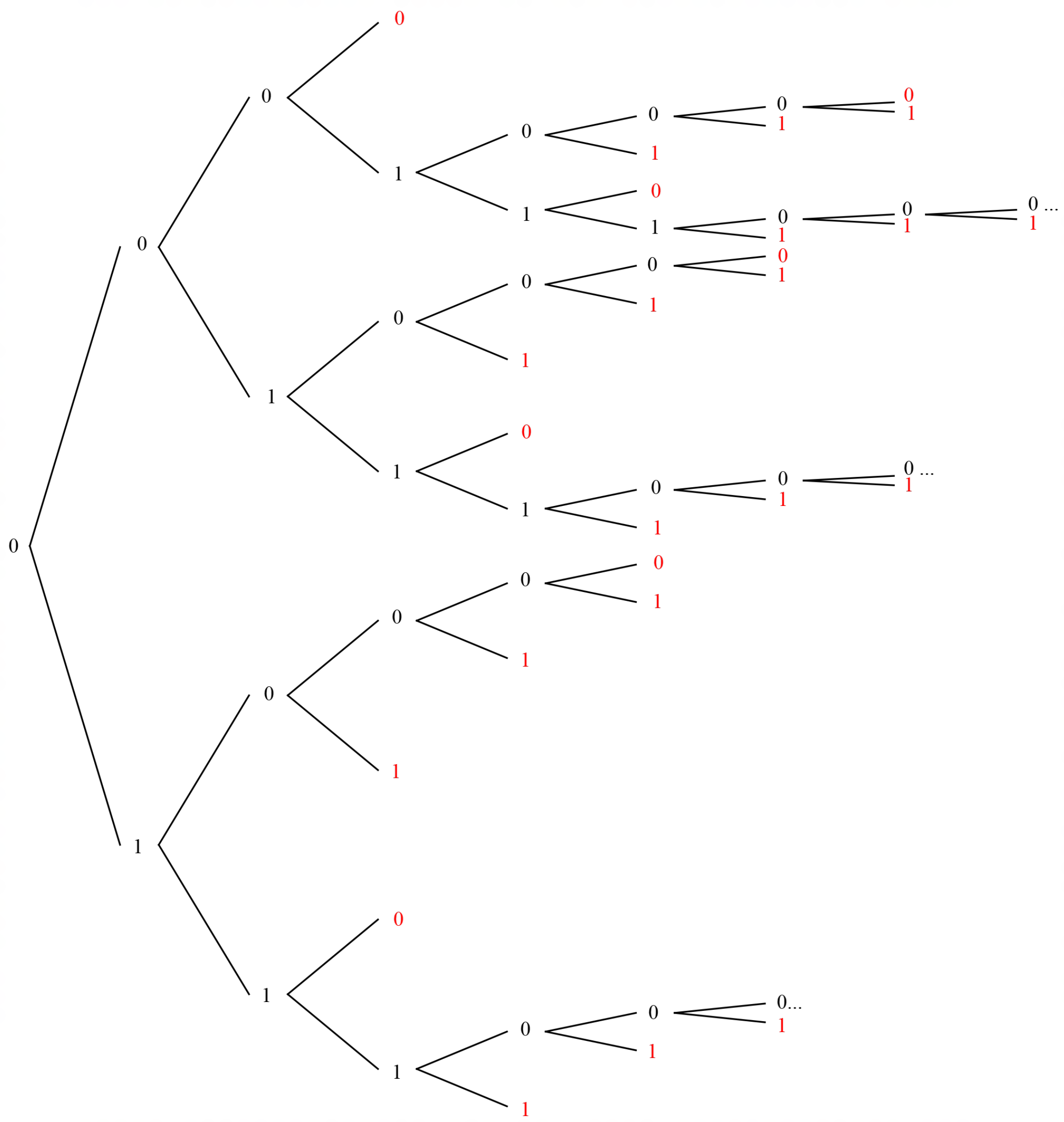
But $a_{i+3} = a_{i+4} = a_{i+5} = a_{i+6}$ is contrary to (1). Thus the original assumption is false.

Hence $a_{i+3} \neq a_i$. In particular, the terms $a_1, a_4, a_7, a_{10}, \ldots$ alternate, with $a_i = a_1$ for $i \equiv 1 \pmod 6$ and $a_i \neq a_1$ for $i \equiv 4 \pmod 6$. As $2020 \equiv 4 \pmod 6$ and $a_1 = 0$, it follows that $a_{2020} = 1$.

