MATHCOUNTS®

2020 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2020 MATHCOUNTS® State Competition. These solutions provide creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

2020 State Competition Sprint Round Solutions

1. The angle referred to initially as $\angle$B is more specifically $\angle$ABC after the bisection. Due to the bisection, $m\angle\text{DBC} = \frac{1}{2} m\angle\text{ABC} = 100/2 = 50$ degrees. Because the sum of the measures of the three angles of a triangle is always 180°, we have:
   $180 = m\angle\text{DBC} + m\angle\text{BCD} + m\angle\text{BDC} = 50 + 20 + m\angle\text{BDC} = 70 + m\angle\text{BDC}$, so:
   $m\angle\text{BDC} = 180 - 70 = \mathbf{110}$ degrees.

2. We can divide both the numerator and the denominator by the numerator 1.2×10^2, resulting in a numerator of 1 (dividing any nonzero number by itself yields 1), and a denominator of $\frac{4.8 \times 10^5}{1.2 \times 10^2} = 4 \times 10^{5-2} = 4 \times 10^3 = 4000$. Thus, we end up with the simplification $\frac{1.2 \times 10^2}{4.8 \times 10^5} = \frac{\mathbf{1}}{\mathbf{4000}}$.

3. I will use a instead of the airplane symbol and b instead of the ball symbol, making our two equations:
   $b + a = 14$;
   $b - a = \;\; 4$. Subtracting this lower equation from the above equation yields:
   $2a = 10$, so $a = \mathbf{5}$.

4. Starting at A we go 4 units right to get to B, then 2 units up to get to C, then 2 units left to get to D, then 2 units up to get to E, then 2 units left to get to F, and finally 4 units down to get back to the starting point A. The total number of units traveled is $4 + 2 + 2 + 2 + 2 + 4 = \mathbf{16}$.

5. We want to partition the 40 ft × 72 ft play area into squares of size n ft × n ft. To avoid gaps and overlap without cutting tiles, both 40 and 72 must be divisible by n; since we want the largest possible tiles, we need n to be the greatest common divisor of 40 and 72, which is 8. We will need $40/8 = 5$ rows each with $72/8 = 9$ tiles for a total tile count of $5 \times 9 = \mathbf{45}$ square tiles.

6. Let s and b be the distance that Rose and Robyn, respectively, ran. We are given:
   $s + b = 29$ mi and $b = 2s - 4$ mi, with the latter equation being equivalent to
   $2s - b = \;\; 4$ mi. Since we want s and not b, let's add the two equations to get:
   $3s \qquad = 33$ mi, so that $s = \mathbf{11}$ miles.

7. The problem does not state whether the two folds are parallel or perpendicular to each other, nor if parallel whether the fold is along the short or the long edge of the original paper, so we consider all options. If the two folds are perpendicular, then each dimension of the final paper is 1/2 of the original, making the original sheet 8 in × 10 in, with a perimeter double the sum of the two side lengths for 36 in. If the two folds are parallel, the effect of the two folds is to make one side the paper 1/4 as along as the original while keeping the other side unchanged, making the original sheet either 4 in × 20 in or 16 in × 5 in, depending on which original side the folds were made, resulting in a perimeter of 48 in or 42 in, respectively. The greatest of these three possible values is **48** inches.

8. In order for $k!$ to have 7 factors of 2, we must have $8 \le k \le 9$; in order to have 2 factors of 3, we must have $6 \le k \le 8$. These two pairs of constraints together tell us that $k = \mathbf{8}$, which is consistent with having exactly 1 factor of each of 5 and 7.

9. $200(200 + x) = 207^2 - 49 = 207^2 - 7^2 = (207 + 7)(207 - 7) = (214)200 = 200(200 + 14)$ so $x = \mathbf{14}$.

10. Let's put the 0 point of our number line at A and put B on the positive side of A, thus at 24. C is at the midpoint of AB, thus 12. E is at the midpoint of CB, thus $\frac{12+24}{2} = 18$. D is at the midpoint of AE, thus at

$\frac{0+18}{2} = 9$. F is at the midpoint of AC, thus $\frac{0+12}{2} = 6$. G is at the midpoint of EB, thus $\frac{18+24}{2} = 21$. The distance between F and G is $|21 - 6| = \mathbf{15}$ units.

11. Let l be the duration of a long movie and s be the duration of a short movie. The total time duration of the movie watching was 5 hours + 15 minutes = (5 × 60 + 15) minutes = 315 minutes. Other given information includes: $l + 2s = 315$ minutes and $l = 1.50s = \frac{3}{2}s$, so $s = \frac{2}{3}l$.
Thus, $315 \text{ min} = l + \frac{4}{3}l = \frac{7}{3}l$ and $l = \frac{3}{7}(315) = 3(45) = \mathbf{135}$ minutes.

12. $d = (1 + 0.10)c = 1.10c$; $c = (1 - 0.25)b = 0.75b$; $a = (1 + 0.50)b = 1.50b = \frac{3}{2}b$ so $b = \frac{2}{3}a$. We need to determine $\frac{a-d}{a}$ as a percentage. Thus, $d = 1.1c = 1.1(0.75b) = 1.1\left(0.75\left(\frac{2}{3}a\right)\right) = 1.1(0.5a) = 0.55a$.
Then $\frac{a-d}{a} = \frac{a-0.55a}{a} = \frac{0.45a}{a} = 0.45 = 0.45 \times 100\ \% = \mathbf{45}\ \%$.

13. $|(A + G) - (A - G)| = |2G| = 2|G|$.
$A = (A + F + G) - (F + G) = 11 - 10 = 1$.
$B + C = (A + B + C) - A = 5 - 1 = 4$.
$D = (B + C + D) - (B + C) = 7 - 4 = 3$.
$E + F = (D + E + F) - D = 11 - 3 = 8$.
$G = (E + F + G) - (E + F) = 13 - 8 = 5$.
Therefore, the desired value is $2|5| = \mathbf{10}$.

