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MATH 2243: **Linear Algebra** & **Differential Equations**

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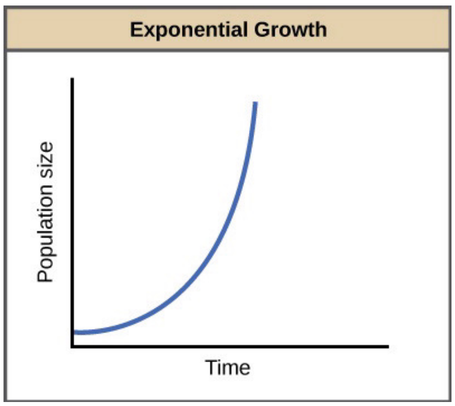
2.1: Mathematical Models and Numerical Methods

Population Models

Our previous (exponential) population model was: $\frac{dP}{dt} = kP$, with $P(t) = P_0 e^{kt}$.
where the birth and death rates were constant.



However, the following model is more realistic, with variable death and birth rates (but still assumes no immigration or emigration).

Let $\beta(t)$ and $\delta(t)$ be the (non-constant) number of births and deaths (respectively) per person (or animal), per unit of time. Then the change in population over time is:
$\frac{\Delta P}{\Delta t} = [In] - [Out] = \frac{[Born]}{\Delta t} - \frac{[Die]}{\Delta t} \approx \beta(t)P(t) - \delta(t)P(t) = (\beta(t) - \delta(t))P(t)$.
Taking the limit as $\Delta t \to 0\ldots$

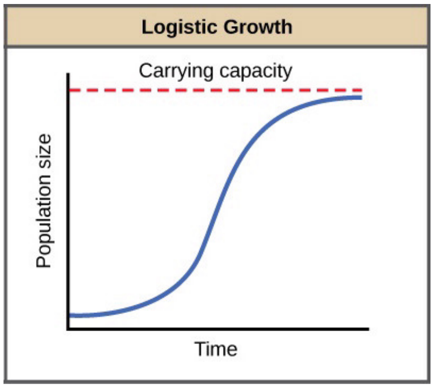
$\frac{dP(t)}{dt} = (\beta(t) - \delta(t))P(t)$, notice that when β and δ are constant, $\frac{dP}{dt} = kP$, where $k = \beta - \delta$.

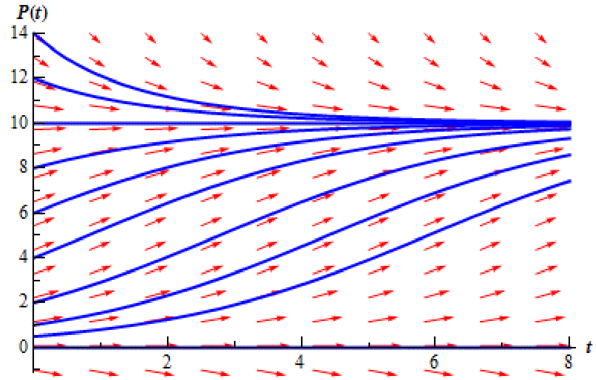
Bounded Populations

A common form (called the **logistic equation**) occors when β is linearly decreasing with respect to P (which means: $\beta = c - kP$, with $c, k > 0$), and the death rate δ is a constant:
$\frac{dP}{dt} = (\beta - \delta)P = (c - kP - \delta)P = kP\left(\frac{c-\delta}{k} - P\right) = kP(M - P)$, where $M = \frac{c-\delta}{k}$.

The reason we notate it this way is to isolate an important quantity M, which we can see on a graph, and is the "**limiting population**" or "**carrying capacity**."

Logistic Equation: $\frac{dP}{dt} = kP(M - P)$.



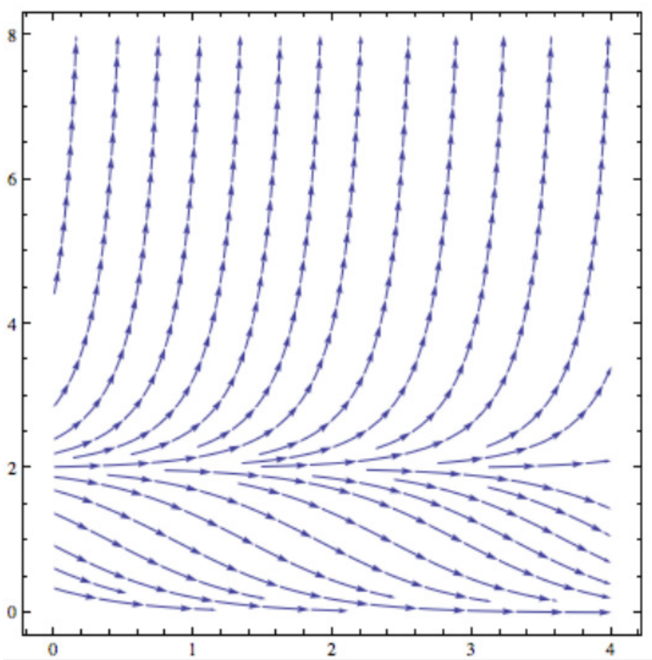


Using your newly acquired skill of separation-of-variables, you can now show that the initial value problem: $\frac{dP}{dt} = kP(M-P),\ P(0) = P_0$ has the solution $P(t) = \frac{MP_0}{P_0+(M-P_0)e^{-kMt}}$. (try this as an exercise!)