Solution 10 (Thanom Shaw)

I think it's somehow easier to see everything in a tree diagram, so I've drawn one up to help see the solution to this problem.

The tree diagram below shows the options for each term in the sequence starting with $a_1 = 0$. A 0 or 1 in red indicates that that term breaks one of the given conditions and so such a sequence is would not be valid.



From the tree diagram we can see the only valid sequences begin:

$$0, 0, 0, 1, 1, 1, 0, 0, 0, \ldots$$
$$0, 0, 1, 1, 1, 0, 0, 0, \ldots$$
$$0, 1, 1, 1, 0, 0, 0, \ldots$$

Finally, note that a sequence containing the six consecutive terms $1, 1, 1, 0, 0, 0$ can only continue repeating this sequence $1, 1, 1, 0, 0, 0$.

Hence the only sequences satisfying both conditions begin with one, two, or three 0s, and then repeating $1, 1, 1, 0, 0, 0$.

Consider now the 2020th term. Since

$$2020 - 1 \equiv 2019 \equiv 3 \bmod 6,$$
$$2020 - 2 \equiv 2018 \equiv 2 \bmod 6,$$
$$2020 - 3 \equiv 2017 \equiv 1 \bmod 6,$$

the 2020th term is the 3rd, 2nd, or 1st term of the repeating sequence $1, 1, 1, 0, 0, 0$, and hence is a 1.

6. Let $ABCD$ be a square. For a point P inside $ABCD$, a *windmill* centred at P consists of two perpendicular lines ℓ_1 and ℓ_2 passing through P, such that

   - ℓ_1 intersects the sides AB and CD at W and Y, respectively, and
   - ℓ_2 intersects the sides BC and DA at X and Z, respectively.

   A windmill is called *round* if the quadrilateral $WXYZ$ is cyclic.

   Determine all points P inside $ABCD$ such that every windmill centred at P is round.

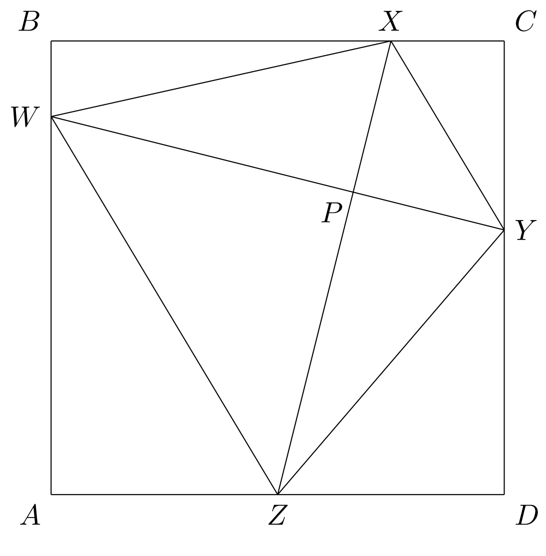
Solution 1 (Norman Do)

Since $\angle WBX = \angle WPX = 90°$, we know that the quadrilateral $WBXP$ is cyclic. Similarly, since $\angle YDZ = \angle YPZ = 90°$, we know that the quadrilateral $YDZP$ is cyclic.

Suppose that the quadrilateral $WXYZ$ is cyclic. Then we have the following equal angles.

$$\angle ABP = \angle WBP = \angle WXP = \angle WXZ = \angle ZYW = \angle ZYP = \angle ZDP = \angle ADP$$

Therefore, triangles ABP and ADP share the common side AP, have the equal sides $AB = AD$, and have the equal angles $\angle ABP = \angle ADP$. It follows that either $\angle APB = \angle APD$ or $\angle APB + \angle APD = 180°$. In the first case, triangles ABP and ADP are congruent, so P must lie on the segment AC. In the second case, P must lie on the segment BD. Therefore, P lies on one of the diagonals of the square $ABCD$.



Conversely, suppose that P lies on one of the diagonals of the square $ABCD$. In fact, we may assume without loss of generality that P lies on AC. Then the triangles ABP and ADP are congruent and we have the following equal angles.

$$\angle WXZ = \angle WXP = \angle WBP = \angle ABP = \angle ADP = \angle ZDP = \angle ZYP = \angle ZYW$$

Since $\angle WXZ = \angle ZYW$, it follows that the quadrilateral $WXYZ$ is cyclic.