14. To get from square 4 to square 32 using only downward motions, one must pass through exactly one of squares 18, 19 and 20. For each of those 3 options, we can determine how many ways lead from square 4 to the square in question, and then how many ways lead from the square in question to square 32; since the ways from square 4 to the intermediate square are independent of the ways from the intermediate square to square 32, the number of ways from square 4 through the intermediate square under assessment to square 32 is the product of those two values. The total count of ways is the sum of those products over the 3 intermediate squares being examined. For square 18 there is 1 way from square 4 (4..8..11..15..18) and 1 way to square 32 (18..23..27..32). For square 19 there are 3 ways from square 4 (4..8..11..15..19; 4..8..11..16..19; 4..8..12..16..19) and 3 ways to square 32 (19..23..27..32; 19..24..27..32; 19..24..28..32). For square 20 there are 2 ways from square 4 (4..8..11..16..20; 4..8..12..16..20) and 2 ways to square 32 (20..24..27..32; 20..24..28..32). Therefore, the total number of qualifying ways from square 4 to square 32 is $1 \times 1 + 3 \times 3 + 2 \times 2 = \mathbf{14}$ ways.

15. We have an uncut length, an uncut width, a shortened length, a shortened width, and a hypotenuse of the right triangular cutoff representing the values 10, 2, 11, 10 and 5 [not in the same order]. Because all values are integers, the cutoff piece sides need to be a Pythagorean triple. The only options for the hypotenuse are 5 (3:4:5) and 10 (6:8:10). A hypotenuse of 5 does not work because there are no two sides differing by 3 and no two sides differing by 4; 10 does work because $11 - 5 = 6$ and $10 - 2 = 8$. Therefore, we have an 11×10 rectangle with a right triangular piece with legs of 6 and 8 removed, so the area of the pentagon is $11 \times 10 - \frac{6\times 8}{2} = 110 - 24 = \mathbf{86}$ units^2.

16. The arithmetic mean is the sum divided by the count, so $A = \frac{200}{9}$. The median of 9 values in increasing order is the fifth value. To allow the fifth value to be larger with a fixed sum of 200, let's make the 4 smallest values as small as allowed [nonnegative], namely 0, and the 4 largest values as small as possible, so they should be equal to the fifth value. Thus, we have 4 values of 0 and 5 values that are equal and whose sum is 200. Thus, each of those values must be $\frac{200}{5} = 40$, so $M = 40$ and $M - A = 40 - \frac{200}{9} = \frac{360-200}{9} = \mathbf{\frac{160}{9}}$.

17. The base of the pyramid will be the 3 congruent original outer edges of the square, thus forming an equilateral triangle with sides whose length is $\sqrt{2}$ times the length of each dotted segment, for $4\sqrt{2}$ in. The area enclosed by an equilateral triangle with side s is $\frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}\left(4\sqrt{2}\right)^2 = \mathbf{8\sqrt{3}}$ in^2.

18. $(x^3 + x^2 - 3x + b)(2x^4 + cx^3 + x^2 - x + 1) = 2x^7 + 8x^6 + x^5 - 4x^4 + 39x^3 + 11x^2 - 10x + 7$.
Substitute two values for x to be able to solve for a and b—keep the values simple. Two values that are commonly good to use for x are 0 and 1 in such problems:
$x = 0$: $(b)(1) = 7$, so $b = 7$.
$x = 1$: $(6)(3 + c) = 54$, so $c = 6$.
Therefore, $b + c = \mathbf{13}$.

19. $a_0 = 10 = 10^1$; $a_1 = 10^2$; $a_2 = 10^4$; $a_3 = 10^8$; .... In general, $a_k = 10^{2^k}$.
Therefore, the product of a_1 through a_n is $10^{2^1} \cdot 10^{2^2} \cdot 10^{2^3} \cdot \ldots \cdot 10^{2^n} = 10^{2^1+2^2+2^3+\cdots 2^n}$.
The exponent in the last expression is a geometric series with first term 2, last term 2^n, and common ratio 2, so the sum is $\frac{2-2^{n+1}}{1-2} = 2^{n+1} - 2$. This makes the product to be $10^{2^{n+1}-2}$. Now, 10^k has $k + 1$ digits. Since we want at least 100 digits, the exponent needs to be 1 less. Therefore, $2^{n+1} - 2 \geq 99$, so $2^{n+1} \geq 101$. The powers of 2 in that region are $2^6 = 64$ and $2^7 = 128$, so $n + 1 \geq 7$. The least value that works for n is **6**.

20. $1024 = 2^{10} = (2^5)^2 = 32^2 = (2^2)^5 = 4^5$. Thus, there are 32 perfect squares and 4 perfect fifth powers for 1 to 1024, inclusive. Of these, 2 values overlap in the two sets: $1^{10} = 1$ and $2^{10} = 1024$. (A number is both a perfect square and a perfect fifth power if and only if it is a perfect tenth power.) Thus, we start off with 1024 integers, subtract 32 as being perfect squares and subtract 4 more as being perfect fifth powers, leaving 988 to consider. However, in doing so, we have deleted 1 and 1024 twice, so we need to add 2, to make sure any deleted value is deleted exactly once. Thus, there are **990** such positive integers.

