

29 Functions and their Graphs

The concept of a function was introduced and studied in Section 7 of these notes. In this section we explore the graphs of functions. Of particular interest, we consider the graphs of linear functions, quadratic functions, cubic functions, square root functions, and exponential functions. These graphs are represented in a coordinate system known as the **Cartesian coordinate system** which we explore next.

The Cartesian Plane

The Cartesian coordinate system was developed by the mathematician René Descartes in 1637. The **Cartesian coordinate system**, also known as the rectangular coordinate system or the xy-plane, consists of two number scales, called the **x-axis** and the **y-axis**, that are perpendicular to each other at point O called the **origin**. Any point in the system is associated with an **ordered pair** of numbers (x, y) called the **coordinates** of the point. The number x is called the **abscissa** or the **x-coordinate** and the number y is called the **ordinate** or the **y-coordinate**. The abscissa measures the distance from the point to the y-axis whereas the ordinate measures the distance of the point to the x-axis. Positive values of the x-coordinate are measured to the right, negative values to the left. Positive values of the y-coordinate are measured up, negative values down. The origin is denoted as $(0, 0)$.

The axes divide the coordinate system into four regions called **quadrants** and are numbered counterclockwise as shown in Figure 29.1

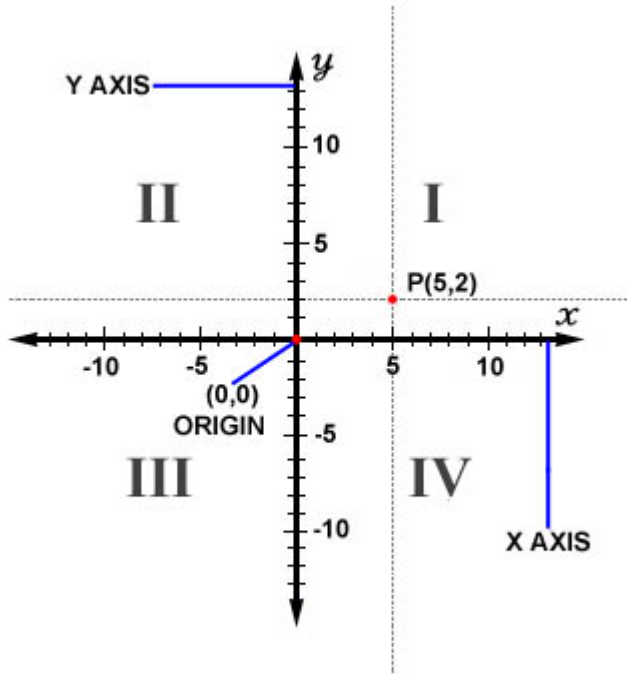
To **plot a point** $P(a, b)$ means to draw a dot at its location in the xy-plane.

Example 29.1

Plot the point P with coordinates $(5, 2)$.

Solution.

Figure 29.1 shows the location of the point $P(5, 2)$ in the xy-plane. ■



2 DIMENSIONAL CARTESIAN COORDINATE SYSTEM

Figure 29.1

Example 29.2

Complete the following table of signs of the coordinates of a point $P(x, y)$.

	x	y
Quadrant I		
Quadrant II		
Quadrant III		
Quadrant IV		
Positive x-axis		
Negative x-axis		
Positive y-axis		
Negative y-axis		

Solution.

	x	y
Quadrant I	+	+
Quadrant II	-	+
Quadrant III	-	-
Quadrant IV	+	-
Positive x-axis	+	0
Negative x-axis	-	0
Positive y-axis	0	+
Negative y-axis	0	-

When Does a Graph Represent a Function

To say that y is a function of x means that for each value of x there is exactly one value of y . Graphically, this means that each vertical line must intersect the graph at most once. Hence, to determine if a graph represents a function one uses the following simple visual test:

Vertical Line Test: A graph is a function if and only if every vertical line crosses the graph at most once.

According to the vertical line test and the definition of a function, if a vertical line cuts the graph more than once, the graph could not be the graph of a function since we have multiple y values for the same x -value and this violates the definition of a function.

Example 29.3

Which of the graphs (a), (b), (c) in Figure 29.2 represent y as a function of x ?