Solution 2 (Alice Devillers)

Assume P has the property. Then in particular the quadrilateral with ℓ horizontal is cyclic. Let $PW = a$, $PZ = b$ and the square have side L. Then $\tan(\angle WZP) = a/b$, $\tan(\angle YZP) = (L-a)/b$, so $\tan(\angle YZW) = \frac{bL}{b^2-a(L-a)}$. Similarly $\tan(\angle WXY) = \frac{(L-b)L}{(L-b)^2-a(L-a)}$. Since $WXYZ$ is cyclic, the two angles must add up to 180, so the tangents must be opposite numbers. Hence $\frac{bL}{b^2-a(L-a)} = -\frac{(L-b)L}{(L-b)^2-a(L-a)}$, which simplifies to $L(a-b) = a^2 - b^2$, so either $a = b$ and P is on AC, or $a = L - b$ and P is on BD.

Conversely, assume P is on a diagonal, WLOG say AC. Because of the two right angles the two quadrilaterals $WAZP$ and $XCYP$ are cyclic. It follows that $\angle WZP = \angle WAP = 45$, so the right-angled triangle WZP is isosceles, and $PZ = PW$. Similarly $PY = PX$. It follows that the triangles PYZ and PXW are congruent, and $\angle YZP = 90 - \angle PXW$. Now we see that $\angle YZW + \angle YXW = \angle YZP + \angle PZW + \angle YXP + \angle PXW = (90 - \angle PXW) + 45 + 45 + \angle PXW = 180$. Thus the quadrilateral $WXYZ$ is cyclic.

Solution 3 (Angelo Di Pasquale)

The following is an alternative to the "hard part" of the problem only.

If $WXYZ$ is cyclic, then we deduce as in the official solution that $\angle ABP = \angle ADP$. Then the sine rule in triangles ABP and ADP yields

$$\frac{AD}{\sin\angle APD} = \frac{AP}{\sin\angle ADP} = \frac{AP}{\sin\angle ABP} = \frac{AB}{\sin\angle APB}.$$

Since $AB = AD$ we obtain either $\angle APB = \angle APD$ or $\angle APB + \angle APD = 180^\circ$, then continue as in the official solution.

Solution 4 (Angelo Di Pasquale)

The following is an alternative to the "hard part" of the problem only.

If P does not lie on the diagonal BD, then without loss of generality, P lies inside triangle ABD. If $WXYZ$ is cyclic, then we deduce as in the official solution that $\angle ABP = \angle ADP = \alpha$, say. Thus we also have $\angle PBD = \angle PDB = 45^\circ - \alpha$. Therefore, triangle PDB is isosceles with $PD = PB$. Hence P lies on the perpendicular bisector of BD — that is, P lies on AC. Thus if $WXYZ$ is cyclic, then P must lie on one of the diagonals.

Solution 6 (Ivan Guo)

Set $A(1,1), B(1,-1), C(-1,-1), D(-1,1)$ and $P(a,b)$. Let the gradient of WY be m, then very quickly you can work out the following:

$$PW^2 = (1+m^2)(1-a)^2, PY^2 = (1+m^2)(1+a)^2, PX^2 = (1+m^2)(1+b)^2, PY^2 = (1+m^2)(1-b)^2.$$

By power of a point, $WXYZ$ is necessarily cyclic if and only if

$$(1-a^2)^2 = (1-b^2)^2.$$

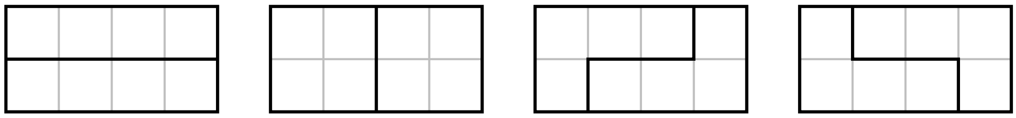
Since both a, b are between -1 and 1, the solution of this equation is $a = \pm b$ which corresponds to the diagonals of the square.

Solution 6 (Kevin McAvaney)

Let P be any point inside the square $ABCD$, where $A = (0, s)$, $B = (s, s)$, $C = (s, 0)$, and $D = (0, 0)$. Suppose that all of the quadrilaterals $WXYZ$ are cyclic and consider the quadrilateral for which XZ is horizontal and WY is vertical. Let P have coordinates (a, b) and note that the triangles XPY and WPZ are similar. This leads to the equation $\frac{b}{a} = \frac{s-a}{s-b}$. Hence, we obtain $s(a - b) = a^2 - b^2$, which yields $a = b$ or $a = s - b$. Either way, P lies on a diagonal of $ABCD$.