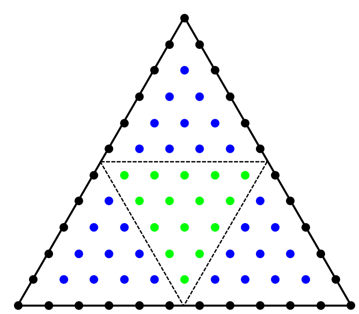
21. We would like to be able to use all 10 digits in our number, and the leftmost digit should be a 9, if possible, to achieve the largest qualifying 10-digit number. Let's start with 9. The next digit needs to divide 9 or be divisible by 9—only 3, 1 and 0 work, so try the largest, 3. Then the next digit needs to divide 3 or be divisible by (and not already be used)—only 6, 1 and 0 work, so try the largest, 6. Then the next digit must be 2, 1 or 0, so try the largest, 2. The next digit needs to be 8, 4, 1 or 0, so try the largest, 8. The next digit needs to be 4, 1 or, 0, so try the largest, 4. The next digit needs to be 1 or 0, so try the largest, 1. Since 1 divides anything, it will work we whatever we have left: 7, 5 or 0, so try the largest, 7. Only 0 and 5 remain, and, of these two, 7 divides only 0, so that must come next. For the last digit, 5 will work, because 5 divides 0. We have successfully completed the largest possible qualifying number: **9,362,841,705**.

22. [1]: $x + \ \ y = a$
[2]: $3x + 2y = a + 15$
[3]: $4x + 5y = a + 19$
Notice how adding equations [2] and [3] yields:
[4]: $7x + 7y = 2a + 34$, in which the coefficients of x and y are equal, just as in [1].
Let's subtract 7 times equation [1] from equation [4] to yield:
[5]: $0 = -5a + 34$, so $a = \mathbf{\frac{34}{5}}$.

23. There are 5 digits to permute, so $5! = 120$ total permutations, with 13,579 being the 1st and 97,531 being the 120th in increasing order by size of number. The leftmost digit is the one used for comparison first to decide which of two numbers is greater. Once the leftmost digit has been chosen for a given permutation, there are 4 digits remaining to permute, for a total of $4! = 24$ permutations. Thus, there are 24 permutations for each possible leftmost digit. Thus, permutations 1 through 24 start with 1, 25 through 48 start with 3, 49 through 72 start with 5, 73 through 96 start with 7, and 97 through 120 start with 9. Since we want the 100th permutation, that means we start with 9. For each choice of digit we might choose second, we will have $3! = 6$ permutations of the remaining 3 digits. Thus, permutations 97 through 102 start with 91—this includes our desired 100th permutation. We keep iterating this process: permutations 97 and 98 start with 913, while 99 and 100 (which is what we want) start with 915. Then we want the second of 37 and 73, so we end up with **91,573**.

24. Note that each digit A through G occurs once in each of the 1s, the 10s, the 100s and the 1000s. That means for each place value we will end up with A + B + C + D + E + F + G, even though not necessarily occurring in that order, addition is commutative and associative. We will get 1000 of these for the leftmost column of digits, 100 for the second column, 10 for the third, and 1 for the fourth. Thus the total sum will be:
$(1000 + 100 + 10 + 1)(A + B + C + D + E + F + G) = 1111(A + B + C + D + E + F + G)$.
Now, $1111 = 11 \times 101$, with 11 and 101 both prime, so 101 is the larger; the sum of the 7 digit values cannot exceed $7 \times 9 = 63$, so this part contributes no prime factor greater than 101. Thus, the greatest prime factor of the sum of the four numbers must be **101**, regardless the value of each letter representing some digit 0 through 9.

25. The figure shows the 12th triangular number of lattice points, thus $\frac{12(12+1)}{2} = 78$. However, we must read the question very carefully. Only "the lattice points *inside* the triangle" [my italic emphasis added] are to be considered, that is, the blue and green points in the figure. This eliminates the 33 points on the triangle (technically, the definition of a triangle includes only the 3 sides and not the interior, though we often speak loosely of "the area of a triangle", when we really mean the area enclosed by a triangle). Therefore, 45 points are under consideration for selection. The triangle inequality requires that the longest side of a triangle must have length less than half the sum of the lengths of all 3 sides. In an equilateral triangle the sum of the distances from any one point interior to the triangle to the 3 sides of the triangle equals the altitude of the triangle [Viviani's theorem]. Therefore, a point is desired if and only if for each vertex of the triangle, the point is farther from the vertex than from the opposite side. This region of points is interior to the triangle formed with vertices at the midpoints of the sides of the original triangle, and the qualifying points are colored green in the figure. The probability is the number of green points divided by the number of points that are blue or green. There are 15 points interior to the new triangle, making the probability $\frac{15}{45} = \mathbf{\frac{1}{3}}$. [NOTE: The 4 triangles formed by the 3 auxiliary lines are congruent, so if we were working in continuous space, the probability of any (x, y) point being within the middle smaller equilateral triangle would be only $\frac{1}{4}$; however, we cannot assume with a discrete lattice that the lattice points distribute evenly across the different subregions, even though they might enclose the same physical area.]



26. The circle being tangent to the x-axis at $x = 3$ means that a diameter of the circle is perpendicular to the x-axis at $x = 3$, meaning that the x-component of the circle center is 3, and, when combined with the knowledge that the circle is at or above the x-axis, the y-component is equal to the radius R of the circle. Thus, the equation of the circle is of the form $(x-3)^2 + (y-R)^2 = R^2$, so $(x-3)^2 + y^2 - 2Ry = 0$. Substitute a helpful (x, y) coordinate of a known point on the circle to determine the radius: we know $(3; 0)$ is on the circle, but that is not helpful as we need $y \neq 0$ to determine R, so use $(6; 6)$: $0 = 9 + 36 - 12R$, so $R = \frac{45}{12} = \frac{15}{4}$, making the circle have the equation $(x-3)^2 + y^2 - \frac{15}{2}y = 0$. Another point on the circle is (p, p), so let's substitute p for x and y in the equation and solve for p: $0 = (p-3)^2 + p^2 - \frac{15}{2}p = p^2 - 6p + 9 + p^2 - \frac{15}{2}p = 2p^2 - \frac{27}{2}p + 9$. Before applying the quadratic formula, multiply the equation by 2 to simplify the fractions to be handled: $4p^2 - 27p + 18 = 0$, which has solutions $p = \frac{27 \pm \sqrt{729-288}}{8} = \frac{27 \pm 21}{8}$. The $+$ yields 6, which we were already given; the $-$ yields $\mathbf{\frac{3}{4}}$.

