2D Electrostatics and the Density of Quantum Fluids

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References

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E.H. Lieb, N. Rougerie, JY, *A universal density bound for perturbations of the Laughlin liquid*, **preprint**

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See also:

N. Rougerie, S. Serfaty, J.Y., *Quantum Hall states of bosons in rotating anharmonic traps, Phys. Rev. A* **87**, 023618 (2013); arXiv:1212.1085

N. Rougerie, S. Serfaty, J.Y., *Quantum Hall Phases and the Plasma Analogy in Rotating Trapped Bose Gases, J. Stat. Phys*, **154**, 2–50 (2014), arXiv:1301.1043

The Laughlin wave function

The Laughlin wave function, suggested by Laughlin in 1983 as a variational ansatz for the ground state of a 2D electron gas in a strong perpendicular magnetic field, has the form

$$\Psi^{(\ell)}_{\rm Laugh} = C_{N,\ell} \prod_{i<j} (z_i - z_j)^\ell e^{-\sum_{i=1}^N |z_i|^2/2}$$

with ℓ odd ≥ 3 and $C_{N,\ell}$ a normalization constant. The factors $(z_i - z_j)^\ell$ strongly suppress a repulsive interaction between the particles.

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This function is basic for the understanding of the FQHE. One can also consider such functions for bosons with ℓ is even and ≥ 2.

Such a function describes a highly correlated state of a quantum fluid. It is important to understand the robustness of the built-in correlations when the system is perturbed by an external potential V.

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Methaphoric picture of the N-particle density (not due to Laughlin!):

The particles change places randomly but in a correlated way, as tightly packed as the factors $(z_i - z_j)^\ell$ allow, like huddling emperor penguins during an Antarctic winter. Each "penguin" claims on the average an area $\ell\pi$.

The plasma analogy

Laughlin's argument for the density $(\ell\pi)^{-1}$ is more mathematical. It is based on the "plasma analogy":

The N-particle density $|\Psi^{(\ell)}_{\rm Laugh}|^2$ can be interpreted as the Boltzmann-Gibbs factor at temperature $T = N^{-1}$ of classical 2D jellium, i.e., a 2D Coulomb gas in a uniform neutralizing background. A mean field approximation leads to the claimed density.

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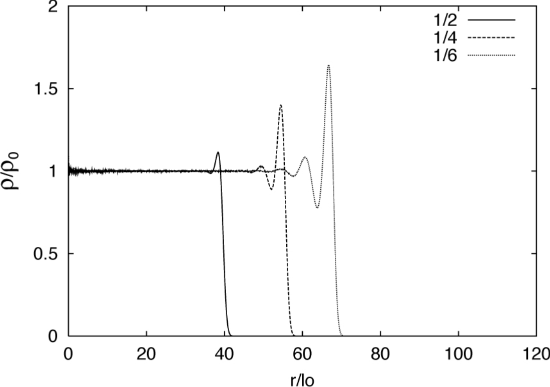
Numerical calculations (O. Ciftja) show, however, that the density may be considerably larger than $(\ell\pi)^{-1}$ close to the edge. The result can thus only hold in a suitable weak sense in the limit $N \to \infty$.



1/2
1/4
1/6
2
1.5
1
0.5
0
ρ/ρ0
0
20
40
60
80
100
120
r/lo

The density as a Boltzmann-Gibbs factor

We denote $(z_1, \ldots, z_N)$ by Z for short and consider the scaled N particle probability density (normalized to 1)

$$\mu^{(N)}(Z) = N^N \left|\Psi^{(\ell)}_{\rm Laugh}(\sqrt{N}Z)\right|^2 .$$

We can write

$$\mu^{(N)}(Z) \quad = \quad \mathcal{Z}_N^{-1} \exp\left(-N\sum_{j=1}^{N}|z_j|^2 + 2\ell\sum_{i<j}\log|z_i - z_j|\right)$$