The converse is the same as in the official solution.

7. A *tetromino tile* is a tile that can be formed by gluing together four unit square tiles, edge to edge. For each positive integer n, consider a bathroom whose floor is in the shape of a $2 \times 2n$ rectangle. Let T_n be the number of ways to tile this bathroom floor with tetromino tiles. For example, $T_2 = 4$ since there are four ways to tile a 2×4 rectangular bathroom floor with tetromino tiles, as shown below.



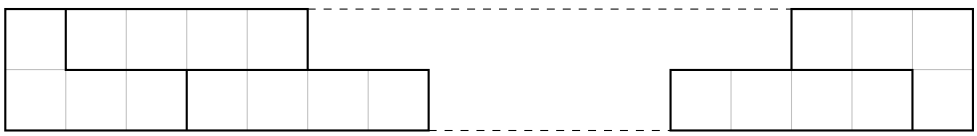
Prove that each of the numbers $T_1, T_2, T_3, \ldots$ is a perfect square.

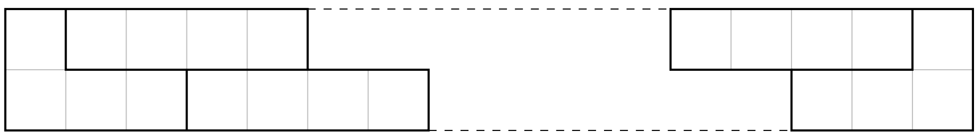
Solution 1 (Sean Gardiner)

Let F_n denote the Fibonacci sequence, defined by $F_0 = 1$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. We will prove that $T_n = F_n^2$.

Consider a tiling of a $2 \times 2n$ rectangle with tetrominoes, where $n \geq 2$. Consider the behaviour of the tiling in the leftmost column of the rectangle. Exactly one of the following three cases must arise.

- The leftmost column is covered by a 2×2 square, whose removal leaves one of the T_{n-1} tilings of the $2 \times 2(n-1)$ rectangle.
- The leftmost column is covered by two 1×4 rectangles, whose removal leaves one of the T_{n-2} tilings of the $2 \times 2(n-2)$ rectangle.
- The leftmost column is covered by an L-tetromino and resembles one of the following diagrams, or their reflections in a horizontal axis. The areas outlined by dashed lines are tiled with 1×4 rectangles. (Observe that there may actually be any non-negative integer number of 1×4 rectangles appearing in the diagram.) The top case occurs when the area covers a number of columns that is 0 modulo 4, while the bottom case occurs when the area covers a number of columns that is 2 modulo 4. The removal of these areas leaves one of the T_{n-k} tilings of the $2 \times 2(n-k)$ rectangle, where $k = 2, 3, \ldots, n$. The case $k = n$ leaves zero columns, so we set $T_0 = 1$ to allow for this case.





Hence, we have shown that the following recursion holds for $n \geq 2$.

$$T_n = T_{n-1} + T_{n-2} + 2(T_0 + T_1 + \cdots + T_{n-2}).$$

Using this equation, one can directly verify that for $n \geq 3$,

$$T_n - 2T_{n-1} - 2T_{n-2} + T_{n-3} = 0.$$

Now observe that

$$F_n^2 + F_{n-3}^2 = (F_{n-1} + F_{n-2})^2 + (F_{n-1} - F_{n-2})^2 = 2F_{n-1}^2 + 2F_{n-2}^2.$$

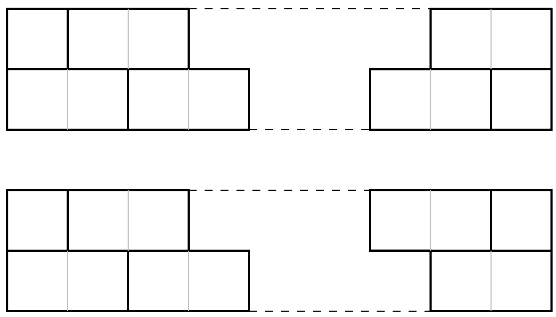
We can check that $T_n = F_n^2$ for small values of n and then use these two matching recursions to deduce that $T_n = F_n^2$ for all positive integers n.

Solution 2 (Norman Do)

We use the known fact that a tiling of a $1 \times n$ rectangle with unit squares and dominoes is enumerated by the Fibonacci number F_n, as defined in the official solution. Therefore, the number of tilings of a $2 \times n$ rectangle with unit squares and horizontal dominoes is simply F_n^2. We will provide a bijection between the tilings of a $2 \times n$ rectangle with unit squares and horizontal dominoes and the tilings of a $2 \times 2n$ rectangle with tetrominoes.