27. The probability of an integer being not divisible by 3 is $\frac{2}{3}$, of being not divisible by 5 is $\frac{4}{5}$, and of not being divisible by 7 is $\frac{6}{7}$. Over a count of consecutive integers that is a multiple of the least common multiple of 3, 5 and 7, these three probabilities are independent, so the probability that an integer in such a band is not divisible by any of 3, 5 and 7 is $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} = \frac{16}{35}$. The least common multiple of 3, 5, and 7 is the product of the three values, so 105. Thus, out of any 105 consecutive integers, exactly $\frac{16}{35} \times 105 = 48$ will fail to be divisible by each of 3, 5, and 7. That means that in the range 1 through $19 \times 105 = 1995$, there are exactly $19 \times 48 = 912$ such integers. Itemize the remaining values and cull out those divisible by 3, 5 or 7:
1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012 2013 2014 2015 2016 2017 2018 2019.
There remain 11 values not shaded out, so the answer is $912 + 11 =$ **923** positive integers.

28. The only calculator you are allowed to use is the one inside your skull augmented with pencil and scratch paper. The product has many digits (16) but its value is not that hard to determine given the values of the factors. The *hard* part would be to extract the square root of a 16-digit integer, even though it is a perfect square. This is MATHCOUNTS, so if the answer does not say what form to use, it has to be an integer, so if we can make good, easy approximations and get close to an expected form with expected properties and value, then the nearest integer of that form is very likely the answer. As $\frac{1}{9} = 0.111\ldots$, with the 1s repeating forever, so 11,111,111 is close to [slightly less than] $\frac{1}{9} \times 10^8$, while 100 000 011 is close to [slightly greater than] 10^8, so the product is close to $\frac{1}{9} \times 10^{16}$. The square root must be close to $\frac{1}{3} \times 10^8 \approx 33{,}333{,}333$. The last 2 digits of the product come from $11 \times 11 = 121$, thus 21 [and an electronic calculator would be very likely be unable to tell you that]; adding the 4 yields 25, which is compatible with the number whose square root to be extracted having a units digit of 5. An excellent guess is 33,333,335. If you have time to spare, you can enhance your confidence in this guess by noticing your estimate for the left factor is too high by $\frac{1}{9}$, which out of $\frac{1}{9} \times 10^8$ is a relative error of only 1 part in 10^8. Your estimate of 10^8 for 100,000,011 is 11 too low for a relative error of -11 parts in 10^8. The relative error of a product is close to the sum of the relative errors of the factors for small relative errors, so you are close to -10 parts in 10^8. Adding 4 to a 16-digit number has a negligible effect on relative error. Finally, taking a square root cuts the relative error in half to -5 parts in 10^8. Thus, you expect your answer to be very close to 5 parts in 10^8 too low, which would be $\frac{5}{10^8} \times \frac{1}{3} \times 10^8 = \frac{5}{3}$, so we need to raise our estimated answer of $33{,}333{,}333\frac{1}{3}$ by about $1\frac{2}{3}$, which results in our predicted **33,333,335**.

29. The triangle is isosceles with base of length 6 and two congruent sides of length 5, so it can be partitioned into to congruent right triangles with the two congruent sides (length 5) being the hypotenuse of the two right triangles and the two halves of the base (length 3) each being one leg of the two right triangles. Thus, the two right triangles are 3:4:5 right triangles, so the height of the dartboard is the length of the second leg, 4. By symmetry we can tell the minimum sum of the squares of the three distances will occur at the centroid of the dartboard, so that the origin of the coordinate system should be put at the centroid of the dartboard. (Note: It is not critical to know to put the origin of the coordinate system at the centroid, but it avoids some completions of squares to solve that slow you down a bit.] To make solving the problem faster we will place the x-axis parallel to and above the base of the dartboard and the y-axis passing through the midpoint of the base. The x-component of the base vertices are −3 and +3, and of the apex is 0. The centroid of a triangle occurs 2/3 of the way from a vertex to the midpoint of the opposite side; the distance between the apex and the midpoint of the base is 4, so 2/3 of that distance (that is, $\frac{8}{3}$) needs to be above the centroid and the remaining distance of $\frac{4}{3}$ needs to be below the centroid. Thus, the two vertices for the base are at $\left(-3; -\frac{4}{3}\right)$ and $\left(+3; -\frac{4}{3}\right)$, and the apex is at $\left(0; +\frac{8}{3}\right)$. The sum of the squares of the distances from a point $(x; y)$ to each of these 3 vertices is:

$$(x+3)^2 + \left(y+\frac{4}{3}\right)^2 + (x-3)^2 + \left(y+\frac{4}{3}\right)^2 + (x-0)^2 + \left(y-\frac{8}{3}\right)^2$$
$$= x^2 + 6x + 9 + y^2 + \frac{8}{3}y + \frac{16}{9} + x^2 - 6x + 9 + y^2 + \frac{8}{3}y + \frac{16}{9} + x^2 + y^2 - \frac{16}{3}y + \frac{64}{9} = 3x^2 + 3y^2 + \frac{86}{3},$$

which is to be less than 30. Thus, $3x^2 + 3y^2 < \frac{4}{3}$, so $x^2 + y^2 < \frac{4}{9} = \left(\frac{2}{3}\right)^2$ and $\frac{x^2+y^2}{(2/3)^2} < 1$, representing the interior of a circle of radius $\frac{2}{3}$ centered at the origin, lying well within the dartboard. This region has area $\pi\left(\frac{2}{3}\right)^2 = \frac{4}{9}\pi$. The dartboard's area is $\frac{1}{2}(6)(4) = 12$, of which the circular region is the fraction $\frac{\frac{4}{9}\pi}{12} = \boldsymbol{\frac{\pi}{27}}$.