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$$\begin{aligned}\mu^{(N)}(Z) &= \mathcal{Z}_N^{-1} \exp\left(-N\sum_{j=1}^{N}|z_j|^2 + 2\ell\sum_{i<j}\log|z_i - z_j|\right)\\ &= \mathcal{Z}_N^{-1}\exp\left(-\tfrac{1}{T}\mathcal{H}_N(Z)\right),\end{aligned}$$

with $T = N^{-1}$ and

$$\mathcal{H}_N(Z) = \sum_{j=1}^{N}|z_j|^2 + \frac{2\ell}{N}\sum_{i<j}\log\frac{1}{|z_i - z_j|}.$$

The free energy functional

The probability measure $\mu^{(N)}(Z)$ minimizes the free energy functional

$$\mathcal{F}(\mu) = \int \mathcal{H}_N(Z)\mu(Z) + T \int \mu(Z) \log \mu(Z)$$

for this Hamiltonian at $T = N^{-1}$.

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for this Hamiltonian at $T = N^{-1}$.

The $N \to \infty$ limit is in this interpretation a mean field limit where at the same time $T \to 0$. It is thus not unreasonable to expect that for large N, in a suitable sense

$$\mu^{(N)} \approx {\rho^{\mathrm{mf}}}^{\otimes N}$$

with a one-particle density ρ^{mf} minimizing a mean field free energy functional.

$$\mathcal{F}^{\mathrm{mf}}[\rho] = \int_{\mathbb{R}^2} |z|^2\rho + \ell \int\int \rho(z)\log\frac{1}{|z-z'|}\rho(z') + N^{-1}\int_{\mathbb{R}^2}\rho\log\rho.$$

The mean field limit

A rigorous estimate proving the convergence of the true 1-particle density to the mean field density ρ^{mf} was derived by Rougerie, Serfaty and JY in 2013:

Theorem (Comparison of true density and mean field density)

There exists a constant $C > 0$ such that for large enough N and any $U \in H^1(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$

$$\left| \int_{\mathbb{R}^2} \left(\mu^{(1)} - \rho^{\mathrm{mf}} \right) U \right| \leq C (\log N/N)^{1/2} \|\nabla U\|_{L^1} + C N^{-1} \|\nabla^2 U\|_{L^\infty}.$$

Properties of the Mean Field Density

The picture of the 1-particle density arises from asymptotic formulas for the mean-field density. The latter is, for large N, well approximated by a density $\hat{\rho}^{\rm mf}$ that minimizes the mean field functional without the entropy term.

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The variational equation satisfied by this density is:

$$|z|^2 - 2\ell\hat{\rho}^{\mathrm{mf}} * \log|z| - C \geq 0$$

with "$=$" where $\hat{\rho}^{\mathrm{mf}} > 0$ and "$>$" where $\rho^{\mathrm{mf}} = 0$.

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Applying the Laplacian gives

$$1 - \ell\pi\,\hat{\rho}^{\mathrm{mf}}(z) = 0$$

where $\hat{\rho} > 0$. Hence $\hat{\rho}^{\mathrm{mf}}$ takes the constant value $(\ell\pi)^{-1}$ on its support.

The Extended Laughlin phase

To allow for a response of the Laughlin state to an external potential V we consider the extended Laughlin phase, $\mathcal{L}_{\ell,N}$, defined as the space of square integrable wave functions of the form

$$\Psi = \phi(z_1, \ldots, z_N)\Psi^{(\ell)}_{\rm Laugh}$$

with ϕ holomorphic and symmetric.

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$$\Psi = \phi(z_1, \dots, z_N)\Psi^{(\ell)}_{\mathrm{Laugh}}$$

with ϕ holomorphic and symmetric.

The space $\mathcal{L}_{\ell,N}$ consist exactly of functions that

- belong to the LLL
- vanish at least as $(z_i - z_j)^\ell$ as z_i and z_j come together, to avoid a strong repulsive interaction.

The effect of the prefactor ϕ

Minimizing in $\mathcal{L}_{\ell,N}$ the expectation value of the physical many-body Hamiltonian with a contact interaction leads to a variational problem of a special kind where only the integral of the density against the potential enters. Changing V changes the optimal prefactor ϕ, and thus deforms the density distribution. However, we claim that

The density of any function in $\mathcal{L}_{\ell,N}$ is still bounded above by $(\pi\ell)^{-1}$ (in a suitable weak sense).