If initial population $P_0 = M$?

If initial population $P_0 > M$? If initial population $P_0 < M$?

Explosion or Extinction Model: $\frac{dP}{dt} = kP(P-M)$.



Problem: **#8** Separate variables and use partial fractions to solve the initial value problem:

$$\frac{dx}{dt} = 7x(x-13),\ x(0) = 17.$$

$\frac{1}{x(x-13)}dx = 7dt$, if $x \neq 13$ and $x \neq 0$.

Can $x(t) \equiv 13$ or $x(t) \equiv 0$, with our initial condition $x(0) = 17$?

$$\int \frac{1}{x(x-13)}dx = \int 7dt$$

We would prefer to integrate something more like $\int \frac{A}{x} + \frac{B}{x-13}\,dx = \int 7\,dt$.

Partial fractions:

Observe that $\frac{1}{x(x-13)} = \frac{A}{x} + \frac{B}{x-13}$, when $1 = A(x-13) + Bx$.

Collecting powers of x : $(A+B)x - 13A = 1$,

Comparing powers of x : $A + B = 0$ and $-13A = 1$,

so $A = -\frac{1}{13}$, and $B = \frac{1}{13}$.

Therefore: $\int \frac{1}{x(x-13)}\,dx = \int \frac{-\frac{1}{13}}{x} + \frac{\frac{1}{13}}{x-13}\,dx$.

And our equation becomes: $-\frac{1}{13}\int \frac{1}{x} - \frac{1}{x-13}\,dx = \int 7\,dt$

$\Rightarrow \int (\frac{1}{x} - \frac{1}{x-13})\,dx = -91\int dt$

$\Rightarrow \ln|x| - \ln|x-13| = -91t + c$

$\Rightarrow \ln\left|\frac{x}{x-13}\right| = -91t + c$

$\Rightarrow \frac{x}{x-13} = Ce^{-91t}$, where $C \neq 0$. Then ...?

Since $x(0) = 17$, $\frac{17}{17-13} = Ce^{-91\cdot 0} = C$, $\Rightarrow C = \frac{17}{4}$.

So, $\frac{x}{x-13} = \frac{17}{4}e^{-91t}$.

Solve explicitly for x?

$\Rightarrow 4x = 17(x-13)e^{-91t}$

$\Rightarrow 4x = 17xe^{-91t} - 221e^{-91t}$, $\Rightarrow 4x - 17xe^{-91t} = -\frac{221}{e^{91t}}$

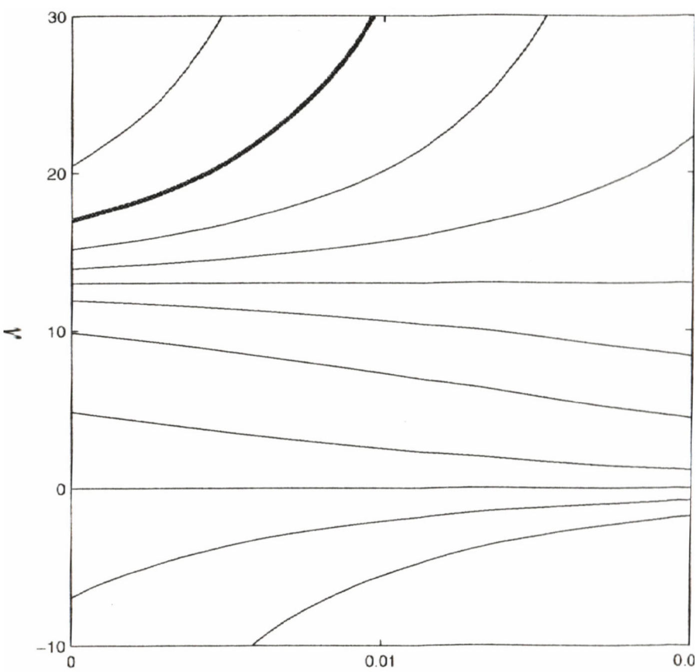
$\Rightarrow x(4 - 17e^{-91t}) = -\frac{221}{e^{91t}}, \qquad \Rightarrow x = -\frac{221}{(4-17e^{-91t})e^{91t}},$

(when $4 - 17e^{-91t} \neq 0$, which occurs when $t \neq -\frac{1}{91}\ln\frac{4}{17} \approx 0.016$)

$\Rightarrow x(t) = \frac{221}{17-4e^{91t}}.$

Use either the exact solution or a computer-generated slope field to sketch the graphs of several solutions of the given differential equation, and highlight the indicated particular solution.