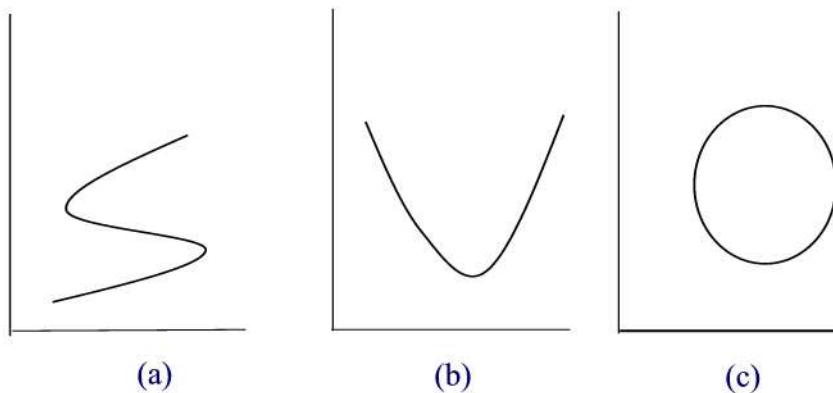


Figure 29.2

Solution.

By the vertical line test, (b) represents a function whereas (a) and (c) fail to represent functions since one can find a vertical line that intersects the graph more than once. ■

The **domain** of a function is the collection of all possible x-coordinates that can be used in the formula of the function. For example, $x = 1$ is in the domain of $f(x) = x + 1$ since $f(1) = 1 + 1 = 2$ whereas $x = 1$ is not in the domain of $f(x) = \frac{1}{x-1}$ since $f(1) = \frac{1}{0}$ which is undefined.

The collection of all values of y-coordinates that correspond to the x-coordinates is called the **range** of the function. For example, the range of $f(x) = \sqrt{x-1}$ is the interval $[0, \infty)$ whereas that of the function $f(x) = \frac{1}{x-1}$ is the set $\mathbb{R} - \{0\}$.

Practice Problems

Problem 29.1

Plot the points whose coordinates are given on a Cartesian coordinate system.

- (a) $(2, 4), (0, -3), (-2, 1), (-5, -3)$.
- (b) $(-3, -5), (-4, 3), (0, 2), (-2, 0)$.

Problem 29.2

Plot the following points using graph papers.

(a) (3,2) (b) (5,0) (c) (0,-3) (d) (-3,4) (e) (-2,-3) (f) (2,-3)

Problem 29.3

Complete the following table.

(x,y)	$x > 0, y > 0$	$x < 0, y > 0$	$x > 0, y < 0$	$x < 0, y < 0$
Quadrant				

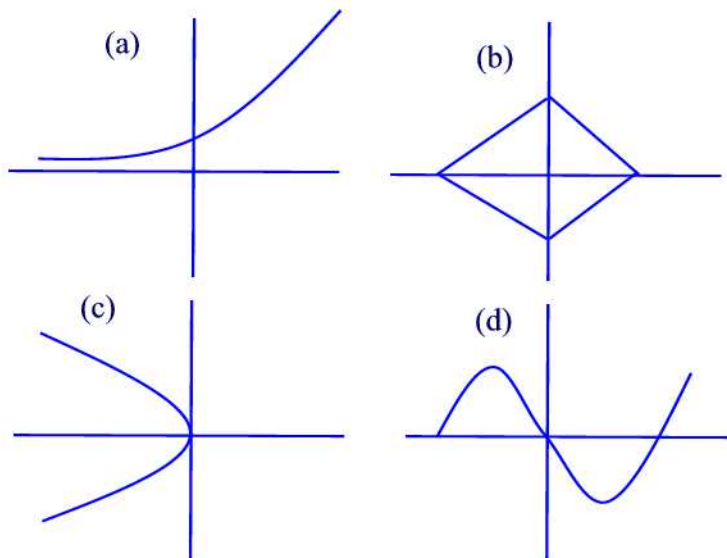
Problem 29.4

In the Cartesian plane, shade the region consisting of all points (x, y) that satisfy the two conditions

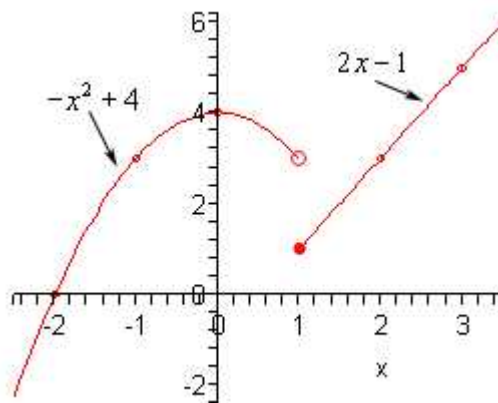
$$-3 \leq x \leq 2 \text{ and } 2 \leq y \leq 4$$

Problem 29.5

Determine which of the following graphs represent a function.