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The density of any function in $\mathcal{L}_{\ell,N}$ is still bounded above by $(\pi\ell)^{-1}$ (in a suitable weak sense).

In other words: **The prefactor can only shift or decrease the density, not increase it.**

We refer to this property as the **incompressibility** of the extended Laughlin phase.

The Main Theorem

Define

$$E_\ell(V,N) = \inf\left\{ \int V(z)\mu^{(1)}(z)dz \ : \Psi \in \mathcal{L}_\ell^N \right\}$$

and the **'bathtub energy'**

$$E_\ell^{\rm bt}(V) = \inf\left\{ \int V(z)\rho(z)dz \ : 0 \le \rho \le (\ell\pi)^{-1}, \ \int \rho = 1 \right\}.$$

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Theorem (Optimal incompressibility bound)

For any $V\in C^2(\mathbb{R}^2)$

$$\liminf_{N\to\infty} E_\ell(V,N) \ge E_\ell^{\rm bt}(V).$$

Moreover, V *is radially symmetric, monotonously increasing and polynomially bounded, equality holds and is asymptotically achieved for the Laughlin wave function.*

Why is a density bound important?

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- demonstrates the strong resistance of the Laughlin wave function against compression
- justifies the neglect of disorder and/or small external electric field
- justifies Laughlin's argument for Hall conductivity $1/\ell$ since an electric current moves charges transversally but cannot accumulate them
- provides a clear sign for a transition from a BEC regime to a FQHE regime in a rotating Bose gas

The general Gibbs-Boltzmann factor

The first step in a proof of the theorem is to write the N-particle density as

$$\mu^{(N)}(Z) = \frac{1}{\mathcal{Z}_N} \exp\left(-\frac{1}{T}\mathcal{H}(Z)\right)$$

where $T = \frac{1}{N}$.

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where $T = \frac{1}{N}$.

The classical Hamiltonian is now of the form

$$\mathcal{H}(Z) = \sum_{j=1}^{N} |z_j|^2 + \frac{2\ell}{N} \sum_{1\leq i<j\leq N} \log \frac{1}{|z_i - z_j|} + \mathcal{W}(Z).$$

with

$$\mathcal{W}(Z) := -\frac{2}{N} \log \left|\phi\left(\sqrt{N}\, Z\right)\right|.$$

Special ϕ

A mean field analysis, using a theorem of Diaconis and Freedman, is possible for a subclass of wave functions in $\mathcal{L}_\ell^N$, namely for ϕ of the form

$$\prod_{j=1}^{N} f_1(z_j) \prod_{(i,j)\in\{1,\ldots,N\}} f_2(z_i, z_j),$$

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The method extends, with some additional complications, to more general ϕ:

$$\prod_{j=1}^{N} f_1(z_j) \prod_{(i,j)\in\{1,\dots,N\}} f_2(z_i, z_j) \dots \prod_{(i_1,\dots,i_n)\in\{1,\dots,N\}} f_n(z_{i_1}, \dots, z_{i_n}).$$

with n fixed, or not growing too fast with N).

General ϕ

For general $\phi(Z)$, the potential $\mathcal{W}(Z)$ is a genuine N-body interaction, and a mean field approximation is at present out of reach.

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However, $\mathcal{W}(Z)$ has the important property of being superharmonic in each variable, i.e..

$$-\Delta_{z_i}\mathcal{W}(Z) \geq 0$$

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However, $\mathcal{W}(Z)$ has the important property of being superharmonic in each variable, i.e..

$$-\Delta_{z_i}\mathcal{W}(Z) \geq 0$$

for each z_i.

Using this property one can apply a different method to obtain an optimal incompressibility bound for all functions in $\mathcal{L}_\ell^N$.

The method is based on 2D electrostatics and the analysis of a Thomas-Fermi model of a special kind.