Typical solution curves…



Problem: #13 Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$.

(a) Show that $P(t) = \frac{P_0}{1-kP_0t}$, with k constant: Note that $P(t) \to +\infty$ as $t \to \frac{1}{kP_0}$. This is doomsday.

$\beta := c_1P$ and $\delta := c_2P$, so $\beta - \delta = (c_1 - c_2)P = kP$.

Using the equation $\frac{dP}{dt} = (\beta - \delta)P$ from the book, and substituting from above gives us: $P' = kP^2$ with k positive.

Using separation of variables: $\int \frac{1}{P^2}dp = k\int dt \quad \Rightarrow \quad -P^{-1} = kt + C$

$\Rightarrow \quad P(t) = \frac{1}{C-kt}.$

The initial condition $P(0) = P_0$ then gives $P_0 = \frac{1}{C}$ or $C = \frac{1}{P_0}$. So $P(t) = \frac{1}{\frac{1}{P_0} - kt} = \frac{P_0}{1 - kP_0 t}$.

(**b**) **Suppose that** $P_0 = 6$ **and that there are nine rabbits after ten months**. **When does doomsday occur**?

If $P_0 = 6$, then $P(t) = \frac{6}{1-6kt}$.

Now the fact that $P(10) = 9$ implies that $9 = \frac{6}{1-60k}$, or $k = \frac{1}{180}$.

So $P(t) = \frac{6}{1-\frac{t}{30}} = \frac{180}{30-t}$. Hence it is clear that $P \to \infty$ as $t \to 30$ months (doomsday of infinite bunnies).

Problem: #22 Logistic Equation, $\frac{dP}{dt} = kP(M-P)$.

Suppose that at time $t = 0$, half of a "**logistic**" **population** of 100,000 persons have heard a certain rumor, and that the number of those who have heard it is then increasing at the rate of 1,000 persons per day. How long will it take for this rumor to spread to 80% of the population?

$P' = kP(M-P)$

What does P stand for? What is the carrying capacity?

We'll work in thousands of persons, so $M = 100$ and $P' = kP(100-P)$

Try substituting $P'(0) =$??

$P'(0) = 1$. So,

$$1 = P'(0) = kP_0(100 - P_0)$$

$$= 50k(100 - 50) = 2,500k$$

$$1 = 2,500k \quad \text{or} \quad k = 0.0004 \text{ and } P' = (0.0004)P(100 - P).$$

"**How long will it take for this rumor to spread to** 80% **of the population**?"

If t denotes the number of days until 80,000 people have heard the rumor, then Equation 7 in the text ($P(t) = \frac{MP_0}{P_0+(M-P_0)e^{-kMt}}$) gives…

$$80 = \frac{100 \cdot 50}{50+(100-50)e^{-0.04t}} = \frac{5000}{50+50e^{-0.04t}} \quad \Rightarrow \quad 50 + 50e^{-0.04t} = \frac{5000}{80} = 62.5$$

$$\Rightarrow e^{-0.04t} = \frac{12.5}{50} = \frac{1}{4}, \quad \Rightarrow \quad -0.04t = \ln\frac{1}{4} = -\ln 4$$

$$\Rightarrow t = \frac{\ln 4}{0.04} \approx 34.66 \text{ days.}$$

Thus the rumor will have spread to 80% of the population in a little less than 35 days.

Problem: **#34**

If $P(t)$ satisfies the logistic equation ($P' = kP(M - P)$), use the chain rule to show that…

$$P''(t) = 2k^2P(P - \tfrac{1}{2}M)(P - M).$$

Differentiation of both sides of the logistic equation $P' = kP \cdot (M - P)$ yields

$$P'' = \frac{dP'}{dP} \cdot \frac{dP}{dt} = [k(M - P) + kP(-1)] \cdot kP(M - P)$$

$$= k[M - P - P] \cdot kP(M - P) = k^2P(M - 2P)(M - P)$$

$= 2k^2P(P - \frac{1}{2}M)(P - M)$ as desired.

The question continues…"**Conclude that**:

- $P'' > 0$ if $0 < P < \frac{1}{2}M$
- $P'' = 0$ if $P = \frac{1}{2}M$
- $P'' < 0$ if $\frac{1}{2}M < P < M$
- $P'' > 0$ if $P > M$

In particular, it follows that any solution curve that crosses the line $P = \frac{1}{2}M$ has an inflection point where it crosses that line, and therefore resembles one of the lower S shaped curves in the graph of the logistic equation below…"