Consider a tiling of a $2 \times n$ rectangle with unit squares and horizontal dominoes. Divide it into pieces by cutting along all vertical lines that do not pass through a domino.

- If a piece consists of two unit squares on top of each other, then we replace it with a 2×2 rectangle.
- If a piece consists of two dominoes on top of each other, then we replace it with two 1×4 rectangles on top of each other.
- Otherwise, the piece must resemble one of the following diagrams, or theire reflefc-tions in a horizontal axis. The areas outlined by dashed lines are tiled with dominoes. (Observe that there may actually be any positive integer number of dominoes appearing in the diagram.) In this case, we replace the diagram with the respective diagrams from the official solution.



The construction gives the necessary bijection between the tilings of a $2 \times n$ rectangle with unit squares and horizontal dominoes and the tilings of a $2 \times 2n$ rectangle with tetrominoes. Therefore, we conclude that $T_n = F_n^2$.

Solution 3 (Jamie Simpson)

Say that T_n is the number of ways of using n tetrominoes to make a $2n \times 2$ rectangle, U_n the number of ways using n tetrominoes to make a shape ending in a length 2 upper prong (as kindly illustrated by Norman) and L_n the number of ways of making a shape ending in a lower prong. We can make a $2 \times (n+1)$ rectangle by adding a 2×2 square to an n tetromino rectangle, by adding two 4×1 tetrominoes to an $n-1$ tetromino rectangle, by adding an L-shaped tetromino to an n tetromino shape ending in an upper prong or an n tetromino shape ending in a lower prong. This means we have

$$T_{n+1} = T_n + T_{n-1} + U_n + L_n.$$

An $n+1$ tetromino upper prong shape can be made be adding an L shaped tetromino to an n tetromino rectangle or by adding a 1×4 tetromino to an n tetromino lower prong shape. This gives

$$U_{n+1} = T_n + L_n.$$

Similarly, by considering making lower prong shapes we get

$$L_{n+1} = T_n + U_n.$$

It's easily checked that

$$\begin{array}{lll} T_1 = 1 & U_1 = 1 & L_1 = 1 \\ T_2 = 4 & U_2 = 2 & L_2 = 2. \end{array}$$

Then using the recurrence equations we get

n	T_n	U_n	L_n
1	1	1	1
2	4	2	2
3	9	6	6
4	25	15	15

This suggests that $T_n = F_{n+1}^2$, $U_n = F_nF_{n+1}$ and $L_n = F_nF_{n+1}$ where F_n is the nth Fibonacci number. We use induction to prove this. From the table we see it holds for low values of n. Assume it holds for n. Then

$$\begin{aligned} T_{n+1} &= T_n + T_{n-1} + U_n + L_n \\ &= F_{n+1}^2 + F_n^2 + 2F_nF_{n-1} \\ &= (F_n + F_{n+1})^2 \\ &= F_{n+2}^2. \end{aligned}$$

$$\begin{aligned} U_{n+1} &= T_n + L_n. \\ &= F_{n+1}^2 + F_nF_{n+1} \\ &= F_{n+1}(F_{n+1} + F_n) \\ &= F_{n+1}F_{n+2}. \end{aligned}$$

Since $U_n = L_n$ the result follows by induction and we see that T_{n+1} is a perfect square for all n.

Solution 4 (Ian Wanless)

Alternative way to get from the first recurrence of the official solution to the answer:

$$
\begin{aligned}
F_n^2 &= (F_{n-1} + F_{n-2})^2 = F_{n-1}^2 + F_{n-2}^2 + 2F_{n-1}F_{n-2} \\
&= F_{n-1}^2 + F_{n-2}^2 + 2F_{n-2}(F_{n-2} + F_{n-3}) \\
&= F_{n-1}^2 + 3F_{n-2}^2 + 2(F_{n-3} + F_{n-4})F_{n-3} \\
&= F_{n-1}^2 + 3F_{n-2}^2 + 2F_{n-3}^2 + 2F_{n-3}F_{n-4} \\
&\vdots \\
&= F_{n-1}^2 + 3F_{n-2}^2 + 2F_{n-3}^2 + 2F_{n-4}^2 + \cdots + 2F_0
\end{aligned}
$$

8. Prove that for each integer k satisfying $2 \le k \le 100$, there are positive integers $b_2, b_3, \ldots, b_{101}$ such that
$$b_2^2 + b_3^3 + \cdots + b_k^k = b_{k+1}^{k+1} + b_{k+2}^{k+2} + \cdots + b_{101}^{101}.$$

Solution 1 (Angelo Di Pasquale)

Lemma. Consider the equation
$$x^k = \sum_{i=1}^{n} x_i^{k_i}, \tag{1}$$
where $k, k_1, \ldots, k_n$ are given positive integers. If $\gcd(k, k_i) = 1$ for $i = 1, \ldots, n$, then equation (1) has a solution in positive integers $x, x_1, \ldots, x_n$.