A crucial property of $\mathcal{H}(Z)$

With these methods it can be proved that the density of the points z_i^0 in a minimizing configuration Z^0 for $\mathcal{H}(Z)$ is, for large N, bounded by $(\pi\ell)^{-1}$.

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With these methods it can be proved that the density of the points z_i^0 in a minimizing configuration Z^0 for $\mathcal{H}(Z)$ is, for large N, bounded by $(\pi\ell)^{-1}$.

More precisely: Define

$$\rho^0(z) = \frac{1}{N}\sum_{j=1}^{N}\delta(z - z_i^0).$$

There exists a nonnegative bounded function $\tilde{\rho}^0$ of integral 1 such that

$$\tilde{\rho}^0(z) \leq \frac{1}{\pi\ell}(1 + o(1))$$

and for any differentiable function f on $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} (\rho^0 - \tilde{\rho}^0)\, f \to 0$$

in the limit $N \to \infty$.

An auxiliary TF model

The proof of the *crucial property* is based on a study of an auxiliary Thomas-Fermi model.

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For fixed points $x_i \in \mathbb{R}^2$ ("nuclei") we define a functional of functions $\sigma(\cdot)$ on $\mathbb{R}^2$ ("electron density") by

$$\mathcal{E}[\sigma] = -\int V_{\mathrm{nucl}}(x)\sigma(x)\,dx + D(\sigma,\sigma)$$

with

$$V_{\mathrm{nucl}}(x) = \sum_{i=1}^{n} \log \frac{1}{|x - x_i|},$$

$$D(\sigma,\sigma') = \tfrac{1}{2}\int\int \sigma(x)\log\frac{1}{|x - x'|}\sigma'(x')\,dx\,dx'$$

and the conditions

$$\int \sigma(x)\log(1+|x|)dx < \infty, \qquad 0 \leq \sigma(x) \leq 1, \qquad \int \sigma(x)dx = n.$$

Basic properties

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- The TF equation holds:

$$\Phi^{\mathrm{TF}}(x) = \begin{cases} \geq 0 & \text{if } \sigma^{\mathrm{TF}}(x) = 1 \\ 0 & \text{if } \sigma^{\mathrm{TF}}(x) = 0 \end{cases}$$

where

$$\Phi^{\mathrm{TF}}(x) = V_{\mathrm{nucl}}(x) - \log \frac{1}{|\cdot|} * \sigma^{\mathrm{TF}}(x)$$

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is the electrostatic potential.

The derivation of these properties requires some effort because the TF model is of a singular type and standard methods have to be modified.

The support of σ^{TF}

According to the TF equation the support of σ^{TF} is the same as the support of the potential Φ^{TF} which is continuous away from the 'nuclei'. Denote by

$$\mathcal{B}(x_1, \dots, x_n)$$

the open set where Φ^{TF} is strictly larger than 0. The support is the closure $\bar{\mathcal{B}}$ and Φ^{TF} is zero on the boundary $\partial\mathcal{B}$.

The support of $\sigma^{\rm TF}$

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the open set where $\Phi^{\rm TF}$ is strictly larger than 0. The support is the closure $\bar{\mathcal{B}}$ and $\Phi^{\rm TF}$ is zero on the boundary $\partial\mathcal{B}$.

Important properties:

- The area of $\mathcal{B}(x_1, \dots, x_n)$ is equal to n.

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- The area of $\mathcal{B}(x_1, \dots, x_n)$ is equal to n.
- $\mathcal{B}(x_1, \dots, x_{n-1}) \subset \mathcal{B}(x_1, \dots, x_n)$.
- For a single nucleus at x_i, $\mathcal{B}(x_i)$ is the disc with center at x_i and radius $\pi^{-1/2}$.

The "exclusion rule" for minimizing configurations

Consider a classical jellium Hamiltonian with an additional plurisuperharmonic term:

$$H(x_1, \ldots, x_N) = \frac{\pi}{2} \sum_{i=1}^{N} |x_i|^2 + \sum_{1 \leq i < j \leq N} \log \frac{1}{|x_i - x_j|} + W(x_1, \ldots, x_N)$$

with W symmetric and superharmonic in each variable x_i.

