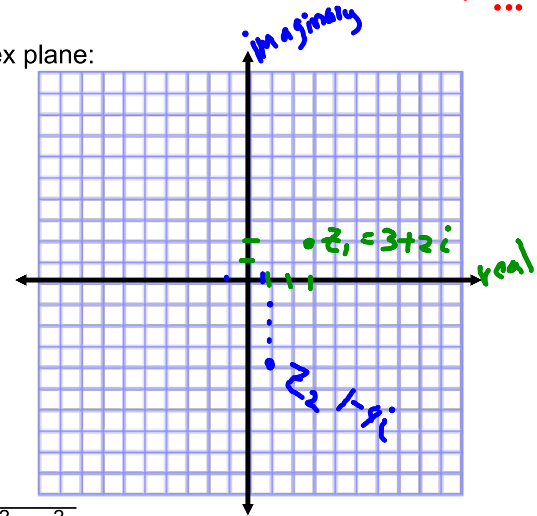
3.5 Trigonometric Form of Complex Numbers

- Plot complex numbers in a complex plane.
- Determine the modulus and argument of complex numbers and write them in trigonometric form.
- Multiply and divide two complex numbers in trigonometric form.
- Use DeMoivre's Theorem to find powers of complex numbers.
- Determine the nth roots of complex numbers.
- What is the square root of i? Are there more than one of them?

Review : What is i ?

$i = \sqrt{-1}$ $\quad i^2 = -1$ $\quad i^3 = -i$ $\quad i^4 = 1$ $\quad i^5 = i$ $\quad i^6 = -1$ ...

Complex plane:



Rectangular form of a complex number: a + bi

real imaginary

$z_1 = 3+2i$

$z_2 = 1-4i$

Absolute value of a complex number: $|a+bi| = \sqrt{a^2 + b^2}$

$|z_1| = \sqrt{9+4} = \sqrt{13}$

$|z_2| = \sqrt{1+16} = \sqrt{17}$

Add two complex numbers: $z_1 + z_2 = 3+2i + 1-4i$

$4-2i$

Multiply two complex numbers: $z_1 \cdot z_2 = (3+2i)(1-4i)$

$3 - \underbrace{12i + 2i}_{} - 8i^2$ $\quad -8(-1)$

$11 - 10i$

Trigonometric form of a complex number.

$z = a + bi$ becomes $z = r(\cos\theta + i\sin\theta)$

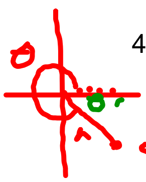
modulus

$r = |z|$ and the reference angle, θ is given by $\tan\theta' = |b/a|$

Note that it is up to you to make sure θ is in the correct quadrant.

argument

Example: Put these complex numbers in Trigonometric form.



$4 - 4i$

$r = \sqrt{16+16}$
$= \sqrt{32}$
$= 4\sqrt{2}$

$\theta' = \tan^{-1}\left|\frac{-4}{4}\right|$
$\tan^{-1}|1| = \frac{\pi}{4}$

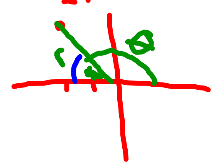
argument θ $\frac{7\pi}{4}$

$4-4i =$

$4\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$-2 + 3i$

-2+3i



$r = \sqrt{4+9} = \sqrt{13}$

$\theta' = \tan^{-1}\frac{3}{2} = 56.3°$

$\theta = 180° - 56.3°$
$123.7°$

$\sqrt{13}(\cos 123.7° + i\sin 123.7°)$

$-2 + 3i$

Writing a complex number in standard form:

Example: Write each of these numbers in a + bi form.

$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$

$\sqrt{2}\,(\cos 2\pi/3 + i \sin 2\pi/3)$

$\sqrt{2}\left(-\frac{1}{2}\right) + \sqrt{2}\, i\, \frac{\sqrt{3}}{2}$

$-\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2} i$

$20\,(\cos 75^{\circ} + i \sin 75^{\circ})$

$5.2 + 19.3i$

Multiplying and dividing two complex numbers in trigonometric form:

$z_1 z_2 = r_1 r_2(\cos(ø_1+ø_2) + i \sin(ø_1+ø_2))$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(ø_1 - ø_2) + i \sin(ø_1 - ø_2))$

$z_1 = 3(\cos 120° + i \sin 120°)$ $z_2 = 12 (\cos 45° + i \sin 45°)$

To multiply two complex numbers, you multiply the moduli and add the arguments.