**Problem 29.6**

Consider the function f whose graph is given below.



(a) Complete the following table

x	-2	-1	0	1	2	3
f(x)						

(b) Find the domain and range of f .

(c) For which values of x is $f(x) = 2.5$?

Graphs of Linear Functions

A **linear** function is any function that can be written in the form $f(x) = mx + b$. As the name suggests, the graph of such a function is a straight line.

Example 29.4

The sales tax on an item is 6%. So if p denotes the price of the item and C the total cost of buying the item then if the item is sold at \$ 1 then the cost is $1 + (0.06)(1) = \$1.06$ or $C(1) = \$1.06$. If the item is sold at \$2 then the cost of buying the item is $2 + (0.06)(2) = \$2.12$, or $C(2) = \$2.12$, and so on. Thus we have a relationship between the quantities C and p such that each value of p determines exactly one value of C . In this case, we say that C is a function of p . Find a formula for p and graph.

Solution.

The chart below gives the total cost of buying an item at price p as a function of p for $1 \leq p \leq 6$.

p	1	2	3	4	5	6
C	1.06	2.12	3.18	4.24	5.30	6.36

The graph of the function C is obtained by plotting the data in the above table. See Figure 29.3.

The formula that describes the relationship between C and p is given by

$$C(p) = 1.06p. \blacksquare$$

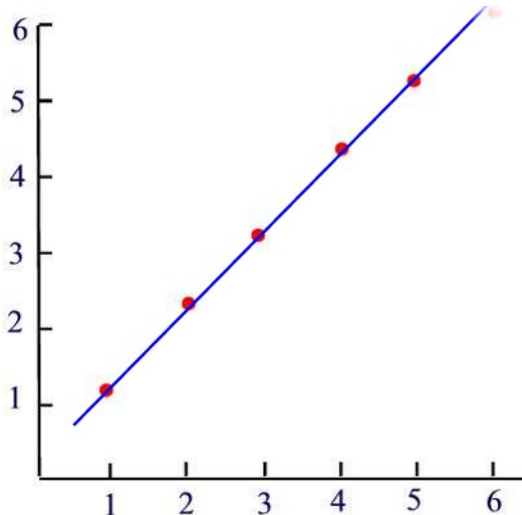


Figure 29.3

Graphs of Quadratic Functions

You recall that a linear function is a function that involves a first power of x . A function of the form

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

is called a **quadratic function**. The word "quadratus" is the latin word for a square.

Quadratic functions are useful in many applications in mathematics when a linear function is not sufficient. For example, the motion of an object thrown either upward or downward is modeled by a quadratic function.

The graph of a quadratic function is a curve called a **parabola**. Parabolas may open upward or downward and vary in "width" or "steepness", but they all have the same basic "U" shape.

All parabolas are symmetric with respect to a line called the **axis of symmetry**. A parabola intersects its axis of symmetry at a point called the **vertex** of the parabola.

Many quadratic functions can be graphed easily by hand using the techniques of stretching/shrinking and shifting (translation) the parabola $y = x^2$.

Example 29.5

Sketch the graph of $y = \frac{x^2}{2}$.

Solution.

Starting with the graph of $y = x^2$, we shrink by a factor of one half. This means that for each point on the graph of $y = x^2$, we draw a new point that is one half of the way from the x-axis to that point. See Figure 29.4 ■

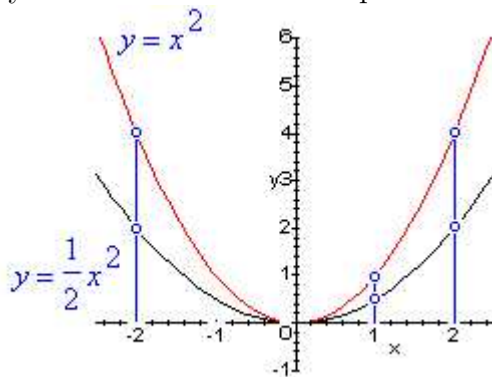


Figure 29.4

When a quadratic function is in standard form, then it is easy to sketch its graph by reflecting, shifting, and stretching/shrinking the parabola $y = x^2$.