30. Hmm! 729 is a peculiar value to be used in a problem like this; 729 happens to be the only 3-digit integer that is both a perfect square and a perfect cube, thus perfect sixth power: $729 = 3^6$. Let's keep that in the back of our mind. First, extend the list of values in the sequence to look for some kind of pattern: 1, 2, 4, 5, 10, 11, 13, 14, 28. [The following sequences show why the excluded values do not work: 1-2-*3*, 2-4-*6*, 1-4-*7*, 2-5-*8*, 1-5-*9*, 10-11-*12*, 13-14-*15*, 12-14-*16*, 5-11-*17*, 10-14-*18*, 1-10-*19*, 2-11-*20*, 5-13-*21*, 4-13-*22*, 5-14-*23*, 4-14-*24*, 1-13-*25*, 2-14-*26*, 1-14-*27*.] There is a gap between 5 and 10, with 10 being double 5, and between 14 and 28, with 28 being double 14. How about powers of 3 (remembering 729)? Well, $10 = 3^2 + 1$ and $28 = 3^3 + 1$. How about $3^1 + 1 = 4$? Sure enough, we have a gap between 2 (half of 4) and 4—just too short to stand out at first. Since powers of 3 seem to be important, let's try writing the numbers in our sequence using base 3: 1_3, 2_3, 11_3, 12_3, 101_3, 102_3, 111_3, 112_3, 1001_3. The rightmost place seems to be always 1 or 2; the other places seem to be always 0 or 1. This actually makes some sense. The pattern for the rightmost place in a 3-term arithmetic sequence, using base 3, would be one of 0-1-2, 0-2-1, 1-2-0, 1-0-2, 2-0-1, 2-1-0, 0-0-0, 1-1-1, or 2-2-2. We started off with 1-2 so we do not allow 0 to be generated, and if 0 is never generated we never break out of the cycle of keeping only 1s and 2s. A similar argument holds for the other places with 0s and 1s but is more complicated by carries/regroupings from places to the right. We have not proven that this is the correct pattern (a positive integer is a term in our sequence if and only if its base 3 representation has a 1 or 2 in the rightmost place, and a 0 or 1 in all other places) but for the sake of time during the competition, it seems reasonably plausible, so let's go for it. The endpoint is 729 = $1\,000\,000_3$ and is not in our sequence because the rightmost place ends in 0 rather than 1 or 2. All positive integers less than 729 have 6 [or less] base 3 places. [A number like 28 can be represented as $001\,001_3$ (normally we do not write leading 0s but mathematically nothing forbids us to do so) or we can use $1\,001_3$ regard the missing 5th and 6th places from the right as blank and implicitly 0.] Each of the 6 place values has 2 options (1 or 2 for the rightmost, and 0 or 1 for the rest) independently of the values in the other places, so the total number of accepted values is $2^6 = \mathbf{64}$.

2020 State Target Round Solutions

1. Joel works 8 hours per day and gets to keep 80% of his \$7.25/h wages. Therefore, to earn and keep \$2400 requires Joel to work: $\frac{\$2400}{8\frac{\text{h}}{\text{d}} \times 0.8 \times \$7.25/\text{h}} = \frac{\$2400}{\$46.40/\text{d}} = 51.72 \ldots$ d, which rounds to **52** days.

2. We have already been given that the output of machine 2 is 3, so we pick up from there. For machine 3, the x input is from the original upper input, 5, and the y input is the output of machine 2, so 3, and we are to divide the x value by the y value to result in an output of $\frac{5}{3}$. For machine 4, the x input is output of machine 3, so $\frac{5}{3}$, and the y input is the output of machine 2, so 3, and we are to add the two inputs and then divide by 2 to result in an output of $\frac{7}{3}$. For machine 5, the x input is from the original upper input, 5, and the y input is the output of machine 4, so $\frac{7}{3}$, and we are to divide the x value by the y value to result in an output of $\frac{15}{7}$. For machine 6, the x input is output of machine 5, so $\frac{15}{7}$, and the y input is the output of machine 4, so $\frac{7}{3}$, and we are to add the two inputs and then divide by 2 to result in an output of $\frac{47}{21}$, which is equal to the mixed number $\mathbf{2\frac{5}{21}}$.

3. We need a fraction whose value in standard decimal form is between 0.60 and $0.6\overline{6}$. Sixths don't work because $\frac{3}{6} = 0.5$ is too small and $\frac{4}{6} = \frac{2}{3}$ is too large. Sevenths don't work since $\frac{4}{7} = 0.571 \ldots$ is too small and $\frac{5}{7} = 0.714 \ldots$ is too large. For eighths, $\mathbf{\frac{5}{8}} = 0.625$ works.

4. $S_k = 1 + 2 + 3 + \cdots + (k-1) + k = \frac{k(k+1)}{2}$, the sum of the first k positive integers, but this quantity for k iterating from n through $2n - 1$ is always paired with a factor 2, as $2S_k = k(k+1)$, in P_n. Therefore, $P_n = [n(n+1)][(n+1)(n+2)][(n+2)(n+3)] \ldots [(2n-2)(2n-1)][(2n-1)(2n)]$, in which we a matching pair of factors in the right factor of one bracket and the left factor of the next bracket, for every bracket except the last. Therefore, $(n+1)$ through $(2n-1)$ all occur as squares that can be simplified out of the radical. The only remaining factors are the left factor of the first bracket and the right factor of the last bracket: $n(2n) = 2n^2$, whose square root is $n\sqrt{2}$. Upon simplification 2 is the only portion left in the radical, so $b = \mathbf{2}$.