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with W symmetric and superharmonic in each variable x_i.

Proposition (Exclusion rule for minimizers)

If $(x_1^0,\dots,x_N^0)$ is a minimizing configuration for H, then for all $1\le n\le N-1$,

$$x_i^0\notin\mathcal{B}(x_1^0,\dots,x_n^0)\quad \textit{for}\quad i=n+1,\dots,N.$$

Proof of the exclusion rule

Fix x_j^0, $j \neq i = n+1$ and consider the function

$$F(x) = H(x_1^0, \ldots, x_n^0, x, x_{n+2}^0, \cdots x_N^0)$$

We show that if $x \in \mathcal{B}(x_1^0, \ldots, x_n^0) \equiv \mathcal{B}$ then there is a $\hat{x} \in \partial\mathcal{B}$ such that $F(\hat{x}) < F(x)$.

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Adding and subtracting a term $\log \frac{1}{|x|} * \mathbb{1}_{\mathcal{B}}$ we can write

$$F(x) = \Phi(x) + R(x)$$

with

$$\Phi(x) = \sum_{i=1}^{n} \log \frac{1}{|x - x_i^0|} - \int_{\mathcal{B}} \log \frac{1}{|x - x'|} dx'$$

and

$$R(x) = \frac{\pi}{2}|x|^2 + \int_{\mathcal{B}} \log \frac{1}{|x - x'|} dx' + \sum_{i=n+2}^{N} \log \frac{1}{|x - x_i^0|} + W(x) + \text{const.}$$

Proof of the exclusion rule (cont.)

Now Φ is precisely the TF potential corresponding to ‘nuclear charges’ at $x_i^0, \ldots x_n^0$. Hence $\Phi > 0$ on $\mathcal{B}$ and zero on the boundary $\partial\mathcal{B}$.

Proof of the exclusion rule (cont.)

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The first two terms in R are harmonic on $\mathcal{B}$ when taken together. (The Laplacian applied to the first term gives 2π and to the second term -2π on $\mathcal{B}$.) The other terms are superharmonic on $\mathcal{B}$. Thus, R takes its minimum on the boundary, so there is a $\hat{x} \in \partial\mathcal{B}$ with $R(x) \geq R(\hat{x})$. On the other hand, $\Phi(x) > 0 = \Phi(\hat{x})$ so $F(x) > F(\hat{x})$.

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Altogether we have shown that in a minimizing configuration $(x_1^0, \dots, x_N^0)$ of H, no x_i^0 can lie in any TF set $\mathcal{B}$ defined by other points in the configuration. □

A density bound

The essential fact needed for the proof of our Main Theorem is the following density bound for configurations satisfying the exclusion rule:

Proposition (Exclusion rule implies a density bound)

For $R > 0$ let $n(R)$ denote the maximum number of nuclei that a ball $B(R)$ of radius R can accomodate while respecting the exclusion rule, i.e., such that

$\{x_1, \ldots, x_n\} \subset B(R)$ implies $x_i \notin \mathcal{B}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Then

$$\limsup_{R \to \infty} \frac{n(R)}{\pi R^2} \leq 1.$$

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Then

$$\limsup_{R\to\infty} \frac{n(R)}{\pi R^2} \leq 1.$$

The proof is somewhat tricky. It is indirect and makes use of bounds on the gradient of Φ^{TF} and of Newton's theorem for its circular averages.

Consequences for $\mathcal{H}(Z)$

After scaling,

$$x \to z = \sqrt{\frac{\pi \ell}{N}}\, x,$$

the density bound applies to the Hamiltonian

$$\mathcal{H}(Z) = \sum_{j=1}^{N} |z_j|^2 + \frac{2\ell}{N} \sum_{1 \leq i < j \leq N} \log \frac{1}{|z_i - z_j|} + W(Z).$$