The quadratic function $f(x) = a(x - h)^2 + k$, a not equal to zero, is said to be in **standard form**. If a is positive, the graph opens upward, and if a is negative, then it opens downward. The line of symmetry is the vertical line $x = h$, and the vertex is the point (h, k) .

Any quadratic function can be rewritten in standard form by completing the square. Note that when a quadratic function is in standard form it is also easy to find its zeros by the square root principle.

Example 29.6

Write the function $f(x) = x^2 - 6x + 7$ in standard form. Sketch the graph of f and find its zeros and vertex.

Solution.

Using completing the square method we find

$$\begin{aligned}
 f(x) &= x^2 - 6x + 7 \\
 &= (x^2 - 6x) + 7 \\
 &= (x^2 - 6x + 9 - 9) + 7 \quad (\text{Just square } \frac{6}{2}) \\
 &= (x^2 - 6x + 9) - 9 + 7 \\
 &= (x - 3)^2 - 2
 \end{aligned}$$

From this result, one easily finds the vertex of the graph of f is $(3, -2)$. To find the zeros of f , we set f equal to 0 and solve for x .

$$\begin{aligned}
 (x - 3)^2 - 2 &= 0 \\
 (x - 3)^2 &= 2 \\
 x - 3 &= \pm\sqrt{2} \\
 x &= 3 \pm \sqrt{2}
 \end{aligned}$$

Finally, the graph of f is given in Figure 29.5■

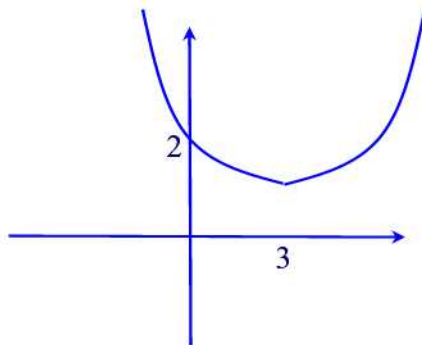


Figure 29.5

Graphs of Exponential Functions

An **exponential function** is a function that can be written in the form

$$f(t) = b \cdot a^t$$

where a is positive and different from 1. We call a the **base** of the function. Figure 29.6 shows the graph of an exponential function.

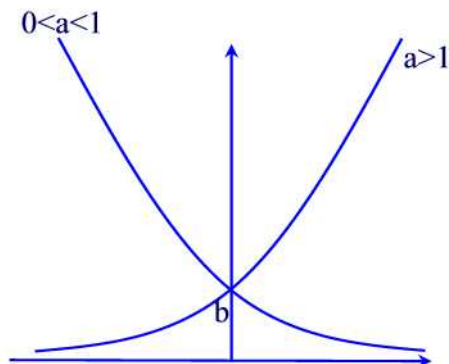


Figure 29.6

For $a > 1$ the function is increasing. In this case, we say that the function represents an exponential **growth**. If $0 < a < 1$ then the function represents an exponential **decay**.

Remark 29.1

Why a is restricted to $a > 0$ and $a \neq 1$? Since t is allowed to have any value then a negative a will create meaningless expressions such as \sqrt{a} (if $t = \frac{1}{2}$). Also, for $a = 1$ the function $P(t) = b$ is called a **constant function** and its graph is a horizontal line.

Miscellaneous Functions

We consider some of the graphs of some important functions.

Square Root Function

The square root function is the function $f(x) = \sqrt{x}$. To get the graph well just plug in some values of x and then plot the points.

x	0	1	4	9
f(x)	0	1	2	3

The graph is given in Figure 29.7

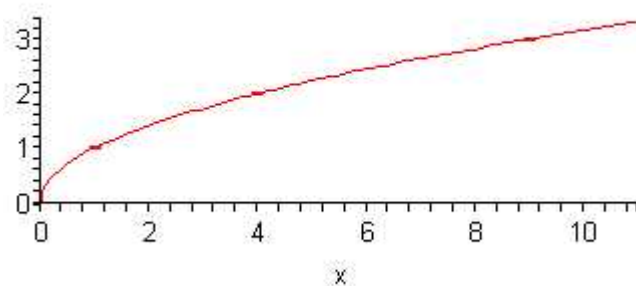


Figure 29.7

Absolute Value Function

We've dealt with this function several times already. It's now time to graph it. First, let's remind ourselves of the definition of the absolute value function.