5. The truth-tellers will correctly state how many truth-tellers there are, whereas the liars will state a wrong count of truth-tellers. The group of truth-tellers will be that group whose answer regarding the number of truth-tellers matches the number of people in the group. The only such match is 11, so the group claiming 11 truth-tellers are truth-tellers and they told the truth when they said there are **61** liars.

6. We work backwards. There's only 1 way to get to ◯ from ◯-you're already there! And there's 1 way to get to ◯ from each of the spaces adjacent to ◯. We place a blue 1 in each cell to show that there's 1 way to get to ◯ from the corresponding cell:

		Y	1	①
			X	1
				Z
⋆				

Next, consider the square with the X. From there, we can take one step right (and then there is 1 way to finish). Or we can take one step up (and then there is 1 way to finish). Combining these, we have 1 + 1 = 2 ways to finish from X.

Then, consider the square marked *Y*. From there, we can take one step right (1 way to finish from there), or two steps right (1 way to finish from there-already done!). Combining these, we have 1 + 1 = 2 ways to finish from *Y*. The situation at *Z* is identical to that at *Y*, except that we are going up instead of to the right. We'll keep exploiting this symmetry throughout this solution! Now, we have

	A	2	1	①
		B	2	1
			C	2
				D
⋆				

From *A*, we can go one step right (2 ways to finish), two steps right (1 way to finish), or three steps right (1 way to finish). So, there are 2 + 1 + 1 = 4 ways to finish from *A*. As with *Y* and *Z* above, we use symmetry to see that there are 4 ways to finish from Das well.

From *B*, we can go one step up (2 ways to finish), one step right (2 ways to finish), or two steps right (1 way to finish). So, there are 2 + 2 + 1 = 5 ways to finish from *B*. And a corresponding 5 ways to finish from *C*.

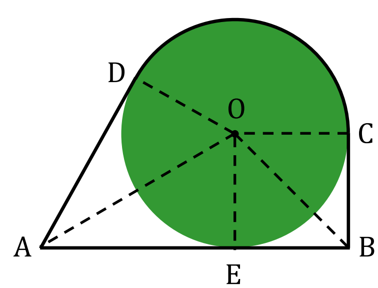
	4	2	1	①
		5	2	1
			5	2
				4
⋆				

We keep going in this manner. At each square as we work backwards, we find the number of paths from that square by adding all the numbers in the squares directly to the right of that square and all the numbers directly above that square. We use symmetry as above to cut down on the calculations, and we finally find:

8	4	2	1	①
28	12	5	2	1
94	37	14	5	2
289	106	37	12	4
838 ⋆	289	94	28	8

We see that the number sequences of moves to get from ★ to ◯ is **838** sequences.

7. Let's draw in the figure some more auxiliary lines connecting the center of circle O points of tangency of line segments to O. A radius to a point of tangency is perpendicular to the tangent line, so $\overline{OC} \perp \overline{CB}$, $\overline{OE} \perp \overline{AB}$, and $\overline{OD} \perp \overline{DA}$. Thus, △OBC is a right triangle. The hypotenuse OB is given as $10\sqrt{2}$ cm and radius OC is given as 10 cm, making BC to be 10 cm and △OBC a right isosceles triangle. Similar analysis of △OBC shows BE to be 10 cm as well. With 4 congruent sides and some right angles, we know that OCBE is a square, making the measures of ∠OEC and arc EC be 90 degrees. Triangles OAD and OAE are right triangles with legs OD and OE, respectively, being half the hypotenuse OA, so the two triangle are 30-60-90, with angles DOA and AOE measuring 60 degrees connecting the respective short leg and the hypotenuse, so the combined angle DOE and corresponding arc DE measuring 120 degrees. When we combine arcs DE and EC into major arc DEC, the resulting measure is 210 degrees.



This is where the rope is *not* in contact with the watermelon. The rope is in contact with the watermelon over the remaining part of the circle O, which is the minor arc DC, which has measure $360 - 210 = 150$ degrees, for $\frac{5}{12}$ of the circumference of O, making an arc length of $\frac{5}{12} \times 2\pi(10) = \frac{25}{3}\pi$ cm. Segment DA is the long leg of 30-60-90 △AOD, and thus has length $\sqrt{3}$ times as long as the 10 cm of the short leg OD for $10\sqrt{3}$ cm. Therefore, the total length of the rope is $\left(10 + \frac{25}{3}\pi + 10\sqrt{3}\right)$ cm $= 53.5004\ldots$ cm, which rounds to the nearest 0.1 cm as **53.5** cm.

8. We have information about the powers that start with 8, and we want to learn about the powers that start with 9. So, we think about how these powers get produced in Tyrell's list. Each power is double the power before it in the list. We get a number that starts with 8 or 9 when we double a number that starts with 4. That's promising-we found something in common between the powers we know something about (the ones that start with 8) and the ones we want to know about (the ones that start with 9).

   We can only get a power that starts with 4 if we are doubling a number that starts with 2. But we don't always get a power that starts with 4 when we double a number that starts with 2! Sometimes we get a power that starts with 5. Hmmm ... And we can only get a power that starts with 2 if we're doubling a power that starts with 1. But sometimes doubling a power that starts with 1 will give us a power that starts with 3.

   That's enough working backwards. Now that we have some idea of how we can get a power that starts with 8 or 9 (we need a power that starts with 4), let's take a look at the first several powers of 2 to see if we find anything interesting. We'll group the powers by the number of digits the powers have, since we have a little information now about the patterns we expect as we double numbers starting with a number that begins with 1.