Consequences for $\mathcal{H}(Z)$

After scaling,

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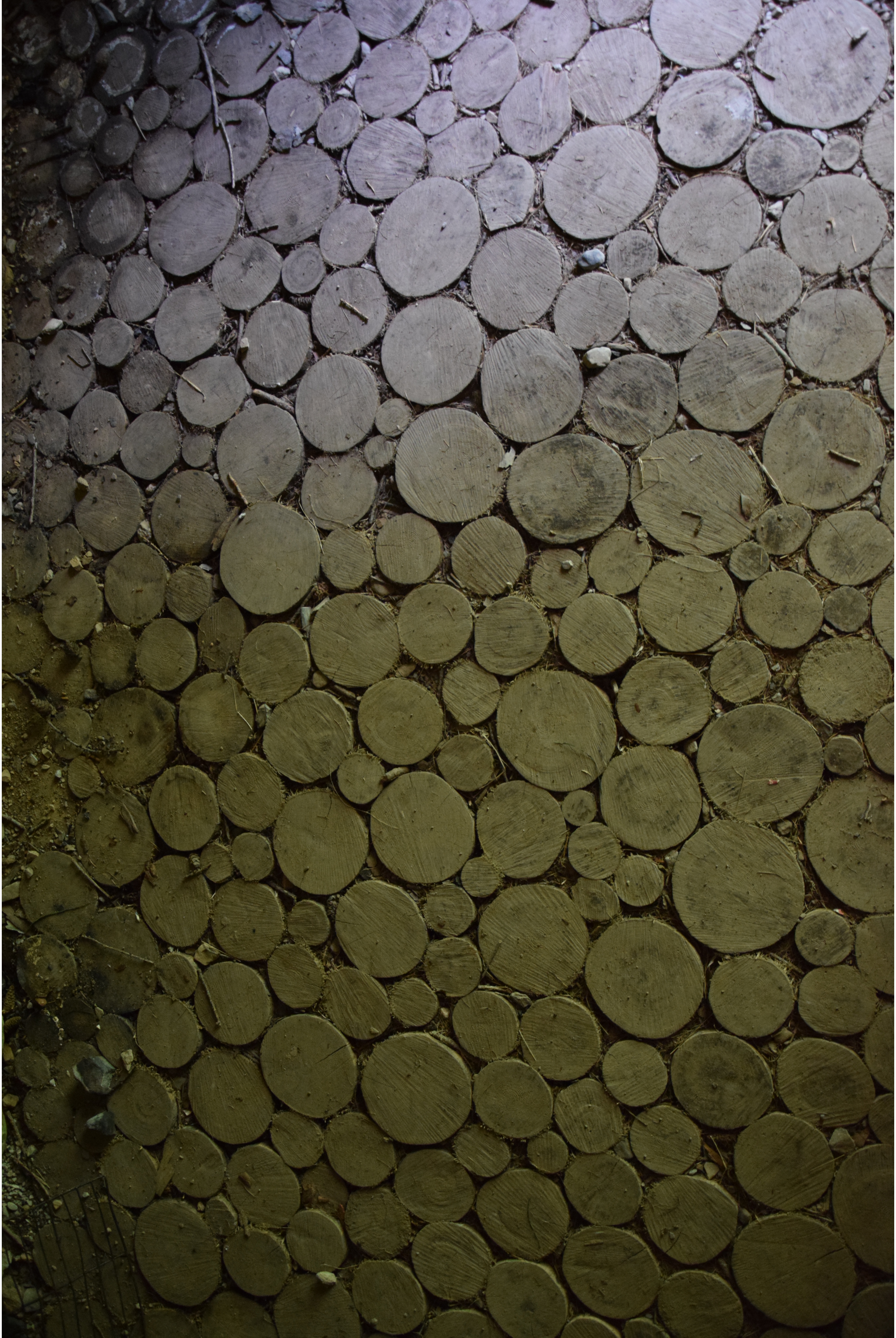
Using the “cheese theorem” one shows that for a minimizing configuration $Z^0 = (z_1^0, \dots, z_N^0)$ the empirical measure

$$\rho^0(z) = \frac{1}{N} \sum_{j=1}^{N} \delta(z - z_i^0)$$

is approximated in the weak sense by by an absolutely continuous distribution $\tilde{\rho}^0$ of integral 1 satisfying the bound

$$\tilde{\rho}^0(z) \le \frac{1}{\pi\ell}(1 + o(1)).$$



The perturbed Hamiltonian

To prove the Main Theorem (energy lower bound in terms of the 'bathtub' energy) we have to consider a perturbed Hamiltonian

$$\mathcal{H}^{\varepsilon}(Z) = \mathcal{H}(Z) + \varepsilon \sum_{i=1}^{N} U(z_i)$$

with $\varepsilon > 0$ and $U \in C^2(\mathbb{R}^2)$ of compact support.