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Finding some points to plot we get

x	-2	-1	0	1	2
f(x)	2	1	0	1	2

The graph is given in Figure 29.8

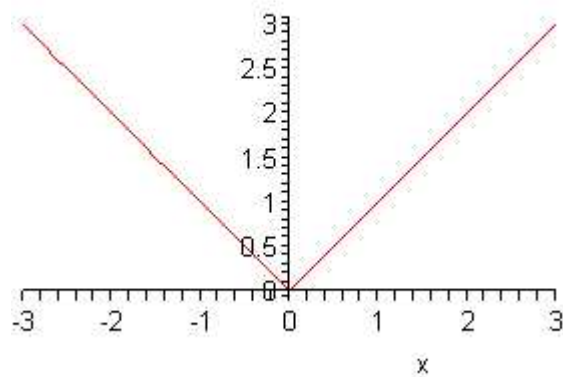


Figure 29.8

Cubic Function

We will consider the following simple form of a cubic function $f(x) = x^3$. First, we find some points on the graph.

x	-2	-1	0	1	2
f(x)	-8	-1	0	1	8

The graph is given in Figure 29.9

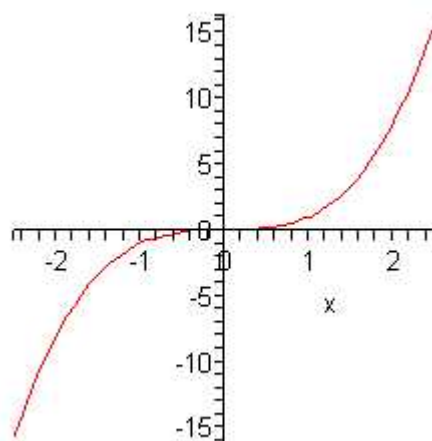


Figure 29.9

Step Functions

The far we have considered functions whose graphs are "continuous", that is, the graphs have no holes or jumps. Step functions are functions that are not continuous.

Example 29.7

The charge for a taxi ride is \$1.50 for the first $\frac{1}{5}$ of a mile, and \$0.25 for each additional $\frac{1}{5}$ of a mile (rounded up to the nearest $\frac{1}{5}$ mile).

- Sketch a graph of the cost function C as a function of the distance traveled x , assuming that $0 \leq x \leq 1$.
- Find a formula for C in terms of x on the interval $[0, 1]$.
- What is the cost for a $\frac{4}{5}$ - mile ride?

Solution.

(a) The graph is given in Figure 29.10.

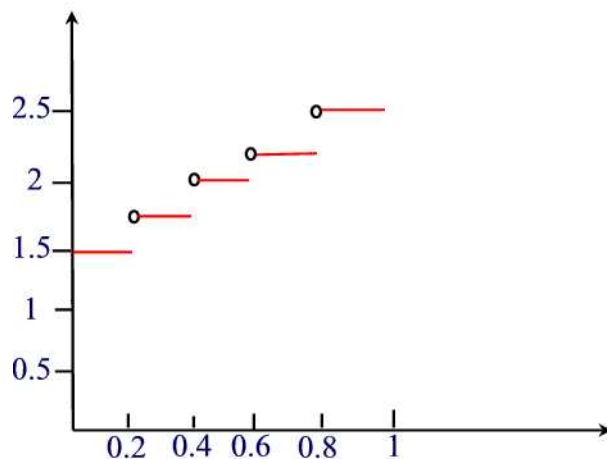


Figure 29.10

(b) A formula of $C(x)$ is

$$C(x) = \begin{cases} 1.50 & \text{if } 0 \leq x \leq \frac{1}{5} \\ 1.75 & \text{if } \frac{1}{5} < x \leq \frac{2}{5} \\ 2.00 & \text{if } \frac{2}{5} < x \leq \frac{3}{5} \\ 2.25 & \text{if } \frac{3}{5} < x \leq \frac{4}{5} \\ 2.50 & \text{if } \frac{4}{5} < x \leq 1. \end{cases}$$

(c) The cost for a $\frac{4}{5}$ ride is $C(\frac{4}{5}) = \$2.25$. ■

Practice Problems

Problem 29.7

Make a table of five values of the function $f(x) = 2x + 3$ and then use the points to sketch the graph of $f(x)$.

Problem 29.8

Make a table of five values of the function $f(x) = \frac{1}{2}x^2 + x$ and then use the points to sketch the graph of $f(x)$.