   1 digit: 1,2,4,8
   2 digits: 16, 32, 64
   3 digits: 128,256,512
   4 digits: 1024, 2048, 4096, 8192

   Right away, we see some of the patterns that we expected. We only have a number that starts with 8 or 9 in groups that have a power that starts with 4. We also see that each group has a power that starts with 1. That's because any k-digit power that starts with a number greater than 1 is double another k-digit power. So, there must be a power that starts with 1 in each group.

   Doubling a number that starts with 1 gives a number that starts with 2 or 3, so the second power in each group starts with 2 or 3.

   Doubling a number that starts with 2 or 3 gives a number that starts with 4, 5, 6 or 7. So, the third power in each group starts with one of these four digits.

   If the third power in the group starts with 4, then there will be a fourth power in the group, and that power will start with 8 or 9. Otherwise, the group only has three powers.

   Aha! We now know that every group has three or four powers, and that the powers starting 8 or 9 must the be the fourth powers in the four-power groups. We have enough information to tackle the problem now!

   Tyrell's list has the first digits of 1001 powers. Because the last power has 302 digits, our group-by-digit approach above produces 302 groups. From the given information about the end of Tyrell's list, we know that the last group has 1 power. The rest of the groups have 3 or 4 powers. Removing that last group, we have 301 groups that together have 1000 powers. Removing the first 3 powers of each group (so that many end up empty), we take away $3 \cdot 301 = 903$ powers, leaving $1000 - 903 = 97$ powers, each of which is the last power in a group of 4 powers. That is, each of these remaining 97 powers starts with 8 or 9. We are told that 52 of these powers start with 8, so the other $97 - 52 = 45$ start with 9. So, 9 appears **45** times.

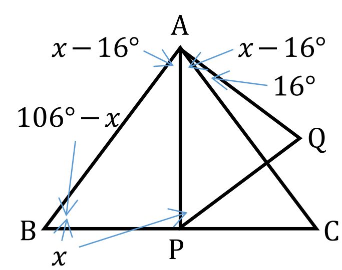
2020 State Team Round Solutions

1. The leftmost digit must be nonzero and even, and we want that digit as small as possible, which means 2. That leaves a needed sum of 12 for the remaining 4 digits, which has to be split across at least 2 digits. We must not forget that we may now use 0, and may use as many as two of them (remembering we need 2 more nonzero digits to contribute 12 to the sum), so let's make the next 2 digits 00. With the restriction of even and a sum of 12, the only options for the remaining digits are 48, 66 and 84, of which 48 is the smallest. Therefore, our desired number is **20,048**.

2. There are $2(20+12)-4=60$ posts (subtracting the 4 compensates for double counting the 4 corner posts, and, because the posts are placed in the form of a closed figure, there need to be 60 connections between posts for each wire. For each interpost and circumpost pair, $12+0.5=12.5$ of wire is needed. There are 60 such pairs for each wire and there are 3 wires. Therefore, the total length of wire needed is $3\times 60\times 12.5=\mathbf{2250}$ meters.

3. Let t represent the fraction of attempts at field goals worth 3 points; then $1-t$ represents the fraction of attempts at field goals worth 2 points, since all field goal are in one of those two categories. The average field goal point effectiveness is the sum over the two categories of the product of the point value of the category times the fraction of success for that category time the fraction of field goal attempts in that category: $1.0389=2\times 0.4993(1-t)+3\times 0.3651t=0.9986+0.0967t$, so $0.0967t=0.0403$ and $t=\frac{0.0403}{0.0967}=\frac{403}{967}=0.4167\ldots\times 100\ \%=41.67\ldots\ \%$, which rounds to **42**%.

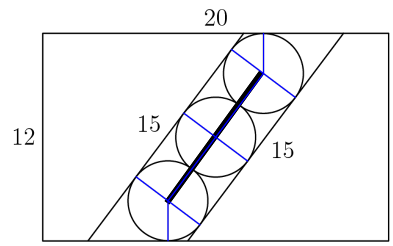
4. The isosceles triangles BAC and AQP are similar, so their base angles (PAQ, APQ, CBA and BCA) are congruent, and it the measure of the first, PAQ, that we are to determine. Let's call its measure x. Since we are given the measure of $\angle$CAQ as 16 degrees, the measure of $\angle$CAP is $x-16$. Now, since $\overline{\text{AP}}$ bisects the apex $\angle$CAB, that means that the measure of $\angle$CAB is $2x-32$ and each base angle has measure $90-\frac{1}{2}(2x-32)=106-x$; however, we also set the measure of the base angles to be x. Therefore, $x=106-x$, so $2x=106$ and $x=\mathbf{53}$ degrees.



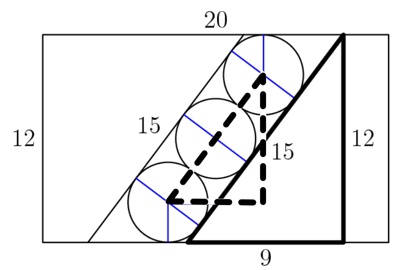
5. $(-8x;-8y)=☆(x,y)=(x+3y;9x-5y)$. For both components, the equality implies $9x+3y=0$, so $x=-\frac{1}{3}y$ and $\frac{x}{y}=-\mathbf{\frac{1}{3}}$.

6. When a point is rotated about a second point, the first point maintains a constant distance from the second point through the rotation from the pre-rotation position to the post-rotation. That means the pre-rotation position and the post-rotation position lie on a circle centered at the point about which the rotation is occurring. Therefore, the center point of the rotation lies on the perpendicular bisector of the line segment between the pre-rotation and post-rotation positions. A and A′ lie on the line $x=1$ and their midpoint is at $(1,6)$, so the perpendicular bisector is $y=6$. B and B′ lie on the line $y=8$ and their midpoint is at $(11,8)$, so the perpendicular bisector is $x=11$. The only point satisfying both equations is **(11, 6)**.