$z_1 \cdot z_2 = 36(\cos \underline{165}° + i \sin 165°)$

$0 < \theta < 360$

To divide two complex numbers, you divide the moduli and subtract the arguments.

$\frac{z_1}{z_2} = \frac{3}{12}(\cos 75° + i \sin 75°)$

$\frac{1}{4}(\cos 75° + i \sin 75°)$

Please note that you must be sure your that in your answer
r is positive and 0< θ <360º .

$z_1 z_2 = r_1 r_2(\cos(ø_1+ø_2) + i \sin(ø_1+ø_2))$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(ø_1 - ø_2) + i \sin(ø_1 - ø_2))$

Here is an example. Find the product and quotient of these two complex numbers.

z_1 = 3(cos 150º + i sin 150º) and z_2 =12 (cos 275º + i sin 275º)

$z_1 \cdot z_2 = 36(\cos 425 + i \sin 425°)$

$= 36(\cos 65° + i \sin 65°)$

$\frac{z_1}{z_2} = \frac{3}{12}(\cos(-125°) + i \sin(-125°))$

$= \frac{1}{4}(\cos 235° + i \sin 235°)$

Powers of complex numbers

DeMoivre's Theorem: If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$z^n = r^n(\cos n\theta + i\sin n\theta)$

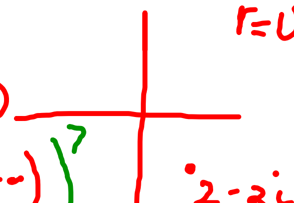
$z^3 = r(\cos\theta + i\sin\theta)\cdot r(\cos\theta + i\sin\theta)\cdot r(\cos\theta + i\sin\theta)$

$= r^3(\cos 3\theta + i\sin 3\theta)$

$r = \sqrt{4+4} = \sqrt{8}$

$2\sqrt{2}$

Example: Use DeMoivre's Theorem to find $(2-2i)^7$



$z^7 = \left(2\sqrt{2}(\cos 315^\circ + i\sin 315^\circ)\right)^7$

$\leftarrow (2\sqrt{2})^7(\cos 2205^\circ + i\sin 2205^\circ)$

$2^7 \cdot 2^{7/2}$ $\quad 1024\sqrt{2}(\cos 45^\circ + i\sin 45^\circ)$

$2^{10}\cdot\sqrt{2}$ $\quad 1024\sqrt{2}\left(\frac{\sqrt{2}}{2}\right) + 1024\sqrt{2}\, i\left(\frac{\sqrt{2}}{2}\right)$

$1024\sqrt{2}$ $\quad 1024 + 1024i$

Roots of complex numbers

Every number has two square roots.

The square roots of 16 are: ± 4

The square roots of 24 are: $\pm\sqrt{24} = \pm 2\sqrt{6}$

The square roots of -81 are: $\pm\sqrt{-81} = \pm\sqrt{81}\cdot\sqrt{-1} = \pm 9i$

The square roots of -75 are: $\pm\sqrt{75} = \pm 5\sqrt{3}\,i$

Likewise, every number has three cube roots, four fourth roots, etc. (over the complex number system.)

So if we want to find the four fourth roots of 16 we solve this equation.

$x^4 = 16$

$x^2 = \pm 4$

$x^2 = 4 \qquad x^2 = -4$

$x = \pm 2 \qquad x = \pm\sqrt{-4}$

$= \pm i\sqrt{4}$

$\pm 2i$

$\{2, -2, 2i, -2i\}$

$\rightarrow \quad x^4 - 16 = 0$

$(x^2-4)(x^2+4) = 0$

$(x-2)(x+2)(x^2+4) = 0$

$2, -2$

$x^2 + 4 = 0$

$x^2 = -4$

$x = \pm\sqrt{-4} = \pm 2i$

If we solve $x^6 - 1 = 0$ we can do some fancy factoring to get six roots.

Do you remember how to factor the sum/difference of two cubes?