Problem 29.9

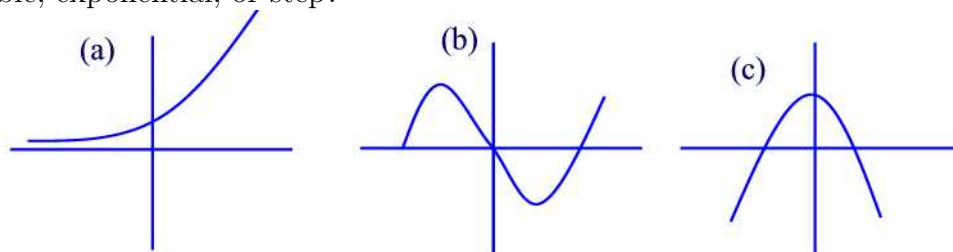
Make a table of five values of the function $f(x) = 3^x$ and then use the points to sketch the graph of $f(x)$.

Problem 29.10

Make a table of five values of the function $f(x) = 2 - x^3$ and then use the points to sketch the graph of $f(x)$.

Problem 29.11

Which type of function best fits each of the following graphs: linear, quadratic, cubic, exponential, or step?

**Problem 29.12**

Suppose that a function f is given by a table. If the output changes by a fixed amount each time the input changes by a constant then the function is linear. Determine whether each of the following functions below are linear.

x	0	2	4	6
f(x)	20	40	80	160

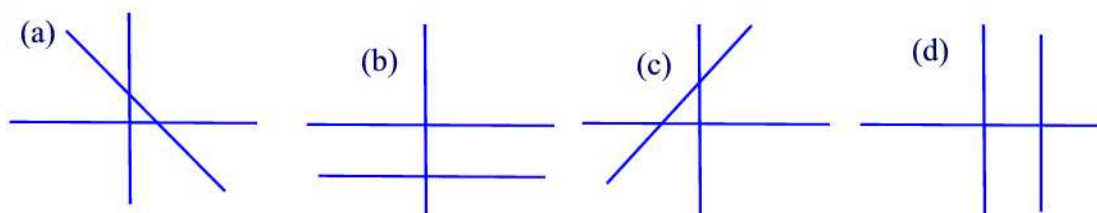
x	10	20	30	40
g(x)	6	12	18	24

Problem 29.13

Show how to solve the equation $2x + 3 = 11$ using a calculator.

Problem 29.14

In the linear function $f(x) = mx + b$ the parameter m is called the **slope**. The slope of the line determines whether the line rises, falls, is vertical or horizontal. Classify the slope of each line as positive, negative, zero, or undefined.



Problem 29.15

Algebraically, one finds the slope of a line given two points (x_1, y_1) and (x_2, y_2) on the line by using the formula

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Would the ratio

$$\frac{y_1 - y_2}{x_1 - x_2}$$

give the same answer? Explain.

Problem 29.16

Water is being pumped into a tank. Reading are taken every minutes.

Time(min)	0	3	6	9	12
Quarts of water	0	90	180	270	360

- Plot the 5 points.
- What is the slope of the line joining the 5 points?
- Estimate how much water is in the tank after 1 minute.
- At what rate is the water being pumped in?

Problem 29.17

- Graph $f(x) = x^2 - 2$ by plotting the points $x = -2, -1, 0, 1, 2$.
- How is this graph related to the graph of $y = x^2$?

Problem 29.18

- Graph $f(x) = x^2 + 3$ by plotting the points $x = -2, -1, 0, 1, 2$.
- How is this graph related to the graph of $y = x^2$?

Problem 29.19

- Graph $f(x) = (x - 1)^2$ by plotting the points $x = -2, -1, 0, 1, 2$.
- How is this graph related to the graph of $y = x^2$?

Problem 29.20

- Graph $f(x) = (x + 2)^2$ by plotting the points $x = -2, -1, 0, 1, 2$.
- How is this graph related to the graph of $y = x^2$?

Problem 29.21

Graph each equation by plotting points that satisfy the equation.

(a) $x - y = 4$.

(b) $y = -2|x - 3|$.

(c) $y = \frac{1}{2}(x - 1)^2$.

Problem 29.22

Find the x- and y-intercepts of each equation.

(a) $2x + 5y = 12$.

(b) $x = |y| - 4$.

(c) $|x| + |y| = 4$.