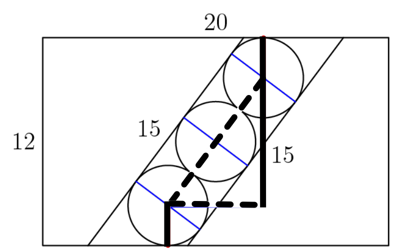
7. Let's call the radius of the circle r, and let's draw in all of the radii that might be useful:



Notice that the segment in the middle connecting the center of the bottom circle to the center of the top circle has length $4r$ and is parallel to the diagonal segments of length 15. And in particular, it is the hypotenuse of a right triangle that's similar to a 9-12-15 right triangle, as shown here:



This means that the dashed triangle has a height of $\frac{12}{15} \cdot 4r = \frac{16}{5}r$.



The total length of the solid vertical lines is $r + \frac{16}{5}r + r = \frac{26}{5}r$. But this must equal the height of the box, so $\frac{26}{5}r = 12$. Solving for r, we get $r = 12 \cdot \frac{5}{26} = \frac{60}{26} = \mathbf{\frac{30}{13}}$ feet. (Note that the length of the box was totally irrelevant. As long as the box is wide enough to fit the diagonal stripe, its length doesn't play any role in the solution.)

8. There are 2 ways for the addition of distinct primes to result in 10: 2 + 3 + 5 and 3 + 7.
For 2 + 3 + 5 we need numbers of the form $2^i 3^j 5^k$ where i, j and k are positive integers. The least such value is 30 and, indeed, all suitable numbers must be appropriate multiples of 30, that is of the form $30m$; m must consist only of factors of 2, 3 and 5 (not necessarily all, or even any, of those), and must be no more than 33 so that we do not exceed 1000. The acceptable values for m are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30 and 32—a count of 19.
For 3 + 7 we need numbers of the form $3^i 7^j$ where i and j are positive integers. The least such value is 21 and, indeed, all suitable numbers must be appropriate multiples of 21, that is of the form $21m$; m must consist only of factors of 3 and 7 (not necessarily both, or even either, of those), and must be no more than 47 so that we do not exceed 1000. The acceptable values for m are 1, 3, 7, 9, 21 and 27—a count of 6.
Therefore, the total count of integers from 2 to 1000, inclusive, with prime-sum radical being 10 is $19 + 6 = \mathbf{25}$ integers.

9. The tallest cheerleader must go in the middle of the back row. Randomly distribute the remaining 13 cheerleaders in the remaining 13 slots—that is 13! possible arrangements. Compare the leftmost cheerleaders in the two rows; there is a probability of $\frac{1}{2}$ that the taller is properly in the back row. Move one file to the right and again compare front and back row cheerleaders, with probability $\frac{1}{2}$ the taller of the two is in back. The same $\frac{1}{2}$ occurs for the third file. We must be careful now for the fourth (middle file), because we have already forced the tallest cheerleader to be in back at this location, so that file is guaranteed to have the taller in back. For each of the fifth, sixth, and rightmost files we have the probability $\frac{1}{2}$ of having the taller in the back row. Thus, we have a probability of $\left(\frac{1}{2}\right)^3 \times 1 \times \left(\frac{1}{2}\right)^3 = \frac{1}{64}$ of a taller cheerleader behind a shorter in each of the files. Now, there are $3! = 6$ arrangements of the leftmost 3 cheerleaders in the back row, with only 1 having the appropriate height drop-off going away from center, the probability of the correct arrangement for the back left is $\frac{1}{6}$. The same is true for the back right. Each of these considerations is independent of the others. Therefore, the number of distinct acceptable arrangements is $\frac{13!}{64\times6\times6} = \mathbf{2,702,700}$ ways.

10. Because we are dealing with remainders when products are divided by 30 and the count 900 of integers we have in scope is divisible by 30, we learn everything we need from modulo 30 arithmetic and processing only the integers 1 through 30; every other set of 30 consecutive integers behaves identically. Now, $30 = 2 \times 3 \times 5$ and of those prime factors, both 3 and 5 are relatively prime to the desired remainder value 4; if any of the 6 numbers multiplied together have 3 or 5 as a factor, then the product will have 3 or 5, respectively, as a factor, and that just does not happen with a remainder of 4 upon division by 30. Thus, we can throw away all the multiples of 3 or 5 from further consideration and analyze only 1, 2, 4, 7, 8, 11, 13, 14, 16, 17, 19, 22, 23, 26, 28 and 29 (a count of 16 values) rather than all 30 values from 1 through 30. Picking 6 of these 16 values randomly with replacement yields $16^6 = 16\,777\,216$ possible arrangements of factors that could yield a product leaving a remainder of 4 when divided by 30. The only way to get an odd remainder is if all 6 picked values are odd, which has a probability of $\frac{1}{2^6} = \frac{1}{64}$, so $\frac{16{,}777{,}216}{64} = 262{,}144$ cases can be thrown out, leaving 16 515 072 potential patterns leaving the desired remainder of 4 upon division by 30. However, it is equally likely that a product of 6 values taken (repetitions allowed) from our 16, with at least 1 of the 6 being even, will yield any of our even options—2, 4, 8, 14, 16, 22, 26, 28 as the remainder when dividing the product of the 6 values by 30. Therefore, $\frac{1}{8}$ of the 16,515,072 candidates for a total of 2,064,384 will yield the actual remainder of 4, out of $30^6 = 729{,}000{,}000$ total sequences of numbers that can be multiplied. This yields a probability of randomly picking 6 integers from 1 to 30 (or, equivalently, 1 through 900) yielding a product that leaves a remainder of 4 when divided by 30 of $\frac{2{,}064{,}384}{729{,}000{,}000} = 0.0028318 \ldots \times 100\ \% = \mathbf{0.283}\%$ when rounded to the nearest 0.001 %.