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with $\varepsilon > 0$ and $U \in C^2(\mathbb{R}^2)$ of compact support.

We claim that if Z^{ε} is a minimizing configuration of $\mathcal{H}^{\varepsilon}$, then the empirical measure

$$\rho^{\varepsilon}(z) = \frac{1}{N} \sum_{j=1}^{N} \delta(z - z_i^{\varepsilon})$$

is approximated in the weak sense by by a continuous distribution $\tilde{\rho}^{\varepsilon}$ of integral 1 satisfying the modified bound

$$\tilde{\rho}^{\varepsilon}(z) \leq \frac{1}{\pi \ell}(1 + o(1))(1 + \tfrac{1}{4}\varepsilon \|\Delta U\|_{\infty}).$$

The perturbed Hamiltonian (cont.)

The proof of the last statement is essentially the same as for the unperturbed Hamiltonian. We add and subtract

$$\frac{\varepsilon}{4}\sum_{i=1}^{N}\|\Delta U\|_\infty |z_i|^2$$

and use that $U(z) - \frac{1}{4}\|\Delta U\|_\infty |z|^2$ is superharmonic, so that

$$\varepsilon\sum_{i=1}^{N}\left(U(z_i) - \tfrac{1}{4}\|\Delta U\|_\infty |z_i|^2\right)$$

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The change of $|z|^2$ to $\left(1 + \frac{\varepsilon}{4}\|\Delta U\|_\infty\right)|z|^2$ has the same effect as dividing ℓ by $\left(1 + \frac{\varepsilon}{4}\|\Delta U\|_\infty\right)$.

Proof of the Main Theorem

The free energy of the perturbed Hamiltonian is

$$F_N^\varepsilon := \inf \left\{ \mathcal{F}_N^\varepsilon[\mu], \quad \mu \in \mathcal{P}(\mathbb{R}^{2N}) \right\}$$

where the free energy functional on the space $\mathcal{P}(\mathbb{R}^{2N})$ of probability measures on $\mathbb{R}^2$ is

$$\mathcal{F}_N^\varepsilon[\mu] := \int_{\mathbb{R}^{2N}} \mathcal{H}^\varepsilon(Z)\mu(Z)dZ + N^{-1} \int_{\mathbb{R}^{2N}} \mu \log \mu$$

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Formally, $\int U\mu^{(1)}$ is the derivative at $\varepsilon = 0$ of the free energy. The Main Theorem is proved via estimates on F_N^ε. The entropic part tends to zero as $N \to \infty$ so $\int U\mu^{(1)}$ is essentially the integral of U against the empirical measure for a minimizing configuration of $\mathcal{H}^\varepsilon(Z)$.

Conclusions and outlook

- We have derived a rigorous density bound showing that the Laughlin wave function behaves as an incompressible liquid whose response to perturbations by external fields is very rigid.

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Conclusions and outlook

- We have derived a rigorous density bound showing that the Laughlin wave function behaves as an incompressible liquid whose response to perturbations by external fields is very rigid.
- The method of proof is based on the study of a 2D Thomas Fermi model of a special kind. The method has potential applications for 2D Coulomb systems and in random matrix theory.
- For certain radial potentials, upper bounds for the energy have also been derived, using trial functions of the form

$$\prod_{i=1}^{N} f(z_i)\Psi^{(\ell)}_{\mathrm{Laugh}}(z_1,\ldots,z_N).$$

An important challenge is to prove, for all reasonable external potentials, that the ground state in the extended Laughlin phase always has this form.