Proof. Let $d = \prod_{i=1}^{n} k_i$, and let $x_i = y^{\frac{d}{k_i}}$ for some positive integer variable y. Equation (1) becomes
$$x^k = ny^d. \tag{2}$$
Let $y = n^z$ and $x = n^w$ for some positive integer variables z and w. Equation (2) becomes
$$n^{kw} = n^{dz+1} \quad \Rightarrow \quad kw = dz + 1. \tag{3}$$
Since $\gcd(k, d) = 1$, by the Euclidean algorithm, (3) has solutions in positive integers w and z.

We use the lemma extensively in the problem at hand..

If $k \ge 52$, then let $b_{100} = b_{99} = \cdots = b_{k+1} = b_k = b_{k-1} = \cdots = b_{2k-99} = 1$. It remains to solve $b_{101}^{101} = \sum_{i=2}^{2k-100} b_i^i$. Since 101 is prime, and all the superscripts on the right are less than 101, we may apply the lemma to dispose of this case.

If $k = 51$, then let $b_i = 1$ for all $2 \le i \le 101$.

If $11 \le k \le 50$, then at most 8 of the terms on the RHS have superscripts that are multiples of 11. Thus we may set all b_i on the LHS equal to 1 except for b_{11}, and the same number of b_i on the RHS equal to 1, and making sure that all b_i with $11 \mid i$ on the RHS are set equal to 1. What is left is an equation of the form $b_{11}^{11} = \sum b_{j_i}^{j_i}$ where none of the j_i are divisible by 11. Hence we may apply the lemma to dispose of this case.

If $3 \le k \le 10$, let $b_i = 1$ for every $i > k$ with $3 \mid i$ (there are at least 30 and at most 32 of such i). Also let $b_1 = 6$. Set the remaining b_i on the LHS equal to 1 except for b_3 and the correct number of remaining b_i on the RHS equal to 1 so that what is left is an equation of the form $b_3^3 = \sum b_{j_i}^{j_i}$ where none of the j_i are divisible by 3. Hence we may apply the lemma to dispose of this case.

Finally, if $k = 2$, then let $b_1 = 11$, $b_3 = b_4 = 2$, and $b_i = 1$ for $5 \le i \le 101$.

Hence we have covered all required k.

Solution 2 (Andrew Elvey Price)

Let x and ρ be positive integers to be determined later and set

- $b_2 = x\rho^{\frac{100!}{2}}$
- $b_{101} = \rho^{\frac{100!+1}{101}}$

- $b_j = \rho^{\frac{100!}{j}}$ for $3 \le j \le 100$.

Then the equation that we want to solve becomes

$$x^2\rho^{100!} + \rho^{100!} + \ldots + \rho^{100!} = \rho^{100!} + \ldots + \rho^{100!} + \rho^{100!+1}.$$

which, dividing by $\rho^{100!}$ is equivalent to

$$x^2 + k - 2 = 100 - k + \rho.$$

This evidently has many solutions, for example we can set $x = 10$ and $\rho = 2k - 2$.

Solution 3 (Ivan Guo)

Consider the equation

$$b_2^2 + \cdots + b_k^k - b_{k+1}^{k+1} - \cdots - b_{100}^{100} = L.$$

First choose $b_2, \ldots, b_{100}$ arbitrarily so that L is positive (e.g., by making b_2 very big). Since 101 is coprime to 100!, there exists positive integers c, d such that $100!c + 1 = 101d$ (in fact, by Wilson's theorem, $c = 1$ works). The multiplying both sides by $L^{100!c}$, we have

$$(b_2L^{100!c/2})^2 + \cdots + (b_kL^{100!c/k})^k - (b_{k+1}L^{100!c/(k+1)})^{k+1} - \cdots - (b_{100}L^{100!c/100})^{100} = (L^d)^{101},$$

as required.