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

$$(x^3-1)(x^3+1) = 0$$

$$(x-1)(x^2+x+1)(x+1)(x^2-x+1) = 0$$

$$1, -1, \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$x^2 + x + 1 = 0$$

$$\frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2}$$

$$x^2 - x + 1 = 0$$

$$x = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\,i}{2}$$

Later we will solve this using a variation of DeMoivre's Theorem.

We can extend DeMoivre's Theorem for roots as well as powers.

$z = r(\cos\theta + i\sin\theta)$ has n distinct nth roots.

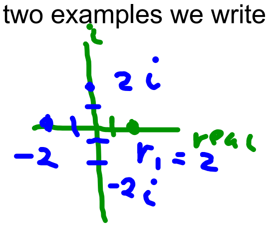
The first is $\sqrt[n]{r}\,(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$

and the others are found by adding 360º/n or $2\pi/n$ n-1 times to the angle of the first answer.

Thus for the previous two examples we write:

$x^4 = 16$

$\frac{360^\circ}{4} = 90^\circ$



$x^6 - 1 = 0$

We will do this on the next page.

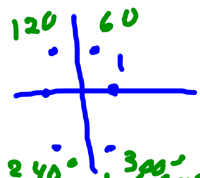
Two more:

Find the three cube roots of -8.

Find the five fifth roots of unity (1).

Now to solve the previous problem, $x^6 - 1 = 0$, we can use this theorem.

Start with $x^6 = 1$ We are looking for the six sixth roots of unity (1)



$$\frac{360^\circ}{6} = 60^\circ$$

$$1(\cos 0^\circ + i \sin 0^\circ) = 1$$
$$1(\cos 60^\circ + i \sin 60^\circ)$$
$$1(\cos 120^\circ + i \sin 120^\circ)$$
$$1(\cos 180^\circ + i \sin 180^\circ)$$
$$\vdots$$
$$1(\cos 240^\circ + i \sin 240^\circ)$$
$$1(\cos 300^\circ + i \sin 300^\circ)$$

These are the six sixth roots of 1.

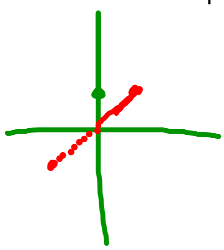
If you put them in rectangular form you will have:

$1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1$

$-\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$

The same ones we got on page 9 by factoring.

Finally we can answer the question: What are the two square roots of i ?



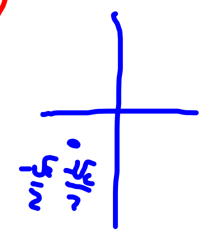
$0+1i$

$$\left(1(\cos 90^\circ + i\sin 90^\circ)\right)^{\frac{1}{2}}$$

$\frac{360}{2}=180$

$\sqrt{1}(\cos 45^\circ + i\sin 45^\circ)$

$\sqrt{1}(\cos 225^\circ + i\sin 225^\circ)$



$1\,\frac{\sqrt{2}}{2} + 1i\frac{\sqrt{2}}{2}$

$\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

In summary ~ Powers and roots of a complex number in trigonometric form:

$$z = 3(\cos 115^\circ + i \sin 115^\circ)$$

$z^n = r^n(\cos(n\phi) + i \sin(n\phi))$

$z^{1/n} = \sqrt[n]{r}\,(\cos(\phi/n) + i \sin(\phi/n))$
for the first root, with others 360º/n apart.

The cube of z (z to the third power):

$$z^3 = 3^3 (\cos 345^\circ + i \sin 345^\circ)$$
$$= 27(\cos 345^\circ + i \sin 345^\circ)$$

The five fifth roots of z:

$$z^{1/5} = \sqrt[5]{3}\,(\cos 23^\circ + i \sin 23^\circ)$$
$$= \sqrt[5]{3}\,(\cos 95^\circ + i \sin 95^\circ)$$
$$= \sqrt[5]{3}\,(\cos 167^\circ + i \sin 167^\circ)$$
$$= \sqrt[5]{3}\,(\cos 239^\circ + i \sin 239^\circ)$$
$$= \sqrt[5]{3}\,(\cos 311^\circ + i \sin 311^\circ)$$

first $\theta \rightarrow \frac{115^\circ}{5} = 23^\circ$

Second $\theta \rightarrow 23^\circ + \frac{360}{5}$
$= 23^\circ + 72^\circ$
$= 95^\circ$